A Review and Evaluation of Multiobjective Programing Techniques

JARED L. COHON

Department of Geography and Environmental Engineering, Johns Hopkins University, Baltimore, Maryland 21218

DAVID H. MARKS

Department of Civil Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Three criteria are established for the evaluation of the utility of multiobjective programing techniques for water resource planning. The criteria are computational efficiency, explicitness of trade offs among objectives, and the amount of information generated for decision making. The multiobjective approaches are classified into generating techniques, techniques which rely on the prior articulation of preferences, and techniques which foster iterative definition of preferences. The methods in the various classes are reviewed and evaluated in terms of the hypothesized criteria. The evaluations are then used in establishing conclusions about the applicability of the multiobjective approaches to water resource problems.

The water resource planning profession is currently engrossed in a period of reformulation of its project evaluation procedures and development of corresponding mathematical techniques. The ongoing transformation is from traditional benefit-cost analysis and its associated uniobjective planning models to multiobjective analysis, which has promoted a field of mathematical programing called vector optimization. Vector optimization theory, or the solution of mathematical programing models with more than a single objective function, cannot be characterized as new, since *Kuhn and Tucker* [1951] and *Koopmans* [1951] must be credited with its discovery. However, vector optimization theory remained relatively undeveloped from 1951 until the 1960's when multiobjective public investment problems became more common and 'trade off' became a favorite word of managers, planners, and decision makers in both the private and the public sector.

In the last 10 yr a great deal of effort has been devoted to the development of solution techniques for vector optimization problems. The origins of this effort have been varied: techniques have been developed by systems analysts and decision theorists for private and public sector problems, by control theorists for engineering (guidance and design) problems, and by water resource economists and systems analysts for water resource (public sector) planning problems. All of the contributors to the recent development of vector optimization theory shared one or two common goals: the formulation of methods which are theoretically operational and which attempt to avoid the large computational effort associated with multiobjective problems.

The existing techniques have been developed very rapidly: at least 20 different methods for solving vector optimization problems have been formulated in the last 10 yr, and most of those in the last five. This impressive proliferation of methods in a very short period of time has left little time for the consideration of some very important issues related to the multiobjective analysis of public investment alternatives. It is the purpose of this paper to discuss these issues and to apply this discussion to an evaluation of proposed multiobjective solution techniques. The intended results of this analysis are conclusions on the applicability of vector optimization techniques to water resource planning problems and the identification of useful directions for future research on multiobjective problems.

EVALUATION CRITERIA

A prerequisite for the assessment of the utility of the various proposed solution techniques is the development of evaluation criteria. *Loucks* [1975] evaluated multiobjective solution techniques in terms of four criteria, based on his belief that a technique should capture the nature of the decision-making process; i.e., it should simulate the bargaining process, recognize uncertainty in trade offs and preferences, be sufficiently general, and be able to predict the outcome of the decision-making process.

The criteria suggested by Loucks are useful for the evaluation of techniques which are to be used for the prediction of political decisions. This analytical goal is valid, but it should perhaps be reserved until prediction techniques are more fully developed.

The criteria which are proposed here have an operational orientation: the fundamental determinant of the utility of a technique is the extent to which it is consistent with multiobjective benefit-cost analysis. This implies the following criteria for a useful solution technique. (1) The technique must be computationally feasible and relatively efficient, (2) it must foster the explicit quantification of the trade offs among objectives, and (3) it must provide sufficient information that an informed decision can be made.

The above criteria are admittedly subjective. Quantifying such things as 'sufficient information' will be at times an awkward task. Nevertheless, the exercise will be useful, since it will bring into focus some of the important considerations which analysts must face when they are confronted with multiobjective planning problems. The following discussion will lend support to the proposed criteria.

Computational feasibility, as well as efficiency, is an important criterion both because of its practical implications and because it has motivated much of the research devoted to vector optimization. The practical implications of computational efficiency are obvious: planners must perform analyses with limited budgets. Many of the techniques discussed here have been motivated by researchers' recognition of the large computational requirements associated with previously existing techniques. Thus the goal of developing many of the methods was to decrease computational requirements. In many cases, however, computational burden remained large. It is the purpose of this criterion to quantify how large the burden is.

There are probably several alternative measures of computational efficiency, two of which are dollar costs and computer time. Neither of these will be used, however, since they are both problem and project specific; i.e., they may depend on the size of the problem, the nature of the model, type and size of computer, optimization strategy, and other factors. A general measure is required. Thus solution techniques will be assessed in this paper in terms of the number of solutions of a scalarized version of the original multiobjective problem which they require. As will be shown, the number of solutions will generally depend on the number of objectives and on the degree of approximation which is selected.

The number of solutions required by a technique is an adequate indicator of computational efficiency. It is desirable, in addition, to have some notion of computational feasibility, which implies some knowledge of the size of the optimization model. Although 100 solutions of a model with 1000 constraints and decision variables may be clearly infeasible (for a given computer budget), the same number of solutions of a 50-constraint model is feasible. The basic concern here is with river basin optimization (screening) models for which there is no 'typical' size. One need only examine a portion of the literature to substantiate this fact [*Blanchard*, 1964; *Wallace*, 1966; *Rogers*, 1969; *Loucks*, 1969; *Problete and McLaughlin*, 1970; *Cohon et al.*, 1974].

The variability of model size is demonstrated in Table 1, in which model dimensions are shown for four different configurations of the Rio Colorado in Argentina [*Cohon et al.*, 1974]. Since feasibility is the concern, the largest model sizes to be expected should be the analytical guide. Thus solution techniques should be applicable to optimization models with 1000 constraints and 1000 decision variables. This choice of dimensions should include a sufficient safety factor.

The remaining two criteria derive from the practical significance of multiobjective benefit-cost analysis of public investment problems as originally formulated by *Marglin* [1967]. The essence of Marglin's analysis is the explicit quantification of the trade offs among objectives. This is also the recurring theme of *Marglin et al.* [1972]. Explicit value judgments are considered crucial by other economists as well, since a relaxation of this requirement would enhance the possibilities of abuse of the planning process resulting in a divergence between governmental action and the 'public interest.' *Cicchetti et al.* [1973, p. 724], for example, opposed the recent guidelines recommended by the *U.S. Water Resources Council* [1973] because 'they are open to abuse and will likely lead to a deterioration in the quality of water resource investment decisions.'

TABLE 1. Various Sizes of Screening Models as a Function of Sites and Seasons

Sites	Seasons	Number of Constraints	Number of Decision Variables
13	2	134	132
13	3	196	187
28	3	435	422
38	3	629	665

The statistics shown are for a screening model formulated for the analysis of the Rio Colorado, Argentina [see *Cohon et al.*, 1974].

These reservations about multiobjective analysis are perhaps misdirected, since traditional benefit-cost analysis has been a target of the same political abuse with which Cicchetti et al. are concerned, as *Wildavsky* [1967] has pointed out. Adding new objectives to the analysis would not necessarily increase the likelihood of political transgressions. (Indeed *Maass* [1966, p. 212] has argued that traditional benefit-cost analysis breeds misuse because of its narrow and rigid adherence to the objective of economic efficiency.) What is really needed is a responsive decision-making process which fosters explicit consideration of value judgments. *Cicchetti et al.* [1972, p. i] have, in fact, stated that 'economic analysis can best serve this [decision-making] process by making explicit the tradeoffs involved in a given situation.' Thus a multiobjective solution technique which is to be used by or for the political decision-making process must force the explicit consideration of trade offs.

The explicit quantification of the trade offs can be done only if the problem is approached in a meaningful, easily understood manner which provides sufficient information. This means that the multiobjective technique must be consistent with a clear and simple approach to the problem. It is futile, and antithetical to the essence of planning, to complicate the analysis with all sorts of esoteric terms and terminology. Yet, as a subsequent discussion will show, some of the multiobjective techniques rely on the collection of abstruse, sometimes exotic data from the decision maker, thereby producing meaningless results.

The criterion of sufficient information, although it is critical for the decision-making process, is particularly difficult to quantify. In general, the most information that can be provided is an exact representation of the noninferior set, which captures the entire range of feasible trade offs among all objectives, and the set of noninferior solutions, which indicates the alternatives that correspond to points in the noninferior set. (These terms are defined below.) The sufficiency of information supplied by a technique will be judged in relation to this maximum of information available. The general rule of thumb, 'the more the better,' will be used as a guideline for evaluation.

CLASSIFICATION OF TECHNIQUES

This paper does not represent the first attempt at bringing some order to the somewhat disordered field of vector optimization. Others have tried; among them are *Terry* [1963], *Kapur* [1970], *Roy* [1971], and *Loucks* [1975]. Loucks' work is of particular importance, since he shared many of the concerns expressed herein. The classification of techniques used in this paper will resemble that proposed by Loucks, although some important differences will be apparent.

The classification outlined by *Loucks* [1975] is

Techniques using assumed preference functions
Techniques for reducing incomparability and uncertainty
  Deterministic methods
  Probabilistic methods

The techniques in each of the classes were evaluated in terms of the criteria suggested by Loucks, which were mentioned previously.

