A Multi-objective Decomposition Algorithm Based in Variable Space Diversity

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***Abstract*—In the last decade the presence of Multi-Objective Evolutionary Algorithms (MOEAs) has grown remarkably. Mostly, the MOEAs incorporate components to directly improve the quality in the objective space, which implicitly involves the decision variable space. Nevertheless, to attain even better quality solutions, the decision variable space should be explicitly taken into account as part of the design process. In this way the balance between exploration and intensification could be directly induced. A quite popular strategy that provides such balance yields in the incorporation of a novel replacement operator. This operator explictly promotes diversity in the decision variable space through several stages along the optimization process. In this work a decomposition based MOEA (VSD-MOEA/D), which incorporates this replacement operator, is proposed. In addition, through several experiments, is empirically proved that as the number of obejctives is increased, the diversity degree required in the decision variable space should be less. Even more, our proposal attained better results than the state-of-the-art MOEAs increasing the number of decision variables.**

***Index Terms*—Evolutionary Algorithms, Multi-objective, Decomposition, Diversity.**

I. Introduction

MULTI-OBJECTIVE Evolutionary Algorithms (MOEAs) are one of the most popular approaches to deal with Multi-objective Optimization Problems (MOPs). MOEAs are usually employed in problems whose formulation is complicated or inaccessible. A continuous box-constrained minimization MOP involves two or more conflicting objectives and are defined in Eq. (1)

$$\begin{aligned} &min \quad F(x) = (f_1(x), ..., f_M(x)) \\ &s.t. \quad x \in \Omega. \end{aligned} \tag{1}$$

where $\Omega \subseteq \Re^D$ denotes the decision space, $F : \Omega \rightarrow Y \subseteq \Re^M$ consists of M objectives and Y is the objective space. Given two solutions $x_1, x_2 \in \Re^D$ is said that x_1 dominates x_2 denoted as $x_1 \prec x_2$ if and only if $f_i(x_1) \leq f_i(x_2)$ for all $i \in \{1, ..., M\}$ and $f_i(x_1) < f_i(x_2)$ for at least one objective. A solution $F(x^*)$ is called a Pareto-optimal solution if there does not exist $F(x) \in Y$ such that $x \prec x^*$. The set of all $x^* \in Y$ is called the Pareto-optimal solution set (PS), and their image is the Pareto Front (PF). The goal of the MOEAs is to find a set of solutions that are well-distributed and converged to the PF in the objective space [1].

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The Evolutionary Algorithms (EAs) are popular meta-heuristics to deal with MOPs due its capability to approximate several solutions in a single run. In the last decade, several strategies that take into account executions in long-term have been quite successfull mainly in the most complex problems [2]. These strategies explicitly preserves the diversity in the population incorporating the stopping criterion and elapsed time to attain a properly balance between exploration and exploitation [3].

The mechanisms designed to deal with diversity have turned to be essential to attain quality solutions in single-EAs. Perhaps, one of the most critical issues of promoting diversity is that it provides a way to deal with premature convergence and stagnation. Diversity can be taken into account in the design of several components such as in the variation stage [4], [5], replacement phase [3] and/or popultions models [6].

Recently, several remarkable EAs incorporates a replacement phase, which maintains a balance between exploration and exploitation. Such transition is gradually imposed taking into account the stopping criterion and the elapsed time. Those strategies that incorporate such replacement phase has attained remarkable results, mainly in long-term executions. For instance, in combinatorial domains new best-known solutions for some well-known variants of the frequency assignment problem [2], and for a two-dimensional packing problem [7]. In addition, this principle guided the design of the winning strategy at the Second Wind Fram Layout Optimization Competition, that was held in the Genetic and Evolutionary Conference. Recently, in the case of continuous domains this replacement phase has been incorporated to Differential Evolution (DE) [8], which attained remakably superior results than the winners of the competition carried out in IEEE Congress of Evolutionary Computation (CEC) of the years 2016 and 2017. Therefore, this novel principle of incorporing the replacement phase to explicitly control the diversity has given quite good results in both discrete and continuous domains. Remarkably, the incorporation of this paradigm in EAs have allowed to discover new solutions in several NP-hard problems, which required long-term executions which tend to be even more feasible caused by the constatn growing of computational power.

The usual design of MOEAs is especifically build to attain well-spread solutions and to cover the PF (coverage), therefore some of them incorporate different mechanisms to achieve such goal. However, most of the MOEAs disregard the variable space even though those algorithms can suffer the same drawbacks raisen in single-objective space, e.g. premature convergence and stagnation. Perhaps one of the main challenges to incorporate strategies to control the diversity in variable space is that in multi-objective spaces inducing diversity in variable spaces does not guarantee diversity in the objective space. Following the goal of MOEAs where is desired to attain solutions well-spread in the objective space the MOEAs implicilty inherit some degree of diversity in the variable space, therefore convergence does not appear in the variable space. Nevetheless, the implicit diversity induced in the variable space by the objective space might not be enough an the reproduction operators could lose its exploratiory strength.

