Special Relativity

Lorentz-Einstein transformations between coordinate systems S and S' with S' moving in $+x$ direction at speed v relative to S

Lorentz factor: $\gamma \equiv (1 - v^2/c^2)^{-1/2}$

$$
\begin{aligned}
t &= \gamma(t' + vx'/c^2) \\
x &= \gamma(x' + vt') \\
y &= y' \\
z &= z'
\end{aligned}
\qquad
\begin{aligned}
t' &= \gamma(t - vx/c^2) \\
x' &= \gamma(x - vt) \\
y' &= y \\
z' &= z
\end{aligned}
$$

$$
\begin{aligned}
&\Delta x = \gamma(\Delta x' + v\Delta t') \\
&\Delta t = \gamma(\Delta t' + v\Delta x'/c^2) \\
&\Delta x/\Delta t = (\Delta x' + v\Delta t')/(\Delta t' + v\Delta x'/c^2) \\
&\Delta x' = 0 \Longrightarrow \Delta x/\Delta t = v
\end{aligned}
\qquad
\begin{aligned}
&\Delta x' = \gamma(\Delta x - v\Delta t) \\
&\Delta t' = \gamma(\Delta t - v\Delta x/c^2) \\
&\Delta x'/\Delta t' = (\Delta x - v\Delta t)/(\Delta t - v\Delta x/c^2) \\
&\Delta x = 0 \Longrightarrow \Delta x'/\Delta t' = -v
\end{aligned}
$$

S measures the speed of S' to be v, and S' measures the speed of S to be $-v$

Relativity of Time Dilation

S *duration*: $\Delta x = 0 \Longrightarrow \Delta t = \Delta t'/\gamma$

S' *duration*: $\Delta x' = 0 \Longrightarrow \Delta t' = \Delta t/\gamma$

Relativity of Length Contraction

S *length*: $\Delta t = 0 \Longrightarrow \Delta x = \Delta x'/\gamma$

S' *length*: $\Delta t' = 0 \Longrightarrow \Delta x' = \Delta x/\gamma$

S' clock is stationary in the S' frame with $\Delta x' = 0$, and moves at speed v relative to S. That means $\Delta x = \gamma(\Delta x' + v\Delta t') = v\Delta t$, so $\Delta x = \gamma v\Delta t' = v\Delta t$ which leaves $\Delta t' = \Delta t/\gamma$. The time between two events $\Delta t'$ as measured by the S' clock is smaller than the time duration Δt measured by the S clock by a factor of γ :

$$\Delta t' = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right) = \gamma\left(\Delta t - \frac{v^2\Delta t}{c^2}\right) = \gamma\Delta t\left(1 - \frac{v^2}{c^2}\right) = \gamma\Delta t\frac{1}{\gamma^2} = \frac{\Delta t}{\gamma}$$

As seen by S, the S' clock runs slower than the S clock by a factor of γ. As seen by S, any physical process in S' takes longer to complete by a factor of γ than in S (time dilation). As seen by S' any physical process in S takes longer to complete by a factor of γ than in S' (time dilation is symmetric in SR).

S' ruler oriented parallel to the x-axis moves at speed v. The distance between the end points of the ruler as measured by S is Δx with $\Delta t = 0$ between the two end-point events, which means

$$\Delta x' = \gamma(\Delta x - v\Delta t) = \gamma\Delta x$$

S' measures its co-moving ruler length to be bigger than what S measures it to be by a factor of γ. In other words, as seen by S the S' ruler is shorter than an identical S ruler by a factor of γ. Any object co-moving with S' is shortened in the direction of motion by a factor of γ relative to what its measured length is in S (Lorentz contraction). Any object co-moving with S is shortened in the direction of motion by a factor of γ relative to what its measured length is in S' (Lorentz contraction is symmetric in SR).

