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# Constrained Subproblems in Decomposition based Multiobjective Evolutionary Algorithm

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***Abstract*—A decomposition approach decomposes a multiobjective optimization problem into a number of scalar objective optimization subproblems. It plays a key role in decomposition based multiobjective evolutionary algorithms. However, many widely-used decomposition approaches, originally proposed for mathematical programming algorithms , may not be very suitable for evolutionary algorithms. To help decomposition based multiobjective evolutionary algorithms balance the population diversity and convergence in an appropriate manner, this letter proposes to impose some constraints on the subproblems. Experiments have been conducted to demonstrate that our proposed constrained decomposition approach works well on most test instances. We further propose a strategy for adaptively adjusting constraints by using information collected from the search. Experimental results show that it can significantly improve the algorithm performance.**

***Index Terms*—Evolutionary multiobjective optimization; decomposition approach; constraint.**

## I. Introduction

MANY real world problems can be formulated as *multiobjective optimization problems (MOPs)*. An MOP can be defined as:

$$\begin{array}{ll} \text{minimize} & F(x) = (f_1(x), f_2(x), \cdots, f_m(x))^T \\ \text{subject to} & x \in \Omega \end{array} \quad (1)$$

where $\Omega \subset R^n$ is the feasible search region, $x = (x_1, x_2, \cdots, x_n)^T$ is the decision variable vector, $f_i : R^n \to R, i = 1, \cdots, m$ are the $m$ objective functions, and $R^m$ is the objective space.

It should be noted that, in this letter, we only consider an MOP whose objectives are continuous and the feasible region is closed and bounded in $R^n$.

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Given two vectors $u, v \in R^m$, $u$ is said to *dominate* $v$, denoted as $u \prec v$, if and only if $u_i \leq v_i$ for every $i \in \{1, \cdots, m\}$ and $u \neq v$. A solution $x^* \in \Omega$ is called *Pareto optimal* to *(1)* if there does not exist a solution $x \in \Omega$ such that $F(x) \prec F(x^*)$. All the Pareto optimal solutions comprise the *Pareto set (PS)* in the feasible region and the *Pareto front (PF)* is defined as $PF = \{F(x)|x \in PS\}$ [1].

Without a priori information, a decision maker often requires a good approximation to the *PF* for further decision making in practice. Classical optimization methods may fail to do so especially when the objective functions are non-differentiable and without closed forms. For this reason, people resort to heuristic optimization methods such as *evolutionary algorithms (EAs). Multiobjective evolutionary algorithms (MOEAs)* have been attracting much attention since these algorithms require very few assumptions on the problem. A number of MOEAs have been proposed over the last two decades [2]. These MOEAs can be classified as three categories: the Pareto-dominance based approaches [3], [4], the indicator based approaches [5], [6], and the decomposition based approaches [7]–[9].

*Multiobjective evolutionary algorithm based on decomposition (MOEA/D)* [8], [10] decomposes an MOP into $N$ scalar optimization subproblems and optimizes all of them in a single run [8]. $N$ procedures (or agents) are used in MOEA/D, and each procedure is for a different subproblem. These procedures exchange information with one another for improving their search efficiency. To design an efficient and effective MOEA/D for a particular MOP, we need to consider several issues such as the population size, recombination operators and decomposition approaches. This letter focuses on decomposition approaches in MOEA/D. A decomposition approach defines subproblems and then guides the selection process. It also somehow determines how different search procedures collaborate with one another. The weighted sum approach, the weighted Tchebycheff approach and the penalty based boundary interaction (*PBI*) approach are the three most commonly-used decomposition approaches in MOEA/D [1], [11], [12]. The objective function in each subproblem in these approaches is a linear or nonlinear weighted aggregation of $f_1, \ldots, f_m$. The weighted sum approach works very well for convex PFs, but is unable to approximate a nonconvex *PF* [1]. The weighted Tchebycheff approach can deal with nonconvex PFs, but its objective function is not smooth [1]. The *PBI* approach is able to approximate a *PF* very well if the weight vectors are set in an appropriate manner [8], however, its penalty factor is not always easy to set. Some effort has been

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made to study and improve decomposition approaches. Use of different decomposition approaches at different search stages, and combinations of different decomposition approaches have been studied in [13], [14]. Decomposition approaches have been hybridized with the R2 indicator [15]. Some methods for setting weight vectors have been developed (e.g. [11], [12], [16]–[19]). The effect of dynamically adjusting weight vectors has also been studied in [12]. An adaptive epsilon comparison approach has been proposed in [20] to balance the convergence and the diversity.

