VSD-MOEA: A Dominance-Based Multi-Objective Evolutionary Algorithm with Explicit Variable Space Diversity Management

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***Abstract*—Most state-of-the-art Multi-objective Evolutionary Algorithms (MOEAs) promote the preservation of the diversity in the objective space, whereas the information about the diversity in the decision variables is usually neglected. In this paper, the Variable-Space-Diversity based MOEA (VSD-MOEA) is presented. VSD-MOEA is a dominance-based MOEA that considers explicitly the diversity in the decision variables and the objective space. The information gathered of both spaces is used simultaneously with the aim of properly adapting the balance between exploration and intensification during the optimization process. Particularly, at the initial stages, decisions taken by the approach are more biased by the information of the diversity in the decision variables, whereas in the last stages decisions are only based on the information of the objective space. The latter is achieved through a novel density estimator. The new method is compared with state-of-art MOEAs using several benchmarks with two and three objectives. The novel proposal attains much better results than state-of-the-art schemes, showing a more stable and robust behavior.**

I. INTRODUCTION

MULTI-OBJECTIVE Optimization Problems (MOPs) involve the simultaneous optimization of several objective functions that are usually in conflict [1]. A continuous box-constrained minimization MOP, which is the kind of problem addressed in this paper, can be defined as follows:

$$\begin{aligned} &minimize \quad \vec{F} = [f_1(\vec{\mathbf{x}}), f_2(\vec{\mathbf{x}}), ..., f_M(\vec{\mathbf{x}})] \\ &subject \;\; to \quad x_i^{(L)} \leq x_i \leq x_i^{(U)}, \quad i = 1, 2, ..., n. \end{aligned} \tag{1}$$

where n corresponds to the dimension of the variable space, $\vec{\mathbf{x}}$ is a vector of n decision variables $\vec{\mathbf{x}} = (x_1, ..., x_n) \in R^n$, which are constrained by $x_i^{(L)}$ and $x_i^{(U)}$, i.e. the lower bound and upper bound, and M is the number of objective functions to optimize. The feasible space bounded by $x_i^{(L)}$ and $x_i^{(U)}$ is denoted by Ω, each solution is mapped to the objective space with the function $F : \Omega \rightarrow R^M$, which consist of M real-valued objective functions and R^M is called the *objective space*.

Given two solutions $\vec{\mathbf{x}}$, $\vec{\mathbf{y}} \in \Omega$, $\vec{\mathbf{x}}$ dominates $\vec{\mathbf{y}}$, mathematically denoted by $\vec{\mathbf{x}} \prec \vec{\mathbf{y}}$, iff $\forall m \in 1, 2, ..., M : f_m(\vec{\mathbf{x}}) \leq f_m(\vec{\mathbf{y}})$ and $\exists m \in 1, 2, ..., M : f_m(\vec{\mathbf{x}}) < f_m(\vec{\mathbf{y}})$. The best solutions of a MOP are those whose objective vectors are not dominated by any other feasible vector. These solutions are known as the Pareto optimal solutions. The Pareto set is the set of all Pareto optimal solutions, and the Pareto front are the images of the Pareto set. The goal of most multi-objective optimizers is to obtain a proper approximation of the Pareto front, i.e., a set of well distributed solutions that are close to the Pareto front.

One of the most popular metaheuristics used to deal with MOPs is the Evolutionary Algorithm (EA). In single-objective EAs, it has been shown that taking into account the diversity of the variable space to properly balance between exploration and exploitation is highly important to attain high quality solutions [2]. Diversity can be taken into account in the design of several components such as in the variation stage [3], [4], replacement phase [5] and/or population model [6]. The explicit consideration of diversity leads to improvements in terms of premature convergence avoidance, meaning that taking into account the diversity in the design of EAs is specially important when dealing with long-term executions. Recently, some diversity management algorithms that combine the information of diversity, stopping criterion and elapsed generations have been devised. They have allowed to provide a gradual loss of diversity that depends on the time or evaluations granted to the execution [5]. Particularly the aim of such a methodology is to promote exploration in the initial generations and gradually alter the behavior towards intensification. These schemes have provided really promising results. For instance, new best-known solutions for some well-known variants of the frequency assignment problem [7], and for a two-dimensional packing problem [5] have been attained using the same principles. Additionally, this principle guided the design of the winning strategy of the Second Wind Farm Layout Optimization Competition[1], which was held in the Genetic and Evolutionary Computation Conference. Thus, the benefits of such kind of design patterns have been shown in several different single-objective optimization problems.

One of the goals in the design of Multi-objective Evolutionary Algorithms (MOEAs) is to obtain a well-spread set of solutions in the objective space. The maintenance of some degree of diversity in the objective space implies that complete convergence does not appear in the variable space [8]. In some way, the variable space inherits some degree of diversity

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Digital Object Identifier xxx

Manuscript received XX, XX 2019; revised XX XX, XX.

[1]https://www.irit.fr/wind-competition/

due to the way in which the objective space is taken into account. However, this is just an indirect way of preserving the diversity in the variable space, so in some cases the level of diversity might not be large enough to ensure a high degree of exploration. For instance, it has been shown that with some of the WFG benchmarks, in most of the state-of-the-art MOEAs the *distance parameters* quickly converge, meaning that the approach focuses just on optimizing the *position parameters* for a long period of the optimization process [8]. Thus, while some degree of diversity is maintained, a similar situation to premature convergence is presented meaning that genetic operators might not be able to generate better trade-offs.

Attending to the differences between state-of-the-art single-objective EAs and MOEAs, this paper proposes a novel MOEA, the Variable-Space-Diversity based MOEA (VSD-MOEA), that is based on controlling the amount of diversity in the variable space in an explicit way. Similarly to the successful methodology applied in single-objective optimization, the stopping criterion and the amount of evaluations performed are used to vary the amount of desired diversity. The main difference with respect to the single-objective case is that the objective space is simultaneously considered, which is performed with a novel objective-space density estimator. Particularly, the approach grants more importance to the diversity of the variable space in the initial stages, whereas as the generations evolve, it gradually grants more importance to the diversity of the objective space. In fact, at the last period of the execution, the diversity of the variable space is neglected, so in the last phases the proposal is quite similar to current state-of-the-art approaches. To our knowledge, this is the first MOEA whose design follows this adaptive principle. Since there exist currently a quite large amount of different MOEAs [9], three popular schemes have been selected to validate our proposal. This validation has been performed with several well-known benchmarks and proper quality metrics. The important benefits of properly taking into account the diversity of the variable space is clearly shown in this paper. Particularly, the advantages are clearer in the most complex problems. Note that this is consistent with the single-objective case, where the most important benefits have been obtained in complex multi-modal cases [7].

The rest of this paper is organized as follows. Section II provides a review of related papers. Some key components related to diversity and the VSD-MOEA design are discussed. The VSD-MOEA proposal is detailed in section III. Section IV is devoted to the experimental validation of the novel proposal. Finally, conclusions and some lines of future work are given in Section V. Note also that some supplementary materials are given. They include details of the experimental results with additional metrics as well as some explanatory videos.

II. LITERATURE REVIEW

This section is devoted to review some of the most important papers that are closely related to our proposal. First, some of the most popular ways of managing diversity in EAs are presented. Then, the state-of-the-art in MOEAs is summarized.

A. Diversity Management in Evolutionary Algorithms

The proper balance between exploration and exploitation is one of the keys to success in the design of EAs. In the single-objective domain it is known that properly managing the diversity in the variable space is a way to control such balance, and as a consequence, a large amount of diversity management techniques have been devised [10]. Particularly, these methods are classified depending on the component(s) of the EA that is modified to alter the amount of maintained diversity. A popular taxonomy identifies the following groups [11]: *selection-based*, *population-based*, *crossover/mutation-based*, *fitness-based*, and *replacement-based*. Additionally, the methods are referred to as *uniprocess-driven* when a single component is altered, whereas the term *multiprocess-driven* is used to refer to those methods that act on more than one component.

Among the previous proposals, the replacement-based methods have attained very high-quality results in last years [7], so this alternative was selected with the aim of designing a novel MOEA incorporating an explicit way to control the diversity in the variable space. The basic principle of these methods is to bias the level of exploration in successive generations by controlling the diversity of the survivors [7]. Since premature convergence is one of the most common drawbacks in the application of EAs, modifications are usually performed with the aim of slowing down the convergence. One of the most popular proposals belonging to this group is the *crowding* method which is based on the principle that offspring should replace similar individuals from the previous generation [12]. Several replacement strategies that do not rely on crowding have also been devised. In some methods, diversity is considered as an objective. For instance, in the hybrid genetic search with adaptive diversity control (HGSADC) [13], individuals are sorted by their contribution to diversity and by their original cost. Then, the rankings of the individuals are used in the fitness assignment phase. A more recent proposal [7] incorporates a penalty approach to alter gradually the amount of diversity maintained in the population. Particularly, initial phases preserve a larger amount of diversity than the final phases of the optimization. This last method has inspired the design of the novel proposal put forth in this paper for multi-objective optimization.

It is important to remark that in the case of multi-objective optimization, few works related to the maintenance of diversity in the variable space have been developed. The following section reviews some of the most important MOEAs and introduces some of the works that consider the maintenance of diversity in the variable space.