The classification suggested here differs from the one shown above in one major respect: the classification will be directed primarily at techniques which can be used to aid the decision-making process, but they are not necessarily useful for predicting the outcome of the process. Thus in terms of the criteria of *Loucks* [1975] the techniques do not simulate the bargaining process, and they are not able to predict the outcome of the decision-making process. The impact of this distinction between Loucks' approach and the current classification is twofold. First, some methods considered by *Loucks* [1975] will not be mentioned here because they are primarily tools for prediction. Second, the methods which are used solely to identify 'noninferior solutions' (defined in a subsequent section on terminology) are recognized here as a class of solution methods. These 'generating' techniques are particularly powerful when there are no more than three planning objectives.

The classification of multiobjective solution techniques follows. The techniques which have been proposed in the literature and which will be evaluated below are displayed by classes. The methods appear in approximate chronological order within each class.

- Generating techniques
  - Weighting method
  - Constraint method
  - Derivation of a functional relationship for the noninferior set
  - Adaptive search
- Techniques which rely on prior articulation of preferences
  - Goal programing
  - Assessing utility functions
  - Estimation of optimal weights
  - Electre method
  - Surrogate worth trade off method
- Techniques which rely on progressive articulation of preferences
  - Step method
  - Iterative weighting method
  - Sequential multiobjective problem solving (Semops)

VECTOR OPTIMIZATION PROBLEM

The term 'vector optimization' is a contradiction in terms, since one cannot in general optimize a vector. Nevertheless, the phrase describes the problem at hand quite precisely:

$$\max \mathbf{Z}(\mathbf{x}) = [Z_1(\mathbf{x}), Z_2(\mathbf{x}), \cdots, Z_p(\mathbf{x})] \tag{1}$$

subject to

$$g_i(\mathbf{x}) \leq 0 \qquad i = 1, 2, \cdots, m \tag{2}$$

$$x_j \geq 0 \qquad j = 1, 2, \cdots, n \tag{3}$$

where $\mathbf{Z}(\mathbf{x})$ is the p-dimensional objective function; i.e., there are p objectives; $\mathbf{x}$ is an n-dimensional vector of decision variables; and the $g_i(\mathbf{x})$ represent the constraints associated with the problem. The region defined by the constraint set (2) and (3) in n-dimensional Euclidean vector space

$$\mathbf{X} = \{\mathbf{x}/g_i(\mathbf{x}) \leq 0, \forall_i \quad x_j \geq 0, \forall_j\}$$

will be referred to here as the feasible region in decision space. Every feasible solution to the problem in (1)–(3), i.e., all $\mathbf{x} \in \mathbf{X}$, implies a value for each objective, i.e., $Z_k(\mathbf{x})$, $k = 1, 2, \cdots, p$. The p-dimensional objective function maps the feasible region in decision space $\mathbf{X}$ into the feasible region in objective space $\mathbf{Z}(\mathbf{X})$, defined on the p-dimensional Euclidean vector space.

The statement that 'a vector cannot be optimized' is simply another way of saying that without information about preferences which provide a rule for combining the objectives the objectives are incommensurable and at least some solutions (alternatives) are therefore incomparable. Such incomplete orderings, which are characteristic of multiobjective planning problems, imply that in the absence of preference information an optimal solution cannot be found to the problem in (1)–(3), since all feasible solutions are not ordered (comparable). A complete ordering, which is characteristic of scalar (single-objective) optimization problems, can be obtained for a vector optimization only by introducing value judgments into the solution process. This observation has provided the motivation for the techniques grouped in the second and third classes above.

Some of the feasible solutions to the vector optimization problem can be eliminated from further consideration by the incomplete ordering associated with the p-dimensional objective function, with no knowledge of preferences. If it can be assumed that more of all objectives is considered good by everyone, then only noninferior solutions are of interest. A noninferior solution is a feasible solution, $\mathbf{x} \in \mathbf{X}$, for which there exists no other feasible solution, $\mathbf{x}' \in \mathbf{X}$, such that

$$Z_r(\mathbf{x}') > Z_r(\mathbf{x}) \tag{4a}$$

for some $r = 1, 2, \cdots, p$, and

$$Z_k(\mathbf{x}') \geqq Z_k(\mathbf{x}) \tag{4b}$$

for all $k \neq r$.

The collection of all noninferior solutions is referred to here as the set of noninferior solutions: $\mathbf{X}^* = \{\mathbf{x}/\mathbf{x} \in \mathbf{X}$ and $\mathbf{x}$ noninferior by the definition in (4)$\}$. The set $\mathbf{X}^*$ is defined in decision space and is a subset of the feasible region in decision space $\mathbf{X}$, i.e., $\mathbf{X}^* \subseteq \mathbf{X}$.

Each noninferior solution $\mathbf{x} \in \mathbf{X}^*$ implies values for each of the p objectives $\mathbf{Z}(\mathbf{x})$. The collection of all of the $\mathbf{Z}(\mathbf{x})$ for $\mathbf{x} \in \mathbf{X}^*$ yields the noninferior set $\mathbf{Z}(\mathbf{X}^*)$. The noninferior set is defined in objective space, and it is a subset of the feasible region in objective space, i.e., $\mathbf{Z}(\mathbf{X}^*) \subseteq \mathbf{Z}(\mathbf{X})$.

These definitions are easily grasped for a simple example. Consider the following two-objective two-decision variable problem:

$$\max \mathbf{Z}(\mathbf{x}) = [Z_1(\mathbf{x}), Z_2(\mathbf{x})] \tag{5}$$

$$Z_1(\mathbf{x}) = 5x_1 - 2x_2 \tag{6}$$

$$Z_2(\mathbf{x}) = -x_1 + 4x_2 \tag{7}$$

subject to

$$g_1(\mathbf{x}) = -x_1 + x_2 - 3 \leq 0 \tag{8}$$

$$g_2(\mathbf{x}) = x_1 + x_2 - 8 \leq 0 \tag{9}$$

$$g_3(\mathbf{x}) = x_1 - 6 \leq 0 \tag{10}$$

$$g_4(\mathbf{x}) = x_2 - 4 \leq 0 \tag{11}$$

$$g_5(\mathbf{x}) = -x_1 \leq 0 \tag{12}$$

$$g_6(\mathbf{x}) = -x_2 \leq 0 \tag{13}$$

The feasible region in decision space $\mathbf{X}$ and the set of noninferior solutions $\mathbf{X}^*$ are shown in Figure 1. The feasible region in objective space $\mathbf{Z}(\mathbf{X})$ and the noninferior set $\mathbf{Z}(\mathbf{X}^*)$ are shown in Figure 2.

In the case of this simple example with two decision variables and two objectives the feasible region in objective space was found by the enumeration of all of the extreme points and the computation of the values of each objective at each of these corner solutions. The noninferior set $\mathbf{Z}(\mathbf{X}^*)$ was then found by applying the definition of noninferiority. The extreme points with objective values in the noninferior set, i.e., $\mathbf{Z}(\mathbf{x}^1)$, $\mathbf{Z}(\mathbf{x}^2)$, $\mathbf{Z}(\mathbf{x}^3)$, and $\mathbf{Z}(\mathbf{x}^4)$, are also in the set of nonin-

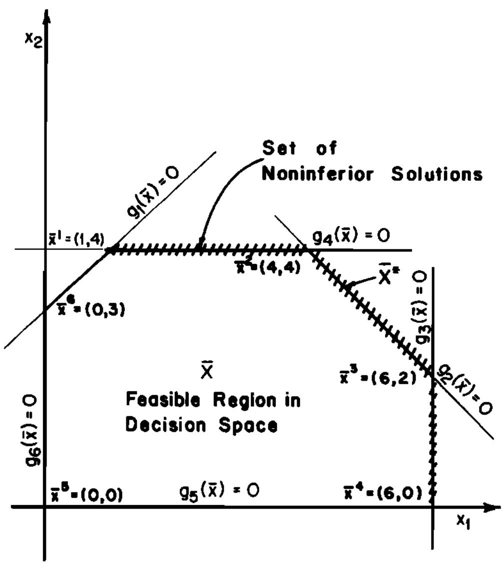

Fig. 1. The feasible region in decision space X and the set of noninferior solutions $\mathbf{X}^*$.

ferior solutions $\mathbf{X}^*$. Of course, this enumeration procedure is computationally feasible for only very simple (and unrealistic) problems.

A final bit of terminology is the notion of a best-compromise solution. In the absence of preference information, not one of the noninferior solutions can be considered (by the analyst) to be preferable to any other noninferior solution. However, when preferences are known, as represented by an indifference surface, for example, then one of the noninferior solutions can be identified as the best-compromise solution, as shown in Figure 2 for the example problem. Most authors have called the solution which is 'best' in the above context the optimal solution. Best-compromise solution, which was coined by *Belenson and Kapur* [1973], seems preferable, however, since it will suggest to the reader that a noninferior solution so identified is optimal only in terms of a particular set of value judgments.