In spite of the amazing amount of MOEAs that have been developed, this paper proposes a novel and different MOEA *the Variable Space Diversity MOEA based in Decomposition* (VSD-MOEA/D), which explicitly induces diversity in the variable space through several stages to obtain a proper balance from exploration toward exploitation. Particularly, the MOEA/D-DE that attained the first place in the CEC-09 is taken into account [9]. This MOEA is transformed incorporating an original replacement phase that takes into consideration the stopping criterion and the number of function evaluations. In this way, this algorithm grants more importance to the diversity of variable space in the initial stages, and as the function evaluations evolve, it gradually grants more importance to the diversity of the objective space. Thus in the last functions evaluations the algorithm has a similar behavior than the state-of-the-art MOEAs. In addition, VSD-MOEA/D employes three populations and deals with the diveristy problems caused by the mating selection and replacement mechanisms of the MOEA/D-DE. Since that in the literature exists a broad kind of MOEAs based in decomposition the validation of our proposal is carried out taking into consideration the MOEA/D [10] and MOEA/D-DE [9] that are based in decomposition and R2-EMOA that is based in indicators. This paper clearly shows the remarkable benefits of properly taking into account the diversity of the variable space.

The rest of this paper is organized as follows. Section II provides a widely review of decomposition algorithms, diversity in EAs and related works. The VSD-MOEA/D proposal is detailed in section III. Section IV is devoted to the experimental validation of the novel proposal. Finally, conclusions and some lines of future work are given in section V.

II. Literature Review

A. Diversity in Evolutionary Algorithms

The proper balance between exploration and exploitation is one of the keys to designing a successful EAs. In the single-objective domain, it is known that properly managing the diversity of the variable space is a way to achieve this balance, and as a consequence, a large number of diversity management techniques have been devised [11]. Specifically, these methods are classified depending on the component(s) of the EA that is modified to alter how much diversity is maintained. A popular taxonomy identifies the following groups [12]: *selection-based*, *population-based*, *crossover/mutation-based*, *fitness-based*, and *replacement-based*. Additionally, the methods are referred to as *uniprocess-driven* when a single component is altered, whereas the term *multiprocess-driven* is used to refer to those methods that act on more than one component.

Among the previous proposals, the replacement-based methods have yielded very high-quality results in recent years [2], so this alternative was selected with the aim of designing a novel MOEA that explicitly incorporates a way to control the diversity of the variable space. The basic principle of these methods is to bias the level of exploration in successive generations by controlling the diversity of the survivors [2]. Since premature convergence is one of the most common drawbacks in the application of EAs, modifications are usually performed with the aim of slowing down the convergence. One of the most popular proposals belonging to this group is the *crowding* method, which is based on the principle that offspring should replace similar individuals from the previous generation [13]. Several replacement strategies that do not rely on crowding have also been devised. In some methods, diversity is considered as an objective. For instance, in the hybrid genetic search with adaptive diversity control (HGSADC) [14], individuals are sorted by their contribution to diversity and by their original cost. Then, the rankings of the individuals are used in the fitness assignment phase. A more recent proposal [2] incorporates a penalty approach to gradually alter the amount of diversity maintained in the population. Specifically, the initial phases preserve a higher amount of diversity than the final phases of the optimization. This last method has inspired the design of the novel proposal put forth in this paper for multi-objective optimization.

It is important to remark that in the case of multi-objective optimization, little work related to maintaining the diversity of the variable space has been done. The following section reviews some of the most important MOEAs and introduces some of the works that consider the maintenance of diversity of the variable space.

B. MOEAs *and Diversity in Decision Variable Space*

In the last decade few MOEAs were specifically designed to address the diversity in the decision variables space. Although that the diversity in single-objectives EAs is a matter of importance refs, in multi-objective optimization usually the diversity in the variable space is ignored. This might occurs, since the objectives are usually in conflict, therefore often is maintained a diversity level in the decision space. Also, the decision space is disregarded, since that at the end of the optimization process the quality of the solutions relies only in the objective space. In single-objective optimization, high quality solutions have been provided since that a balance between exploration and exploitation is reached through the optimization process refs. In this way, the premature convergence, which is considered as a drawback, can be avoided. Some strategies in single-objective optimization to avoid this drawback is explicitly induce a the diversity considering the criteria stop, thus at first stages the exploration levels are promoted and at the end the exploitation of the promising regions is induced. A similar issue is addressed in multi-objective problems, in such a way that the evolutionary search is stagnated, and only are explored the same region. Particularly, the idea to integrate decision space diversity into the optimization has been proposed in 1994 with the first NSGA work REF. In this last work the decision vectors are considered into the fitness sharing procedure. Thereafter, the most algorithms concentrated in the objectives space only. Alternatively, several approaches with MOPs related directly in decision space has arisen. These approaches, further as the usual MOPs, also aims to provide diverse solutions in the decision space. Principally, based in that there exists a variety of problems where the image of the Pareto Front corresponds to several distributions in the Pareto Set.

