Modeling low pressure wave damping due to thermoviscous effects in isotropic fluids.

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Abstract

When modeling waves in fluids it is often accurate enough to use the wave equation, especially in multi dimensional cases for small pressures. However, the canonical wave equation only accounts for attenuation due to dispersive effects and not due to damping caused by physical processes such as friction or energy loss. When considering the wave equation doesn't account for damping, it shows the maximum value a wave may experience at a point in time and space, or that the wave can continue on forever in 1 dimensional or bounded cases. In order to get a more accurate description of modeling waves, a linear term is added to the wave equation allowing it to show how a wave is damped due to viscous effects. When considering a solution to the wave equation with the damping term, results suggest the wave equation is not an accurate model for more viscous fluids, and that pressure waves decrease in amplitude and increase in wavelength due to damping in time which increases with their frequency. However, the canonical wave equation appears to hold for fluids with a low kinematic viscosity, allowing them to retain about 90% of their amplitude after traveling 90 kilometers in the case of water.

Introduction

This paper seeks to answer one question: "how do waves attenuate in water due to viscous effects" and how far can they travel in viscous media. Viscous effects can be described as the "friction" that pressure waves feel when traveling through a fluid. Viscous effects are damping, they reduce the total pressure of the wave as it travels through the substance. This paper will focus on pressure wave attenuation for low amplitude pressure waves in a constant temperature isotropic fluidic medium. Attenuation is the process of a wave decreasing in amplitude or strength from a combination of dispersive and damping effects. While damping is defined as the loss of wave energy or strength from physical processes as the wave travels through a media. While dispersion is the act of the wave splitting apart which can be shown by the solution to the wave equation.[5] This paper will first present what is known about pressure waves and what is commonly done to model their attenuation. Afterwards, a basic model will be presented that will help us understand how we can adjust the wave equation to incorporate damping due to viscous effects. Next, the damped wave equation will be presented and solved as a boundary value problem in order to get an understanding of how the damping affects the pressure waves. Finally, a formula showing the amplitude of a wave decreasing in time will be derived, which will answer the question, how far can waves travel in viscous media.

Motivation/What is known about modeling pressure waves

There are two main types of pressure waves, shocks and sound waves. Sound waves are just pressure waves with low enough pressure to propagate through a material at the speed of sound, while shocks can exceed the speed of sound due to their larger pressures, making them capable of inducing permanent deformation, fractures, cavitation and even sonic booms which create discontinuities in many models. In time both shock and sound waves will attenuate, meaning their amplitude will decrease as a result of both dispersion and damping. In this paper, dispersion is the act of the wave dissipating or spreading out, while damping is the effect of the wave losing energy due to physical processes.

One of the simplest models for sound wave propagation is just the acoustic wave equation below

$$-p_{tt} + c^2 \Delta p = 0$$

$$\boldsymbol{x} \in R^n \qquad p(\boldsymbol{x},0) = \alpha(\boldsymbol{x})$$

$$p = (\boldsymbol{x}, t) \qquad p_t(\boldsymbol{x},0) = \beta(\boldsymbol{x})$$

The solution to this equation can be found for any initial conditions α and β with Kirchhoff's formula[5], which shows in higher dimensions waves will naturally disperse and decrease in strength. However, the right handside to the equation is always equal to zero, meaning the wave equation doesn't include any form of damping, it models a wave free of any physical damping. Meaning it isn't fully applicable to our question in the introduction.

Commonly the field of acoustics models pressure waves by deriving PDEs from conservation laws. Generally these PDEs come from low order taylor series approximations coming from constitutive relations and conservation laws. When pressure increases more terms are needed which are typically nonlinear terms. A nonlinear PDE derived by Rasmussen that shows how waves dampen in time is presented below.