EVALUATION OF SOLUTION TECHNIQUES

Generating Techniques

Some of the methods included in this class were among the first multiobjective solution procedures developed, since they follow directly from the Kuhn-Tucker conditions for non-

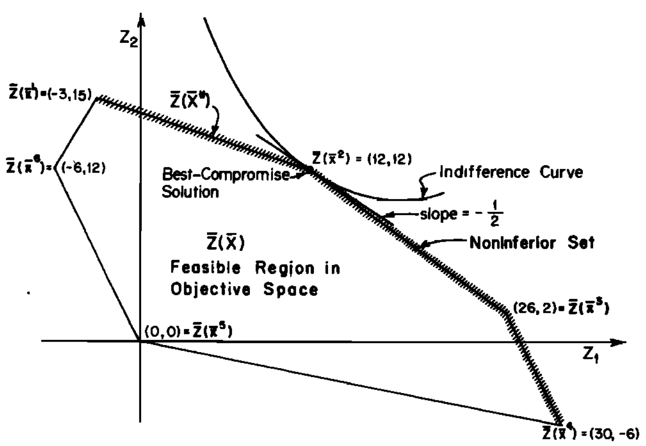

Fig. 2. The feasible region in objective space $\mathbf{Z}(\mathbf{X})$, the noninferior set $\mathbf{Z}(\mathbf{X}^*)$, and the best-compromise solution.

inferior solutions. The purpose of all of the generating techniques is to identify the set of noninferior solutions $\mathbf{X}^*$, as well as the noninferior set $\mathbf{Z}(\mathbf{X}^*)$. Thus the generating techniques provide all of the information that one can extract from a multiobjective model. This is accomplished without preference information from decision makers.

Weighting and constraint methods. The weighting method was the first technique developed for the generation of noninferior solutions for vector optimization. This is to be expected, since the approach was implied by *Kuhn and Tucker* [1951]. In the same paper in which they presented their conditions for optimality for scalar optimization problems, Kuhn and Tucker also presented noninferiority conditions: if a solution $\mathbf{x}$ to the vector maximization problem in (1)–(3) is noninferior, then there exist $w_k \geq 0$, $k = 1, 2, \cdots, p$ (w_r strictly positive for some $r = 1, 2, \cdots, p$), and $\lambda_i \geq 0$, $i = 1, 2, \cdots, m$, such that

$$\mathbf{x} \in \mathbf{X} \tag{14}$$

$$\lambda_i g_i(\mathbf{x}) = 0 \qquad i = 1, 2, \cdots, m \tag{15}$$

and

$$\sum_{k=1}^{p} w_k \nabla Z_k(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i \nabla g_i(\mathbf{x}) = 0 \tag{16}$$

These conditions are necessary for a noninferior solution, and when all of the $Z_k(\mathbf{x})$ are concave and $\mathbf{X}$ is a convex set, they are sufficient as well. The mathematical programer will recognize that only (16) differs from the Kuhn-Tucker conditions for optimality.

Zadeh [1963] pointed out that the third condition in (16) implies that noninferior solutions can be obtained by solving a scalar optimization problem, in which the objective function is a weighted sum of the components of the original vector-valued objective function $\mathbf{Z}(\mathbf{x})$. That is, the solution to the following problem is, in general, noninferior:

$$\max \sum_{k=1}^{p} w_k Z_k(\mathbf{x}) \tag{17}$$

subject to

$$\mathbf{x} \in \mathbf{X} \tag{18}$$

where $w_k \geq 0$ for all k and strictly positive for at least one objective. This important observation showed that noninferior solutions are obtainable through the use of existing linear programing packages (when all of the functions are linear or admit to appropriate linearization). The noninferior set and the set of noninferior solutions can be generated by parametrically varying the weights w_k in the objective function. This was initially demonstrated by *Gass and Saaty* [1955] for a two-objective problem.

The constraint method is in a sense the dual of the weighting method. It also follows directly from the Kuhn-Tucker conditions for noninferiority. Rewriting the third condition in (16) gives

$$w_r \nabla Z_r(\mathbf{x}) + \sum_{\substack{k=1 \\ k \neq r}}^{p} w_k \nabla Z_k(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i \nabla g_i(\mathbf{x}) = 0 \tag{19}$$

Since only relative values of the weights are of significance, the rth objective can be selected as the numeraire so that $w_r = 1$. The condition in (19) becomes

$$\nabla Z_r(\mathbf{x}) + \sum_{\substack{k=1 \\ k \neq r}}^{p} w_k \nabla Z_k(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i \nabla g_i(\mathbf{x}) = 0 \tag{20}$$

This rewritten condition allows the second term to be interpreted as a weighted sum of the gradients of $p - 1$ lower-bound constraints, since there is a plus sign before the summation. This interpretation implies that noninferior solutions can be found by solving

$$\max Z_r(\mathbf{x}) \tag{21}$$

subject to

$$\mathbf{x} \in \mathbf{X} \tag{22}$$

$$Z_k(x) \geq L_k \qquad \text{all } k \neq r \tag{23}$$

in which L_k is a lower bound on objective k. Note that this formulation yields a scalar objective function, so that the problem may be solved with existing techniques. Parametric variation of the L_k in (23) traces out the noninferior set.

It should be pointed out that preliminary analysis is required to identify starting values for the L_k in (23). Solving p scalar problems in which the objective function is one of the $Z_k(\mathbf{x})$, using a different one in each problem, will yield a lower bound for L_k for all k, which will guarantee noninferiority. That is, the following problem should be solved for $k = 1, 2, \cdots, p$:

$$\max Z_k(\mathbf{x}) \tag{24}$$

subject to

$$\mathbf{x} \in \mathbf{X} \tag{25}$$

The optimal solutions to these p problems, $\mathbf{x}^k$, $k = 1, 2, \cdots, p$, are then used to compute corresponding values for all objectives other than k, $Z_j(\mathbf{x}^k)$, all $j \neq k$. The result of all of the solutions will be p values for each objective. The minimum of all of these values for a particular objective provides a good value of L_k with which to begin the constraint method.

Weighted objective functions such as (17) were introduced to public investment planning by Marglin in the work by *Maass et al.* [1962, pp. 78–81] and again by *Marglin* [1967, pp. 23–24] and by *Major* [1969]. The use of constraints to represent objectives in public investment problems was introduced by Marglin in the book by *Maass et al.* [1962, pp. 70–81, 81–84] and again by *Marglin* [1967, pp. 24–26]. The intent of these authors was not to generate the entire noninferior set, since they identified only a single set of w_k or L_k which were in some sense socially optimal. That is, Marglin and Major proceeded directly to the definition of the best-compromise solution. This approach is discussed further in the next section, as 'the identification of optimal weights.' The 'switching value' method presented by *Marglin et al.* [1972, pp. 141–148] is another limited version of the weighting method.

The intent of the weighting and constraint methods is the identification of the noninferior set and the set of noninferior solutions within which the best-compromise solution will lie. In terms of the criteria formulated in previous sections, the weighting and constraint methods are particularly applicable to public investment problems, the latter having been applied to water resource problems by *Miller and Byers* [1973] and *Cohon and Marks* [1973]. The trade offs among objectives are explicitly considered, and the totality of all noninferior alternatives are found and displayed as in Figure 2. The approaches, in addition, portray the problems in a manner which is consistent with intuition by graphically displaying trade offs (for $p \leq 3$) and by showing the impact of selecting values for weights on the alternatives.

The major weakness of the weighting and constraint methods is computational efficiency when there are several objectives. The number of solutions of the scalarized problem in (17)–(18) or (21)–(23) required to identify completely or even approximate the noninferior set increases exponentially with the number of objectives. If K_k is the number of values of the weight w_k or the lower bound L_k on objective k which are used to find noninferior solutions, then

$$S = \prod_{\substack{k=1 \\ k \neq r}}^{p} K_k \tag{26}$$

where the rth objective is the numeraire and S is the number of solutions which must be found. Depending on the size of the scalarized problem and on the number of objectives, S can represent a substantial computational requirement. If $K_k = K$, $\forall_k$, then

$$S = K^{p-1} \tag{27}$$

and S is an exponential function of the number of objectives. For example, if $K = 5$ and $p = 6$, then $S = 5^5 = 3125$ solutions of a problem which may be as large as 1000 constraints.

Thus the utility of the weighting and constraint methods may be confined principally to problems for which $p = 2$ or 3. In addition to the computational constraints on the methods when $p > 3$, display of results also becomes a problem for higher-dimensional problems. That is, when $p \geq 4$, the analyst can no longer draw the noninferior set as in Figure 2 for the two-objective problem. Thus even if the noninferior set (or an approximation of it) can be generated when $p \geq 4$, the power of the method in graphically capturing the essence of the multiobjective problem may be lost.

The sensitivity of computational burden to the number of objectives is typical of most multiobjective solution techniques. This should be expected, however, since in general high-dimensional problems are simply more complex than lower-dimensional problems.