In 2003 GDEA [15] integrated diversity into the search as an additional objective. This MOEA introduced by Toffolo and Benini invoked two selection criteria, non-dominated sorting as the primary one and a metric for decision space diversity as the secondary one. In 2005, Chan and Ray [16] suggested to use two selection operators in MOEAs; one encourages the diversity in the objective space and the other does so in the decision space. They implemented KP1 and KP2, two algorithms using these two selection operators. After that, in 2008, the Omni-optimizer [17] was developed, which extends the original idea of the NSGA. Particularly, the diversity measure take both the decision and the objective space diversity into account, Omni-optimizer first uses a rank procedure, were the objective space measure is always considered first, and only if there are ties the diversity in decision space is taken into consideration. However, the drawback of this approach is that the diversity plays an inferior role and there is no possibility to change the tradeoff between the diversity measures. In 2009, were proposed the CMA-ES niching framework [18] , and the probabilistic Model-based Multi-objective Evolutionary Algorithm (MMEA)[19] . The first, extend a niching framework to include the diversity in the space diversity. The second, applies a clustering procedure in objective space and then builds a model from the solutions in these clusters. In 2010, was proposed the Diversity Integrating Hypervolume-based Search Algorithm (DIVA) [20], this algorithm introduces a method to integrate decision space diversity into the hypervolume indicator, such that these two set measures can be optimized simultaneously.

C. Decomposition-Based MOEAs

In the last decades MOEAs have gained enough popularity dealing with MOPs, given their outstanding performance a vast amount of variants have been designed. To better classify the different schemes, several taxonomies have been proposed [21]. A well known classification can be based on Pareto dominance, indicators and/or decomposition [1]. The domination-based MOEAs are based on the application of the Pareto dominance relation and techniques to promote the diversity in the objective space [22]. Similarly, the indicator-based MOEAs incorporate a measure of quality of the approximations attained by the MOEAs [23]. A recent MOEA that belongs to this category is the R2-Indicator-Based Evolutionary Multi-objective Algorithm (R2-EMOA) [24], whose performance in MOPs has been quite promising.

Finally, the decomposition based algorithms which implements scalarizing functions to transform the MOP into several single-objective optimization subproblems and are simultaneously solved in a single run. The transformation can be in several ways, some of the most known are the weighted sum approach, the weighted tchebycheff approach and the penalty-based boundary intersection approach. Therefore, a set of weight vectors defines different single-objective functions, which are optimized by the MOEA. In addition, the weight vectors are selected a priori with the aim to obtain well-spread solutions among the Pareto front, however the optimal selection of weights depends of each problem and its Pareto-Geometry.

Although that one of the most popular decomposition-based algorithms is the MOEA/D proposed by Zhang et. al [10] there are several antecedents of metaheuristics that implements the idea of decomposition for solving MOPs [25] and [26]. Particularly, MOEA/D has its origins in the cellular multi-objective genetic algorithm (C-MOEA) proposed by Murata and Gen [26]. One special feature of the MOEA/D is the definition of neighborhoods. Each subproblem is associated with the k-nearest subproblems in terms of the distance to the weight vectors conforming a neighborhood. Another feature of the MOEA/D is the mating selection and replacement mechanism incorporated in each neighborhood. Those features have implications in the diversity preserved in the population by the algorithm. Specifically, a single good solution can replace several inferior neighboring solutions that can result in deterioration of the population diversity [27]. To alliviate the previous shortcomming and stablishing several improvements Li and Zhang proposed MOEA/D-DE [28]. The main changes imposed by the MOEA/D-DE are the incorporation of DE operators, computational resource allocation strategy, mating selection and replacement mechanism. In this decomposition algorithm, the simulated binary crossover (SBX) that is employed in the general MOEA/D is replaced by the DE operators. However to promote variation in the population the polynomial mutation [29] is still used. This adjustment is given that the DE operator can be invariant of any orthogonal coordinate rotation being ideal for dealing with complicated PS. In addition, in the MOEA/D all the subproblems are treated equally reciving the same computational effort by generation. However, depending in the complexity and the Pareto shape of the functions, each subproblem might require a different computational effort, therefore MOEA/D-DE incorporates a computational resource allocation strategy. In order, given that the replacement mechanism of the MOEA/D is too elitist, the MOEA/D-DE incorporates an extra measure for maintaining the population diversity in the mating selection, thus given a probability this mechanism selects three parent solutions from the neighborhood or the whole population, enhancing the exploration ability. Finally, in the MOEA/D the maximal number of solutions replaced by a solution could be as large as the neighborhood size reducing dramatically the diversity among the population. To overcome this shortcoming the MOEA/D-DE incorporates a parameter to limit the maximal number of solutions replaced by a child solution, avoiding to many copies in the population.