[3]$\psi_{tt} - 2\psi_{xt}\psi_x + \psi_x\psi_x\psi_{xx} - b\psi_{xxt} = c^2\psi_{xx}$. Eq 0.1

$\psi(x,0)=\mu(x)$,$\psi_t(x,0)=g(x)$ and $x \in (-\infty, \infty)$

$\psi_{tt} - c^2\psi_{xx} = 2\psi_{xt}\psi_x - b\psi_{xxt} - \psi_x\psi_x\psi_{xx}$.(canonical form)

Where $\psi=\psi(x,t)$ is the velocity potential which is related to pressure and b is the diffusivity of sound.

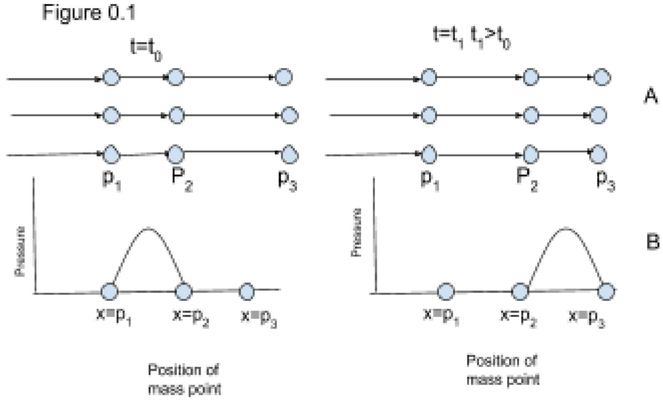
While this PDE forms a strong model for modeling attenuation due to damping, it is weakly nonlinear which will make it difficult to find a solution in both bounded and unbounded cases. Looking at Rasumussen's[3] numerical solutions, it appears Eq0.1 behaves very similarly to the canonical wave equation, but accounts for how the waves structure changes in time while it is damped, rather than just the damping reducing the amplitude of the wave. Avraham, and S. W. Rienstra[2] in their book often solve these nonlinear acoustic PDEs using method of characteristics for unbounded cases, since all pressure waves travel at a constant speed based on the medium.

While there appears to be a great deal of research available on modeling sound wave attenuation in solids and liquids, there is not a whole lot of information available on modeling sound attenuation with a linear model. Most of the research is concerned with local effects, such as how the waves redistribute or form smaller waves, rather than how far they can go before being weakened due to damping. This is likely due to the practicality of the problem in the introduction, most of the time dispersion greatly weakens waves to the point where their amplitude approaches zero very quickly. However, in one dimensional or bounded cases this isn't possible, so our primary source of attenuation is damping. In the next section we will begin forming a model for pressure wave propagation, based on physical intuition of longitudinal waves.

Developing a model

This section discusses how we can develop a model for pressure wave propagation. A simple way to think about pressure wave propagation is that particles are being compressed and their neighboring particles are being decompressed. In order to reach equilibrium or to disperse the energy evenly, the neighboring particles then become compressed while the compressed particles decompress, which continues transports the pressure until it hits the end of the substance.(Figure 4) How quickly these particles return to their equilibrium position is based on their elasticity and density. For more inelastic substances such as solids, the deformed particles return to their equilibrium position quickly, while for something like a gas it may take longer. Assuming all particles return to their equilibrium position after a pressure wave passes through them the speed at which this is done is the speed of sound in the medium, which is the value of "c" in the wave equation. We are interested in modeling the pressure wave propagation as shown in "B", we want points "x" on the wave is passing through to be fixed as the pressure goes through it in our model. Implying we should start with the wave equation below. When considering what c is, it is how quickly the pressure is transported with respect to mass, which is stated below. This can also be found by deriving the wave equation from the Navier stokes equation.[1]

Figure 0.1



[1]Equivalently $c^2=p_\rho(x_0)=dp/d\rho=1/(d\rho/dp)=(d\rho/dp)^{-1}$. Eq0

$c=(d\rho/dp)^{-(1/2)}$ the rate of change of density with respect to pressure is measured as young's modulus or it is how much an object will compress under pressure. The speed of sound or wave speed can be calculated by the square root of young's modulus divided by the material's density. We start by listing Eq1 or Eq2 below since our model involves a transport of energy at a constant speed.

