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The Use of Reference Objectives in Multiobjective Optimization - Theoretical Implications and Practical Experience

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SUMMARY

The paper presents a survey of known results and some new developments in the use of reference objectives -- that is, any reasonable or desirable point in the objective space -- instead of weighting coefficients in multiobjective optimization. The main conclusions are as follows:

-- Any point in the objective space -- no matter whether it is attainable or not, ideal or not -- can be used instead of weighting coefficients to derive scalarizing functions which have minima at Pareto points only. Moreover, entire basic theory of multiobjective optimization -- necessary and sufficient conditions of optimality and existence of Pareto-optimal solutions, etc. -- can be developed with the help of reference objectives instead of weighting coefficients or utility functions.

-- Reference objectives are very practical means for solving a number of problems such as Pareto-optimality testing, scanning the set of Pareto-optimal solutions, computer-man interactive solving of multiobjective problems, group assessment of solutions of multiobjective optimization or cooperative game problems, or solving dynamic multiobjective optimization problems.

THE USE OF REFERENCE OBJECTIVES IN MULTIOBJECTIVE OPTIMIZATION -- THEORETICAL IMPLICATIONS AND PRACTICAL EXPERIENCE

A.P. Wierzbicki

1. INTRODUCTION

This paper is aimed at a revaluation of some basic assumptions in multicriteria optimization and decision-making from a pragmatical point of view, addressing the question why the known, highly developed methods and techniques in multicriteria analysis are not fully operational in applications. It is assumed that the reader is well acquainted with the state-of-the-art in multicriteria analysis as represented, for example, by [2,3,7,9,12] and that he has also encountered some of the vexing problems in the applications of this highly developed theory. The basic question in applications of multicriteria analysis is, in fact, only one though it may take various forms:

-- What is more valuable? - the perfection of a compromise based on a model which is never perfect, or the time of a top-rank decision maker? If confronted with a multitude of questions "would you prefer this alternative to the other one?", would not the decision maker simply send the analyst back to where he belongs?

-- Does a decision maker think in terms of trade-offs and weighting coefficients or is he rather concerned with aspiration levels and values?

-- Has a decision maker consistent preferences, which under known assumptions could be revealed in the form of a utility function, or does he simply want to attain certain goals?

-- Is it easier in applications to determine marginal rates of substitution between various objectives, or to choose reasonable ranges or scales for those objectives?

-- Is a compromise in a group of decision makers attained through a balance of their preferences or rather by an agreement on goals?

-- Is not the term "a decision maker" an abstraction, convenient for the analyst? Or do we rather deal with decision making organizations as usual, where the top-level decisions are based on a careful and extremely well prepared, but highly intuitive appraisal of a few crucial indices?

Many similar questions can be stated and all these questions have puzzled many researchers. In fact, some recent papers -- see Ackoff 1979, [1] -- go as far as questioning entirely the practical value of decision analysis and optimization. Some authors prefer a retreat to purely heuristical procedures for decision making to psychological, "soft-science" approaches. Though having much respect for careful, logical analysis of a problem, for deep intuition and psychology, I am not entirely convinced. I would rather address another question, which in a sense summarizes all the above doubts:

-- What is wrong with the basic tools of multicriteria analysis? Should we not reexamine its basic axioms?

Historical reflection can help us in reaching this goal. When, in 1896, Pareto [17] has formulated the foundations of multicriteria optimization and used weighting coefficients to this end, he opened an entire field of research. Weighting coefficients play therefore a central role in the contemporary paradigm of multicriteria analysis -- all necessary and sufficient conditions of multiobjective optimality, all equilibria and trade-offs, all utility maximization is basically related to weighting coefficients. When the foundations of the general economic equilibrium theory were formulated, a consumer was assumed to maximize a utility function representing his preference ordering of commodity bundles -- what, in the equilibrium, directly corresponds to Pareto weighting coefficients forming a linear approximation of the utility function. This was a most satisfactory development of economic theory and still is a contemporary part of its basic paradigm. It has also found confirmation in empirical studies of the free market -- as far as any market is fully free -- and resulted in further deep theoretical studies providing for an axiomatic basis of preference orderings and utility theory at a high mathematical level (see, e.g., Debreu 1959 [5]).

But here is a place for reflection: while a nameless agent on a free market may be well described by his utility function, no individual thinks in terms of preferences of commodity bundles when buying in a supermarket. When I am going to do some shopping, I know that I have to buy, for example, a quantity of milk, sugar, bread, and a shirt for my son; if I have enough money, I might also buy a toy for him and a tool for my gardening. In fact, I am thinking in terms of goals; if they are attainable, I might want to improve them. Moreover, my way of thinking does not change very much when I have to make decisions as a science manager.

However, further extensive studies [2,9,12] on decision making with multiple objectives were related strongly to preferences and utility theory. Identification methods for individual and group preferences as well as utility functions have been developed; statistical approaches have been considered to take into account uncertainty and risks; and even interactive procedures devised to involve a decision maker more directly into decision analysis have been based on learning about his preferences. Moreover, most of the applied studies in multiobjective optimization and decision making are implicitly or explicitly formulated in terms of weighting coefficients, tradeoffs and utility functions.

On the other hand, many researchers have realized the need of an alternative approach. Savlukadze [20,21] and others considered the use of utopia points as unattainable objective values representing some aspiration levels. Dyer [6], Kornbluth [13] and others introduced goal programming -- the use of variable bounds on objective values in an interactive process of multicriteria optimization. Yet these and related works have not had the impact they deserved because of several reasons.

First, it was not clear whether it is possible to develop a consistent, basic theory of multiobjective optimization and decision making based on the use of reference objectives --

that is, any desirable aspiration levels for objectives -- rather than weighting coefficients. In other words, the necessary and sufficient conditions, existence conditions, relations to preference orderings, etc., had to be formulated in terms of reference objectives. This question has been attached to some of my earlier works [22,23,25]; a synthesis and further development of relevant results is presented in the next chapter of this paper.

Second, the use of reference objectives implies a choice of distance or norm in objective space and this choice has been considered, erroneously, as being equivalent to the choice of weighting coefficients. In order to work with reference objectives one has, admittedly, to choose reasonable scales or ranges for all criteria. But the choice of a reasonable range is inherent to any computation or measurement and does not necessarily imply the choice of trade-offs. After having made a decision based on reference objectives, the corresponding weighting coefficients can be a posteriori determined (see next chapter) and examined. This is one of the links between the theory based on reference objectives and the more classical theory, but it does not impede the practical usefulness of reference objectives.

Third, the use of reference objectives has not been widely tested in applications, and various problems related to consideration of uncertainties, to group decision making, to interactive procedures of decision making, etc., have not been solved yet. Another chapter of this paper is devoted to these problems.

2. BASIC THEORY

Fundaments

Let $E_0 \subset E$ be a set of admissible decisions or controls or alternatives to be evaluated. We do not specify yet the nature of space E. Let G be the space of objective values or performance indices or goals. We assume that G is a Hilbert space, out of several reasons. First, some abstract properties of the Hilbert space -- mostly the properties of a projection on a cone -- simplify the reasoning and proofs. Second, a Hilbert space is the least abstract one that includes trajectories of dynamical systems or probability distributions and we would like to consider also dynamical trajectories or probability distributions as possible goals of multiobjective optimization. Third, the Euclidean space E^n is a (finite-dimensional) Hilbert space, and we can therefore use graphical illustrations and intuition to comment on results.

Let a mapping $Q : E_0 \to G$ be given, defining numerically the consequences of each decision or alternative. Let $Q_0 = Q(E_0) \subset G$ be the set of attainable objectives. To choose between them, suppose a *partial preordering* in G is given by means of a *positive cone* (any closed, convex, proper cone) $D \subsetneq G$:

$$(1) \qquad q_1, q_2 \in G \quad , \qquad q_1 \leqslant q_2 \iff q_2 - q_1 \in D \quad .$$

A corresponding strong partial preordering in G can be defined by:

$$(2) \qquad q_1, q_2 \in G \quad , \qquad q_1 < q_2 \iff q_2 - q_1 \in \tilde{D} \overset{\mathrm{df}}{=} D \setminus (D \cap -D) \quad .$$

Suppose, to simplify the exposition, that we are interested in minimizing all the objectives (losses, risks, etc.). In the Hilbert space G, we define correspondingly a *minimal element of* Q_0 *with respect to the partial preordering* (1) or a D-*minimal element of* Q_0:

$$(3) \qquad \hat{q} \in Q_0 \text{ is D-minimal} \iff Q_0 \cap (\hat{q} - \tilde{D}) = \phi \quad .$$

Let us denote by $\hat{Q}_0$ the set of all D-minimal points in Q_0. If $G = R^2$ and $D = R^2_+ = \{(q^1, q^2) \in R^2 : q^1 \geq 0 \;,\; q^2 \geq 0\}$, then a D-minimal point of Q_0 is Pareto-minimal, see Figure 1. In fact, in finite-dimensional cases we are mostly concerned with Pareto-minimal points; the possibility of using other positive cones illustrates only possible generalizations of infinite-dimensional spaces.

