DACC Simulation: A Fair? Two-Sided Matching Algorithm

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Introduction

The men-proposing Deferred Acceptance (DA) algorithm of Gale and Shapley [3] shows the existence of a stable matching in any marriage problem. Stability has proven to be important in applications using extensions of men-proposing DA. For instance, in school choice [1], matching doctors to residencies [4], and more recently matching cadets with branches of the U.S. military [5]. In his paper, "Deferred Acceptance With Compensation Chains", Piotr Dworczak provides an algorithm, Deferred Acceptance With Compensation Chains (DACC), with an interesting result: "A matching is stable if and only if it is the outcome of a DACC algorithm" [2]. Moreover, Dworczak states that DACC could be an attractive algorithm when the market designer is concerned about procedural fairness, with reference to Klaus and Klijn, 2006 [6]. It is not clear why a market designer would be concerned with this definition of "procedural fairness" that depends on intermediate steps of the algorithm; rather we would suspect that a market designer would care about the distribution of outcomes given a data generating process for the economy, if the market designer cares about any *ex-ante* notion of fairness at all.

Hence, I use DACC as a case study for "procedurally-fair" algorithms. I simulate matching markets with random preferences, and I run DACC(Φ) repeatedly using different orderings Φ. For each generation of Φ, I report whether DACC(Φ) converges to the men-optimal stable matching, the women-optimal stable matching, or an intermediate one. The result is that, under my preference structure and randomly drawing the ordering Φ, if there is a large gap between the number of agents on each side, DACC(Φ) tends to pick the stable matching optimal for the side of the market with more agents, if multiple stable matchings exist. In addition, markets with a similar number of men and women tend to have multiple stable matchings, often including at least three, whereas if there is a large discrepency between the number of men and women, there is often only one stable matching.

DACC Algorithm

In this algorithm agents make offers one at a time in a pre-specified order. Agents who receive offers can accept if the offer is better than their current match or reject if not. Because agents from both sides are allowed to propose, unlike in men-prosing DA, in DACC it is possible for agents to renege on their proposal—one agent i may propose to an agent j, but upon receiving a better offer i divorces j. As Dworczak describes, this reneging on a proposal, called deception, initiates to compensation chain (CC). These CC's serve the purpose of guaranteeing convergence of DACC to a matching.

DACC works as follows:

1. Agents (from either side) propose in a pre-specified order Φ
2. When proposing, agents make a proposal to best available partner
3. When receiving, agents (tentatively) accept an offer if the proposer is preferred to their current partner
4. Partners become unavailable to i when they reject or divorce i. Partners become

available to i when they propose to i.

5. When an agent i is divorced by j who previously proposed to i, we say that i is *deceived*, and we *compensate* i by allowing i to make an additional offer out of turn.

A few points about DACC should be help to understand the following sections:

- The algorithm depends on a pre-specified ordering Φ of agents.
- Given any stable matching μ, there exists ordering Φ such that DACC(Φ) (DACC run using ordering Φ) converges in finite time to μ. Hence, if Φ is constructed randomly, DACC(Φ) converges to each stable matching with positive probability.