B. Multi-objective Evolutionary Algorithms

In recent decades, several MOEAs have been proposed. While the purpose of most of them is to provide a well-spread set of solutions close to the Pareto front, several ways of facing this purpose have been devised. Therefore, several taxonomies have been proposed with the aim of better classifying the different schemes [14]. Particularly, a MOEA can be designed based on Pareto dominance, indicators and/or decomposition [15]. Since none of the groups has a remarkable superiority over the others, in this work all of them are taken into account to validate our proposal. This section introduces the three types of schemes and some of the most popular approaches belonging to each category. Then, one MOEA of each category is selected to carry out the validation of VSD-MOEA.

The dominance-based category includes those schemes where the Pareto dominance relation is used to guide the design of some of its components such as the fitness assignment, parent selection and replacement phase. The dominance relation does not inherently promotes the preservation of diversity in the objective space, therefore additional techniques such as objective-space density estimators are usually integrated with the aim of obtaining a proper spread and convergence to the Pareto front.

In order to assess the performance of MOEAs, several quality indicators have been devised. In the indicator-based MOEAs, the use of the Pareto dominance relation is substituted by some quality indicators to guide the decisions performed by the MOEA. An advantage of this kind of algorithms is that the indicators usually take into account both the quality and diversity in objective space, so incorporating additional mechanisms to promote diversity in the objective space is not required. Among the different indicators, hypervolume is a widely accepted Pareto-compliance quality indicator. The Indicator-Based Evolutionary Algorithm (IBEA) [16] was the first method belonging to this category. A more recent one is the R2-Indicator-Based Evolutionary Multi-objective Algorithm (R2-EMOA) [17], which has reported a quite promising performance in MOPs. Its most important feature is the use of the R2 indicator.

Finally, decomposition-based MOEAs [18] are based on transforming the MOP into a set of single-objective optimization problems that are tackled simultaneously. This transformation can be performed in several ways, e.g. with a linear weighted sum or with a weighted Tchebycheff function. Given a set of weights to establish different single-objective functions, the MOEA searches for a single high-quality solution for each of them. The weight vectors should be selected with the aim of obtaining a well-spread set of solutions [1]. The Multi-objective Evolutionary Algorithm Based on Decomposition (MOEA/D) [19] is the most popular decomposition-based MOEA. Its main principles include problem decomposition, weighted aggregation of objectives and mating restrictions through the use of neighborhoods.

It is important to stand out that none of the most popular algorithms in the multi-objective field introduce special mechanisms to promote diversity in variable space. However, some efforts have been dedicated to this principle. A popular approach to promote the diversity in the decision space is the application of fitness sharing [20] in a similar way than in single-objective optimization. Although, fitness sharing might be used to promote diversity both in objective and decision variable space, most popular variants consider only distances in the objective space. Another MOEA designed to promote diversity in both the decision and the objective space is the Genetic Diversity Evolutionary Algorithm (GDEA) [21]. In this case, each individual is assigned with a diversity-based objective which is calculated as the Euclidean distance in the genotype space to the remaining individuals in the population. Then, a ranking that considers both the original objectives and the diversity objective is used to sort individuals. Another somewhat popular approach is to calculate distances between candidate solutions by taking into account both the objective and variable space [22], [23] with the aim of promoting diversity in both spaces. A different proposal combines the use of two selection operators [24]. The first one promotes diversity and quality in the objective space whereas the second one promotes diversity in the decision space. A different approach was to modify the hypervolume to integrate the decision space diversity in a single metric [25]. In this approach, the proposed metric is used to guide the selection in the MOEA. Finally, some indirect mechanisms that might affect the diversity have also been taken into account. Probably, the most popular one is the use of mating restrictions [26], [18].

Attending to the results of previous described approaches, it is clear that taking into account the decision space diversity in the design phase might bring benefits to decision makers because the final solutions obtained by these methods present a larger decision space diversity than the ones obtained by traditional approaches [22], [27]. Thus, while clear improvements are obtained when taking into account metrics related to the Pareto set, the benefits in terms of the obtained Pareto front are not so clear. We claim that one of the reasons of this behavior might be that the diversity in the variable space is considered in the whole optimization process. However, in a similar way that in the single objective domain, reducing the importance granted to the diversity in the decision space as the generations progress [5] might be really important to attain better approximations of the Pareto front. Currently, no MOEA considers this idea, so this principle has guided the design of our novel MOEA.

III. PROPOSAL

This section is devoted to fully describe our novel proposal. There are two main contributions in this paper. First, a novel objective-space density estimator is proposed. Second, a novel replacement phase that takes into account the diversity in the variable space is devised. The replacement phase does not only considers variable space diversity. Instead it integrates information of both the variable and objective space. Particularly, the novel objective-space density estimator is used inside the replacement phase. The main principle behind the design of the novel replacement is to use the stopping criterion and elapsed generations with the aim of gradually moving from exploration to exploitation during the search process. Note that this principle might be incorporated in any of the three categories of MOEAs. In this paper, our decision was to incorporate it in a dominance-based approach and apply it to problems with a low number of objectives. Particularly, our proposal is quite standard except for the replacement phase. Algorithm 1 shows the pseudocode of VSD-MOEA. The parent selection is performed with binary tournament based on the dominance raking with ties broken randomly. The variation stage is based on applying the well-known Simulated Binary

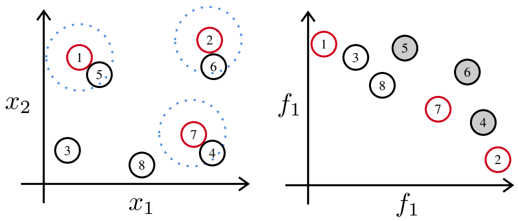


Fig. 1. Penalty Method of the Replacement Phase - The left side represents the variables space and the right side the objective space

Algorithm 1 Main procedure of VSD-MOEA

```
Initialization: Generate an initial population P_0 with N individuals.
Assign t = 0
while (not stopping criterion) do
    Evaluation: Evaluate all individuals in the population.
    Mating selection: Perform binary tournament selection based in the rank (ties
    are broken randomly) on P_t in order to fill the mating pool.
    Variation: Apply SBX crossover and Polynomial mutation to the mating pool to
    create a child population Q_t.
    Survivor selecction: Combine P_t and Q_t, and apply the replacement scheme
    (Algorithm 2) to create P_{t+1}.
    t = t + 1
```

Crossover (SBX) and polynomial mutation [28], [29]. The rest of this section are devoted to describe the replacement phase and the novel objective-space density estimator.

A. Replacement Phase of VSD-MOEA

The replacement phase of EAs is in charge of deciding in each generation which are the survivors among the members of the previous population and offspring. The novel replacement promotes a gradual movement from exploration to exploitation, which has been a quite beneficial principle in the design of single-objective optimizers [5]. Particularly, the replacement phase operates as follows. First, the members of the previous population and offspring are joined in a multi-set with $2 \times N$ individuals. Then, N individuals must be selected to survive, which is performed with an iterative process that selects an additional individual at each step. In order to take into account the diversity in the decision space, the Distance to Closest Neighbor (DCN) of each individual is calculated at each step. Thus, if the multi-set containing the currently selected survivors is called S, then the DCN of an individual I is calculated as $\min_{s \in S} Distance(I, s)$. Normalized Euclidean distances are considered, so in order to calculate distances between any two individual A and B, Eq. (2) is applied.

$$Distance(A, B) = \left(\frac{1}{n} \sum_{i=1}^{n} \left(\frac{A_i - B_i}{x_i^{(U)} - x_i^{(L)}} \right)^2 \right)^{1/2} \quad (2)$$

Note that individuals with large DCN values contribute significantly to promote exploration. In order to avoid an excessive decrease of the exploration degree, individuals with a DCN value lower than a threshold value are penalized and they can only be selected if non-penalized individuals do not exist. Then, among the non-penalized individuals, an objective-space density estimator is used to select the additional survivor of the iteration. In our case, the novel density estimator described in the next subsection is used.

Algorithm 2 Replacement Phase of VSD-MOEA

```
1:  Input: P_t (Population of current generation), Q_t (Offspring of current Generation)
2:  Output: P_{t+1}
3:  R_t = P_t ∪ Q_t
4:  P_{t+1} = ∅
5:  Penalized = ∅
6:  D_t = D_I − D_I * G_Elapsed / (0.9 * G_End)
7:  for k ∈ 1...M do
8:      Move to P_{t+1} the individual that optimize AWF_k (Eq. 3)
9:  while |P_{t+1}| ≤ N do
10:     Compute DCN of individuals in R_t with P_{t+1} used as reference set
11:     Move to Penalized the individuals in R_t with DCN < D_t
12:     if R_t is empty then
13:         Compute DCN of individuals in Penalized with P_{t+1} used as reference
            set
14:         Move to R_t the individual in R_t with largest DCN
15:     non − dominated − rank − assignment(R_t ∪ P_{t+1})
16:     Use the novel density estimator to select a new survivor from R_t and move it
        to P_{t+1}
17: return P_{t+1}
```

In order to better visualize the penalty method, it can be considered that after selecting each survivor, a hyper-sphere centered in such candidate solution — in the variable space — is created. Then, all the individuals that are inside a hyper-sphere are penalized and the objective-space estimator takes into account only the non-penalized individuals. This is illustrated in Fig. 1, which represents a state where three individuals have been selected to survive and an additional survivor must be picked up. The left side shows individuals in the variable space. Current survivors are marked with a red border and each one of them is surrounded by a blue dash circle with radius D_t. In this situation, the penalized individuals are the number 4, 5, and 6. In the objective space —right side — penalized individuals are shown with gray background, indicating that the objective-space density estimator can not select them.

