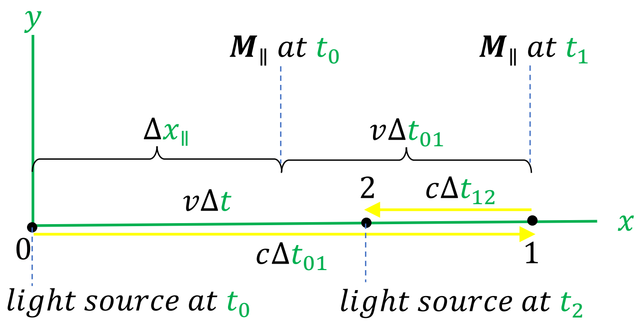
Length Contraction

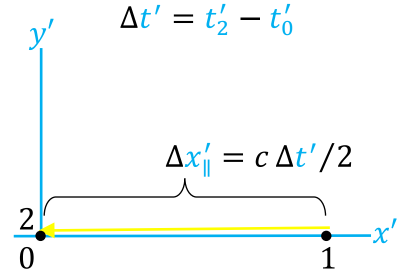
Derivation of the length contraction formula from the time dilation formula,
using the light path reflected off mirror $\boldsymbol{M}_{\parallel}$
which is offset from the light source parallel to the direction of motion
(example has $v = c/2$)

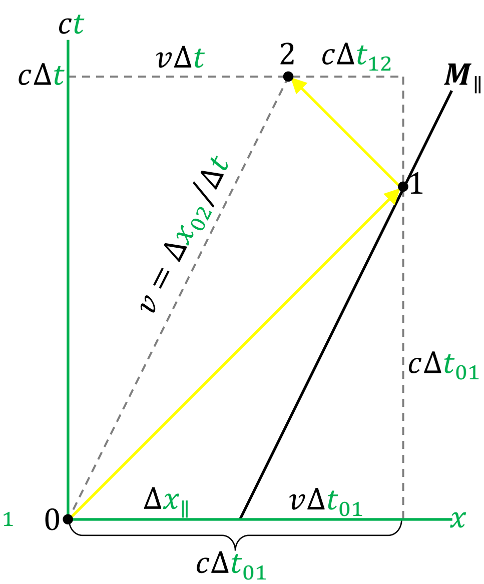
event 0: send light pulse
event 1: bounce at mirror $\boldsymbol{M}_{\parallel}$ (moving along x axis)
event 2: receive light pulse (also send next light pulse)

$$\Delta t = \Delta t_{01} + \Delta t_{12} = \gamma \Delta t'$$



$$\Delta t'/2 = t'_2 - t'_1 = t'_1 - t'_0$$
$$\Delta t' = t'_2 - t'_0$$

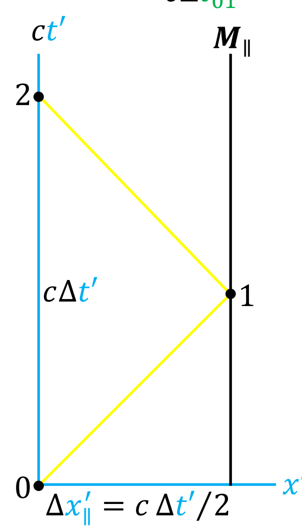




mirror $\boldsymbol{M}_{\parallel}$ is ahead of event 0 in the direction of motion
S' moves at speed v in the $+x$ direction relative to S
let $\Delta t'$ be S' time from event 0 to event 2 = one S' clock tick
let $\Delta x'_{\parallel}$ be S' (proper) distance from event 0 to $\boldsymbol{M}_{\parallel} = c\,\Delta t'/2$
let $\Delta x_{\parallel}$ be instantaneous S distance from event 0 to $\boldsymbol{M}_{\parallel}$
let Δt be S time from event 0 to event 2 = one S clock tick
let Δt_{01} be S time from event 0 to event 1
let Δt_{12} be S time from event 1 to event 2

S distance from event 0 to event 1 $= c\Delta t_{01} = \Delta x_{\parallel} + v\Delta t_{01}$
$\Delta x_{\parallel} = \Delta t_{01}(c - v) \qquad \Delta t_{01} = \Delta x_{\parallel}/(c - v) \qquad \Delta t_{12} = \Delta t - \Delta t_{01}$

S distance from event 0 to event 2 $= v\Delta t = c\Delta t_{01} - c\Delta t_{12}$
$v\Delta t = c\Delta t_{01} - c(\Delta t - \Delta t_{01}) = 2c\Delta t_{01} - c\Delta t$
$\Delta t(c + v) = 2c\Delta t_{01} = 2c\,\Delta x_{\parallel}/(c - v)$
$\Delta x_{\parallel} = \Delta t(c + v)(c - v)/(2c) = \Delta t(c^2 - v^2)\,c/(2c^2)$



use time dilation formula: $\Delta t = \gamma \Delta t'$
$\Delta x_{\parallel} = \gamma \Delta t'(c^2 - v^2)\,c/(2c^2)$
$\quad = \gamma(c\Delta t'/2)(1 - v^2/c^2) = \gamma \Delta x'_{\parallel}\gamma^{-2} = \Delta x'_{\parallel}/\gamma$

Generalize to any $\Delta x'$ length moving relative to S:

$$\boxed{\Delta x = \Delta x'/\gamma}$$

$\gamma \geq 1 \Rightarrow$ length contraction as seen by S frame:
S length Δx is less than S' length $\Delta x'$ (by a factor of γ in the direction of motion)
S sees S' rulers to be shorter than S rulers (by a factor of γ in the direction of motion)