III. PROPOSAL

In this section a decomposition MOEA that promotes diversity in the decision variable space is proposed. Initially the main framework of the propose MOEA, which follows the decomposition MOEAs style is described. Thereafter, the incorpored replacement phase to preserve diversity in decision variable space is explained in detail. Finally, several considerations –probed empirically– related to the DE operator are disscused.

The idea behind our proposal is to boost a proper balance between exploration and exploitation. In particular the *Variable Space Diversity* MOEA *based in Decomposition* (VSD-MOEA/D) explores properly the search space decreasing gradually the diversity requirements of the population in function of the elapsed generations and stopping criterion. In this way, VSD-MOEA/D explicitly grants diversity in the variable space and preservers quality solutions in the objectives space.

The two novelties of VSD-MOEA/D are the inclusion of a replacement phase and Elite population. While the former preserves diversity keeping the neighbourhoods definitions, the latter records the best individual in each subproblem. Particularly, the Parent population (P), Offspring population (Q) and the Elite population (E) are taken into consideration. The role of the Parent population (P) is to provide diverse solutions at several stages in function of the elapsed time. In contrast, the Elite population (E) records the best values attained in each subproblem among all the optimization process.

The general framework of the VSD-MOEA/D is taken from MOEA/D-DE [28] which employes DE operators. Algorithm 1 shows the main procedure of VSD-MOEA/D. Similarly than MOEA/D-DE the initialization create a random population, assign an adequately ideal vector $\mathbf{z}$, and the neighborhoods of each subproblem are associated (line 1). Then iterativelly the generations evolve until the stopping criterion is met (lines 3 - 10). Particularly, in each generation and for each subproblem the following steps take place. In the mating selection three different indexes (r_1, r_2, r_3) are randomly selected, those indexes belong to the neighborhood $B(i)$ with probability δ or to the entire population with probability $(1-\delta)$ (line 5). In the variation stage (line 6) a new solution Q_i^t is created employing the individuals $P_{r_1}^t, P_{r_2}^t, P_{r_3}^t$ with the application of DE and polynomial mutation CITA. In line 8 the reference vector $\mathbf{z}$ is updated as the lowest objective values. Given the new individual Q_i^t all the elite individuals belonging to the neighborhood $B(i)$ are updated according to the a specific approach i.e. Tchebycheff approach. Finally, the Parents (P^t), Elite (E_t) and Offspring (Q^t) populations are joined to select the parents of the next generation (P^{t+1}) in the replacement (line 9). Although that the three populations are joinned and treated in the replacement stage (detailed in the following section) the order of the individuals is important, therefore the parents P^{t+1} of next generations preserves the order that is related with each subproblem.

Algorithm 1 Main procedure of VSD-MOEA/D

1: **Initialization**: Generate an initial population P^0 with N individuals, initialize $\mathbf{z} = (z_1, ..., z_m)^T$ far away from the front, initialize the weight vectors $\lambda^1, ..., \lambda^N$ and neighbourhoods $B(i)$.
2: Assign $t = 0$
3: **while** (not stopping criterion) **do**
4:   **for** each subproblem $i \in N$ **do**
5:     **Mating selection**: Select randomly three indexes ($r_1 \neq r_2 \neq r_3 \neq i$) from neighborhood $B(i)$ with probability δ or from the entire population with probability $(1-\delta)$.
6:     **Variation**: Generate a solution y from $P_{r_1}^t$, $P_{r_2}^t$ and $P_{r_3}^t$ by DE operator, and perform a mutation operator on y with probability p_m to produce a new solution Q_i^t.
7:     **Update reference z**: For each $j = 1, .., m$, set $z_j = min\{z_j, F_j(Q_i^t)\}$
8:     **Survivor selection**: Update all the elite vectors (E^t) from the neighborhood $B(i)$ according $g(E_i^t|\lambda_i, z)$.
9:   **Replacement**: Generate P^{t+1} by applying the replacement scheme described in Algorithm 2, using P^t, Q^t and E^t as inputs.
10:   $t = t + 1$