Most decomposition approaches [1] were originally proposed for mathematical programming algorithms. Direct use of them in MOEA/D may not always be very suitable for balancing the population diversity and convergence on some MOPs [21], [22]. Using these approaches, MOEA/D can assign the same solution to several different subproblems if no other extra measures are taken. To overcome this shortcoming, this letter proposes a method to impose constraints on subproblems. To further improve the algorithm performance, We also develop a way to adaptively adjust the constraints during the search.

The remainder of this letter is organized as follows. Section II briefly introduces the MOEA/D framework and some commonly used decomposition approaches. Section III presents the *constrained decomposition approach* and an adaptive strategy for adjusting the constraints. In Section IV, the experimental studies are given to demonstrate the efficiency of the proposed method as well as some discussions in this letter. Finally, the letter is concluded in Section V.

## II. Multiobjective Evolutionary Algorithm based on Decomposition (MOEA/D)

In this section, the framework of MOEA/D and the three commonly used decomposition approaches are introduced in details.

### A. Framework

MOEA/D decomposes an MOP into a set of $N$ scalar objective optimization subproblems. Their optimal solutions as a whole can approximate the *PF*. MOEA/D evolves a population of $N$ candidate solutions. Solution $x^i$ is for subproblem $i$. The $T$ neighborhoods of subproblem $i$ are the $T$ subproblems whose weight vectors are the $T$ closest ones to its weight vector $\lambda^i$. $B^i(T)$ is used in this letter to denote the set of the $T$ indexes of neighboring subproblems of subproblem $i$. Since the objective of a subproblem is continuous of its weight vector, a small change in its weight vector should not lead to a big change in its optimal solution in most cases. For this reason, we can assume that the optimal solutions of two neighboring subproblems are close. MOEA/D makes use of this assumption and optimizes all the $N$ subproblems in a single run. MOEA/D can be regarded as a generalized and improved framework of cMOEA proposed in [7]. Many different MOEA/D variants have been proposed [21], [23]–[25]. The MOEA/D framework [10] used in this letter is given in Algorithm 1.

**Algorithm 1:** MOEA/D Framework

**Input**:
- $N$: population size.
- $T$: neighborhood size.
- $n_r$: maximal number of updated subproblems.
- $B^i(T)$: index set of the neighbors of subproblem $i$, $i = 1, 2, \cdots, N$.
- $\delta$: a control parameter.
- $g^i(\cdot)$: objective function to minimize in subproblem $i$, $i = 1, 2, \cdots, N$.
- termination condition.

**Output**: an approximation to the PF (PS).

`/* initialization */`

Initialize the population $x^i$, weight vectors $\lambda^i$, neighbor index set $B^i(T)$ and set $F^i = F(x^i)$ for $i = 1, 2, \cdots, N$.

`/* main loop */`

**while** *termination condition is not met* **do**
  **for** *each subproblem* $i = 1, 2, \cdots, N$ **do**
    `/* determine the mating/update pool */`
    Set $P = \begin{cases} B^i(T) & rand() \leq \delta \\ \{1, 2, \cdots, N\} & \text{otherwise} \end{cases}$.
    `/* offspring recombination */`
    Generate a new solution $y$ by using recombination operators on some solutions randomly selected from $\{x^i | i \in P\}$.
    `/* population replacement */`
    Set $c = 0$. **while** $c < n_r$ *and* $P \neq \emptyset$ **do**
      Randomly pick an index $j$ from $P$.
      $P := P \setminus \{j\}$
      **if** $y$ *is better than* $x^j$ *for subproblem* $j$ **then**
        Set $x^j = y, F^j = F(y)$.
        $c = c + 1$.
      **end**
    **end**
  **end**
**end**

**return** $x^1, \cdots, x^N$ *and their objective function values* $F^1, \cdots, F^N$.

In this framework, whether or not a new solution $y$ replaces $x^j$ is determined by the objective function of subproblem $j$. Thus the definition of subproblems, i.e., the decomposition approach, is crucial in MOEA/D.

