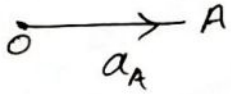
Unit Vector

A unit vector along 'A' is defined as a vector whose magnitude is unity and its direction is along A.

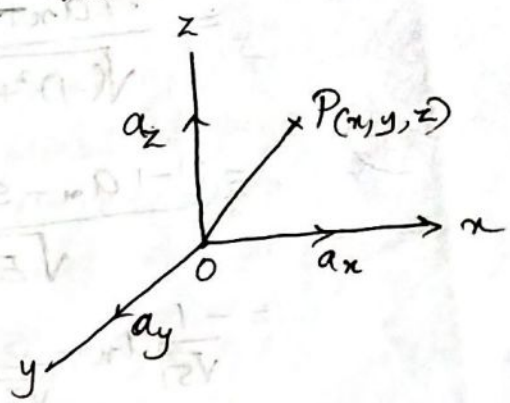
$$a_A = \frac{\vec{OA}}{|\vec{OA}|} = \frac{\text{vector}}{\text{magnitude}}$$



Position vector

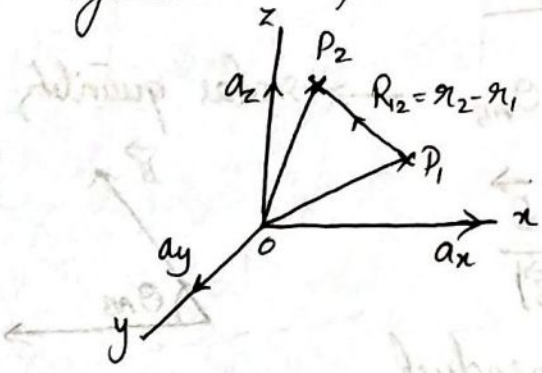
Position vector of point P is a vector directed from origin O to point P ie;

$$\vec{OP} = x a_x + y a_y + z a_z$$



Relative Position vector

Given two points P_1, P_2



$$R_{ij} = r_j - r_i$$

Q: Given two points M(-1, 2, 1) & N(3, -3, 0) & P(-2, -3, -4). Find a) R_{MP} b) $|r_M|$ c) a_{MP}

$$\vec{r_M} = \vec{OM} = -1a_x + 2a_y + 1a_z$$

$$\vec{r_P} = \vec{OP} = -2a_x - 3a_y - 4a_z$$

$$R_{MP} = \vec{r}_P - \vec{r}_M$$

$$= (-2--1)a_x + (-3-2)a_y + (-4-1)a_z$$

$$= -1a_x - 5a_y - 5a_z$$

b) $|r_M| = \sqrt{(-1)^2 + 2^2 + 1^2}$

$$= \sqrt{6}$$

$$= \underline{\underline{2.449}}$$

c) $a_{MP} = \dfrac{\vec{R}_{MP}}{|R_{MP}|}$

$$= \frac{-1a_x + 5a_y - 5a_z}{\sqrt{(-1)^2 + (-5)^2 + (-5)^2}}$$

$$= \frac{-1a_x - 5a_y - 5a_z}{\sqrt{51}}$$

$$= \frac{-1}{\sqrt{51}}a_x - \frac{5}{\sqrt{51}}a_y - \frac{5}{\sqrt{51}}a_z$$

$a_x \cdot a_x = a_y \cdot a_y = a_z \cdot a_z = 1$

$a_x \cdot a_y = a_y \cdot a_z = a_z \cdot a_x = 0$

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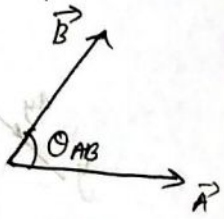
Dot product or Scalar product

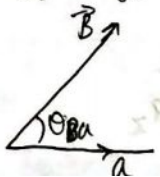
$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta_{AB} \longrightarrow \text{scalar quantity}$$

$$\cos\theta_{AB} = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|}$$



Application of dot product

The most important application of dot product is that it can be used for finding the component of vector in a given direction.

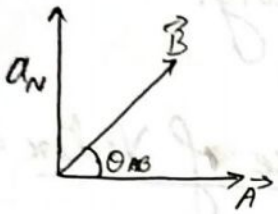


The scalar component of $\vec{B}$ ~~vector~~ in the direction of the unit vector a is $\vec{B}\cdot a$.

The vector component of $\vec{B}$ in the direction of unit vector a is $(\vec{B}\cdot a)a$

Cross product or Vector product

$$\vec{A}\times\vec{B} = |\vec{A}||\vec{B}|\sin\theta_{AB}\cdot a_N$$



$a_N \rightarrow$ normal unit vector

$$A\times(B+C) = A\times B + A\times C$$

$$A\times A = 0$$

$$A\times B = -B\times A$$

$$\vec{A} = A_x a_x + A_y a_y + A_z a_z$$

$$\vec{B} = B_x a_x + B_y a_y + B_z a_z$$

$$\vec{A}\times\vec{B} = \begin{vmatrix} a_x & a_y & a_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$a_x\times a_x = a_y\times a_y = a_z\times a_z = 0$$

$$a_x\times a_y = a_z$$

$$a_y\times a_z = a_x$$

$$a_z\times a_x = a_y$$

Rectangular Coordinate System (RCS)

$$\vec{A} = A_x a_x + A_y a_y + A_z a_z$$

Differential displacement or line element,

$$\boxed{dl = dx\,a_x + dy\,a_y + dz\,a_z} \rightarrow \text{vector qty}$$

Differential normal surface area,

$$\left.\begin{aligned} ds &= dy\,dz\,a_x \\ &\quad dx\,dz\,a_y \\ &\quad dx\,dy\,a_z \end{aligned}\right\} \rightarrow \text{vector qty}$$

Differential volume,

$$dV = dx\,dy\,dz \rightarrow \text{scalar qty}$$

Range of coordinate variables x, y, z

$$-\infty < x < \infty$$
$$-\infty < y < \infty$$
$$-\infty < z < \infty$$

Cylindrical Coordinate S/m

Coordinate variables are ρ, ϕ, z

$\rho \rightarrow$ radius of the cylinder

$\phi \rightarrow$ azimuthal angle

$z \rightarrow$ vertical distance

Any vector $\vec{A}$ can be represented by $\vec{A} = A_\rho a_\rho + A_\phi a_\phi + A_z a_z$

unit vectors are a_ρ, a_ϕ, a_z

Differential displacement, dl

$$\boxed{dl = d\rho\,a_\rho + \rho\,d\phi\,a_\phi + dz\,a_z} \rightarrow \text{vector qty}$$

Differential normal surface area,

$$\left.\begin{aligned} ds &= \rho\,d\phi\,dz\,a_\rho \\ &= d\rho\,dz\,a_\phi \\ &= \rho\,d\rho\,d\phi\,a_z \end{aligned}\right\} \text{vector qty}$$

Differential volume, dV

$$dV = d\rho\cdot\rho\,d\phi\cdot dz$$

Range of coordinate variables ρ, ϕ, z

$0 \le \rho \le \infty$

$0 \le \phi \le 2\pi$

$-\infty \le z \le \infty$

Spherical coordinate s/m (SCS)

Coordinate variables are r, θ, ϕ

$\vec{A} = A_r a_r + A_\theta a_\theta + A_\phi a_\phi$

Differential displacement,

$$\boxed{dl = dr\, a_r + r\, d\theta\, a_\theta + r \sin\theta\, d\phi\, a_\phi}$$

→ vector

Differential normal surface area,

$$\left.\begin{aligned} dS &= r\,d\theta \sin\theta \cdot a_r \\ &= r\,dr \sin\theta\, d\phi\, a_\theta \\ &= r\,dr\,d\theta\, a_\phi \end{aligned}\right\} \text{vector}$$

Differential volume,

$$dV = dr \cdot r\,d\theta \cdot r \sin\theta\, d\phi$$

→ scalar

Range; $0 \le \theta \le \pi$

$0 \le \phi \le 2\pi$

Vector Operators

Del operator or gradient operator.

$$\nabla = a_x \frac{\partial}{\partial x} + a_y \frac{\partial}{\partial y} + a_z \frac{\partial}{\partial z}$$

grad V or ∇V

In RCS

$$\boxed{\nabla V = a_x \frac{\partial V}{\partial x} + a_y \frac{\partial V}{\partial y} + a_z \frac{\partial V}{\partial z}}$$

In CCS

$$\boxed{\nabla V = a_\rho \frac{\partial V}{\partial \rho} + a_\phi \frac{1}{\rho} \frac{\partial V}{\partial \phi} + a_z \frac{\partial V}{\partial z}}$$

In SCS

$$\boxed{\nabla V = a_r \frac{\partial V}{\partial r} + a_\theta \frac{1}{r} \frac{\partial V}{\partial \theta} + a_\phi \frac{1}{r \sin\theta} \frac{\partial V}{\partial \phi}}$$

V is a scalar quantity and the resultant of grad V is a vector quantity.