Since penalizing with a large threshold value — radius of the hyperspheres — induces a higher degree of exploration, it makes sense to reduce this value during the optimization process. This is precisely one of the keys of our proposal. The sizes of the hyper-spheres are modified dynamically by taking into account the stopping criterion and elapsed generations. Particularly, the radius is decreased in a linear way starting from an initial distance. This means that in the initial phases exploration is promoted. However, as the size of the radius decreases only very close individuals are penalized, meaning that more exploitation is performed. Note that this method requires a parameter which is the initial radius of the hyper-spheres which is denoted as D_I. Setting this parameter with a large value might provoke the penalization of a lot of individuals, thus non-useful diversity might be maintained. However, too small values might not prevent fast convergence and therefore the approach with such a parameterization might behave as a traditional non-diversity based approach. The robustness of the proposal with respect to this additional parameter is studied in our experimental validation.

Algorithm 2 fully describes the replacement phase of VSD-MOEA. First, the population of the previous generation (P_t) and the offspring (Q_t) are joined in R_t (line 3). The multiset R_t contains, in each iteration, the remaining non-penalized individuals that might be selected to survive. The population of survivors (P_{t+1}) and the set containing the penalized individuals are initialized to the empty set (lines 4 and 5). Then, the threshold value (D_t) that is used to penalize too close individuals is calculated (line 6). Note that D_I denotes the initial radius or threshold value, $G_{Elapsed}$ is the amount of generations that have been evolved, and G_{End} is the stopping criterion, i.e. the number of generations that are to be evolved in the execution of VSD-MOEA. The linear decrease is calculated so that after the 90% of the generations, the D_t value is lower than 0, meaning that no penalties are performed. This means that in the first 90% of the generations, more exploration than in traditional MOEAs is induced, whereas in the final stages, a traditional MOEA is applied. Finally, for each objective a high-quality candidate solution is selected to survive. Note that selecting the best solution for each objective might provoke some drawbacks related to accepting small improvement in an objective at the cost of important worsening in other objectives [30]. To solve this issue augmented functions can be applied, which has been the alternative used in this paper. Particularly, for each objective k, the candidate solution that minimizes the Augmented Weighted Function (AWF) given in Eq. 3 is selected and consequently moved to P_{t+1} (line 8). Note that, augmented functions usually take into account weight vectors with the aim of dealing with objectives that present very different scales. Since benchmarks that have similar scales in each objective have been used in this paper, there was no need to apply such weight vectors.

$$AWF_k(\vec{x}) = f_k(\vec{x}) + 10^{-4} \times \sum_{j=1}^{m} f_j(\vec{x}) \tag{3}$$

Then, an iterative process that selects an individual at each iteration is executed until the survivors set contains N individuals (line 9). The iterative process works as follows. First, the DCN value of each reamining non-penalized individual is calculated (line 10). Then, those individuals with a DCN value lower than D_t are moved to the set of penalized individuals (line 11). If all the remaining individuals are penalized (line 12), it means that the amount of exploration is lower than expected. Thus, the individual with largest DCN values is recovered, i.e. moved to the non-penalized individuals set (lines 13 and 14). Finally, the objective space is taken into account. Specifically, the candidate individuals and the current survivors are joined. Then, the *fast-non-dominated-sort* procedure is executed with such a set, stopping as soon as a front with a candidate individual is found (line 15). Then, for each candidate individual that belongs to the lowest front, the individual with higher contribution to the diversity in the objective space is selected (line 16). The specific way in which the diversity in the objective space is measured is described in the next section.

B. A Novel Density Estimator for the Objective Space

Since the dominance definition is not related to the preservation of diversity in the objective space, dominance-based MOEAs incorporate special procedures to maintain diverse solutions, such as clustering and/or crowding. In this paper, we define a novel distance metric, and an iterative heuristic selection approach, which selects an individual of the best front and with the largest defined distance. Specifically, the novel distance is called "Improvement Distance" (ID) and it follows the same principles that guided the design of both indicators the IGD+ and $I_{\epsilon+}$ [31], [16], [32]. The main idea is to prefer those individuals whose quality in all objectives is similarly preserved. Particularly, a non-dominated individual can be very distant to the Pareto Front due that such individual could be the best in one objective but mainly deteriorated in the rest of objectives, so as result it has high diversity in objective space. In fact, high improvements in one objective value are related to larger selection probabilities and not the opposite, so this behavior should be avoided.

The key idea takes into account the dominance relation between the candidate and reference individuals. Consequently, the reference and the candidate individuals are compared. If the reference individual is dominated by the candidate individual, then the euclidean distance with no modification is implemented. However if they are non-dominated with each other, then is calculated the minimum distance from the reference individual to the dominated region by the candidate individual. Additionally, if the candidate individual is dominated by the reference individual, then is computed the I_ϵ indicator, that gives the minimum distance by which the candidate individual needs to or can be translated in each dimension in objective space such that the reference individual is dominated. Therefore, this distance can be viewed as an amount of inferiority of the solution in comparison with the reference individual. The improvement distance is defined in Equation (4) which incorporates the I_ϵ indicator (Equation 5) where R and C are the reference and candidate solutions respectively.

$$ID(R,C) = \left(\sum_{i=1}^{M} (max(0, R_i - C_i))^2\right)^{1/2} - I_\epsilon(R,C) \tag{4}$$

$$I_\epsilon(R,C) = \begin{cases} min_\epsilon\{f_i(C) - \epsilon \leq f_i(R)\} & R \preceq C \\ 0 & otherwise \end{cases} \tag{5}$$

Specifically, this distance is considered as a Weakly Pareto Compliant Indicator. In addition, this metric relaxes some difficulties encountered when the number of objectives is increased, given that the solutions in many objectives are usually non-dominated with each other by using the Pareto dominance relation. This means a very low selection pressure toward the Pareto front in Pareto-dominance-based MOEAS [33]. Principally, the improvement distance is effective with over-prioritization of dominance-resist solutions i.e., solutions with exceptional performance in one objective and extremely poor performance in many others [34].

Given that the design of this MOEA is considered for long-term executions, the Algorithm 2 is implemented efficiently. Thus, the distances are pre-computed and are updated in each iteration, the same for the dominance count information that is used in the *conditionally-non-dominated-sort*. In fact, the worst case complexity of this algorithm is $O((3N)^2 \times n)$, since that the dimension of the decision variable space is usually bigger than the number of objectives.

IV. Experimental Validation

In this section the experimental validation is carried out, showing that controlling the diversity in the variable space is a way to improve further some of the results obtained by the state-of-art-MOEAs. Firstly, several technical specifications taken into account in our comparison outline are explained. Thereafter, to have a broad perception of the VSD-MOEA some experiments are driven. Between them an analyzes which is designed to test the scalability in the decision variable space of each MOEA. This analyzes is narrowed through some specific and well known problems. In the same line, with the intention to have a better understanding of the critical parameter that induces the initial amount of diversity is taken into account. This is carried out through several settings and with all the test-problems. Finally, since that the mechanism imposed in our proposal, which avoid the premature convergence, highly depends on the elapsed time, we reported the benefits of it considering short-term and long-term executions. The latter experiments shows that the VSD-MOEA has a decent performance in midterm executions.

To validate the proposed MOEA, in this work are considered some of the most popular benchmark in the multi-objective field. Particularly, the WFG [35], DTLZ [36], and UF [37] test problems have been used for our purpose. In addition, out experimental validation includes the VSD-MOEA, as well three well-known state-of-the-art MOEAs. There are the NSGA-II [38] , the MOEA/D [39], and the R2-EMOA [40], that can be classified as based-dominance, based-decomposition, and based-indicators respectively. The MOEA/D implementation taken into account belongs to the first place in the “Congress on Evolutionary Computation 2009” (CEC) [41]. Given that all the considered algorithms are stochastic, each execution was repeated 35 times with different seeds. The common configuration in all of them was the following: the stopping criterion was set to $250,000$ generations, the population size was fixed to 100, the WFG test problems were configured with two and three objectives, which are set to 24 parameters, where 20 of them are distance parameters, and 4 are position parameters. Additionally, in the DTLZ test instances, the number of decision variables is set to $n = M + r - 1$, where $r = \{5, 10, 20\}$ for DTLZ1, DTLZ2 to DTLZ6 and DTLZ7 respectively, as is suggested by the authors [36]. The UF benchmark is composed of ten test instances, which is categorized in two groups that are based in the number of objective functions to solve, thus the first seven consists of two objectives, and the reaming of three, also the number of decision variables taken into account in each one is $n = 30$. In general, the crossover and mutation operators are the Simulated Binary Crossover (SBX), and polynomial [28], [29], which probabilities are set to 0.9 and $1/n$ respectively. Also, the crossover and mutation distribution indexes were assigned to 2 and 50 respectively. The extra-parameterization of each algorithm is showed in the Table I.

