VSD-MOEA: A Dominance-Based Multi-Objective Evolutionary Algorithm with Explicit Variable Space Diversity Management

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***Abstract*—Most state-of-the-art Multi-objective Evolutionary Algorithms (MOEAs) promote the preservation of the diversity in the objective space, whereas the information about the diversity in the decision variables is usually neglected. In this paper, the Variable-Space-Diversity based MOEA (VSD-MOEA) is presented. VSD-MOEA is a dominance-based MOEA that considers explicitly the diversity in the decision variables and the objective space. The information gathered of both spaces is used simultaneously with the aim of properly adapting the balance between exploration and intensification during the optimization process. Particularly, at the initial stages, decisions taken by the approach are more biased by the information of the diversity in the decision variables, whereas in the last stages decisions are only based on the information of the objective space. The latter is achieved through a novel density estimator. The new method is compared with state-of-art MOEAs using several benchmarks with two and three objectives. The novel proposal attains much better results than state-of-the-art schemes, showing a more stable and robust behavior.**

I. Introduction

MULTI-OBJECTIVE Optimization Problems (MOPs) involve the simultaneous optimization of several objective functions that are usually in conflict [1]. A continuous box-constrained minimization MOP, which is the kind of problem addressed in this paper, can be defined as follows:

$$\begin{aligned} &minimize \quad \vec{F} = [f_1(\vec{\mathbf{x}}), f_2(\vec{\mathbf{x}}), ..., f_M(\vec{\mathbf{x}})] \\ &subject \quad to \quad x_i^{(L)} \leq x_i \leq x_i^{(U)}, \quad i = 1, 2, ..., n. \end{aligned} \tag{1}$$

where n corresponds to the dimension of the variable space, $\vec{\mathbf{x}}$ is a vector of n decision variables $\vec{\mathbf{x}} = (x_1, ..., x_n) \in R^n$, which are constrained by $x_i^{(L)}$ and $x_i^{(U)}$, i.e. the lower bound and upper bound, and M is the number of objective functions to optimize. The feasible space bounded by $x_i^{(L)}$ and $x_i^{(U)}$ is denoted by Ω, each solution is mapped to the objective space with the function $F : \Omega \rightarrow R^M$, which consist of M real-valued objective functions and R^M is called the *objective space*.

Given two solutions $\vec{\mathbf{x}}$, $\vec{\mathbf{y}} \in \Omega$, $\vec{\mathbf{x}}$ dominates $\vec{\mathbf{y}}$, mathematically denoted by $\vec{\mathbf{x}} \prec \vec{\mathbf{y}}$, iff $\forall m \in 1, 2, ..., M : f_m(\vec{\mathbf{x}}) \leq f_m(\vec{\mathbf{y}})$ and $\exists m \in 1, 2, ..., M : f_m(\vec{\mathbf{x}}) < f_m(\vec{\mathbf{y}})$. The best solutions of a MOP are those whose objective vectors are not dominated by any other feasible vector. These solutions are known as the Pareto optimal solutions. The Pareto set is the set of all Pareto optimal solutions, and the Pareto front are the images of the Pareto set. The goal of most multi-objective optimizers is to obtain a proper approximation of the Pareto front, i.e., a set of well distributed solutions that are close to the Pareto front.

One of the most popular metaheuristics used to deal with MOPs is the Evolutionary Algorithm (EA). In single-objective EAs, it has been shown that taking into account the diversity of the variable space to properly balance between exploration and exploitation is highly important to attain high quality solutions [2]. Diversity can be taken into account in the design of several components such as in the variation stage [3], [4], replacement phase [5] and/or population model [6]. The explicit consideration of diversity leads to improvements in terms of premature convergence avoidance, meaning that taking into account the diversity in the design of EAs is specially important when dealing with long-term executions. Recently, some diversity management algorithms that combine the information of diversity, stopping criterion and elapsed generations have been devised. They have allowed to provide a gradual loss of diversity that depends on the time or evaluations granted to the execution [5]. Particularly the aim of such a methodology is to promote exploration in the initial generations and gradually alter the behavior towards intensification. These schemes have provided really promising results. For instance, new best-known solutions for some well-known variants of the frequency assignment problem [7], and for a two-dimensional packing problem [5] have been attained using the same principles. Additionally, this principle guided the design of the winning strategy of the Second Wind Farm Layout Optimization Competition[1], which was held in the Genetic and Evolutionary Computation Conference. Thus, the benefits of such kind of design patterns have been shown in several different single-objective optimization problems.