Other generating methods. Vector optimization has received considerable attention from control theorists who have developed other methods which belong in the first class. The decision-making implications of the techniques do not differ from the two methods discussed previously, since they operate on a similar premise, i.e., generation of the noninferior set which embodies all of the relevant information for decision makers and which forces explicit quantification of trade offs in a clear manner. The particular weakness of the methods discussed below is that they are not applicable to public investment problems of large or even moderate sizes; i.e., they are 'computationally infeasible.'

Reid and Vemuri [1971], repeated by *Vemuri* [1974], showed that for a certain class of problems a functional relationship between each objective and the set of weights on all objectives could be derived. Given these relationships, the value of any objective can be found by a simple calculation for any value of the weights. The idea is very powerful, and it would be very useful for the analysis of public investment problems, except that the method is severely limited computationally. The application of the technique requires objectives expressed in the mathematical form

$$Z_k(\mathbf{x}) = \prod_{j=1}^{n} (x_j)^{a_{jk}} \qquad k = 1, 2, \cdots, p \tag{28}$$

the special properties of which allow the derivation of the functional relationship between $Z_k(\mathbf{x})$ and w_k, $\forall_k$. The applicability of the method is limited, since this specific form may not be observed frequently in water resource planning problems. A further weakness is that the method is limited to unconstrained optimization problems. A further computational limitation of the method stems from the necessity of differentiating each objective with respect to each variable. This procedure, and the subsequent solution of a set of simultaneous linear equations, would be computationally infeasible for realistic planning problems.

Another approach suggested by *Beeson* [1971] and *Beeson and Meisel* [1971] is an adaptive search procedure which avoids the repetitive solutions of a scalar version of the multiobjective problem which are required by the weighting and constraint methods. The search proceeds from an initial noninferior solution to approximations of other noninferior solutions, the direction of the search being determined by the gradients of the objective functions. The search is restarted p times to insure good coverage of the set of noninferior solutions.

The developers found the adaptive search method to be particularly useful for control problems in which there were few variables and many objectives. That is, since the analysis takes place in decision space and not objective space, it is the dimensionality of the former which has the greatest impact on the computational requirements of the method. This feature, however, detracts from its applicability to water resource problems, since there are typically at least several hundred variables. Thus in general the technique is computationally infeasible for the analysis of river basin planning problems.

Techniques Which Rely on Prior Articulation of Preferences: Noninterative Methods

The generating techniques are in general computationally intensive because alternatives can be eliminated only on the basis of noninferiority, since the methods are based on the incomplete orderings associated with the p objectives in the original multiobjective problem. The methods in this class are based on the observation that if a complete ordering or at least a 'more complete' ordering could be derived, then the computational burden can perhaps be reduced since some or most of the noninferior solutions can be eliminated by the new ordering. The basis for the orderings is the articulation of preferences prior to the solution of the multiobjective problem.

Goal programming. *Charnes and Cooper* [1961, Appendix B] developed goal programing, and it has been applied by *Charnes et al.* [1969] and by *Lee and Jaaskelainen* [1971]. Goal programing is based on the minimization of weighted absolute deviations from targets for each objective. For the general vector maximization problem in (1)-(3), the goal programing formulation is

$$\min \sum_{k=1}^{p} |d_k| \tag{29}$$

subject to

$$\mathbf{x} \in \mathbf{X} \tag{30}$$

$$Z_k(\mathbf{x}) - d_k = T_k \qquad k = 1, 2, \cdots, p \tag{31}$$

where d_k is the deviation from the target for the kth objective and is unrestricted in sign and T_k is the target for the kth objective.

The absolute value in (29) is nonlinear, but an equivalent linear formulation [*Wagner*, 1969, p. 558] is

$$\min \sum_{k=1}^{p} b_k + e_k \tag{32}$$

subject to

$$\mathbf{x} \in \mathbf{X} \tag{33}$$

$$Z_k(\mathbf{x}) - b_k + e_k = T_k \qquad k = 1, 2, \cdots, p \tag{34}$$

$$b_k, e_k \geq 0 \tag{35}$$

where b_k and e_k are the positive and negative deviations, respectively, from the kth target for the kth objective.

Goal programing allows the deviations to be weighted according to priorities on each objective P_k. Furthermore, one objective j may be identified as a high-priority objective, implying a value of P_j such that $P_j >>> P_k$. Any desirable relationship among the priorities may be specified. The objective function in (32) with priorities on the deviations becomes

$$\min \sum_{k=1}^{p} P_k(d_k + e_k) \tag{36}$$

Goal programing is computationally efficient in relation to the generating methods. Fewer than the K^{p-1} solutions will generally be required, since most noninferior solutions are eliminated by the targets T_k on each objective. However, extensive sensitivity analysis may be required to insure the noninferiority of the goal programing solution. Consider Figure 3 in which the (unknown) noninferior set for an arbitrary two-objective problem is drawn. Suppose that the objective Z_2 is a 'high-priority' objective, i.e., $P_2 >>> P_1$. If the specified targets are T_1 and T_2, then the goal programing solution of the problem will be that which gives point A in Figure 3. This is easy to see because the priorities force the minimization of deviations from $T_2 - Z_2$ before objective 1 is considered. Note also that for any target greater than T_2 in Figure 3 the solution will be at point A. The solution at point A is noninferior and furthermore, as long as the target on objective 2 is greater than or equal to T_2'', the solution will be noninferior. If, however, the target on objective 2 is less than T_2'', e.g., T_2''', the goal programing formulation will yield an inferior solution.

Goal programing has been shown to be a very useful tool for multiobjective decision making in private sector problems; a setting in which the decision maker commands the system in question and in which a clear notion of targets and priorities exists. Public sector problems, on the other hand, are characterized by a sprawling, complex decision-making

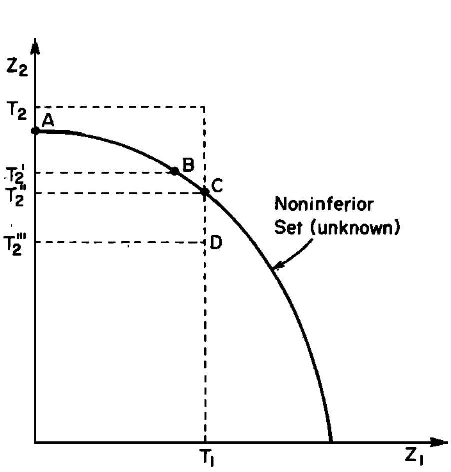

Fig. 3. Goal programing and noninferiority.

process in which no one may be able to state targets for the objectives. Indeed it would be a mistake to talk about a target for some national objectives: public projects are not designed to attain a prespecified level of economic efficiency benefits. Relative levels of most social objectives, not absolute magnitude, are of importance. A second obstacle to the quantification of targets and priorities is that decision makers are expected to supply the necessary data with no knowledge of what the feasible trade offs embodied in the noninferior set are.

Goal programing, although it is of great potential use in the private sector, is limited in its utility for the solution of public investment problems. It is computationally efficient, but the value judgments that it elicits, while they are certainly explicit, are the wrong ones, and they are requested from decision makers without prior knowledge of the alternatives.

Utility functions and optimal weights. When the noninferior set is known, the best-compromise solution is defined as the point at which the noninferior set (or transformation curve) and one social indifference curve (a contour of equal utility) of the social indifference map are tangent. This has been shown by *Marglin* [1967, p. 27] and by *Major* [1969, p. 1175]. Thus if a utility function can be formulated for a particular problem, then the best-compromise solution may be found.

Utility functions have been used extensively in consumer demand theory in economics, and they have also been applied to private and public decision-making problems. *Aumann* [1964] appears to be the first to have considered utility functions for multiobjective problems. His basic consideration was the impact of the partial order associated with multiple-criterion functions on the completeness axiom of utility theory. *Keeney* [1969] and *Raiffa* [1969] investigated the development and application of multiattribute utility functions. *Briskin* [1966] showed how indifference curves which are described by an exponential function could be used to solve a two-dimensional problem. The example which Briskin considered was a routing problem for which minimization of time and cost were important objectives.

Approaches such as Briskin's require the prior generation of the noninferior set. Thus one of the techniques in the previous class, e.g., the weighting method, must first be used. The computational effort associated with such an approach is therefore the K^{p-1} solutions of the problem required by a generating method plus the computations for developing the utility functions and for identifying the best-compromise solution.

Geoffrion [1967] developed a method for proceeding more or less directly from a specification of a utility function to the best-compromise solution, bypassing the generation of the entire noninferior set in most cases. Geoffrion considered bicriterion problems of the form

$$\max U[Z_1(\mathbf{x}), Z_2(\mathbf{x})] \tag{37}$$

subject to

$$\mathbf{x} \in \mathbf{X} \tag{38}$$

where the only restriction on the utility function U is that it be a monotonically nondecreasing ordinal function (in each objective) and 'preferably quasi-concave.' The author then shows how a 'relevant portion' of the noninferior set, as well as the best-compromise solution, is found.