Despite the fact that the MOEA/D, and R2-EMOA can be employed with the same utility function –in this case the Tchebycheff function– each one of them is designed through a notorious dissimilar paradigm. Thus, the weight vectors taken

TABLE I
PARAMETERIZATION OF EACH MOEA CONSIDERED

Algorithm	Configuration
MOEA/D	Max. updates by sub-problem (η_r) = 2, tour selection = 10, neighbor size = 10, period utility updating = 30 generations probability local selection (δ) = 0.9,
VSD-MOEA	$D_I = 0.4$
R2-EMOA	$\rho = 1$, offspring by iteration = 1

into consideration for each one of them were different. The main reason of this, is that the R2-EMOA can be configured with a different population size than the number of weight vectors without been significantly affected. Particularly, the R2-EMOA employs 501 and 496 weight vectors for two and three objectives respectively. Contrarily, in the MOEA/D each weight vector is identified as a sub-problem, therefore the population should correspond to the same number of weight vectors. In addition, the weight vectors used in the MOEA/D should be uniformly scattered on the unit-simplex, however it can be a drawback since that the number of vectors required for this task increases non-linearly according the number of objectives. Therefore, in this version is applied the method proposed in [42], [43] where the uniform design (UD) [44] and good lattice point method (GLP) are combined. In this way, the number of weight vectors that is required by this MOEA is not affected by the number of objectives.

Mainly, the experimental analyzes is carried out considering the hypervolume indicator (HV). The HV metric measures the size of the objective space dominated by the approximated solutions given a reference point, so the solutions dominated by the reference point are not considered. Particularly, the reference point is chosen to be a vector which values are sightly larger (ten percent) than the nadir point as is suggested in [45]. Similarly that in [46], and to have a fair comparison the normalized HV is taken into account. Specifically, the HV reported is computed as the ratio between the HV reached by a set of solutions and the HV of the optimal Pareto Front. In this way, the more approximated to the unity this metric is, the more converged are the solutions to the Pareto Front.

In order to statistically compare the HV results, a similar guideline than the proposed in [47] was used. First a Shapiro-Wilk test was performed to check whatever or not the values of the results followed a Gaussian distribution. If, so, the Levene test was used to check for the homogeneity of the variances. If samples had equal variance, an ANOVA test was done; if not, a Welch test was performed. For non-Gaussian distributions, the non-parametric Kruskal-Wallis test was used to test whether samples are drawn from the same distribution. An algorithm X is said to win algorithm Y when the differences between them are statistically significant, if the mean and median obtained by X are higher than the mean and median achieved by Y.

In the Tables II, III are showed the normalized hypervolume with two an three objectives respectively. From this empirical results, it is clear that the VSD-MOEA achieves the highest general mean HV with both two and three objectives (last row). Even more, the standard deviation is the lowest in almost all the problems, therefore this MOEA shows to be stable, consequently it is able to attain similar results through several runs. The best general mean considering two objectives is achieved by the VSD-MOEA with 0.955. Also, the second general mean is obtained by the NSGA-II with 0.886. It is important to remark that the general mean can be unsteady to atypical measurements, thus very low values could affect highly the general mean. However, all the state-of-the-art-MOEAs achieved a general mean of 0.88 considering two objectives, whilst the VSD-MOEA obtained 0.95. Considering three objectives the performance of the NSGA-II is seriously affected, this might occurs since that the density estimator employed in the NSGA-II highly depends on the dominance-relation, thus in some problems (e.g. multi-frontal problems UF10) the solutions do not converge adequately to the Pareto front. In the same line, the VSD-MOEA achieved the best HV values in almost all the problems, in fact such values that are higher than 0.9 are close enough to the Pareto Front. The second best MOEA based in the general mean is the R2-EMOA with 0.855, despite that it adopts the same utility function that the MOEA/D, the latter is lightly lower with 0.835, this might occurs given that the MOEA/D has several parameters to be tuned, and the selected configuration could be insufficient to long-term executions.

In the Tables IV and V are showed the statistical tests with two and three objectives respectively. In the column tagged "Diff" is computed the difference between the mean of each algorithm and the best mean achieved. Taking into account two objectives the VSD-MOEA attains the best score of 52 wins, the second best score is attained by the R2-EMOA with 34 wins. In spite that the NSGA-II achieves a better general mean, it wins less times that the remaining MOEAs. Meaning that the NSGA-II obtains values close enough than the best MOEA, this can be viewed in the last row where is computed the total sum of all the problems. Generally speaking, the best results are obtained by the VSD-MOEA, since that the total sum of the "Diff" column is 0.061, thus when this algorithm does not achieved the best results, it obtained near solutions to the best results. Furthermore, the algorithms R2-EMOA and MOEA/D achieved a similar total "Diff" values. Particularly, the worst "Diff" value achieved by the VSD-MOEA is with the problem WFG6. This problem is uni-modal and non-separable, and this might occurs since that the initial factor distance is very high. In fact through other experimental analyzes this problem was correctly solved with $D_I = 0.1$ whose mean achieved was of 0.917, and 0.868 for two and three objectives respectively. In addition, a similar behavior can be seen with three objectives (Table V), where the VSD-MOEA improves significantly in comparison to the state-of-the-art-MOEAs. Particularly, the most complicated problems are better solved by the VSD-MOEA as are UF3, UF4, UF5, and UF6 composed by two objectives, and UF9, UF10 with three objectives. The UF5 is considered as one of the most difficult problems since that the optimal Pareto front is conformed by 21 points, also it has several sub-optimal regions where the solutions could suffer stagnation. Nevertheless, since that it is a disconnected Pareto front, the MOEAs that considers weight vectors faces several difficulties (e.g. knee regions), consequently they have more chances to stagnate. In fact in the latter problem the NSGA-II has a lower "Diff" value (0.048) than the MOEA/D and R2-EMOA (0.205 and 0.122). Diversely, the UF10 is a multi-frontal problem, this means that there exists different sub-optimal non-dominated fronts that correspond to different locally optimal values [48], this characteristic increases the difficulties of the problem as the number of objectives increases. However, the latter problem has notably converged better in VSD-MOEA (0.627) than in the remaining MOEAs (second best 0.413).

A. Decision Variable Scalability Experiments

The scalability of each MOEA is also evaluated respect with the number of decision variables [49]. The figures 2 and 3 show the mean of HV attained with 50, 100, 250, 500, and 1000 variables respectively. The scalability study was taken into consideration with some problems conformed by easy and difficult characteristics. Particularly, the problems considered in this analyzes were the DTLZ4, UF5 which are conformed by two objectives, and DTLZ4, UF10 with three objectives. In addition, each selected problem was repeated 35 times, thus the mean of all the HV values are reported. The VSD-MOEA attained the best results in two objectives with both problems, however it is remarkable to notice that the HV value reported by the NSGA-II improves as the number of decision variables is increased. Specifically, the DTLZ4 is uni-modal, and its Pareto shape is concave, also it has a polynomial bias. This interesting behavior is also showed by the R2-EMOA, and can be explained as follows. The probability of stagnation in certain sub-optimal regions can be avoided increasing the decision variables space. It highly depends in the operators that are taken into account (in this case the SBX and polynomial mutation), thus in some circumstances the more bigger the variable decision space is, the better quality solutions are reached. This can be lightly seen in the MOEA/D after 250 variables, however we believe that this MOEA is highly parameter-sensitive. A differently effect is presented in the UF5, in this problem the more bigger the variable space is, the less quality solutions are obtained. Despite this the VSD-MOEA still attains the best HV values. However, the VSD-MOEA is not the best MOEA in all the circumstances, it can be seen considering three objectives (Figure 3). Specifically, where the number of variables is increased to one thousand in both problems (DTLZ4 an d UF10) its performance is highly degraded. This might occurs for several reasons, perhaps the most important is related with the measurement employed in the variable space (Euclidean distance), this inconvenient is popularly known as *The Curse of Dimensionality* [50], [51]. This drawback means that under certain broad conditions, as dimensionality increases, the distance to the nearest neighbor approaches the distance to the farthest neighbor. In other words, the contrast in distances to different data points becomes no-existent. The remaining MOEAs show a similar behavior with the DTLZ4 with two and three objectives. As previously mentioned the performance of VSD-MOEA is seriously deteriorated, however it is still better than the NSGA-II. Also, the R2-EMOA seems to be enough stable with three objectives in the problems DTLZ4 and UF10.

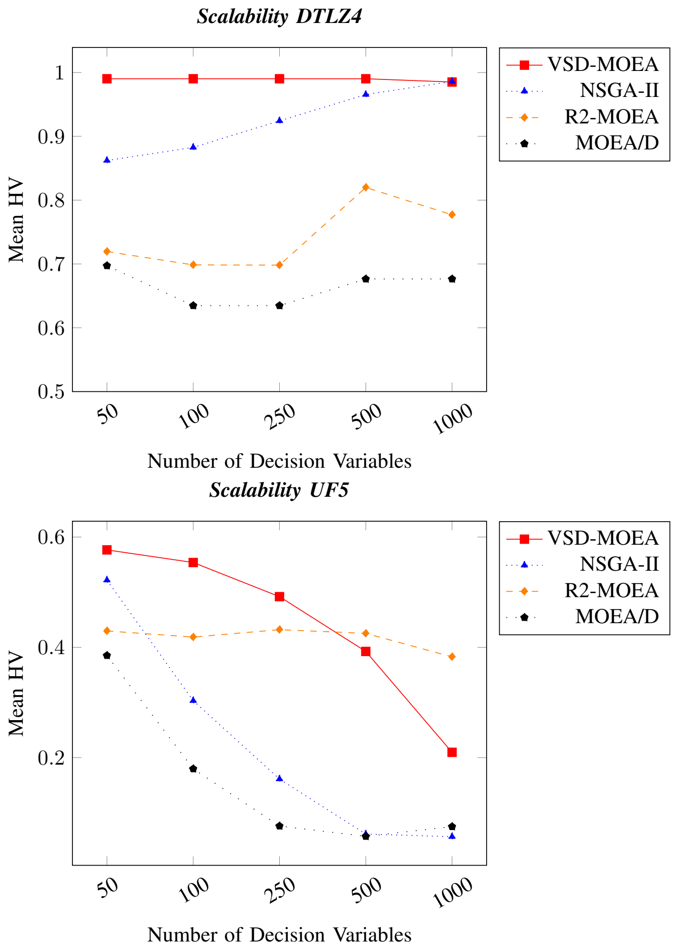


Fig. 2. Mean of the HV (35 runs) considering two objectives.