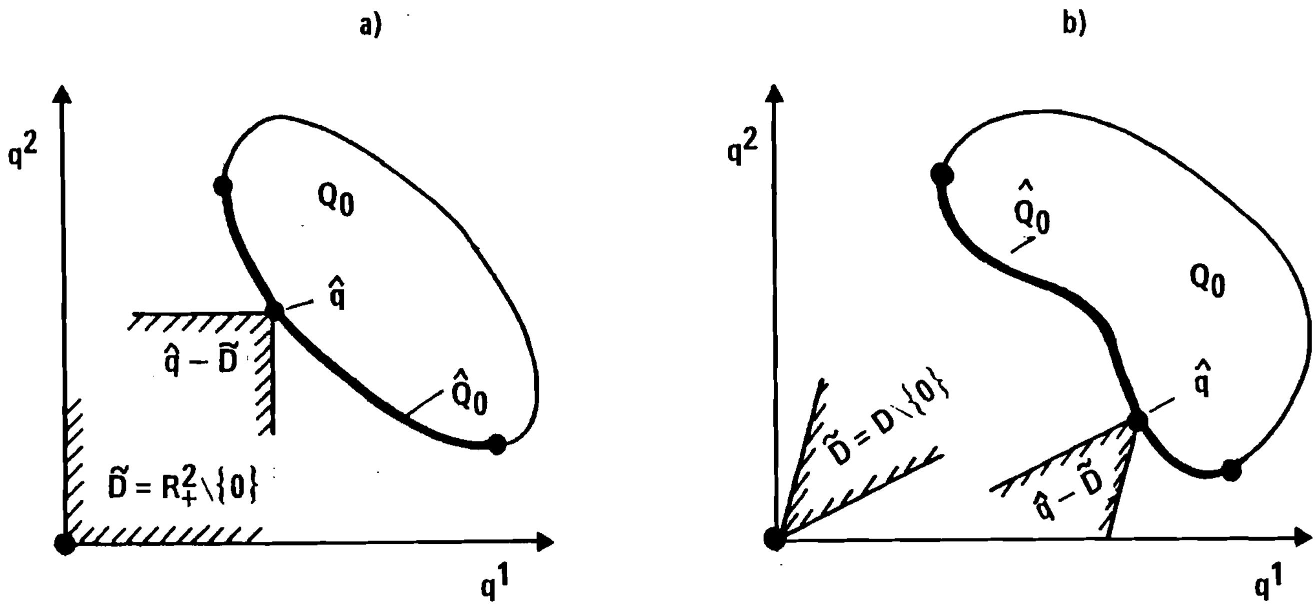

FIGURE 1. D-MINIMAL POINTS AND SETS:
a) PARETO-MINIMAL POINTS
b) MORE GENERAL D-MINIMAL POINTS

Projections on Cones

One of the most important properties of partial preorderings induced by a positive cone in Hilbert space is that we can give a precise answer to the typical question: given two outcomes q_1, q_2 which are incomparable (that is, neither $q_1 \preccurlyeq q_2$ nor $q_2 \preccurlyeq q_1$), what is the part of q_2 that has improved with respect to q_1? The answer results from the following lemma.

Lemma 1 (Projections on Cones in a Hilbert Space -- Moreau 1962 [16]). *Given a Hilbert space* G *and a closed, convex cone* $D \subset G$, *each element* $q \in G$ *can be uniquely and orthogonally decomposed into its projections on the cones* $-D$ *and* $D^* = \{q^* \in G : \langle q^*, q\rangle \geq 0 \;\; \forall q \in D\}$:

$$(4) \qquad (q = \bar{q} + \bar{\bar{q}}\,,\ \bar{q} \in -D\,,\ \bar{\bar{q}} \in D^*\,,\ \langle \bar{q}, \bar{\bar{q}}\rangle = 0) \iff (\bar{q} = q^{-D}\,,\ \bar{\bar{q}} = q^{D^*})$$

where the projections q^{-D} *and* q^{D^*} *are defined by:*

$$(5) \qquad q^{-D} = \operatorname*{arg\,min}_{\bar{q} \in -D} \|q - \bar{q}\| \quad ; \qquad q^{D^*} = \operatorname*{arg\,min}_{\bar{\bar{q}} \in D^*} \|q - \bar{\bar{q}}\| \quad .$$

and $\langle \cdot, \cdot \rangle$ *denotes the scalar product,* $\|\cdot\|$ *denotes the norm.*

The cone D* is called the dual cone; -D and D* are called mutually polar. If $D = R^2_+$, then $D^* = D = R^2_+$, and $q^{D^*} = (\max(0,q^1),\max(0,q^2))$ is just the vector composed of the positive components of the vector q. This is interpreted in Figure 2, where Lemma 1 is applied to the difference $q_2 - q_1 = q$ in order to discern the part of q_2 that has improved when compared to q_1 and the other part that is worse than q_1.

The projection on a cone has several additional useful properties of norm-minimality, Lipschitz-continuity, Fréchét-differentiability of its square norm, etc. - see Wierzbicki and Kurcyusz 1977 [24].

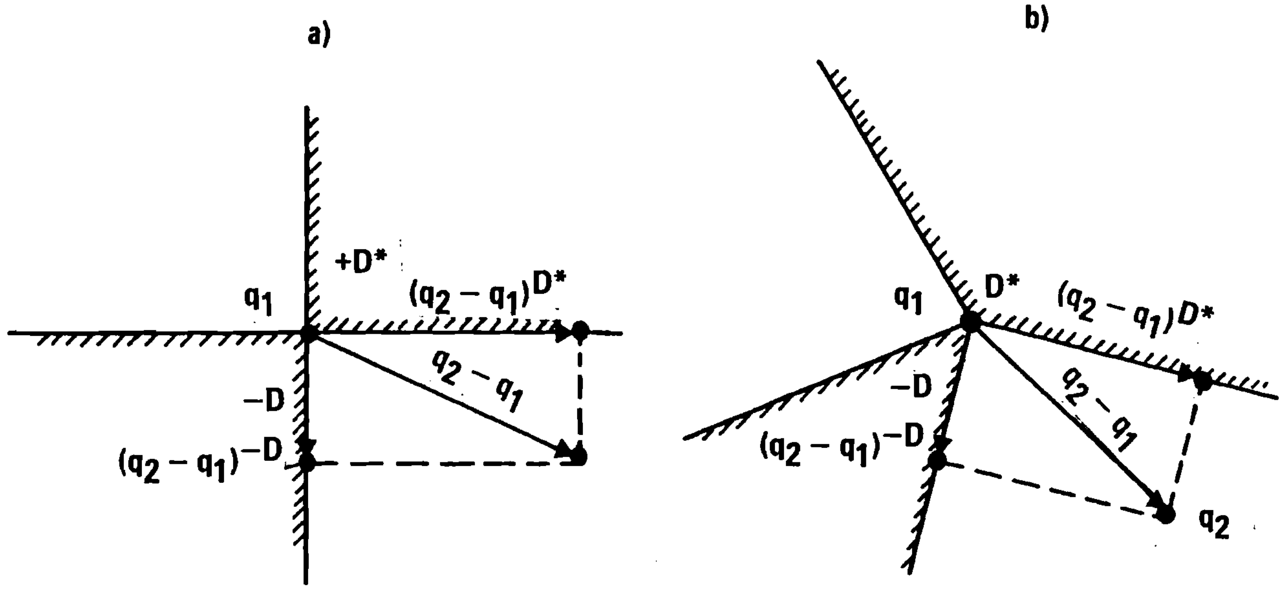

FIGURE 2. DETERMINATION OF THE NEGATIVE AND THE POSITIVE PART OF $q_2 - q_1$ VIA PROJECTION ON CONES: a) $D = R^2_+$; b) MORE GENERAL CASE

Order-Preserving Functions and Penalty Scalarization; Sufficient Conditions for Multicriteria Optimality

Now we approach the basic question in the use of reference objectives in multicriteria optimization: given *any* aspiration level or reference objective $\bar{q} \in G$, can we construct a scalarizing function $s : G \times Q_0 \to R^1$ which is strictly order-preserving in its second argument (thus can be considered as a type of arbitrarily chosen utility function)? Recall that $s(\bar{q},q)$ is strictly order-preserving in q, iff:

(6) $$q_2 \prec q_1 \Rightarrow s(\bar{q},q_2) < s(\bar{q},q_1)$$

and that each minimal point of a strictly D-order-preserving function is a D-minimal point (see, e.g., [5,23]):

(7) $$\hat{q} = \arg\min_{q \in Q_0} s(\bar{q},q) \Rightarrow (\hat{q} - \tilde{D}) \cap Q_0 = \phi \quad .$$

The answer to the above question is not only positive but also vague: there are many scalarizing functions that are strictly order-preserving. For example, choose any vector of positive weighting coefficients -- or, more generally, Lagrange multipliers $\lambda \in \mathring{D}^* = \{q^* \in G : \langle q^*,q \rangle > 0 \ \forall q \in \tilde{D}\}$, where $\mathring{D}^*$ is called the quasi-interior of D^* - and define the known linear function $s(\bar{q},q) = \langle \lambda, q - \bar{q} \rangle$ which in the simplest case is just the sum of weighted objective differences $\sum_{i=1}^{n} \lambda^i (q^i - \bar{q}^i)$. This function is strictly order-preserving, and each of its minimal points is D-minimal, or Pareto-minimal. But the minimal points of this function do not depend on the information contained in $\bar{q}$ and require the information contained in λ. Therefore, we should look for nonlinear strictly order-preserving functions that do not require the specification of weighting coefficients λ and have minima dependent on the reference objective $\bar{q}$. One such function has the following form:

(8) $$s(\bar{q},q) = -\|q - \bar{q}\| + \rho \|(q - \bar{q})^{D^*}\|$$

or to provide for differentiability

(8a) $$s(\bar{q},q) = -\|q - \bar{q}\|^2 + \rho \|(q - \bar{q})^{D^*}\|^2$$

where $\rho > 1$ is an arbitrary scalar coefficient. These functions are called penalty scalarizing functions. One of the basic properties of these functions is the following:

Lemma 2 (Wierzbicki 1975, [22]). *If* G *is a Hilbert space,* $D \subset G$ *is a closed convex cone satisfying the condition* $D \subseteq D^*$, *and* $\rho > 1$, *then, for any* $\bar{q} \in G$, *the function* $s(\bar{q},q)$ *defined by (8) or (8a) is strictly order-preserving.*