Invariant Spacetime Interval

$$-\Delta\tau^2 = \Delta\sigma^2 = -c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = -c^2\Delta t'^2 + \Delta x'^2 + \Delta y'^2 + \Delta z'^2$$

timelike interval ($\Delta\tau =$ proper time): $\Delta\tau^2 > 0 \quad \Delta\sigma^2 < 0$
lightlike interval (null separation): $\Delta\tau^2 = \Delta\sigma^2 = 0$
spacelike interval ($\Delta\sigma =$ proper distance): $\Delta\tau^2 < 0 \quad \Delta\sigma^2 > 0$

$$\gamma^2 = \left(1 - \frac{v^2}{c^2}\right)^{-1}$$

$$\Delta t' = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right) \qquad \Delta x' = \gamma(\Delta x - v\Delta t) \qquad \Delta y' = \Delta y \qquad \Delta z' = \Delta z$$

$$-c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = -c^2\gamma^2\left(\Delta t - \frac{v\Delta x}{c^2}\right)^2 + \gamma^2(\Delta x - v\Delta t)^2 + \Delta y^2 + \Delta z^2$$

$$-c^2\Delta t^2 + \Delta x^2 = -c^2\gamma^2\left(\Delta t^2 - 2\frac{v\Delta x\Delta t}{c^2} + \frac{v^2\Delta x^2}{c^4}\right) + \gamma^2(\Delta x^2 - 2v\Delta t\Delta x + v^2\Delta t^2)$$

$$-c^2\Delta t^2 + \Delta x^2 = \gamma^2\left(-c^2\Delta t^2 + 2v\Delta x\Delta t - \frac{v^2\Delta x^2}{c^2} + \Delta x^2 - 2v\Delta t\Delta x + v^2\Delta t^2\right)$$

$$-c^2\Delta t^2 + \Delta x^2 = \left(1 - \frac{v^2}{c^2}\right)^{-1}\left[-c^2\Delta t^2\left(1 - \frac{v^2}{c^2}\right) + \Delta x^2\left(1 - \frac{v^2}{c^2}\right)\right]$$

$$-c^2\Delta t^2 + \Delta x^2 = -c^2\Delta t^2 + \Delta x^2 \ \blacksquare$$

$$\Delta t = \gamma\left(\Delta t' + \frac{v\Delta x'}{c^2}\right) \qquad \Delta x = \gamma(\Delta x' + v\Delta t') \qquad \Delta y = \Delta y' \qquad \Delta z = \Delta z'$$

$$-c^2\gamma^2\left(\Delta t' + \frac{v\Delta x'}{c^2}\right)^2 + \gamma^2(\Delta x' + v\Delta t')^2 + \Delta y'^2 + \Delta z'^2 = -c^2\Delta t'^2 + \Delta x'^2 + \Delta y'^2 + \Delta z'^2$$

$$-c^2\gamma^2\left(\Delta t'^2 + 2\frac{v\Delta x'\Delta t'}{c^2} + \frac{v^2\Delta x'^2}{c^4}\right) + \gamma^2\left(\Delta x'^2 + 2v\Delta t'\Delta x' + v^2\Delta t'^2\right) = -c^2\Delta t'^2 + \Delta x'^2$$

$$\gamma^2\left(-c^2\Delta t'^2 - 2v\Delta x'\Delta t' - \frac{v^2\Delta x'^2}{c^2} + \Delta x'^2 + 2v\Delta t'\Delta x' + v^2\Delta t'^2\right) = -c^2\Delta t'^2 + \Delta x'^2$$

$$\left(1 - \frac{v^2}{c^2}\right)^{-1}\left[-c^2\Delta t'^2\left(1 - \frac{v^2}{c^2}\right) + \Delta x'^2\left(1 - \frac{v^2}{c^2}\right)\right] = -c^2\Delta t'^2 + \Delta x'^2$$

$$-c^2\Delta t'^2 + \Delta x'^2 = -c^2\Delta t'^2 + \Delta x'^2 \ \blacksquare$$

Transverse Light Bounce (*used to derive time dilation*)

Direction of motion of S' *wrt* S *at speed* v *is from source event to destination event*
Let mirror $M_\perp$ *be offset from source event perpendicular to direction of motion*
Let point P *be midway between source and destination events*
S *distance in direction of motion from source to* P: $v\Delta t$
S' (*proper*) *distance from source to mirror* $M_\perp$ *at time of source event*: $\Delta y' = c\Delta t'$
S *and* S' *transverse distances are equal* (*no length contraction*), *so*
S *distance from* P *to mirror* $M_\perp$ *at time of bounce*: $\Delta y' = c\Delta t'$
S *light path* (*diagonal*) *distance from source to mirror* $M_\perp$ *at time of bounce*: $\Delta y = c\Delta t$

