Linear Transformations
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October 21, 2017

Let V, W be vector spaces over a field F. A function $\varphi : VW$ is a linear transformation if

$$\varphi(ru + sv) = r\varphi(u) + s\varphi(v)$$

where $r, s \in F$ and vectors $u, v \in V$. The set of all linear transformations from V to W is namely $\mathcal{L}(V, W)$ and if $V = W$, then φ is called **linear operator**, namely $\mathcal{L}(V)$. Let $\varphi \in \mathcal{L}(V, W)$. The subspace

$(\varphi) = \{v \in V \mid \varphi v = 0\}$, is called the kernel of φ. $(\varphi) = \{\varphi v \mid v \in V\}$, is called the image of φ.

The dimension of (φ) is known as the **nullity** () and the dimension of (φ) is known as the **rank** ().

Theorem 1 (Rank-nullity theorem) *Let V, W be vector spaces over a field F, and let $\varphi \in \mathcal{L}(V, W)$ be a linear transformation, then*

$$\dim(V) = (\varphi) + (\varphi).$$

Proof: Obviously $(\varphi) \subseteq V$, $(\varphi) \subseteq W$. Let $_x = \{x_1, \ldots, x_s\},_y = \{y_0, \ldots, y_r\}$ be the bases for the vector spaces (φ) and $\varphi(V)$ with dimension s and r respectively. Let m be the dimension of V, so we must prove that $s + r = m$. If $y_i \in \varphi(V)$ then there exists $z_i \in V$ such that $\varphi(z_i) = y_i$. We claim that $=_x \cup \varphi^{-1}(_y) = \{x_1, x_s\} \cup \{z_1, \ldots z_r\}$ because $\{z_1, \ldots, z_r\} = \varphi^{-1}(_r)$ is a linearly independent subset of V, then should be a subset of V which is linearly independent. Notice that $_x \cap \varphi^{-1}(_y) = \emptyset$. Now, we must show that spans V. Let $v \in V$ such that $\varphi(v) = w$, for any $w \in W$. There exists $\alpha_1, \ldots, \alpha_r$ such that $w = \alpha_1 y_1 + \ldots + \alpha_r y_r$ because $_y$ forms a bases of $\varphi(V)$. We see that $T(v - \alpha_1 z_1 - \ldots - \alpha_r z_r) = \alpha(v) - w = 0$, thus

$$v = \alpha_1 z_1 + \ldots + \alpha_r z_r +_1 x_1 + \ldots_s x_s,$$

since $v - \alpha_1 z_1 - \ldots - \alpha_r z_r =_1 x_1 + \ldots +_s x_s$, thus $v \in \mathit{span}()$ and indeed $\mathit{span}() = V$ and forms a bases from B. Hence $s + r = m$.

From the proof of theorem **??**, we have the next one theorem

Theorem 2 *Let $\varphi \in \mathcal{L}(V, W)$. Then*

[1)] φ is surjective iff $(\varphi) = W$, φ is injective iff $(\varphi) = \{0\}$.

See problem 1.

1 Isomorphisms

Let $\varphi : VW$ be a linear transformation, we say that $V \approx W$ are **isomorphic** if φ is bijective.

Theorem 3 *Let V and W be vector spaces over F. Then $V \approx W$ iff* $\dim(V) = \dim(W)$.

Proof: We suppose that $\dim(V) = \dim(W)$. Let $_v = \{v_1, \ldots, v_k\}$ and $_w = \{w_1, \ldots, w_k\}$ be bases for V and W respectively. If $\varphi : VW$ is defined as $v_i \mapsto w_i$ for all i, then $[\varphi]_w^v = I$ which is invertible, thus φ is invertible and indeed an isomorphism.

Let $V = (\varphi) \oplus (\varphi)^c$, where $(\varphi)^c$ is the complement of (φ). It follows that

$$\dim(V) = \dim((\varphi)^c) + \dim((\varphi))*, \tag{1}$$

from theorem **??**, we know that $\dim(V) = (\varphi) + (\varphi)$, then it follows that in (*), $(\varphi)^c \approx (\varphi)$.

2 Linear Transformations from F^n to F^m

Let A be a $n \times m$ matrix over F, and let $\varphi_A \in \mathcal{L}(F^n, F^m)$ such that $\varphi_A(v) = Av$, thus

$$A = (\varphi e_1 | \ldots | \varphi e_n),$$

where $\{e_1, \ldots, e_n\}$ is a base for A.