Observe, first, that the condition $D \subseteq D^*$ is not very restrictive, since if $D = E^n_+$, then $D^* = D \subseteq D^*$; generally, the condition means that the cone D should not be "too broad". Secondly, observe that the lemma is valid for *any* $\bar{q} \in G$ and, therefore, generalizes and puts two known approaches into a common frame: utopia point approach, where $\bar{q} \notin Q_0$ and $Q_0 \subset \bar{q} + D$ (a point $\bar{q}$ satisfying the last requirement is called D-preceding Q_0), and goal programming approach, where $\bar{q} \in Q_0$. In fact, observe that if $Q_0 \subset \bar{q} + D$, then $q - \bar{q} \in D \subseteq D^*$ for all $q \in Q_0$, and $(q - \bar{q})^{D^*} = q - \bar{q}$; thus, function (8a) takes the form $s(\bar{q},q) = (\rho - 1)\|q - \bar{q}\|^2$ and we *minimize* the distance from point $\bar{q}$ to Q_0, see Figure 3a. If $\bar{q} \in Q_0$ is attainable, then there are always points $q \in Q_0$ such that $q \in \bar{q} - D$, $(q - \bar{q})^{D^*} = 0$, and $s(\bar{q},q) = -\|q - \bar{q}\|^2$, see Figure 3b. Now, minimizing the minus norm or *maximizing* the norm of the objective improvement $q - \bar{q}$, subject to the constraint $q - \bar{q} \in -D$ is a variant of goal-programming: we would like to get the best point ($\hat{q}_\infty$ in Figure 3b) we can once the aspiration levels are satisfied. But the basic property of the scalarizing function (8) or (8a) is that the additional constraint $q - \bar{q} \in -D$ need not be treated as a hard constraint; its violation is expressed by the penalty term $\rho\|(q - \bar{q})^{D^*}\|^2$, as a soft constraint. Even if the aspiration levels $\bar{q}$ are slightly violated (depending on the penalty coefficient ρ, see Figure 3b) at a minimal point $\hat{q}$ of $s(\bar{q},q)$, the point $\hat{q}$ is D-minimal. And, finally, if neither $\bar{q} \in Q_0$ nor $Q_0 \subset \bar{q} + D$, see Figure 3c, then the known approaches could not use the information contained in $\bar{q}$, whereas the minimization of function (8) and (8a) still results in a D-minimal point.

Thus, *any* desirable reference objective point $\bar{q}$ can be used to determine a corresponding D-minimal point $\hat{q}$. The latter depends clearly not only on reference objective $\bar{q}$, but also on the penalty coefficient ρ and the particular norm chosen (or on the scaling of separate objectives). But this dependence has only technical character: we do not assume that a scalarizing function of the form (8) or (8a) represents *the utility function* of a given decision maker, we rather use this scalarizing function to approximate locally his preferences (and his utility function, if he actually has one) via an interactive procedure, through asking him questions he understands well. An illustration of such a procedure is represented in Figure 3d. The corresponding question is: "You have asked us to attain objective levels $\bar{q}_i = (\bar{q}_i^1, \bar{q}_i^2, \ldots)$. The best we can do under the limitations of our model is $\hat{q}_i = (\hat{q}_i^1, \hat{q}_i^2, \ldots)$. Do you accept this, or would you like to modify your desired levels to some $\bar{q}_{i+1} = (\bar{q}_{i+1}^1, \bar{q}_{i+1}^2, \ldots)$? In the latter case, please specify new desired levels." Obviously, this procedure can have many variants: the analyst can respond with more than one $\hat{q}_i$ to a given $\bar{q}_i$ by varying the coefficient ρ, or the norm, or even by applying specially designed variations $\Delta\bar{q}_i$ in order to present the decision maker with more than one alternative. But the basic idea remains the same: to ask the decision maker about aspiration levels and not about preferences.

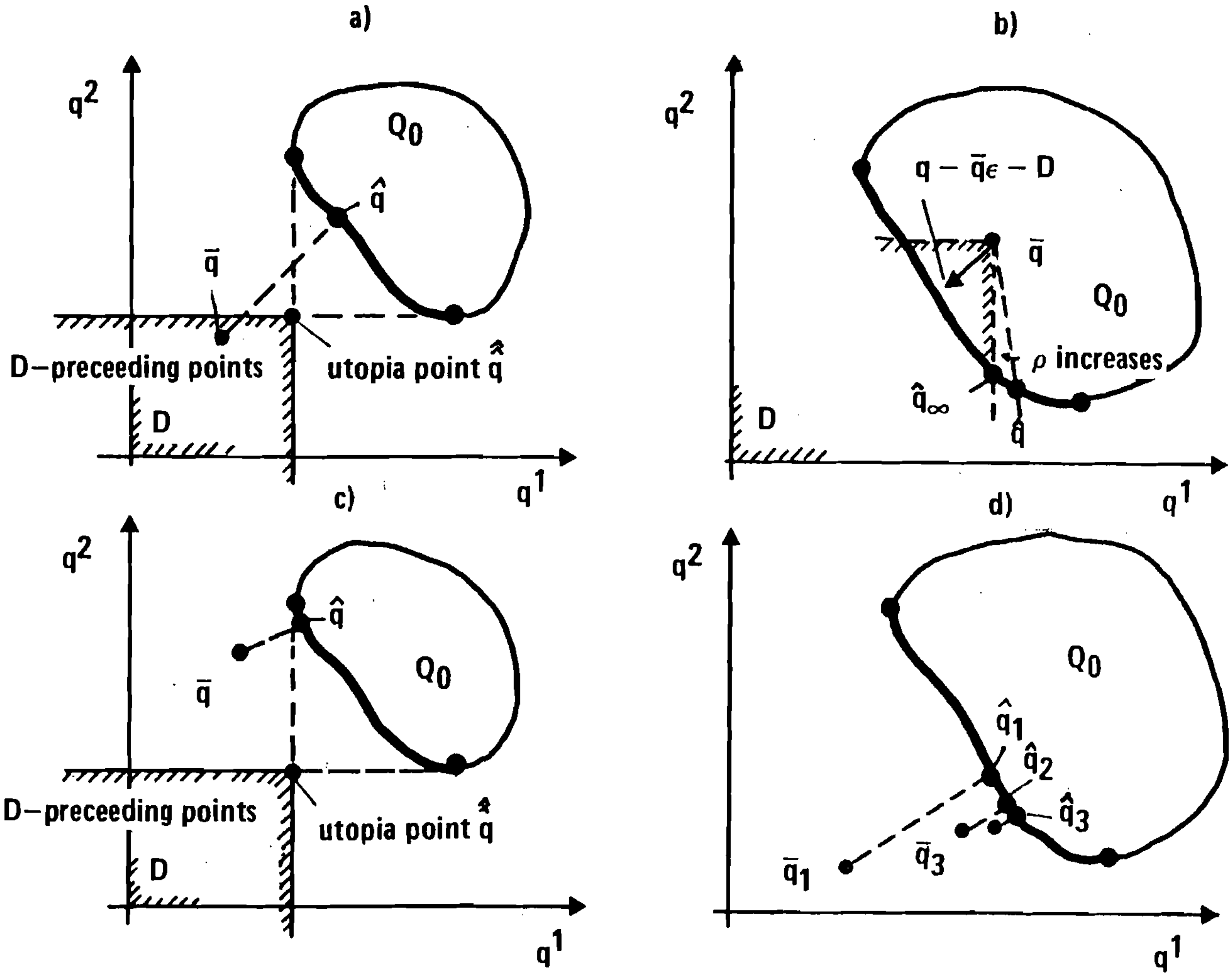

FIGURE 3. MINIMAL POINTS OF THE PENALTY SCALARIZING FUNCTION (8):
a) WHEN $\bar{q}$ IS D-PRECEDING Q_0; b) WHEN $\bar{q}$ IS ATTAINABLE;
c) WHEN q IS NEITHER D-PRECEDING NOR ATTAINABLE;
d) AN INTERACTIVE PROCEDURE

Necessary Conditions, Relation to Weighting Coefficients, and Existence of Multicriteria Optimal Solutions

The scalarizing function (8) or (8a) has other useful properties. The most important one is that of order-approximation: the level set $S_0 = \{q \in G : s(\bar{q},q) \leq 0\}$ approximates the set of improvement $\bar{q} - D$ from above and arbitrarily closely for sufficiently large ρ - see Figure 4a. More precisely, the following lemma holds:

Lemma 3 (Wierzbicki 1977, [23]). *Denote* $D_\varepsilon = \{q \in G : \text{dist}(q,D) = \|q^{-D^*}\| < \varepsilon\|q\|\}$. *For arbitrarily small* ε, *choose* $\rho > \varepsilon^{-2}$. *Then the level set* S_0 *of the function (8) or (8a) satisfies the following relation:*

$$(9) \qquad \bar{q} - D \subset S_0 = \{q \in G : s(\bar{q},q) \leq 0\} \subset \bar{q} - D_\varepsilon \quad .$$

From this lemma, the following necessary condition of D-minimality can be easily deduced:

Lemma 4 (General Necessary Condition of Multicriteria Optimality). *If* G *is a Hilbert space with a positive cone* $D \subseteq D^*$, *and if* $\hat{q}$ *is a* D_ε*-minimal point of* $Q_0 = Q(E)$ *(that is, if* $(\hat{q} - \tilde{D}_\varepsilon) \cap Q_0 = \phi$ *with* $\tilde{D}_\varepsilon = D_\varepsilon \setminus (D_\varepsilon \cap -D_\varepsilon)$ *and* D_ε *defined as in Lemma 3), then*

$$(10) \qquad \min_{q \in Q_0} s(\hat{q},q) = 0$$

where $s(\hat{q},q)$ *is defined as in (8) or (8a) with* $\rho > \max(1,\varepsilon^{-2})$ *and the minimum in (10) is attained at* $q = \hat{q}$. *Moreover, if* $\hat{q} \in Q_0$ *is attainable but not* D_ε*-minimal, then* $\min_{q\in Q_0} s(\hat{q},q) < 0$. *If* $\hat{q} \notin Q_0$ *is not attainable, then* $\min_{q\in Q_0} s(\hat{q},q) > 0$.