Algorithm 2 Replacement Phase of VSD-MOEA/D

Input: P^t (Parent population), Q^t (Offspring population), E^t (Elite population).
Output: P^{t+1}
$R^t = P^t \cup Q^t \cup E^t$ (Keeping their associated subproblems)
$P^{t+1} = \emptyset$
$Penalized = \emptyset$
$D^t = D_I - D_I * \frac{G_{Elapsed}}{0.5 * G_{End}}$
while A subproblem is not assigned **do**
  Compute DCS of individuals in R^t, using P^{t+1} as a reference set
  Move the individuals in R^t with $DCS < D^t$ to $Penalized$
  if R^t is empty **then**
    Compute DCS of individuals in $Penalized$, using P^{t+1} as a reference set
    Move the individual in $Penalized$ with the largest DCS to R^t
  Select a new survivor from R^t with the best function value (associated with its weigth vector) and move it to its related subproblem in P^{t+1}.

return P^{t+1}

A. *Replacement Phase of* VSD-MOEA/D

B. *Additional considerations*

*Describe some about the mating and the replacement and get in detail about the diversity drawbacks with the proposed strategies of the state-of-the-art algortihms. *Introduce in some way the three populations. *Diagram support. *Explain in detail the pseudocode. *Subsection for the replacment phase. *

$$Distance(A, B) = \left(\frac{1}{n} \sum_{i=1}^{n} \left(\frac{A_i - B_i}{x_i^{(U)} - x_i^{(L)}} \right)^2 \right)^{1/2} \tag{2}$$

IV. Experimental Validation

This section describes the experimental validation carried out to study the performance and gain a clear understanding of the specifics of VSD-MOEA. Our results clearly show that controlling the diversity of the variable space provides a way to further improve the results obtained by the state-of-art MOEAs. First, we discuss some technical specifications involving the benchmark problems and algorithms implemented. We then present a comparison between VSD-MOEA and state-of-the-art algorithms when used on the long-term. Then, three additional experiments to fully validate VSD-MOEA are included. These analyses are designed to test the scalability in the variable space, the performance with different stopping criteria, and the behavior with different initial penalty thresholds.

This work takes into account some of the most popular and widely used benchmarks in the multi-objective field. These problems are the WFG [?], DTLZ [?], and UF [?] configured in a standard way. The WFG test problems were used with two and three objectives and were configured with 24 parameters, 20 of them corresponding to distance parameters and 4 to position parameters. In the DTLZ test problems, the number of variables was set to $n = M + r - 1$, where $r = \{5, 10, 20\}$ for DTLZ1, DTLZ2 to DTLZ6 and DTLZ7, respectively. The UF benchmark comprises seven problems with two objectives (UF1-7) and three problems with three objectives (UF8-10). All of them were configured with 30 variables. Note that the experiment used to analyze the scalability in the variables considers different numbers of variables.

The experimental validation includes three well-known state-of-the-art MOEAs and VSD-MOEA. The MOEAs that are considered are NSGA-II [?], MOEA/D [?], and R2-EMOA [?], which can be classified as dominance-based, decomposition-based, and indicator-based, respectively. In the case of MOEA/D, several variants have been devised. The MOEA/D implementation considered is the one that obtained first place in the Congress on Evolutionary Computation's 2009 MOP Competition [9]. The common configuration in all the experiments was as follows: the population size was set to 100, and the genetic operators were the Simulated Binary Crossover (SBX) and polynomial mutation [?], [?]. The crossover probability was set to 0.9 and the crossover distribution index was set to 2. Similarly, the mutation probability and distribution index were fixed to $1/n$ and 50, respectively. The additional parameterization required by each algorithm is shown in Table I. Note that scalarization functions are required in MOEA/D and R2-EMOA. In both cases, the Tchebycheff approach is used. The procedure for generating the weight vectors differs in MOEA/D and R2-EMOA. R2-EMOA was applied with 501 and 496 weight vectors for two and three objectives, respectively [24]. In contrast, MOEA/D requires the same number of weight vectors as the population size. They were generated with the uniform design (UD) and the good lattice point (GLP) method [?], [?].

