PETER SYMONDS COLLEGE

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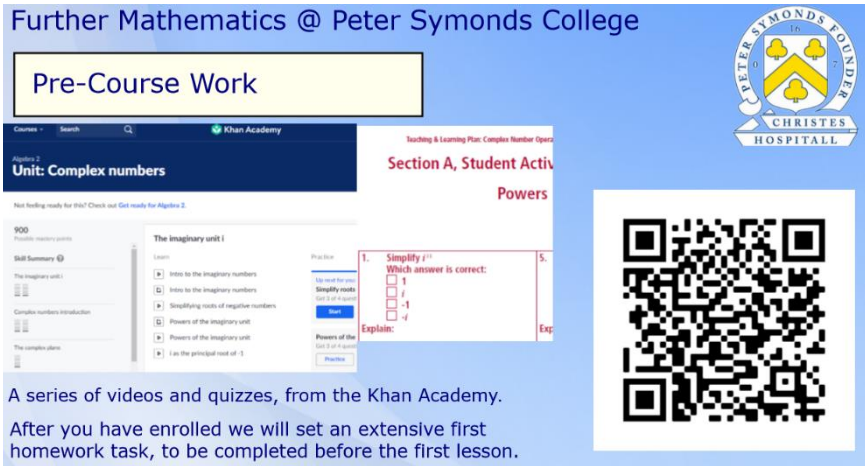
A-Level Further Mathematics

We are delighted that you have enrolled for the most exciting course at Peter Symonds College.

Post Enrolment Task

After enrolment, please click on our slide below to access the Khan Academy, or use the URL: https://www.khanacademy.org/math/algebra2/x2ec2f6f830c9fb89:complex



Work through the series of videos, notes, quizzes and practice questions within the Khan Academy unit on Complex Numbers, then complete the questions on the following pages. The Khan Academy resources will help you to complete our post enrolment task and also to prepare for the first half term of the course, where we investigate this topic in more depth.

Project Maths
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Development Team

Real Life Context

Complex Numbers are useful in representing a phenomenon that has two parts varying at the same time, for example an alternating current. Also, radio waves, sound waves and microwaves have to travel through different media to get to their final destination. There are many instances where, for example, engineers, doctors, scientists, vehicle designers and others who use electromagnetic signals need to know how strong a signal is when it reaches its destination. The two parts in this context are: the rotation of the signal and its strength. The following are examples of this phenomenon:

- A microphone signal passing through an amplifier
- A mobile phone signal travelling from the mast to a phone a couple of miles away
- A sound wave passing through the bones in the ear
- An ultrasound signal reflected from a foetus in the womb
- The song of a whale passing through miles of ocean water

Complex Numbers are also used in:

- The prediction of eclipses
- Computer game design
- Computer generated images in the film industry
- The resonance of structures (bridges, etc.)
- Analysing the flow of air around the wings of a plane in aircraft design

Appendix 1

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Introduction to Complex Numbers Using simple equations to see the need for different number systems

1.NATURAL NUMBERS (N)

N = 1,2,3,4,...are positive whole numbers.

Solve the following equations using only Natural Numbers:

(a) $x + 2 = 5$
(b) $x + 5 = 2$

Solution (a):

$x + 2 = 5$
$x + 2 - 2 = 5 - 2$
$x = 3$

The solution: 3 is an element of N or $3 \in$ N.

Solution (b)

$x + 5 = 2$
$x = 2 - 5$
$x = ?$

We have a problem, since there is no natural number solution for 2 - 5. Therefore we must invent new numbers to solve for 2 – 5.

2.INTEGERS (Z)

Z = ...-3,-2,-1, 0, 1, 2, 3,...are positive and negative whole numbers.

Solve the following equations using only Integers:

(a) $x + 5 = 2$
(b) $3x = 4$

Solution (a):

$x + 5 = 2$
$x + 5 - 5 = 2 - 5$
$x = -3$

The solution: – 3 is an element of Z or $-3 \in$ Z.

Solution (b)

$3x = 4$
$x = 4 \div 3$
$x = ?$

We have a problem since there is no integer which solves $4 \div 3$. Therefore we must invent new numbers to solve $4 \div 3$.

3.RATIONAL NUMBERS (Q)

These are numbers that can be written in the form a/b (fraction) where a, b $\in$ Z and b $\neq 0$.

