BONUS Week 3 Homework

ⓘ This is a preview of the published version of the quiz

Started: Jul 2 at 7:53am

Quiz Instructions

Question 1

1 pts

(Lesson 2.10: Conditional Expectation.) BONUS: Suppose that

$f(x, y) = 6x$ for $0 \le x \le y \le 1$. Hint (you may have seen this someplace): the marginal p.d.f. of X turns out to be

$f_X(x) = 6x(1 - x)$ for $0 \le x \le 1$. Find the conditional p.d.f. of Y given that $X = x$.

$$f(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{6x}{6x(1-x)} = \frac{1}{1-x}$$

- ✓ a. $f(y|x) = \frac{1}{1-x}, \quad 0 \le x \le y \le 1$ ✓
- b. $f(y|x) = \frac{1}{1-x}, \quad 0 \le x \le 1$
- c. $f(y|x) = \frac{1}{1-y}, \quad 0 \le y \le 1$ ✗
- d. $f(x|y) = \frac{1}{1-x}, \quad 0 \le x \le y \le 1$ ✗

Question 2

1 pts

(Lesson 2.10: Conditional Expectation.) BONUS: Again suppose that

$f(x, y) = 6x$ for $0 \le x \le y \le 1$. Hint (you may already have seen this someplace): the the marginal p.d.f. of X turns out to be $f_X(x) = 6x(1 - x)$ for $0 \le x \le 1$. Find $\mathsf{E}[Y|X = x]$.

$$E[Y|X=x] = \int_{\mathbb{R}} y f(y|x)\,dy$$

- a. $\mathsf{E}[Y|X = x] = 1/2, \quad 0 \le x \le 1$

$$= \int_{\mathbb{R}} y \frac{1}{1-x}\,dy = \int_x^1 \frac{y}{1-x}\,dy$$

$$= \left[\frac{y^2}{2(1-x)}\right]_x^1 = \frac{1}{2(1-x)} - \frac{x^2}{2(1-x)}$$

- b. $\mathsf{E}[Y|X=x] = \frac{1+x}{2}, \quad 0 \le x \le 1$ ✓

$= \frac{1-x^2}{2(1-x)} = \frac{\cancel{(1-x)}(1+x)}{2\cancel{(1-x)}} = \frac{1+x}{2}$

- c. $\mathsf{E}[Y|X=x] = \frac{1+y}{2}, \quad 0 \le y \le 1$
- d. $\mathsf{E}[X|Y=y] = \frac{1+y}{2}, \quad 0 \le y \le 1$

Question 3

1 pts

(Lesson 2.10: Conditional Expectation.) BONUS: Yet again suppose that $f(x,y) = 6x$ for $0 \le x \le y \le 1$. Hint (you may already have seen this someplace): the the marginal p.d.f. of X turns out to be $f_X(x) = 6x(1-x)$ for $0 \le x \le 1$. Find $\mathsf{E}[\mathsf{E}[Y|X]]$.

Method 1

$E[E[Y|X]] = E[Y]$

$= \int_{\mathbb{R}} E(Y|x) f_X(x)\, dx$

$= \int_{\mathbb{R}} \frac{1+x}{2} 6x(1-x)\, dx = \frac{1}{2}\int_{\mathbb{R}} 6x(1-x^2)\, dx$

$= \frac{1}{2}\int_0^y 6x - 6x^3\, dx = \frac{1}{2}\left[\frac{6x^2}{2} - \frac{6x^4}{4}\right]_0^1$

$= \frac{1}{2}\left[\frac{6}{2} - \frac{6}{4}\right] = \frac{3}{4}$

Method 2

$f_Y(y) = \int_{\mathbb{R}} f(x,y)\, dx$

$= \int_0^y 6x\, dx$

$= \left[\frac{6x^2}{2}\right]_0^y = 3y^2$

$E[Y] = \int_{\mathbb{R}} y f_Y(y)\, dy$

$= \int_0^1 3y^3\, dy = \left[\frac{3y^4}{4}\right]_0^1 = \frac{3}{4}$

- a. 1/2
- b. 2/3
- c. 3/4 ✓
- d. 1

Question 4

1 pts

(Lesson 2.14: Estimation.) BONUS: Consider two estimators, T_1 and T_2, for an unknown parameter θ. Suppose that the $\mathsf{Bias}(T_1) = 0$, $Bias\,(T_2) = \theta$, $\mathsf{Var}(T_1) = 4\theta^2$, and $\mathsf{Var}(T_2) = \theta^2$. Which estimator might you decide to use and why?