$$-p_{tt} + c^2 p_{xx} = 0 \qquad \text{Eq1}$$

$$-p_{tt} + c^2 \Delta p = 0 \qquad \text{Eq2}$$

However, every time the particles return to equilibrium there is energy lost due to friction and many other processes. This pressure loss can be dependent upon many variables, particularly ones related to the pressure at a given time. Meaning the wave equation is not always equal to zero, but a function describing the net amount of this energy loss, in order to account for the conservation of energy. The output of the function f represents pressure, so it serves to show damping due to physical processes. The damping is dependent on the pressure at a given time, so in extreme cases when damping is high, Eq2 will behave less like the wave equation and more like an overdamped spring. However, damping will be relatively low since we're restricted to low amplitude pressure waves. We are also assuming the wave equation terms have the most influence in the system, so the total value of "f" should be small compared to that of the wave equation, and its input are the same order or less of derivatives of the wave equation.

$$-p_{tt} + c^2 p_{xx} = -f(t, p, p_{xxt}, p_x, p_t, p_{xx}, p_{xt}) \qquad \text{Eq2}$$
max(f)≤max(p(x,t))

$$-p_{tt} + c^2 \Delta p = -f(t, p, \Delta p_t, p_x, p_t, p_y, \ldots) \qquad \text{Eq3}$$
max(f)≤max(p(**X**,t))

When considering this function f there are many terms it could depend on. Theoretically Eqn 2 and 3 can model a wide variety of pressure waves, however it cannot model shocks. We are going to only consider terms that appear linearly, so we can find a solution when treating it as a boundary value problem. The term we will be most interested in is the p_{xxt} term since it shows how the wave is damped in time with respect to pressure, however doing this requires we set all other terms equal to zero except p_{xxt} in the function f, meaning we must assume small pressure, otherwise the impact of these terms could increase and make our model inaccurate. Adding a constant coefficient on p_{xxt} we get eq3 below. Eqn3 can also be derived from the navier stokes equations in either lagrangian or eulerian coordinates.

$$-p_{tt} + c^2 p_{xx} = -f(p_{xxt}, 0, 0, 0, 0, 0, 0) = \eta(x, t) = -\frac{4v}{3} p_{txx} \quad \text{Eq3[1,2]}$$

$$-p_{tt} + c^2 \Delta p = -f(\Delta p_t, 0, 0, 0, 0, ...) = \eta(X, t) = -\frac{4v}{3} \Delta p_t \quad \text{Eq4[1,2]}$$

$X = \langle x_1, x_2, x_3 \ldots \rangle$

$$-p_{tt} + c^2 p_{xx} = -f(t, p, p_{xxt}, p_x, p_t, p_{xx}) = -bp_{txx} - 2p_{xt}p_x - p_x p_x p_{xx} \quad \text{Eq5[3]}$$

η only exists for when f is linear by superposition of solutions, meaning we can find a forcing function in terms of position and time and not pressure. However, for Eq5 above this is not guaranteed. The constant "v" is the kinematic viscosity of the fluid the pressure wave is traveling through. This can be derived from the Navier Stokes equations by following Schreyer's derivation of Eq3[1], as well as Eq4,Eq3 which also shows that the equations are the lowest order approximation of attenuation. Qualitatively this new term should capture roughly how a wave decreases given the wave speed for the fluid and its initial velocity, assuming its pressure is low. However, neglecting the other terms means our solution is not displaying the full attenuation of the wave, but rather the minimum. Eq5 adapted from [3] shows a more accurate description of attenuation, but is nonlinear and has an experimental constant "b".

In brief, a linear term was added to the wave equation to help it describe the minimum dampening giving us the damped wave equation(Eq 3,4). This term dampens the wave equation, implying the wave equation shows the maximum value a wave may have, and when adding In the next part, dampening occurs on that value. However, we had to neglect many other possible terms, restricting us to low pressure values. We will find a solution to this PDE by considering it as a boundary value problem in the next section.

