Optimal Hedging with Advanced Delta Modelling

Joseph Kirk

August 8, 2024

Contents

1 Introduction

1.1 Preamble

This report is prepared to satisfy the requirements of the CQF assignment *Optimal Hedging with Advanced Delta Modelling*. Three topics are studied.

The first investigation uses Geometric Brownian Motion as an example to introduce Monte Carlo methods and variance reduction techniques. Specifically we study the effect of antithetic variates, sobol sequences and the brownian bridge to reduce the number of paths required to converge on the Black Scholes price of a European Call Option.

The second study looks at the affect of hedging with actual and implied volatility under the condition that future realised volatility is above implied volatility. We demonstrate through simulation and prove with mathematical workings that hedging with the actual volatility leads to a known value of total P&L.

We conclude the report by partially replicating the study conducted by John C. Hull and Alan White [1]. Using 2023 implied volatility for SPX options (provided by OptionsDX) we fit the minimum variance delta (MVD) model. This model anticipates the change in magnitude of implied volatility and applies a correction to the Black Scholes delta.

1.2 Code

The Python code used to generate the data and plots has been provided alongside this report. Core functionality has been coded as reusable modules and the experiments are available as Jupyter notebooks. There are more details in the README of the code package. External packages used are

- numpy / pandas - SIMD vectorized operations and data manipulation
- scikit-learn - Linear regression
- matplotlib - Plot generation for reports
- scipy - Normal distribution and Sobol sequences

All other methods have been implemented from scratch. The scientific libraries are so commonly used that they are considered part of the standard library in Python.

2 Monte Carlo Variance Reduction

Monte Carlo methods are a class of computational algorithms that rely on repeated random sampling to obtain numerical results. It is particularly prevelant in the field of finance where many financial models do not have closed form solutions. While European options can be priced using the Black-Scholes formula (which is derived from Geometric Brownian Motion), more complex products such as American options, that can be exercised at any point, require a numerical approach.

To price an option using Monte Carlo methods we must compute the fair value of said option. This is the present value of the expected payoff at expiry under a risk-neutral random walk.

1. Simulate the risk-neutral random walk starting at today's value of the asset S0 over the required time horizon. This gives one realization of the underlying price path.
2. For this realization calculate the option payoff.
3. Perform many more such realizations over the time horizon.
4. Calculate the average payoff over all realizations.
5. Take the present value of this average, this is the option value.

Monte Carlo is a widely accepted method for pricing options since it is simple to implement and can be used for products with complex path dependency. However, it can be computationally expensive. If we can reduce the variance of the simulated path then we can reduce the number of paths we need to calculate. This is the aim of variance reduction techniques. To study variance reduction techniques we use the motivating example of pricing a European Call Option whose underlying follows Geometric Brownian Motion. Here is the stochastic differential equation (SDE) for GBM in the risk neutral world.

$$dS(t) = rS(t)\,dt + \sigma S(t)\,dW(t)$$

The analytical solution for GBM is given by

$$S(t) = S(0)\exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right)$$

It is possible to generate the asset price at time T using a single step. Since we wish to investigate variance reduction techniques we opt to focus our investigation on the numerical solutions discussed in the following section (Euler Maruyama and Milstein). These numerical methods require the full path to be simulated. Monte carlo simulations using the analytical solution have been provided in the code.

2.1 Numerical Solutions

In this section we present numerical schemes for the solution of the GBM SDE.

2.1.1 The Euler Scheme

The Euler-Maruyama discretization is a numerical method used to approximate solutions to stochastic differential equations (SDEs). The Euler-Maruyama method approximates the solution of an SDE by discretizing time into small steps of size Δt. The discretized form of the SDE is given by:

$$X_{t_{n+1}} = X_{t_n} + a(X_{t_n}, t_n)\Delta t + b(X_{t_n}, t_n)\Delta W_n$$

where $\Delta W_n = W_{t_{n+1}} - W_{t_n} \sim \mathcal{N}(0, \Delta t)$ is a normally distributed random variable with mean 0 and variance Δt.

