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FARTHEST POINTS IN $W^\star$-COMPACT SETS

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We show that while farthest points always exist in $w^\star$-compact sets in duals to Radon-Nikodym spaces, this is generally not the case in dual Radon-Nikodym spaces. We also show how to characterise weak compactness in terms of farthest points.

The purpose of this note is to find under what conditions the Edelstein-Asplund-Lau results on the existence of farthest points in weakly compact sets can be extended to $w^\star$-compact sets.

To fix our notation, let C be a norm closed bounded subset of a real Banach space X and x be an element of X. We define

$$r(x) = r(x, C) = \sup\{\|x - z\| \mid z \in C\}$$

and call $r(x)$ the farthest distance from x to C. Equivalently, $r(x)$ is the radius of the smallest ball of centre x, containing C. The function r is convex as supremum of such functions, and continuous since $|r(x) - r(y)| \leqslant \|x - y\|$, for all x, $y \in X$. A point $z \in C$ is called a farthest point of C if there exists $x \in X$ such that $\|x - z\| = r(x)$. The existence of a farthest point of C is equivalent to the fact that the set

$$D = \{x \in X | (\exists z \in C)(\|x - z\| = r(x))\}$$

is non-empty. Since it follows that any farthest point of a convex set C in a locally uniformly rotund space is a strongly exposed point of C, the notion of a farthest point is widely used in the study of the extreme structure of sets. In fact, it was the first discovered method of obtaining exposed points in sets [**10**]. We refer the reader to [**3**] and [**5**] for unexplained notions.

Extending the results of [**6**] and [**1**], Lau showed [**9**] that if C is a weakly compact set in a Banach space X, then the set D defined above is dense in X and thus, in particular, C has farthest points. Other extensions of the results of [**6**] and [**1**] may be found in [**11**]. The connection between farthest points and strongly exposed points in

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mentioned above led Edelstein and Lewis [7] to ask the following question: Does the existence of (strongly) exposed points in C imply the existence of farthest points in C?

Proposition 1 gives a negative answer to this question. Moreover, it shows that Lau's result on weakly compact sets does not always extend to $w^\star$-compact sets in a dual space $X^\star$. However, our Proposition 3 shows that if X has the Radon-Nikodym property then any $w^\star$-compact set in $X^\star$ contains a farthest point, thus extending Proposition 1 of [**11**]. Finally, we point out in Proposition 4 that Lau's result actually characterises weakly compact sets.

PROPOSITION 1. *Let $\ell^1(\mathbb{N})$ be the Banach space of all real summable sequences $x = (x_n)$, equipped with its usual norm $\|x\| = \sum_{n=1}^{\infty} |x_n|$. Let*

$$C = \{(a_n) \in \ell^1(\mathbb{N}) \colon \sum_{n=1}^{\infty} \left(|a_n| + |a_n|^2\right) \leqslant 1\}.$$

Then C is a weak*-compact convex of $\ell^1(\mathbb{N})$ and C has no farthest points.

PROOF OF PROPOSITION 1: To prove that C has no farthest points, it is enough to show that, given $x \in \ell^1(\mathbb{N})$:

i) $r(x) = 1 + \|x\|$ and
ii) $\|x - z\| < 1 + \|x\|$ for all $z \in C$.

To do so, first notice that if $z \in C$ then $\|z\| < 1$. Therefore, $\|x - z\| < \|x\| + \|z\| < 1 + \|x\|$ and ii) is proved. This also shows that $r(x) \leqslant 1 + \|x\|$. Hence, to prove i), it is enough to construct a sequence (u_n) of elements of C such that

$$\lim_{n\to\infty} \|x - u_n\| = 1 + \|x\|.$$

For each $n \in \mathbb{N}$, let $\delta_n > 0$ be such that $n\left(\delta_n + \delta_n^2\right) = 1$. Note that $0 < \delta_n < \frac{1}{n}$ and $\lim_{n\to\infty} n\delta_n = 1$. Let $u_n = (\delta_n, \dots, \delta_n, 0, 0, \dots)$ where δ_n is repeated n times. By our choice of δ_n, $u_n \in C$.

