Week 11 Homework

ⓘ This is a preview of the published version of the quiz

Started: Jul 2 at 7:50am

Quiz Instructions

Question 1

1 pts

(Lesson 8.1: Introduction to Input Modeling.) It's GIGO time! Let's consider an $M/M/1$ queueing system with Exp(λ) interarrivals and Exp(μ) FIFO services at a single server. You may recall from some class (either this one or stochastic processes) that the steady-state expected cycle time (i.e., the time that the customer is in the system, including wait + service) is $w = 1/(\mu - \lambda)$.

If you were to try this out in Arena, let's say with $\texttt{EXPO}(10 = 1/\lambda)$ interarrivals and $\texttt{EXPO}(8 = 1/\mu)$ services (note the notation change between my usual "Exp" and Arena's " $\texttt{EXPO}$ "), then we'd get $w = 1/(0.125 - 0.1) = 40$. Go ahead, see for yourself in Arena, but make sure that you run the system for 100,000 or so customers so that you can be sure that you're in steady-state!

Finally, here's the GIGO question, which will show what can happen when you mis-model a component of your process: What is the (approximate) steady-state expected cycle time if you have i.i.d. **UNIF(5, 15)** interarrivals instead of $\texttt{EXPO}(10 = 1/\lambda)$ interarrivals? Note that both interarrival distributions have the same mean (10), but that doesn't necessarily imply that they'll have the same expected cycle times. Hint: You may want to use Arena as described above.

- a. about 1
- b. about 10
- c. about 23
- d. about 40
- e. about 62

Question 2

1 pts

(Lesson 8.2: Identifying Distributions.) Let's play Name That Distribution!

The number of times a "3" comes up in 10 dice tosses.

- a. Bernoulli
- b. Binomial
- c. Geometric
- d. Negative Binomial
- e. Pareto

$X \sim Bin(n, p)$, $n = 10$, $p = 1/6$

Question 3

1 pts

(Lesson 8.2: Identifying Distributions.) Name That Distribution!

The number of dice tosses until a 3 comes up.

number of trials till first success.

- a. Bernoulli
- b. Binomial
- c. Geometric
- d. Negative Binomial
- e. Pareto

Question 4

1 pts

(Lesson 8.2: Identifying Distributions.) Name That Distribution!

The number of dice tosses until a 3 comes up for the 4th time.

Neg Binomial is a discrete probability distribution that models the number of trials needed to achieve a specified number of successes in a sequence of iid Bernoulli trials.

- a. Bernoulli
- b. Binomial
- c. Geometric
- d. Negative Binomial
- e. Pareto

Question 5

1 pts

(Lesson 8.2: Identifying Distributions.) Name That Distribution!

IQs are typically...

- a. Uniform
- b. Normal
- c. Exponential
- d. Weibull
- e. Pareto

Question 6

1 pts

(Lesson 8.2: Identifying Distributions.) Name That Distribution!

Cases in which you have limited information, e.g., you only know the min, max, and "most likely" values that a random variable can take.

- a. Bernoulli
- b. Poisson
- c. Triangular
- d. Weibull

e. Pareto

Question 7

1 pts

(Lesson 8.3: Unbiased Point Estimation.) Find the sample variance of $-3, -2, -1, 0, 1, 2, 3$

$\bar{X} = 0, \; n = 7$

- a. 0
- b. 1
- c. 4
- d. 14/3 ✓
- e. 6

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

$$= \frac{1}{6}(3^2 + 2^2 + 1^2 + 1^2 + 2^2 + 3^2)$$

$$= \frac{1}{6}(28)$$

$$= \frac{14}{3}$$

Question 8

1 pts

$X \sim Pois(6) \quad \mu = 6$ and $\sigma^2 = 6$

(Lesson 8.3: Unbiased Point Estimation.) If $X_1, \ldots, X_{10}$ are i.i.d. Pois(6), what is the expected value of the sample variance S^2?

- a. 1/6
- b. 1/36
- c. 6 ✓
- d. 36
- e. 60

Theorem: Suppose $X_1, \ldots, X_n$ are i.i.d. any-thing (ie any distribution) with mean μ & variance σ^2. Then $E[S^2] = Var(X_i) = \sigma^2$

Recall: an estimator is said to be unbiased if its expected value is equal to the true parameter it estimates. Formally, if θ is a parameter and $\hat{\theta}$ is an estimator for θ, then $\hat{\theta}$ is unbiased if $E[\hat{\theta}] = \theta$.

$E[S^2] = \sigma^2 = 6$ [S^2 is always unbiased for the variance of X_i]

Question 9

1 pts

(Lesson 8.4: Mean Squared Error.) Suppose that estimator A has bias = 3 and variance = 12, while estimator B has bias -2 and variance = 14. Which estimator (A or B) has the lower mean squared error?