Theorem 4 *Let V and W be finite-dimensional vector spaces over F, with* ***ordered bases*** *$_b = \{b_1, \ldots, b_n\}$ and $_c = \{c_1, \ldots, c_m\}$, respectively.*

[1.] The map $\mu : \mathcal{L}(V, W)\mathcal{M}_{m,n}(F)$ defined by

$$\mu(\varphi) = [\varphi]_{b,c},$$

is an isomorphism and so $\mathcal{L}(V, W) \approx \mathcal{M}_{m,n}(F)$. Hence,

$$\dim(\mathcal{L}(V, W)) = \dim(\mathcal{M}_{m,n}(F)) = m \times n$$

If $\phi \in \mathcal{L}(U, V)$ and $\varphi \in \mathcal{L}(V, W)$ and if $_{b,c}$ and $_d$ are ordered bases for

U, V and W repectivey. Then

$$[\varphi\phi]_{b,d} = [\varphi]_{c,d}[\phi]_{b,c}$$

Thus $\varphi\phi$ is the product of the matrices φ and ϕ.

Proof: First, let's prove that μ is linear

$$\begin{aligned}[s\phi + r\varphi]_{b,c}[b_i]_b &= [(s\phi + r\varphi)(b_i)]_c \\ &= [s\phi(b_i) + r\varphi(b_i)]_c \\ &= s[\phi(b_i)]_c + r[\varphi(b_i)]_c \\ &= (s[\phi]_{b,c} + r[\varphi]_{b,c})[b_i]_b\end{aligned}$$

Since $[b_i]_b = e_i$ is a basis vector, we conclude that

$$[s\phi + r\varphi]_{b,c} = s[\phi]_{b,c} + r[\varphi]_{b,c},$$

thus μ is linear. If $A \in \mathcal{M}_{m,n}$, we define φ by the condition $[\varphi b_i]_c = A^{(i)}$, so $\mu(\varphi) = A$ and μ is surjective. Since $(\mu) = \{0\}$ because $[\varphi]_b = 0$ implies that $\varphi = 0$. Hence, the map μ is an isomorphism.

3 Invariant subspaces

Let $S \subset V$ be a subspace, which is said to be φ-**invariant** if $\varphi S \subseteq S$ (i.e. $\varphi s \in S$ for all $s \in S$), where $\varphi \in \mathcal{L}(V)$ (See problem 4). Let $\varphi \in \mathcal{L}(V)$. If $V = S \oplus T$ and if both S and T are φ-invariant, we say that the pair (S, T) **reduces** φ. Let $V = S \oplus T$. The linear operator $\rho_{S,T} : VV$ defined by

$$\rho_{S,T}(s + t) = s,$$

where $s \in S$ and $t \in T$ is called **proyection** onto S **along** T

Theorem 5 *Let $V = S \oplus T$. Then (S, T) reduces $\varphi \in \mathcal{L}(V)$ if and only if φ commutes with $\rho_{S,T}$.*

Proof: Suppose that there exists a projection $\rho = \rho_{S,T}$ for which

$$\rho\varphi\rho = \varphi\rho,$$

then S and T are φ-invariant if and only if

$$\rho_{S,T}\varphi\rho_{S,T} = \rho_{S,T}\varphi \quad \textit{and} \quad (\iota - \rho_{S,T})\varphi(\iota - \rho_{S,T}) = (\iota - \rho_{S,T})\varphi$$

that is equivalent to

$$\rho_{S,T}\varphi\rho_{S,T} = \rho_{S,T}\varphi \quad \textit{and} \quad \rho_{S,T}\varphi = \varphi\rho_{S,T},$$

then follows that $\rho_{S,T}\varphi = \varphi\rho_{S,T}$.

4 Problems

[1.] Proof theorem 2. (Sylvester's rank inequality) Let $\varphi, \phi \in \mathcal{L}(V, W)$, and let n be the dimension of V. Show that

$$(\varphi) + (\phi) - n \leq (\varphi\phi).$$

Let A be a $n \times m$ matrix over F. Show that$\varphi_A : F^n F^m$ is injective iff $(A) = n$. $\varphi_A : F^n F^m$ is surjective iff $(A) = m$.

Let S and T be subspaces of V, such that $V = S \oplus T$ and let $\varphi \in \mathcal{L}(V)$. Show that if S is φ-invariant then does not imply that T is also φ-invariant. Let V be a vector space over a field F of characteristic $\neq 2$ and let ρ and σ be projections. Show that the difference $\rho - \sigma$ is a projection if and only if

$$\rho\sigma = \sigma\rho = \sigma.$$

5 Further Links

Don Monk's notes on Advanced linear algebra: http://euclid.colorado.edu/ monkd/m5151.html

Prof. Monk has used the book Advanced linear algebra by Steve Roman to take notes on the course, as well as the author of this handout did it, specifically Chapter 2.