$$\text{Pythagorean formula: } (c\Delta t)^2 = (v\Delta t)^2 + (c\Delta t')^2$$

$$c^2\Delta t^2 = v^2\Delta t^2 + c^2\Delta t'^2$$
$$\Delta t^2(c^2 - v^2) = c^2\Delta t'^2$$
$$\Delta t^2 = \frac{c^2}{c^2 - v^2}\Delta t'^2 = \left(1 - \frac{v^2}{c^2}\right)^{-1} \Delta t'^2$$
$$\Delta t = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \Delta t'$$

$$\Delta t = \gamma\Delta t' \quad \blacksquare$$

$\gamma > 1$: S *time duration* Δt *is greater* (*by a factor of* γ) *than* S' *time duration* $\Delta t'$
time dilation: S *sees* S' *clock ticks slower than* S *clock*

Direction of Motion Light Bounce (used with time dilation to derive length contraction)

Direction of motion of S' wrt S at speed v is from source event to destination event
Let mirror $M_\parallel$ be ahead of source event in direction of motion
Let $M_\perp$ and $M_\parallel$ be at the same S' (proper) distance from source event: $\Delta y' = \Delta x' = c\Delta t'$
Let Δx be S distance from source to $M_\parallel$ (at source event time)
Let Δt_1 be S time from source event to bounce event at $M_\parallel$
Let Δt_2 be S time from bounce at $M_\parallel$ to destination event
Let Δt be S time from source event to destination event: $\Delta t = 2\gamma\Delta t' = \Delta t_1 + \Delta t_2$

(here Δt is full roundtrip tick, $\Delta t'$ is halftick)

$$S \text{ distance from source event to bounce at } M_\parallel = \Delta x + v\Delta t_1 = c\Delta t_1$$
$$\Delta x = \Delta t_1(c - v)$$
$$\Delta t_1 = \Delta x/(c - v)$$

$$S \text{ distance from source event to destination event} = v\Delta t = c\Delta t_1 - c\Delta t_2$$
$$v\Delta t = c\Delta t_1 - c(\Delta t - \Delta t_1) = 2c\Delta t_1 - c\Delta t$$
$$\Delta t(c + v) = 2c\Delta t_1 = 2c\,\Delta x/(c - v)$$
$$\Delta t = 2\gamma\Delta t' = 2c\,\Delta x/(c^2 - v^2) = 2c^{-1}\,\Delta x\Big/\left(1 - \frac{v^2}{c^2}\right)$$
$$\Delta x = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}\left(1 - \frac{v^2}{c^2}\right)c\Delta t' = \left(1 - \frac{v^2}{c^2}\right)^{1/2}\Delta x'$$

$$\Delta x = \gamma^{-1}\Delta x'$$

$\gamma > 1$: S length Δx is less (by a factor of γ) than S' length $\Delta x'$
length contraction in direction of motion
S sees S' measuring rods shorter (in direction of motion) than S measuring rods

$\boldsymbol{Light\ Bounce\ at\ Angle\ from\ Direction\ of\ Motion}$ $(use\ time\ dilation\ and\ length\ contraction)$

$Direction\ of\ motion\ of\ S'\ wrt\ S\ at\ speed\ v\ is\ from\ source\ event\ to\ destination\ event$
$Let\ mirror\ M_\theta\ be\ offset\ from\ source\ event\ at\ S'\ angle\ \theta'\ to\ direction\ of\ motion$
$Let\ \Delta t'\ be\ S'\ time\ duration\ from\ source\ event\ to\ bounce\ at\ mirror\ M_\theta$
$Let\ M_\perp, M_\parallel\ and\ M_\theta\ be\ at\ the\ same\ S'(proper)\ distance\ from\ source\ event: r' = c\Delta t'$
$Let\ \Delta x'\ and\ \Delta y'\ be\ S'(proper)\ coordinates\ of\ M_\theta$

$$\Delta x' = r' \sin\theta' \qquad \Delta y' = r' \cos\theta' \qquad \tan\theta' = \sin\theta' / \cos\theta' = \Delta y'/\Delta x'$$
$$r'^2 = \Delta x'^2 + \Delta y'^2 = r'^2 \sin^2\theta' + r'^2 \cos^2\theta'$$