Geoffrion's approach can require no more computations than the weighting or constraint methods. When the utility function has an appropriate shape, then computations are considerably reduced, since parts of the noninferior set can be eliminated without their generation. Like the generating techniques this method is also sensitive to the number of objectives; i.e., computational burden increases (approximately exponentially) with the number of objectives. The number of solutions for Geoffrion's approach will be estimated as $<K^{p-1}$.

The motivation for assessing utility functions was provided by the theoretical definition of the best-compromise solution as discussed by Marglin and Major. The same two authors have also shown that the line which passes through the point of tangency of the noninferior set and the social indifference curve is a surrogate for preferences: the slope of this line is proportional to the ratio of the weights on the objectives. These weights, called the 'optimal weights,' give the relative value which society holds for the objectives.

If the optimal weights $\mathbf{w}^*$ are known, then the best-compromise solution can be found by solving the problem

$$\max Z(\mathbf{x}; \mathbf{w}^*) = \sum_{k=1}^{p} w_k{}^* Z_k(\mathbf{x}) \tag{39}$$

subject to

$$\mathbf{x} \in \mathbf{X} \tag{40}$$

where the solution to this problem must be the best-compromise solution, $\mathbf{x}^*$. This is obvious from the previous discussion of the best-compromise solution.

When the optimal weights are provided to the analyst, the computational burden is extremely light, as light as it can be. Only one solution is necessary, although sensitivity analysis on the weights would be desirable. There is no computational sensitivity to the number of objectives.

The optimal weight method and the use of utility functions are both computationally efficient. Both approaches also insure the explicit consideration of trade offs. The major difficulty with these approaches is the insufficiency of information which is supplied to decision makers. The use of utility functions or optimal weights requires decision makers to articulate value judgments in an information void. They will have no knowledge of the feasible trade offs between objectives (the noninferior set) or the implications of their decision for project design. *Freeman* [1969] has pointed out that inferring optimal weights from past decisions is inappropriate because those decisions were made in the absence of knowledge of the noninferior set. This argument may be extended to the present problem: explicit quantification of trade offs does not insure the optimality of decisions when the information upon which decisions are made is insufficient.

Electre method. The Electre method, described by *Roy* [1971, pp. 250–257], attempts to structure a partial ordering of alternatives which is stronger than the incomplete ordering implied by noninferiority and which still allows some incomparability to remain.

The method is based on what Roy calls an 'outranking relationship' R. The relationship is analogous to a preference ordering of alternatives; however, transitivity is not required. The statement $\mathbf{x}^1 R \mathbf{x}^2$ means $\mathbf{x}^1$ is preferred to $\mathbf{x}^2$. However, $\mathbf{x}^1 R \mathbf{x}^2$ and $\mathbf{x}^2 R \mathbf{x}^3$ does not necessarily imply that $\mathbf{x}^1 R \mathbf{x}^3$.

In Electre a specific outranking relationship is developed for the set of noninferior solutions. Inferior solutions are not considered, and one must begin with the generated set of noninferior solutions $\mathbf{X}^*$. Much of the method is concerned with building the outranking relationship from value judgments supplied by the decision maker. In Electre the value judgments take the form of weights on the objective, a 'concordance condition' and a 'discordance condition.' The two latter pieces of information are attempts to quantify the decision maker's range of comparability. That is, in comparing two alternatives $\mathbf{x}^1$ and $\mathbf{x}^2$, the concordance condition captures the decision maker's tolerance for 'bad scores' in the comparison of $\mathbf{Z}(\mathbf{x}^1)$ with $\mathbf{Z}(\mathbf{x}^2)$ which will still allow $\mathbf{x}^1 R \mathbf{x}^2$ to be valid. The discordance condition gives limits of comparability; e.g., $\mathbf{Z}(\mathbf{x}^1) = (1000, 1)$ and $\mathbf{Z}(\mathbf{x}^2) = (1, 1000)$ may be incomparable because of the extreme variation of the two objectives. Given the necessary information, the outranking relationship is formulated and used to construct a graph G in which a node represents a noninferior alternative. The arcs of the graph are drawn such that an arc directed from node $\mathbf{x}^1$ to node $\mathbf{x}^2$ implies $\mathbf{x}^1 R \mathbf{x}^2$. The next step is to find the kernel G^* of the graph G. The nodes contained in the kernel represent those alternatives which are preferred on the basis of R. The nodes (noninferior solutions) not in the kernel may be eliminated from further consideration.

The Electre method is not applicable to water resource problems, since it is not computationally attractive and because trade offs are obscured by the analysis. Recall that the approach begins with the set of noninferior solutions or a representative sample of it. This implies that prior computations must be performed with a generating technique, so that the number of solutions is $>K^{p-1}$. The Electre method is more suitable for problems with discrete alternatives. Water resource problems, on the other hand, are usually characterized by an infinite number of possible alternatives due to the continuous nature of sizing problems in water resources.

Another criticism is that the nature of the value judgments does not seem to be appropriate in a public decision-making setting. It would be difficult enough to get a large group of decision makers to produce a relatively simple item such as a weight. Complicating the task of the decision makers by requiring some vague quantity such as a concordance condition does not seem to be appropriate. Indeed it may defeat the whole purpose of making the trade offs among objectives explicit so that the implications of decisions can be understood clearly. Thus although the method supplies sufficient information, the value judgments which it requires tend to obscure the real problem.

Surrogate worth trade off method. The motivation for this method presented by *Haimes and Hall* [1974] is that the choice of optimal weights should be made with the knowledge that trade offs are a function of the levels of the objectives. Decision makers may wish to put a relatively higher weight on one objective if, say, two units rather than 10 units of that objective were generated.

The distinguishing feature of the surrogate worth trade off method is the generation of 'trade off functions' which show the relationship between a weight on one objective (when another objective is the numeraire) and the values of that objective. A set of trade off functions may be interpreted as a disaggregated noninferior set, in which the objectives are considered in pairs.

The computational procedure, as described by Haimes and Hall, is first to transfer the multiobjective problem into

$$\max Z_r(\mathbf{x}) \tag{41}$$

subject to

$$\mathbf{x} \in \mathbf{X} \tag{42}$$

$$Z_k(\mathbf{x}) \geq L_k \tag{43}$$

for all $k \neq r$, where L_k is the lower bound on the kth objective. One of the objectives expressed as a constraint, say, objective s, is then varied over K values of L_s, keeping the other objectives, all $k \neq r, s$, fixed at L_k. The problem in (41)–(43) is solved for each value of L_s, producing at most K noninferior solutions. The dual variable associated with the constraint for the sth objective when the rth objective is in the objective function is T_{rs}. Haimes and Hall have shown that T_{rs} is the trade off between objectives r and s. There are (possibly as many as) K values of T_{rs} generated by solving the modified problem with K values of L_s. The trade off T_{rs}, taken as a function of $Z_s(\mathbf{x})$, is the previously referred to trade off functions.

With generated values of T_{rs} and $Z_s(\mathbf{x})$ the authors suggest using regression analysis to get the function $T_{rs}[Z_s(\mathbf{x})]$. After this is performed, another of the $k \neq r$ objectives is selected as objective s, and the procedure is repeated until $T_{rk}[Z_k(\mathbf{x})]$ is generated for all $k \neq r$. The next step is to replace the rth objective and repeat the procedure until all $T_{jk}[Z_k(\mathbf{x})]$ are generated for all $j = 1, 2, \cdots, p$ and all $k = 1, 2, \cdots, j-1, j+1, \cdots, p$. The result is a set of functions which relate the weights to the levels of the objectives and which can be displayed graphically. The number of trade off functions is in general equal to p^2 where p is the number of objectives. If, however, the relationships that $T_{kk} = 1$ and $T_{jk} = 1/T_{kj}$ [*Haimes and Hall,* 1974] are exploited, then the number of relevant trade off functions N is reduced, i.e.,

$$N = \binom{p}{2} = \frac{p!}{(p-2)!\,2!} = \frac{p(p-1)(p-2)!}{(p-2)!\,2!} = \frac{p(p-1)}{2} \tag{44}$$

The trade off functions give the analyst the required information to extract 'surrogate worth functions' W_{jk} from the decision maker. There is one surrogate worth function for every trade off function; thus the intent of constructing the W_{jk} is to attach values to the previously computed trade offs. The W_{jk} are ordinal, varying between -10 and $+10$, with some arbitrary but predetermined value which indicates an acceptable ('optimal') trade off. The set of optimal trade offs or weights found by this method are then used to identify the best-compromise solution.

In terms of decision-making implications the surrogate worth trade off method is similar to the optimal weight and utility function methods which were described previously. The emphasis of all three methods is the same: the identification of optimal weights which leads to a specification of the best-compromise solution. All three methods are also favorable in terms of the criterion of explicit quantification of trade offs.

The surrogate worth trade off method provides a great deal more information than the optimal weight or utility function methods, although less than the maximum information associated with the generating methods is supplied. The information supplied is not 'complete' in that trade off functions are generated between two objectives, assuming fixed values for all of the remaining objectives. Thus the variation of trade offs with the level of objectives is captured in only a limited sense.