Preference Generation and DACC Implentation

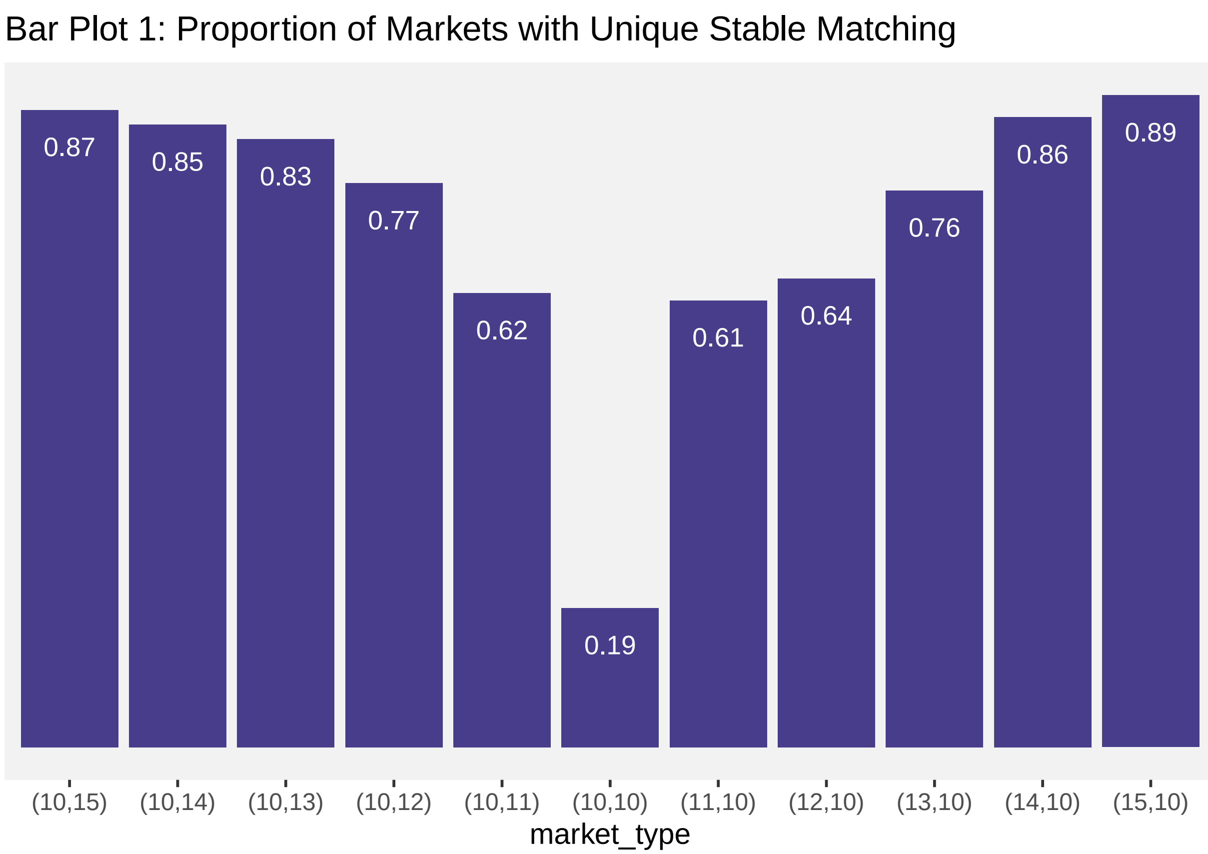
I generate eleven types of market with 10 men and 15 women, 10 men and 14 women, ..., 10 men and 10 women, 11 men and 10 women, ..., 15 men and 10 women, respectively. For each market type, I generate random preferences for the agents 100 times, where I require that agents find all agents on the other side acceptable. Thus, in total I generate 1100 markets. For each market, I randomly draw agents from the men and women to construct an ordering of sufficient length Φ, and I run the Deferred Acceptance with Compensation Chains algorithm, and record the result DACC(Φ) $= \mu$.

Taking a step back, some of the 1100 markets I have generated have a unique stable matching. To check whether this is the case, before running DACC for a particular market L, I run men-proposing DA (M-DA) and women-proposing DA (W-DA) on that market L. If M-DA and W-DA result in different outcomes, there is not a unique stable matching, so I proceed to calculate 100 orderings Φ for that market, and I record the results DACC(Φ) as: μ_M, if DACC(Φ) results in the same outcome as M-DA; μ_W if DACC(Φ) results in the same outcome as W-DA; and μ^* if DACC(Φ) results in a matching besides the men- and women-prosing DA outcomes, an intermediate one.

If M-DA and W-DA result in the same matching, the market has a unique stable matching, so DACC always converge to the same result, regardless of the ordering Φ. In this case, for the sake of efficiency I do not generate the 100 orderings Φ or run DACC.

Unique Stable Matchings

For each of the 11 market types, I record the proportion of markets with a unique stable matching by market type.



As shown, the proportion significantly decreases as the number of men and women in the market grow nearer. It is difficult to diagnose why this is true.