Using the Euler-Maruyama method, the discretized form of GBM is:

$$S_{t_{n+1}} = S_{t_n} + \mu S_{t_n}\Delta t + \sigma S_{t_n}\Delta W_n$$

2.1.2 The Milstein Scheme

The Milstein scheme improves upon the Euler-Maruyama method by adding a correction term involving the derivative of the diffusion coefficient. The discretized form of the SDE using the Milstein scheme is given by:

$$X_{t_{n+1}} = X_{t_n} + a(X_{t_n}, t_n)\Delta t + b(X_{t_n}, t_n)\Delta W_n + \frac{1}{2} b(X_{t_n}, t_n)\frac{\partial b(X_{t_n}, t_n)}{\partial X}\left((\Delta W_n)^2 - \Delta t\right)$$

For GBM, the derivative of the diffusion term $b(S_t, t) = \sigma S_t$ with respect to S_t is $\frac{\partial b(S_t,t)}{\partial S} = \sigma$. Using the Milstein scheme, the discretized form is:

$$S_{t_{n+1}} = S_{t_n} + \mu S_{t_n}\Delta t + \sigma S_{t_n}\Delta W_n + \frac{1}{2} S_{t_n}\sigma^2\left((\Delta W_n)^2 - \Delta t\right)$$

2.2 Variance Reduction Techniques

The accuracy of a Monte Carlo simulation is measured by the standard error $\sigma/\sqrt{N}$. To improve the accuracy of a simulation we can increase the number of random paths N or reduce the standard deviation σ. Variance reduction techniques aim to reduce the standard error by reducing the variance of the simulated paths.

2.2.1 Antithetic Variates

Antithetic variates leverage the fact that for every gaussian drawn there is a corresponding negative variate which has the same probability of being drawn. If we draw variate $v_i = v(z_i)$ then we also use $v_i' = v(-z_i)$. v_i' is negatively correlated to v_i but has the same variance.[3]

$$\text{Var}\left(\frac{v_i + v_i'}{2}\right) = \frac{1}{2}\left(\text{Var}(v_i) + \text{Cov}(v_i, v_i')\right)$$

Because v_i and v_i' are negatively correlated, $\text{Cov}(v_i, v_i') < 0$.

$$\text{Var}\left(\frac{v_i + v_i'}{2}\right) < \frac{1}{2}\,\text{Var}(v_i)$$

2.2.2 Sobol Sequences

Sobol sequences are a type of low-discrepancy sequence used in numerical methods for generating quasi-random points. Unlike purely random sequences, which can have clusters and gaps, quasi-random sequences like Sobol sequences are designed to fill the space more evenly. This results in more reliable and faster convergence in numerical simulations. We illustrate the difference between random numbers and sobol sequences in Figure 1.

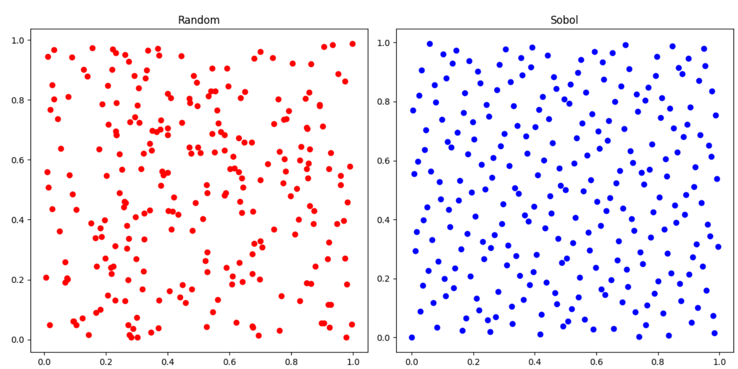


Figure 1: Sobol Sequence

2.2.3 Brownian Bridge

Brownian Bridge is a path construction method. Rather than building the Wiener path incrementally, the Brownian Bridge uses the first variate z_1 to determine the terminal point $W_{t_n} = \sqrt{t_n} z_1$. The next variate z_2 is used to determine a point on the path W_j conditional on points W_{t_n} and $W_0 = 0$. This is repeated until the entire path is constructed.