Deriving a solution from a boundary value problem

While the wave equation has an unbounded solution that can be found through kirchhoff's formula or d'alembert's formula, the extra term requires that we find η as a function of position and time, which we can then find an unbounded solution by extending the solution to include a forcing function. Since we seek to understand how the waves attenuate as they move we can treat it as a boundary value problem with reflective conditions. We start with the boundary and initial value problem below.

$$-p_{tt} + c^2 p_{xx} = -\frac{4v}{3} p_{txx}$$ Eq5

$p(x,0) = Z(x)$ $\quad$ $p(a,t) = 0$

$p_t(x,0) =$ ��(x) $\quad$ $p(b,t) = 0$

x∈(a,b)

Since we have a homogenous linear bounded PDE we can use separation of variables or assume the function “p” is a product of 2 functions.

$p =$ ��$(x)G(t)$ Eq6

Substituting Eq6 into Eq5 and rearranging we get

��$''$/�� $= G''/(c^2 G + \frac{4}{3} v G')$

Since **Φ**=**Φ**(x) and G=G(x) they are equal to some constant ƛ. We now get two ODEs below.

��$''$/��ƛand $G''/(c^2 G + \frac{4}{3} v G') =$ƛ

Both of these ODEs have solutions that can be found with a characteristic equation.

Φ=a*sin($\sqrt{ƛ}$ x)+b*cos($\sqrt{ƛ}$ x)

Now considering boundary conditions P(a,t)=P(b,t)=**Φ**(a)=**Φ**(b)=0 we can find the eigenvalues or ƛ.

ƛ= $\frac{n^2\pi^2}{(b-a)^2}$ and **Φ**=a*sin($\sqrt{ƛ}$). Eq10

Someone could also use cosine as a solution as well, however we’re sticking with sine. Now that we have a solution for **Φ**, next we look at finding a solution to G. Writing G as a characteristic equation and solving for r with the quadratic formula we get ** below

$$G'' + ƛ\,[\,c^2 G + \tfrac{4}{3} v G'] = 0$$

$$r^2 + ƛ[c^2 + \tfrac{4}{3} v r] = 0 \;\; where\ G = Aexp(r_1 t) + Bexp(r_2 t)$$

$r = \frac{1}{2}v\lambda \pm \sqrt{(\frac{4}{3}v)^2\lambda^2 - 4c^2\lambda}$

$\boldsymbol{\alpha} = (\frac{4}{3}v)^2\lambda^2 - 4c^2\lambda$

$G = e^{-3/2v\lambda t}[A sin(\sqrt{-\alpha}t/2) + B cos(\sqrt{-\alpha}t/2)]$ or n<Q

$G = e^{-3/2v\lambda t}[Ae^{\sqrt{-\alpha}t/2} + Be^{\sqrt{-\alpha}t/2}]$ for n>Q. Eq11

Where Q is a natural number that makes the eigenvalues large enough to make the solution for r real giving us two solutions. Going back to Eq6 and multiplying **Φ** and G we get the eigenfunctions, since our equation is linear we can represent the general solution as a combination of these.

$P_n = \boldsymbol{\Phi}_n G_n$

$$P = \sum_{n=0}^{Q} e^{-3/2v\lambda}[A_n sin(\sqrt{-\alpha}/2) + B_n cos(\sqrt{-\alpha}/2)]sin(\sqrt{\lambda}x) + \sum_{n=Q}^{\infty}[R_n e^{\sqrt{-\alpha}/2t} + L_n e^{-\sqrt{-\alpha}/2t}]sin(\sqrt{\lambda}x)e^{-3/2v\lambda} \quad \text{Eq12}$$