Now given $x \in \ell^1(\mathbb{N})$ and $\varepsilon > 0$, let $p \in \mathbb{N}$ be such that $\sum_{n=p+1}^{\infty} |x_i| < \frac{\varepsilon}{3}$. We have for $n > p$

$$\begin{aligned}\|x - u_n\| &= \sum_{i=1}^{p} |x_i - \delta_n| + \sum_{i=p+1}^{n} |x_i - \delta_n| + \sum_{i=n+1}^{\infty} |x_i| \\ &\geqslant \sum_{i=1}^{p} |x_i| - p\delta_n - \sum_{i=p+1}^{n} |x_i| + (n-p)\delta_n \\ &\geqslant \|x\| - \frac{\varepsilon}{3} - p\delta_n - \frac{\varepsilon}{3} + (n-p)\delta_n.\end{aligned}$$

Choose $n_0 > p$ big enough so that $(n-2p)\delta_n > 1 - \frac{\varepsilon}{3}$ for $n \geqslant n_0$. We have that for all $n \geqslant n_0$:

$$\|x - u_n\| \geqslant 1 + \|x\| - \varepsilon.$$

Therefore,

$$\lim_{n\to\infty} \|x - u_n\| = 1 + \|x\|.$$

∎

Remarks. (a) Since $\ell^1(\mathbb{N})$ has the Radon-Nikodym property, C is the norm closed convex hull of its strongly exposed points [**3**]. Hence, Proposition 1 gives a negative answer to the question of Edelstein and Lewis mentioned above.

(b) The proof of Proposition 1 shows that the intersection of all the balls containing C is the unit ball of $\ell^1(\mathbb{N})$. Note also that the sequence $\{u_n\}$ in the proof does not depend on x.

Before proceeding, let us recall that the subdifferential of a convex function f defined on a Banach space X is defined by

$$\partial f(x) = \{x^\star \in X^\star | (\forall y \in X)(\langle x^\star, y - x\rangle \leqslant f(y) - f(x))\}.$$

It was shown in [**9**] that if C is a closed bounded subset of a Banach space X and $r(x)$ is the farthest distance function for C, then for any $x \in X$ and $x^\star \in \partial r(x)$, we have that $\|x^\star\| \leqslant 1$ and thus $\sup\{\langle x^\star, x - z\rangle : z \in C\} \leqslant \|x^\star\| r(x) \leqslant r(x)$. Moreover, Lau showed that the set

$$G = \{x \in X | (\forall x^\star \in \partial r(x))(\sup\{\langle x^\star, x - z\rangle : z \in C\} = r(x))\},$$

is dense G_δ in X.

We shall need the following.

LEMMA 2. *Let C be a closed bounded subset of a Banach space X and r the farthest distance function on X associated with C. Assume that r is Fréchet differentiable at $x_0 \in X$. Then*

$$r(x_0) = \sup\{\langle x_0^\star, x_0 - z\rangle \mid z \in C\}, \qquad \text{where } \partial r(x_0) = \{x_0^\star\}.$$

PROOF: According to Lau's result mentioned before the statement of Lemma 2 it is enough to show that, under our assumptions, the following implication holds:

If $r(x_0) - \sup\{\langle x_0^\star, x_0 - z\rangle | z \in C\} = \alpha > 0$, then there is a neighbourhood U of x_0 such that

$$r(y) - \sup\{\langle y^\star, y - z\rangle | z \in C\} > 0,$$

whenever $y \in U$ and $y^\star \in \partial r(y)$.

To prove that the implication holds, take $y \in X$, $y^\star \in \partial r(y)$ and $z \in C$. We have:

$$\begin{aligned} |\langle y^\star, y - z\rangle - \langle x_0^\star, x_0 - z\rangle| &\leqslant |\langle y^\star, y - z\rangle - \langle y^\star, x_0 - z\rangle| + |\langle y^\star, x_0 - z\rangle - \langle x_0^\star, x_0 - z\rangle| \\ &\leqslant \|y^\star\|\,\|y - x_0\| + \|y^\star - x_0^\star\|\,\|x_0 - z\|. \end{aligned}$$

Therefore,

$$|\sup\{\langle y^\star, y - z\rangle | z \in C\} - \sup\{\langle x_0^\star, x_0 - z\rangle | z \in C\}| \leqslant \|y - x_0\| + r(x_0)\|y^\star - x_0^\star\|.$$