In contrast to the known necessary conditions of multicriteria optimality via weighting coefficients λ, Lemma 4 is easily applicable and valid even for nonconvex sets Q_0 of attainable objectives. Lemma 4, in fact, corresponds to supporting the set Q_0 at $\hat{q}$ by the set S_0 contained in the cone $\hat{q} - D_\varepsilon$, while the known necessary conditions of multicriteria optimality correspond to supporting the set Q_0 at $\hat{q}$ by a hyperplane, cf. Figure 4a,b.

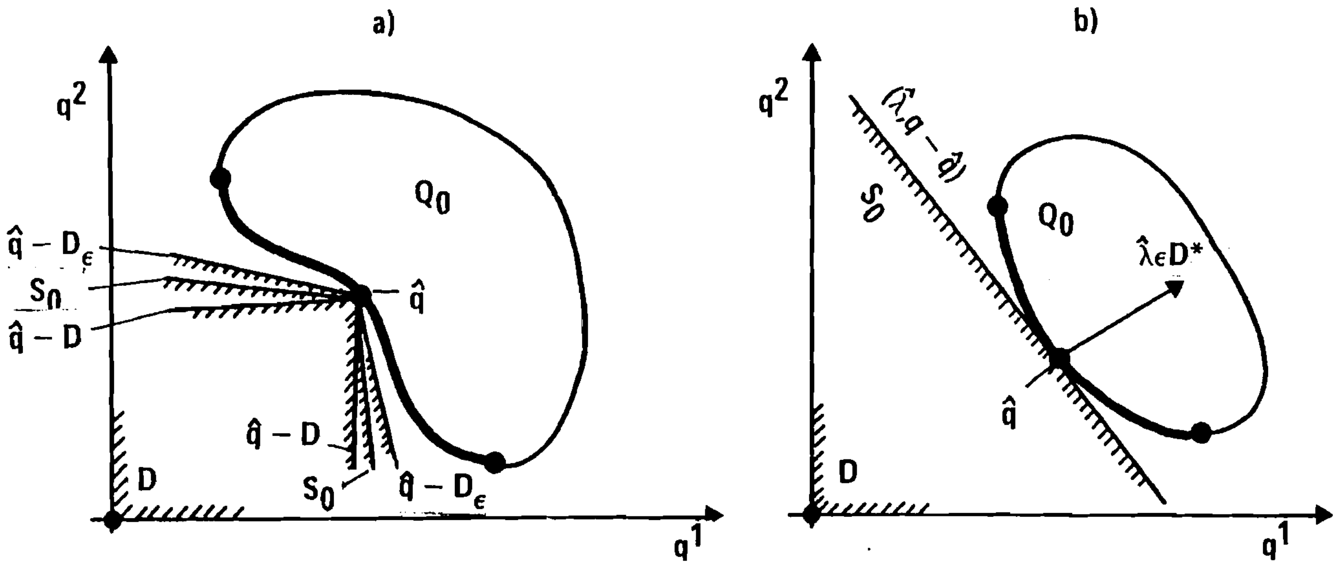

FIGURE 4. NECESSARY CONDITIONS OF MULTICRITERIA OPTIMALITY:
a) GENERAL CASE, WITH THE USE OF PENALTY SCALARIZING FUNCTIONS;
b) CONVEX CASE, WITH THE USE OF WEIGHTING COEFFICIENTS $\hat{\lambda}$.

It is also interesting to note that, if the reference objective $\bar{q}$ is not D-minimal, the corresponding minimal point $\hat{q}$ of the scalarizing function (8a) defines uniquely a related vector $\hat{\lambda}$ of weighting coefficients:

Lemma 5 (A Posteriori Determination of Weighting Coefficients, [25]). *Suppose the assumptions of Lemma 2 are satisfied and let $\hat{q}$ be a minimal point of the function (8a), $\hat{q} \neq \bar{q}$. Suppose Q_0 is locally convex in a neighborhood of $\hat{q}$. Then:*

$$(11) \qquad \hat{\lambda} = \frac{\bar{q} - \hat{q} + \rho(\hat{q} - \bar{q})^{D^*}}{\|\bar{q} - \hat{q} + \rho(\hat{q} - \bar{q})^{D^*}\|}$$

is a (normalized) vector of weighting coefficients at $\hat{q}$, that is, the set $\tilde{S}_0 = \{q \in G : \langle\hat{\lambda}, q - \hat{q}\rangle \leq 0\}$ supports locally the set Q_0 at $\hat{q}$.

Another result of rather theoretical importance, related to the notion of reference objectives, is the following simple lemma:

Lemma 6 (Sufficient Conditions for the Existence of Multicriteria Optimal Solutions). *Suppose there exists a reference objective* $\bar{q}$ *such that the set* $(\bar{q}-D)\cap Q_0$ *is nonempty and (weakly) compact. Then there exist* D-*minimal points* $\hat{q}$ *of the set* Q_0.

This lemma has been given in [22] under the additional assumption that the cone D* has nonempty quasi-interior $\overset{\circ}{D}{}^* = \{q^* \in G : \langle q^*,q\rangle > 0 \ \forall q \in \tilde{D}\}$, and was proved via consideration of the linear form $\langle \lambda,q\rangle$, $\lambda \in \overset{\circ}{D}{}^*$. But we can omit the additional assumption, since the function (8a) is weakly lower semicontinuous, see [24], and thus has a minimum in $(\bar{q}-D)\cap Q_0$ under the assumptions of the lemma. This minimum is a D-minimal point of the set $(\bar{q}-D)\cap Q_0$, hence also of the set Q_0.

Practical Forms of Penalty Scalarizing Functions

If G is finite-dimensional with the Euclidean norm, $G = E^n$, and $D = E^n_+$, then the penalty scalarizing function (8a) takes the form:

$$(12) \qquad s(\bar{q},q) = -\sum_{i=1}^{n}(q^i-\bar{q}^i)^2 + \rho\sum_{i=1}^{n}(\max(0,q^i-\bar{q}^i))^2$$

which might be convenient for nonlinear dependence of q^i on the decision variables $x \in E_0$, but is not convenient for multicriteria linear programming problems. However, penalty scalarizing functions based on other norms in R^n, that is, the sum of absolute values norm:

$$(13) \qquad s(\bar{q},q) = -\sum_{i=1}^{n}|q^i-\bar{q}^i| + \rho\sum_{i=1}^{n}\max\ (0,q^i-\bar{q}^i)$$

or the maximum (Chebychev) norm

$$(14) \qquad s(\bar{q},q) = -\max_i|q^i-\bar{q}^i| + \rho\max_i\ (0,q^i-\bar{q}^i)$$

possess almost all properties of the function (12): if $\rho > 1$, then the function (13) is strictly order-preserving in q for any $\bar{q}$, and the function (14) is order-preserving (hence her minimal points are Pareto-minimal except in some degenerate cases). These functions are also order-approximating, see [25]. If the dependence of q^i on decision variables $x \in E_0$ is linear, then the minimization of functions (13), (14) can be reduced after typical transformations into linear programming problems. For practical applications of reference objectives in multi-criteria linear programming, a combination of functions (14) and (15) might be also useful, see [14].

Another practical form of penalty scalarizing functions is related to a typical procedure in goal programming, where one of the objectives is minimized, subject to variable attainable levels of aspiration for other objectives treated as constraints. The use of penalty scalarizing functions results in a more universal procedure of this type, since the assumed levels for other objectives do not necessarily have to be attainable when using penalty terms. To represent this method, it is necessary to split the space of objectives in a Cartesian product of the space R^1 of values of the first objective, and a space G_r for other objectives, $G = R^1 \times G_r$, with $D = R^1_+ \times D_r$ and $q = (q^1, q_r)$. Then the corresponding penalty scalarizing function is:

$$s(\bar{q},q) = q^1 - \bar{q}^1 + \rho \| (q_r - \bar{q}_r)^{D^*_r} \| \tag{15}$$

or, if differentiability is important:

$$s(\bar{q},q) = q^1 - \bar{q}^1 + \tfrac{1}{2}\rho \| (q_r - \bar{q}_r)^{D^*_r} \|^2 \quad . \tag{15a}$$

If the space G_r is Hilbert and $D_r \subseteq D^*_r$, then the functions (15), (15a) are order-preserving for any $\rho > 0$ and the function (15) is order-approximating (to obtain the order approximation property in the function (15a), one had to square also $q^1 - \bar{q}^1$). The reference level $\bar{q}^1$ matters actually only in the order-approximation property since it does not influence the minimum of the functions (15), (15a). The reference objective $\bar{q}_r \in G_r$ is not necessarily attainable and ρ can be small, provided it is positive; nevertheless, each minimal point of the functions (15), (15a) is a D-minimal point.

If $G_r = R^{n-1}$ and $D_r = R_+^{n-1}$, then any norm can be used in (15), (15a). The functions:

$$s(\bar{q},q) = q^1 - \bar{q}^1 + \rho\left(\sum_{i=2}^{n} (\max(0,q^i - \bar{q}^i))^2\right)^{\frac{1}{2}} \tag{16}$$

$$s(\bar{q},q) = q^1 - \bar{q}^1 + \tfrac{1}{2}\rho \sum_{i=2}^{n} (\max(0,q^i - \bar{q}^i))^2 \tag{16a}$$

$$s(\bar{q},q) = q^1 - \bar{q}^1 + \rho \sum_{i=2}^{n} \max\ (0,q^i - \bar{q}^i) \tag{17}$$

are strictly order-preserving, whereas (16) and (17) are also order-approximating, and the function

$$s(\bar{q},q) = q^1 - \bar{q}^1 + \rho \max_{i\geq 2} \max\ (0,q^i - \bar{q}^i) \tag{18}$$

is order-preserving and order-approximating, see [25]. All these functions actually express a simple approach to goal programming: treat the objectives $q^2,\ldots,q^n$ as constraints, given aspiration levels $\bar{q}^2,\ldots,\bar{q}^n$, and introduce penalty components for them. But new, compared to typical goal programming, is the fact that $\bar{q}^2,\ldots,\bar{q}^n$ need not be attainable and that the penalty coefficient ρ need not be increased to infinity, nor other iterations on penalty terms need to be performed: even if some or all of the constraints $q^2 \leq \bar{q}^2,\ldots,q^n \leq \bar{q}^n$ are violated, all minimal points of the functions (16), (16a), (17), (18) are Pareto-minimal.