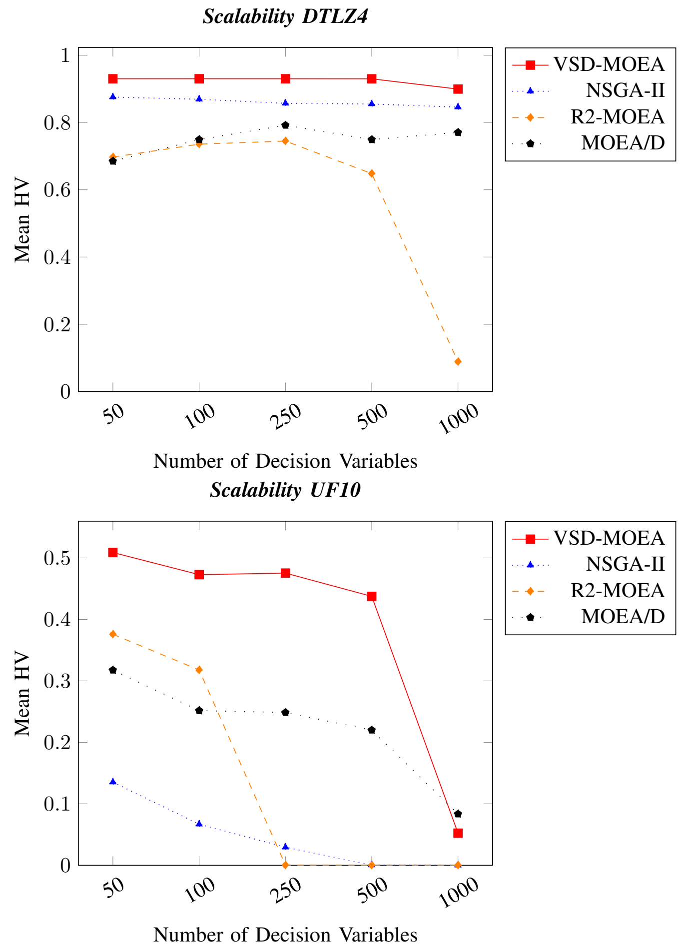


Fig. 3. Mean of the HV (35 runs) considering three objectives.

B. Analyzes of the Initial Factor Distance

The VSD-MOEA induces the diversity of the decision variable space through the initial distance factor (D_I), such parameter is decreased as the number of generations elapses. Given that this parameter influences the performance of the algorithm, a detailed empirical analyzes is taken into account as follows. Since that this parameter is computed as a fraction of the main normalized diagonal which belongs to the unitary hyper-cube, the portions considered are $D_I = \{0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$. In the Figure 4 is shown the general mean of the HV for each configuration with two and three objectives respectively. Particularly, the initial parameter $D_I = 0.0$, which does not promote diversity, should have a similar performance than a classic MOEA. In spite that none diversity is promoted, the VSD-MOEA achieves better general mean values than the remaining MOEAs, those values are 0.905, 0.895 for two and three objectives respectively. Even more, the benefits of promoting diversity are outstanding from 0.909 and 0.895 of two and three objectives to 0.936 and 0.899 with $D_I = 0.1$. It seems that such benefits are more notorious with two objectives than with three. This might occurs given that the population size might not been enough to cover the entire objective space. Based on this empirical results, the most suitable parameter configuration should be set with $D_I = 0.4$.

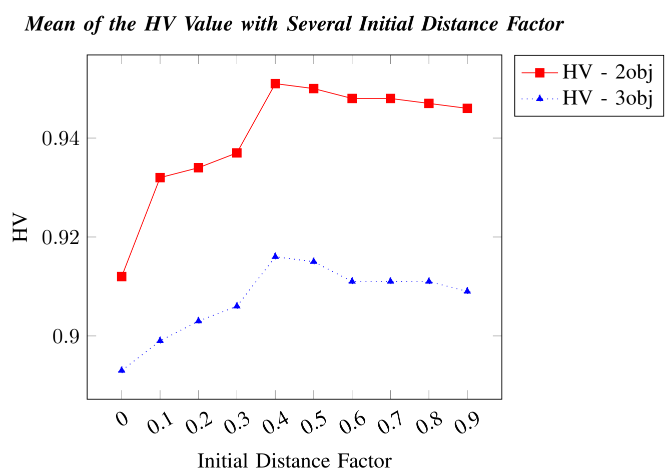


Fig. 4. Mean of Inidicator Considering All Instances with Several Initial Distance Factors

C. Diversity of the MOEAs Through Generations

In order, to have a better understanding of the diversity behavior some WFG problems have been selected. The WFG problems divide the decision variables in two kinds of parameter: the distance parameters and the position parameters. A parameter x_i is a distance parameter when for all parameter vectors $\vec{F}(\mathbf{x})$, modifying x_i in $\vec{F}(\mathbf{x})$ results in a parameter vector that dominates $\vec{F}(\mathbf{x})$, is equivalent to $\vec{F}(\mathbf{x})$, or is dominated by $\vec{F}(\mathbf{x})$. However, if x_i is a position parameter, modifying x_i in $\vec{F}(\mathbf{x})$ always results in a vector that is incomparable or equivalent to $\vec{F}(\mathbf{x})$ [52].

In this section we show that state-of-the-art-MOEAS do not always maintain high enough diversity. Particularly, the selected problems are used to show that premature convergence appears in the set of distance parameters. Consequently, the operators involved lose its exploratory strength. We select the WFG1, WFG5, and WFG6 problems, because they have simple definition, but most MOEAs faces difficulties with them. In addition, those problems were taken into account given that the WFG1 and WFG5 were better solved by the VSD-MOEA. Generally speaking, the WFG1 converged to the Pareto Front with our proposal, this since that the accuracy obtained was of 0.993. In contrast, the WFG5 was still far away of the Pareto Front which HV mean was of 0.923. Contrarily, the WFG6 was chosen since that the VSD-MOEA achieved the worst results. Whereas the WFG1 and WFG6 are uni-modal, the WFG5 is a highly deceptive problem. Moreover, the WFG1 and WFG5 are conformed by the separable properties of its objective functions. In fact, the distance parameters values associated to Pareto optimal solutions for WFG1-WFG7 have exactly the same values in the distance parameters. This values is shown as follows:

$$x_{i=k+1:n} = 2i \times 0.35 \tag{6}$$

Taking into account the stochastic behavior of MOEAS, 35 independent executions were run by each selected problem. In all of them, the stopping criterion was set to $250,000$ generations. In order to analyze the diversity, the average Euclidean distance among individuals (ADI) is calculated, i.e. the mean value of all pairwise distances among individuals in the population is reported. In the Figures 5, 6, and 7 are showed the evolution of diversity of the WFG1, WFG5, and WFG6 respectively. In those figures each MOEA is showed by dashed and solid lines that represents two and three objectives respectively. Particularly, the VSD-MOEA properly maintains diversity in both kind of parameters, while in the remaining algorithms the diversity in the distance parameters is lost after the 10% (generation 2500) of total generations. Thus, after those MOEAS converges, they are basically modifying the position parameters, so the majority of the time is improving further the diversity in the objectives space and the convergence is neglected. In fact, this diversity issue is also present in the WFG5. However, all the MOEAs have a minimum lower bound of diversity in the position parameters of those selected problems. Contrastively, considering the WFG5 (Figure 6) the VSD-MOEA does not converge at all in the distance parameters with three objectives, this might be an effect of promoting too much diversity. Besides this issue, the VSD-MOEA still achieves the best HV values. In addition, this drawback is more notorious in the problem WFG6. In fact in two and three objectives this MOEA did not converges in the distance