$$x' = \gamma(x - vt)$$

$$x' = (x - vt)\left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

$$(x')^2 = (x - vt)^2\left(1 - \frac{v^2}{c^2}\right)^{-1}$$

$$(x')^2\left(1 - \frac{v^2}{c^2}\right) = x^2 - 2vt + v^2t^2$$

$$(x')^2 - (x')^2\frac{v^2}{c^2} - x^2 + 2vt - v^2t^2 = 0$$

$$v^2t^2 + (x')^2\frac{v^2}{c^2} - 2vt + x^2 - (x')^2 = 0$$

$$v^2\left(t^2 + \frac{(x')^2}{c^2}\right) - 2vt + x^2 - (x')^2{}^{\;2} = 0$$

$$v = \frac{t \pm \sqrt{t^2 - \left(t^2 + \frac{(x')^2}{c^2}\right)(x^2 - (x')^2)}}{t^2 + \frac{(x')^2}{c^2}}$$

$$x = \gamma(x' + vt')$$

$$x = (x' + vt')\left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

$$x^2 = (x' + vt')^2\left(1 - \frac{v^2}{c^2}\right)^{-1}$$

$$x^2\left(1 - \frac{v^2}{c^2}\right) = (x')^2 + 2vt' + v^2(t')^2$$

$$x^2 - x^2\frac{v^2}{c^2} - (x')^2 - 2vt' - v^2(t')^2 = 0$$

$$v^2(t')^2 + x^2\frac{v^2}{c^2} + 2vt' + (x')^2 - x^2 = 0$$

$$v^2\left((t')^2 + \frac{x^2}{c^2}\right) + 2vt' + (x')^2 - x^2 = 0$$

$$v = \frac{-t' \pm \sqrt{(t')^2 - \left((t')^2 + \frac{x^2}{c^2}\right)((x')^2 - x^2)}}{(t')^2 + \frac{x^2}{c^2}}$$

$\frac{dx'}{dt'} = \gamma\left(\frac{dx}{dt'} - v\frac{dt}{dt'}\right)$	$\frac{dx}{dt} = \gamma\left(\frac{dx'}{dt} - v\frac{dt'}{dt}\right)$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \qquad x' = \gamma(x - vt) \qquad y' = y \qquad z' = z$$
$$t = \gamma\left(t' + \frac{vx'}{c^2}\right) \qquad x = \gamma(x' + vt') \qquad y = y' \qquad z = z'$$
$$\gamma \equiv \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

Expanding sphere in S' : $\qquad (r')^2 = (ct')^2 = (x')^2 + (y')^2 + (z')^2$
$$\left[c\gamma\left(t - \frac{vx}{c^2}\right)\right]^2 = [\gamma(x - vt)]^2 + y^2 + z^2$$
$$c^2\gamma^2\left(t - \frac{vx}{c^2}\right)^2 = \gamma^2(x - vt)^2 + y^2 + z^2$$
$$\left(ct - \frac{vx}{c}\right)^2 = (x - vt)^2 + y^2\left(1 - \frac{v^2}{c^2}\right) + z^2\left(1 - \frac{v^2}{c^2}\right)$$
$$c^2t^2 - 2tvx + \frac{v^2x^2}{c^2} = x^2 - 2xvt + v^2t^2 + y^2\left(1 - \frac{v^2}{c^2}\right) + z^2\left(1 - \frac{v^2}{c^2}\right)$$
$$t^2(c^2 - v^2) = x^2\left(1 - \frac{v^2}{c^2}\right) + y^2\left(1 - \frac{v^2}{c^2}\right) + z^2\left(1 - \frac{v^2}{c^2}\right)$$
$$c^2t^2\left(1 - \frac{v^2}{c^2}\right) = x^2\left(1 - \frac{v^2}{c^2}\right) + y^2\left(1 - \frac{v^2}{c^2}\right) + z^2\left(1 - \frac{v^2}{c^2}\right)$$

