# Assignment 2

Continuous Time Models

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August 31, 2022

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## Pop Song Model

There is a striking resemblance one could make about how a songs popularity changes with time, a ironically relevant one. Imagine the scenario: A new song has just released and nobody has heard it yet, meaning the total population is *susceptible* to the song, if one should use epidemiological lingo. Yes, the analogy of a spreading virus to the spreading of a song is not only comically pleasing, the nature of it also seems to fit quite well. Following the definition of a "pop song", we assume that everyone in the population will listen to the song at some point. So after a while it begins to catch on, and *infected* people are listening inexhaustibly to the song and referring to it as a "banger" to their friends. The virus, so to speak, has begun to spread. As more people finds out about the song, and put on in their headphones and speakers, less of the total population will have yet to have heard the song: the susceptible population decreases. Up until the point of saturation, when everyone has heard it, meaning all have been infected. Inevitably the pop song which so many people have used to define their identity with, has lost its touch. Not as attractive anymore, it's played less often and seldom talked about. The share of the population who no longer listens to it can be represented as *recovered*. The number of plays continues to decrease, until it dies off and is buried in the archives along with all other so-called "bangers". The human race has survived another wave of a "smash-hit" and will hopefully be better prepared the next time one comes along.

The system of differential equations used to describe the spread of an infectious disease is called the SIR model, which is an abbreviation for Susceptible, Infected and Recovered.

$$\begin{aligned}
\frac{dS}{dt} &= -\beta \frac{S(t)I(t)}{N(t)} \\
\frac{dI}{dt} &= \beta \frac{S(t)I(t)}{N(t)} - \gamma I(t) \\
\frac{dR}{dt} &= \gamma I(t)
\end{aligned}$$

To simulate this system it need to be discretized in time. By approximating the derivatives as $\dot{S} = (S_{t+1} - S_t)/\Delta t$ the system is expressed as

$$\begin{aligned}
S_{t+1} &= S_t - \beta \frac{S_t I_t}{N_t} \Delta t \\
I_{t+1} &= I_t + \left( \beta \frac{S_t I_t}{N_t} - \gamma I_t \right) \Delta t \\
R_{t+1} &= R_t + \gamma I(t) \Delta t
\end{aligned}$$

And we can choose to vectorize it to make it easy to handle while coding. We

$$\vec{x} = \begin{bmatrix} S \\ I \\ R \end{bmatrix} \qquad \dot{\vec{x}} = \begin{bmatrix} \dot{S} \\ \dot{I} \\ \dot{R} \end{bmatrix}$$

$$A = \begin{bmatrix} -\frac{\beta}{N} & 0 & 0 \\ \frac{\beta}{N} & -\gamma & 0 \\ 0 & \gamma & 0 \end{bmatrix} \qquad K = \begin{bmatrix} I & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Lastly putting them together to form the system of linear differential equations. The subscript $t$ denotes the matrices and vectors change of value for each time step. Additionally, following the discretized system above, we are inserting the previous value of each quantity. Note that $\mathbb{1}$ is the identity matrix.

$$\vec{x}_{t+1} = (A_t K_t + \mathbb{1}) x_t$$

Applying this in python we yield the following result seen in figure1 (Code is attached in the appendix below)

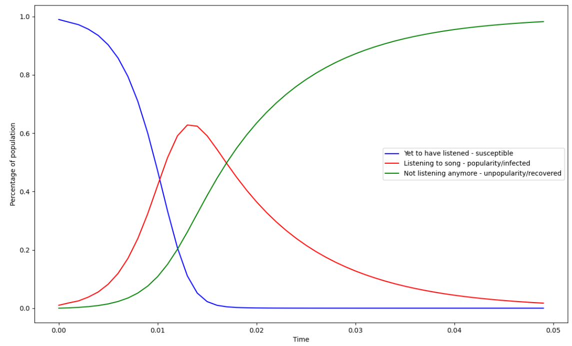



Figure 1: The popularity of the pop song evolves over time similar to the spread of a virus

**Equilibrium points and Stability analysis**

The equilibrium points for $I$ are defined as when $\dot{I} = 0$.

$$0 = \dot{I} = \beta \frac{SI}{N} - \gamma I \Rightarrow I \left( \beta \frac{S}{N} - \gamma \right) = 0 \tag{1}$$

