50.034 Probability and Statistics
Week 11 Review Problems

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1 Problem 1: Affine mappings from intervals

1. $\dfrac{x+7}{4}+1$
2. $\dfrac{5(x+1)}{13}-5$
3. $\dfrac{7(x-3)}{2}+10$
4. $\dfrac{x-2}{2}+2$

2 Problem 2: Uniform distribution

Let the set of data points $x_1, \cdots, x_n$ be A.

1. For this uniform probability model, $f(x|c) = \dfrac{1}{c-1}$ for $1 \leq x \leq c$ and 0 elsewhere. The model is therefore defined for $c > 1$.
2. The data $x_1, \cdots, x_n$ has values $x_i > 1$. Intuitively, c has to take the value of the largest x_i, in order to maximize the likelihood function. This is akin to fitting the uniform probability model from values 1 to $max(A)$, which gives us the maximum likelihood of the dataset.

3 Problem 3: Inverse Gaussian distribution

For i.i.d. data, $x_1, \cdots, x_n$,

$$L(x, u) = \prod_{i=1}^{n} f(x|u) \tag{1}$$

Using the log likelihood, products are converted to sums:

$$\ln \prod_{i=1}^{n} f(x|u) = \sum_{i=1}^{n} \ln(f(x|u)) \tag{2}$$

Substituting in $f(x|u)$, we get

$$\frac{u}{\sqrt{2\pi}}\sum_{i=1}^{n}\left(\ln\left(x_i^{-\frac{3}{2}}\right)+\ln\left(e^{-\frac{(x_i-u)^2}{x_i}}\right)\right) \tag{3}$$

Using $\log a^b = b\log a$, the expression simplifies to

$$\frac{u}{\sqrt{2\pi}}\sum_{i=1}^{n}\left(-\frac{3}{2}\ln(x_i)-\frac{(x_i-u)^2}{x_i}\right) \tag{4}$$

Expanding the terms for easier differentiation:

$$-\frac{3u}{2\sqrt{2\pi}}\sum_{i=1}^{n}\ln(x_i)-\frac{u}{\sqrt{2\pi}}\sum_{i=1}^{n}x_i+\frac{2u^2}{\sqrt{2\pi}}-\frac{u^3}{\sqrt{2\pi}}\sum_{i=1}^{n}\frac{1}{x_i} \tag{5}$$

Computing the partial derivative w.r.t. u and equating to 0,

$$\frac{\partial\log(L)}{\partial u}=-\frac{3}{2\sqrt{2\pi}}\sum_{i=1}^{n}\ln(x_i)-\frac{1}{\sqrt{2\pi}}\sum_{i=1}^{n}x_i+\frac{4u}{\sqrt{2\pi}}-\frac{3u^2}{\sqrt{2\pi}}\sum_{i=1}^{n}\frac{1}{x_i}=0 \tag{6}$$

Now we have a quadratic equation in u and can solve for u_{MLE}.

4 Problem 4: Expectations and Correlation Coefficients

1. The normalizing constant c can be found by equating the following triple integral to 1:

$$\int_2^3\int_0^1\int_{\frac{1}{2}}^{\frac{3}{2}} c(x+4y+z^2+0.5)\,dx\,dy\,dz=1 \tag{7}$$

   Evaluating, $c\left(\frac{7}{2}+\frac{19}{3}\right)=1$. Solving for c gives $c=\frac{6}{59}\approx 0.1016$.

2. The marginal f_x is the integral of the joint w.r.t. y and z, which was previously calculated in the process of the triple integral,

$$f_x=c\left[xy+2y^2+\frac{41y}{6}\right]_0^1=c\left(x+\frac{53}{6}\right)=\frac{6x}{59}+\frac{53}{59} \tag{8}$$

   The marginal f_y is the integral of the joint w.r.t. x and z,

$$f_y=\int_{0.5}^{1.5}c\left(x+4y+\frac{41}{6}\right)dx=c\left(4y+\frac{47}{6}\right) \tag{9}$$

   Hence the conditional $f_{X|Y}$ can be evaluated:

$$f_{X|Y}=\frac{f(x,y)}{f(y)}=\frac{x+4y+\frac{41}{6}}{4y+\frac{47}{6}} \tag{10}$$

3. Computed values of $E[X \cdot Y]$, $E[X]$, $E[Y]$, $E[X^2]$, $E[Y^2]$, σ_X, σ_Y:

$$E[X \cdot Y] = \int_0^1 \int_{0.5}^{1.5} x \cdot y \cdot f(x,y)\, dx\, dy = \frac{127}{236} \tag{11}$$

$$E[X] = c \int_{0.5}^{1.5} x^2 + \frac{53x}{6}\, dx = \frac{119}{118} \tag{12}$$

$$E[Y] = c \int_0^1 4y^2 + \frac{47y}{6}\, dy = \frac{63}{118} \tag{13}$$

$$E[X^2] = c \int_{0.5}^{1.5} x^3 + \frac{53x^2}{6}\, dx = \frac{779}{708} \tag{14}$$

$$E[Y^2] = c \int_0^1 4y^3 + \frac{47y^2}{6}\, dy = \frac{65}{177} \tag{15}$$

$$\sigma_X = \sqrt{E[X^2] - (E[X])^2} = 0.2886 \tag{16}$$

$$\sigma_X = \sqrt{E[Y^2] - (E[Y])^2} = 0.2867 \tag{17}$$

$$Cov(X,Y) = E[X \cdot Y] - E[X] \cdot E[Y] = -\frac{1}{3481} \tag{18}$$

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \cdot \sigma_Y} = -3.47 \times 10^{-3} \tag{19}$$

4. It is unknown if X and Y are independent, since the covariance or correlation coefficient does not tell us anything about it.