Convergence of an Interactive Procedure of Multicriteria Optimization with Variable Goals

Consider now a practical interactive procedure for choosing a Pareto-minimal point, where the actual decisions are made by a decision maker and the mathematical model of a given problem and the optimization techniques serve only as a tool to help him to recognize quickly a relevant part of the Pareto-minimal set.

At the beginning, the decision maker is presented with all the information about the model of the problem he desires -- for example, with the minimal levels of objective functions when minimized separately, and with the corresponding decisions. After that, he is asked to specify the vector of the desired levels for all objective functions, $\bar{q}_0 = (\bar{q}_0^1,\ldots,\bar{q}_0^n) \in R^n$ (only the finite-dimensional case is considered here, although generalizations to the infinite-dimensional case are possible and even have applicational value).

For each desired objective vector $\bar{q}_i$, the mathematical model and the optimization technique respond with:

1) The Pareto-minimal attainable objective vector $\hat{q}_i$, obtained through a minimization of the function (12), and the corresponding decision variable levels (any other penalty scalarizing function from the previous paragraph can also be used, depending on the particular nature of the model);

2) n other Pareto-minimal attainable objective vectors $\hat{q}_{i,j}$, $j = 1,\ldots,n$, obtained through minimization of the function (12) with perturbed reference points:

$$\bar{q}_{i,j} = \bar{q}_i + \alpha d_i e_j ; \quad e_j = (0,\ldots 1_j,\ldots 0) ; \quad d_i = \|\bar{q}_i - \hat{q}_i\| ; \quad \alpha \in (0;1] \tag{19}$$

where d_i is the distance between the desired objective vector $\bar{q}_i$ and the attainable one $\hat{q}_i$, e_j is the jth unit basis vector, and α is a scalar coefficient. Only the case $\alpha = 1$ is considered in the sequel, which corresponds to the widest-spread additional information for the decision maker and is also more difficult to obtain convergence of the procedure.

To obtain any additional information at the beginning of the procedure, the decision maker can change $\bar{q}_0$ several times (without counting it as iterations, i is kept equal 0) and analyze the responses. Once he is ready for "real bargaining", he specifies a desired objective vector $\bar{q}_1$, $i = 1$, and the iteration count begins. Now his modifications of the desired vector to $\bar{q}_{i+1}$ from $\bar{q}_i$ are limited by the responses $\hat{q}_{i,j}$ corresponding to $\bar{q}_{i,j}$ through two requirements:

$$(20) \qquad \bar{q}_{i+1} - \bar{q}_i \in S_i = \left\{ \Delta q \in R^n : \Delta q = \sum_{j=1}^{n} \eta_j (\hat{q}_{i,j} - \bar{q}_i), \eta_i \geq 0, \sum_{i=1}^{n} \eta_i \leq 1 \right\}$$

$$(21) \qquad \| \bar{q}_{i+1} - \bar{q}_i \| \geq \beta d_i = \beta \| \hat{q}_i - \bar{q}_i \| \quad ; \qquad \beta \in (0;1]$$

where β is a prespecified parameter. The requirement (21) states that the decision maker has to move at least some part of the distance to the Pareto set, the requirement (20) limits his directions of movement to the simplex spanned by $\bar{q}_i$ and $\hat{q}_{i,j}$. Actually, the decision maker should not be bothered by technicalities (20), (21); it is sufficient that he is informed about them and, after he has specified any $\bar{q}_{i+1}$, a complementary automatic procedure projects $\bar{q}_{i+1} - \bar{q}_i$ on S_i to satisfy (20) and adjusts its length to satisfy (21), if necessary.

The above procedure and limitations of the adjustments of the desirable objective vector $\bar{q}_i$ are depicted in Figure 5.

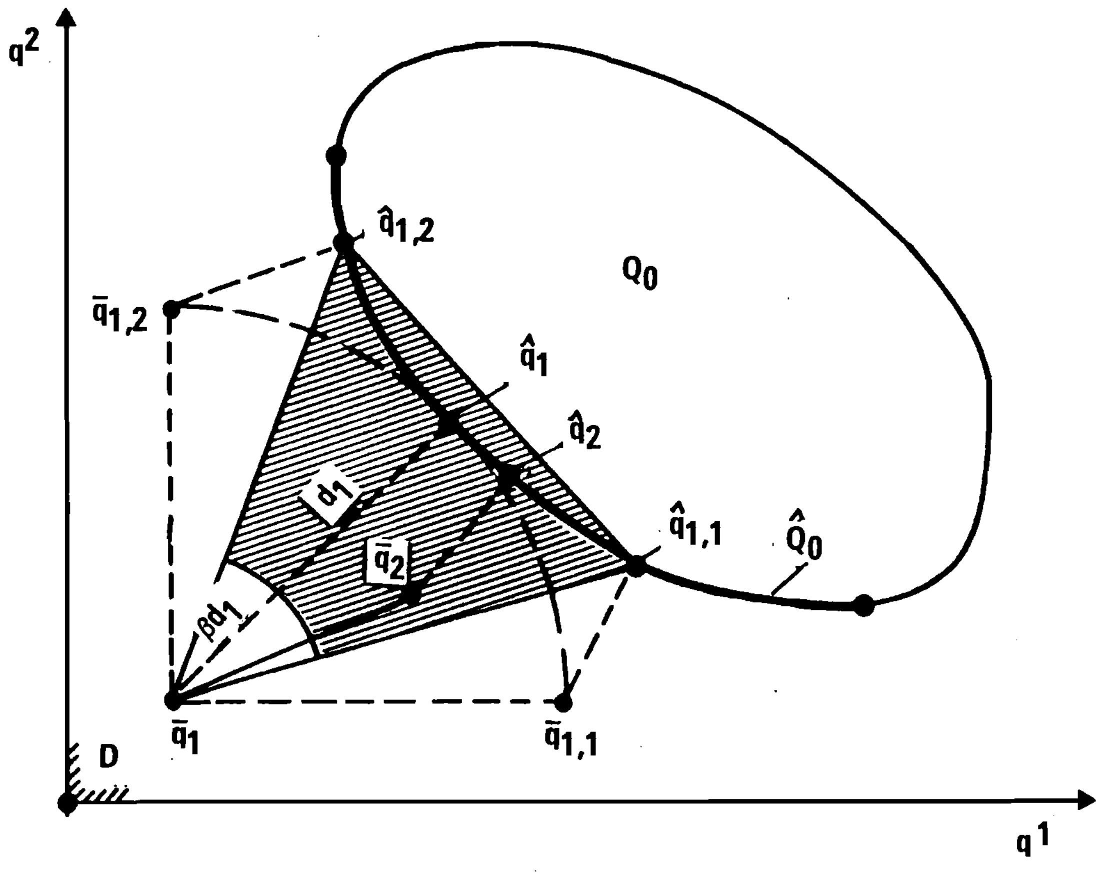

FIGURE 5. ILLUSTRATION OF THE INTERACTIVE PROCEDURE OF MULTIOBJECTIVE OPTIMIZATION. SHADED REGION DENOTES THE SET OF ADMISSIBLE $\bar{q}_2$.

It is clear from Figure 5 that, once the decision maker has specified $\bar{q}_1$, he can usually obtain from this point only a limited part of the Pareto-minimal set Q_0. This is both weakness and strength of the procedure. The weakness can be compensated by the initial, exploratory part of the procedure, where the decision maker can gather as much information as he wishes. The strength consists precisely of the limitation of the procedure to the region of interest for the decision maker. Observe that this region would decrease if α were smaller than 1.

The conditions of the convergence for this procedure are given by the following lemma:

Lemma 7 (Convergence of the Interactive Multicriteria Optimization Procedure). *Suppose the set of attainable objectives* Q_0 *is convex,* $G = E^n$, $D = E^n_+$ *(the norm used in scalarization is Euclidean). Then, for any* $\alpha, \beta \in (0;1]$, *the procedure described above with requirements (19),(20), (21) is convergent, that is,* $\lim_{i \to \infty} d_i = 0$.

The proof of this new though not very astonishing result is given in the Appendix. The lemma can probably be proved for other than Euclidean norms in R^n. Observe that if the requirement (20) were substituted by a simpler one, for example, $\bar{q}_{i+1} - \bar{q}_i \in R^n_+$, one could devise moves for the decision maker which would result in divergence. But these moves would also be unreasonable from his point of view and, counting on his reasonability, we can simplify the requirements (20), (21), or even simply drop them asking the decision maker to move generally in the direction of the Pareto-set.

In the lemma we did not assume any preference-ordering or underlying utility function describing the behavior of the decision maker, and we did not conclude anything about the final point of the procedure, $\hat{q}_\infty = \lim_{i \to \infty} \hat{q}_i$, although the existence of such a limit is easy to prove. From a purely mathematical point of view, it would be interesting to examine under which assumptions on the decision maker's behavior we can prove that $\hat{q}_\infty$ actually maximizes (or minimizes) his utility function. From a pragmatical point of view, such an investigation would only confuse the issue since the underlying motivation of the interactive procedure is to find a compromise directly in terms of goals, not in terms of utility functions. Also, we do not expect the decision maker playing with the interactive procedure until $i \to \infty$; experiments show that he very soon accepts some $\hat{q}_i$, putting $\bar{q}_{i+1} = \hat{q}_i$ and thus stopping the procedure. At this point, he can also be informed on the trade-offs implied by his decision: weighting coefficients $\hat{\lambda}_i$ related to the point $\hat{q}_i$ can be computed from equation (11).