### B. Three Commonly Used Decomposition Approaches

Let $\lambda^i = (\lambda^i_1, \ldots, \lambda^i_m)^T$ be a weight vector for the $i^{th}$ subproblem, i.e., $\sum_{j=1}^{m} \lambda^i_j = 1$ and $\lambda^i_j \geq 0$ for all $j = 1, 2, \cdots, m$. Let $z^* = (z^*_1, \ldots, z^*_m)^T$ be a utopian point [1]. In our implementation, $z^*_i$ is set to be a value slightly smaller than the smallest $f_i$ value found so far. The three most commonly used decomposition approaches are given in the following.

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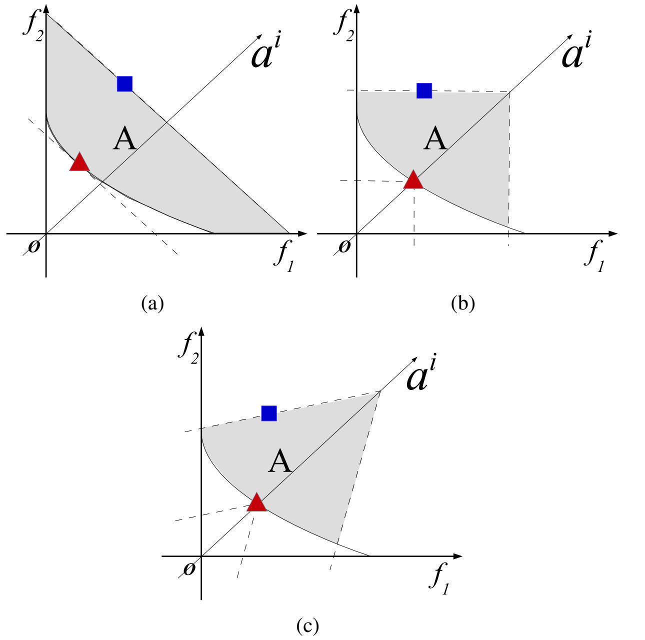


Fig. 1. Illustrations of the improvement regions for the three commonly used decomposition approaches: (a) *The Weighted Sum Approach*, (b) *The Weighted Tchebycheff Approach*, and (c) *The PBI Approach*. In each sub-figure, region $A$ is the improvement region. The *square* point is the current solution of subproblem $i$ with the direction vector $a^i$, the *triangle* point is its optimal solution and the *dash line* is its contour.