The surrogate worth trade off method has the potential for bringing clarity and meaning to a multiobjective problem. It can be a powerful tool for decision makers who experience difficulty in evaluating trade offs. Its greatest utility would appear to be for problems with several objectives ($p > 3$), since it leads decision makers through a systematic comparison of objectives, two at a time. This approach may decrease the confusion associated with high dimensionality in objective space when it is administered properly. Unfortunately, the surrogate worth trade off method is vulnerable to computational sensitivity to the number of objectives, a generic characteristic of multiobjective solution techniques.

As mentioned previously, $p(p - 1)/2$ trade off functions based on K-generated points are to be found by the surrogate worth trade off method. Thus the number of solutions to be found is $\{K[p(p - 1)/2]\}$. However, Haimes and Hall suggest that during the trade off function generation phase, the relationship that $T_{jk} = 1/T_{kj}$ should not be used so that a consistency check may be included. Therefore if one assumes that $T_{jk} \neq 1/T_{kj}$, the number of solutions is $[Kp(p - 1)]$. The total requirements are $Kp(p - 1)$, the solutions of the optimization model; $+p(p - 1)/2$, the regression analyses to find the trade off functions; $+p(p - 1)/2$, the derivations (performed with decision makers) of the surrogate worth functions; plus the solution of a system of equalities or of the original multiobjective problem to find the best-compromise solution. The last three requirements do not fit into the computational efficiency criterion, so that the total computational burden will be stated as $>Kp(p - 1)$. The computational requirements of the surrogate worth trade off method increase approximately as the number of objectives squared, p^2.

Techniques Which Rely on Progressive Articulation of Preferences

A great deal of effort has been invested in the development of techniques which are based on the progressive articulation of preferences. The methods which fall into this class can be characterized by a general algorithmic approach: (1) find a noninferior solution, (2) get decision makers' reactions to this solution and modify the problem accordingly, and (3) repeat steps (1) and (2) until satisfaction is attained or until some other termination rule is applicable.

Methods which follow this general procedure have been proposed by *Klahr* [1958], *Savir* [1966], *Maier-Rothe and Stankard* [1970], *Benayoun et al.* [1971], *Belenson and Kapur* [1973], and *Monarchi et al.* [1973]. The step method or Stem, which was proposed by *Benayoun et al.* [1971], will be discussed here, since it is representative of this class of techniques.

The step method proceeds with the construction of a 'payoff table' which is found by solving

$$\max Z_k(\mathbf{x}) = \sum_{i=1} c_i^k x_i \tag{45}$$

subject to

$$\mathbf{x} \in \mathbf{X} \tag{46}$$

for $k = 1, 2, \cdots, p$.

The solution to this problem $\mathbf{x}^k$ gives by definition the maximum value for the kth objective which is called M_k, i.e., $Z_k(\mathbf{x}^k) = M_k$. The values of the other $p - 1$ components of the objective function implied by $\mathbf{x}^k$ are Z_j^k for $j = 1, 2, \cdots, k - 1, k + 1, \cdots, p$, where $Z_j^k = Z_j(\mathbf{x}^k)$. These values are then used to construct a payoff table shown as Table 2. Row k in Table 2 gives the values of the p objectives for the solution $\mathbf{x}^k$.

Step 1 in the general algorithm is accomplished by finding a noninferior solution 'which is the "nearest" in the minimax sense to the ideal solution $\mathbf{x}^*$' [*Benayoun et al.*, 1971, p. 369]. The ideal solution $\mathbf{x}^*$, which would yield the diagonal cells M_k, $k = 1, 2, \cdots, p$, in the payoff table in Table 2 is assumed to be infeasible, since otherwise the objectives would not conflict. Thus the problem to be solved is

$$\min F \tag{47}$$

subject to

$$\pi_k[M_k - Z_k(\mathbf{x})] - F \leq 0 \qquad k = 1, 2, \cdots, p \tag{48}$$

$$\mathbf{x} \in \mathbf{X}^i \qquad F \geq 0 \tag{49}$$

where $\mathbf{X}^i = \mathbf{X}$ for $i = 1$ and $\mathbf{X}^i$ for $i > 1$ is a modified form of $\mathbf{X}$ which incorporates the decision makers' reactions to the solution found at the $(i - 1)$th iteration.

The F in (47) and (48) is the maximum weighted deviation of an objective from the ideal solution, and it is to be minimized. The weights π_k in (48) are defined as

$$\pi_k = \alpha_k \bigg/ \sum_{k=1}^{p} \alpha_k \tag{50}$$

where

$$\alpha_k = \frac{M_k - m_k}{M_k} \left[\sum_{j=1}^{p} (c_j^k)^2 \right]^{-1/2} \tag{51}$$

in which m_k is the minimum value of the kth objective found by finding the smallest cell in the kth column of Table 2 and in which c_j^k is the coefficient for the jth decision variable for the kth component of the objective function as in (45). Thus the π_k represent normalized weights on the various objectives which by (51) depend on the variation of the value of the objective from the 'ideal solution' M_k.

Step 2 of the general algorithm in the step method consists of asking the decision makers to indicate which objectives in the solution can be decreased so that unsatisfactory levels of objectives may be increased. The problem is then modified by redefining $\mathbf{X}^i$ at the ith iteration in a manner which incorporates the decision makers' reactions.

The iterations continue until the decision makers are satisfied with the results, an indication that a best-compromise solution has been found. If at any iteration the decision maker feels that not one of the objectives is satisfactory, then Stem stops with the conclusion that no best-compromise solution exists. At most, p iterations are performed where p is the number of objectives. After p iterations, either the decision makers are satisfied or it is concluded that no best-compromise solution can exist.

TABLE 2. Payoff Table for the Step Method

Solution Which Optimizes kth Objective	Value of kth Objective					
	z_1	z_2	$\cdots$	z_k	$\cdots$	z_p
$\mathbf{x}^1$	M_1			z_k^1		
$\mathbf{x}^2$		M_2		z_k^2		
$\vdots$			$\ddots$	$\vdots$		
$\mathbf{x}^k$	z_1^k	z_2^k	$\cdots$	M_k	$\cdots$	z_p^k
$\vdots$				$\vdots$	$\ddots$	
$\mathbf{x}^p$				z_k^p		M_p

TABLE 3. Summary of the Evaluation of Multiobjective Solution Techniques

	Computational Efficiency (Number of Solutions)	Are Trade Offs Explicit?	Information Supplied for Decision Making
Generating techniques			
Weighting and constraint methods	K^{p-1}	yes	maximum
Derivation of functional relationship, adaptive search	infeasible		
Prior articulation of preferences			
Goal programing	1+	no	insufficient
Assessing utility functions	$\leq K^{p-1}$	yes	insufficient
Estimation of optimal weights	1	yes	insufficient
Electre method	$>K^{p-1}$	no	sufficient
Surrogate worth trade off method	$>Kp(p - 1)$	yes	sufficient
Progressive articulation of preferences			
Step method, etc.	$\leq p$	no	sufficient

Here p is the number of objectives, and K is the number of intervals of each objective considered in parametric analysis of noninferior set.

The step method is typical of interactive multiobjective solution techniques. The maximum number of solutions (iterations) is usually equal to the number of objectives. Most of the methods also exhibit a similar stopping condition with the associated conclusion that a best-compromise solution does not exist. The major differences among the techniques are the ways in which the noninferior solutions are found and the manner in which preferences are represented in the model at step 2.

In spite of the computational efficiency of Stem and most of the other interactive methods they do not appear to be particularly useful for water resource problems. For example, one curious result of the step method is the assertion that a best-compromise solution does not exist if the decision maker is not satisfied after p iterations. This implies that he is not willing to forfeit any of the satisfactory objectives to improve the unsatisfactory ones. The step method proposes that when this situation exists, then no decision can be made. This is not realistic, however, as decisions in these situations must be reached whether or not the decision maker is satisfied. Unfortunately, Stem is of little assistance in these cases because the unsatisfied decision maker may not have sufficient information to reach a decision.

The interactive methods also do not explicitly capture the trade offs between objectives. The weights in no way reflect a value judgment on the part of the decision maker. They are artificial quantities, concocted by the analyst to reflect deviations from an ideal solution, which is itself an artificial quantity. This definition of the weights serves to obscure rather than capture the normative nature of the multiobjective problem.

SUMMARY OF THE EVALUATION

The evaluation of the solution techniques is summarized in Table 3. A comparison of the techniques is itself a multiobjective problem with three criteria. As happens with any multiobjective problem, the first step is to exclude clearly inferior alternatives from further consideration. Thus the Electre method is dominated by the weighting and constraint methods, whereas goal programing and utility functions are dominated by the estimation of optimal weights. The latter conclusion implies that if value judgments are expected from the decision makers prior to analysis, then the analyst should pursue a simple and meaningful quantity such as a weight rather than targets and priorities (as in goal programing) or multiattribute utility functions.

The techniques in the progressive articulation of preference class are not inferior to the estimation of optimal weights because decision makers receive slightly more information from the latter methods. However, it is concluded that the marginal increase in information is not sufficient to make up for the loss of an explicit consideration of trade offs. An honest analyst must point out, however, that this last comparison constitutes a value judgment. The implied ordering places a relatively high weight on explicitness.