Detailed DACC Results

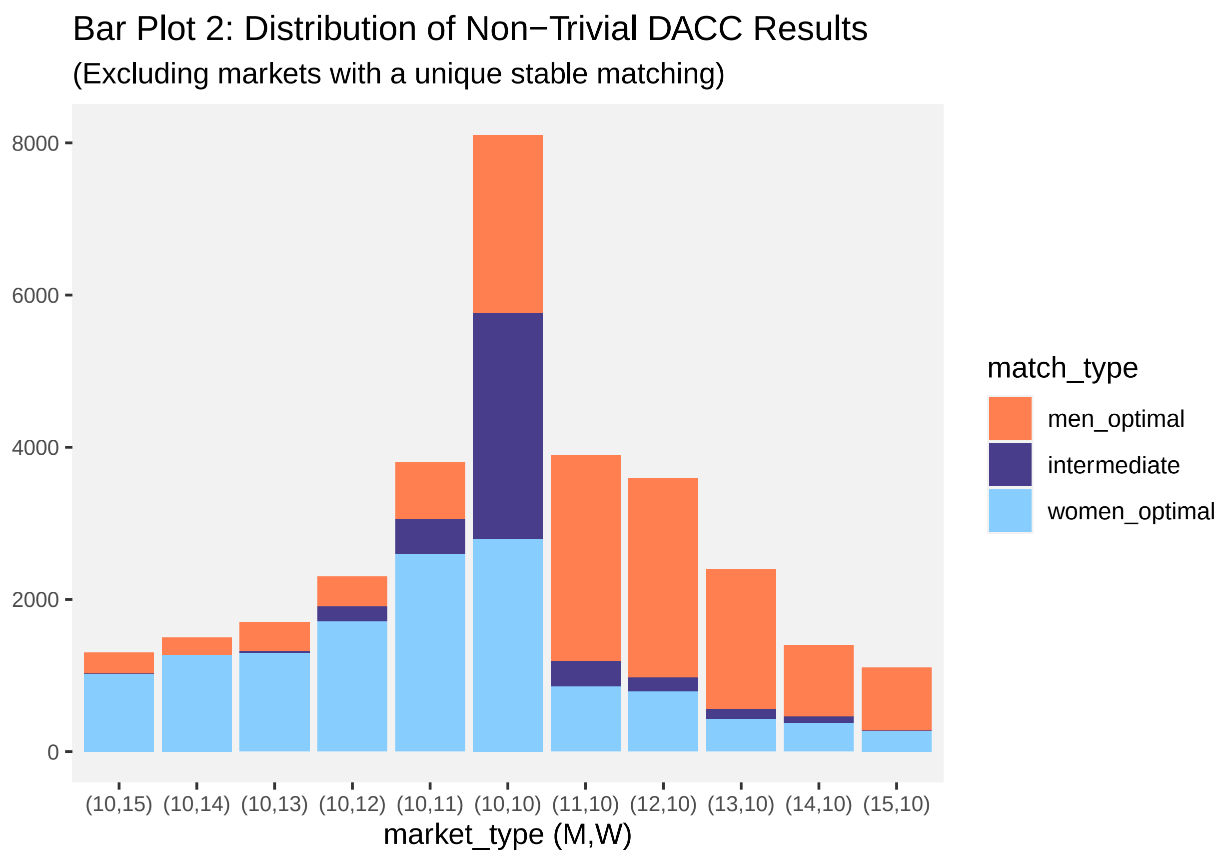
Table 1: DACC Matching Results for 25/100 Markets per Type (M,W)

	(M,W) = (10,15)			(M,W) = (10,10)			(M,W) = (15,10)		
	μ_M	μ^*	μ_W	μ_M	μ^*	μ_W	μ_M	μ^*	μ_W
1	100	0	100	28	49	23	100	0	100
2	100	0	100	16	0	84	100	0	100
3	6	0	94	0	13	87	100	0	100
4	100	0	100	11	0	89	100	0	100
5	100	0	100	10	0	90	100	0	100
6	100	0	100	9	25	66	77	0	23
7	100	0	100	25	0	75	100	0	100
8	100	0	100	100	0	100	100	0	100
9	19	0	81	2	89	9	100	0	100
10	100	0	100	81	0	19	100	0	100
11	100	0	100	10	77	13	100	0	100
12	100	0	100	21	79	0	100	0	100
13	100	0	100	38	43	19	100	0	100
14	100	0	100	95	0	5	100	0	100
15	100	0	100	2	0	98	100	0	100
16	100	0	100	100	0	100	100	0	100
17	24	0	76	38	36	26	100	0	100
18	100	0	100	30	0	70	100	0	100
19	100	0	100	23	69	8	100	0	100
20	100	0	100	60	0	40	100	0	100
21	100	0	100	44	0	56	100	0	100
22	100	0	100	10	73	17	100	0	100
23	100	0	100	2	49	49	100	0	100
24	100	0	100	100	0	100	100	0	100
25	100	0	100	1	93	6	100	0	100

[1] Row values black if market has unique stable matching.

The first column numbers the markets for reference. Each of the three tables give the DACC results for three of the eleven market types, which differ by number of men M and number of women W.