We see that this is true when

$$\begin{aligned} I &= 0 \\ S &= \frac{N\gamma}{\beta} \Rightarrow I > 0 \end{aligned}$$

With this in mind, we now inspect the eigenvalues of the Jacobian matrix for $f(S, I) = \beta \frac{SI}{N} - \gamma I$. With a normalized population $N = 1$ we have that the eigenvalues are given by

$$\lambda_{1,2} = -\frac{1}{2}(\beta I - \beta S + \gamma) \pm \sqrt{\frac{1}{4}(\beta I - \beta S + \gamma)^2 - (\gamma)^2} \tag{2}$$

We can simplify this by realizing that if $\lambda_{1,2} < 0$ then The first term on the RHS must be larger in size than the square root

$$-\frac{1}{2}(\beta I - \beta S + \gamma) < \sqrt{\frac{1}{4}(\beta I - \beta S + \gamma)^2 - (\gamma)^2} \tag{3}$$

Here we present the different cases for what defines the stability based on the equation above

$$\begin{aligned} \text{Stable if } \lambda_{1,2} < 0 &\Rightarrow \beta I \gamma < 0 \\ \text{Lyapunov stable if } \lambda_{1,2} = 0 &\Rightarrow \beta I \gamma = 0 \\ \text{Unstable if } \lambda_{1,2} > 0 &\Rightarrow \beta I \gamma > 0 \end{aligned}$$

With the equilibrium points given above, $I = 0$ and $I > 0$, we can see that when $I = 0$ we have that $\beta I \gamma = 0$, meaning its Lyapunov stable, and when $I < 0$, $\beta I \gamma > 0$ corresponding to an unstable equilibrium. Indeed, looking at the popularity curve in red in figure 1, we see that $\dot{I} = 0$ at the peak of popularity $I \approx 0.68$, but will decrease, as it is unstable. However as time reaches infinity, when $I = 0$, it remains there, at $I = 0$, thus it is stable, or upon closer inspection, Lyaponov-stable. Further this can be verified by looking at the phasespace of S and I in figure 2

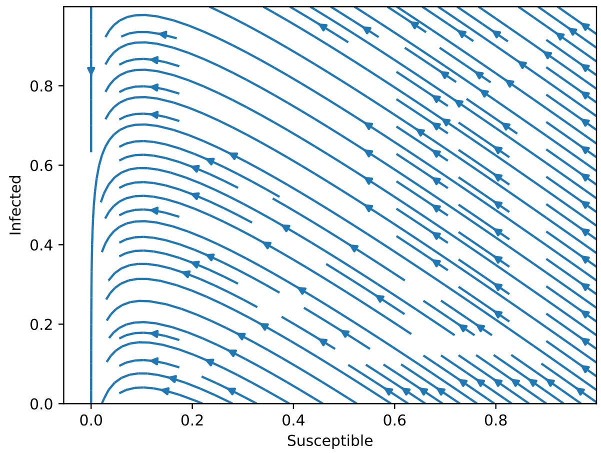
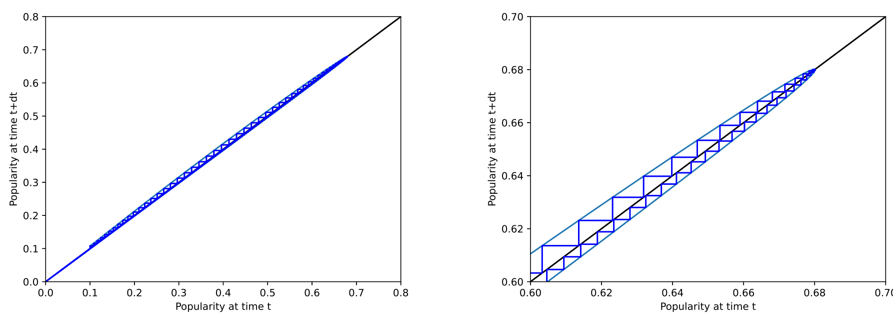