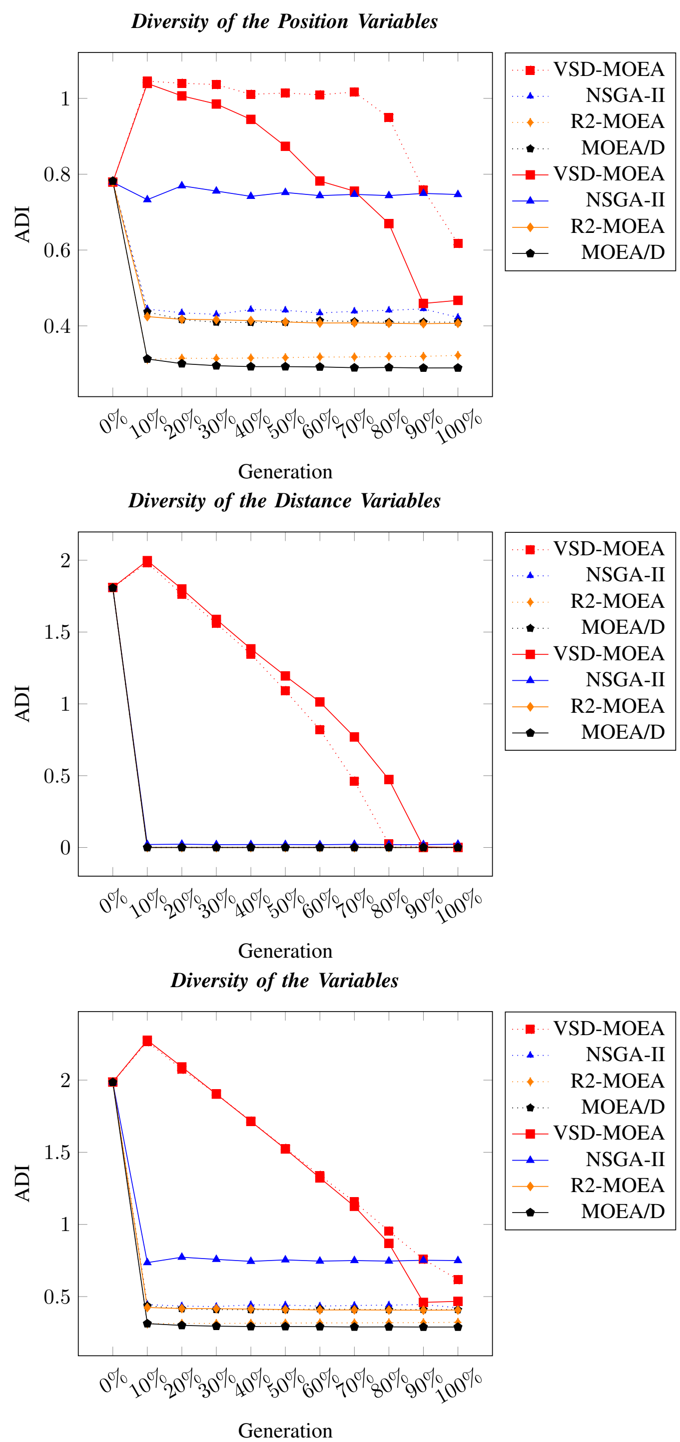


Fig. 5. Evolution of the diversity in the problem WFG1

parameters, and the position variables are still more diversified as in WFG1.

D. Performance of the MOEAs Related with the Criteria Stop

In spite that the VSD-MOEA is specially designed to attain quality solutions in long-term executions. In this section is showed the performance of the MOEAs modifying the criteria stop. Mainly, three ranges of criteria stop were reviewed. Each range was split in ten intervals, such ranges considered were

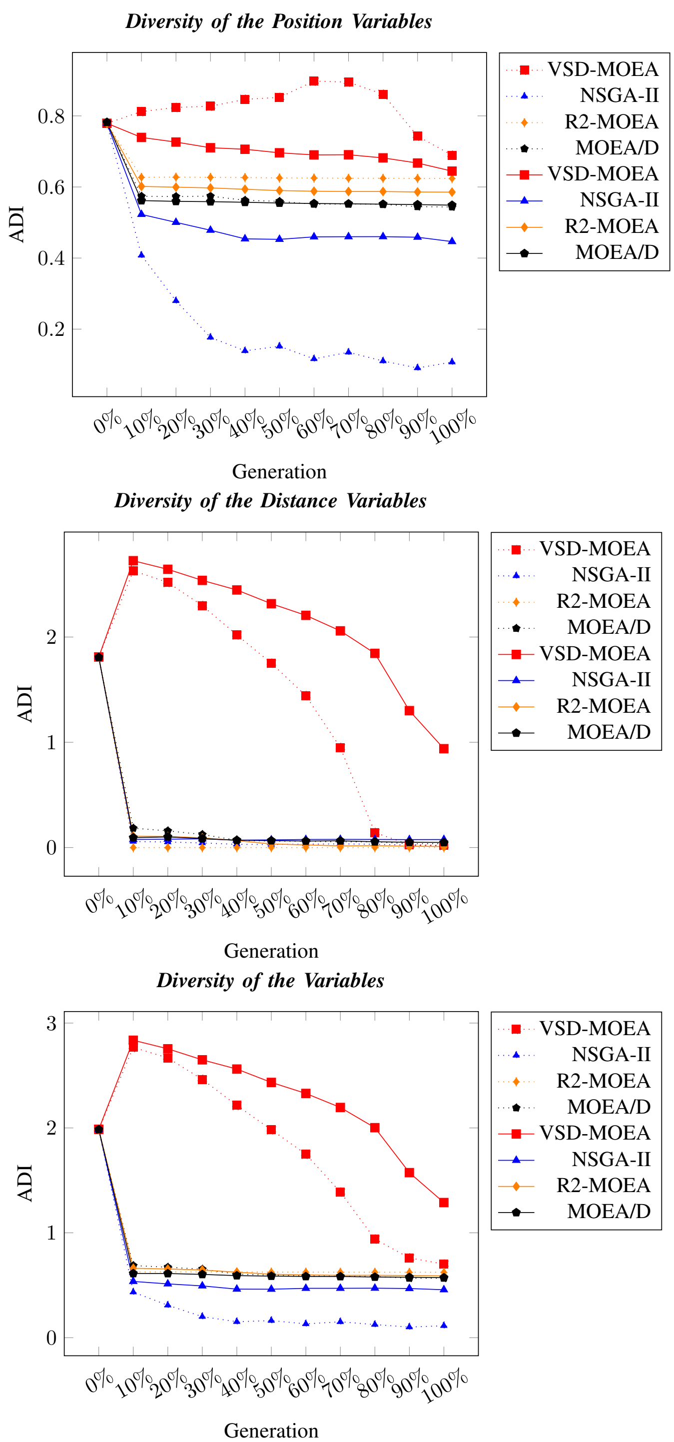


Fig. 6. Evolution of the diversity in the problem WFG5

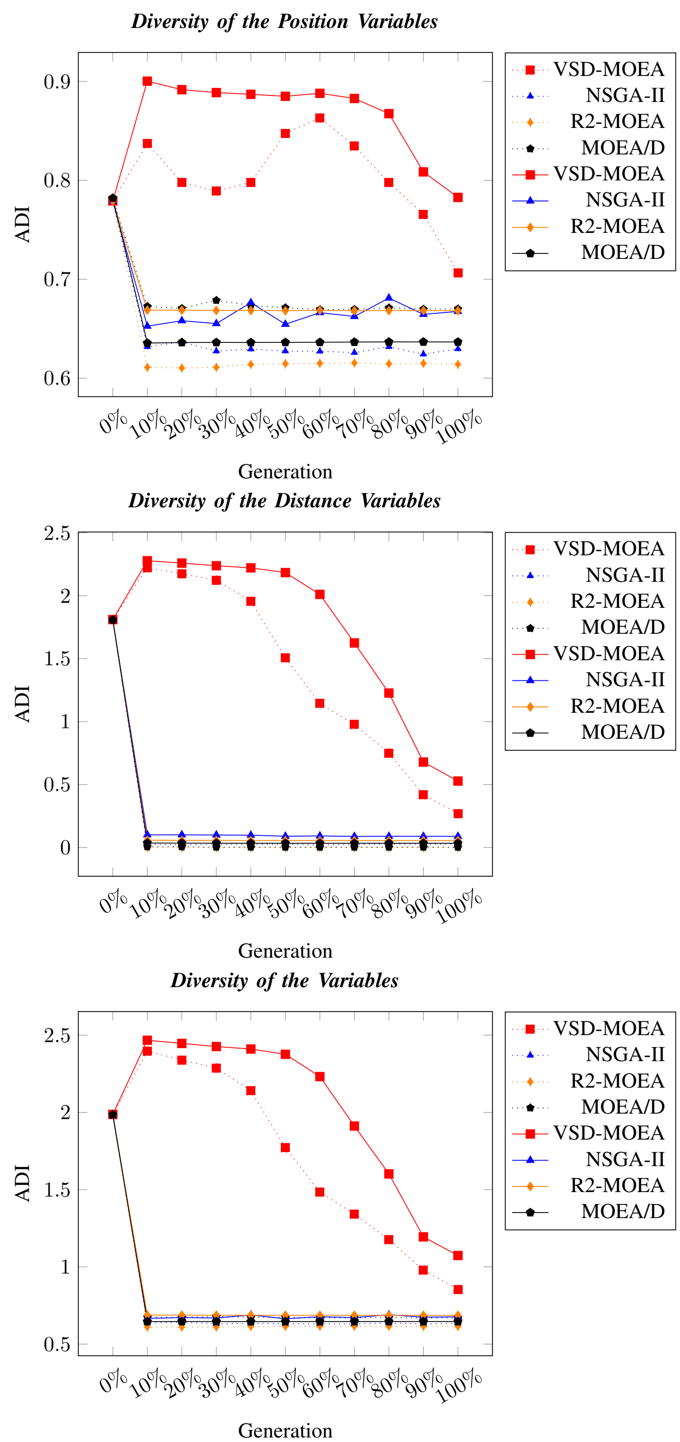
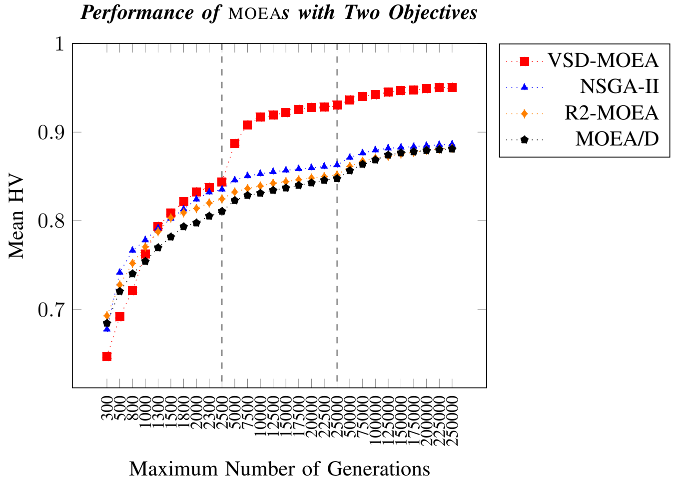