5. $$f(x,z) = c \int_0^1 x + 4y + z^2 + 0.5\, dy = c\left(x + z^2 + \frac{5}{2}\right) \tag{20}$$

   The expectation $E[X + 2Z]$ is thus:

$$c \int_2^3 \int_{0.5}^{1.5} (x+2)\left(x + z^2 + \frac{5}{2}\right) dx\, dz \tag{21}$$

   Evaluating the double integral w.r.t. x and z, we obtain,

$$E[X + 2Z] = \frac{719}{118} \tag{22}$$

5 Problem 5: Marginals

1. The normalizing constant c can be found by equating the following double integral to 1:

$$\int_0^{0.5} \int_0^2 cxe^{x^2} \sin(2\pi y)\, dx\, dy = 1 \tag{23}$$

   Integrating w.r.t. x,

$$c \int_0^{0.5} \sin(2\pi y) \left[\frac{e^{x^2}}{2}\right]_0^2 dy = 1 \tag{24}$$

Integrating w.r.t. y,

$$c\left(\frac{e^4-1}{2}\right)\left[\frac{-\cos(2\pi y)}{2\pi}\right]_0^{\frac{1}{2}} = 1 \tag{25}$$

$$c\left(\frac{e^4-1}{2}\right)\left(\frac{1}{\pi}\right) = 1 \tag{26}$$

Solving for c,

$$c = \frac{2\pi}{e^4-1} \tag{27}$$

2. The marginal $f_y(y)$ is the integral of the joint w.r.t x.

$$f_y(y) = \int_0^2 f(x,y)\,dx = c\sin(2\pi y)\int_0^2 xe^{x^2}\,dx \tag{28}$$

Evaluating the integral,

$$f_y(y) = c\sin(2\pi y)\frac{e^4-1}{2} \tag{29}$$

Substituting the value of c found in Part 1,

$$f_y(y) = \pi\sin(2\pi y) \tag{30}$$

6 Problem 6: Expectations

1. The expectation $E[e^{x^3}]$ is the value of

$$\int_0^2 e^{x^3}\cdot f_x(x)\,dx \tag{31}$$

The marginal $f_x(x)$ is equal to

$$c\int_0^{0.25} x^2e^{3x^3}\sin(2\pi y)\,dy \tag{32}$$

$$= \frac{c}{2\pi}x^2e^{3x^3} \tag{33}$$

Evaluating the integral w.r.t. x,

$$\int_0^2 e^{x^3}\frac{c}{2\pi}x^2e^{3x^3}\,dx \tag{34}$$

$$= \frac{c}{24\pi}(e^32-1) \tag{35}$$

To solve for c, equate the double integral of $f(x,y)$ to 1:

$$c\int_0^{0.25}\int_0^2 x^2e^{3x^3}\sin(2\pi y)\,dx\,dy \;=\; 1 \tag{36}$$

Evaluating the integral w.r.t x,

$$c \int_0^{0.25} \left[\frac{e^{3x^3}}{9} \right]_0^2 \sin(2\pi y)\, dy \;=\; 1 \tag{37}$$

Evaluating the integral w.r.t. y,

$$c \left(\frac{e^{24} - 1}{9} \right) \left[\frac{1}{2\pi} \right] \tag{38}$$

Solving for c,

$$c = \frac{18\pi}{e^{24} - 1} \tag{39}$$

Substituting c to obtain $E[e^{x^3}]$,

$$E[e^{x^3}] = \frac{18\pi}{e^{24} - 1} \left(e^{32} - 1 \cdot \frac{1}{24\pi} \right) = \frac{3(e^{32} - 1)}{4(e^{24} - 1)} \tag{40}$$

2. The expectation $E[Y]$ is the value of

$$\int_0^{0.25} \int_0^2 x^2 e^{3x^3} y \sin(2\pi y)\, dx\, dy \tag{41}$$

Substituting the integral w.r.t. x from previous parts,

$$c \left(\frac{e^{24} - 1}{9} \right) \int_0^{0.25} y \sin(2\pi y)\, dy \tag{42}$$

Evaluating the integral by parts w.r.t. y,

$$\left[c \left(\frac{e^{24} - 1}{9} \right) \left[\frac{-y \cos(2\pi y)}{2\pi} \right]_0^{0.25} - \int_0^{0.25} \frac{-\cos(2\pi y)}{2\pi}\, dy \right] \tag{43}$$

Simplifying,

$$c \left(\frac{e^{24} - 1}{9} \right) \left[\frac{\sin(2\pi y)}{4\pi^2} - \frac{y \cos(2\pi y)}{2\pi} \right]_0^{0.25} \tag{44}$$

$$= c \left(\frac{e^{24} - 1}{9} \right) \left(\frac{1}{4\pi^2} \right) \tag{45}$$

Substituting c,

$$E[Y] = \frac{18\pi}{e^{24} - 1} \left(\frac{e^{24} - 1}{9} \right) \left(\frac{1}{4\pi^2} \right) = \frac{1}{2\pi} \tag{46}$$