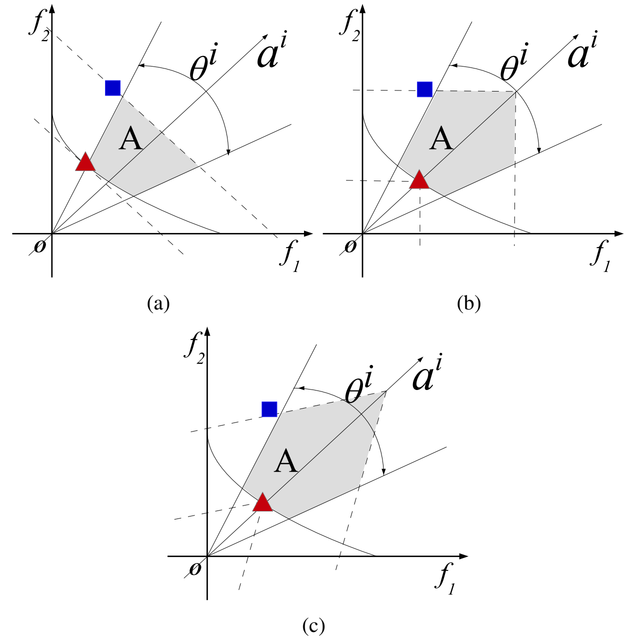


Fig. 2. Illustrations of the improvement regions for the three constrained decomposition approaches: (a) *The Constrained Weighted Sum Approach*, (b) *The Constrained Weighted Tchebycheff Approach*, and (c) *The Constrained PBI Approach*. In each sub-figure, region $A$ is the improvement region. The *square* point is the current solution of subproblem $i$ with the direction vector $a^i$, the *triangle* point is its optimal solution and the *dash line* is its contour.

*1) Weighted Sum Approach:* The $i^{th}$ subproblem is defined as

$$\begin{aligned} \text{minimize} \quad & g^{ws}(x|\lambda^i) = \sum_{j=1}^{m} \lambda_j^i f_j(x) \\ \text{subject to} \quad & x \in \Omega \end{aligned} \tag{2}$$

Its direction vector $a^i$ is defined to be $\lambda^i$. To do decomposition, users need to select a number of different weight vectors.

*2) Weighted Tchebycheff Approach:* The $i^{th}$ subproblem is:

$$\begin{aligned} \text{minimize} \quad & g^{te}(x|\lambda^i, z^*) = \max_{1\leq j\leq m} \{\lambda_j^i |f_j(x) - z_j^*|\} \\ \text{subject to} \quad & x \in \Omega \end{aligned} \tag{3}$$

Its direction vector $a^i$ is defined to be $(1/\lambda_1^i, \cdots, 1/\lambda_m^i)^T$.

*3) Penalty-based Boundary Intersection (PBI) Approach:* This approach is a variant of *Normal-Boundary Intersection* approach [8]. The $i^{th}$ subproblem is defined as

$$\begin{aligned} \text{minimize} \quad & g^{pbi}(x|\lambda^i, z^*) = d_1^i + \beta d_2^i \\ & d_1^i = (F(x) - z^*)^T \lambda^i / \parallel \lambda^i \parallel \\ & d_2^i = \parallel F(x) - z^* - (d_1^i/||\lambda^i||)\lambda^i \parallel \\ \text{subject to} \quad & x \in \Omega \end{aligned} \tag{4}$$

where $||\cdot||$ denotes the $L_2$-norm, and $\beta$ is a control parameter. Its direction vector $a^i$ is defined to be $\lambda^i$.

To use the above three approaches to decompose problem (1) into $N$ subproblems, users need to choose $N$ different weight vectors for defining $N$ subproblems. Some work has been done on how to select weight vectors such that the optimal solutions of the subproblems can produce a good approximation to the *PF* [11], [16]–[18].

Subproblems in these three approaches are unconstrained. Let $x^i$ be the current solution for subproblem $i$. When these approaches are used in the MOEA/D algorithm in Section II, $y$ is better than $x^i$ in *Line 9* in Algorithm 1 means that $y$ has a lower objective function value than $x^i$ for this subproblem. As illustrated in Fig. 1, the *improvement region* of $x^i$ for subproblem $i$ in the objective space can be defined as the set:

$$\{F(x)|x \text{ is better than } x^i \text{ for subproblem } i\}$$

We should point out that some Pareto dominance operators [26]–[28] such as cone-e dominance [27] can be regarded as approaches for adjusting improvement regions in Pareto dominance based algorithms.

Any new solutions in this region can improve $x^i$ and replace $x^i$ for subproblem $i$. The improvement regions in the three widely used decomposition approaches may be too large for some problems. As a result, a single new solution can replace several different old solutions and thus the population diversity will be reduced. If this new solution is close to a local *PF*, the search could be trapped in it.

In the following, we will study how to add constraints to these subproblems and reduce the volumes of the improvement regions for controlling the balance between the population diversity and the convergence.

## III. Constrained Decomposition Approach

In this section, the constrained decomposition approaches and an adaptive strategy for adjusting the constraints are introduced in detail.