$$W(T) = \sqrt{T} z_1$$

$$W(T/2) = \frac{1}{2} W(T/2) + \frac{1}{2}\sqrt{T} z_2$$

$$W(T/4) = \frac{1}{2}(W(T/2) + W(T)) + \frac{1}{2}\sqrt{T} z_4$$

etc. The general formula for the Brownian Bridge is

$$W(t_i) = (1-\gamma)(W(t_l)) + \gamma W(t_m) + \sqrt{\gamma(1-\gamma)(m-l)\Delta t} z_i$$

Incremental and Brownian Bridge path constructions have the same variance so we'd expect a similar rate of convergence. We show in our results that the Brownian Bridge combined with a Sobol sequence has superior convergence rate. It was suggested by Markowitz[4] that the improvement in convergence is due to reduction of the effective dimension of the problem.

2.3 Results

In this section we present the results of pricing a European Call Option using the Euler approximation with the following variance reduction / path construction techniques

- Antithetic Variates
- Sobol Sequences
- Brownian Bridge
- Sobol Sequences with Brownian Bridge

We replicate the workings of Sergei Kucherenko and Nishant Shah [2] and use the following parameters for the simulation: $S_0 = 100$, $K = 100$, $r = 0.05$, $\sigma = 0.2$, $T = 0.5$. The Black-Scholes price of the option is 6.89. 128 steps are computed and the number of simulations is varied from 2 to 256. As expected there is little difference in the convergence rates of incremental and brownian bridge path construction in the absence of variance reduction techniques. If we combine the Sobol sequence with the Brownian Bridge we see a significant improvement in convergence rate. Figure 2 shows the evolution of the price estimate price as the number simulations is increased.

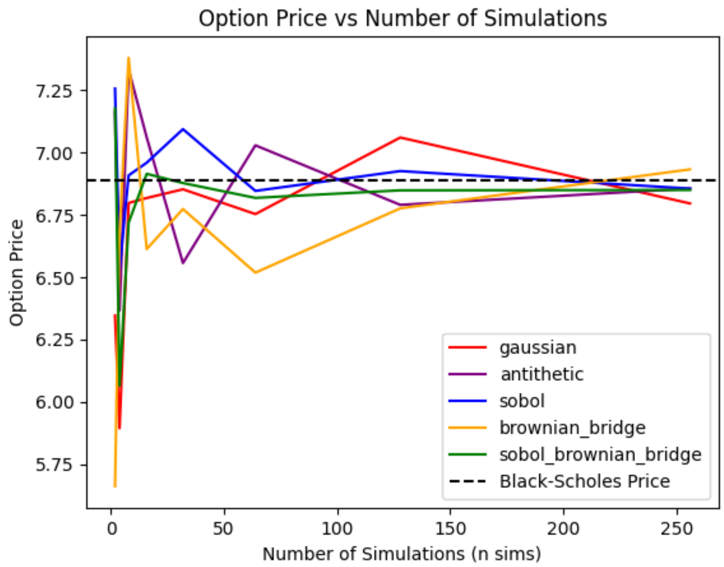


Figure 2: European Call Option Price Convergence

We also look at the root mean square error (RMSE). Each simulation is run $L = 50$ times.

$$RMSE = \sqrt{\frac{1}{L}\sum_{l=1}^{L}(V^l - V^{BS})^2}$$

RMSE is plotted against the number of simulations in Figure 3. When variance reduction techniques are applied the accuracy of the results is significantly improved reducing the number of simulations needed to achieve a desired level of accuracy. According to the simulations the convergence rate of the Brownian Bridge with Sobol sequences decreases as $1/N^{0.95}$.