TABLE 4. Noninferior Multiobjective Solution Techniques

	Computational Efficiency (Number of Solutions)	Information Supplied for Decision Making
Weighting and constraint methods	K^{p-1}	maximum
Estimation of optimal weights	1	insufficient
Surrogate worth trade off method	$>Kp(p - 1)$	sufficient

All of the above methods foster the explicit consideration of trade offs.

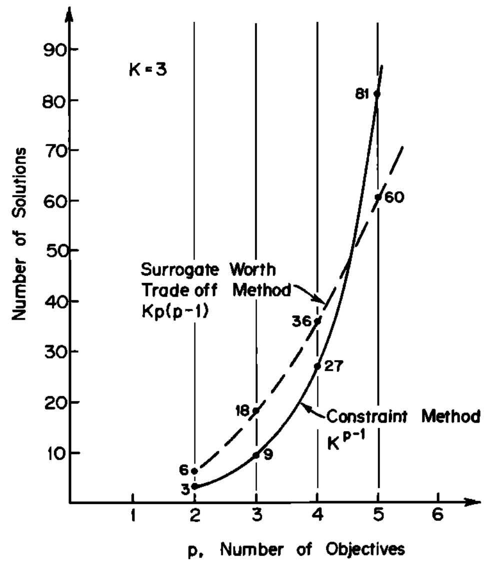

Fig. 4. Number of solutions for the constraint (weighting) and surrogate worth trade off methods when $K = 3$.

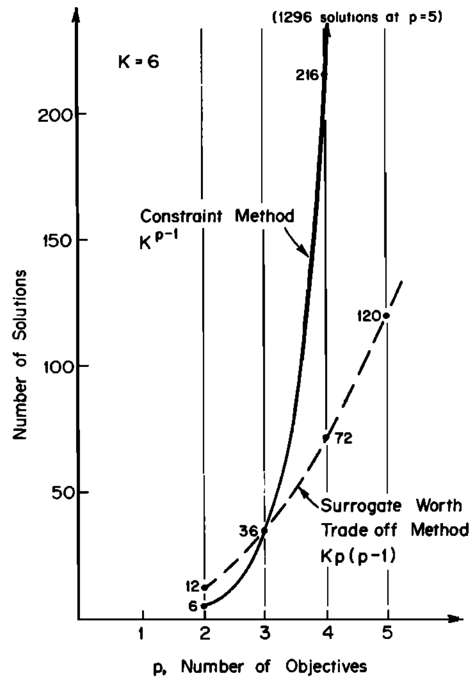

Fig. 5. Number of solutions for the constraint (weighting) and surrogate worth trade off methods when $K = 6$.

The above preliminary screening of inferior approaches leaves the following noninferior methods: the weighting and constraint methods, the estimation of optimal weights, and the surrogate worth trade off method. These four approaches all foster an explicit consideration of trade offs, differing only in their computational efficiency and information supply, as shown in Table 4. Choosing from among the remaining methods must involve value judgments on the relative importance of the two criteria, since the approaches are noninferior.

The set of preferred approaches will be further reduced by arguing against the use of an optimal weight method. The argument is that decision making in an informational void is to be avoided, even when there are obvious computational advantages. The conclusion is that 'top-down' planning, in the terminology of *Marglin et al.* [1973], should not be pursued because the decision maker is placed in the position of articulating value judgments without knowledge of the implications of his decision.

The final task before concluding the evaluation is to contrast the weighting and constraint methods with the surrogate worth trade off method. Table 4 indicates that the weighting and constraint approaches provide in general more information than the surrogate worth trade off method. If in addition, the computational burden of the weighting and constraint methods is less than that of the surrogate worth trade off method, then the former will dominate the latter.

Expressions for the computational requirements of the methods are presented in Table 4. The number of solutions required is a function of K, the number of values of each objective which is considered in the application of the technique, and p, the number of objectives. The requirements of the methods are graphed in Figure 4 for $K = 3$ and in Figure 5 for $K = 6$. The conclusion is that for some range of p, given a value for K, the computational requirements of the weighting and constraint methods are lower than those for the surrogate worth trade off method. This range can be generalized with the aid of Table 5. The results in Table 5 and the observation that the computational requirements of the surrogate worth method are actually greater than $Kp(p - 1)$ lead to the conclusion that the weighting and constraint methods are preferred to the surrogate worth trade off method when $p \leq 3$, i.e., when there are fewer than four objectives. The surrogate worth method is useful for problems with four or more objectives. By limiting the region of analysis with its pairwise comparisons of objectives, the surrogate worth method develops a decreased set of information, but it decreases computational requirements. Even for this method, however, computational burden remains large when $p \geq 4$ (see Figures 4 and 5).

CONCLUSIONS

The goal of the preceding discussion was to bring some order to the rapidly expanding field of multiobjective programing. Several of the proposed techniques for solving multiobjective problems were reviewed and criticized. It was shown in the context of three evaluation criteria, which are ad-

TABLE 5. Range of K for Which Computational Requirements of Weighting and Constraint Methods Are lower Than Those of Surrogate Worth Trade Off Method

Number of Objectives p	Solutions for Weighting and Constraint Methods $S_1 = K^{p-1}$	Solutions for Surrogate Worth Trade Off Method $S_2 = Kp(p - 1)$	Range of K for Which $S_1 \leq S_2$
2	K	$2K$	any
3	K^2	$6K$	$K \leq 6$
4	K^3	$12K$	$K \leq 3$
5	K^4	$20K$	$K \leq 2$

mittedly subjective but hopefully defensible, that (1) many techniques should not be considered applicable to multiobjective water resource problems; (2) when there are fewer than four objectives, a generating technique such as the weighting method or constraint method should be used in order to capture the essence of the multiobjective problem; and (3) when there are four or more objectives, a technique which restricts the size of the feasible region such as the surrogate worth trade off method should be used. In high-dimensional problems, however, large computational burdens cannot be avoided.

The conclusions should be tempered with a few more comments. First, although several techniques were found to be inappropriate for use in large water resource planning problems, there are situations for every approach to which it is applicable. Even a technique such as the Electre method may be useful when the analyst is confronted by a decision maker who is best served by a highly mechanized solution process.

Second, the weighting and constraint methods were considered as one throughout this analysis. This was appropriate, since in terms of the stated criteria the two methods are identical. There are, however, some important practical considerations which support the choice of the constraint method over the weighting method. These issues are discussed by *Cohon and Marks* [1973].

Third, an analysis of this type is particularly useful in defining areas which are in need of further research. A particularly difficult task for multiobjective programing is the development of efficient generating techniques for problems with four or more objectives. Considerable effort has been devoted to this problem [e.g., *Philip*, 1972; *Holl*, 1973], but further research is required to develop an efficient procedure.

Finally, an important aspect of multiobjective planning which was not considered here should be mentioned. A current direction of research into public investment decision is directed at the assessment of the political feasibility of investment alternatives. *Loucks'* [1974] review was developed with political feasibility as a primary consideration. Research in this area is rapidly leading toward the definition of the next problem in public planning: the multiobjective multiple-decision maker problem. Before following this relatively new research path, it is important to synthesize formally its forerunner: the multiobjective issues in planning. This was the intent of the preceding analysis.

NOTATION

- a_{jk} exponent of the jth decision variable for kth polynomial objective.
- b_k positive deviation from the target on objective k.
- c_j^k objective function coefficient for the jth decision variable for objective k.
- d_k deviation from the target on objective k.
- e_k negative deviation from the target on objective k.
- F maximum weighted deviation of an objective from the ideal solution.
- $g_i(\ )$ ith constraint.
- G graph consisting of nodes (noninferior alternatives) and arcs, the direction of which implies preference.
- G^* kernel of the graph G.
- K_k number of different weights or lower bounds for objective k which are considered during the solution process.
- L_k lower bound on objective k.
- m_k minimum value for objective k in a payoff table.
- M_k maximum value for objective k in a payoff table.
- p number of objectives.
- P_k priority on objective k.
- R outranking relationship.
- S number of solutions of the multiobjective problem which are found during the solution process.
- T_{jk} trade off between objectives j and k.
- T_k target on objective k.
- $U(\ )$ utility function.
- w_k weight on objective k.
- w_k^* optimal weight for objective k; the weight for objective k which yields the best-compromise solution.
- W_{jk} surrogate worth function for objectives j and k.
- $\mathbf{x}$ $n \times 1$ vector of decision variables; the jth decision variable x_j is the jth component of $\mathbf{x}$.
- $\mathbf{X}$ feasible region in decision space; $\mathbf{X} = \{\mathbf{x}/\mathbf{x}$ feasible$\}$.
- $\mathbf{X}^*$ set of noninferior solutions.
- $\mathbf{Z}(\ )$ $1 \times p$ vector of objectives; the kth objective $Z_k(\ )$ is the kth component of $\mathbf{Z}(\ )$.
- $\mathbf{Z}(\mathbf{X})$ feasible region in objective space.
- $\mathbf{Z}(\mathbf{X}^*)$ noninferior set.
- Z_j^k entry in the kth row and jth column of a payoff table; the value of the jth objective for the solution which optimizes the kth objective.
- α_k weight on the distance of the kth objective from the ideal solution.
- λ_i dual variable associated with the ith constraint.
- π_k normalized weight on the distance of the kth objective from the ideal solution.