Now choose a neighbourhood U of x_0 such that

$$2\|y - x_0\| + r(x_0)\|y^\star - x_0^\star\| < \alpha$$

whenever $y \in U$ and $y^\star \in \partial r(y)$. This is possible due to the Fréchet differentiability of r at x_0 [**2**, Lemma 5]. If $y \in U$ and $y^\star \in \partial r(y)$, then

$$\begin{aligned} &(r(x_0) - \sup\{\langle x_0^\star, x_0 - z\rangle | z \in C\}) - (r(y) - \sup\{\langle y^\star, y - z\rangle | z \in C\}) \\ &\qquad \leqslant |r(x_0) - r(y)| + |\sup\{\langle y^\star, y - z\rangle | z \in C\} - \sup\{\langle x_0^\star, x_0 - z | z \in C\}| \\ &\qquad \leqslant \|x_0 - y\| + \|y - x_0\| + r(x_0)\|y^\star - x_0^\star\| < \alpha \end{aligned}$$

and this implies, by the definition of α, that

$$r(y) - \sup\{\langle y^\star, y - z\rangle z \in C\} > 0$$

whenever $y \in U$ and $y^\star \in \partial r(y)$. Our implication is proven and the proof of Lemma 2 is finished. ∎

Proposition 3. *Let X be a Banach space with the Radon-Nikodym property, let $X^\star$ be its dual space in its usual dual norm and C be a $w^\star$-compact subset of $X^\star$. Then the set D of all points in $X^\star$ which have farthest points in C contains a subset D_1 dense and G_δ in $X^\star$.*

Proof: First notice that the farthest distance function r associated with C is $w^\star$-lower semicontinuous as supremum of such functions. Since X has the Radon-Nikodym property it follows from [**4**] that r is Fréchet differentiable on a dense G_δ subset D_1 in $X^\star$. So, to finish the proof of Proposition 3 we show that if r is Fréchet-differentiable at x, then x admits a farthest point in C. If $x \in D_1$, denote by $x^\star$ the only element of $\partial r(x)$. Since r is $w^\star$-lower semicontinuous and Fréchet differentiable at x, $x^\star \in X$ [**2**, Corollary 1]. Since C is $w^\star$-compact, there is a $z_0 \in C$ such that $\langle x^\star, x - z_0\rangle = \sup\{\langle x^\star, x - z\rangle | z \in C\}$. Using Lemma 2 we have $r(x) = \langle x^\star, x - z_0\rangle \leqslant \|x^\star\| \cdot \|x - z_0\| \leqslant \|x - z_0\| \leqslant r(x)$. Thus $\|x - z_0\| = r(x)$ and the proof is finished. ∎

Proposition 4. *Let X be a Banach space and C be a closed convex bounded subset of X. The following are equivalent:*

(i) *C is weakly compact;*

(ii) *for every equivalent norm $\| \|_1$ on X, $D = \{x \in X | r(x) = \|x - z\|_1$ for some $z \in C\}$ is dense in X, where $r(x) = \sup\{\|x - z\|_1 | z \in C\}$.*

Proof: (i) $\Rightarrow$ (ii) was proved by Lau in [**9**]. Conversely, let $\|\cdot\|$ be the norm of X and let $B(x,r)$ denote the $\|\cdot\|$-ball centred at x and radius r. Assume without loss of generality that $C \subset B(0,1)$. By James' theorem [**5**] there is a functional $f \in X^\star$, $\|f\| = 1$ which does not attain its supremum on C. Let $B_1 = B(0,6) \cap \{x \in X |_{-1} \leqslant f(x) \leqslant 1\}$ and $\|\cdot\|_1$ be the Minkowski functional of B_1. Note that if $B_1(x,r)$ is the ball cenetred at x with radius δ with respect to the norm $\|\cdot\|_1$, then

$$B_1(x,\delta) = 6B(x,\delta) \cap \{y \in X \mid -\delta \leqslant f(x) - f(y) \leqslant \delta\}.$$

Pick $x_0 \in B(0,4) \cap \{x \in X | f(x) < -3\}$. We claim that if $x \in B(x_0,1)$, then x has no farthest point in C when X is equipped with the new norm $\| \|_1$, hence D is not dense in X.

