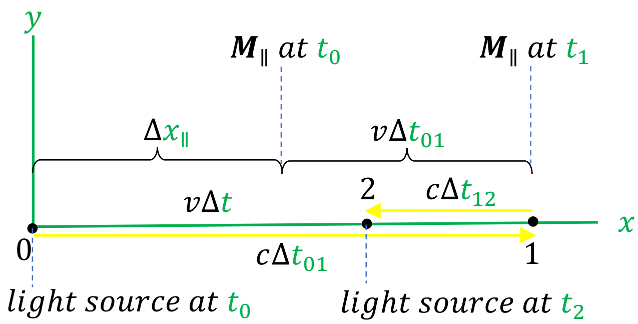
Length Contraction

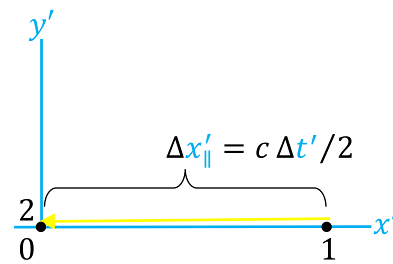
Derivation of the length contraction formula from the time dilation formula,
using the light path reflected off mirror $\boldsymbol{M}_\parallel$
which is offset from the light source parallel to the direction of motion
(example has $v = c/2$*)*

event 0: *send light pulse*
event 1: *bounce at mirror* $\boldsymbol{M}_\parallel$ *(moving along* x *axis)*
event 2: *receive light pulse (also send next light pulse)*

$$\Delta t = \Delta t_{01} + \Delta t_{12} = \gamma \Delta t'$$

$$\Delta t' = t'_2 - t'_0$$
$$\Delta t'/2 = t'_2 - t'_1 = t'_1 - t'_0$$





mirror $\boldsymbol{M}_\parallel$ *is ahead of event* 0 *in the direction of motion*
S' *moves at speed* v *in the* $+\,x$ *direction relative to* S
let $\Delta t'$ *be* S' *time from event* 0 *to event* 2 = *one* S' *clock tick*
let $\Delta x'_\parallel$ *be* S' *(proper) distance from event* 0 *to* $\boldsymbol{M}_\parallel = c\,\Delta t'/2$
let $\Delta x_\parallel$ *be instantaneous* S *distance from event* 0 *to* $\boldsymbol{M}_\parallel$
let Δt *be* S *time from event* 0 *to event* 2 = *one* S *clock tick*
let Δt_{01} *be* S *time from event* 0 *to event* 1
let Δt_{12} *be* S *time from event* 1 *to event* 2

S *distance from event* 0 *to event* 1 $= c\Delta t_{01} = \Delta x + v\Delta t_{01}$
$\Delta x_\parallel = \Delta t_{01}(c - v) \qquad \Delta t_{01} = \Delta x_\parallel/(c - v) \qquad \Delta t_{12} = \Delta t - \Delta t_{01}$

S *distance from event* 0 *to event* 2 $= v\Delta t = c\Delta t_{01} - c\Delta t_{12}$
$v\Delta t = c\Delta t_{01} - c(\Delta t - \Delta t_{01}) = 2c\Delta t_{01} - c\Delta t$
$\Delta t(c + v) = 2c\Delta t_{01} = 2c\,\Delta x_\parallel/(c - v)$
$\Delta x_\parallel = \Delta t(c + v)(c - v)/(2c) = \Delta t(c^2 - v^2)\,c/(2c^2)$

use time dilation formula: $\Delta t = \gamma \Delta t'$
$\Delta x_\parallel = \gamma \Delta t'(c^2 - v^2)\,c/(2c^2) = \gamma(c\Delta\, t'/2)(1 - v^2/c^2) = \gamma \Delta x'_\parallel \gamma^{-2} = \Delta x'_\parallel/\gamma$

Generalize to any $\Delta x'$ *length moving relative to* S

$$\boxed{\Delta x = \Delta x'/\gamma}$$

$\gamma \geq 1 \;\Rightarrow$ *length contraction as seen by* S *frame*:
S *length* Δx *is less than* S' *length* $\Delta x'$ *(by a factor of* γ *in the direction of motion)*
S *sees* S' *rulers to be shorter than* S *rulers (by a factor of* γ *in the direction of motion)*