Divergence of Vector quantity

$\vec{A} = A_x a_x + A_y a_y + A_z a_z$ {In RCS}

div A or $\nabla \cdot A$

$$\nabla \cdot A = \left(a_x \frac{\partial}{\partial x} + a_y \frac{\partial}{\partial y} + a_z \frac{\partial}{\partial z}\right) \cdot (A_x a_x + A_y a_y + A_z a_z)$$

$$\begin{aligned} &= a_x \cdot a_x \frac{\partial A_x}{\partial x} + a_x \cdot a_y \frac{\partial A_y}{\partial x} + a_x \cdot a_z \frac{\partial A_z}{\partial y} \\ &+ a_y \cdot a_x \frac{\partial A_y}{\partial y} + a_y \cdot a_y \frac{\partial A_y}{\partial y} + a_y \cdot a_z \frac{\partial A_z}{\partial y} \\ &+ a_z \cdot a_x \frac{\partial A_x}{\partial z} + a_z \cdot a_y \frac{\partial A_y}{\partial z} + a_z \cdot a_z \frac{\partial A_z}{\partial z} \end{aligned}$$

$$\boxed{\nabla \cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}$$

↳ scalar qty

In CCS

$$\boxed{\nabla \cdot A = \frac{1}{\rho} \frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}}$$

In SCS,

$$\boxed{\nabla \cdot A = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}A_\phi}$$

Curl of a Vector

del cross A

In RCS,

$$\nabla \times A = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\vec{A} = A_x a_x + A_y a_y + A_z a_z$$

In CCS,

$$\vec{A} = A_\rho a_\rho + A_\phi a_\phi + A_z a_z$$

$$\nabla \times A = \frac{1}{\rho}\begin{vmatrix} a_\rho & \rho a_\phi & a_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

In SCS,

$$\vec{A} = A_r a_r + A_\theta a_\theta + A_\phi a_\phi$$

$$\nabla \times A = \frac{1}{r^2 \sin\theta}\begin{vmatrix} a_r & r a_\theta & r\sin\theta\, a_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & r A_\theta & r\sin\theta\, A_\phi \end{vmatrix}$$

The circulation of a vector field A around a closed path L is equal to the surface integral of the curl of A over the open surface S bounded by L.

Stokes Theorem

Relates line integral & surface integral.

$$\oint_L A \cdot dl = \int_S (\nabla \times A)\, ds$$

Divergence theorem

total outward flux of a vector field A through the closed surface S is the same as volume integral of the divergence of A

$$\oint_S A \cdot ds = \int_V \nabla \cdot A \, dV$$

Static Electric Field. [Electrostatics]

Electrostatics deals with behavior of stationary electric charge.

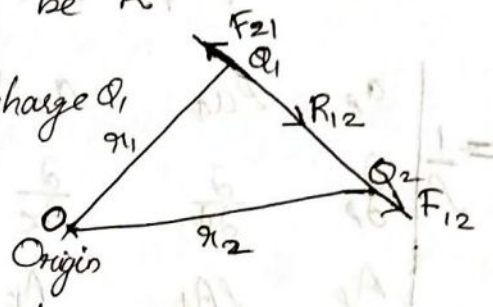
Coulomb's law

Coulomb's law states that the force F b/w two point charges Q_1 and Q_2 is directly proportional to the product Q_1Q_2 of two charges and inversely proportional to the ~~distanc~~ square of the distance R b/w ~~them~~ them and is along the line joining b/w ~~them~~.

~~Let the two~~

$$\boxed{F = \frac{k\,Q_1Q_2}{R^2}}$$

Let the two point charges be Q_1 and Q_2 and the distance b/w them be R



$\vec{r_1}$ → position vector of point charge Q_1

~~$\vec{r_1} = \vec{OQ_1}$~~

$\vec{r_1} = \vec{OQ_1}$

$\vec{r_2}$ - position vector of point charge Q_2

$\vec{r_2} = \vec{OQ_2}$

$\vec{R_{12}}$ = relative position vector of Q_1 & Q_2 : $R_{12} = \vec{r_2} - \vec{r_1}$

$$F = \frac{k\,Q_1Q_2}{R^2}$$

$$k = \frac{1}{4\pi\varepsilon_0} = 9\times10^9 \text{ m/F}$$

$\varepsilon_0 = 8.8542\times10^{-12}$ permittivity of free space.

$$F_{12} = k\,\frac{Q_1Q_2}{|\vec{r_2}-\vec{r_1}|^2}\,a_{R12}$$

$$F_{12} = k\,\frac{Q_1Q_2}{|\vec{r_2}-\vec{r_1}|^2}\,\frac{(\vec{r_2}-\vec{r_1})}{|\vec{r_2}-\vec{r_1}|} = \frac{k\,Q_1Q_2\,(\vec{r_2}-\vec{r_1})}{|\vec{r_2}-\vec{r_1}|^3}\ \text{N}$$

F_{21} on Q_1 due to Q_2 is $F_{21} = -F_{12}$

$a_{R_{21}} = -a_{R_{12}}$

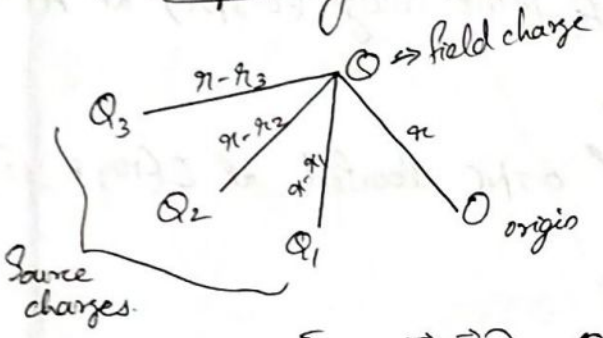
Here $Q_2 \rightarrow$ source charge

$Q_1 \rightarrow$ field charge (charge on which the force is required is called field charge)

The charge which produces force is called source charge.

$$\left\{ F_{21} \begin{array}{l} \leftarrow \text{field} \\ \uparrow \\ \text{sourc} \end{array} \right\}$$

Principle of Superposition



$$F = \frac{1}{4\pi\varepsilon_0}\left[\frac{QQ_1(\vec{r}-\vec{r}_1)}{|\vec{r}-\vec{r}_1|^3} + \frac{QQ_2(\vec{r}-\vec{r}_2)}{|r-r_2|^3} + \frac{QQ_3(\vec{r}-\vec{r}_3)}{|r-r_3|^3}\right]$$

$$F = \frac{1}{4\pi\varepsilon_0}\left[\sum_{k=1}^{3} \frac{QQ_k(\vec{r}-\vec{r}_k)}{|r-r_k|^3}\right]$$

Q: A charge of $-0.3\,\mu C$ is located at $A(25, -30, 15)$ in cm & a second charge of $0.5\,\mu C$ is at $B(-10, 8, 12)$ cm

Find $\underset{\text{(electric field)}}{E}$ and force at (a) origin (b) P at $(15, 20, 50)$ cm

Electric field Intensity or Electric field Strength

Electric field ~~Ex~~ It is a region in which test charge ~~produces~~ experiences a force.

It is the force per unit charge when placed in an electric field.