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### A. The Constrained Decomposition Approach

To reduce the improvement regions, we propose the following constrained optimization subproblem for the $i^{th}$ subproblem in MOEA/D:

$$\begin{array}{ll} \text{minimize} & g(x|\lambda^i, z^*) \\ \text{subject to} & \langle a^i, F(x) - z^* \rangle \leq 0.5\theta^i \\ & x \in \Omega, \end{array} \quad (5)$$

where $g(\cdot)$ is the aggregated objective function defined by a decomposition approach, $\lambda^i$ is a weight vector and $z^*$ is a utopian point, $\langle a^i, F(x) - z^* \rangle$ is the acute angle between $a^i$ and $F(x) - z^*$. $a^i$ is the direction vector of the subproblem, $a^i = \lambda^i$ when the weighted sum approach and the PBI approach are used, and $a^i = (1/\lambda_1^i, \cdots, 1/\lambda_m^i)^T$ when the weighted Tchebycheff approach is used. $\theta^i$ is a control parameter for defining the improvement region. Fig. 2 illustrates the improvement regions of the constrained decomposition approaches for subproblem $i$.

When the above constrained decomposition approach is used in MOEA/D, $x$ is said to be better than $y$ for the $i^{th}$ subproblem with weight vector $\lambda^i$ if

- neither $x$ nor $y$ meets the constraint, and $\langle a^i, F(x) - z^* \rangle < \langle a^i, F(y) - z^* \rangle$,
- neither $x$ nor $y$ meets the constraint, $\langle a^i, F(x) - z^* \rangle = \langle a^i, F(y) - z^* \rangle$ and $g(x|\lambda^i, z^*) < g(y|\lambda^i, z^*)$,
- $x$ meets the constraint but $y$ does not, or
- both $x$ and $y$ meet the constraint, and $g(x|\lambda^i, z^*) < g(y|\lambda^i, z^*)$.

The improvement region of the constrained decomposition approach is controlled by $\theta^i$. Different subproblems at different stages should need different $\theta^i$ values for balancing population diversity and convergence. In the following, we propose an adaptive way for adjusting $\theta^i, i = 1, 2, \cdots, N$ in an online manner.

### B. Adaptive Strategy

Let $x^i$ be the current solution to subproblem $i$ ($i = 1, \ldots, N$), and $a^i$ be the direction vector of this subproblem. We call

$$\alpha(x^i) = \langle a^i, F(x^i) - z^* \rangle$$

the divergence of $x^i$ to $a^i$ in subproblem $i$. We can further define the relative divergence of $x^i$ in subproblem $i$ as

$$\alpha_r(x^i) = [N \times \alpha(x^i)] / \sum_{k=1}^{N} \alpha(x^k)$$

If all the $\alpha(x^i)$ are very small, it implies a good population diversity. When all the subproblems have been solved, i.e., $x^i$ is optimal to subproblem $i$ for $i = 1, \ldots, N$. $\alpha(x^i)$ becomes zero. It is desirable that all the $\alpha(x^i)$ values converge to zero at about the same speed. In other words, every $\alpha_r(x^i)$ is close to one during the evolution. A small $\theta^i, i = 1, 2, \cdots, N$ in a constrained subproblem will guide the search to reduce the divergence value of its solution, but it may not be good for reducing its objective function value, and vice versa. Therefore, one can adjust the volume of the improvement region of each subproblem according to the divergence of the subproblem. If the divergence is large, the improvement region should be reduced and vice versa.

Based on the above considerations, we propose the following adaptive strategy for adjusting the $\theta^i$ value:

$$\theta^i(t+1) = \begin{cases} \max\{\theta^i(t) - \theta^i_{min}, \theta^i_{min}\} & \text{if } \alpha_r(x^i) > 1 \\ \theta^i(t) & \text{if } \alpha_r(x^i) = 1 \\ \min\{\theta^i(t) + \theta^i_{min}, \theta^i_{max}\} & \text{if } \alpha_r(x^i) < 1 \end{cases} \quad (6)$$

where $\theta^i(t)$ is the $\theta$ value for subproblem $i$ at generation $t$. In our experimental studies, all the $\theta^i(1)$ are initialized to be $\theta^i_{min}$. For simplicity, $\theta^i_{min}$ is set to be $\langle a^i, a^j \rangle$ where $j$ is the index of the closest subproblem of subproblem $i$, and $\theta^i_{max}$ is set to be twice the maximal angle between $a^i$ and the $m$ coordinate axes.

