Assignment 6

Network Analysis

Finn Joel Bjervig[*], August Forsman[†], and Erik Turesson[‡]

1MA256 - Modeling Complex Systems,
Department of Mathematics,
Uppsala university, Sweden

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[*]Electronic address: `jobj8920@student.uu.se`
[†]Electronic address: `aufo8456@student.uu.se`
[‡]Electronic address: `ertu2293@student.uu.se`

1 Coupled Harmonic Oscillators in a network structure

The governing equation for determining the synchronizability of linearly dynamical nodes is

$$\frac{dx_i}{dt} = R(x_i) + \alpha \sum_{i \in N_j} \big(H(x_j) - H(x_i) \big) \tag{1}$$

The system at hand constitutes a complex valued system

$$\frac{dx_i}{dt} = \mathbf{i}\omega x_i + \alpha \sum_{i \in N_j} \left(x_j^\gamma - x_i^\gamma \right) \tag{2}$$

x_i are the nodes state and N_i the neighbouring nodes. R is the local reaction term that determines the inherent behaviour of the system. $H = x^\gamma$ is called the output function that applies to all nodes.

The above equation can be simplified by utilizing the Laplacian matrix

$$\frac{dx_i}{dt} = R(x_i) - \alpha L \begin{pmatrix} H(x_1) \\ H(x_2) \\ \vdots \\ H(x_n) \end{pmatrix} \tag{3}$$

This system can be synchronized if and only if the trajectory x_i is stable $\forall i$. Thus we apply linear stability analysis on around this stable state $x_s(t)$

$$x_i(t) = x_s(t) + \Delta x_i(t) \tag{4}$$

Plugging into equation 3 we yield the final result of the linearization

$$\frac{d\Delta x}{dt} = \big(R'(x_s)\mathbf{I} - \alpha H'(x_s)\mathbf{L} \big) \Delta x = \mathbf{\Lambda} \Delta x \tag{5}$$

$\mathbf{I}$ is the identity matrix. Since x_s changes over time, the eigenvalues of the coefficient matrix must be negative in order for the system to be stable. It is known that the matrix $a\mathbf{X}+b\mathbf{I}$ has eigenvalues $a\lambda_i+b$, where λ_i are the eigenvalues of $\mathbf{X}$. So the eigenvalues of the coefficient matrix $\mathbf{\Lambda}$ are thus $-\alpha\lambda_i H'(x_s(t))+R'(x_s(t))$, where λ_i are the eigenvalues of the Laplacian, and is different for different graph networks. We shall have a look at the karate club network. The eigenvalues of interest are the two smallest ones λ_1, λ_2 and the largest one λ_{34}

$$\left(\lambda_1 = -3.5597 \cdot 10^{-15} \quad \lambda_2 = 0.4685 \quad \dots \quad \lambda_{34} = 18.1366\right) \tag{6}$$

The smallest eigenvalue λ_1 is equal to R' and cant be changed. So we look for the next smallest one and the largest one, to sandwich all other eigenvalues since $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ and analyze the stability with the two following criteria

$$\alpha\lambda_2 H'(x_s(t)) > R'(x_s(t)) \tag{7}$$

$$\alpha\lambda_n H'(x_s(t)) > R'(x_s(t)) \tag{8}$$

Since H' can take on negative values also, we need to look at both criteria.

$$\alpha\lambda H'(x_s(t)) > R'(x_s(t)) \tag{9}$$

In this case, $R = i\omega x_s(t)$, meaning $R'(x_s(t)) = i\omega$, resulting in

$$\alpha\lambda H'(x_s(t)) > i\omega \tag{10}$$

However - to make this valid for complex state values of x_s - we look at the real parts of the equality

$$\mathrm{Re}(\alpha\lambda H'(x_s(t))) > \mathrm{Re}(i\omega) = 0 \tag{11}$$

Since $\alpha, \lambda_2, \lambda_n > 0$, the equality depends only on $\mathrm{Re}(H'(x_s))$. Thus, we look at where $\mathrm{Re}(H'(x_s)) > 0 \ \forall x_s$. Since x_s is a complex number, we write

$$\mathrm{Re}(H'(x_s)) = \mathrm{Re}(\gamma x^{\gamma-1}) = \gamma\,\mathrm{Re}(r^{\gamma-1} e^{i\theta(\gamma-1)}) \tag{12}$$

