University of Colorado
Boulder

THE HAMILTONIAN PRINCIPLE, LAGRANGIAN MECHANICS AND THE LAGRANGEPOINTS

FINN JOEL BJERVIG

August 31, 2022

Abstract

This paper focuses on select contributions of Joseph Louise Lagrange such as Lagrangian mechanics and the solution to the restricted three body problem known as the Lagrange points. Additionally, a brief history of the prior works of physicists, mathematicians and philosophers leading to the fundamental principle of least time/energy/action is presented. No rigorous derivation is delivered, but the arguments and instructions for the process is, thereby focusing on the result of the merging and extrapolations of postulates.

CONTENTS

1 INTRODUCTION

Solving mechanical problem in a classical Newtonian fashion by summing up all forces can get quite tedious as systems get more complex. To describe the motion of particles subject to a set of constraint and forces, the methodology of Lagrangian mechanics is a very elegant one and is less time consuming for said complex systems.

The nature of physics has long been recognized by physicists and philosophers to follow the course that will minimize certain quantities in order to maximize the systems entropy. Pierre de Fermat (1607-1665) and Rene Descartes (1596-1650) both had different opinions regarding the speed of light through a medium as opposed to in free space (vacuum). In Fermat's publication *Principle of Least Time* in 1657, he claimed that light travels slower through a more dense matter while Descartes argued that the light would be traveling faster. Much later in 1740 Pierre Louis Moreau de Maupertuis (1698-1759), without any rigorous mathematical proof, designed the *Principle of Least Action*, which stated that a mass subject to a conservative force: $F = -\nabla(U)$ (as gravity or electrostatic forces) will travel the path of least action : $\delta(S) = 0$, where S is defined though the action integral$S = \int m\boldsymbol{v} \cdot d\boldsymbol{s}$, and is v expressed through the total and potential energy through equation of kinetic energy. As Carl Gustav Jacob Jacobi (1804-1851) later emphasized this postulated works in accordance with Fermat's principle of least time, which was later mathematically proved by Leonard Euler (1707-1783) and thus settled the controversy between Fermat and Descartes. These principles are so called variational principles: an alternative method for finding a description of state by recognizing it as an extreme point (maximum or minimum) of a function. As we shall see, the culmination of these principles is the Hamiltonian principle which elegantly delivers the equation of motion for practically any system in a generalized coordinate system. All these terms will be explained. [1] [2]

1.1 *Virtual Work*

The principle of Virtual work is an old one, traced back to Aristotle's (384-322 B.C). It states that A system on N particles is at equilibrium if the "virtual work"

$$\sum_{i=1}^{N} F_i \delta x^i = 0 \tag{1}$$

is zero for all virtual displacements that obey this physical constraint.

Jean Le Rond d'Alembert (1717-1783) generalized the principle of virtual work in 1742 by taking into account newtons second law of accelerating forces F = ma

$$\sum_{i=1}^{N} (F_i - m_i \frac{d^2 x^i}{dt^2}) \delta x^i = 0. \tag{2}$$

By introducing newtons law of motion, the principle of virtual work became a analytical expression from which the equations of motion of a generalized system could be derived. D'Alemberts principle thus states that the sum of all forces on a system is equal to the work done by the corresponding acceleration of that system. Joseph Louise Lagrange(1736-1813) modified

D'alemberts principle to what has become one of classical mechanics greatest generalized equations of motion. Consider the infinitesimal work identity

$$0 = \left(\boldsymbol{F} - m_i \frac{d^2 \boldsymbol{x}^i}{dt^2} \right) \delta \boldsymbol{x} = \delta W - \frac{d}{dt} \left(m \frac{d\boldsymbol{x}}{dt} \delta \boldsymbol{x} \right) + m \frac{d\boldsymbol{x}}{dt} \frac{d\delta \boldsymbol{x}}{dt} \tag{3}$$

F is the force acting on a particle of mass m such that $\delta W = \boldsymbol{F} \delta \boldsymbol{x}$ represents the virtual work along the virtual displacement $\delta \boldsymbol{x}$. The position $\boldsymbol{x} = \boldsymbol{x}(q_1, ...q_k, t)$ is a time dependent function of k generalized coordinates. Upon further derivation the general force Q_i is also introduced along with the kinetic energy of the system. Hence a modification of D'alembert's principle has been made described by general coordinates. Considering the general force as a conservative force, it can be written as the gradient of a potential energy U

$$\frac{d}{dt} \left(\frac{\partial T}{\dot{q}_i} \right) - \frac{\partial T}{\partial q_i} = -\frac{\partial V}{\partial q_i} \tag{4}$$

Leaving this equation for now, we shall derive a second result from d'alemberts principle, now written as

$$\delta T + \delta V = \frac{d}{dt} \left(m \frac{d\boldsymbol{x}}{dt} \delta \boldsymbol{x} \right) \tag{5}$$

The virtual work is $\delta W = -\delta U$ if the force can be expressed as a energy potential $\boldsymbol{F} = -\nabla U$. From that, integrating equation 4 will yield the very profound result named the *Hamiltonian Principle*. It reads:

Of all the possible paths along which a dynamical system move from one point to another within a specified time interval (consistent with any constraints), the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energy.