## IV. Experimental Study

To investigate the effects of our proposed constrained decomposition approach and adaptive strategy, we compare the three variants of MOEA/D:

- MOEA/D with adaptive constrained decomposition approach (MOEA/D-ACD).
- MOEA/D with the constrained decomposition approach and a fixed $\theta$ value (MOEA/D-CD).
- MOEA/D proposed in [10].

Both MOEA/D-ACD and MOEA/D-CD are the same as MOEA/D in [10] except for their decomposition approaches.

The weighted Tchebycheff approach is used in the above three algorithms.

### A. Experimental Settings

*1) Test Instances:* Three test sets from [10], [21], [29] are used. They are ZDT1-ZDT4, ZDT6, F1-F9 and MOP1-MOP7. F6, MOP6 and MOP7 have three objectives, and the other instances have two objectives. The dimensionality of the search space $n$ is 30 for ZDT1-ZDT3, ZDT6, F1-F5, F7 and F9; 10 for ZDT4, F6, F8 and MOP1-MOP7.

*2) Recombination Operator and Parameter Settings:* In our implementation, the differential evolution operator [30] and the polynomial mutation [31] are used as the recombination operator. The settings of all the common parameters in these three algorithms are the same as in [10]:

- Population size $N$: 100 for ZDT1-ZDT6; 200 for F1-F5, F7-F9 and MOP1-MOP5; and 300 for F6, MOP6 and MOP7.
- Maximal function evaluations: $20,000$ for ZDT1-ZDT3 and ZDT6; $100,000$ for ZDT4; and $150,000$ for all other test instances.
- Independent runs: 30 for all the test instances.

*3) Performance Metric:* In our study, the metric $I_H^-$ [32] is used as the performance metric.

**Relative Hypervolume Indicator** $I_H^-$ [32]:

$$I_H^-(P, P^*, F^*) = I_H(P^*, F^*) - I_H(P, F^*)$$

where

$$I_H(P, F^*) = Vol(\bigcup_{v \in P} [v_1, f_1^*] \times \cdots [v_m, f_m^*]),$$

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$Vol(\cdot)$ is the Lebesgue measure. $F^* = (f_1^*, \ldots, f_m^*)^T$ is the nadir point. $P$ is the final set of objective vectors obtained by an algorithm, $v = (v_1, \ldots, v_m)^T$ is an objective vector in $P$. $P^*$ is a set of uniformly distributed solutions on the $PF$. The lower metric $I_H^-$ is, the closer $P$ to the true PF $P^*$ is.

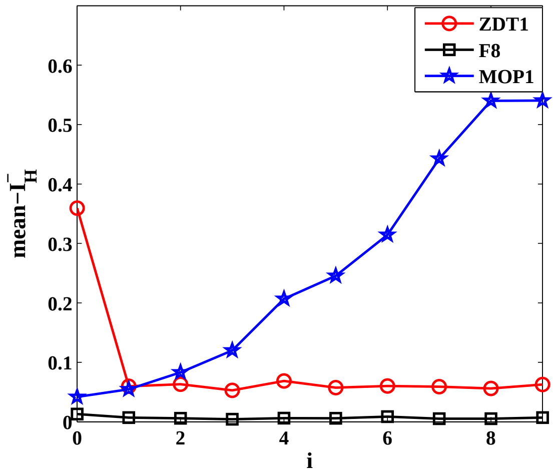


Fig. 3. The mean $I_H^-$ metric value of the final population versus different groups of $\theta$ values in MOEA/D-CD for ZDT1, F8 and MOP1. The horizontal axis is for the index $k$ of $(\theta_k^1, \theta_k^2, \cdots, \theta_k^N)$.