To solve for A_n, B_n, R_n and L_n orthogonality of eigenfunctions is used. This process is very tedious and is attached in the reference document. Applying this process we get

$$P = \sum_{n=1}^{Q} e^{-3/2v\lambda}A_n[cos(\sqrt{-\alpha}/2t) + \frac{3k(n)+3v\lambda A_n}{\sqrt{-\alpha}}B_n sin\sqrt{-\alpha}/2t)]sin(\sqrt{\lambda}x) + \sum_{n=Q}^{\infty}[A_n - \frac{k_n}{(A_n+1)(-3/2v\lambda+\sqrt{\alpha}/2)}]e^{\sqrt{\alpha}/2t}\, [A_n + \frac{k_n}{(A_n+1)(-3/2v\lambda+\sqrt{\alpha}/2)}]e^{-\sqrt{\alpha}/2t}\, sin(\sqrt{\lambda}x). \quad \text{Eqn 13}$$

Where

$A_n = 2\int_a^b Z(x)sin(\sqrt{\lambda}\,x)dx$ Eqn13.a

$k_n = 2\int_a^b u(x)sin(\sqrt{\lambda}\,x)dx$ Eqn13.b

$\lambda = \frac{n^2\pi^2}{(b-a)^2}$

Eqn13.c

$\boldsymbol{\alpha} = \frac{16}{9}v^2\lambda^2 - 4c^2\lambda$

Eqn13.d

Q=Step[$\sqrt{\frac{36Lc^2}{16v^2\pi^2}}$].

Eqn14.e

Eqn is the solution for all boundaries located at a,b with reflective conditions and initial conditions u(x) and Z(x). The number "Q" is determined by the kinematic viscosity constant and wave speed. As you increase viscosity, it will make the value of Q smaller, making the overall solution behave more like hyperbolic sinhs and coshs and less like sines and cosines. However, for most problems the finite sum in the first part to Q is an accurate approximation since it converges pretty quickly and Q is often very large for most kinematic viscosities. It can also be noted that when setting v=0 Q becomes infinitely large and our solution reduces down the wave equation solution. It can be shown that EQN is a solution with matlab, it is very long and tedious attempting to show it by hand.

Using numerical integration to find coefficients for A_n treating A=A[n] as a vector evaluated at "n" in the sum and letting u(x)=0 and Z(x)=sech^2(x) we get the three videos. Also note viscosity is very dependent on temperature, so we note the temperatures below. Also the damping is independent of the wave speed, so we can pick an arbitrary wave speed if since we're concerned about the dampening effect.

Note the pressure loss is the graph of $\eta = \eta\,(x,t)$ which tells us the effect of adding the kinematic viscosity term(v). This is found by finding the difference between the damped and undamped wave equation. Graphs were made in matlab using experimental constants obtained from source [4]. η is a plane wave just like the solution to the wave equation.

1. (Room temperature)Water [4] $v = 10^{-6}\ m^2/s$
2. (Cooler temperature)Oil [4] $v = 0.09 m^2/s$
3. (Room temperature)Kerosene [4] $v = 2.4 * 10^{-4} m^2/s$

It appears that using the standard wave equation for modeling pressure waves in water yields accurate results. However, when applying it to more viscous substances like oil and kerosene there is significantly more damping occuring and the difference between the values becomes substantial. Meaning that it isn't accurate to model pressure waves with the canonical wave equation in oil, kerosene or more viscous substances or when temperatures are low. Looking at the bottom graph in the video for η, it can be observed that the solution for η is a traveling wave that increases in time that "damps" the main wave. Moreover, it appears that as more damping occurs, the length of the wave pulse(Z(x)) increases while its amplitude decreases, implying it would increase the wavelength of soundwave and decrease its amplitude.