Observe also that the interactive procedure does not depend on the scaling or ranges for separate objective functions. Naturally, the scaling must be reasonable in order not to impede computational efficiency nor exposition of the results to the decision maker, and it is advisable to use scales that correspond to approximately equal ranges of attainable values of objective functions. But this requirement of a reasonable scaling does not imply an a priori specification of a vector of weighting coefficients, and the results are relatively invariant to the scaling transformation (after changing scales, both $\hat{q}_i$ and a posteriori determined $\hat{\lambda}_i$ -- if not normalized -- change proportionally).

3. APPLICATION AREAS OF REFERENCE OBJECTIVES AND PENALTY SCALARIZATION

Multiobjective Optimization Problems

In typical multiobjective optimization, penalty scalarization can be used not only in interactive procedures of decision making but also in analyzing possible outcomes. For example, a typical question: is a given decision Pareto-efficient, or not? - can be conveniently resolved by applying Lemma 4, while an application of weighting coefficients results in rather complicated procedures.

One must bear in mind however, that a decision that is not Pareto-efficient in the optimization model might be Pareto-efficient for the decision maker, for various reasons. First, the decision maker might consider other criteria -- for example, of aesthetical or political nature -- than those expressed by the model. Second, the decision maker might have intuitively a more precise assessment of various constraints, etc., only inadequately expressed by the model. Consequently, by looking at the optimization model only as a tool to aid the decision maker, it is possible to analyze these interesting questions further, and reference objectives are certainly better suited than weighting coefficients for such an analysis. However, much has to be done yet in this direction of research.

Another convenient application of penalty scalarizing functions in the analysis of multicriteria problems is the scanning of the Pareto set, naturally under the assumption that the number of criteria is not too large. Scalarizing functions of the type (16),(16a),(17),(18) can be used for this purpose. An example of application to control engineering, see Wierzbicki 1978 [25], shows that the use of weighting coefficients for that purpose can lead to disastrous results, while reference objectives give reliable answers. This is depicted in Figure 6.

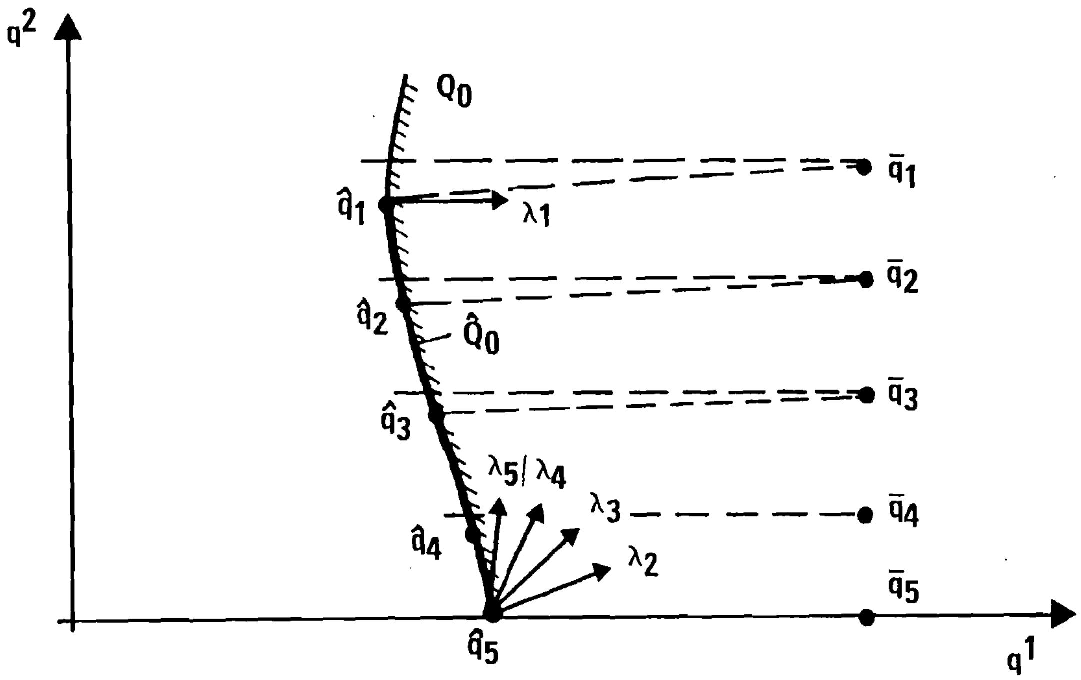

FIGURE 6. REFERENCE OBJECTIVES VERSUS WEIGHTING COEFFICIENTS WHEN SCANNING AN IRREGULARLY SHAPED PARETO - SET

Reference Trajectories

In many applications of dynamic modelling, scalar-valued objective functions do not precisely express the goals of a decision maker or a modeller. Of primary interest is often a function of time, a trajectory of the model. For example, an economist might want to compare the trajectories of inflation rates and of GNP while not being ready to average them and to use scalar indices. Thus, *a function of time is an equally reasonable goal in decision making as a scalar index*, and analysts avoided the use of functions as goals only because of the lack of appropriate techniques. However, the possibility of using reference objectives in a Hilbert space provides for an appropriate technique. This is explained in Figure 7 where, as a goal, an economist specified a desirable GNP and a reasonable inflation rate as functions of time. A model after an optimization, say, in respect to taxes, responds by attainable (and, in a sense, Pareto-optimal) functions of GNP and inflation, and the economist can modify then his reference functions in order to influence the outcomes.

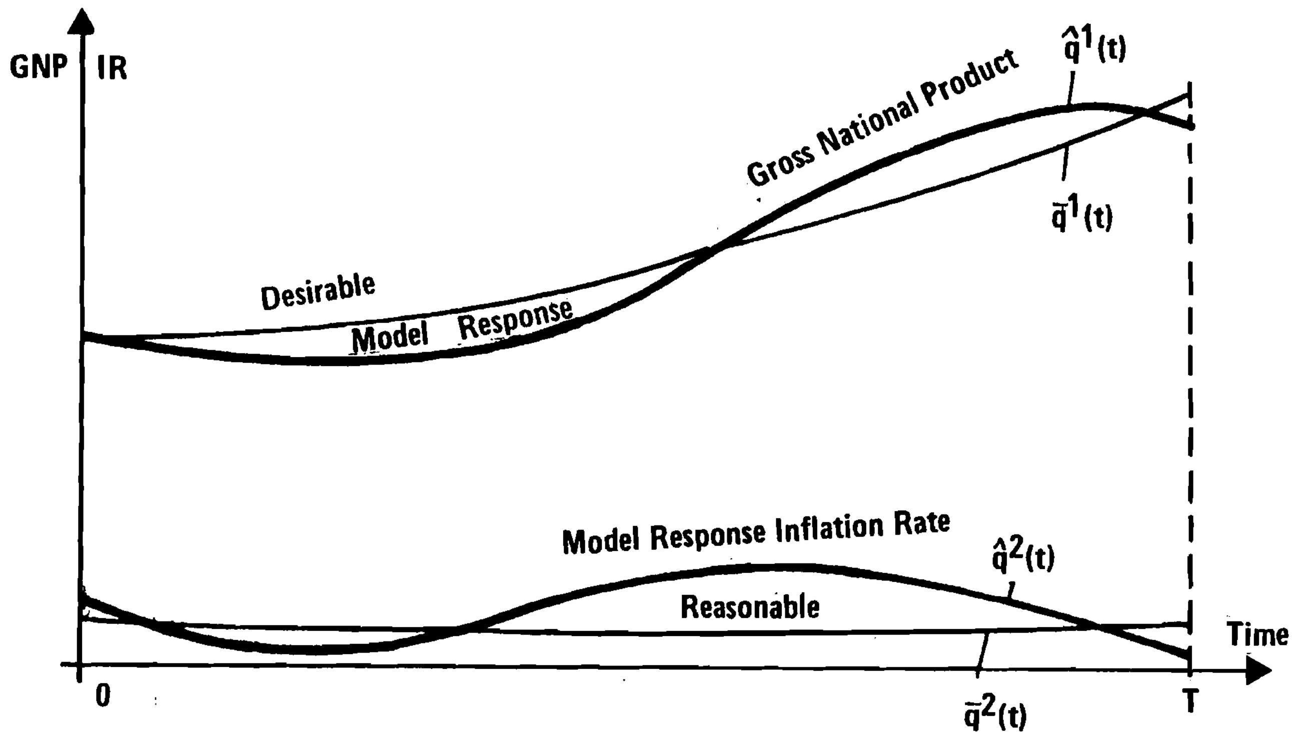

FIGURE 7. FUNCTIONS OF TIME OR TRAJECTORIES AS REFERENCE OBJECTIVES

If the dynamic model is time-continuous, as depicted in Figure 7, then we might choose, for example, the $L^2[0;T]$ space for analysis, which results in the following expression for the scalarizing function (8a):

$$(22) \qquad s(\bar{q},q) = -\int_0^T ((q^1(t) - \bar{q}^1(t))^2 + (q^2(t) - \bar{q}^2(t))^2)dt + \rho\int_0^T ((\bar{q}^1(t) - q^1(t))_+^2 + (q^2(t) - \bar{q}^2(t))_+^2)dt$$

where $(\cdot)_+ = \max(0,\cdot)$ and the change to $(\bar{q}^1(t) - q^1(t))_+$ instead of $(q^1(t) - \bar{q}^1(t))_+$ results from the fact that we maximize GNP and penalize the GNP-trajectory if it stays below the desirable level. We can also use more general spaces and other norms -- for example, the Chebychev norm -- if we take into account the results presented in Wierzbicki 1977, [23].