$MSE = Var + (Bias)^2$

- a. A

$MSE(A) = Var(A) + [Bias(A)]^2 = 12 + 3^2 = 21$

- b. B ✓

$MSE(B) = Var(B) + [Bias(B)]^2 = 14 + (-2)^2 = 18$

Question 10

1 pts

Suppose $X_1, \dots, X_n \overset{iid}{\sim} Norm(\mu, \sigma^2)$

The simultaneous MLEs for μ & σ^2 are

$\hat{\mu} = \bar{X} \qquad \hat{\sigma}^2 = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n}$

(Lessons 8.5 and 8.6: Maximum Likelihood Estimators.) If $X_1 = 2$, $X_2 = -2$, and $X_3 = 0$ are i.i.d. realizations from a Nor(μ, σ^2) distribution, what is the value of the maximum likelihood estimate for the variance σ^2?

$\bar{X} = (2 + (-2) + 0) \div 3 = 0$

$\hat{\sigma}^2 = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n} = \frac{2^2 + (-2)^2 + 0^2}{3} = \frac{4+4}{3} = \frac{8}{3}$

- a. 0
- b. 1
- c. 8/3 ✓
- d. 4
- e. None of the above

Question 11

1 pts

Poisson pmf $\quad P(X=k) = \frac{\lambda^k e^{-\lambda}}{k!}$

likelihood function $L(\theta) \equiv \prod_{i=1}^n f(x_i)$ where $f(x_i)$ is pdf/pmf

(Lessons 8.5 and 8.6: Maximum Likelihood Estimators.) Suppose we observe the Pois(λ) realizations $X_1 = 5, X_2 = 9$ and $X_3 = 1$. What is the maximum likelihood estimate of λ?

$L(\lambda) = P(X_1 = x_1, X_2 = x_2, X_3 = x_3 \mid \lambda) = \prod_{i=1}^n P(X_i = x_i \mid \lambda)$

- a. 0

$L(\lambda) = \prod_{i=1}^n \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} = \left[\prod_{i=1}^n \left(\frac{\lambda^{x_i}}{x_i!}\right)\right] e^{-n\lambda} = \frac{\lambda^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!} e^{-n\lambda}$

- b. 5 ✓

In this case, $L(\lambda) = \frac{\lambda^{x_1} e^{-\lambda}}{x_1!} \times \frac{\lambda^{x_2} e^{-\lambda}}{x_2!} \times \frac{\lambda^{x_3} e^{-\lambda}}{x_3!}$

- c. 25

$L(\lambda) = \frac{\lambda^5 e^{-\lambda}}{5!} \cdot \frac{\lambda^9 e^{-\lambda}}{9!} \cdot \frac{\lambda e^{-\lambda}}{1!} = \frac{\lambda^{15} e^{-n\lambda}}{5!\,9!}$

- d. 1/5

$\ln[L(\lambda)] = \ln\left[\frac{\lambda^5 e^{-\lambda}}{5!}\right] + \ln\left[\frac{\lambda^9 e^{-\lambda}}{9!}\right] + \ln\left[\frac{\lambda e^{-\lambda}}{1}\right]$

$$\frac{d}{d\lambda}\ln[L(\lambda)] = \frac{d}{d\lambda}\left[\ln\left[\frac{\lambda^5 e^{-\lambda}}{5!}\right] + \ln\left[\frac{\lambda^9 e^{-\lambda}}{9!}\right] + \ln\left[\frac{\lambda e^{-\lambda}}{1}\right]\right]$$

$$= \left[\frac{5\lambda^4(-1)e^{-\lambda}}{5!} \div \frac{\lambda^5 e^{-\lambda}}{5!}\right] + \left[\frac{9\lambda^8(-1)e^{-\lambda}}{9!} \div \frac{\lambda^9 e^{-\lambda}}{9!}\right] + \left[\frac{(-1)e^{-\lambda}}{1} \div \frac{\lambda e^{-\lambda}}{1}\right]$$

$$= \left(-\frac{5}{\lambda}\right) + \left(-\frac{9}{\lambda}\right) + \left(-\frac{1}{\lambda}\right)$$

$$= \frac{-15}{\lambda} \times$$

Careless. Use Ln property to calculate in a smarter way.