Expanding sphere in S: $\qquad r^2 = (ct)^2 = x^2 + y^2 + z^2$
$$c^2\left[\gamma\left(t' + \frac{vx'}{c^2}\right)\right]^2 = [\gamma(x' + vt')]^2 + (y')^2 + (z')^2$$
$$c^2\gamma^2\left(t' + \frac{vx'}{c^2}\right)^2 = \gamma^2(x' + vt')^2 + (y')^2 + (z')^2$$
$$\left(ct' + \frac{vx'}{c}\right)^2 = (x' + vt')^2 + (y')^2\left(1 - \frac{v^2}{c^2}\right) + (z')^2\left(1 - \frac{v^2}{c^2}\right)$$
$$c^2(t')^2 + 2t'vx' + \frac{v^2(x')^2}{c^2} = (x')^2 + 2x'vt' + v^2(t')^2 + (y')^2\left(1 - \frac{v^2}{c^2}\right) + (z')^2\left(1 - \frac{v^2}{c^2}\right)$$
$$(t')^2(c^2 - v^2) = (x')^2\left(1 - \frac{v^2}{c^2}\right) + (y')^2\left(1 - \frac{v^2}{c^2}\right) + (z')^2\left(1 - \frac{v^2}{c^2}\right)$$
$$c^2(t')^2\left(1 - \frac{v^2}{c^2}\right) = (x')^2\left(1 - \frac{v^2}{c^2}\right) + (y')^2\left(1 - \frac{v^2}{c^2}\right) + (z')^2\left(1 - \frac{v^2}{c^2}\right)$$
$$(r')^2 = (ct')^2 = (x')^2 + (y')^2 + (z')^2$$

$Expanding\ circle\ centered\ at\ origin\ of\ S:\quad (ct)^2 = x^2 + y^2$

$Moving\ ellipse\ centered\ at\ (vt, 0):\qquad \frac{(x - vt)^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{b}{a} = \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad a^2 = b^2\left(1 - \frac{v^2}{c^2}\right) \qquad \frac{v^2}{c^2} = 1 - \frac{a^2}{b^2} \qquad \frac{v}{c} = \frac{\sqrt{b^2 - a^2}}{b}$$

Solve for $x(t)$ by eliminating y :

$$y^2 = c^2t^2 - x^2 = b^2\left[1 - \frac{(x - vt)^2}{a^2}\right] = \frac{b^2}{a^2}[a^2 - (x - vt)^2]$$

$$\frac{a^2}{b^2}(c^2t^2 - x^2) = a^2 - x^2 + 2xvt - v^2t^2$$

$$\left(1 - \frac{a^2}{b^2}\right)x^2 - 2vtx + \frac{a^2}{b^2}c^2t^2 + v^2t^2 - a^2 = 0$$

$$x = \frac{2vt \pm \sqrt{4v^2t^2 - 4\left(1 - \frac{a^2}{b^2}\right)\left(\frac{a^2}{b^2}c^2t^2 + v^2t^2 - a^2\right)}}{2\left(1 - \frac{a^2}{b^2}\right)}$$

$$= \frac{vt \pm \sqrt{v^2t^2 - \frac{v^2}{c^2}\left[\left(1 - \frac{v^2}{c^2}\right)c^2t^2 + v^2t^2 - a^2\right]}}{\frac{v^2}{c^2}}$$

$$= \frac{c^2}{v^2}\left(vt \pm \sqrt{v^2t^2 - \frac{v^2}{c^2}[c^2t^2 - a^2]}\right)$$

$$\boxed{x = \frac{c^2}{v}\left(t - \frac{a}{c}\right) \qquad use\ minus\ sign\ so\ initial\ contact\ is\ on\ the\ left}$$

Solve for $y(t)$

$$y^2 = c^2t^2 - x^2 = c^2t^2 - \left[\frac{c^2}{v}\left(t - \frac{a}{c}\right)\right]^2 = c^2t^2 - \frac{c^4}{v^2}\left(t - \frac{a}{c}\right)^2 = c^2\left[t^2 - \frac{c^2}{v^2}\left(t - \frac{a}{c}\right)^2\right]$$