But most practical dynamic models are time-discrete and an economist might be interested only in $q^1(t)$ and $q^2(t)$ for $t = 0,1,\ldots,T$. Then the problem is in fact finite-dimensional and we do not need the Hilbert space formulation; all forms of penalty scalarizing functions described in previous paragraphs are applicable in such a case. On the other hand, the number of objectives $q^1(0),q^1(1),\ldots,q^1(T)$, $q^2(0),q^2(1),\ldots,q^2(T)$ might be quite large and it is convenient to think then in terms of discrete-time trajectories, not in terms of separate objectives.

The idea of reference trajectories has been applied and found useful in a study on the Finnish forestry industrial sector (Kallio and Lewandowski, 1979, [14]).

Compromises in Cooperative Games

There are many approaches to finding Pareto-equilibria or compromises in cooperative games. Motivated by the observation that, in reality, a group of decision makers prefers, first, to discuss, bargain and agree about goals, an agreement-aiding procedure based on reference points in the space of objectives has been devised.

This procedure is explained in Figure 8 in the case of two decision makers and two objective functions. Each decision maker has its "own" objective function, the levels of which however do not only depend on his decisions: $q^1 = q^1(x^1,x^2)$ and $q^2 = q^2(x^1,x^2)$. It is also assumed that this is a case of full information and each decision maker fully understands the objective and possible actions of the other one. Moreover, the choice of model decisions x^1,x^2 is delegated to an optimization procedure based on the mathematical model; the decision makers only state and try to agree on reference objective levels. Each of them is asked thus to specify his reference point $\bar{q}_{i0} = (\bar{q}^1_{i0},\bar{q}^2_{i0})$, $i = 1,2$, which expresses his judgment about his own aspiration level and that of his partner; later, these reference points are modified in an iterative procedure to $\bar{q}_{ij}$ until either $\lim_{j\to\infty} \bar{q}_{1j} = \lim_{j\to\infty} \bar{q}_{2j}$ or the negotiations break down displaying an adversary situation.

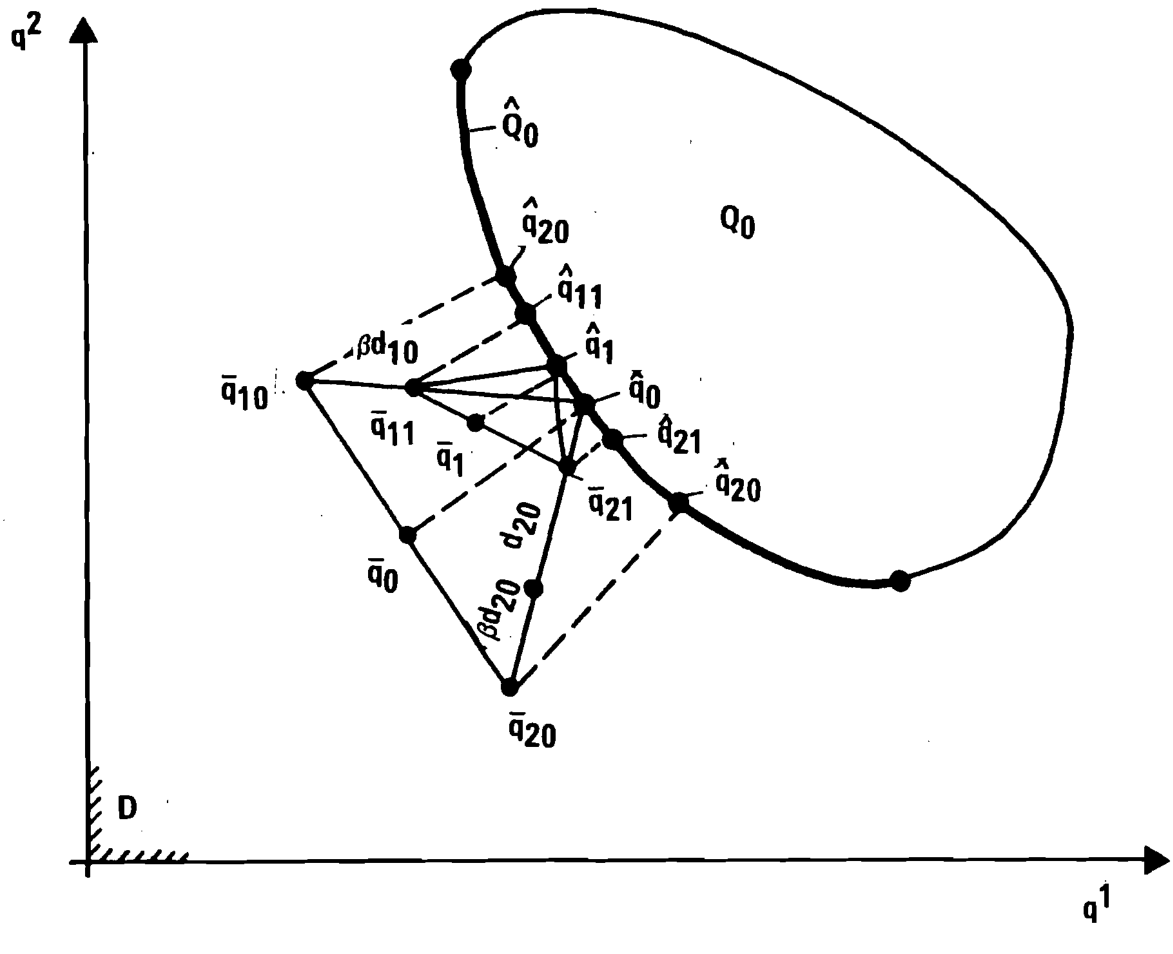

FIGURE 8. AN AGREEMENT-AIDING PROCEDURE BASED ON REFERENCE OBJECTIVES

Suppose, the first decision maker with a dominant objective to minimize q^1, has specified $\bar{q}_{10} = (\bar{q}^1_{10}, \bar{q}^2_{10})$ naturally allotting a relatively lower level for $\bar{q}^1_{10}$, his "own" objective than for $\bar{q}^2_{10}$, the one of his partner. Correspondingly, the point $\bar{q}_{20} = (\bar{q}^1_{20}, \bar{q}^2_{20})$ specified by the second decision maker has $\bar{q}^1_{20} > \bar{q}^1_{10}$ and $\bar{q}^2_{20} < \bar{q}^2_{10}$, because he is interested in minimizing his "own" objective q^2.

Since we can assume, at the beginning, nothing else than the equity of each decision maker's requirements, the agreement-aiding procedure simply determines $\bar{q}_0$ as the middle-point of the segment $[\bar{q}_{10}; \bar{q}_{20}]$ (or of a corresponding simplex in case of more decision makers) and responds through a minimization of one of the penalty scalarizing functions from previous paragraphs by a Pareto point $\hat{q}_0$ corresponding to $\bar{q}_0$ as well as by Pareto points $\hat{q}_{10}, \hat{q}_{20}$ corresponding to $\bar{q}_{10}, \bar{q}_{20}$. This way both decision makers have a proposition of compromise and information about attainable levels of objectives. The distances $d_{10} = \|\hat{q}_0 - \bar{q}_{10}\|$ and $d_{20} = \|\hat{q}_0 - \bar{q}_{20}\|$ are also determined.

Now both decision makers have to make concessions in terms of two scalars $\alpha_1, \alpha_2 \in [\beta; 1]$, where $\beta \in (0;1]$ is a prespecified minimal concession level. The modified reference points $\bar{q}_{11}$, $\bar{q}_{21}$ are determined by

$$(23) \quad \bar{q}_{1,j+1} = \bar{q}_{1,j} + \alpha_1(\hat{q}_j - \bar{q}_{1,j}); \quad \bar{q}_{2,j+1} = \bar{q}_{2,j} + \alpha_2(\hat{q}_j - \bar{q}_{2,j}).$$

Thus, both decision makers have to move in the direction of $\hat{q}_j$, at least β times the distance d_{ij}. In Figure 8, it was assumed that the first decision maker made only the minimal concession $\alpha_1 = \beta$, while the second decided to make a bigger one, $\alpha_2 > \beta$. When $\bar{q}_{1,j+1}$ and $\bar{q}_{2,j+1}$ are determined, the procedure is repeated.

The mechanism of this procedure very strongly urges both decision makers to reach an agreement. Therefore, at some stage of the procedure, one or both of the decision makers can decide if he should break the negotiations, that is, not making any further concessions. Two further possibilities can be envisaged:

- -- either both decision makers agree on entering negotiations with modified reference points $\bar{q}_{i0}$;
- -- or an additional influence-revealing procedure is called for. This procedure can consist for example of a one-sided game of the dissident decision maker "against the computer", where the dissident decision maker defines his decisions, say x_1, on his own while the optimization procedure tries to represent the other (or others) decision maker and choose x_2 to obtain the best bargain. Various rules concerning the sequence of decision making and the use of outcomes in restarting negotiations can be introduced here.

Much has to be done yet in investigating various aspects of the agreement-aiding procedure. An application to the study of the Finnish forestry industrial sector has given interesting results -- see Kallio and Lewandowski, 1979. [14].

Other Applications and Possible Extensions

There are several pragmatical problems in applications of optimizing techniques that call for the use of reference points. Many single-objective optimization models represent problems in which other objectives occur but are deemed not very important until the single-objective solution is presented to the decision maker and found unacceptable for various reasons. A classical example of such a situation is the application of dynamic linear programming problems for economic planning. Since the solutions of linear programming problems correspond to vertices of a simplex, some crucial decision variables often tend to take on-off character; exaggerating, the "optimal" solution can be often interpreted as "first invest all GNP for two years and do not consume, then do not invest for three years and consume all GNP." Clearly, such an "optimal" solution would be never accepted by a decision maker; the tendency of linear programming models to produce such solutions is one of the reasons of a wide-spread critique of using optmization models at all. But there is also the explanation that linear models, however convenient in handling, describe real problems inaccurately. The remedy is not necessarily to introduce nonlinearities into the model; sometimes even a linear model can be adequate if constructed accurately to express actual goals and constraints.