Generally, $E = \frac{F}{q}$ NC^{-1} or V/m

or $$\boxed{E = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{2} \frac{Q_i(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|^3}}$$

Soln:

$\cancel{F = \frac{1}{4\pi\varepsilon_0} \sum}$

Let the source charges be $-0.3\mu C$ located at $A(25,-30,15)$ cm and $0.5\mu C$ located at $B(-10,8,12)$ cm

$\vec{r}_1$ - position vector of point charge $(-0.3\mu C)$ at $A(25,-30,15)$

$= 25a_x - 30a_y + 15a_z$

$\vec{r}_2$ - position vector of $0.5\mu C$ located at $B(-10,8,12)$

$= -10a_x + 8a_y - 12a_z$

Case c) ~~r is the position~~ At origin $(0,0,0)$

$\vec{r} = 0a_x + 0a_y + 0a_z$

$$E = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{2} \frac{Q_i(r-r_i)}{|r-r_i|^3}$$

$$E = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q_1(\vec{r}-\vec{r}_1)}{|\vec{r}-\vec{r}_1|^3} + \frac{Q_2(\vec{r}-\vec{r}_2)}{|\vec{r}-\vec{r}_2|^3}\right]$$

$\vec{r}-\vec{r}_1 = 0a_x + 0a_y - 0a_z - (25a_x - 30a_y + 15a_z)$

$= -25a_x + 30a_y - 15a_z$

$|\vec{r}-\vec{r}_1| = \sqrt{(-25)^2 + 30^2 + (-15)^2} = 41.83$

$\vec{r}-\vec{r}_2 = 0a_x + 0a_y + 0a_y - (-10a_x + 8a_y + 12a_z)$

$= 10a_x - 8a_y - 12a_y$

$|\vec{r}-\vec{r}_2| = \sqrt{10^2 + (-8)^2 + (-12)^2} = 17.55$

~~$E = 9\times10^9 \times 10^{-2}$ coulomb $[25a_x - 30a_y$~~

$$E = 9\times10^{9}\times10^{2}\left[\frac{-0.3\times10^{-6}(-25a_x+30a_y+15a_z)}{41.83^{3}} + \frac{0.5\times10^{-6}(10a_x-8a_y-12a_z)}{(17.55)^{3}}\right]$$

$$= 9\times10^{5}\left[-\frac{0.3(-25a_x+30a_y-15a_z)}{41.83^{3}} + \frac{0.5(10a_x-8a_y-12a_z)}{17.55^{3}}\right]$$

$$= 9\times10^{5}\left[-4.098\times10^{-6}(-25a_x+30a_y-15a_z) + 9.25\times10^{-5}(10a_x-8a_y-12a_z)\right]$$

$$= 92.205a_x - 110.646a_y + 55.323a_z + 832.5a_x - 666a_y - 999a_z$$

$$E = 924.7a_x - 776.64a_y - 943.67a_z$$

Electric field due to continuous charge distribution

If the charges are distributed along a line or surface or in volume (region)
Eg: in cables, spherically charged conductor etc
Then the total charge 'Q' due to these distributions are obtained as

$Q = \int_L \rho_L \, dL$; ρ_L - line charge density in C/m

$Q = \int_V \rho_V \cdot dV$; ρ_V - volume charge density in C/m^3

$Q = \int_S \rho_S \cdot dV$; ρ_S - surface charge density in C/m^2

For line charge, $E = \int_L \frac{\rho_L \cdot dl}{4\pi\varepsilon_0 R^2} a_R$

For surface charge, $E = \int_S \frac{\rho_S \cdot dS}{4\pi\varepsilon_0 R^2} a_R$

For volume charge, $E = \int_V \frac{\delta_V \cdot dV}{4\pi\varepsilon_0 R^2} a_R$

Electric flux (Ψ)

Electric flux (Ψ) through a closed surface is equal to the charge enclosed by the surface $\Psi = Q$

Electric flux density (D)

Electric flux per square meter is called electric flux density.
unit - C/m^2
It is a vector quantity and it is a fraction of

charge and ~~fraction~~ position only.
It is independent of medium.

$$\psi = \oint_S D \cdot ds$$

$\boxed{D \cdot ds}$

$\int A \cdot dl = \int_S \nabla \times A$

$\int_S A \cdot ds = \iiint_V \nabla \cdot A \, dV$

Gauss law

Gauss law states that the total electric flux through any closed surface is equal to the total charge enclosed by that surface.

$$\psi = Q$$

$ds \rightarrow$ differential surface area

$$\psi = \oint_S D \cdot ds \longrightarrow (1)$$

$$Q = \iiint_V \rho_v \cdot dv \longrightarrow (2)$$

$$\boxed{\oint_S D \cdot ds = \iiint_V \rho_v \cdot dv} \longrightarrow \text{Integral form}$$

By applying divergence theorem

$$\boxed{\oint_S D \cdot ds = \iiint_V (\nabla \cdot D) dv} \leftarrow \text{Integral form}$$

$\nabla \cdot A$
$\nabla \times A$

$$\psi = Q.$$

$$\iiint_V (\nabla \cdot D) dv = \iiint_V \rho_v \, dv$$

$\boxed{\nabla \cdot D = \rho_v}$

maxwell's first

$$\leftarrow \boxed{\nabla \cdot D = \rho_v} \rightarrow \text{Differential form (point form)}$$

Application of Gauss law.

This is used for the calculation of electric field due to symmetric charge distribution.

a) Point Charge

Suppose a point charge Q is placed at the origin of spherical coordinate s/m. To determine D at P_1 choose a spherical surface (Gaussian surface) centered

at the origin containing P.D points along unit vector a_r.

$$\vec{D} = D_r a_r + D_\theta a_\theta + D_\phi a_\phi$$

$$dl = dr\, a_r$$

$$ds = r^2 \sin\theta\, d\theta\, d\phi \cdot a_r$$

$$Q = \phi$$

$$= \oint_s D \cdot ds$$

$$= \oint_s (D_r a_r + D_\theta a_\theta + D_\phi a_\phi)(r^2 \sin\theta\, d\theta\, d\phi \cdot a_r)$$

$$= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} D_r r^2 \sin\theta\, d\theta\, d\phi$$

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$$Q = D_r \cdot r^2 \left[(-\cos\theta)_0^{\pi} \cdot (\phi)_0^{2\pi} \right]$$

$$= D_r r^2 \left[-\cos\pi + \cos 0 \right] \left[2\pi - 0 \right]$$

$$= D_r r^2\, 2 \cdot 2\pi$$

$$Q = 4\pi r^2 D_r$$

$$\boxed{D_r = \frac{Q}{4\pi r^2}}$$

$$D = D_r a_r$$

$$\boxed{D = \frac{Q}{4\pi r^2} a_r}$$

$$\boxed{E = \frac{D}{\varepsilon} = \frac{Q}{4\pi r^2 \varepsilon} a_r}$$

$D = D_r a_r + D_\theta a_\theta + D_\phi a_\phi$

only r component

$\therefore D = D_r a_r$.

$$D = \varepsilon E$$

$$\boxed{\varepsilon = \varepsilon_0 \varepsilon_r}$$

$$\left\{ E = \frac{Q}{4\pi r^2 \varepsilon_0 \varepsilon_r} a_r \right\}$$

b) Electric field due to Infinite line charge.

Suppose the infinite line charge having line charge density ρ_L C/m lies along the z-axis. To determine D at a point P, we choose a cylindrical surface containing P. ~~The electric~~

The electric flux density vector D is normal to the cylindrical Guassian surface.

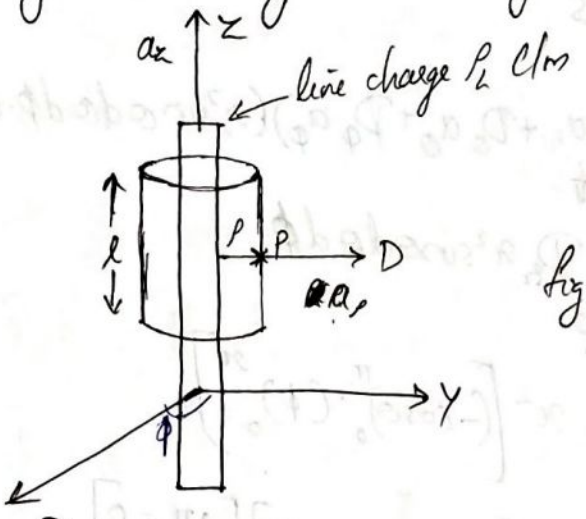

fig: Guassian surface about an infinite line charge.

$$D = D_\rho a_\rho + D_\phi a_\phi + D_z a_z \; ; \; dl = d\rho\, a_\rho + \rho\, d\phi\, a_\phi + dz\, a_z$$

