Differential Transformations

$c = G = 1 \qquad r, \theta, \phi$ *assumed to be constant*

$$v = \sqrt{\frac{2M}{r}} = \text{escape velocity at distance } r \text{ from } M \qquad \gamma \equiv \frac{1}{\sqrt{1-v^2}} = \frac{1}{\sqrt{1-\frac{2M}{r}}} = \left(1-\frac{2M}{r}\right)^{-1/2}$$

$$\begin{aligned} \boldsymbol{ds^2} &= \boldsymbol{-d\tau^2} \\ &= -dt_{rain}^2 + dr_{rain}^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 \\ &= -dt_{shell}^2 + dr_{shell}^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 \\ &= -\left(1-\frac{2M}{r}\right) dt_{far}^2 + \left(1-\frac{2M}{r}\right)^{-1} dr_{far}^2 = -\gamma^{-2} dt_{far}^2 + \gamma^2 dr_{far}^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 \end{aligned}$$

$-dt_{rain}^2 + dr_{rain}^2$	$=$	$-dt_{shell}^2 + dr_{shell}^2$	$=$	$-\left(1-\frac{2M}{r}\right) dt_{far}^2 + \left(1-\frac{2M}{r}\right)^{-1} dr_{far}^2$
$\boldsymbol{dt_{rain}}$	$=$	$\left(1-\frac{2M}{r}\right)^{-1/2}\left(dt_{shell} + \sqrt{\frac{2M}{r}}\, dr_{shell}\right)$	$=$	$dt_{far} + \left(1-\frac{2M}{r}\right)^{-1}\sqrt{\frac{2M}{r}}\, dr_{far}$
$\boldsymbol{dr_{rain}}$	$=$	$\left(1-\frac{2M}{r}\right)^{-1/2}\left(\sqrt{\frac{2M}{r}}\, dt_{shell} + dr_{shell}\right)$	$=$	$\sqrt{\frac{2M}{r}}\, dt_{far} + dr_{far}\left(1-\frac{2M}{r}\right)^{-1}$
$\left(1-\frac{2M}{r}\right)^{-1/2}\left(dt_{rain} - \sqrt{\frac{2M}{r}}\, dr_{rain}\right)$	$=$	$\boldsymbol{dt_{shell}}$	$=$	$\left(1-\frac{2M}{r}\right)^{1/2} dt_{far}$
$\left(1-\frac{2M}{r}\right)^{-1/2}\left(dr_{rain} - \sqrt{\frac{2M}{r}}\, dt_{rain}\right)$	$=$	$\boldsymbol{dr_{shell}}$	$=$	$\left(1-\frac{2M}{r}\right)^{-1/2} dr_{far}$
$\left(1-\frac{2M}{r}\right)^{-1}\left(dt_{rain} - \sqrt{\frac{2M}{r}}\, dr_{rain}\right)$	$=$	$\left(1-\frac{2M}{r}\right)^{-1/2} dt_{shell}$	$=$	$\boldsymbol{dt_{far}}$
$dr_{rain} - \sqrt{\frac{2M}{r}}\, dt_{rain}$	$=$	$\left(1-\frac{2M}{r}\right)^{1/2} dr_{shell}$	$=$	$\boldsymbol{dr_{far}}$
Local Rain (dt_{rain}, dr_{rain}) Raindrop coordinates Free-Falling Observer		**Local Shell** (dt_{shell}, dr_{shell}) Stationary coordinates Fiducial Observer		**Far-Away** (dt_{far}, dr_{far}) Global coordinates Schwarzschild Bookkeeper/Mapmaker
$\boldsymbol{dt_{rain}}$	$=$	$\gamma(dt_{shell} + v\,dr_{shell})$	$=$	$dt_{far} + \gamma^2 v\,dr_{far}$
$\boldsymbol{dr_{rain}}$	$=$	$\gamma(v\,dt_{shell} + dr_{shell})$	$=$	$v\,dt_{far} + \gamma^2 dr_{far}$
$\gamma(dt_{rain} - v\,dr_{rain})$	$=$	$\boldsymbol{dt_{shell}}$	$=$	$\frac{dt_{far}}{\gamma}$
$\gamma(dr_{rain} - v\,dt_{rain})$	$=$	$\boldsymbol{dr_{shell}}$	$=$	$\gamma\,dr_{far}$
$\gamma^2(dt_{rain} - v\,dr_{rain})$	$=$	$\gamma\,dt_{shell}$	$=$	$\boldsymbol{dt_{far}}$
$dr_{rain} - v\,dt_{rain}$	$=$	$\frac{dr_{shell}}{\gamma}$	$=$	$\boldsymbol{dr_{far}}$

Note: The Far and Shell frames are both stationary, yet they exhibit time dilation ($dt_{far} = \gamma dt_{shell}$) and length contraction ($dr_{far} = \gamma^{-1} dr_{shell}$). Unlike SR, these kinematic relativistic effects are not symmetric. This can be modeled by warped spacetime or by flowing space (at speed v past the stationary Shell frame toward M).