In markets with 10 men and 15 women, DACC favors the men-optimal matching. This is intuitive, as DACC is the same as men-proposing DA when only men are in Φ (and when women are also excluded from the stopping criterion). Since there are mostly men in this type of market, the ordering Φ tends to include men more often than women, so DACC(Φ) is closer to men-proposing DA than it is to women-proposing DA. When there are 15 men and 10 women, the results are symmetric, this time favoring the women-optimal stable matching μ_W. When there are 10 men and 10 women, the incidence of markets with a unique stable matching drastically decreases, a stark difference from the other two market types shown in the table. This trend is also apparent in Bar Plot 1. Additionally, in these markets DACC(Φ) rarely converges to either extreme stable matching.



The final figure reveals the distribution of DACC results excluding markets with a unique stable matching. Because there are 100 drawings of Φ for each of the 100 markets in each market type, each bar is bounded by 10,000. When a market has significantly more men than women, or women than men, DACC(Φ) tends to converge to the best stable matching for the side of the market with more agents, and rarely converges to an intermediate matching, if one exists. When a market has a similar number of men and women, DACC(Φ) tends to converge to an intermediate stable matching, the men-optimal stable matching, and the women-optimal stable matching with similar likelihood. See Table 2 in the Appendix for exact proportions of match type, for each market type.

Linear Programming

A *matching* is a one-to-one mapping μ from the set of men and women $M \cup W$ to itself, such that:

- $\mu(m) = w$ if and only if $\mu(w) = m$, in which case m is matched to w.
- If $\mu(m)$ is not in W, then $\mu(m) = m$, in which case m is unmatched.
- If $\mu(w)$ is not in M, then $\mu(w) = w$, in which case w is unmatched.

The *incidence vector* of a matching μ is a vector $x \in \{0,1\}^{|M|\cdot|W|}$ such that $x_{m,w} = 1$ if $\mu(m) = w$ and $x_{m,w} = 0$ otherwise.

According to Roth, Alvin E., Uriel G. Rothblum and John H. Vande Vate (1993) [8], the stable matchings of a market are precisely the *integer* solutions of the following set of linear equations:

1. $\sum_j x_{m,j} \leq 1$ for all m in M,
2. $\sum_i x_{i,w} \leq 1$ for all w in W,
3. $x_{m,w} \geq 0$ for each $(m,w) \in M \times W$,
4. $\sum\limits_{\{j \in W:\, j \succ_m w\}} x_{m,j} + \sum\limits_{\{i \in M:\, i \succ_w m\}} x_{i,w} + x_{m,w} \geq 1$ for all $m \in M$ and $w \in W$.

Moreover, if we let C be the convex polyhedron of solutions to the linear constaints (1)-(4), then the integer points of C are precisely its extreme points. Thus, we may use a linear programming routine to find the vertices of the convex polytope, and span the entire set of stable matchings. Knowing the full set of stable matching will allow us to better assess the outcomes of DACC, because we will know for certain whether intermediate matchings exist. In particular, we may focus on markets known to have intermediate stable matchings, and report what proportion of our simulations converge to such matchings.

Bootstrap % Accuracy of Market Structure Estimates

In markets that have at least one intermediate matching, on average what % of the intermediate matchings does our simulation process find? To answer this question, we will look at only a few heuristic markets, which we know to have at least one intermediate matching. We will use a cumbersome, and hence slow, algorithm to span all of the stable matching of these markets. Then we will run our simulation process repeatedly to see what proportion of the intermeidate stable matchings our simulations find.

Conclusion

Using DACC, any stable matching can be reached by choice of Φ. However, depending on the structure of the economy, i.e. number of agents on each side and the preference-generating process, DACC with random uniform construction of Φ can tend towards particular stable matchings unevenly. It would likely be a mistake to consider the label of DACC as "procedurally fair" as a desirable feature of the algorithm for a market designer. Moreover, the *ex-ante* notion of procedural fairness does seem to be useful for a market designer concerned with outcomes. Perhaps a better *ex-ante* notion of fairness in matching markets should only apply to algorithms that would pass such an analysis as this with results that better fit our intuition of fairness. Constraining an *ex-ante* notion of fairness to a particular method of generating the market may serve as a better alternative.

References

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Appendix

Table 2: Table Companion for Bar Plot 2

market_type	prop_m	prop_int	prop_w
(10,15)	0.21	0.00	0.79
(10,14)	0.15	0.00	0.85
(10,13)	0.22	0.02	0.76
(10,12)	0.17	0.09	0.74
(10,11)	0.19	0.12	0.68
(10,10)	0.29	0.37	0.35
(11,10)	0.69	0.09	0.22
(12,10)	0.73	0.05	0.22
(13,10)	0.77	0.06	0.18
(14,10)	0.67	0.06	0.27
(15,10)	0.75	0.00	0.25