Ideas for deriving an unbounded solution

This section briefly discusses an idea for finding an unbounded solution. While we derived a solution through separation of variables, this only works in the case of a bounded problem, however if one writes the initial condition as a superposition of the eigenfunctions, then the unbounded solution can be found. The solution will diverge as the interval gets wider due to gibbs phenomena. To get a deeper understanding of the behavior of the PDE an unbounded solution is needed. Unfortunately I was unable to derive an unbounded solution without using its eigenfunctions, since I could not figure out how to solve for $\eta = \eta\,(x,t)$ in Eqn6. A possible unverified solution for η is given below. The solution is presented in Eqn15 if someone can solve for η.

$$\eta(x,t) = (1 - e^{-\frac{4}{3}vt})(f(\tfrac{4v}{3}x + ct) + g(\tfrac{4v}{3}x - ct)) = (1 - e^{-\frac{4}{3}vt})P_{undamp}(\tfrac{4vx}{3}, t)$$

$$p = \tfrac{1}{2}(z(x+ct) + z(x-ct)) + \tfrac{1}{2c}\int_{x-ct}^{x+ct} \mu(s)ds - \tfrac{1}{2c}\int_0^t \int_{x-ct}^{x+ct} \eta(s,t)dsdt \;\; for\; t > 0\,. \qquad \text{Eqn15}$$

If a higher dimensional solution is needed, a modified form of Kirchhoff's formula that includes a forcing term should be used. This shows our unbounded solution is only well posed for t>0 and that there will be a primary plane wave with a damping wave that reduces the value of the primary one. This can be observed when looking at the videos presented in the last section. The solution of η could be postulated based on this.

Amplitude decrease in time

This section will derive a relationship for describing wave attenuation due to damping in time, by looking at the maximum value of a wave pulse as time progresses. In a simplified case we can consider how the maximum value of the wave changes in time, we can do this by setting **Φ** to the max of the wave pulse or max($Z(x)$) and using Eqn11 and Eqn6. Note the max(Z(x)) is also referred to as the amplitude. Applying it to Eq15 we can find the constants A,B below.

p=max($Z(x)$)* $e^{-3/2v\lambda t}[Asin(\sqrt{-??}/2t) + Bcos(\sqrt{-??}/2t)]$=max($Z(x)$)(A+B) $e^{-3/2v\lambda t}$. Eq15

p(0)=max(Z(x)) Eq15.a

$p_t(0)$=0 Eq15.b

$1 = Asin(\sqrt{-��}/2 * 0) + Bcos(\sqrt{-��}/2 * 0)$ B=1

0=Acos(0)+sin(0) A=0

p=max($Z(x)$)* $e^{-3/2v\lambda t} cos(\sqrt{-��}/2t) \leq$ max($Z(x)$) $e^{-3/2v\lambda t}$. Eq16

We note that the amplitude is the maximum value that cosine can take, we are not interested in seeing the oscillations so we look at it in terms of its maximum giving Eq17.

$$max(Z(x))e^{-3/2v\lambda t} = \frac{��}{100} max(Z(x)) .$$ Eq17

Eq17 shows the maximum impulse at a given time and H is a % attenuation solving for t we get Eq18 and Eq19 below.

$t_{damp} = ln(\frac{��}{100})\frac{-2}{3v\lambda}$. Eq18

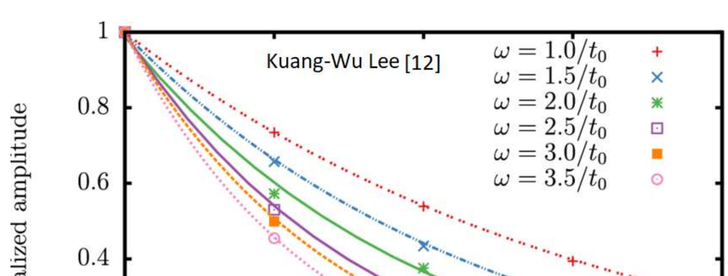
D_{damp}=c $ln(\frac{��}{100})\frac{-2}{3v\lambda}$. Eq19

This is just a more specific form of the commonly used one in medical imaging below

[11]A= $A_0 e^{-az}$

where z is the distance, A_0 is the unattenuated value and a is the attenuation constant.