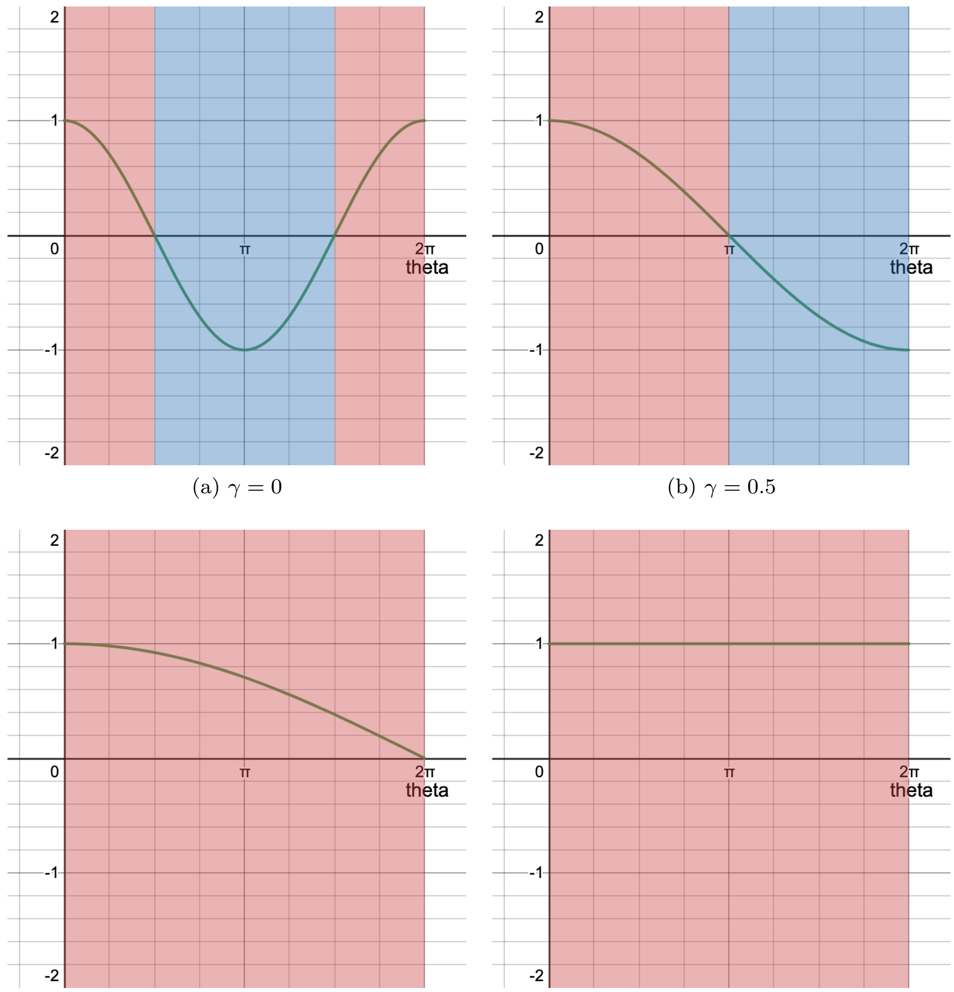
Using Euler's formula, $e^{ix} = \cos x + i \sin x$, we can get the real part of the expression above as

$$\gamma r^{\gamma-1} \cos\left(\theta(\gamma-1)\right) = \gamma r^{\gamma-1} \cos\left(\theta\gamma - \theta\right)$$

We see that the entire criteria can be summarized as the condition for when

$$\cos\left(\theta\gamma - \theta\right) > 0, \quad \forall\theta \tag{13}$$

To solve this for $0 < \theta < 2\pi$, we note that the above expression is true when $-\frac{\pi}{2} < \theta(\gamma-1) < \frac{\pi}{2}$. At $\theta = 2\pi$ we have $-\frac{\pi}{2} < 2\pi(\gamma-1) < \frac{\pi}{2} \Rightarrow -\frac{1}{4} < \gamma - 1 < \frac{1}{4} \Rightarrow 0.75 < \gamma < 1.25$ for unconditional stability. For some selected values of γ, equation 13 is plotted in 1. We see that $\gamma = 0.75$ just about fulfills the stability condition for all values of θ.



(a) $\gamma = 0$

(b) $\gamma = 0.5$

(c) $\gamma = 0.75$

(d) $\gamma = 1$

Figure 1: The stability regions for some selected values of γ depending on the value of θ. Red denotes the stable region where the green curve is above 0 and blue the unstable region where the curve is below.

2 Numerical Experiments

The verification of the calculations made in Section 1 is linked to as animations in Appendix A with varying $\gamma = \{0, 0.5, 0.75, 1\}$ and fixed values of $\omega = 0.8$ and $\alpha = 0.8$. As expected, no synchronisation occurs for $\gamma = 0$, since this will just lead to $\dot{x} = i\omega x$.

In the animations it can bee seen that the system seems to be stable even for some γ outside of the stability criteria derived above. A possible reason for this may be that any unstable regions are a sufficiently small part of the period that any disturbance created is not great enough to cause the entire system to scatter.

Appendix

A Links to animations

1. $\gamma = 0$
2. $\gamma = 0.5$
3. $\gamma = 0.75$
4. $\gamma = 1$
5. $\gamma = 3$ (unstable)

B Python Code

Code files are attached with submission.