Invariant Spacetime Interval

$$-\Delta\tau^2 = \Delta\sigma^2 = -c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = -c^2\Delta t'^2 + \Delta x'^2 + \Delta y'^2 + \Delta z'^2$$

timelike interval ($\Delta\tau$ = proper time): $\Delta\tau^2 > 0 \quad \Delta\sigma^2 < 0$
lightlike interval (null separation): $\Delta\tau^2 = \Delta\sigma^2 = 0$
spacelike interval ($\Delta\sigma$ = proper distance): $\Delta\tau^2 < 0 \quad \Delta\sigma^2 > 0$

$$\gamma^2 = \left(1 - \frac{v^2}{c^2}\right)^{-1}$$

$$\Delta t' = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right) \qquad \Delta x' = \gamma(\Delta x - v\Delta t) \qquad \Delta y' = \Delta y \qquad \Delta z' = \Delta z$$

$$-c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = -c^2\gamma^2\left(\Delta t - \frac{v\Delta x}{c^2}\right)^2 + \gamma^2(\Delta x - v\Delta t)^2 + \Delta y^2 + \Delta z^2$$

$$-c^2\Delta t^2 + \Delta x^2 = -c^2\gamma^2\left(\Delta t^2 - 2\frac{v\Delta x\Delta t}{c^2} + \frac{v^2\Delta x^2}{c^4}\right) + \gamma^2(\Delta x^2 - 2v\Delta t\Delta x + v^2\Delta t^2)$$

$$-c^2\Delta t^2 + \Delta x^2 = \gamma^2\left(-c^2\Delta t^2 + 2v\Delta x\Delta t - \frac{v^2\Delta x^2}{c^2} + \Delta x^2 - 2v\Delta t\Delta x + v^2\Delta t^2\right)$$

$$-c^2\Delta t^2 + \Delta x^2 = \left(1 - \frac{v^2}{c^2}\right)^{-1}\left[-c^2\Delta t^2\left(1 - \frac{v^2}{c^2}\right) + \Delta x^2\left(1 - \frac{v^2}{c^2}\right)\right]$$

$$-c^2\Delta t^2 + \Delta x^2 = -c^2\Delta t^2 + \Delta x^2 \quad \blacksquare$$

$$\Delta t = \gamma\left(\Delta t' + \frac{v\Delta x'}{c^2}\right) \qquad \Delta x = \gamma(\Delta x' + v\Delta t') \qquad \Delta y = \Delta y' \qquad \Delta z = \Delta z'$$

$$-c^2\gamma^2\left(\Delta t' + \frac{v\Delta x'}{c^2}\right)^2 + \gamma^2(\Delta x' + v\Delta t')^2 + \Delta y'^2 + \Delta z'^2 = -c^2\Delta t'^2 + \Delta x'^2 + \Delta y'^2 + \Delta z'^2$$

$$-c^2\gamma^2\left(\Delta t'^2 + 2\frac{v\Delta x'\Delta t'}{c^2} + \frac{v^2\Delta x'^2}{c^4}\right) + \gamma^2\left(\Delta x'^2 + 2v\Delta t'\Delta x' + v^2\Delta t'^2\right) = -c^2\Delta t'^2 + \Delta x'^2$$

$$\gamma^2\left(-c^2\Delta t'^2 - 2v\Delta x'\Delta t' - \frac{v^2\Delta x'^2}{c^2} + \Delta x'^2 + 2v\Delta t'\Delta x' + v^2\Delta t'^2\right) = -c^2\Delta t'^2 + \Delta x'^2$$

$$\left(1 - \frac{v^2}{c^2}\right)^{-1}\left[-c^2\Delta t'^2\left(1 - \frac{v^2}{c^2}\right) + \Delta x'^2\left(1 - \frac{v^2}{c^2}\right)\right] = -c^2\Delta t'^2 + \Delta x'^2$$

$$-c^2\Delta t'^2 + \Delta x'^2 = -c^2\Delta t'^2 + \Delta x'^2 \quad \blacksquare$$