T_{damp} is the amount of time it takes for the wave to lose a % of its amplitude, while D_{damp} is how far the wave travels before it loses a % of its strength. Note λ represents a constant related to 1/wavelength(Eq13.d), which can be found for a given wave frequency and a material. Increasing the frequency will increase lambda since it will decrease the wavelength. Showing that higher frequency waves experience more damping while lower frequency waves experience less based on Eq19. Eq19 is continuous with Lee's[12] results shown on the figure below, which shows attenuation based on frequency.



Choosing λ=170(square of the speed of sound in water over a wave with acoustic frequency) we get the results in figure 0.4 below. Eq9 can be used to calculate the wave speed, also notice that it is highest in solids, less in liquids and the least in gases.

Attenuation for different mediums(Figure 0.4)

Substance	t_{damp}(seconds)	Wave Speed(m/s)	D_{damp}
Water(90% attenuation)	413	1500	61km
Oil(cold)(50% attenuation)	0.03	1500	45m
Kerosene(50% attenuation)	27	1324	35km
Aluminum[7](50% attenuation)	2.8	6420	18km
Air(50% attenuation)	271	343	93km

It appears damping is stronger in solids and in more viscous fluids. Water appears to experience almost no attenuation, this is due to its extremely low viscosity, while oil and kerosene are much more viscous substances. However Eq was created under many assumptions and has not been verified experimentally and has not had much analysis done on it, there is a high probability that it could be an inaccurate model. Results show sound waves travel farther in fluids rather than solids, dampen quicker in solids than fluids and waves travel the fastest in solids, slower in liquids and slowest in gases, and that higher frequency waves dampen quicker than lower frequency ones.

Conclusion

While the wave equation forms a foundation for understanding wave behavior, it doesn’t form a good model for wave propagation in viscous media. This is due to the wave equation not including damping effects due to physical processes. In this paper we modified the wave equation to account for damping due to viscous effects in a liquid. After solving this new PDE as a boundary value problem, it was shown that the wave equation becomes very inaccurate in viscous fluids and that over time pressure waves decrease in amplitude while increasing in wavelength. Moreover, we found a relationship showing how the amplitude of the wave decreases in either time or distance with the damping being increased by the wave frequency. With this we found results in Figure0.4 showing that pressure waves can travel upwards of 20-90km before significantly decreasing in strength. However, it is worth noting that this modified wave equation will only hold under small pressure and doesn't show how the lost energy disperses in time.