Q =...-4.6, -4, -3.5, -2.07, -1, 0, 0.82,...
Q =... $-\frac{46}{10}$, $-\frac{4}{1}$, $-\frac{7}{2}$, $-\frac{207}{100}$, $-\frac{1}{1}$, $\frac{0}{1}$, $\frac{82}{100}$...

All repeating decimals can be written as rational numbers:

$0.\dot{3} = 0.333... = \frac{1}{3}$
$0.1\dot{6} = 0.1666... = \frac{1}{6}$
$0.\dot{1}4285\dot{7} = 0.142857142857... = \frac{1}{7}$

Solve the following equation using only Rational Numbers:

(a) $3x = 4$
(b) $x^2 = 5$

Solution (a)

$3x = 4$
$x = \frac{4}{3}$
Solution: $\frac{4}{3}$ is an element of Q, or $\frac{4}{3} \in$ Q.

Solution (b)

$x^2 = 5$
$x = \sqrt{5}$
$x = ?$

We have a problem, since there is no rational number for the $\sqrt{5}$. Therefore we must invent new numbers to solve $\sqrt{5}$.

Appendix 1

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Introduction to Complex Numbers Using simple equations to see the need for different number systems (continued)

IRRATIONAL NUMBERS

These are numbers that cannot be written in the form a/b where $a, b, \in Z$ and $b \neq 0$. Irrational numbers are non terminating, non repeating decimals such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{4}$, π, e. Pythagoras came across the existence of these numbers around 500 BC.

$c^2 = a^2 + b^2$

$c^2 = 1^2 + 1^2$

$c^2 = 2$

$c = \sqrt{2}$

4.REAL NUMBERS (R)

This is the number system we get when we put all the Rational Numbers together with all the Irrational Numbers. The Rationals and Irrationals form a continuum (no gaps) of Real Numbers provided that the Real Numbers have a one to one correspondence with points on the Number Line.

Solve the following equations using Real Numbers:

$x^2 – 1 = 0$

$x^2 - 1 = 0$ or	$x^2 – 1 = 0$
$(x - 1)(x + 1)= 0$	$x = \pm\sqrt{1}$
$x = 1$ or $x = -1$	$x = \pm 1$

No problem since $-1, +1 \in R$

$x^2 - 3 = 0$

$x = \pm \sqrt{3}$

No problem since $-\sqrt{3}, +\sqrt{3} \in R$

Solve this equation using only Real Numbers

$x^2 + 1 = 0$

$x^2 = -1$

$x = \sqrt{-1}$

What number when multiplied by itself (squared) gives -1? What does your calculator say when you try it: ERROR. We have a problem since there is no Real Number for $\sqrt{-1}$. A number whose square is negative cannot be Real. Therefore we must invent new numbers to solve $\sqrt{-1}$.

5.COMPLEX NUMBERS (C)

Complex Numbers can be written in form

$z = a + ib$, where $a, b \in R$

$i^2 = -1$ and $i = \sqrt{-1}$

Re $(z) = a$ and Im $(z) = b$

Appendix 2

A History of Complex Numbers

Cardano and Tartaglia

Complex Numbers arose from the need to solve cubic equations and not as is commonly believed from the need to solve quadratic equations. Gerolamo Cardano and Tartaglia in the 16th century devised formulae to solve cubic equations of a "REDUCED" form. In their solution to cubic equations with Real roots which could be guessed, the square root of negative numbers turned up in the derivation of these solutions and there was no way to avoid them.

Solve the cubic equation:

$x^3 - x = 0$

$x(x^2 - 1) = 0$

$x(x - 1)(x + 1) = 0$

$x = 0, \quad x = 1, \quad x = -1.$

But when Tartaglia used his formula he got this solution:

$$x = \frac{1}{\sqrt{3}}\left(\left(\sqrt{-1}\right)^{\frac{1}{3}} + 1/\left(\sqrt{-1}\right)^{\frac{1}{3}}\right)$$

At first this looks like nonsense; however, formal calculations with Complex Numbers show that

$z^3 = i$

$z = \left(\sqrt{-1}\right)^{\frac{1}{3}}$ the cube roots of i has solutions

$$z = -i, \quad z = \frac{\sqrt{3}}{2} + \frac{1}{2}i, \quad z = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$$

Substituting these in for $z = \left(\sqrt{-1}\right)^{\frac{1}{3}}$ in Tartaglia's cubic formula and simplifying we get the solutions

$x = 0, \quad x = 1, \quad x = -1.$

This forced an investigation on the square root of negative numbers which has continued to the present day.