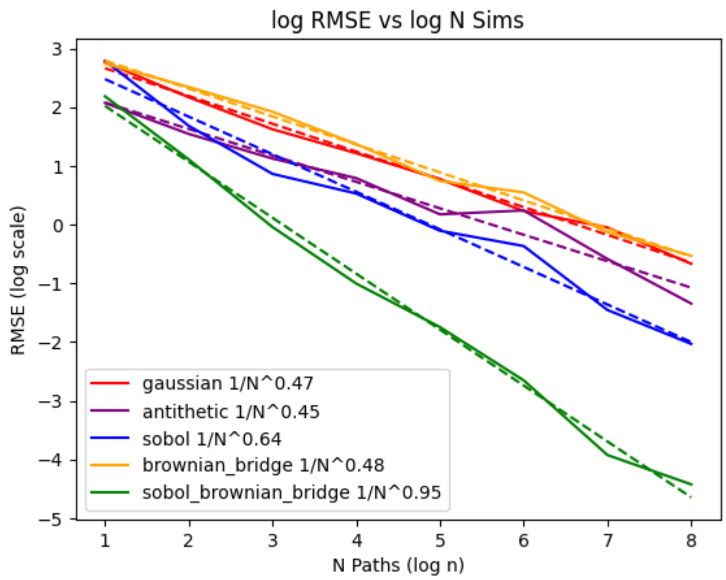


Figure 3: Root mean square error vs the number of simulations

3 Hedging with actual or implied volatility

In this section we study the differences between hedging a portfolio of options with implied and actual volatility. The figures that follow have been created by studying the P&L of a long option, short stock portfolio. The stock price is assumed to follow Geometric Brownian Motion and is generated using the Milstein approximation using Sobol / Brownian Bridge variance reduction. It has initial price $S_0 = 100$. The option is a European Call with strike $K = 100$, risk free rate $r = 0.02$. Actual volatility is taken to be $\sigma_a = 0.35$ and implied volatility $\sigma_i = 0.2$. The time to maturity is $T = 1$ and the number of steps to produce the stock path is 256. If the option is hedged with the actual volatility it has a theoretical P&L of $5.85

Much of the following analysis is taken from CQF lecture notes and Quantitative Finance books authored by Wilmott [5].

3.1 Actual volatility

By hedging with actual volatility we are replicating a short position in a correctly priced option. The payoffs for the long option and the short replicated option will exactly cancel. The total profit from hedging with actual volatility is known and can be calculated as the difference between the actual and implied option price

$$V(S,t;\sigma^a) - V(S,t;\sigma^i)$$

To prove this setup a long option short stock portfolio. The option is valued at V^i, and we hedge with $-\Delta^a S$ of the stock. The portfolio is left for a short time period δ_t and revalued. In the notation superscript a refers to actual and superscript i refers to implied. Dividends are ignored for brevity, but it is a small extension.

'Today' at time t

Component	**Value**
Option	V^i
Stock	$-\Delta^a S$
Cash	$-V^i + \Delta^a S$

Table 1: Portfolio at time t

'Tomorrow' at time $t + \delta_t$

Component	**Value**
Option	$V^i + dV^i$
Stock	$-\Delta^a S - \Delta^a dS$
Cash	$(-V^i + \Delta^a S)(1 + rdt)$

Table 2: Portfolio at time $t + \delta_t$

Mark to market we have made the following profit over one time step

$$dV^i - \Delta^a dS - (V^i - \Delta^a S)rdt \tag{1}$$

If the option is priced using the actual volatility then the option is correctly valued at V^a so we can write

$$dV^a - \Delta^a dS - (V^a - \Delta^a S)rdt = 0 \tag{2}$$

Subtracting 1 from 2 we can rewrite the mark-to-market profit from t to $t + \delta_t$ as

$$\begin{aligned}
1-2 &= dV^i - dV^a + r(V^a - \Delta^a S)dt - r(V^i - \Delta^a S)dt \\
&= dV^i - dV^a - r(V^i - V^a)dt \\
&= e^{rt}d(e^{-rt}(V^i - V^a)) \\
d(e^{-rt}V) &= d(e^{-rt}dV) - re^{-rt}Vdt \\
&= e^{-rt}(dV - rVdt)
\end{aligned}$$