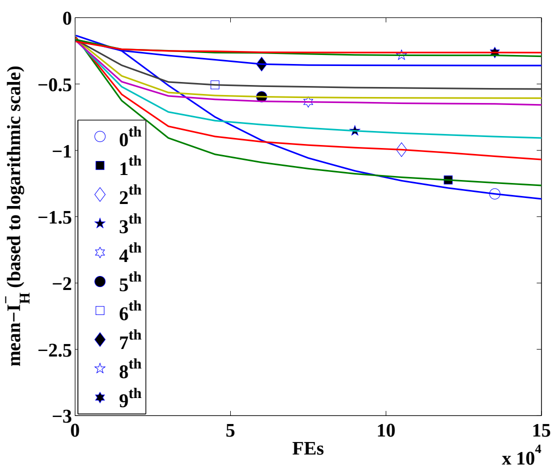


Fig. 4. The mean $I_H^-$ metric value versus the function evaluations $FEs$ value run by MOEA/D-CD with ten different settings of $\theta$ values, i.e.,$(\theta_k^1, \theta_k^2, \cdots, \theta_k^N), k = 0, 1, \cdots, 9$, on MOP1.

In computing the metrics, 2,000 points are uniformly sampled from true PF for the bi-objective test instances and 5,000 for the tri-objective test instances. And, $(1.2, 1.2)^T$ and $(1.2, 1.2, 1.2)^T$ are the nadir points for bi-objective and tri-objective problems respectively.

### *B. Results and Discussions*

*1) The impact of $\theta$ in MOEA/D-CD:* To investigate the impact of $\theta$ in MOEA/D-CD, the following ten different settings of $\theta$: $(\theta_k^1, \ldots, \theta_k^N)$ $(k = 0, \ldots, 9$ ) are tested:

$$\theta_k^i = \theta_{min}^i + (k/9)(\theta_{max}^i - \theta_{min}^i)$$

for $k = 0, \ldots, 9$, where $\theta_{min}^i$ and $\theta_{max}^i$ are the lower and upper bound of $\theta^i$ for subproblem $i$. Actually, MOEA/D-CD with $k = 9$ is the original MOEA/D.

Fig. 3 plots the mean $I_H^-$ metric value of the final population versus $\theta$ in MOEA/D-CD. It is clear that the performance of MOEA/D-CD is not very sensitive to $\theta$ on F8. However, it is not the case on ZDT1 and MOP1. Thus, we can conclude that an appropriate setting of $\theta$ in MOEA/D-CD is problem dependent.

Fig. 4 plots the evolution of the mean $I_H^-$ metric value versus function evaluations in MOEA/D-CD on MOP1. We can observe that different $\theta$ values have different performances. MOEA/D-CD with $k = 0$ performs the worst at the early stage but the best at the late stage. These experiments suggest that different problems at different stages require different $\theta$ values. Therefore, it is very reasonable to use adaptive strategies.

TABLE I
$I_H^-$-METRIC VALUES OF THE FINAL POPULATIONS OBTAINED BY MOEA/D-CD, MOEA/D AND MOEA/D-ACD.