Fig. 7. Evolution of the diversity in the problem WFG6

$[300, 2500]$, $[2500, 25000]$ and $[25000, 250000]$ respectively. In the Figure 8 are showed the mean HV values attained with each MOEA with two and three objectives respectively. This shows that the VSD-MOEA attains the worst HV values considering only 300 generations. However, as the number of generations increases the HV values are improved significantly, in fact after 2500 generations the VSD-MOEA has a notorious improvement respect to the state-of-the-art-MOEAs in both two and three objectives. It can be seen that as the maximum number of generations is higher than 25000 the remaining MOEAS seems to be stagnated, specifically considering two objectives. Moreover, considering three objectives the R2-EMOA shows improvements after 100000 generations, however it still has lower values than the VSD-MOEA. In addition, taking into account three objectives the best general mean value achieved was above 0.9, there is still possible that increasing the number of generations the VSD-MOEA achieves better results. However, in some circumstances, as seems to



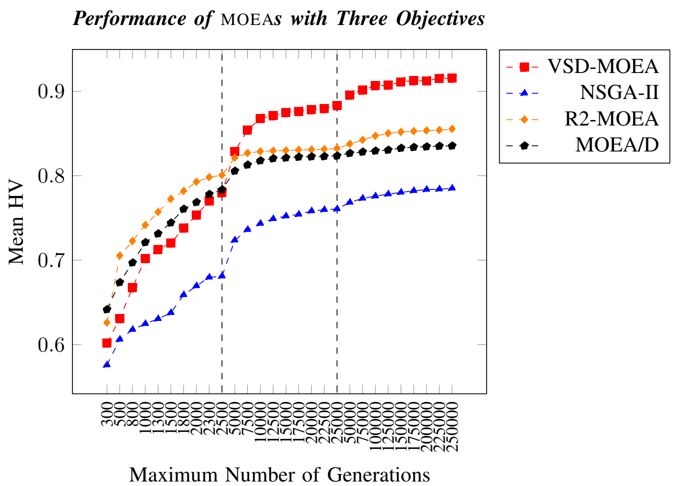


Fig. 8. Performance of the MOEAs considerin several maximum number of generations.

be in the case of two objectives, might not be able to achieve a better accuracy, thus an upper bound could be present. It is significant important to note how the performance of the NSGA-II is degraded in three objectives, differently to the R2-EMOA, and the MOEA/D. Even more, the MOEA/D seems to be stagnated after 25000 generations, this might occurs for the greedy selection approach that this MOEA incorporates. In fact the MOEA/D attained better results than the R2-EMOA in short-term executions.

V. Conclusion

The evolutionary algorithms have been a most popular approaches to deal with complex optimization problems. Given that the quality of the MOEAs is directly measured in the objective space, the majority of them incorporate sophisticated mechanisms with the aim to attain diverse and converged solutions to the Pareto Front. Since this, the diversity in the decision variables tend to be neglected. However, equivalently than in single-objective problems, where the diversity in variable space has a critical role, the multi-objective problems should take into consideration the diversity in both spaces to avoid premature convergence in sub-optimal regions.

In this work we have provided an algorithm with a particular replacement phase. This phase considers the diversity in both spaces, specifically in the variable space the diversity is based in a decremented dynamic concept. Thus, at the first stages the diversity in variables space is promoted, as the generations elapses this diversity is gradually reduced, thus at the last stages the replacement phase works as a classic MOEA.

In addition, is suggested a novel density estimator of the objective space, which is based in the IGD+ indicator being considered as weakly Pareto compliant. In the experimental validation carried out, is showed that our proposal not only improves the state-of-the-art-MOEAs in long-term execution, also offers a reasonable performance in short-term executions. This validation shows that the VSD-MOEA is able to properly solve several test problems, also it attained better results than the remaining MOEAs in the problems with the most difficult characteristics. In addition, some scalability experiments with the decision variables are carry out through long-term executions, results indicate the superiority and stability measured with the hypervolume indicator.

Therefore, is shown the relevance in having a properly management of the diversity in the variable space, and established a way to preserve the quality and diversity in both decision and objective space.

In future, we plan to develop an adaptive scheme to avoid setting the initial factor distance, even though the VSD-MOEA is not highly sensitive to this parameter adapting this parameter could result in a most robust MOEA. In addition, a multi-objective memetic algorithm with this principle should be proposed, since that this approach managements the diversity a local search should provides even better results.

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TABLE II
STATISTICS HV WITH TWO OBJECTIVES

	MOEA/D				**NSGA-II**				**R2-MOEA**				**VSD-MOEA**			
	Min	**Max**	**Mean**	**Std**	**Min**	**Max**	**Mean**	**Std**	**Min**	**Max**	**Mean**	**Std**	**Min**	**Max**	**Mean**	**Std**
WFG1	0.984	0.993	0.992	0.002	0.987	0.993	0.992	0.002	0.946	0.994	0.988	0.012	0.984	0.994	0.992	0.003
WFG2	0.965	0.996	0.967	0.007	0.966	0.998	0.974	0.014	0.965	0.966	0.966	0.000	0.998	0.998	0.998	0.000
WFG3	0.992	0.992	0.992	0.000	0.987	0.988	0.987	0.000	0.991	0.992	0.991	0.000	0.992	0.992	0.992	0.000
WFG4	0.988	0.988	0.988	0.000	0.983	0.987	0.985	0.001	0.991	0.991	0.991	0.000	0.990	0.990	0.990	0.000
WFG5	0.876	0.893	0.882	0.005	0.884	0.899	0.890	0.002	0.886	0.895	0.891	0.003	0.911	0.946	0.926	0.008
WFG6	0.879	0.940	0.914	0.016	0.894	0.942	0.913	0.012	0.875	0.942	0.912	0.015	0.858	0.885	0.869	0.006
WFG7	0.988	0.988	0.988	0.000	0.983	0.987	0.984	0.001	0.991	0.991	0.991	0.000	0.990	0.990	0.990	0.000
WFG8	0.800	0.822	0.811	0.006	0.771	0.801	0.789	0.006	0.803	0.824	0.815	0.005	0.830	0.955	0.947	0.020
WFG9	0.795	0.972	0.883	0.082	0.793	0.966	0.832	0.070	0.797	0.976	0.884	0.079	0.964	0.975	0.970	0.003
DTLZ1	0.993	0.993	0.993	0.000	0.990	0.992	0.991	0.000	0.992	0.992	0.992	0.000	0.992	0.992	0.992	0.000
DTLZ2	0.989	0.989	0.989	0.000	0.986	0.988	0.987	0.000	0.991	0.992	0.992	0.000	0.990	0.990	0.990	0.000
DTLZ3	0.989	0.989	0.989	0.000	0.987	0.989	0.989	0.001	0.991	0.992	0.992	0.000	0.990	0.990	0.990	0.000
DTLZ4	0.259	0.989	0.781	0.330	0.259	0.988	0.863	0.274	0.259	0.992	0.657	0.365	0.990	0.990	0.990	0.000
DTLZ5	0.989	0.989	0.989	0.000	0.986	0.988	0.987	0.000	0.991	0.992	0.992	0.000	0.990	0.990	0.990	0.000
DTLZ6	0.448	0.910	0.700	0.105	0.138	0.511	0.322	0.075	0.510	0.922	0.691	0.107	0.990	0.990	0.990	0.000
DTLZ7	0.996	0.996	0.996	0.000	0.996	0.997	0.996	0.000	0.997	0.997	0.997	0.000	0.996	0.996	0.996	0.000
UF1	0.991	0.993	0.992	0.000	0.986	0.989	0.988	0.000	0.978	0.994	0.990	0.005	0.992	0.995	0.994	0.000
UF2	0.987	0.993	0.991	0.002	0.980	0.983	0.981	0.001	0.984	0.991	0.988	0.002	0.986	0.992	0.989	0.002
UF3	0.481	0.674	0.597	0.043	0.678	0.871	0.784	0.048	0.531	0.704	0.589	0.041	0.805	0.909	0.867	0.025
UF4	0.881	0.917	0.908	0.006	0.875	0.910	0.889	0.008	0.923	0.935	0.929	0.003	0.920	0.930	0.925	0.002
UF5	0.035	0.792	0.484	0.165	0.256	0.766	0.641	0.104	0.123	0.792	0.566	0.192	0.586	0.762	0.658	0.043
UF6	0.255	0.711	0.447	0.114	0.235	0.801	0.635	0.120	0.349	0.767	0.568	0.113	0.668	0.922	0.827	0.080
UF7	0.987	0.991	0.990	0.001	0.980	0.983	0.981	0.001	0.557	0.991	0.910	0.150	0.975	0.991	0.988	0.003
Mean	0.806	0.935	0.881	0.038	0.808	0.927	0.886	0.032	0.801	0.940	0.882	0.048	0.930	0.964	0.951	0.008