$$\boxed{y = \pm c\sqrt{t^2 - \frac{c^2}{v^2}\left(t - \frac{a}{c}\right)^2}}$$

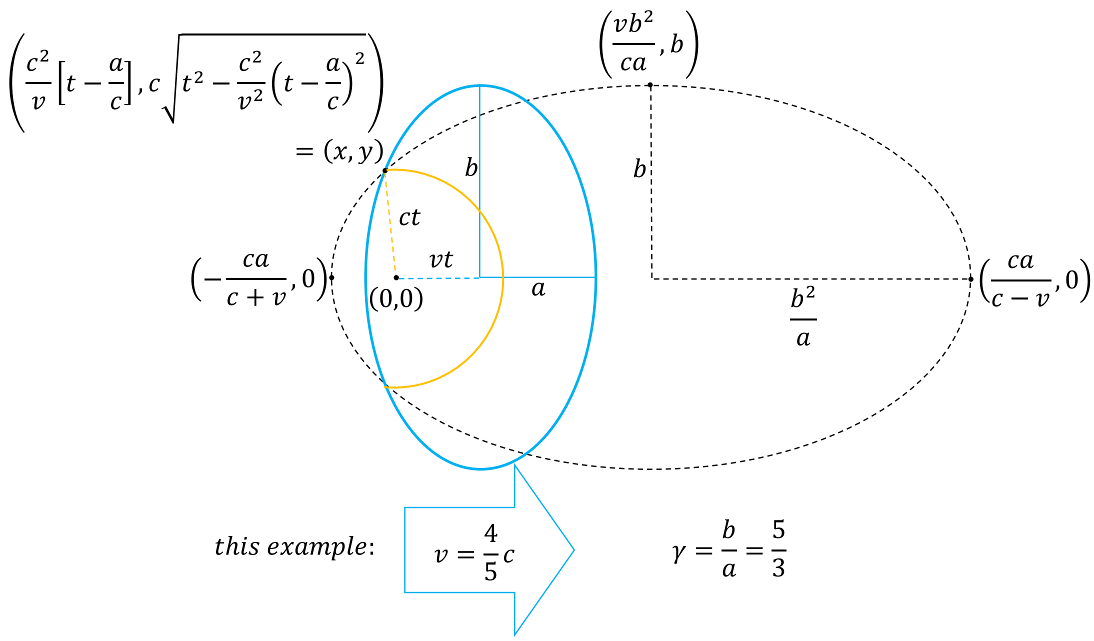
Find ellipse of reflection points by eliminating t

$$x = \frac{c^2}{v^2}\left(vt - \frac{v}{c}a\right) \qquad \frac{v^2}{c^2}x + \frac{v}{c}a = vt$$

$$1=\frac{(x-vt)^2}{a^2}+\frac{y^2}{b^2}=\frac{\left(x-\frac{v^2}{c^2}x-\frac{v}{c}a\right)^2}{a^2}+\frac{y^2}{b^2}=\frac{\left[x\left(1-\frac{v^2}{c^2}\right)-\frac{v}{c}a\right]^2}{a^2}+\frac{y^2}{b^2}$$

$$=\frac{\left[x-\frac{v}{c}a\left(1-\frac{v^2}{c^2}\right)^{-1}\right]^2}{\left[a\left(1-\frac{v^2}{c^2}\right)^{-1}\right]^2}+\frac{y^2}{b^2}$$

$$\boxed{\begin{gathered}1=\frac{\left[x-\frac{v}{c}\frac{b^2}{a}\right]^2}{\left[\frac{b^2}{a}\right]^2}+\frac{y^2}{b^2}\\ \textit{ellipse centered at }\left(\frac{vb^2}{ca},0\right)\textit{ with semiaxes }\frac{b^2}{a}\textit{ and }b,\textit{ foci at }(0,0)\textit{ and }\left(\frac{2vb^2}{ca},0\right)\end{gathered}}$$



$$\textit{Check mid contact }(x,y)=(vt,b):\ (ct)^2=(vt)^2+b^2$$

$$t^2(c^2-v^2)=b^2\qquad t=\frac{b}{\sqrt{c^2-v^2}}=\frac{ca}{c^2-v^2}$$

$$x=\frac{c^2}{v}\left[\frac{b}{\sqrt{c^2-v^2}}-\frac{b}{c}\sqrt{1-\frac{v^2}{c^2}}\right]=\frac{b}{\sqrt{c^2-v^2}}\frac{c^2}{v}\left[1-\frac{\sqrt{c^2-v^2}}{c}\sqrt{\frac{c^2-v^2}{c^2}}\right]$$

$$=t\left[\frac{c^2}{v}-\frac{\sqrt{c^2-v^2}}{v}\sqrt{c^2-v^2}\right]=vt\left[\frac{c^2}{v^2}-\left(\frac{c^2}{v^2}-1\right)\right]=vt$$

$$\text{Check left contact } (x,y) = (-ct, 0):\ \frac{(-ct - vt)^2}{a^2} = 1$$

$$(ct + vt)^2 = a^2 \qquad t(c+v) = a \qquad t = \frac{a}{c+v}$$

$$x = \left[v\left(\frac{a}{c+v}\right) - \frac{v}{c}a\right]\frac{c^2}{v^2} = c\left[\frac{c}{v}\left(\frac{a}{c+v}\right) - \frac{1}{v}a\right] = c\left[\frac{c}{v}\left(\frac{a}{c+v}\right) - \frac{c+v}{v}\left(\frac{a}{c+v}\right)\right]$$

$$= c\left(\frac{a}{c+v}\right)\left[\frac{c}{v} - \frac{c+v}{v}\right] = -c\left(\frac{a}{c+v}\right) = -ct$$