The present value of this profit at t_0 is

$$e^{-r(t-t_0)}e^{rt}d(e^{-rt}(V^i - V^a)) = e^{rt_0}d(e^{-rt}(V^i - V^a))$$

So the total profit from t_0 to expiration is

$$e^{r(t_0)}\int_{t_0}^{T} d(e^{-rt}(V^i - V^a)) = V^a - V^i \tag{3}$$

The profit is guaranteed, but how is it realised? Using

i) Itô's lemma

$$dV = (\frac{dV}{dt} + \frac{1}{2}\sigma_a^2 S^2 \frac{d^2V}{dS^2})dt + \frac{dV}{dS}dS$$

ii) Black-Scholes with $\sigma = \sigma_i$ and

$$\Theta^i + \frac{1}{2}\sigma_i^2 S^2 \Gamma^i + r(S\Delta^i - V^i i) = 0$$

iii) Geometric Brownian Motion as the path for the underlying asset

$$dS = \mu S dt + \sigma_a S dX$$

The profit of a single timestep can also be written as

$$\begin{aligned}
&\Theta^i dt + \Delta^i dS + \frac{1}{2}\sigma^2 S^2 \Gamma^i dt - \Delta^a dS - r(V_i - \Delta^a S)dt \\
&= \Theta^i dt + \mu S(\Delta^i - \Delta^a)dt + \frac{1}{2}\sigma^2 S^2 \Gamma^i dt - r(V^i - V^a)dt + (\Delta^i - \Delta^a)\sigma S dX \\
&= (\Delta^i - \Delta^a)\sigma S dX + (\mu - r)S(\Delta^i - \Delta^a)dt + \frac{1}{2}(\sigma_a^2 - \sigma_i^2)S^2\Gamma^i dt \\
&= \frac{1}{2}(\sigma_a^2 - \sigma_i^2)S^2\Gamma^i dt + (\Delta^i - \Delta^a)((\mu - r)S dt + \sigma_a S dX)
\end{aligned}$$

Figure 4 illustrates how the theoretical P&L ($V^a - V^i = 5.85$) can be realised. A couple of points to note.

- Daily profit is random. This is accounted for by the dX term
- On a mark-to-market it is possible to lose money in spite of the theoretical profit being known
- The final P&L is not exactly the same as the theoretical profit because we hedge discreetly

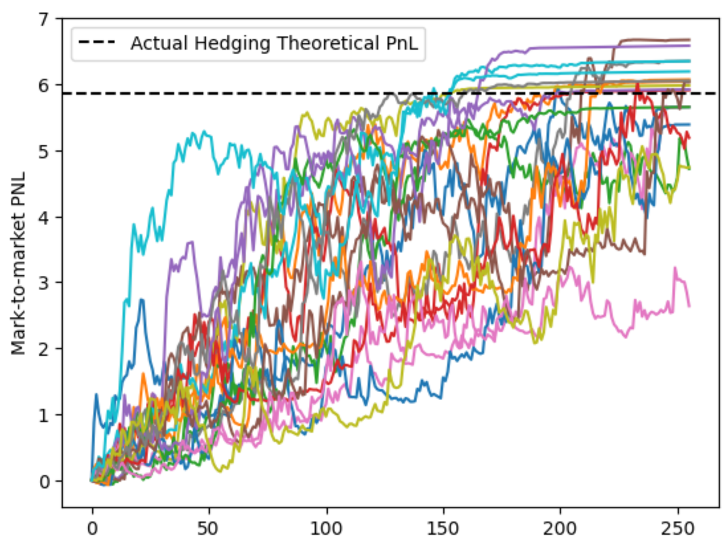


Figure 4: P&L for a delta hedged option on a mark-to-market basis hedged with actual volatility

3.2 Implied volatility

Hedging with implied volatility is a more common practice since the daily profit is deterministic. We balance the fluctuations in stock price with those in the option price.

Consider the same portfolio as before, but we hedge with $-\Delta^i S$ rather than $-\Delta^a S$. The mark-to-market profit over one time step is

$$
\begin{aligned}
&dV^i - \Delta^i dS - (V^i - \Delta^i S)rdt \\
&= \Theta^i dt + \frac{1}{2}\sigma_a^2 S^2 \Gamma^i dt - r(V^i - \Delta^i S)dt \\
&= \frac{1}{2}(\sigma_a^2 - \sigma_i^2) S^2 \Gamma^i dt
\end{aligned}
$$

Adding up the profit over all time steps and discount to present day value we obtain the total P&L.