$$L(\lambda) = \prod_{i=1}^{n}\frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = \frac{\lambda^{\sum_{i=1}^{n}x_i}}{\prod_{i=1}^{n}x_i!}e^{-n\lambda}$$

$$\ln[L(\lambda)] = \ln\lambda^{\sum_{i=1}^{n}x_i} - \ln\left(\prod_{i=1}^{n}x_i!\right) + \ln e^{-n\lambda}$$

$$= \sum_{i=1}^{n}x_i\ln\lambda - \ln\left(\prod_{i=1}^{n}x_i!\right) + (-n\lambda)$$

$$\ln[L(\lambda)] = (5+9+1)\ln\lambda - \ln(5!\times 9!) - 3\lambda$$

$$= 15\ln\lambda - \ln(5!\,9!) - 3\lambda$$

$$\frac{d}{d\lambda}\ln[L(\lambda)] = \frac{15}{\lambda} - 0 - 3 = 0$$

$$\frac{15}{\lambda} = 3$$

$$\lambda = \frac{15}{3} = 5$$

It appears that $\hat{\lambda} = \bar{x} = \frac{9+5+1}{3} = 5$

Note :

Unbiasedness – $\hat{\theta}$ is an unbiased estimator for θ if $E[\hat{\theta}] = \theta$

Maximum Likelihood Estimators (MLEs) – MLEs obtained by maximising the Likelihood Function $L(\theta; X) = P(X = \{x_1, x_2, \dots, x_n\} \mid \theta) = \prod_{i=1}^{n}\underbrace{f(x_i|\theta)}_{\text{pmf/pdf}}$

To maximise the likelihood function, find values of θ for

$$\frac{d}{d\theta}\ln[L(\theta;X)] = \frac{d}{d\theta}\ln\prod_{i=1}^{n}f(x_i|\theta) = \frac{d}{d\theta}\sum_{i=1}^{n}\ln f(x_i|\theta) = 0$$

e. 1/25

Question 12

1 pts

Theorem (Invariance): If $\hat{\theta}$ is the MLE of some parameter θ and $h(\cdot)$ is a one-to-one function, then $h(\hat{\theta})$ is the MLE of $h(\theta)$

(Lesson 8.7: Invariance Property of MLEs) Suppose that we have a number of observations from a Pois(λ) distribution, and it turns out that the MLE for λ is $\hat{\lambda} = 5$. What's the maximum likelihood estimate of $\Pr(X = 3)$?

$P(X=k) = \frac{e^{-\lambda}\lambda^k}{k!}$

- a. 0.1404 ✓
- b. 5
- c. 25
- d. 1/5
- e. 1/25

$P(X=3) = \frac{e^{-\lambda}\lambda^3}{3!}$

$P(\lambda) = \frac{e^{-\lambda}\lambda^3}{3!}$

$P(\hat{\lambda}) = \frac{e^{-5}5^3}{3!} = 0.14037\ldots$

by Invariance.

Question 13

1 pts

$H_0: X_1, X_2, \ldots, X_n \overset{iid}{\sim}$ pmf/pdf $f(x)$ ie good fit

$\alpha \equiv P(\text{Reject } H_0 \mid H_0 \text{ true}) = P(\text{type I error})$

(Lesson 8.9: Goodness-of-Fit Tests.) Suppose we're conducting a χ^2 goodness-of-fit test with Type I error rate $\alpha = 0.01$ to determine whether or not 100 i.i.d. observations are from a lognormal distribution with unknown parameters μ and σ^2. If we divide the observations into 5 equal-probability intervals and we observe a g-o-f statistic of $\chi_0^2 = 11.2$, will we ACCEPT or REJECT the null hypothesis of lognormality?

$k = 5$

they are asking to test if the observations fit a lognormal distribution which is H_0.

$f(x;\mu,\sigma) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}}$

$s = 2$ the μ and σ the number of unknown parameters

$\chi_0^2 = 11.2$ and $\alpha = 0.01$

- Accept
- Reject ✓

Reject H_0 if $\chi_0^2 > \chi^2_{\alpha, k-1-s}$

$5-1-2 = 2 = \nu$

$\chi^2_{\alpha,\nu} \approx \nu\left[1 - \frac{2}{9\nu} + z_\alpha\sqrt{\frac{2}{9\nu}}\right]^3$

$= 2 \times \left[1 - \frac{2}{18} + 2.53\sqrt{\frac{2}{18}}\right]^3 = 10.395$

Question 14

Hence $\chi_0^2 > \chi^2_{\alpha\nu}$, reject H_0

1 pts

Actual value for $\chi^2_{0.01,2} = 9.21$.

Study these calculations.

(Lessons 8.10 and 8.11: χ^2 Goodness-of-Fit Test.) This problem has a really long description, but the question itself will be very short! Be patient!

The number of defects in printed circuit boards is hypothesized to follow a Geometric(p) distribution. A random sample of $n = 70$ printed boards has been collected, and the number of defects observed. Here are the results.