| | MOEA/D-CD | | MOEA/D | | MOEA/D-ACD | |
|---|---|---|---|---|---|---|
| instance | mean | std | mean | std | mean | std |
| ZDT1 | 0.0527 | 0.0217 | 0.1095 | 0.0419 | **0.0204** | 0.0038 |
| ZDT2 | 0.0938 | 0.0313 | 0.2200 | 0.0742 | **0.0112** | 0.0031 |
| ZDT3 | **0.1791** | 0.0646 | 0.3563 | 0.1137 | 0.2111 | 0.1914 |
| ZDT4 | **0.0068** | 0.0010 | 0.0105 | 0.0038 | 0.0140 | 0.0137 |
| ZDT6 | 0.7083 | 0.0000 | 3.0340 | 0.0317 | **0.2863** | 0.3048 |
| F1 | 0.0026 | 0.0000 | **0.0026** | 0.0000 | 0.0026 | 0.0000 |
| F2 | **0.0110** | 0.0022 | 0.0308 | 0.0135 | 0.0243 | 0.0105 |
| F3 | 0.0157 | 0.0055 | 0.0636 | 0.0995 | **0.0151** | 0.0095 |
| F4 | **0.0170** | 0.0198 | 0.0909 | 0.1121 | 0.0258 | 0.0177 |
| F5 | **0.0212** | 0.0050 | 0.0762 | 0.0807 | 0.0519 | 0.0342 |
| F6 | **0.0644** | 0.0087 | 0.0856 | 0.0128 | 0.1796 | 0.0373 |
| F7 | **0.3982** | 0.2270 | 0.9127 | 0.3879 | 1.1007 | 0.3630 |
| F8 | **0.0042** | 0.0031 | 0.0126 | 0.0337 | 0.0064 | 0.0096 |
| F9 | **0.0111** | 0.0023 | 0.0231 | 0.0136 | 0.0251 | 0.0238 |
| MOP1 | 0.0419 | 0.0068 | 0.5405 | 0.1004 | **0.0352** | 0.0027 |
| MOP2 | 0.0545 | 0.0616 | 0.3147 | 0.0300 | **0.0470** | 0.0637 |
| MOP3 | 0.0804 | 0.0955 | 0.5200 | 0.0780 | **0.0475** | 0.0753 |
| MOP4 | 0.0394 | 0.0406 | 0.3599 | 0.0234 | **0.0241** | 0.0229 |
| MOP5 | 0.0443 | 0.0133 | 0.9236 | 0.1519 | **0.0375** | 0.0108 |
| MOP6 | **0.1159** | 0.0248 | 0.3199 | 0.0327 | 0.2431 | 0.0487 |
| MOP7 | **0.1551** | 0.0451 | 0.2532 | 0.0088 | 0.2869 | 0.0284 |

TABLE II
STATISTICS OF COMPARISONS WITH MOEA/D-ACD

| | MOEA/D | MOEA/D-CD |
|---|---|---|
| | $I_H^-$ | $I_H^-$ |
| $+$ | 3 | 9 |
| $-$ | 13 | 8 |
| $\approx$ | 5 | 4 |

"$+$", "$-$" and "$\approx$" denote the number of the performance of corresponding algorithm is better than, worse than and similar to that of MOEA/D-ACD, respectively, according to the Wilcoxon rank sum test with the significance level 0.05.

*2) The effect of the constrained decomposition approach and the adaptive strategy:* To demonstrate the effect of our proposed adaptive strategy, MOEA/D-ACD is compared with MOEA/D and MOEA/D-CD. In our experiments, the value of $\theta$ used in MOEA/D-CD for each instance is the best among the ten settings in terms of the mean $I_H^-$ values.

Tables I shows the mean and standard deviation of the $I_H^-$-metric values of the final populations obtained by the three algorithms. Table II presents the statistical results of the Wilcoxon rank sum test with the significance level 0.05 of the three algorithms. From these two tables, we can observe that MOEA/D is the worst on almost all the test instances. Therefore, we can claim that the constrained decomposition approach does improve the algorithm performance. It is also evident from Table II that MOEA/D-ACD performs better than MOEA/D-CD on 8 instances out of 21 instances in terms of the $I_H^-$-metric. Noting that MOEA/D-CD uses the best setting of $\theta$, thus, we can conclude that the adaptive strategy is beneficial.

## V. Conclusion

To balance diversity and convergence in MOEA/D, this letter has proposed a constrained decomposition approach by imposing a constraint to an unconstrained subproblem. The improvement region of each subproblem is determined by a control parameter $\theta$. An adaptive strategy for adjusting $\theta$ has been proposed. Our experimental results on a set of test instances suggest that this constrained approach helps improve the algorithm performance and the adaptive strategy is useful on most test instances. The future research topic is to develop some other decomposition approaches that suit the MOEA/D framework or study the relationship between the distribution of the population and the performance of an algorithm. The supplementary document and source code can be downloaded from http://www.cs.cityu.edu.hk/~qzhang/publications.html.

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