Figure 2: Phasespace of S and I shows the convergence towards $I = 0$

**Cobweb**




(a) Standard view. plot starts at $I_0 = 0.1$

(b) Zoomed in at unstable equilibrium

Figure 3: The cobwebplot

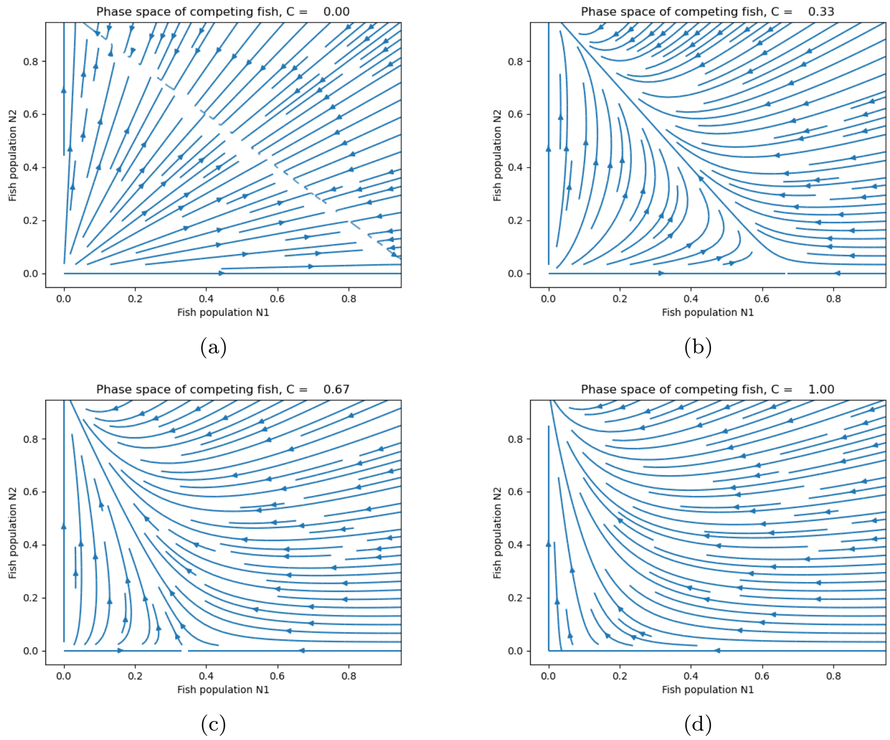
A cobweb is constructed by plotting the solution for time step $I_{t+dt}$ against the solution for the previous time step $I_t$ as $I_{t+dt} = f(I_t)$ Numerically, its done by shifting the solution vector by one index for one of the axes in the plot. It begins with $I_t = I_0$ and with the given function $f$ mapping what the solution in the next time step is $I_{t+dt}$, by moving vertically upward until the line reaches the curve. Then one sets $I_{t+dt} = I_t$ and repeats. The latter operation is done by plotting a diagonal line $I_t = I_t$ and traveling horizontally until that line reaches the diagonal. See figure 3. The stable values are located where the diagonal line and the curve intersect. The cobweb plot seems to agree with the time series plot in that there is an unstable equilibrium at I($t \to$ inf) = 0 . The curves intersect also at $\approx$ 0.68, but is not a stable equilibrium since the trajectory (the dark blue stairs) continues past that point. This is all in agreement with the eigenvalues and the phase space discussed previously. In summary. There is one stable equilibrium point at $I = 0$ and one neutral, or Lyapunov stable equilibrium point at $I \approx 0.68$, which is when $S = \beta/\gamma = 0.1$

## Competing fish

Consider the following system of ODE

$$\frac{dN_1}{dt} = r_1 N_1 \left(1 - \frac{N_1}{K_1}\right) - \lambda_1 N_1 N_2 - C N_1$$
$$\frac{dN_2}{dt} = r_2 N_2 \left(1 - \frac{N_2}{K_2}\right) - \lambda_2 N_1 N_2$$

It describes population growth of two fish species $N_1$ and $N_2$. the parameter $r_i$ is the reproductive rate, $K_i$ how much the environment can sustain the population, and $\lambda_i$ the rate the population is being eaten by the other population. The only thing thats differing is that population $N_1$ are caught by fishers with a rate of $CN_1$.




(a) (b) (c) (d)

Figure 4: The phase space for the fish population for different rates $C \in [0,1]$ of fish population 1 getting caught by fishermen using parameters $r_1 = r_2 = 1$, $K_1 = K_2 = 1$, $\lambda_1 = \lambda_2 = 1$. Link to animation.