Continuing out little derivation endeavor, the postulate stated above can be written in the context of calculus as

$$\int_{t_1}^{t_2} (\delta T - \delta V) dt = \delta \int_{t_1}^{t_2} L \cdot dt = 0 \tag{6}$$

The function $L = K - U$ is called the *Lagrangian function*

2 HAMILTONIAN PRINCIPLE LAGRANGE EQUATIONS

The Hamiltonian principle states:

$$\delta \int_{t_1}^{t_2} (T - U) dt = 0 \tag{7}$$

T is generally known in analytical mechanics to be the kinetic energy of the system, and U the potential energy. δ is a notation of variation. Assigning the system a rectangular coordinate system, the terms can be written as

$$\begin{cases} T = T(x_i) \\ U = U(\dot{x}_i) \end{cases}$$

The difference of the two terms is called the Lagrangian function or the Lagrangian of the particle.

$$L \equiv T(x_i) - U(\dot{x}_i) = L(x_i, \dot{x}_i) \tag{8}$$

Hence equation 7 becomes

$$\delta \int_{t_1}^{t_2} L(x_i, \dot{x}_i) dt = 0 \tag{9}$$

As the postulate states, we need to find the minimum of the Lagrangian L, and this can be achieved by applying the equation of variation

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0, \quad i = 1, 2, 3..$$

(10)

which yields what is known as the Lagrange equations of motion.

generalized coordinates are the set of variables that completely specify the system. They need not be of a specific dimension, as length in Cartesian system, or of angular coordinates which is found in spherical problems. In fact, generalized coordinates can be any variable, of any dimension, or ever dimensionless, as $length^2$ or the variable of friction μ.

3 LAGRANGE POINTS

The solar system consists of eight planets which are all exerting forces of gravity on each other. There is in fact no analytical solution for describing the trajectories of a multitude of celestial objects withing the same, seemingly organized star system. In fact, a system of only three bodies is just as impossible.

The three body problem has no closed analytical solution and can only be described through a chaotic system and computational iterative methods. But by reducing the mass m of one of the three bodies to a negligible amount, one can obtain an analytical solution. The approximation is such that the forces from the two greater bodies M1 M2 are only acting on the small mass, and m is not exerting any forces on them in in turn. The two greater bodies M1 M2 are still influencing each other, just as before, so one can consider the problem of two bodies and solve for when the net force is zero to obtain positions of the Lagrangian points. This modified setup is called the restricted three body problem and the reduced mass approximation is generally valid when objects of relatively small masses are present in the system, as with calculating the trajectory of satellites, since the mass of the satellite is negligible compared to the moon and the earth. By considering this restricted three body problem, one can find solutions for where the small mass object remains stationary relatively to the other two masses. The physical meaning of this is that the potential forces of the heavier masses produces a equilibrium at certain locations in the vicinity of the system, which Lagrange solved for. Hence the name Lagrangian points.

The gravitational force acting on m from M_1 and M_2 are

$$\vec{F} = -\frac{GM_1 m}{\vec{r} - \vec{r}_1{}^3}(\vec{r} - \vec{r}_1) - \frac{GM_2 m}{\vec{r} - \vec{r}_2{}^3}(\vec{r} - \vec{r}_2) \tag{11}$$

Since the system is rotating, it is beneficial to apply a rotating reference frame so that M_1 and M_2 become stationary. The trade-off is that in this setup, pseudo-forces of Coriolis and centrifugal forces become present.

$$\vec{F}_\Omega = \vec{F} - 2m(\vec{\Omega} \times \frac{\partial \vec{r}}{\partial t}) - m\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) \tag{12}$$

The first term $\vec{F}$ is the gravitational force from equation 11, the second term is due to the Coriolis effect and the third term is the centrifugal force.

This expression can also be derived from the generalized potential

$$U_\Omega = U - \vec{v} \cdot (\vec{\Omega} \times \vec{r}) + \frac{1}{2}(\vec{\Omega} \times \vec{r}) \cdot (\vec{\Omega} \times \vec{r}) \tag{13}$$

by applying this equation to the generalized gradient, very similar to the Lagrange equation 2

$$\vec{F}_\Omega = -\nabla_r U_\Omega + \frac{d}{dt}(\nabla_{\vec{v}} U_\Omega)$$

(14)

From the generalized potential, setting transnational velocity $\vec{v} = 0$, we can produce a contour plot as figure 1.