$$
\frac{1}{2}(\sigma_a^2 - \sigma_i^2) \int_{t_0}^{T} e^{-r(t-t_0)} S^2 \Gamma^i dt
$$

Hedging with implied volatility has the following characteristics

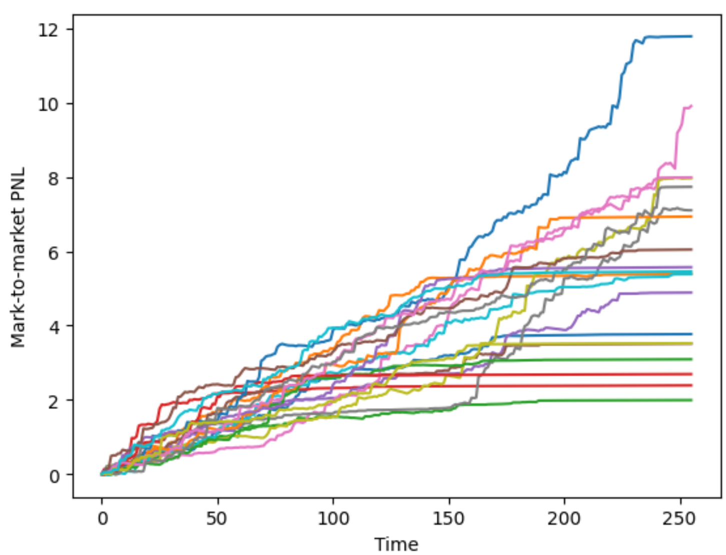


Figure 5: P&L for a delta hedged option on a mark-to-market basis hedged with implied volatility

- The daily profit is deterministic - there are no dX terms
- The total profit is unknown, positive and highly path dependent
- Small fluctuations in the P&L are caused by discrete hedging

These characteristics are visible in Figure 5. The P&L distribution in figure 6 illustrates the different payoff profiles when we hedge with actual and implied volatility.

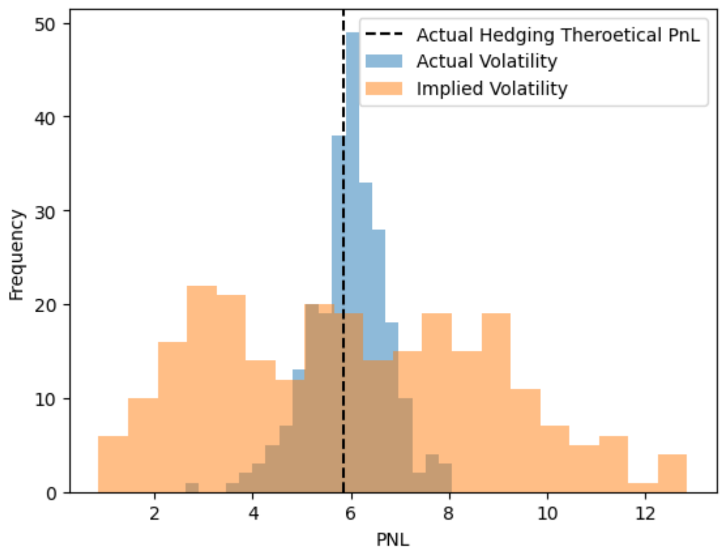
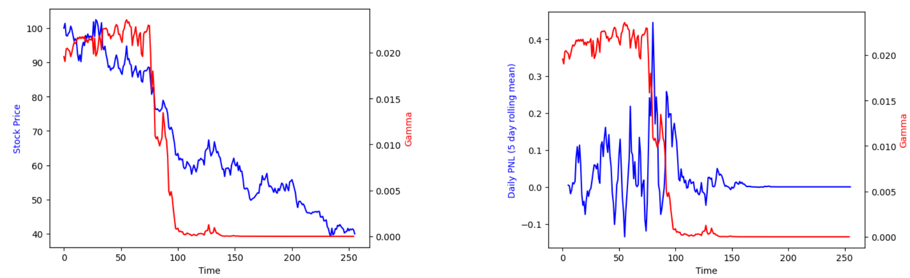


Figure 6: P&L distribution for a delta hedged option on a mark-to-market basis hedged with actual and implied volatility