The "convergence line" that can be seen on each plot in figure 4 has a inclination of $N_2 = 1 - \frac{1}{C-1} N_1$

# Competing fruit flies

The governing system of ODEs reads

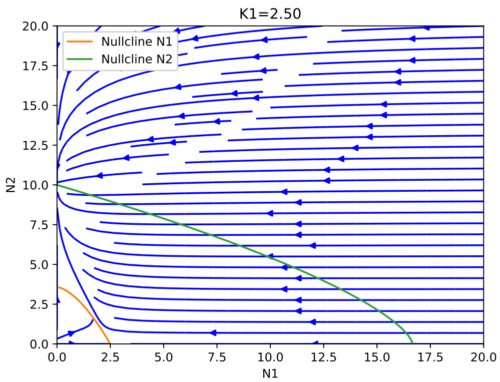
$$\dot{N}_1 = r_1 N_1 \left(1 - \left(\frac{N_1}{K_1}\right)^{\theta_1} - a_{12}\frac{N_2}{K_1}\right)$$
$$\dot{N}_2 = r_2 N_2 \left(1 - \left(\frac{N_2}{K_2}\right)^{\theta_2} - a_{21}\frac{N_1}{K_2}\right)$$

By finding the nullclines of the system, which is defined as one of the temporal derivatives being zero, it is possible to discover equilibrium points. This is achieved by setting one of the populations constant and varying the other. An equilibrium point in the phase space thus occurs when two nullclines intersect.

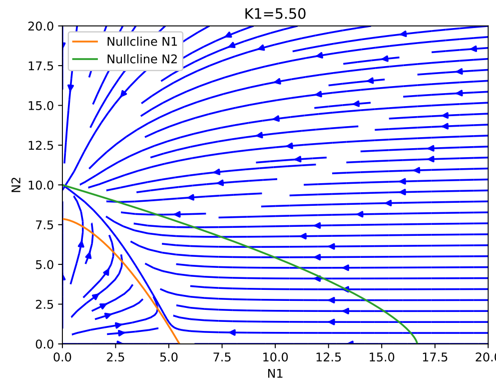
The (non-trivial) nullclines are described by:

$$\dot{N}_1 = 0 \Rightarrow \left(1 - \left(\frac{N_1}{K_1}\right)^{\theta_1} - a_{12}\frac{N_2}{K_1}\right) = 0 \Rightarrow N_2 = \frac{K_1}{a_{21}}\left(1 - \left(\frac{N_1}{K_1}\right)^{\theta_1}\right)$$
$$\dot{N}_2 = 0 \Rightarrow \left(1 - \left(\frac{N_2}{K_2}\right)^{\theta_2} - a_{21}\frac{N_1}{K_2}\right) = 0 \Rightarrow N_1 = \frac{K_2}{a_{12}}\left(1 - \left(\frac{N_2}{K_2}\right)^{\theta_2}\right)$$

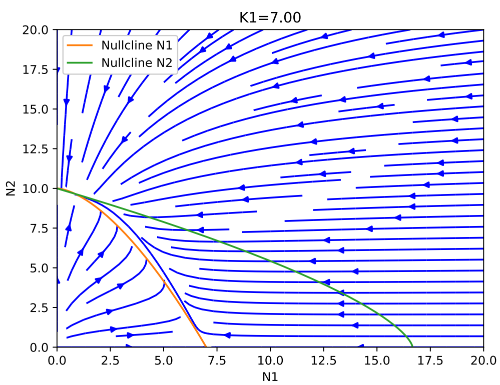
Through using these nullclines in combination with a streamplot we animated the phase space with the nullclines for increasing values for the parameter $K_1$. Some outtakes can be see in figure 5 below.