Acknowledgments. The doctoral dissertation of the first author, upon which much of this paper was based, was supported in part by a Natural Resources fellowship, Resources for the Future, Inc. The authors wish to thank David C. Major of Massachusetts Institute of Technology for his valuable comments and criticisms.

REFERENCES

Aumann, R. J., Subjective programming, in *Human Judgments and Optimality*, edited by M. W. Shelby II and G. L. Bryan, chap. 12, pp. 217–242, John Wiley, New York, 1964.

Beeson, R. M., Optimization with respect to multiple criteria, Ph.D. thesis, Univ. of S. Calif., Los Angeles, June 1971.

Beeson, R. M., and W. S. Meisel, The optimization of complex systems with respect to multiple criteria, in *Proceedings, Systems, Man, and Cybernetics Conference*, Institute of Electronics and Electrical Engineers, Anaheim, Calif., 1971.

Belenson, S. M., and K. C. Kapur, An algorithm for solving multicriterion linear programming problems with examples, *Oper. Res. Quart.*, *24*(1), 65–77, 1973.

Benayoun, R., J. de Montgolfier, J. Tergny, and O. Laritchev, Linear programming with multiple objective functions: Step method (Stem), *Math. Progr.*, *1*(3), 366–375, 1971.

Blanchard, B., A first trial in the optimal design and operation of a water resource system, S.M. thesis, Mass. Inst. of Technol., Cambridge, 1964.

Briskin, L. E., A method of unifying multiple-objective functions, *Manage. Sci.*, *12*(10), B406–B416, 1966.

Charnes, A., and W. W. Cooper, *Management Models and Industrial Applications of Linear Programming*, vol. 1, John Wiley, New York, 1961.

Charnes, A., W. W. Cooper, R. J. Niehaus, and A. Stedry, Static and dynamic assignment models with multiple objectives and some remarks on organization design, *Manage. Sci.*, *15*(8), B365–B375, 1969.

Cicchetti, C. J., et al., Benefits or costs? An assessment of the Water Resources Council's proposed principles and standards, 18 pp., Dep. of Geogr. and Environ. Eng., Johns Hopkins Univ., Baltimore, Md., March 1972.

Cicchetti, C. J., R. K. Davis, S. H. Hanke, and R. H. Haveman, Evaluating federal water projects: A critique of proposed standards, *Science*, *181*, 723–728, 1973.

Cohon, J. L., and D. H. Marks, Multiobjective screening models and water resources investment, *Water Resour. Res.*, *9*(4), 826–836, 1973.

Cohon, J. L., T. B. Facet, A. H. Haan, and D. H. Marks, Mathematical programming models and methodological approaches for river basin planning, technical report, Ralph M. Parsons Lab. for Water Resour. and Hydrodyn., Mass. Inst. of Technol., Cambridge, 1974.

Freeman, A. M., III, Project design and evaluation with multiple objectives, in *Analysis and Evaluation of Public Expenditures: The PPB System,* vol. 1, pp. 565–578, U.S. Government Printing Office, Washington, D. C., 1969.

Gass, S., and T. Saaty, The computational algorithm for the parametric objective function, *Nav. Res. Logistics Quart., 2,* 39–45, 1955.

Geoffrion, A. M., Solving bicriterion mathematical programs, *Oper. Res., 15*(1), 39–54, 1967.

Haimes, Y. Y., and W. A. Hall, Multiobjectives in water resources systems analysis: The surrogate worth trade off method, *Water Resour. Res., 10*(4), 615–624, 1974.

Holl, S., Efficient solutions to a multicriteria linear program, with application to an institution of higher education, Ph.D. thesis, Johns Hopkins Univ., Baltimore, Md., 1973.

Kapur, K. C., Mathematical methods of optimization for multiobjective transportation systems, *Socio-Economic Plann. Sci., 4,* 451–467, 1970.

Keeney, R. L., Multidimensional utility functions: Theory, assessment and application, *Tech. Rep. 43,* Oper. Res. Center, Mass. Inst. of Technol., Cambridge, Oct. 1969.

Klahr, C. N., Multiple objectives in mathematical programming, *Oper. Res., 6*(6), 849–855, 1958.

Koopmans, T. C., Analysis of production as an efficient combination of activities, in *Activity Analysis of Production and Allocation, Cowles Comm. Monogr. 13,* edited by T. C. Koopmans, pp. 33–97, John Wiley, New York, 1951.

Kuhn, H. W., and A. W. Tucker, Nonlinear programming, in *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability,* edited by J. Neyman, pp. 481–492, University of California Press, Berkeley, Calif., 1951.

Lee, S., and V. Jaaskelainen, Goal programming: Management's math model, *Ind. Eng.,* 30–35, Feb. 1971.

Loucks, D. P., Stochastic methods for analyzing river basin systems, *Tech. Rep. 16,* Water Resour. and Mar. Sci. Center, Cornell Univ., Ithaca, N. Y., Aug. 1969.

Loucks, D. P., Conflict and choice: Planning for multiple objectives, in *Economy Wide Models and Development Planning,* edited by C. Blitzer, P. Clark, and L. Taylor, Oxford University Press, New York, 1975.

Maass, A., Benefit-cost analysis: Its relevance to public investment decisions, in *Water Research,* edited by A. V. Kneese and S. C. Smith, Johns Hopkins Press, Baltimore, Md., 1966.

Maass, A., et al., *Design of Water-Resource Systems,* Harvard University Press, Cambridge, Mass., 1962.

Maier-Rothe, C., and M. F. Stankard, Jr., A linear programming approach to choosing between multi-objective alternatives, paper presented at the 7th Mathematical Programming Symposium, The Hague, Sept. 1970.

Major, D. C., Benefit-cost ratios for projects in multiple objective investment programs, *Water Resour. Res., 5*(6), 1174–1178, 1969.

Marglin, S. A., *Public Investment Criteria,* MIT Press, Cambridge, Mass., 1967.

Marglin, S. A., P. Dasgupta, and A. Sen, *Guidelines for Project Evaluation,* U. N. Ind. Develop. Organ., New York, 1972.

Miller, W. L., and D. M. Byers, Development and display of multiple-objective project impacts, *Water Resour. Res., 9*(1), 11–20, 1973.

Monarchi, D. E., C. C. Kisiel, and L. Duckstein, Interactive multiobjective programing in water resources: A case study, *Water Resour. Res., 9*(4), 837–850, 1973.

Philip, J., Algorithms for the vector maximization problem, *Math. Progr., 2*(2), 207–229, 1972.

Poblete, J.-A., and R. T. McLaughlin, Time periods and parameters in mathematical programming for river basins, *Tech. Rep. 128,* Ralph M. Parsons Lab. for Water Resour. and Hydrodyn., Mass. Inst. of Technol., Cambridge, Sept. 1970.

Raiffa, H., Preferences for multi-attributed alternatives, *Memo. Rm-5868-DOT RC,* Rand Corp., Santa Monica, Calif., April 1969.

Reid, R. W., and V. Vemuri, On the noninferior index approach to large-scale multi-criteria systems, *J. Franklin Inst., 291*(4), 241–254, 1971.

Rogers, P., A game theory approach to the problems of international river basins, *Water Resour. Res., 5*(4), 749–760, 1969.

Roy, B., Problems and methods with multiple objective functions, *Math. Progr., 1*(2), 239–266, 1971.

Savir, D., Multi-objective linear programming, *Rep. ORC 66-21,* Oper. Res. Center, Univ. of Calif., Berkeley, Nov. 1966.

Terry, H., Comparative evaluation of performance using multiple criteria, *Manage. Sci., 9*(3), 431–442, 1963.

U.S. Water Resources Council, Water and related land resources, Establishment of principles and standards for planning, *Fed. Regist., 38*(174), 24778–24869, 1973.

Vemuri, V., Multiple-objective optimization in water resource systems, *Water Resour. Res., 10*(1), 44–48, 1974.

Wagner, H. M., *Principles of Operations Research,* Prentice-Hall, Englewood Cliffs, N. J., 1969.

Wallace, J. R., Linear-programming analysis of river-basin development, Sc.D. thesis, Mass. Inst. of Technol., Cambridge, 1966.

Wildavsky, A., Aesthetic power or the triumph of the sensitive minority over the vulgar mass: A political analysis of the new economics, *Daedalus, 96*(4), 1115–1128, 1967.

Zadeh, L. A., Optimality and non-scalar-valued performance criteria *IEEE Trans. Automat. Contr., AC-8*(1), 59–60, 1963.

(Received May 20, 1974;
revised November 11, 1974;
accepted November 22, 1974.)