P&L scales with Γ_i (which is strictly positive for Call options). We wouldn't expect to make much money in a trending market (large positive or negative drift) where the Delta races to 1 or 0. The best we can hope for is for the spot to be close to the strike at expiration causing Gamma to spike. That said the relationship isn't particularly straightforward. Daily PNL is approximated by

$$\frac{1}{2}\Gamma_i S^2((\frac{\Delta S}{S})^2 - \sigma_i^2 dt)$$

In Figure 7 we show how the daily P&L (smoothed over 5 days) evolves with the time dependent Gamma as the stock price nose dives. In the plot, negative daily P&L is seen in conjunction with an increase in Gamma suggesting that there is a corresponding decrease in the local realised volatility. If the realised volatility is lower than the implied volatility then realised P&L will be negative over the timestep.



(a) Caption for Figure 1

(b) Caption for Figure 2

Figure 7: Caption for the combined figure

4 Minimum Variance Delta

In this third and final section of the report we study Minimum Variance Delta (MVD). MVD is the Delta that minimizes the variance of a hedgers positions. by anticipating the change in the magnitude of implied volatility. A correction is then applied to the Black-Scholes Delta. A regression model for computing MVD was introduced by John C. Hull and Alan White in 2017 [1]. In their paper it is shown that

$$\delta_{MV} = \delta_{BS} + \frac{v_{BS}}{S\sqrt{T}}(a + b\delta_{BS} + c\delta_{BS}^2)$$

and the expected change in implied volatility is given by

$$E(\Delta\sigma_i) = \left(\frac{a + b\delta_{BS} + c\delta_{BS}^2}{\sqrt{T}}\right)\frac{\Delta S}{S}$$

To determine the coefficients a, b and c the following regression model is used

$$\Delta f - \delta_{bs}\Delta S = \frac{v_{BS}}{\sqrt{T}}\frac{\Delta S}{S}(a + b\delta_{BS} + c\delta_{BS}^2) + \epsilon$$

Δf is the change in the option price, ΔS is the change in the stock price and ϵ is the error term. δ_{BS} and v_{BS} are the Black-Scholes Delta and Vega respectively.

4.1 Data

We use end of day SPX options data from OptionsDX. The data contains; bid, ask, strike, days to expiry, implied volatility, delta, gamma, theta, vega. It is assumed that the greeks are calculated using the Black-Scholes model. The mid price is calculated and used as the option price for all analysis. The vega quoted is the 1% change in implied volatility, so it is scaled up by 100. The paper suggests that approach is most effective for out of the money options so the data has been restricted to three moneyness levels; 50%, 40% and 30%. The regression is also run across different expiry horizons; 1M, 3M, 6M, 9M and 12M. In spite of these filters the accompanying code can we run on the full dataset (e.g all moneyness levels, calls and puts, and all expiries).

4.2 Results

The regression model is fitted to the data and the coefficients a, b and c are calculated. Figure 8 illustrates how a, b and c evolve using a 15-day rolling window. Rolling estimation accounts for changing environments and can reflect regime changes, for example, the spike in March 2023 is likely caused by the Silicon Valley Bank crash. The parameters are two orders of magnitude larger than those found by Hull and White. This is likely due to do the fact that S&P levels are now 5 times higher than they were in 2014. Similar to Hull and White [1] we uncover that, for call options, a and b are mirror images.

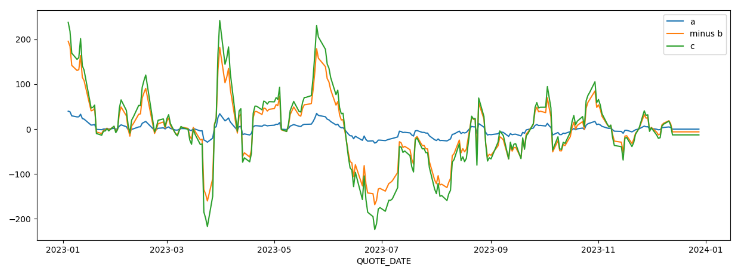


Figure 8: Estimated parameters for a, b, and c for calls in 2023 in the 40 delta, 3M expiry bucket using a 15 day rolling window

As expected we uncover that the Minimum Variance Delta is lower than Black-Scholes Delta in all the cases studied. In Figure 9 we show that δ_{MV} - δ_{BS} gives a parabolic shape (inverted for longer dated options). This suggests the quadratic fitting recipe is a good approximation. We similarly plot the expected change in implied volatility in Figure 10 against Delta which similarly shows a parabolic shape.