Imaginary Numbers

Descartes in the 17th century decided to call them "imaginary" numbers and, unfortunately, the name has stuck even though they are no more imaginary than negative numbers or any other numbers. Euler (around 1777) decided to give the name i to the number which is $\sqrt{-1}$ so now we have:

$i = \sqrt{-1}$ and $i^2 = -1, \quad i^3 = -i, \quad i^4 = 1$

This was to avoid confusion like this:

$$\begin{aligned} 2 &= 1 + 1 \\ &= 1 + \sqrt{1} \\ &= 1 + \sqrt{-1, -1} \\ &= 1 + \sqrt{-1}.\sqrt{-1} \\ &= 1 + i.i \\ &= 1 + i^2 \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

But $2 \neq 0$ So where is the mistake?

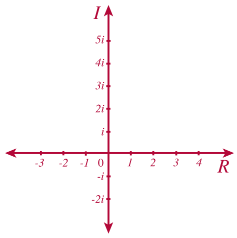
$\sqrt{ab} = \sqrt{a}.\sqrt{b}$ when $a \geq 0$ and $b \geq 0$ $a, b \in R$

or $a \geq 0$ and $b \leq 0$

or $a \leq 0$ and $b \geq 0$

But $\sqrt{ab} \neq. \sqrt{a}.\sqrt{b}$ when $a < 0$ and $b < 0$

In 1806, Argand published a way of representing Real Numbers and Imaginary Numbers on a diagram using two axes at right angles to each other like the Cartesian Plane; this is called the Argand Diagram. On the face of it, the Argand Diagram appears to be identical to the Cartesian Plane. However, it is quite different in that the points which are Complex Numbers can be added, subtracted, multiplied and divided in ways that cannot be done to the points in the Cartesian Plane.



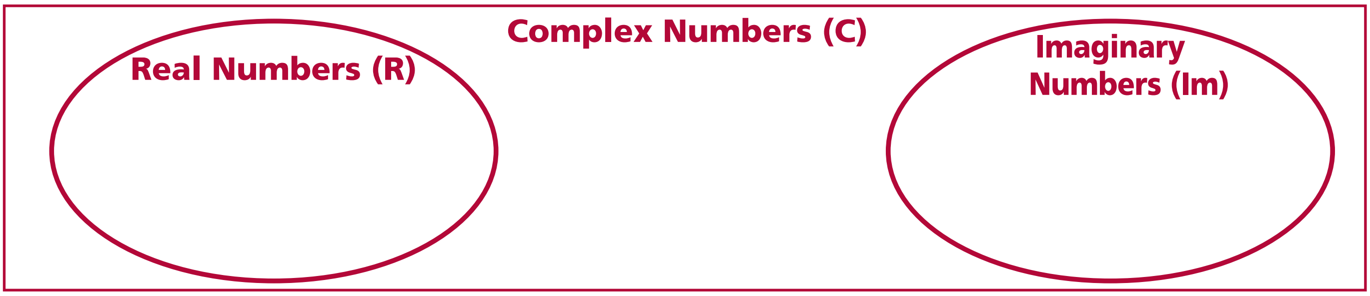
Complex Numbers were further developed in 1843 by William Rowan Hamilton, who invented Quaternions: $i^2 = j^2 = k^2 = ijk = -1$ which extended the study of Complex Numbers to 3-D.

Section A, Student Activity 2

Number Systems (continued)

Question 4: Can a number be real and imaginary at the same time? Can it be either? Place each of these numbers into the appropriate sets below: Imaginary number set, Real number set, Complex Number set

$$\left\{3,\ 0,\ 2+7i,\ 4+0i,\ -5+7i,\ \frac{2}{3}+5i,\ 0+2i,\ i,\ 7-\frac{4}{11}i,\ 5+6i,\ 9,\ 0-\frac{2}{3}i\right\}$$



On graph paper, plot each of the above Complex Numbers on an Argand Diagram. Complete the table below.