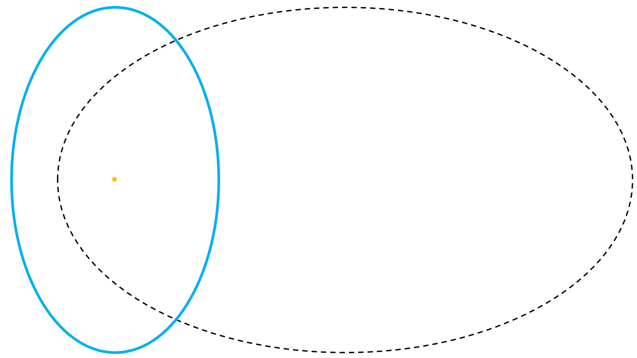
$$\text{Check right contact } (x,y) = (ct, 0):\ \frac{(ct - vt)^2}{a^2} = 1$$

$$(ct - vt)^2 = a^2 \qquad t(c-v) = a \qquad t = \frac{a}{c-v}$$

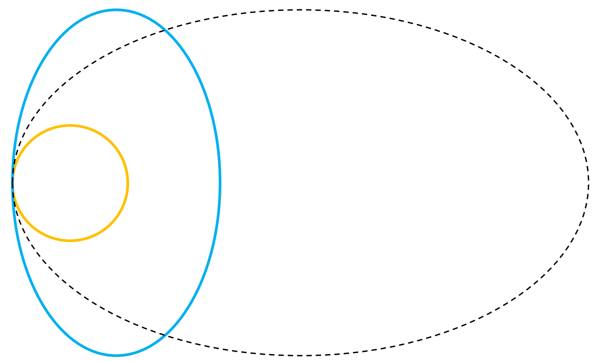
$$x = \left[v\left(\frac{a}{c-v}\right) - \frac{v}{c}a\right]\frac{c^2}{v^2} = c\left[\frac{c}{v}\left(\frac{a}{c-v}\right) - \frac{1}{v}a\right] = c\left[\frac{c}{v}\left(\frac{a}{c-v}\right) - \frac{c-v}{v}\left(\frac{a}{c-v}\right)\right]$$

$$= c\left(\frac{a}{c-v}\right)\left[\frac{c}{v} - \frac{c-v}{v}\right] = c\left(\frac{a}{c-v}\right) = ct$$

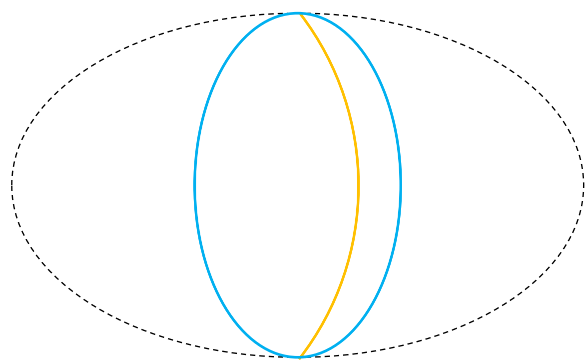
$t = 0$



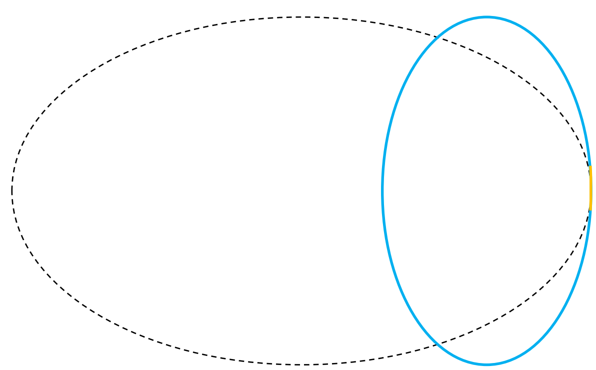
$t = \frac{a}{c+v}$



$t = \frac{ca}{c^2 - v^2}$



$t = \frac{a}{c-v}$



$t = \frac{2ca}{c^2 - v^2}$