Apply Guass law to an arbitrary length l of the line,

$$ds = \rho\, d\phi\, dz \cdot a_\rho$$

$$Q = \oint_S D \cdot ds$$

$$Q = \rho_L \cdot l$$

$$\oint_S D \cdot ds = \oint_S (D_\rho a_\rho + D_\phi a_\phi + D_z a_z) \cdot \rho\, d\phi\, dz \cdot a_\rho$$

$$= \oint_S D_\rho \rho\, d\phi\, dz = \int_{z=0}^{l} \int_{\phi=0}^{2\pi} D_\rho \rho\, d\phi\, dz$$

$$= D_\rho \rho \left[[\phi]_0^{2\pi} [z]_0^{l} \right]$$

$$= D_\rho \rho \left[2\pi \cdot l \right]$$

$$\oint D \cdot ds = 2\pi \rho l D_\rho$$

$$\rho_L \cdot l = 2\pi \rho l D_\rho$$

$$\rho_L = 2\pi \rho D_\rho$$

$$\boxed{D_\rho = \frac{\rho_L}{2\pi\rho}}$$

$$D = D_\rho a_\rho$$

$$\boxed{D = \frac{\rho_L}{2\pi\rho} a_\rho}$$

$$\boxed{E = \frac{D}{\varepsilon} = \frac{\rho_L}{2\pi\rho\varepsilon} a_\rho} \qquad \boxed{\varepsilon = \varepsilon_0 \varepsilon_r}$$

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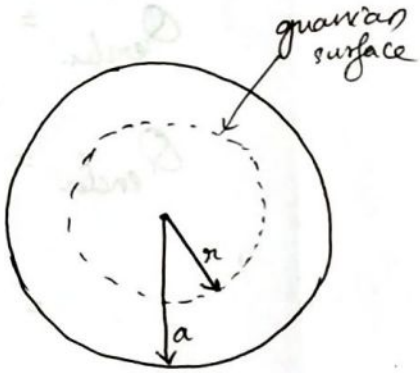
c) <u>Uniformly charged Sphere.</u>

Consider a sphere of radius 'a' filled with volume charge density ρ_v in C/m^3. To determine D every where, construct a gaussian surface for cases $r \le a$ and $r \ge a$ separately.

Case(i) : $r \le a$



$$Q_{enclosed} = \oint_S D \cdot ds \longrightarrow (1)$$

$$Q_{enclosed} = \int_V \rho_v \, dv$$

$$= \int_{r=0}^{r} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \rho_v \, r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$dl = dr\, a_r + r\, d\theta\, a_\theta + r \sin\theta\, d\phi\, a_\phi$$

$$dv = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$ds = r^2 \sin\theta \, d\theta \, d\phi \, a_r$$

$$Q_{enclosed} = \rho_v \left(\frac{r^3}{3}\right)_0^r \left[-\cos\theta\right]_0^\pi \left[\phi\right]_0^{2\pi}$$

$$= \rho_v \cdot \frac{r^3}{3} \times 2 \times 2\pi$$

$$Q_{enclosed} = \frac{4}{3}\pi r^3 \rho_v$$

$$\oint_S D \cdot ds = \oint_S (D_r a_r + D_\theta a_\theta + D_\phi a_\phi) \cdot (r^2 \sin\theta \, d\theta \, d\phi \, a_r)$$

$$= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} D_r \, r^2 \sin\theta \, d\theta \, d\phi$$

$$= D_r r^2 \left[-\cos\theta\right]_0^{\pi} \left[\phi\right]_0^{2\pi}$$

$$= D_r r^2 \times 2 \times 2\pi$$

$$\oint D \cdot ds = 4\pi r^2 D_r$$

$$\text{①} \Rightarrow \quad \frac{4}{3}\pi r^3 \rho_v = 4\pi r^2 D_r$$

$$\boxed{D_r = \frac{1}{3} r \rho_v}$$

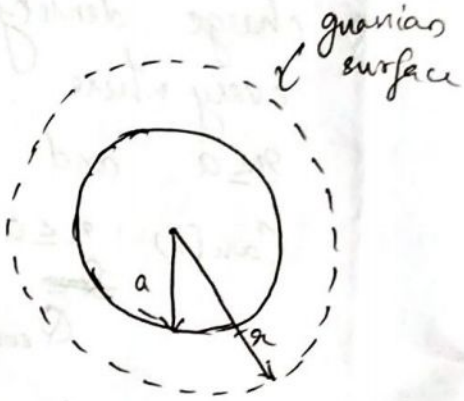
$$D = D_r a_r$$

$$\boxed{D = \frac{1}{3} r \rho_v a_r}$$

$$D = \varepsilon E$$

$$\boxed{E = \frac{D}{\varepsilon} = \frac{r \rho_v a_r}{3\varepsilon}}$$

Case (ii) $r \geq a$



$$Q_{encl} = \oint_S D \cdot ds$$

$$Q_{encl} = \int_V \rho_v \, dv$$

$$= \int_{r=0}^{a} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \rho_v \cdot r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \rho_v \left[\frac{r^3}{3}\right]_0^a \left[-\cos\theta\right]_0^{\pi} \left[\phi\right]_0^{2\pi}$$

$$= \rho_v \cdot \frac{a^3}{3} \times 2 \times 2\pi$$

$$Q_{encl} = \frac{4}{3}\pi a^3 \rho_v$$

$$\oint_S D \cdot ds = \iint_S (D_r a_r + D_\theta a_\theta + D_\phi a_\phi) \cdot (r^2 \sin\theta \, d\theta \, d\phi \, a_r)$$

$$= 4\pi r^2 D_r$$

$$4\pi r^2 D_r = \frac{4}{3}\pi a^3 \rho_v$$

$$\boxed{D_r = \frac{a^3 \rho_v}{3r^2}}$$

$$\boxed{D = D_r a_r = \frac{a^3 \rho_v}{3r^2} a_r}$$

$$\boxed{E = \frac{D}{\varepsilon} = \frac{a^3 \rho_v}{3\varepsilon r^2} a_r}$$

Electrical scalar potential (V)

Potential at a distance 'r' from the point charge is the workdone per unit charge by an external agent in transfering a test charge from infinity to that point.

$$V = -\int_{-\infty}^{r} E \cdot dl$$

$$\boxed{V = -\int_{\infty}^{r} E \cdot dl}$$

For n-point charges

$$\boxed{V(r) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \frac{Q_i}{|r - r_i|}}$$

★ Potential gradient (∇V) or Relation b/w E & V

~~Potential grad~~ Electric field intensity is the negative gradient of V.

$$\boxed{E = -\nabla V}$$

Proof

$$V = -\int E \cdot dl$$

Differentiating

$$dV = -E \cdot dl \longrightarrow (1)$$

$$E = E_x a_x + E_y a_y + E_z a_z$$

$$dV = -E \cdot dl = -(E_x a_x + E_y a_y + E_z a_z) \cdot (d_x a_x + d_y a_y + d_z a_z)$$

$$dV = -E_x d_x - E_y d_y - E_z d_z \longrightarrow (1a)$$

From calculus of multivariables

$$dV = \frac{\partial V}{\partial x} d_x + \frac{\partial V}{\partial y} d_y + \frac{\partial V}{\partial z} d_z \longrightarrow (2)$$

Comparing (1a) & (2)

$$E_x = -\frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}, \quad E_z = -\frac{\partial V}{\partial z} \longrightarrow (2a)$$

$$dV = -E \cdot dl = \left(\frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial y} a_y + \frac{\partial V}{\partial z} a_z\right) \cdot dl$$

$$\cancel{E = \frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial y} a_y + \frac{\partial V}{\partial z} a_z}$$

$$E \cdot dl = \nabla V \cdot dl$$

$$\boxed{E = -\nabla V}$$ Hence proof.

(M-4) Energy Stored in an Electric field.

If there are n point charges, then the total work done in assembling n charges is

$$\boxed{W_E = \frac{1}{2} \sum_{i=1}^{n} Q_i V_i}$$ (in Joules)

If instead of point charges, the region has a continuous charge distribution, then the above equation becomes

$$W_E = \frac{1}{2} \int \rho_v \, dv \cdot V$$

$$\rho_v = \nabla \cdot D$$ (differential form of Gauss law)

$$W_E = \frac{1}{2}\int_V (\nabla\cdot D)V\,dv$$

$$\left\{ (\nabla\cdot A)V = -A(\nabla V) \right.$$

$$W_E = -\frac{1}{2}\int_V D\cdot\nabla V\,dv$$

$D = \varepsilon E$

$$= -\frac{1}{2}\int_V \varepsilon E\cdot\nabla V\,dv$$

$$\boxed{E = -\nabla V}$$

$$= -\frac{1}{2}\int_V -\varepsilon E\cdot E\,dv$$

$$W_E = \frac{1}{2}\int_V \varepsilon E^2\,dV$$

$D^2 = \varepsilon^2 E^2$

$\frac{D^2}{\varepsilon} = \varepsilon E^2$

$$\boxed{W_E = \frac{1}{2}\int_V \frac{D^2}{\varepsilon}\,dV}$$

Electrostatic energy density w_E in J/m^3 is.