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Appendix

Code for one dimensional attenuation:

```
clear
q=65 %sum constant Keep low!!
nx=200 %x steps
nt=500 %time steps
a=0 %left endpoint
b=10 %right endpoint
t0=0 %time start
tf=25 %time end
Ax=linspace(a,b,nx);
At=linspace(t0,tf,nt);
c=0.45;
v=0.0009/3;
f=0;
g=@(k) 15*sech(sqrt(20)/2*((k-3.6))).*sech(sqrt(20)/2*((k-1.6))); %initial condition
%g=@(x) e^(-x^2);  %initial condition

for i=1:q
   fun= @(k) sqrt(c)*sech(sqrt(c)/2*((k-8.6))).*sech(sqrt(c)/2*((k-8.6))).*sin(i*pi*k/(b-a));
   qq = integral(fun,a,b);
   an(i)=2*qq;
   bn(i)=8/((2*i-1)^3*pi^3);
end

R=zeros(nx,nt);

for xx=1:nx
   for tt=1:nt
      x=Ax(xx);
      t=At(tt);
      f=linspace(0,0,q);
      for n=1:q
         lambda=(n)^2*pi^2/(b-a)^2;
         alpha=lambda.^2*16*v^2-4*lambda*c^2;

f(n)=an(n)*((8*n^2*pi^2*v)/((b-a)^2*sqrt(-alpha))*sin(sqrt(-alpha)/2*t)+cos(sqrt(-alpha)/2*t))*sin(s
qrt(lambda)*x)*exp(-lambda*4*v*t);
      end
      R(xx,tt)=sum(f);
   end
```

```
end

for xx=1:nx
    for tt=1:nt
        v=0;
        x=Ax(xx);
        t=At(tt);
        f=linspace(0,0,q);
        for n=1:q
            lambda=(n)^2*pi^2/(b-a)^2;
            alpha=lambda.^2*16*v^2-4*lambda*c^2;

f(n)=an(n)*((8*n^2*pi^2*v)/((b-a)^2*sqrt(-alpha))*sin(sqrt(-alpha)/2*t)+cos(sqrt(-alpha)/2*t))*sin(s
qrt(lambda)*x)*exp(-lambda*4*v*t);
        end
        RR(xx,tt)=sum(f);
    end
end

RRR=abs(R-RR); %energy loss term

for i=1:nt
    AR(i)=sum(RRR(:,i));
end

%for i=2:nx-1
    %for ii=2:nt-1
        %RRR(i,ii)=(RRR(i,ii+1)-RRR(i,ii-1))/(2*(tf-t0));
    %end
%end

v = VideoWriter('oil');
open(v);


for kk = 1:nt
   subplot(2,1,1);
   plot(Ax,transpose(R(:,kk)),Ax,transpose(RR(:,kk))) %,Ax,transpose(RR(:,kk)))
   title 'Oil pressure wave room temp'
   xlabel 'position(meters)'
   ylabel 'pressure(pascals)'
   legend('Viscosity', 'No viscosity')
   xlim([a b])
```

```
   ylim([-45 45])
   subplot(2,1,2);
   plot(Ax,transpose(abs(RR(:,kk)-R(:,kk))))
   ylabel 'Pressure loss'
   xlabel 'position(meters)'
   xlim([a b])
   ylim([0 max(max(abs(RR-R)))])
   frame = getframe(gcf);
   writeVideo(v,frame);
end
close(v);
```

Code for two dimensional attenuation:

```
clear
q=65 %sum constant VERY EXPENSIVE CAREFUL!
nx=20 %x steps
nt=50 %time steps
a=-10 %left endpoint
b=10 %right endpoint
t0=0.01 %time start
tf=100 %time end
Ax=linspace(a,b,nx);
At=linspace(t0,tf,nt);
c=2;
v=0.0009/3;
f=0;
g=@(k) 15*sech(sqrt(20)/2*((k-3.6))).*sech(sqrt(20)/2*((k-1.6))); %initial condition
%g=@(x) e^(-x^2);  %initial condition
syms kk
syms k
syms t
r=exp(-(k-kk)^2/(4*pi*t))*exp(-kk^2);
rr=int(r,kk,-50,50);
rrr=1/2*(exp(-(k-c*t)^2)+exp(-(k+c*t)^2))+(4*v)/(6*sqrt(4*pi*t))*rr;
for i=1:nt
    for j=1:nx
        k=Ax(j);
        t=At(i);
        P(j,i)=(subs(rrr));
    end
end
```

```
U=double(P);

%
%for i=1:nt
    %for i=1:nx
        %Ploss(i,j)=4*v/(6*sqrt(4*pi*At(i)).*integral(r,At(i)),-500,500));
    %end
%end
```

Code for checking solution

```
clear
syms x
syms t
syms v
syms lambda
syms l
syms c
alpha=lambda^2*16*v^2-4*lambda*c;
syms l;
syms pi;w
lambda=pi^2/l^2;
f=((2*lambda/(l^2*sqrt(-alpha))*sin(sqrt(-alpha)/2*t)+cos(sqrt(-alpha)/2*t))*exp(-lambda*4*v*t))*si
n(sqrt(lambda)*x);

r=-diff(f,t,2)+c*diff(f,x,2)+4/3*v*diff(diff(f,x,2),t);

%
x=55
t=1
v=25
lambda=1
l=123
c=1
b=double(subs(r));
```