An introduction of other optimization criteria being accounted for by weighting coefficients does not solve the problem; the weighted objective function remains linear and tends again to produce on-off solutions. Therefore, a widely used approach is to introduce additional constraints, limiting the set of admissible solutions. This is in fact equivalent to goal programming: aspiration levels for other criteria are determined and used as constraints, for example, by demanding that investments and consumption each year should not be less than given levels. But this approach has all drawbacks of goal programming: the aspiration levels must be attainable in order not to make the set of feasible solutions empty, and it is difficult therefore to devise interactive procedures for decision makers setting aspiration levels.

The natural remedy is then to use penalty-scalarizing approach. If any reference trajectory is determined -- for example, concluded from consumption and investments from the past -- then the problem might be formulated as optimizing the original objective criterion plus a penalty term for not attaining the reference trajectory. This is equivalent to the use of a scalarizing function of the type (16), (16a), (17) or (18); as mentioned before, problems with objective functions of the type (17), (18) can be reformulated back to linear programming problems.

Another possible extension of the use of the reference objective approach is the problem of risk evaluation. The typical utility function approach to risk evaluation often fails in applications; see, for example, the paper of Tversky in Bell, Keeney and Raiffa 1977, [2]. One of the reasons is that decision makers seem to intuitively evaluate entire probability distributions instead of just expected utility. But then entire probability distributions can be used as goals either in Hilbert space, if continuous, or in R^n, if discretized.

Consider a problem of standard determination where a given set of standards -- for example, on air pollution -- determines conditional probabilities of hazards -- for example, of mortality and morbidity. Each standard level corresponds also to some costs, and a procedure of standard determination is supposed to compare costs to hazards. Many economic, social and moral issues are involved in the comparison, making it extremely difficult. But the point is that there are alternatives to the classical utility approach of the problem. One of the alternatives is a direct evaluation of probability distributions and costs by determining a desirable shape of the distribution, an acceptable level of the cost, using them as reference objectives in a penalty scalarization, and changing the reference objectives to an interactive procedure.

4. CONCLUSIONS

The motivation of this paper is that of a toolmaker. Systems analysis and decision science can be compared to tools that are applicable to complex problems of modern society. Tools must be checked against real problems. If there are complaints about the efficiency of tools, then the toolmaker should reexamine and redesign them. When he is doing so and finds a new principle of tool construction he should be satisfied -- but not to the extent of forgetting that he is constructing tools which must again be checked in practice and further developed.

While the basic principles of the use of reference objectives and penalty scalarization are rather well developed, as represented in the first chapter of the paper, there are still many tasks to which they can be applied.

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APPENDIX: PROOF OF LEMMA 7

The nature of the lemma calls for a geometrical proof. Consider the intersection of the positive orthant E^n_+, where the origin of the space is shifted to $\bar{q}_i$, with the ball of radius d_i. The ball is then tangent to the convex set Q_0 and to the Pareto set $\hat{Q}_0$ -- cf. Figure 5.

Choose such j that the angle between the vectors $\bar{q}_{i,j} - \bar{q}_i$ and $\hat{q}_i - \bar{q}_i$ is maximal and consider the two-dimensional linear manifold spanned by these vectors (Figure 9).

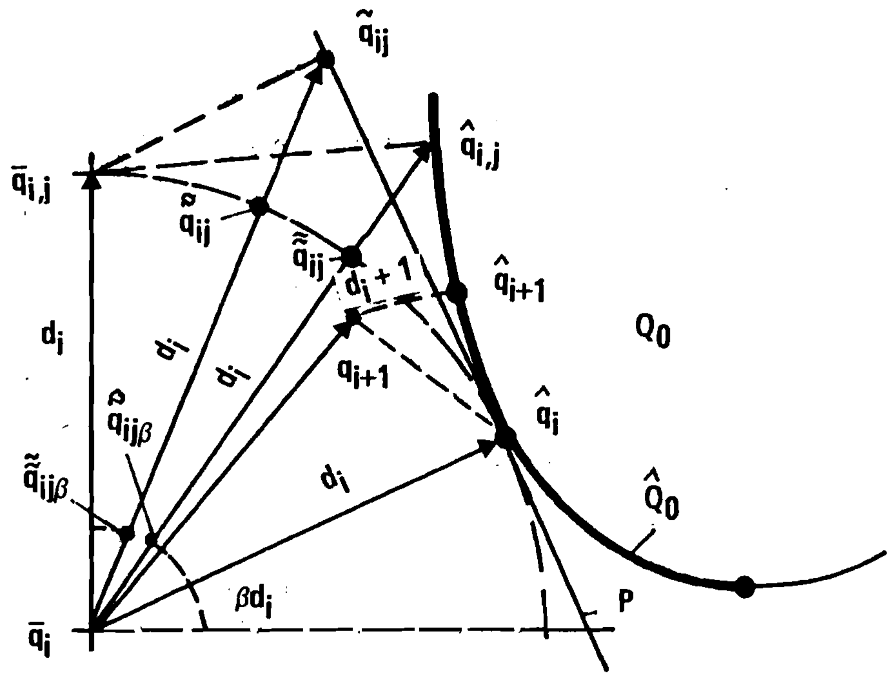

FIGURE 9. A GEOMETRICAL CONSTRUCTION FOR THE PROOF

In this plane, construct a line P separating the cut of the ball with the origin at $\bar{q}_i$ and radius d_i from the cut of the convex set $\hat{Q}_0$. Project the point $\bar{q}_{i,j}$ on this line to obtain $\tilde{q}_{i,j}$. Construct points $\tilde{\tilde{q}}_{ij}$ and $\hat{\tilde{q}}_{ij}$ on the vectors $\tilde{q}_{ij} - \bar{q}_i$ and $\hat{q}_{ij} - \bar{q}_i$ in such a way that their distance from $\bar{q}_i$ is also d_i.

Now, since Q_0 is convex, the angle between the vectors $\tilde{q}_{ij} - \bar{q}_i$ and $\hat{q}_i - \bar{q}_i$ is greater than the angle between $\hat{q}_{ij} - \bar{q}_i$ and $\hat{q}_i - \hat{q}_i$, which in turn was assumed to be the greatest for all j. Moreover, $\tilde{q}_{ij} - \bar{q}_{ij}$ is parallel to $\hat{q}_i - \bar{q}_i$, $\tilde{q}_{ij} - \hat{q}_i$ is orthogonal to them, and $\bar{q}_{ij} - \bar{q}_i$ and $\hat{q}_i - \bar{q}_i$ are both of the length d_i. Therefore, the length of $\tilde{q}_{ij} - \hat{q}_i$ is smaller than d_i and the angle between $\tilde{q}_{ij} - q_i$ and $\hat{q}_i - \bar{q}_i$ smaller than $\pi/4$ radians. Thus, the distance between $\tilde{\tilde{q}}_{ij}$ and $\hat{q}_i$ can be estimated by

$$\text{(A1)} \qquad \|\tilde{\tilde{q}}_{ij} - \hat{q}_i\| \le \sqrt{2-\sqrt{2}}\, d_i$$

which is the length of a secant for a circle, corresponding to the angle $\frac{\pi}{2}$.

Now, consider the point $\tilde{\tilde{q}}_{ij\beta}$ distant βd_i from $\bar{q}_i$ on the vector $\tilde{q}_{ij} - \bar{q}_i$. Again, through purely geometrical consideration the distance from $\tilde{\tilde{q}}_{ij\beta}$ to $\hat{q}_i$ can be estimated by:

$$\text{(A2)} \qquad \|\tilde{q}_{ij\beta} - \hat{q}_i\| \le \sqrt{1+\beta-\sqrt{2}\beta}\, d_i \le \left(1 - \frac{\sqrt{2}-1}{2}\beta\right) d_i \quad .$$

However, if the decision maker chooses his next $\bar{q}_{i+1}$ according to the rules (20), (21), then the distance $d_{i+1} = \|\hat{q}_{i+1} - \bar{q}_{i+1}\|$ clearly satisfies the relations (see Figure 9):

$$\text{(A3)} \quad d_{i+1} \le \|\hat{q}_i - \bar{q}_{i+1}\| \le \max\left(\|\hat{\tilde{q}}_{ij} - \hat{q}_i\|, \|\hat{\tilde{q}}_{ij\beta} - \hat{q}_i\|\right)$$

$$\le \max\left(\|\tilde{\tilde{q}}_{ij} - \hat{q}_i\|, \|\tilde{\tilde{q}}_{ij\beta} - \hat{q}_i\|\right) \le d_i \max\left(\sqrt{2-\sqrt{2}}, \left(1 - \frac{\sqrt{2}-1}{2}\beta\right)\right) .$$

Now, for each $\beta \in (0;1]$ we can choose a scalar $\gamma \in \left[1 - \frac{\sqrt{2}-1}{2}\beta ; \frac{3}{2} - \frac{\sqrt{2}}{2}\right] \subset (0;1)$ such that

$$\text{(A4)} \qquad d_{i+1} \le \gamma d_i \quad .$$

Therefore, $\lim_{i\to\infty} d_i = 0$. It was assumed in the proof that $\bar{q}_i \notin Q_0$. However, the proof for the case $\bar{q}_i \in Q_0$ can be easily supplemented and is, moreover, unnecessary since the decision maker has exploratory moves and will always find $\bar{q}_0 \notin Q_0$ resulting in $\bar{q}_i \notin Q_0$ for all i until the iterations stop.