Given that all the algorithms considered are stochastic, each execution was repeated 35 times with different seeds. The hypervolume indicator (HV) is used to compare the various schemes. Note that in the supplementary material, the results are also compared in terms of the IGD+ metric, with the conclusions being quite similar. The reference point used to calculate the HV is chosen to be a vector whose values are sightly larger (ten percent) than the nadir point, as suggested in [?]. The normalized HV is used to facilitate the interpretation of the results [?], and the value reported is computed as the ratio between the normalized HV obtained and the maximum attainable normalized HV. In this way, a value equal to one means a perfect approximation. Note that a value equal to one is not attainable because MOEAs yields a discrete approximation. Finally, in order to statistically compare the HV ratios, a guideline similar to that proposed in [?] was used. First a Shapiro-Wilk test was performed to check if the values of the results followed a Gaussian distribution. If so, the Levene test was used to check for the homogeneity of the variances. If the samples had equal variance, an ANOVA test was done; if not, a Welch test was performed. For non-Gaussian distributions, the non-parametric Kruskal-Wallis test was used to test whether samples are drawn from the same distribution. An algorithm X is said to beat algorithm Y when the differences between them are statistically significant, and the mean and median HV ratios obtained by X are higher than the mean and median achieved by Y.

TABLE I
PARAMETERIZATION APPLIED TO EACH MOEA

Algorithm	Configuration
MOEA/D	Max. updates by sub-problem (η_r) = 2, tour selection = 10, neighbor size = 10, period utility updating = 30 generations, local selection probability (δ) = 0.9,
VSD-MOEA	$D_I = 0.4$
R2-EMOA	$\rho = 1$, offspring by iteration = 1

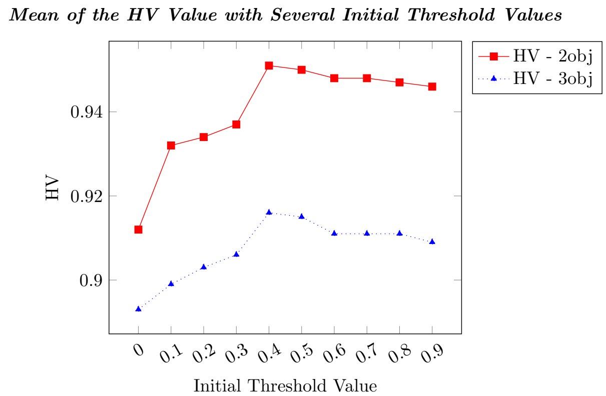

Fig. 1. Mean of HV values taking into account all the problems with several initial threshold values

A. Comparison Against State-of-the-art

B. Effect of the Initial Distance Factor

V. CONCLUSION

APPENDIX A
PROOF OF THE FIRST ZONKLAR EQUATION

Appendix one text goes here.

APPENDIX B

Appendix two text goes here.

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TABLE II
SUMMARY OF THE HYPERVOLUME RATIO RESULTS ATTAINED FOR PROBLEMS WITH TWO OBJECTIVES