$\boxed{\varepsilon = \varepsilon_0\varepsilon_r}$

$$\boxed{w_E = \frac{dW_E}{dV} = \frac{1}{2}\varepsilon_0 E^2 = \frac{D^2}{2\varepsilon_0}}$$

$$D = \frac{\varepsilon_0^2\varepsilon_r^2 E^2}{\varepsilon_0\varepsilon_r}$$

(M4) <u>Capacitance</u>

$$\boxed{C = \frac{Q}{V}}$$ Farad

$$\boxed{W = \frac{1}{2}QV}$$

$Q = CV$

$$W = \frac{C^2V^2}{2C}$$

$$W = \frac{1}{2}CV^2 = \frac{Q^2}{2C}.$$

Capacitance ~~of~~ Coaxial capacitor or Coaxial cable or <u>Coaxial cylindrical Capacitor</u>.

A coaxial capacitor is essentially a coaxial cable or coaxial cylindrical capacitor. Consider length L of 2 coaxial conductors of inner radius (a) & outer radius 'b' (b>a) as shown in fig

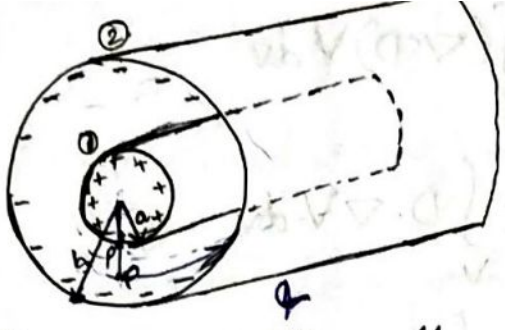


fig: Coaxial capacitor

Let the space b/w the conductor is filled with a homogeneous dielectric with permittivity ε. We assume that conductors 1 & 2 respectively carry $+Q$ & $-Q$ uniformly distributed on them. By applying Gauss's law to an arbitrary Gaussian cylindrical surface of radius ρ $(a < \rho < b)$.

$$Q = \oint_S D \cdot ds \qquad Q = \rho_L \cdot l$$

$$\oint_S D \cdot ds = \oint_S (D_\rho a_\rho + D_\phi a_\phi + D_z a_z) \cdot (\rho \, d\phi \, dz \cdot a_\rho)$$

$$= \int_{z=0}^{l} \int_{\phi=0}^{2\pi} D_\rho \rho \, d\phi \, dz$$

$$= D_\rho \rho \left[[\phi]_0^{2\pi} [z]_0^{l} \right]$$

$$\oint D \cdot ds = 2\pi l \rho D_\rho$$

$$\rho_L \cdot l = 2\pi l \rho D_\rho$$

$$\rho_L = 2\pi \rho D_\rho$$

$\oint D \cdot ds = Q$

$$D_\rho = \frac{\rho_L}{2\pi\rho} \qquad D = D_\rho a_\rho$$

$$D = \frac{\rho_L}{2\pi\rho} a_\rho \qquad D = \varepsilon E$$

$$E = D/\varepsilon \qquad E = \frac{\rho_L}{2\pi\varepsilon\rho} a_\rho$$

Potential at P is, $V = -\int_b E \cdot dl$

$$= -\int_b^{\rho} \frac{\rho_L}{2\pi\varepsilon_0 \rho} a_\rho \cdot dl \qquad \varepsilon = \varepsilon_0 \varepsilon_r, \quad \varepsilon_r = 1$$

$$= \int_{-b}^{\rho} \frac{\rho_L}{2\pi\varepsilon_0 \rho} a_\rho \cdot (d\rho a_\rho + \rho \, d\phi \, a_\phi + dz \, a_z)$$

$$= -\frac{\rho_L}{2\pi\varepsilon_0}\int_b^{\rho}\frac{1}{\rho}\,d\rho$$

$$V = -\frac{\rho_L}{2\pi\varepsilon_0}\left[\ln\rho\right]_b^{\rho}$$

$$= \frac{-\rho_L}{2\pi\varepsilon_0}\left[\ln\rho - \ln b\right]$$

$$= \frac{-\rho_L}{2\pi\varepsilon_0}\ln\left(\frac{\rho}{b}\right)$$

$$\oint E\cdot 2\pi\rho l = Q$$

$$E = \frac{Q}{\varepsilon 2\pi\rho l}$$

$$V = -\int_b^a E\cdot dl$$

$$V = \frac{\rho_L}{2\pi\varepsilon_0}\ln\left(\frac{b}{\rho}\right)$$

$$Q = \rho_L\cdot L$$

$L \rightarrow$ length of cylinder

$$C = \frac{Q}{V} = \frac{\rho_L\cdot L}{\frac{\rho_L}{2\pi\varepsilon_0}\ln(b/\rho)}$$

l.

$Q = \rho_l\cdot l$

$Q = \rho_v, \rho_s, \rho_l$

$$= \frac{L\cdot 2\pi\varepsilon_0}{\ln(b/\rho)}$$

$\iiint \rho_v\, dV$

$$\boxed{C = \frac{2\pi\varepsilon_0 L}{\ln(b/\rho)}}$$

Potential difference b/w the cylinders, put $\rho = a$

$$V = \frac{\rho_L}{2\pi\varepsilon_0}\ln(b/a)$$

$$C = \frac{2\pi\varepsilon_0 L}{\ln(b/a)}$$

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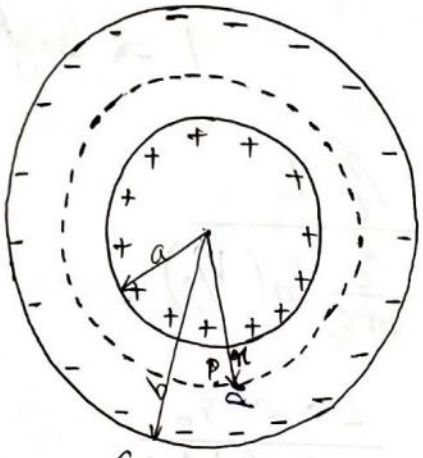
Capacitance of Isolated Sphere.

A spherical capacitor is the case of two concentric spherical conductors. Consider the inner sphere of radius 'a' and outer sphere of radius 'b' ($b > a$) separated by a dielectric medium with permitivity ε. We assume charges $+Q$ & $-Q$ on the inner and outer spheres respectively. By applying Gauss law to an arbitary gaussian spherical surface of radius 'r' ($a < r < b$)



$$Q_{enc} = \oint D \cdot ds$$

$$\oint D \cdot ds = \oint (D_r a_r + D_\phi a_\phi + D_\theta a_\theta) \cdot (r^2 \sin\theta \, d\theta \, d\phi \, a_r)$$

$$= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} D_r r^2 \sin\theta \, d\theta \, d\phi \qquad Q =$$

$$= r^2 D_r \left[-\cos\theta\right]_0^{\pi} \left[\phi\right]_0^{2\pi}$$

$$= r^2 D_r \times 2 \times 2\pi$$

$$= 4\pi r^2 D_r$$

$$Q = 4\pi r^2 D_r$$

$$D_r = \frac{Q}{4\pi r^2} \qquad D = \frac{Q}{4\pi r^2} a_r$$

$$E = \frac{D}{\varepsilon} = \frac{Q}{4\pi\varepsilon r^2} a_r$$

The potential at P is $V = -\int_b^r E \cdot dl$

$$V = -\int_b^r \frac{Q}{4\pi\varepsilon r^2} \cdot a_r \cdot (dr\, a_r + r\, d\theta\, a_\theta + r \sin\theta\, d\phi\, a_\phi)$$

$$= -\int_b^r \frac{Q}{4\pi\varepsilon_0 r^2} dr$$

$$= -\frac{Q}{4\pi\varepsilon} \int_b^r \frac{1}{r^2} dr$$

$$= \frac{Q}{4\pi\varepsilon}\left[\frac{1}{r}\right]_b^r$$

$$V = \frac{Q}{4\pi\varepsilon}\left[\frac{1}{r} - \frac{1}{b}\right]$$

Potential difference b/w the conductors

$r = a$

$$V = \frac{Q}{4\pi\varepsilon}\left[\frac{1}{a} - \frac{1}{b}\right]$$

Letting $b \to \infty$ gives the capacitance of isolated sphere

$$V = \frac{Q}{4\pi\varepsilon}\left[\frac{1}{a} - \frac{1}{\infty}\right]$$

$$V = \frac{Q}{4\pi\varepsilon a}$$

$$\frac{Q}{V} = 4\pi\varepsilon a$$

$$\cancel{C = \frac{Q}{V} = \frac{Q}{Q/4\pi\varepsilon a}}$$

$$C = \frac{Q}{V}$$

$$C = 4\pi\varepsilon a$$

$$\cancel{C = 4\pi a \varepsilon} \quad \boxed{C = 4\pi\varepsilon a}$$

Poisson & Laplace Equation

Poisson eqn

It is a partial differential equation relating the potential at any point in terms of volume density of charge

i.e.,
$$\boxed{\nabla^2 V = -\frac{\rho_v}{\varepsilon}}$$

Proof:-

Gauss law in point form.

