# Response Time Modeling

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## The Exponential Race Model

Suppose that with each response option ($i$), we associate an internal response time $T_i \sim \text{Exp}(\lambda_i)$. We define the rate parameter ($\lambda_i$) that it depends on the estimated value ($v_i$) of the response:

$$\lambda_i = e^{\phi v_i} \tag{1}$$

where $\phi > 0$ is a choice consistency parameter.

Suppose that the response option chosen (which we denote $C$) is determined by which response time ($T_i$) is shortest, i.e. which response "wins the race":

$$P(C = i) = P(T_i = \min\{T_1, \cdots, T_n\}) \tag{2}$$

It can be shown that

$$P(T_i = \min\{T_1, \cdots, T_n\}) = \frac{\lambda_i}{\sum_{j=1}^{n} \lambda_j} \tag{3}$$

This is the Luce choice rule, and when combined with mapping from estimated values ($v_i$) to rate parameters ($\lambda_i$) described by Equation 1 becomes the commonly used softmax response rule:

$$P(C = i) = \frac{e^{\phi v_i}}{\sum_{j=1}^{n} e^{\phi v_j}} \tag{4}$$

## Joint Modeling of Responses and Response Times

Let $C$ denote the observed response, and $T$ denote the observed response time. For each trial, we want to obtain the joint likelihood of $C$ and $T$. Observe that

$$p(C, T) = p(T|C)P(C) \tag{5}$$

For the exponential response time model described above, we have $P(C = i) = \frac{e^{\phi v_i}}{\sum_{j=1}^{n} e^{\phi v_j}}$ and $p(T|C = i) = \lambda_i e^{-\lambda_i t}$, where $\lambda_i = e^{\phi v_i}$.

This makes it very easy to jointly model responses and response times. We simply use the softmax function to obtain the log-likelihood of the response, and the log of that response's exponential pdf to obtain the log-likelihood of the response time, and then add these together:

$$\ln p(C, T) = \ln p(T|C) + \ln p(C) \tag{6}$$

## Relative Times of Correct and Incorrect Responses

So long as the predicted value ($v_i$) of the correct response is greater than that of incorrect alternatives (i.e. so long as learning and perception are doing their jobs properly), the exponential race model predicts that errors will be slower than correct responses. This is because a low $v_i$ leads to a low exponential rate ($\lambda_i = e^{\phi v_i}$) which leads to long response times. If we remove the assumption that $v_i$ is greatest for the correct response option, we can say more generally that response times will be greatest for the least often chosen alternatives.

From what I have observed, errors are often slower than correct responses in the sort of learning tasks we study (though this needs to be confirmed through more careful examination). The slow error property of the exponential race model therefore may not pose a practical problem for modeling such tasks. However, other tasks feature a speed-accuracy tradeoff, which we may want to model. At the moment, I don't see any way to do this using the exponential race model.

## Response Speed and Choice Consistency

The exponential race model makes the further prediction that the frequency of a response will be negatively correlated with response times when $v_i > 0$ but positively correlated when $v_i < 0$. This is because of the way that the softmax choice consistency parameter $\phi$ also governs the size of the exponential rate and hence the average speed of responses.

## The Exponential Race Model is a Special Case of the LBA

The linear ballistic accumulator (Brown & Heathcote, 2008) is also a sort of race model, i.e. each response option has its own time, and the shortest of these determines the response taken and the observed response time. On each trial, the accumulator associated with each response option starts at a position $k$ drawn uniformly from the interval $[0, A]$ ($A$ is the same for each response option). The accumulator then proceeds forward with a constant speed $d$, where $d$ drawn from a distribution with different parameters for each response option. These parameters will be such that $d$ tends to be larger for response options that are more likely to be chosen. The first accumulator to reach the threshold $b$ wins the race, and its corresponding response option is chosen, while the observed response time is equal to $t + t_0$, where $t_0$ is some fixed non-decision time. We have distance = speed $\times$ time, i.e. $b - k = dt$, and thus $t = \frac{b-k}{d}$. We must fix one parameter in order to make the model identifiable, so for convenience we shall fix the decision boundary $b$ at 1. Thus $t = \frac{1-k}{d}$.

The LBA can account for situations in which errors are faster than correct responses as well as those in which errors are slower than correct responses by controlling the parameter $A$, which determines the variability of starting locations. When $A$ is large, then errors will usually occur when the starting point ($k$) of the wrong accumulator happens to be very near the threshold $b$ (which we have fixed at 1). This will lead to fast errors, as occurs when speed is emphasized rather than accuracy. If instead $A$ is small, then variability in starting points ($k$) will have little effect, and responses will be governed instead by variability in the drift speed ($d$). This leads to slow errors, as occur when accuracy is emphasized over speed.

In our learning experiments, so far as I have observed, errors are much slower than correct responses. This makes sense, as participants are not under any pressure to respond quickly. Therefore, let us make the extreme assumption that $A = 0$, i.e. that the starting points for the various accumulators are always the same. Further, let us suppose that the drift speeds ($d$) of the accumulators have inverse exponential distributions with parameters $\lambda_i$ as defined above. Then their response times ($t$) will be distributed exponentially and we have recovered the exponential race model (it is trivial to add in the constant non-decision time $t_0$). Thus, the exponential race model can be considered as a form of the LBA in which we assume a particular form for the drift speed distributions and an extreme emphasis on accuracy over response speed.