	MOEA/D				NSGA-II				R2-EMOA				VSD-MOEA			
	Min	Max	Mean	Std	Min	Max	Mean	Std	Min	Max	Mean	Std	Min	Max	Mean	Std
WFG1	**0.984**	**0.993**	**0.992**	**0.002**	0.987	0.993	0.992	0.002	0.946	0.994	0.988	0.012	0.980	0.994	0.992	0.003
WFG2	0.965	0.996	0.967	0.007	0.966	0.998	0.974	0.014	0.965	0.966	0.966	0.000	**0.998**	**0.998**	**0.998**	**0.000**
WFG3	0.992	0.992	0.992	0.000	0.987	0.988	0.987	0.000	0.991	0.992	0.991	0.000	**0.992**	**0.992**	**0.992**	**0.000**
WFG4	0.988	0.988	0.988	0.000	0.983	0.987	0.985	0.001	**0.991**	**0.991**	**0.991**	**0.000**	0.990	0.990	0.990	0.000
WFG5	0.876	0.893	0.882	0.005	0.884	0.899	0.890	0.002	0.886	0.895	0.891	0.003	**0.894**	**0.928**	**0.914**	**0.010**
WFG6	**0.879**	**0.940**	**0.914**	**0.016**	**0.894**	**0.942**	**0.913**	**0.012**	**0.875**	**0.942**	**0.912**	**0.015**	0.855	0.888	0.868	0.007
WFG7	0.988	0.988	0.988	0.000	0.983	0.987	0.984	0.001	**0.991**	**0.991**	**0.991**	**0.000**	0.990	0.990	0.990	0.000
WFG8	0.800	0.822	0.811	0.006	0.771	0.801	0.789	0.006	0.803	0.824	0.815	0.005	**0.828**	**0.958**	**0.928**	**0.046**
WFG9	0.795	0.972	0.883	0.082	0.793	0.966	0.832	0.070	0.797	0.976	0.884	0.079	**0.963**	**0.975**	**0.970**	**0.004**
DTLZ1	**0.993**	**0.993**	**0.993**	**0.000**	0.990	0.992	0.991	0.000	0.992	0.992	0.992	0.000	0.992	0.992	0.992	0.000
DTLZ2	0.989	0.989	0.989	0.000	0.986	0.988	0.987	0.000	**0.991**	**0.992**	**0.992**	**0.000**	0.990	0.990	0.990	0.000
DTLZ3	0.989	0.989	0.989	0.000	0.987	0.989	0.989	0.001	**0.991**	**0.992**	**0.992**	**0.000**	0.990	0.990	0.990	0.000
DTLZ4	0.259	0.989	0.781	0.330	0.259	0.988	0.863	0.274	0.259	0.992	0.657	0.365	**0.990**	**0.990**	**0.990**	**0.000**
DTLZ5	0.989	0.989	0.989	0.000	0.986	0.988	0.987	0.000	**0.991**	**0.992**	**0.992**	**0.000**	0.990	0.990	0.990	0.000
DTLZ6	0.448	0.910	0.700	0.105	0.138	0.511	0.322	0.075	0.510	0.922	0.691	0.107	**0.990**	**0.990**	**0.990**	**0.000**
DTLZ7	0.996	0.996	0.996	0.000	0.996	0.997	0.996	0.000	**0.997**	**0.997**	**0.997**	**0.000**	0.996	0.996	0.996	0.000
UF1	0.991	0.993	0.992	0.000	0.986	0.989	0.988	0.000	0.978	0.994	0.990	0.005	**0.994**	**0.995**	**0.994**	**0.000**
UF2	**0.987**	**0.993**	**0.991**	**0.002**	0.980	0.983	0.981	0.001	0.984	0.991	0.988	0.002	0.987	0.993	0.990	0.001
UF3	0.481	0.674	0.597	0.043	0.678	0.871	0.784	0.048	0.531	0.704	0.589	0.041	**0.799**	**0.916**	**0.881**	**0.025**
UF4	0.881	0.917	0.908	0.006	0.875	0.910	0.889	0.008	**0.923**	**0.935**	**0.929**	**0.003**	0.923	0.931	0.927	0.002
UF5	0.035	0.792	0.484	0.165	0.256	0.766	0.641	0.104	0.123	0.792	0.566	0.192	**0.582**	**0.763**	**0.647**	**0.040**
UF6	0.255	0.711	0.447	0.114	0.235	0.801	0.635	0.120	0.349	0.767	0.568	0.113	**0.668**	**0.900**	**0.810**	**0.061**
UF7	**0.987**	**0.991**	**0.990**	**0.001**	0.980	0.983	0.981	0.001	0.557	0.991	0.910	0.150	**0.975**	**0.992**	**0.988**	**0.004**
Mean	0.806	0.935	0.881	0.038	0.808	0.927	0.886	0.032	0.801	0.940	0.882	0.048	0.929	0.963	0.949	0.009

TABLE III
STATISTICAL TESTS AND DETERIORATION LEVEL OF THE HV RATIO FOR PROBLEMS WITH TWO OBJECTIVES

	↑	↓	↔	Deterioration
MOEA/D	24	36	9	1.615
NSGA-II	13	49	7	1.496
R2-EMOA	34	21	14	1.597
VSD-MOEA	50	15	4	0.059

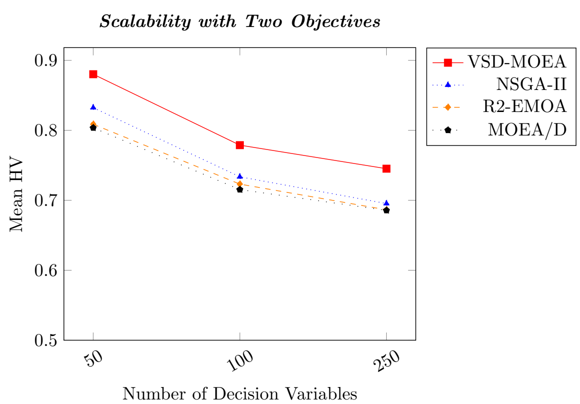

Fig. 2. Mean of the HV ratio for 35 runs for the two-objective problems considering different numbers of variables

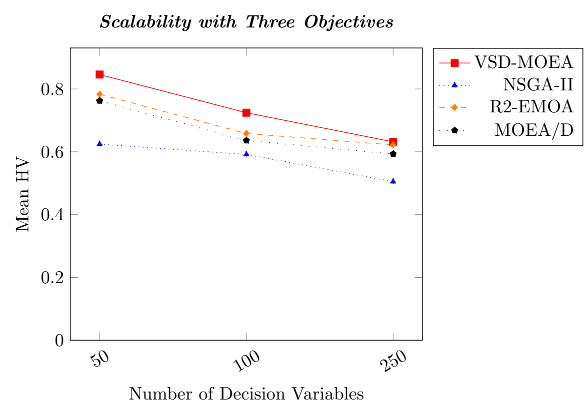

Fig. 3. Mean of the HV ratio for 35 runs for the three-objective problems considering different numbers of variables