TABLE III
STATISTICS HV WITH THREE OBJECTIVES

	MOEA/D				**NSGA-II**				**R2-MOEA**				**VSD-MOEA**			
	Min	**Max**	**Mean**	**Std**	**Min**	**Max**	**Mean**	**Std**	**Min**	**Max**	**Mean**	**Std**	**Min**	**Max**	**Mean**	**Std**
WFG1	0.958	0.969	0.966	0.002	0.925	0.945	0.935	0.005	0.968	0.979	0.975	0.002	0.978	0.984	0.982	0.001
WFG2	0.973	0.978	0.976	0.001	0.959	0.974	0.968	0.004	0.962	0.963	0.963	0.000	0.988	0.991	0.989	0.001
WFG3	0.992	0.992	0.992	0.000	0.976	0.988	0.985	0.002	0.991	0.992	0.992	0.000	0.989	0.989	0.989	0.000
WFG4	0.864	0.865	0.865	0.000	0.854	0.883	0.868	0.007	0.903	0.905	0.904	0.000	0.918	0.920	0.919	0.000
WFG5	0.795	0.804	0.797	0.002	0.806	0.836	0.821	0.008	0.843	0.853	0.848	0.002	0.842	0.861	0.853	0.005
WFG6	0.777	0.832	0.809	0.013	0.788	0.836	0.815	0.011	0.847	0.875	0.857	0.007	0.823	0.848	0.834	0.007
WFG7	0.864	0.865	0.865	0.000	0.858	0.889	0.875	0.008	0.901	0.905	0.904	0.001	0.919	0.920	0.919	0.000
WFG8	0.778	0.785	0.782	0.002	0.697	0.730	0.716	0.008	0.816	0.821	0.819	0.001	0.877	0.912	0.902	0.009
WFG9	0.726	0.851	0.819	0.039	0.720	0.833	0.746	0.027	0.773	0.895	0.872	0.038	0.763	0.881	0.872	0.019
DTLZ1	0.950	0.950	0.950	0.000	0.935	0.950	0.943	0.004	0.939	0.943	0.941	0.001	0.962	0.966	0.964	0.001
DTLZ2	0.899	0.899	0.899	0.000	0.871	0.901	0.886	0.007	0.913	0.916	0.915	0.001	0.928	0.930	0.930	0.000
DTLZ3	0.899	0.899	0.899	0.000	0.876	0.901	0.890	0.006	0.914	0.916	0.915	0.000	0.928	0.931	0.930	0.000
DTLZ4	0.151	0.899	0.813	0.238	0.871	0.904	0.888	0.007	0.151	0.916	0.675	0.298	0.929	0.932	0.930	0.001
DTLZ5	0.978	0.978	0.978	0.000	0.982	0.984	0.983	0.001	0.985	0.986	0.986	0.000	0.986	0.986	0.986	0.000
DTLZ6	0.310	0.889	0.591	0.142	0.183	0.382	0.243	0.056	0.400	0.946	0.672	0.143	0.986	0.986	0.986	0.000
DTLZ7	0.914	0.914	0.914	0.000	0.907	0.935	0.924	0.006	0.837	0.893	0.860	0.014	0.963	0.966	0.964	0.001
UF8	0.151	0.830	0.773	0.107	0.324	0.646	0.463	0.069	0.578	0.917	0.898	0.057	0.876	0.926	0.909	0.010
UF9	0.753	0.916	0.846	0.067	0.368	0.782	0.728	0.096	0.778	0.954	0.844	0.079	0.901	0.973	0.946	0.021
UF10	0.145	0.555	0.341	0.162	0.060	0.391	0.242	0.067	0.143	0.578	0.413	0.166	0.410	0.723	0.591	0.101
Mean	0.730	0.877	0.835	0.041	0.735	0.826	0.785	0.021	0.771	0.903	0.855	0.043	0.893	0.928	0.916	0.009

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TABLE IV
STATISTICAL TESTS OF HV WITH TWO OBJECTIVES

	MOEA/D				NSGA-II				R2-MOEA				VSD-MOEA			
	↑	↓	↔	**Diff**	↑	↓	↔	**Diff**	↑	↓	↔	**Diff**	↑	↓	↔	**Diff**
WFG1	1	0	2	0.000	0	2	1	0.000	0	0	3	0.005	1	0	2	0.000
WFG2	1	2	0	0.032	2	1	0	0.024	0	3	0	0.033	3	0	0	0.000
WFG3	2	1	0	0.001	0	3	0	0.005	1	2	0	0.001	3	0	0	0.000
WFG4	1	2	0	0.003	0	3	0	0.006	3	0	0	0.000	2	1	0	0.001
WFG5	0	3	0	0.044	1	1	1	0.036	1	1	1	0.035	3	0	0	0.000
WFG6	1	0	2	0.000	1	0	2	0.001	1	0	2	0.002	0	3	0	0.045
WFG7	1	2	0	0.003	0	3	0	0.007	3	0	0	0.000	2	1	0	0.001
WFG8	1	2	0	0.136	0	3	0	0.158	2	1	0	0.133	3	0	0	0.000
WFG9	1	1	1	0.087	0	3	0	0.138	1	1	1	0.086	3	0	0	0.000
DTLZ1	3	0	0	0.000	0	3	0	0.002	2	1	0	0.001	1	2	0	0.001
DTLZ2	1	2	0	0.002	0	3	0	0.004	3	0	0	0.000	2	1	0	0.001
DTLZ3	1	2	0	0.002	0	3	0	0.003	3	0	0	0.000	2	1	0	0.001
DTLZ4	0	2	1	0.209	1	1	1	0.128	0	0	3	0.334	2	0	1	0.000
DTLZ5	1	2	0	0.002	0	3	0	0.004	3	0	0	0.000	2	1	0	0.001
DTLZ6	1	1	1	0.291	0	3	0	0.668	1	1	1	0.299	3	0	0	0.000
DTLZ7	0	3	0	0.001	2	1	0	0.001	3	0	0	0.000	1	2	0	0.001
UF1	1	1	1	0.002	0	3	0	0.006	1	1	1	0.004	3	0	0	0.000
UF2	3	0	0	0.000	0	3	0	0.010	1	1	1	0.003	1	1	1	0.002
UF3	0	2	1	0.270	2	1	0	0.084	0	2	1	0.279	3	0	0	0.000
UF4	1	2	0	0.020	0	3	0	0.040	3	0	0	0.000	2	1	0	0.003
UF5	0	3	0	0.175	1	0	2	0.018	1	0	2	0.092	1	0	2	0.000
UF6	0	3	0	0.380	2	1	0	0.192	1	2	0	0.258	3	0	0	0.000
UF7	2	0	1	0.000	1	2	0	0.009	0	3	0	0.079	2	0	1	0.001
Total	23	36	10	1.661	13	49	7	1.542	34	19	16	1.643	48	14	7	0.060

TABLE V
STATISTICAL TESTS OF HV WITH THREE OBJECTIVES

	MOEA/D				NSGA-II				R2-MOEA				VSD-MOEA			
	↑	↓	↔	**Diff**	↑	↓	↔	**Diff**	↑	↓	↔	**Diff**	↑	↓	↔	**Diff**
WFG1	1	2	0	0.017	0	3	0	0.047	2	1	0	0.007	3	0	0	0.000
WFG2	2	1	0	0.014	1	2	0	0.022	0	3	0	0.027	3	0	0	0.000
WFG3	3	0	0	0.000	0	3	0	0.008	2	1	0	0.001	1	2	0	0.004
WFG4	0	3	0	0.055	1	2	0	0.052	2	1	0	0.015	3	0	0	0.000
WFG5	0	3	0	0.056	1	2	0	0.032	2	1	0	0.005	3	0	0	0.000
WFG6	0	2	1	0.048	0	2	1	0.043	3	0	0	0.000	2	1	0	0.024
WFG7	0	3	0	0.055	1	2	0	0.044	2	1	0	0.015	3	0	0	0.000
WFG8	1	2	0	0.121	0	3	0	0.187	2	1	0	0.084	3	0	0	0.000
WFG9	1	2	0	0.053	0	3	0	0.127	2	1	0	0.001	3	0	0	0.000
DTLZ1	2	1	0	0.014	1	2	0	0.022	0	3	0	0.024	3	0	0	0.000
DTLZ2	1	2	0	0.031	0	3	0	0.044	2	1	0	0.015	3	0	0	0.000
DTLZ3	1	2	0	0.031	0	3	0	0.040	2	1	0	0.015	3	0	0	0.000
DTLZ4	0	2	1	0.117	1	1	1	0.042	0	1	2	0.255	3	0	0	0.000
DTLZ5	0	3	0	0.007	1	2	0	0.003	2	0	1	0.000	2	0	1	0.000
DTLZ6	1	2	0	0.395	0	3	0	0.743	2	1	0	0.314	3	0	0	0.000
DTLZ7	1	2	0	0.050	2	1	0	0.041	0	3	0	0.104	3	0	0	0.000
UF8	1	2	0	0.136	0	3	0	0.445	2	0	1	0.011	2	0	1	0.000
UF9	1	1	1	0.100	0	3	0	0.218	1	1	1	0.102	3	0	0	0.000
UF10	0	2	1	0.250	0	2	1	0.349	2	1	0	0.178	3	0	0	0.000
Total	16	37	4	1.551	9	45	3	2.507	30	22	5	1.172	52	3	2	0.027

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