Complex Number	Real part	Imaginary part
3		
0		
$2+7i$		
$4+0i$		
$-5+7i$		
$\frac{2}{3}+5i$	$\frac{2}{3}$	5
$0+2i$		
i		
$7-\frac{4}{11}i$		
$5+6i$		
9		
$0-\frac{2}{3}i$		

Section A, Student Activity 3

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Powers of i

1. Simplify i^{11} Which answer is correct: ☐ 1 ☐ i ☐ -1 ☐ $-i$ Explain:	5. Simplify $4i^3 + 7i^9$ Which answer is correct: ☐ $11i$ ☐ $3i$ ☐ $-3i$ ☐ -11 Explain:
2. Simplify i^{33} Which answer is correct: ☐ 1 ☐ i ☐ -1 ☐ $-i$ Explain:	6. Simplify $(3i^5)^2$ Which answer is correct: ☐ -9 ☐ $-9i$ ☐ 6 ☐ 9 Explain:
3. Simplify $i^{16} + i^{10} + i^8 - i^{14}$ Which answer is correct: ☐ 0 ☐ 1 ☐ 2 ☐ i Explain:	7. Make up a similar question of your own and explain your answer.
4. Simplify $i^{12}.\ 3i^2.\ 2i^8$ Which answer is correct: ☐ $6i$ ☐ -6 ☐ $-6i$ ☐ 6 Explain:	8. Make up a similar question of your own and explain your answer.

Section A, Student Activity 4

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Solving Quadratic Equations

Quadratic Equation	$ax^2 + bx + c = 0$	Solving using the formula (see tables)	Roots
$x^2 + 6x + 13 = 0$	$a =$ $b =$ $c =$ $b^2 - 4ac =$		
$x^2 - 4x + 13 = 0$	$a =$ $b =$ $c =$ $b^2 - 4ac =$		
$2x^2 - 2x + 5 = 0$	$a =$ $b =$ $c=$ $b^2 - 4ac =$		

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Section A, Student Activity 4

Solving Quadratic Equations (continued)

Quadratic Equation	$ax^2 + bx + c = 0$	Solving using the formula (see tables)	Roots
$x^2 - 10x + 34 = 0$	$a =$		
	$b =$		
	$c =$		
	$b^2 - 4ac =$		
$3x^2 - 4x + 10 = 0$	$a =$		
	$b =$		
	$c =$		
	$b^2 - 4ac =$		
$x - \frac{5}{x} = 3$	$a =$		
	$b =$		
	$c =$		
	$b^2 - 4ac =$		

Section A, Student Activity 5

The Modulus of a Complex Number

You will need graph paper with this activity.
Use a different Argand Diagram with labelled axes for each question.

1. What is meant by the absolute value or modulus of $z = 5 + 2i$?

   ____

   Plot z on an Argand Diagram. Write z as an ordered pair of real numbers:

   ____

   ____

   Calculate $|z|$

2. Plot $-4i$ on an Argand Diagram. Write $-4i$ as an ordered pair of real numbers.

   ____

   ____

   Find the distance from (0, 0) to the number $-4i$?

3. Plot as accurately as you can the Complex Number $z = \sqrt{3} + 3i$
   Write this Complex Number as an ordered pair of real numbers.

   ____

   ____

   Calculate $|z|$

Section A, Student Activity 5

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The Modulus of a Complex Number (continued)

4. Find the modulus of the Complex Number $z = a + i\,b$.

   Summarise how you get the modulus or absolute value of a Complex Number by explaining what you do to the real and imaginary parts of the Complex Number.

5. Plot the point $3 + 4i$ on an Argand Diagram. Calculate $|3 + 4i|$
   Give the coordinates of 7 other points which are the same distance from the origin.

   Plot these points on an Argand Diagram.

   What geometric figure contains all the points which are this same distance from the origin?____________________