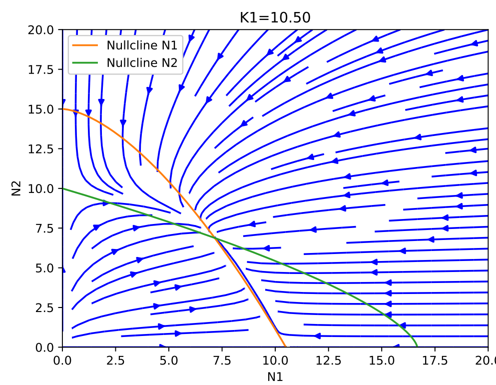

(a)




(b)




(c)




(d)

Figure 5: The phase space with nullclines for varying values of $K_1$. Here, other parameters were set as $r_1 = 4.513$, $r_2 = 1.496$, $K_2 = 10$, $\theta_1 = 1.6$, $\theta_2 = 1.5$, $a_{12} = 0.7$, and $a_{21} = 0.6$. Link to animation over varying values of $K_1$.

In subfigure (a) and (b), with $K_1 = 2.5,\ 5.5$, before the nullclines intersect, we see no positive stable solution (other than the trivial solutions where one species is exterminated). However, once the nullclines intersect around $K_1 = 7$ in subfigure (c), an asymptotically stable equilibrium appears, following the intersection of the nullclines as can be seen in (d).

## Stable Equilibrium

Consider a solution $\vec{x}^* \in \mathbb{R}^n$ to the system of ODEs

$$\vec{x}' = F(\vec{x})$$

where $\vec{x}^*$ is stable if for any neighborhood $\mathcal{O}$ of $\vec{x}^*$ and there is a neighborhood $\mathcal{O}_1$ of $\vec{x}^*$ such that every solution $\vec{x}(t)$ with $\vec{x_0} = \vec{x}_0 \in \mathcal{O}_1$ is defined and remains in $\mathcal{O} \forall t < 0$. For asymptotically stability, we need the above properties of stability to apply and further

$$\lim_{t\to\infty} \vec{x} = \vec{x}^* \in \mathcal{O}$$

For simplicity, introduce $n_1 = \frac{N_1}{K_1}$, $n_2 = \frac{N_2}{K_2}$, $\xi_1 2 = a_{12}\frac{K_2}{K_1}$, and $\xi_2 1 = a_{21}\frac{K_1}{K_2}$. Thus, the model can be rewritten into

$$\begin{aligned}
\dot{n_1} &= r_1 n_1 \left(1 - n_1^{\theta_1} - a_{12} n_2\right) = r_1 n_1 f(n_1, n_2) \\
\dot{n_2} &= r_2 n_2 \left(1 - n_2^{\theta_2} - a_{21} n_1\right) = r_2 n_2 g(n_1, n_2)
\end{aligned}$$

An equilibrium point is asymptotically stable if all eigenvalues of the Jacobian has a real part strictly smaller than zero. The Jacobian for this system is

$$\mathcal{J}(n_1, n_2) = \begin{bmatrix} r_1 f(n_1, n_2) - r_1\theta_1 n_1^{\theta_1} & -r_1 n_1 \xi_{12} \\ -r_2 n_2 \xi_{21} & r_2 g(n_1, n_2) - r_2\theta_2 n_2^{\theta_2} \end{bmatrix}$$

We know that $f, g = 0$ at any equilibrium point $(n_1^*, n_2^*)$, allowing the following simplification to the Jacobian

$$\mathcal{J}(n_1^*, n_2^*) = \begin{bmatrix} -r_1\theta_1 n_1^{\theta_1} & -r_1 n_1 \xi_{12} \\ -r_2 n_2 \xi_{21} & -r_2\theta_2 n_2^{\theta_2} \end{bmatrix}$$

Negative eigenvalues correspond to a negative trace and positive determinant of the matrix. Acknowledging that all constants and values are positive,

$$\begin{aligned}
\mathrm{tr}(\mathcal{J}(n_1^*, n_2^*) &= -r_1\theta_1 n_1^{\theta_1} - r_2\theta_2 n_2^{\theta_2} < 0 \\
\det(\mathcal{J}(n_1^*, n_2^*) &= r_1 r_2 \theta_1 \theta_2 n_1^{\theta_1} n_2^{\theta_2} - r_1 r_2 n_1 n_2 \xi_{12}\xi_{21} > 0 \\
&\Rightarrow {n_1^*}^{\theta_1 - 1} {n_2^*}^{\theta_2 - 1} > \frac{\xi_{12}\xi_{21}}{\theta_1\theta_2}
\end{aligned}$$

Thus, any equilibrium fulfilling this condition is asymptotically stable.

## Python Code

Code files are attached with submission.