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TABLE IV
SUMMARY OF THE HYPERVOLUME RATIO RESULTS ATTAINED FOR PROBLEMS WITH TWO OBJECTIVES

	MOEA/D				NSGA-II				R2-EMOA				VSD-MOEA			
	Min	**Max**	**Mean**	**Std**	**Min**	**Max**	**Mean**	**Std**	**Min**	**Max**	**Mean**	**Std**	**Min**	**Max**	**Mean**	**Std**
WFG1	**0.984**	**0.993**	**0.992**	**0.002**	0.987	0.993	0.992	0.002	0.946	0.994	0.988	0.012	0.980	0.994	0.992	0.003
WFG2	0.965	0.996	0.967	0.007	0.966	0.998	0.974	0.014	0.965	0.966	0.966	0.000	**0.998**	**0.998**	**0.998**	**0.000**
WFG3	0.992	0.992	0.992	0.000	0.987	0.988	0.987	0.000	0.991	0.992	0.991	0.000	**0.992**	**0.992**	**0.992**	**0.000**
WFG4	0.988	0.988	0.988	0.000	0.983	0.987	0.985	0.001	**0.991**	**0.991**	**0.991**	**0.000**	0.990	0.990	0.990	0.000
WFG5	0.876	0.893	0.882	0.005	0.884	0.899	0.890	0.002	0.886	0.895	0.891	0.003	**0.894**	**0.928**	**0.914**	**0.010**
WFG6	**0.879**	**0.940**	**0.914**	**0.016**	**0.894**	**0.942**	**0.913**	**0.012**	**0.875**	**0.942**	**0.912**	**0.015**	0.855	0.888	0.868	0.007
WFG7	0.988	0.988	0.988	0.000	0.983	0.987	0.984	0.001	**0.991**	**0.991**	**0.991**	**0.000**	0.990	0.990	0.990	0.000
WFG8	0.800	0.822	0.811	0.006	0.771	0.801	0.789	0.006	0.803	0.824	0.815	0.005	**0.828**	**0.958**	**0.928**	**0.046**
WFG9	0.795	0.972	0.883	0.082	0.793	0.966	0.832	0.070	0.797	0.976	0.884	0.079	**0.963**	**0.975**	**0.970**	**0.004**
DTLZ1	**0.993**	**0.993**	**0.993**	**0.000**	0.990	0.992	0.991	0.000	0.992	0.992	0.992	0.000	0.992	0.992	0.992	0.000
DTLZ2	0.989	0.989	0.989	0.000	0.986	0.988	0.987	0.000	**0.991**	**0.992**	**0.992**	**0.000**	0.990	0.990	0.990	0.000
DTLZ3	0.989	0.989	0.989	0.000	0.987	0.989	0.989	0.001	**0.991**	**0.992**	**0.992**	**0.000**	0.990	0.990	0.990	0.000
DTLZ4	0.259	0.989	0.781	0.330	0.259	0.988	0.863	0.274	0.259	0.992	0.657	0.365	**0.990**	**0.990**	**0.990**	**0.000**
DTLZ5	0.989	0.989	0.989	0.000	0.986	0.988	0.987	0.000	**0.991**	**0.992**	**0.992**	**0.000**	0.990	0.990	0.990	0.000
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DTLZ7	0.996	0.996	0.996	0.000	0.996	0.997	0.996	0.000	**0.997**	**0.997**	**0.997**	**0.000**	0.996	0.996	0.996	0.000
UF1	0.991	0.993	0.992	0.000	0.986	0.989	0.988	0.000	0.978	0.994	0.990	0.005	**0.994**	**0.995**	**0.994**	**0.000**
UF2	**0.987**	**0.993**	**0.991**	**0.002**	0.980	0.983	0.981	0.001	0.984	0.991	0.988	0.002	0.987	0.993	0.990	0.001
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Mean	0.806	0.935	0.881	0.038	0.808	0.927	0.886	0.032	0.801	0.940	0.882	0.048	0.929	0.963	0.949	0.009

TABLE V
STATISTICAL TESTS AND DETERIORATION LEVEL OF THE HV RATIO FOR PROBLEMS WITH TWO OBJECTIVES

	↑	↓	↔	**Deterioration**
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R2-EMOA	34	21	14	1.597
VSD-MOEA	50	15	4	0.059

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