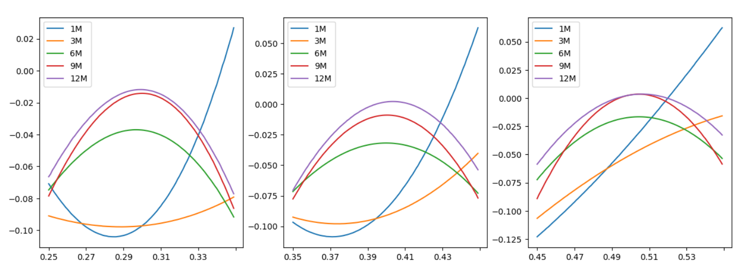


Figure 9: Delta MVD - Delta BS for the 30, 40 and 50 Delta buckets for 1M, 3M, 6M, 9M and 12M tenors

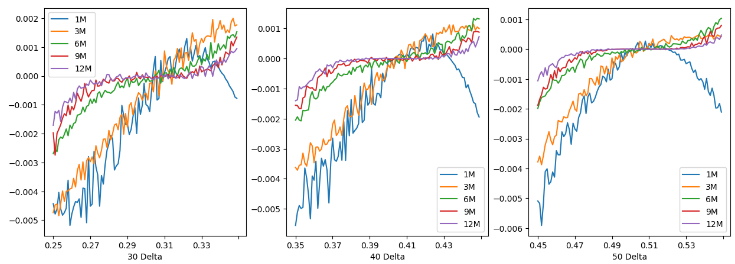


Figure 10: Expected change in implied volatility for the 30, 40 and 50 Delta buckets for 1M, 3M, 6M, 9M and 12M tenors

Sum of square errors (SSE) quantifies how well the regression model fits the data. Specifically, it measures the total deviation of the predicted values from the actual values. A higher SSE indicates a worse fit as we deviate further from the actual values.

The Gain is the measure of improvement of the model. We compare the SSE hedging error for Black-Scholes Delta and Minimum Variance Delta. The hedging error for Minimum Variance Delta is

$$\epsilon_{MV} = \Delta f - \delta_{BS}\Delta S - ltaS = \frac{v_{BS}}{\sqrt{T}}\frac{\Delta S}{S}(a + b\delta_{BS} + c\delta_{BS}^2)$$

and the hedging error for Black-Scholes Delta is

$$\epsilon_{MV} = \Delta f - \delta_{BS}\Delta S$$

The gain is therefore

$$Gain = 1 - \frac{SSE(\epsilon_{MV})}{SSE(\epsilon_{BS})}$$

In Table 3 we show the gain in model accuracy for different moneyness levels and expiries. It further confirms that the Minimum Variance Delta model is superior to the Black-Scholes Delta model reporting gains in all cases. We also confirm that better performance for out of the money options seeing a gain of 18.4% for 50% moneyness options with 1M expiry.

Table 3: Gain in model accuracy

	1M	3M	6M	9M	12M
50%	8.7	5.7	3.9	3.8	1.8
40%	13.3	11.6	7.2	5.4	3.8
30%	18.4	21.5	13.9	10.5	8.4

References

[1] J. Hull, A. White. *Optimal delta hedging for options*. 2017

[2] S. Kucherenko, N. Shah *The Importance of being Global. Application of Global Sensitivity Analysis in Monte Carlo Option Pricing.*

[3] P. Jäckel *Monte Carlo Methods in Finance.*

[4] B. Moskowitz, R.E. Caflisch. *Smoothness and Dimension Reduction in Quasi-Monte-Carlo*

[5] P. Wilmott. *Paul Wilmott on Quantitative Finance. Volume 1.*