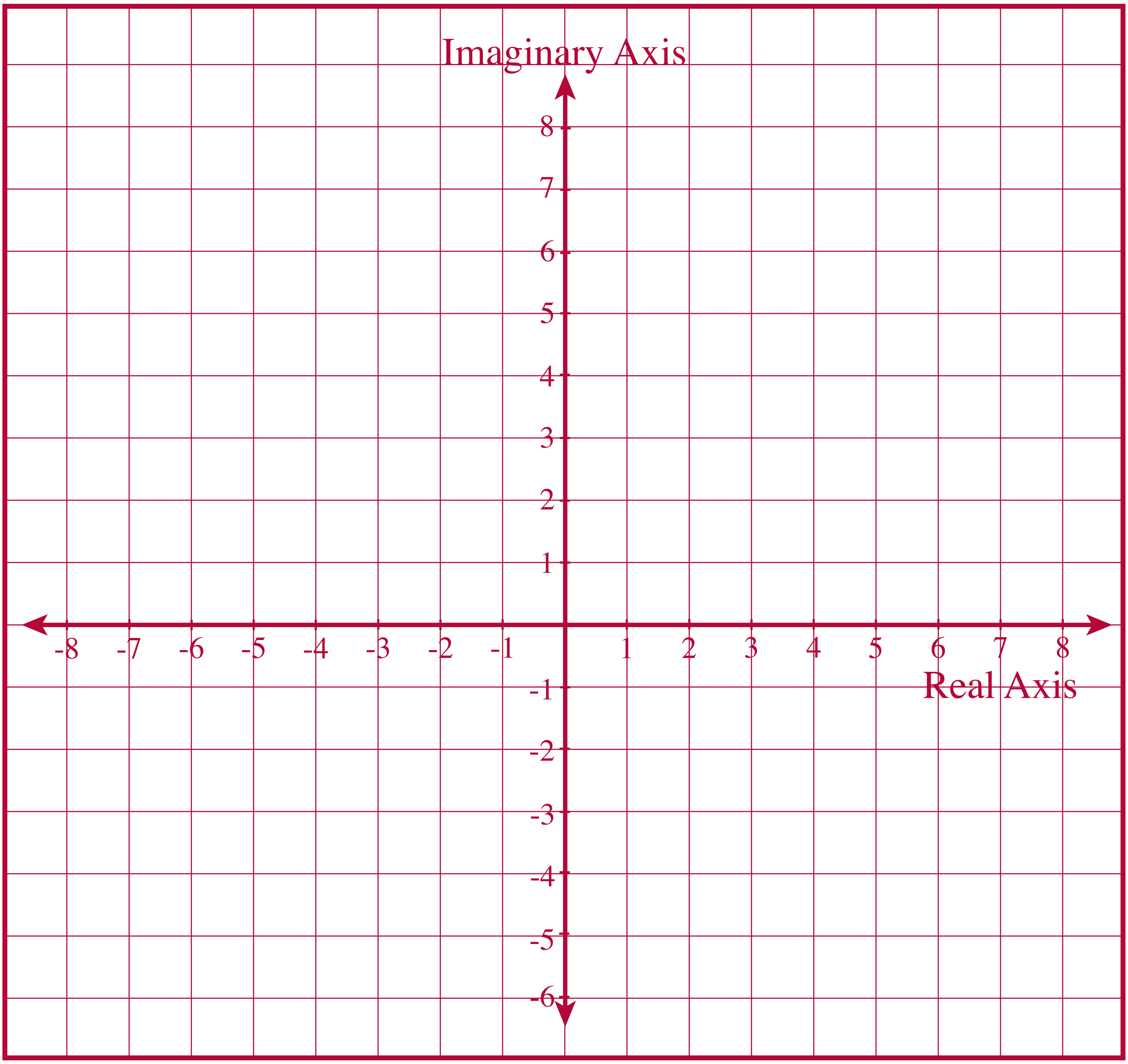
   Draw it on the Argand Diagram.

Section B, Student Activity 2

Addition and Subtraction of Complex Numbers

1. Add $z = 4 + i$ to each of the following complex numbers:
   o $= 0 + 0i$
   $w_1 = 2 + 2i$
   $w_2 = -3 + 2i$
   $w_3 = 0 + 4i$

2. Represent the complex numberso, w_1, w_2, w_3, as points on an Argand Diagram and then show the results from the above exercise using a directed line (a line with an arrow indicating direction) between each w and its corresponding $w + z$. What do you notice?



Section C, Student Activity 1

Adding and Subtracting Complex Numbers: Practice Questions

1	$(12 + 4i) + (7 - 11i)$	
2	$(7 - 2i) + (9 - 4i)$	
3	$(4 - 6i) + (-5 - i)$	
4	$(3 - 8i) - (2 - 4i)$	
5	$(-12 - 5i) - (-2 - 8i)$	
6	$\left(2+\frac{1}{3}i\right)+\left(3-\frac{5}{6}i\right)$	
7	$\left(4+\sqrt{-16}\right)+\left(-5-\sqrt{-25}\right)$	
8	$z_1 = 5+i$ $z_2 = -4 + 6i$ $z_3 = -11 + 2i$ Calculate $(z_1 + z_2) - z_3$	
9	$\left(4-\sqrt{-50}\right)-\left(3+\sqrt{-8}\right)$	
10	$z_1 = a + bi$, $z_2 = c + di$ $z_1 + z_2 =$ $z_2 + z_1 =$ $z_1 - z_2 =$ $z_2 - z_1 =$	

Section D, Student Activity 1

Multiplication by a Real Number

1. If $z = 3 + 4i$, what is the value of $2z$, $3z$, $5z$, $10z$?
2. Represent the origin o = $0 + 0i$, z and $2z$ on an Argand diagram.
3. Find the distance z and $2z$ from o, the origin. Describe at least two methods.
4. Comment on your results.
5. Calculate and plot on an Argand Diagram $\frac{z}{2}, \frac{3z}{4}$. What do you notice?