In the contour plot we can identify where the Lagrange points are located as they corresponds to the extremums of the function. To obtain an useful expression, remember that extemums are the points where the generalized gradient equation 3 is zero, which we showed is the same as setting $\vec{F}_\Omega = 0$. In a more practical context, and quite intuitively so, this means that Lagrangian points are located where there is a equilibrium of forces in the two body system.

$$-\nabla_r U_\Omega + \frac{d}{dt}(\nabla_{\vec{v}} U_\Omega) = \vec{F} - m\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) = 0 \tag{15}$$

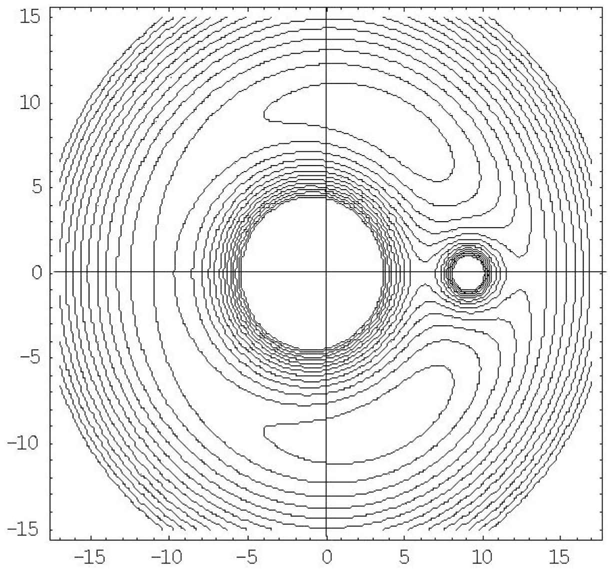

Figure 1: Gravitational potential contour map of a two body system, with five extremums [?]

solving for r we obtain the positions for the Lagrange points in Cartesian coordinates

$$\begin{cases} L_1 = \left(R\left[1-\left(\frac{\alpha}{3}\right)^{\frac{1}{3}}\right], 0\right) \\ \\ L_2 = \left(R\left[1+\left(\frac{\alpha}{3}\right)^{\frac{1}{3}}\right], 0\right) \\ \\ L_3 = \left(-R\left[1+\left(\frac{5\alpha}{12}\right)\right], 0\right) \end{cases} \tag{16}$$

where $\alpha = \frac{M_2}{M_1+M_2}$

In a sun-earth system, The two first Lagrangian points are located on either side of the earth, and the third is at the other side of the sun, at a slightly larger orbit than earth. But there are five Lagrangian points, and the two remaining, L_4 and L_5. We obtain the coordinates by realizing that we need to balance the centrifugal force and the gravitational forces exerted by the earth and sun. The only balancing forces perpendicular to the centrifugal (which is in the positive radial direction) is the gravitational forces from the two bodies. Therefore one should consider the force balance equations parallel and perpendicular to $\vec{r}$. By setting $F_{\Omega}^{\perp} = 0$ and considering the fact that the Lagrange points might not be at the line $y = 0$ as the first three points. With this assumption, projecting the forces and balancing will yield the last two Lagrange points

$$\begin{cases} L_4 = \left(\frac{R}{2}\beta, \frac{\sqrt{3}}{2}R\right) \\ \\ L_5 = \left(\frac{R}{2}\beta, -\frac{\sqrt{3}}{2}R\right) \end{cases} \tag{17}$$

where $\beta = \frac{M_1-M_2}{M_1+M_2}$

Thereby the Lagrangian Points, without any rigorous derivation has been found. [1]

4 DISCUSSION

LAGRANGE POINTS

The Lagrangian points are not only a fascinating solution, it also has practical uses.

L1 and L2 can be shown to be saddle points through the second derivative in the x and y direction along with the mixed derivative. Further, by solving for the eigenvalue in the evolution matrix, a real root appears which means that L1 and L2 are unstable. For this reason, satellites are not actually stationary but are actually oscillating back and fourth around the Lagrange points.

L3 is located on the other side of the sun, it is also a saddle point and a very unstable one, considering the magnitude of the real root of the eigenvalue. In a earth sun system, L3 has no practicality since no communication would be possible since the sun is in the way. One could place satellites at other Lagrange points as a middleman for this sort of communication.

L4 and L5 are both local maximas, which would imply instability. Imagine a ball on a hill, a small push would send it down the hill. But surprisingly, when looking again at the eigenvalues, both are in fact imaginary and thus the points are stable. On these points, you can see accumulations of asteroids and debris which also points toward stability.

Many satellites from the worlds space organizations has been sent to the nearest Lagrange points; 1 and 2. An example is the SOHO (Solar and Heliospheric Observatory) mission to L1. WMAP (Wilkinson Microwave Anisotropy Probe) is located at L2, where the James Webb Telescope will be sent in 2021. [1]

5 CONCLUSION

The idea of natures way of minimizing energy has been the foundation for many physical realizations. The culmination of these postulates has lead to the Hamiltonian principle and the methodology of Lagrangian mechanics. The calculation of the equation of motion through Lagrangian mechanics has proven to be very general and elegant by only taking into account potentials and kinetic energies and the system constraints, all in a general coordinate system. This makes it substantially easier to solve for equation of motion as complexity of the problem setup increases. Not only as a neat solution for the restricted three body problem, but some Lagrange points are useful for space exploration and the much awaited James Webb Telescope will soon be sent there to boldy look where no man has gone before.

REFERENCES

[1] N. J. Cornish, "The lagrange points," *WMAP Education and Outreach*, 1998.

[2] A. R. E. Oliveira, "Lagrange as a historian of mechanics," *Scientific Research*, 2013.