$$\nabla \cdot D = \rho_v$$

$$D = \varepsilon E$$

$$E = -\nabla V$$

$$\nabla \cdot \varepsilon E = \rho_v$$

$$\nabla \cdot \varepsilon(-\nabla V) = \rho_v$$

$$-\nabla^2 \varepsilon V = \rho_v$$

$$\Rightarrow \boxed{\nabla^2 V = \frac{-\rho_v}{\varepsilon}}$$

Laplace Equation

Laplace equation is the special case of Poisson eqn for charge free region.

i.e., $\rho_v = 0$

$$\boxed{\nabla^2 V = 0}$$

In RCS

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

In CCS,

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2}+\frac{\partial^2 V}{\partial z^2}=0$$

In SCS,

$$\nabla^2 V=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}=0$$

Ampere's Circuital law

Ampere's circuital law states that the line integral of H around any closed path is the same as the net current enclosed by the path.

$$\boxed{\oint_l H\cdot dl = I_{enc}}$$

H → magnetic field intensity

Ampere's circuital law is applied to determine H, when the current distribution is symmetrical.

Differential form or point form of Ampere's circuital law

According to Stoke's theorem,

$$\oint_l H\cdot dl = \int_S (\nabla\times H)\cdot ds$$

$\oint H\cdot dl = I_{enc}$

$$I_{enc} = \int_S J\cdot ds$$

J → current density vector

$I = J\cdot A$

A → area of cross section.

$$\oint_l H\cdot dl = \int_S J\cdot ds$$

$$\int_S (\nabla\times H)\cdot ds = \int_S J\cdot ds$$

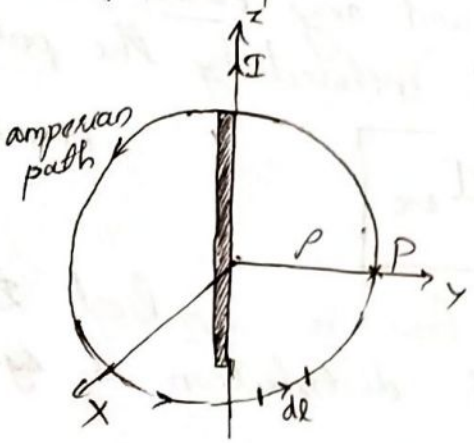
$$\boxed{\nabla\times H = J}$$

Application of Ampere's Circuital law.

(a) Infinite line current

Consider an infinitely long filamentary current I along the z-axis.

To determine H at P we allow a closed path to pass through P and we choose a concentric circle as the amperian path.



$$H = H_\rho a_\rho + H_\phi a_\phi + H_z a_z$$

$$\oint_l H \cdot dl = I_{enc}$$

$$I_{enc} = \oint_l (H_\phi a_\phi) \cdot (d\rho\, a_\rho + \rho\, d\phi\, a_\phi + dz\, a_z)$$

$$= \int_{\phi=0}^{2\pi} \rho H_\phi \, d\phi$$

$$= \rho H_\phi [\phi]_0^{2\pi}$$

$$I_{enc} = 2\pi \rho H_\phi$$

$$\boxed{H_\phi = \frac{I}{2\pi\rho}}$$

$$H = H_\phi a_\phi \Rightarrow \boxed{H = \frac{I}{2\pi\rho} a_\phi}$$

Faraday's law

Faraday established that a time varying magnetic field produces an induced voltage, it is also called emf in a closed ckt which causes flow of current.

Induced emf in any closed circuit is equal to the time rate of change of magnetic flux linkage by the ckt.

$$\boxed{V_{emf} = \frac{-d\lambda}{dt} = -N\frac{d\psi}{dt}}$$

$\lambda \rightarrow$ ~~mag~~ magnetic flux linkage
$N \rightarrow$ no: of turns in the ckt
$\psi \rightarrow$ total magnetic flux

For a ckt with single turn, $N=1$

$$V_{emf} = -\frac{\partial \psi}{\partial t}$$

$$\boxed{\psi = \int_S B \cdot ds}$$ $B \Rightarrow$ magnetic flux density

$$V_{emf} = \oint_L E \cdot dl = -\frac{\partial}{\partial t}\int_S B \cdot ds \rightarrow$$

Integral form

$$\boxed{\oint E \cdot dl = -\int_S \frac{\partial B}{\partial t} ds}$$

According to Stokes theorem

$$\oint E \cdot dl = \int_S (\nabla \times E) \cdot ds$$

$$\therefore \int_S (\nabla \times E) \cdot ds = -\frac{\partial}{\partial t}\int_S B \cdot ds = -\int_S \frac{\partial B}{\partial t} \cdot ds$$

$$\Rightarrow \boxed{\nabla \times E = -\frac{\partial B}{\partial t}}$$

Differential / point form of Faraday's law
Maxwell's eqn for time varying field.

Magnetic scalar & Magnetic Vector Potentials

$V_m \rightarrow$ Magnetic scalar potential

It is defined in a region where $J = 0$

$J \rightarrow$ current density

$A \rightarrow$ magnetic vector potential.

$$\boxed{H = -\nabla V_m, \quad J = 0}$$

$$\boxed{J = \nabla \times H = \nabla \times (-\nabla V_m) = 0}$$

$\rightarrow$ Curl of grad of a scalar fn is zero

$V \rightarrow$ scalar $\boxed{\nabla \times \nabla V = 0}$

$\rightarrow$ Div of grad of a vector valued fn is also zero

$$\boxed{\nabla \cdot (\nabla \times A) = 0}$$

$A \rightarrow$ vector

$\nabla \cdot B = 0$

$B = \mu H$

$\mu \rightarrow$ permeability

$\mu = \mu_0 \mu_r$

$D = \varepsilon E$

$B = \mu H$

$\rightarrow$ Dimensions of V_m are in amperes

$\nabla \cdot B = 0$

$$\boxed{\nabla \cdot B = \nabla \cdot \mu H = \mu \nabla \cdot H = 0}$$

$\boxed{\nabla^2 V_m = 0, \quad J = 0}$ then V_m satisfies laplace's eqn

$$\boxed{V_m = -\int_b^a H \cdot dl}$$

where a specified path or type of path must be selected.