Section E, Student Activity 1

Multiplication by an Imaginary Number

1. If $z = 3 + 4i$, what is the value of iz, i^2z, i^3z, i^4z? Represent your results on an Argand Diagram joining each point to the origin o = 0 + 0i.
2. Investigate what is happening geometrically when z is multiplied by i to get iz? Use geometrical instruments and/or calculation to help you in your investigation.
3. Prove true for the multiplication of iz by i that you get i^2z and the multiplication of i^2z by i that you get i^3z etc.
4. Write your conclusion.
5. Plot on an Argand Diagram $4 + 2i$ and $-i$. Multiply $-i\,(4 + 2i)$. What do you notice?

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Section E, Student Activity 2

Multiplication of Complex Numbers in the form $a + ib$

1. Plot $3 + i$, $1 + 2i$ and their product $1 + 7i$ on an Argand Diagram.
2. Join each point to the origin $o = 0 + 0i$.
3. Measure the angle (by instrument or calculation) made by the line joining $3 + i$ to the origin and the Real Axis and likewise for $1 + 2i$ and $1 + 7i$.
4. What do you notice about the angles?
5. Find the modulus of $3 + i$, $1 + 2i$ and $1 + 7i$.
6. What do you notice?

Section E, Student Activity 3

Multiplying Complex Numbers

When multiplying Complex Numbers all answers are to be given in the form $a+ib$

1	a. Multiply -4 + 3i by 2. b. Plot -4 + 3i and 2 (-4 + 3i) on an Argand Diagram. c. Calculate |-4 + 3i| and |2 (-4 + 3i)|. d. What was the effect of multiplication by 2 on -4 + 3i?	
2	a. Multiply -4 + 3i by i. b. Plot -4 + 3i and i(-4 + 3i) on an Argand Diagram. c. Calculate |i(-4 + 3i)|. d. What was the effect of multiplication by i on -4 + 5i?	
3	a. Plot 4 + 2i and –i on an Argand Diagram. b. Multiply –i (4 + 2i). c. Plot –i (4 + 2i) on an Argand Diagram. d. What was the effect of multiplication by –i on 4 + 2i?	
4	a. Plot 1 + i on Argand Diagram. b. Calculate |1 + i| c. What angle does the line segment joining 1 + i to the origin make with the positive direction of the x axis?	

Section E, Student Activity 3, (continued)

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Multiplying Complex Numbers

4	d. Using what you know about multiplication of one Complex Number by another, what 2 transformations will happen to $1 + i$ if it is multiplied by $(1 + i)$? e. Knowing the modulus of $1 + i$ and the angle it makes with the Real axis, use this information to work out $(1 + i)(1 + i)$. f. Now calculate $(1 + i)(1 + i)$ multiplying them out as you normally would. g. Were you correct in your first answer?	
5	a. Plot $1 + 6i$ and $-1 - 2i$ on an Argand Diagram. b. Multiply $(1 + 6i)(-1 - 2i)$. c. Plot the answer on an Argand Diagram.	
6	If $z_1 = (5 + 4i)$ $z_2 = (3 - i)$ a. Plot z_1 and z_2 on an Argand Diagram. b. Calculate $z_1 \cdot z_2$. c. Plot the answer $z_1 z_2$ on an Argand Diagram.	
7	a. Plot $3 - 2i$ and $3 + 2i$ on an Argand Diagram. What do you notice about both points? b. What angle do you expect the product $(3 - 2i)(3 + 2i)$ to make with the x - axis? Explain. c. Multiply $(3 - 2i)(3 + 2i)$. What do you notice about the answer?	
8	a. Plot $4 + 3i$ on an Argand Diagram b. Plot $(4 + 3i)^2$ on an Argand Diagram	
9	a. Plot on an Argand Diagram: $5 + i(4 - 2i)$	
10	If $z_1 = a + bi$ and $z_2 = c + di$, then a. $z_1 \cdot z_2 =$ b. $z_2 \cdot z_1 =$	