To see it, we first show that $r(x) = \alpha$, where $\alpha = \sup\{f(z) | z \in C\} - f(x)$. Indeed, let $y \in C$; we have $|f(y) - f(x)| \leqslant \sup\{f(z)|z \in C\} - f(x)$ and $\|y - x\| \leqslant \|y\| + \|x_0\| + \|x_0 - x\| \leqslant 6 \leqslant 6\alpha$ (note that for any $y \in C$ and $x \in B(x_0,1)$ we have $f(y) \geqslant -1$ and $f(x) \leqslant f(x_0) + f(x - x_0) < -2$ and thus $\alpha > 1$). Thus $y \in B(x,\alpha)$. Therefore $C \subset B_1(x,\alpha)$ and this shows that $r(x) \leqslant \alpha$. Conversely if $\delta < \alpha$, there exists $y \in C$ such that $f(y) - f(x) > \delta$ and so $y \notin B_1(x,\delta)$. This shows that $r(x) \geqslant \alpha$.

Now for $y \in C$, we have $\|y - x\| \leqslant 6 < 6\alpha$ and $|f(y) - f(x)| < \sup\{f(z)|z \in C\} - f(x) = \alpha$, hence $\|x - y\|_1 < \alpha = r(x)$ and so y is not a farthest point from x in C. ∎

Remark. We would like to point out that, while differentiability properties can be used to show that the set D of points which have at least one farthest point in a $w^\star$-compact C of a dual Banach space X is dense in X, convexity properties can be used to show the uniqueness of farthest points. More precisely, using standard rotundity arguments, we have

Proposition 5. *Let X be a strictly convex Banach space, and C be a norm closed bounded subset of X such that the corresponding set D is dense in X.*

Then the set

$$U = \{y \in X | y \text{ has a unique farthest point in } C\}$$

is also dense in X.

PROOF: First the strict convexity of X implies that for all $x \in X$ and $\lambda > 1$,

$$B(x, \|x\|) \cap S(\lambda x, \|\lambda x\|) = \{0\}$$

where $B(x,r)$ (respectively, $S(x,r)$) denotes the ball (respectively, the sphere) of centre x and radius r.

Now let $x \in D$ and let z be a farthest point from x in C. By translation, there is no loss of generality in assuming $z = 0$. It is enough to show that for any $\lambda > 1$, $y = \lambda x$ has a unique farthest point in C, namely 0. We have

$$B(\lambda x, \|\lambda x\|) \supseteq B(x, \|x\|) \supseteq C$$

and hence $r(\lambda x) \leqslant \|\lambda x\|$. On the other hand,

$$\{0\} \subset C \cap S(\lambda x, \|\lambda x\|) \subset B(x, \|x\|) \cap S(\lambda x, \|\lambda x\|) = \{0\}.$$

Therefore $r(\lambda x) = \|\lambda x\|$ and the only farthest point from y in C is 0. ∎

Note that if X is not strictly convex, the set U of points which admit unique farthest points can be empty. Indeed, let $X = c_0(\mathbb{N})$ and

$$C = \{(z_n) \in c_0(\mathbb{N}) | 0 \leqslant z_n \leqslant \frac{1}{n} \text{ for all } n\}.$$

C is a norm compact convex subset of $c_0(\mathbb{N})$, and therefore $D = c_0(\mathbb{N})$. On the other hand, if $x = (x_n) \in c_0(\mathbb{N})$, we have

$$r(x) = \max\{\max\{|x_n|, |x_n - \frac{1}{n}|\} | n \in \mathbb{N}\}$$

so $r(x) = |x_{n_0}|$ or $r(x) = |x_{n_0} - \frac{1}{n_0}|$, for some $n_0 \in \mathbb{N}$. Define:

$$C_1 = \{(z_n) \in C | z_{n_0} = 0\}$$
$$C_2 = \{(z_n) \in C | z_{n_0} = \frac{1}{n_0}\}.$$

If $r(x) = |x_{n_0}|$ (respectively, $r(x) = |x_{n_0} - \frac{1}{n_0}|$), C_1 (respectively, C_2) is included in the set of farthest points from x in C. In both cases $x \notin U$.

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