(module-3)

Magnetic Vector Potential (A)

A is defined in regions where J is zero or non-zero

$\boxed{\nabla \cdot B = 0}$ unit of A $\rightarrow$ weber/Meter

$\boxed{B = \nabla \times A}$

$$A = \oint \frac{\mu_0}{4\pi R} I \, dl \quad \rightarrow \text{line current}$$

$$A = \oint \frac{\mu_0 k}{4\pi R} ds \quad \rightarrow \text{surface current.}$$

$$A = \oint \frac{\mu_0 J \, dv}{4\pi R} \rightarrow \text{volume current}$$

$$\boxed{\Psi = \int_S B \cdot ds = \int_S (\nabla \times A) \cdot ds = \int_l A \cdot dl}$$ (apply stokes theorem)

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Energy stored in magnetic field

Consider a differential volume in a magnetic field. This volume is covered with conducting sheet at top and bottom surfaces carrying current ΔI. Assume the region is filled with differential volumes. Each volume has an inductance

$$\boxed{\Delta L = \frac{\Delta \Psi}{\Delta I}}$$

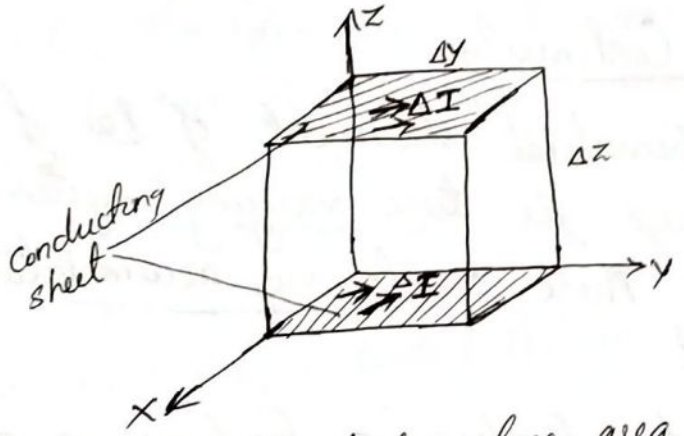


fig: differential volume in a magnetic field

Differential surface area (ΔS) $\Rightarrow \Delta x \cdot \Delta z$

$\Delta I = H \cdot \Delta y$ {using Ampere's circuital law

$I_{enc} = \oint_l H \cdot dl$

$= H \oint dl$

$= H \cdot \Delta y$ effective length

Differential inductance, $\Delta L = \frac{\Delta \Psi}{\Delta I}$

$$= \frac{B \cdot \Delta S}{\Delta I} = \frac{B \cdot \Delta x \Delta z}{H \cdot \Delta y}$$

Energy stored in the inductance of differential volume dv,

$$\Delta W_m = \frac{1}{2} \Delta L \Delta I^2$$

$\{ W_m = \frac{1}{2} L I^2$

$W_E = \frac{1}{2} L I^2$

$$= \frac{1}{2} \frac{B \cdot \Delta x \Delta z}{H \cdot \Delta y} (H \Delta y)^2$$

$$\Delta W_m = \frac{1}{2} B \, \Delta x \Delta z \, H \cdot \Delta y$$

$$\Delta W_m = \frac{1}{2} B \cdot dv\, H$$

$$= \frac{1}{2} B \cdot H \cdot \Delta V$$

The magnetostatic energy density w_m in $J/m^3 = \frac{\Delta W_m}{\Delta V}$

$$w_m = \frac{1}{2} \frac{BH\Delta V}{\Delta V} = \frac{1}{2} BH$$

$$w_m = \frac{1}{2} \mu H^2 = \frac{1}{2\mu} B^2 \quad (B = \mu H) \quad H = \frac{B}{\mu}$$

The total energy in a magnetostatic field

$$W_m = \int w_m dV = \int \frac{1}{2} B \cdot H\, dv$$

$$\boxed{W_m = \frac{1}{2} \int \frac{B^2}{\mu} dV}$$

Equation of Continuity

It is the mathematical statement of law of conservation of electric charge for time varying system ~~and it~~ It states that there can be no accumulation of charges at any point.

The time rate of decrease of charge in a given volume must be equal to the net outward current flow through the surface of the volume.

Let Q_{in} be the total charge enclosed in the closed surface. I_{out} is the outward current flow through the surface

$$\boxed{I_{out} = \oint_S J \cdot ds = -\frac{dQ}{dt}}$$

$$\oint_S J \cdot ds = \iiint_V \nabla \cdot J\, dv \quad \text{(Gauss's divergence theorem)}$$

$$\oint J \cdot ds = -\frac{dQ}{dt} \qquad Q = \iiint_V \rho_v\, dv$$

$$\iiint_V \nabla \cdot J \, dv = -\frac{d}{dt} \iiint_V \rho_v \, dv$$

$$= -\iiint_V \frac{d}{dt} \rho_v \, dv$$

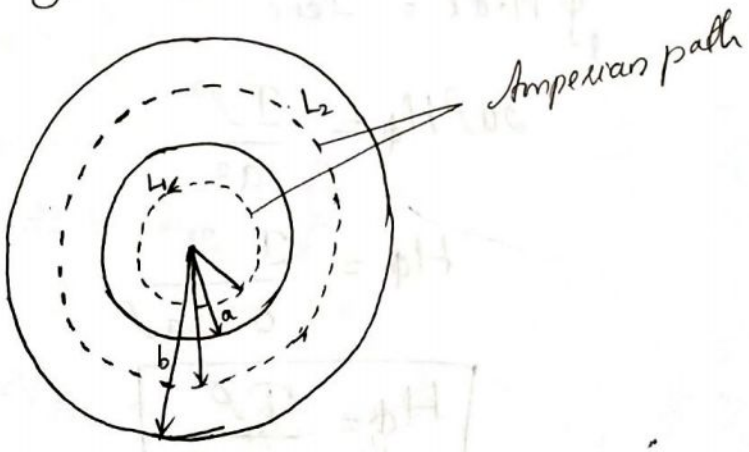
$$\nabla \cdot J = -\frac{d}{dt} \rho_v$$

$$\boxed{\nabla \cdot J + \frac{d}{dt} \rho_v = 0}$$ (differential form)

Application of Ampere's circuital law

(b) Infinitely long transmission line

Consider ~~long~~ an infinitely long transmission line consisting of two concentric cylinders having their axes along z direction. The ~~cross~~ cross section of line is shown in figure. The inner conductor has radius 'a' and carries current I. The outer conductor has radius 'b' and carries ~~curr~~ return current -I. Assume the whole current I is enclosed by L_2 and the current is uniformly distributed in both conductors.



Case i) $0 < \rho < a$

$$\oint_{L_1} H \cdot dl = I_{enc} = \int_S J \cdot ds$$

$J = \frac{I}{A}$

$A = \pi a^2$

$$J = \frac{I}{\pi a^2} a_z$$

$dl = d\rho a_\rho + \rho d\phi a_\phi + dz a_z$

$ds = d\rho \rho d\phi a_z$

$$I_{enc} = \int_S \frac{I}{\pi a^2} \cdot a_z \cdot ds$$

$$I_{enc} = \int_S \frac{I}{\pi a^2} a_z \cdot d\rho\, \rho\, d\phi\, a_z$$

$L_1 \rightarrow$ radius $\rightarrow \rho$

$$= \int_{\rho=0}^{\rho} \int_{\phi=0}^{2\pi} \frac{I}{\pi a^2} \rho\, d\rho\, d\phi$$

$$= \frac{I}{\pi a^2} \left[\frac{\rho^2}{2}\right]_0^{\rho} \left[\phi\right]_0^{2\pi}$$

$$= \cancel{\frac{I}{\pi a^2} \times \frac{\rho^2}{2} \times 2\pi} = \frac{I}{\pi a^2} \frac{\rho^2}{2} \times 2\pi$$

$\cancel{I_{enc} = I}$ $\quad I_{enc} = \frac{I \rho^2}{a^2}$

Now, $\oint H \cdot dl = \oint_l H_\phi a_\phi (dl\, a_\rho + \rho\, d\phi\, a_\phi + dz\, a_z)$

$$= H_\phi \int_{\phi=0}^{2\pi} \rho\, d\phi$$

$$= H_\phi \rho \left[\phi\right]_0^{2\pi}$$

$$\oint H \cdot dl = 2\pi \rho H_\phi$$

$$\oint_l H \cdot dl = I_{enc}$$

$$2\pi \rho H_\phi = \frac{I \rho^2}{a^2}$$

$$H_\phi = \frac{I \rho^2}{a^2 2\pi \rho}$$

$$\boxed{H_\phi = \frac{I \rho}{2\pi a^2}}$$

Case ii) $a < \rho < b$, Choose L_2 as the amperian path

$$\oint_{L_2} H \cdot dl = I_{enc}$$

$$I_{enc} = I$$

$$\oint_0 H \cdot dl = \oint_l H_\phi a_\phi \left(d\rho a_\rho + \rho d\phi a_\phi + dz a_z \right)$$

$$= H_\phi \int_{\phi=0}^{2\pi} \rho \, d\phi$$

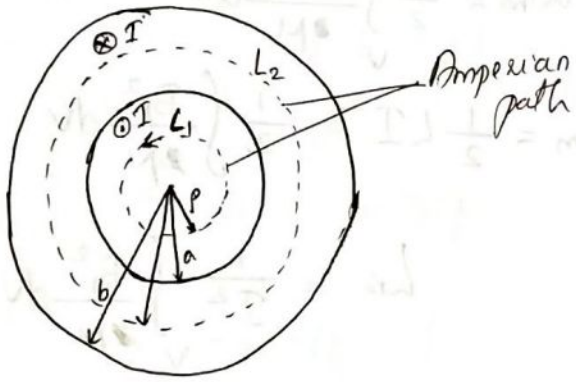
$$= 2\pi \rho H_\phi$$

$$I = 2\pi \rho H_\phi$$

$$\boxed{H_\phi = \frac{I}{2\pi\rho}}$$

Determine the Self Inductance of Coaxial cable of inner radius 'a' & outer radius 'b'.