$MSE(T) = Var(T) + (Bias(T))^2$

$MSE(T_1) = 4\theta^2 + 0^2$

$MSE(T_2) = \theta^2 + \theta^2 = 2\theta^2$

$MSE(T_2) < MSE(T_1)$

- a. T_1 - it has lower expected value
- b. T_1 - it has lower MSE
- c. T_2 - it has lower variance
- d. T_2 - it has lower MSE ✓

Question 5

1 pts

(Lesson 2.15: Maximum Likelihood Estimation.) BONUS: Suppose that $X_1, X_2, \ldots, X_n$ are i.i.d. $\text{Pois}(\lambda)$. Find $\hat{\lambda}$, the MLE of λ. (Don't panic --- it's not that difficult.)

likelihood Function

$$L(\lambda) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \frac{e^{-\lambda} \lambda^{x_i}}{x_i!} = \frac{e^{-n\lambda} \lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n}(x_i!)}$$

$$\ln[L(\lambda)] = \ln \frac{e^{-n\lambda} \lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n}(x_i!)}$$

$$= -n\lambda \ln e + \left[\sum_{i=1}^{n} x_i\right] \ln \lambda - \underbrace{\ln \prod_{i=1}^{n}(x_i!)}_{\text{constant}}$$

$$\frac{d}{d\lambda} \ln[L(\lambda)] = -n + \frac{1}{\lambda} \sum_{i=1}^{n} x_i = 0 \Rightarrow \hat{\lambda} = \frac{1}{n} \sum_{i=1}^{n} x_i = \bar{X}$$

- a. $\bar{X}$
- b. $1/\bar{X}$
- c. $n/\sum_{i=1}^{n} X_i$
- d. S^2

Question 6

1 pts

(Lesson 2.15: Maximum Likelihood Estimation.) BONUS: Suppose that we are looking at i.i.d. Exp(λ) customer service times. We observe times of 2, 4, and 9 minutes. What's the maximum likelihood estimator of λ^2?

$$\frac{2+4+9}{3} = 5$$

$$f(x) = \lambda e^{-\lambda x}, \quad x = 2, 4 \,\&\, 9$$

$$\prod_{x=2,4,9} \lambda e^{-\lambda x} = (\lambda e^{-2\lambda})(\lambda e^{-4\lambda})(\lambda e^{-9\lambda}) = \lambda^3 e^{-15\lambda}$$

$$\hat{\lambda} = \frac{1}{\bar{x}}$$

$$L(\lambda) = \lambda^3 e^{-15\lambda} \Rightarrow \ln[L(\lambda)] = \ln \lambda^3 e^{-15\lambda}$$

$$\ln[L(\lambda)] = 3\ln\lambda - 15\lambda \underbrace{\ln e}_{1} = 3\ln\lambda - 15\lambda$$

$$\frac{d}{d\lambda} 3\ln\lambda - 15\lambda = 0 \Rightarrow \frac{3}{\lambda} - 15 = 0 \Rightarrow \frac{3}{\lambda} = 15 \Rightarrow \lambda = \frac{1}{5}$$

$$\lambda^2 = \frac{1}{25}$$

- a. 5
- b. 1/5
- c. 25
- d. 1/25

Question 7

1 pts

(Lesson 2.16: Confidence Intervals.) BONUS: Suppose we collect the following observations: $7, -2, 1, 6$ (as in a previous question in this homework). Let's assume that these guys are i.i.d. from a normal distribution with *unknown* variance σ^2. Give me a two-sided 95% confidence interval for the mean μ.

$$n = 4, \quad \text{sample mean } \bar{X} = \frac{7-2+1+6}{4} = 3$$

$$\text{sample variance } s^2 = \frac{54}{4-1} = 18, \quad \alpha = 0.05$$

$t_{\frac{\alpha}{2}=0.025,\, n-1=3} = 3.1824$

- ○ a. $[-2, 7]$
- ○ b. $[-3.75,\ 9.75]$
- ○ c. $[-6.75, 6.75]$
- ○ d. $[3.75, 9.75]$

CI: $\bar{X} \pm t_{\frac{\alpha}{2}, n-1} \times \sqrt{\frac{S^2}{n}}$

$= \bar{X} \pm t_{\frac{\alpha}{2}, n-1} \times \frac{S}{\sqrt{n}} = 3 \pm 3.1824 \left(\frac{\sqrt{18}}{\sqrt{4}}\right)$

$= 3 \pm 6.750$

Not saved

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