Number of Defects	Observed frequency
1	34
2	18
3	2
4	9
5	7

It turns out that the MLE of p for the Geom(p) is $\hat{p} = 1/\bar{X}$. (See the following proof if you don't believe me!)

Proof: The likelihood function is

$$L(p) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} (1-p)^{x_i - 1} p = (1-p)^{\sum_{i=1}^{n} x_i - n} p^n.$$

Thus,

$$\ln(L(p)) = \left(\sum_{i=1}^{n} x_i - n\right) \ln(1-p) + n \ln(p),$$

and so,

$$\frac{d \ln(L(p))}{dp} = \frac{-\sum_{i=1}^{n} x_i + n}{1-p} + \frac{n}{p} = 0$$

Solving for p gives the MLE $\hat{p} = \frac{1}{\bar{X}}$.

So in this particular case, we have:

$$\bar{X} = \frac{1(34)+2(18)+3(2)+4(9)+5(7)}{70} = 2.1,$$

and thus $\hat{p} = 0.476$.

Anyhow, we are interested in performing a χ^2 goodness-of-fit test to test the Geometric hypothesis. To this end, let's get the test statistic, χ_0^2? To do so, let's make a little table, assuming that $\hat{p} = 0.476$ is correct. Note that the expected number of observations having a certain value x is $E_x = n \Pr(X = x) = n(1-\hat{p})^{x-1}\hat{p}$. Also note that I've combined the entries in the last row (≥ 5) so the probabilities add up to one.

x	$P(X=x)$	E_x	O_x
1	0.4762	33.33	34
2	0.2494	17.46	18
3	0.1307	9.15	2
4	0.0684	4.79	9
≥ 5	0.0752	5.27	7
	1.0000	70	70

Technically speaking, we really ought to combine the last two cells, since $E_4 = 4.79 < 5$. Let's do so to get the following new-and-improved table.

x	$P(X=x)$	E_x	O_x
1	0.4762	33.33	34
2	0.2494	17.46	18
3	0.1307	9.15	2
≥ 4	0.1436	10.06	16
	1.0000	70	70

Thus, the test statistic is

$$\begin{aligned}\chi_0^2 &= \sum_{x=1}^{4} \frac{(E_x - O_x)^2}{E_x} \\ &= \frac{(33.33-34)^2}{33.33} + \frac{(17.46-18)^2}{17.46} + \frac{(9.15-2)^2}{9.15} + \frac{(10.06-16)^2}{10.06} \\ &= 9.12.\end{aligned}$$

Now, let's use our old friend $\alpha = 0.05$ in our test. Let $k = 4$ denote the number of cells (that we ultimately ended up with) and let $s = 1$ denote the number of parameters we had to estimate. Then we compare against $\chi^2_{0.05,k-s-1} = \chi^2_{0.05,2} = 5.99$. So after all this time, here's my question: Do we ACCEPT or REJECT the Geometric hypothesis? $\left.\chi_0^2 > \chi^2_{0.05,2}\right\}$ Reject

- a. Accept

This means the number of defects probably isn't geometric.

- ✓ b. Reject

This is the exact example from week 11 module 8 lesson 9 video

Question 15

1 pts

(Lesson 8.12: Kolmogorov--Smirnov Test.) Consider the PRN's $U_1 = 0.1, U_2 = 0.9$, and $U_3 = 0.2$. Use Kolmogorov-Smirnov with $\alpha = 0.05$ to test to see if these numbers are indeed uniform. Do we ACCEPT or REJECT uniformity?

Google $D_{0.05,3} = 0.708$

U_i	0.1	0.9	0.2
$U_{(i)}$	0.1	0.2	0.9
i/n	1/3	2/3	1
$\frac{i-1}{n}$	0	1/3	2/3
$\frac{i}{n} - U_{(i)}$	0.233	0.4667	0.1
$U_{(i)} - \frac{i-1}{n}$	0.1	-0.133	0.233

- a. Accept ✓
- b. Reject

Question 16

1 pts

$D^+ \equiv \max_{1 \le i \le n} \left\{ \frac{i}{n} - U_{(i)} \right\} = 0.4667$ | $D^- \equiv \max_{1 \le i \le n} \left\{ U_{(i)} - \frac{i-1}{n} \right\} = 0.233$

$D = \max(D^+, D^-) = 0.4667 < D_{0.05,3} = 0.708$

Hence, fail to reject uniformity.

(Lesson 8.14: Arena Input Analyzer.) You don't have to turn anything in for this, but I'd simply like you to play around with the Arena Input Analyzer. So, did you look at the Input Analyzer? (You should answer YES.)

- a. Yes ✓
- b. No

Not saved

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