Consider a cross section of the cable as shown in figure. Let l be the length of coaxial cable.



Case (i) $0 < \rho < a$

$$J = \frac{I}{A} \qquad A = \pi a^2$$

$$\oint H \cdot dl = I_{enc} = \int_S J \cdot ds$$

$$J = \frac{I}{\pi a^2} \cdot a_z$$

$$I_{enc} = \int_S \frac{I}{\pi a^2} a_z \cdot dz$$

$$= \int_S \frac{I}{\pi a^2} a_z \, \rho d\rho d\phi dz$$

$$= \int_{\rho=0}^{\rho} \int_{\phi=0}^{2\pi} \frac{I}{\pi a^2} \rho d\rho d\phi$$

$$I_{enc} = \frac{I \rho^2}{a^2}$$

$$\oint H\cdot dl = \oint H_\phi a_\phi \left(d\rho a_\rho + \rho d\phi a_\phi + dz a_z\right)$$

$$= H_\phi \int_{\phi=0}^{2\pi} \rho d\phi$$

$$= 2\pi\rho H_\phi$$

$$\oint H\cdot dl = I_{enc}$$

$$2\pi\rho H_\phi = \frac{I\rho^2}{a^2}$$

$$H_\phi = \frac{I\rho}{2\pi a^2}$$

$$H_1 = \frac{I\rho}{2\pi a^2}\cdot d\phi$$

$$B_1 = \mu H_1 = \frac{I\rho\mu}{2\pi a^2} a_\phi$$

$$dl = d\rho a_\rho + \rho d\phi a_\phi + dz a_z$$

$$W_m = \frac{1}{2}\int_V \frac{B_1^2}{2\mu} dV$$

$$dV = d\rho \cdot \rho d\phi \cdot dz = \rho d\rho d\phi dz$$

$$W_m = \frac{1}{2} L I^2 = \frac{1}{2}\int \frac{B_1^2}{2\mu} dV$$

$$L_{in} = \frac{1}{I^2}\int_V \frac{B_1^2}{2\mu} dV$$

$$= \frac{1}{I^2}\int_{\rho=0}^{a}\int_{\phi=0}^{2\pi}\int_{z=0}^{l} \frac{1}{2\mu}\left(\frac{I\rho\mu}{2\pi a^2}\right)^2 \rho d\rho d\phi dz$$

$$= \frac{\mu}{4\pi^2 a^4}\int_{\rho=0}^{a}\int_{\phi=0}^{2\pi}\int_{z=0}^{l} \rho^3 d\rho d\phi dz$$

$$= \frac{\mu}{4\pi^2 a^4}\left[\frac{\rho^4}{4}\right]_0^a \left[\phi\right]_0^{2\pi} \left[z\right]_0^l$$

$$= \frac{\mu}{4\pi^2 a^4} \times \frac{a^4}{4} \times 2\pi \times l$$

$$\cancel{L_{in} = \frac{\mu}{8\pi} a_\phi^2 \times a^2 \times l}$$

$$\frac{1}{I^2}\mu \iiint \frac{\mu^2 I^2 \rho^2}{4\pi^2 a^4}$$

$$\cancel{L_{in} = \frac{\mu a^2 l}{16\pi}}$$

$$\frac{\mu l}{8\pi}$$

$$L_{in} = \frac{\mu}{8\pi} \times l$$

$$\boxed{L_{in} = \frac{\mu l}{8\pi}}$$

<u>Case(ii)</u> $a < \rho < b$

$$\oint_{L_2} H \cdot dl = I_{enc}$$

$$I_{enc} = I$$

$$\oint_l H \cdot dl = \oint H_\phi a_\phi (d\rho a_\rho + \rho d\phi a_\phi + d_z a_z)$$

$$= H_\phi \int_{\phi=0}^{2\pi} \rho \, d\phi$$

$$\oint_l H \cdot dl = 2\pi \rho H_\phi$$

$$2\pi \rho H_\phi = I$$

$$H_\phi = \frac{I}{2\pi\rho}$$

$$H_2 = \frac{I}{2\pi\rho} a_\phi$$

$$B_2 = \mu H_2 = \frac{\mu I}{2\pi\rho} a_\phi$$

$$W_m = \frac{1}{2} \int_V \frac{B_2^2}{\mu} dV$$

$$W_m = \frac{1}{2} L I^2 = \frac{1}{2} \int_V \frac{B_2^2}{\mu} dV$$

$$L_{ext} = \frac{1}{I^2} \int_V \frac{B_2^2}{\mu} dV$$

$$L_{ext} = \frac{1}{I^2}\int_{\rho=a}^{b}\int_{\phi=0}^{2\pi}\int_{z=0}^{l} \frac{1}{\mu}\left(\frac{\mu I}{2\pi\rho}\right)^2 \rho\, d\rho\, d\phi\, dz$$

$$= \frac{\mu}{4\pi^2}\int_{\rho=a}^{b}\int_{\phi=0}^{2\pi}\int_{z=0}^{l} \frac{1}{\rho}\, d\rho\, d\phi\, dz$$

$$= \frac{\mu}{4\pi^2}\left[\ln\rho\right]_a^b \left[\phi\right]_0^{2\pi}\left[z\right]_0^l$$

$$= \frac{\mu}{4\pi^2}(\ln b - \ln a)\times 2\pi \times l$$

$$\boxed{L_{ext} = \frac{\mu l}{2\pi}\ln(b/a)} \quad \text{Henry (H)}$$

Total Inductance, $L = L_{in} + L_{ext}$

$$= \frac{\mu l}{8\pi} + \frac{\mu l}{2\pi}\ln(b/a)$$

$$\boxed{L = \frac{\mu l}{2\pi}\left[\frac{1}{4} + \ln(b/a)\right]} \quad \text{henry (H)}$$

Total inductance per unit length,

$$L' = \frac{L}{l} = \frac{1}{l}\left[\frac{\mu l}{2\pi}\left(\frac{1}{4} + \ln(b/a)\right)\right]$$

$$\boxed{L' = \frac{\mu}{2\pi}\left[\frac{1}{4} + \ln(b/a)\right]} \quad H/m$$

Inductance of two wire transmission line

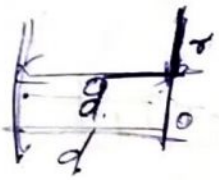
Determine the inductance per unit length of a 2-wire transmission line with separation distance d. Each wire has radius 'a'.

$$L_{in} = \frac{\mu l}{8\pi}$$ (L_{in} is same as that in coaxial cable)

$$L_{ext} = \frac{1}{I^2}\int_V \frac{B_2^2}{\mu}\, dV$$

$$= \frac{1}{I^2}\int_{z=0}^{l}\int_{\phi=0}^{2\pi}\int_{\rho=a}^{d-a} \frac{1}{\mu}\left(\frac{\mu I}{2\pi\rho}\right)^2 \rho\, d\rho\, d\phi\, dz$$

$$= \frac{\mu}{4\pi^2}\int_{z=0}^{l}\int_{\phi=0}^{2\pi}\int_{\rho=a}^{d-a} \frac{1}{\rho}\, d\rho\, d\phi\, dz$$



$$= \frac{\mu}{4\pi^2}\left[\ln\rho\right]_a^{d-a}\left[\phi\right]_0^{2\pi}\left[z\right]_0^{l}$$

$$= \frac{\mu}{4\pi^2}\left[\ln(d-a) - \ln a\right] \times 2\pi \times l$$

$$L_{ext} = \frac{\mu l}{2\pi}\ln\left(\frac{d-a}{a}\right)$$

Since the two wires are symmetrical, total inductance, $L = 2(L_{in} + L_{ext})$

$$L = 2\left[\frac{\mu l}{8\pi} + \frac{\mu l}{2\pi}\ln\left(\frac{d-a}{a}\right)\right]$$

$$\boxed{L = \frac{\mu l}{\pi}\left[\frac{1}{4} + \ln\left(\frac{d-a}{a}\right)\right]} \text{ H}$$

Max