One of the goals in the design of Multi-objective Evolutionary Algorithms (MOEAs) is to obtain a well-spread set of solutions in the objective space. The maintenance of some degree of diversity in the objective space implies that complete convergence does not appear in the variable space [8]. In some way, the variable space inherits some degree of diversity

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due to the way in which the objective space is taken into account. However, this is just an indirect way of preserving the diversity in the variable space, so in some cases the level of diversity might not be large enough to ensure a high degree of exploration. For instance, it has been shown that with some of the WFG benchmarks, in most of the state-of-the-art MOEAs the *distance parameters* quickly converge, meaning that the approach focuses just on optimizing the *position parameters* for a long period of the optimization process [8]. Thus, while some degree of diversity is maintained, a similar situation to premature convergence is presented meaning that genetic operators might not be able to generate better trade-offs.

Attending to the differences between state-of-the-art single-objective EAs and MOEAs, this paper proposes a novel MOEA, the Variable-Space-Diversity based MOEA (VSD-MOEA), that is based on controlling the amount of diversity in the variable space in an explicit way. Similarly to the successful methodology applied in single-objective optimization, the stopping criterion and the amount of evaluations performed are used to vary the amount of desired diversity. The main difference with respect to the single-objective case is that the objective space is simultaneously considered, which is performed with a novel objective-space density estimator. Particularly, the approach grants more importance to the diversity of the variable space in the initial stages, whereas as the generations evolve, it gradually grants more importance to the diversity of the objective space. In fact, at the last period of the execution, the diversity of the variable space is neglected, so in the last phases the proposal is quite similar to current state-of-the-art approaches. To our knowledge, this is the first MOEA whose design follows this adaptive principle. Since there exist currently a quite large amount of different MOEAs [9], three popular schemes have been selected to validate our proposal. This validation has been performed with several well-known benchmarks and proper quality metrics. The important benefits of properly taking into account the diversity of the variable space is clearly shown in this paper. Particularly, the advantages are clearer in the most complex problems. Note that this is consistent with the single-objective case, where the most important benefits have been obtained in complex multi-modal cases [7].

The rest of this paper is organized as follows. Section II provides a review of related papers. Some key components related to diversity and the VSD-MOEA design are discussed. The VSD-MOEA proposal is detailed in section III. Section IV is devoted to the experimental validation of the novel proposal. Finally, conclusions and some lines of future work are given in Section V. Note also that some supplementary materials are given. They include details of the experimental results with additional metrics as well as some explanatory videos.

II. Literature Review

This section is devoted to review some of the most important papers that are closely related to our proposal. First, some of the most popular ways of managing diversity in EAs are presented. Then, the state-of-the-art in MOEAs is summarized.

A. Diversity Management in Evolutionary Algorithms

The proper balance between exploration and exploitation is one of the keys to success in the design of EAs. In the single-objective domain it is known that properly managing the diversity in the variable space is a way to control such balance, and as a consequence, a large amount of diversity management techniques have been devised [10]. Particularly, these methods are classified depending on the component(s) of the EA that is modified to alter the amount of maintained diversity. A popular taxonomy identifies the following groups [11]: *selection-based*, *population-based*, *crossover/mutation-based*, *fitness-based*, and *replacement-based*. Additionally, the methods are referred to as *uniprocess-driven* when a single component is altered, whereas the term *multiprocess-driven* is used to refer to those methods that act on more than one component.

Among the previous proposals, the replacement-based methods have attained very high-quality results in last years [7], so this alternative was selected with the aim of designing a novel MOEA incorporating an explicit way to control the diversity in the variable space. The basic principle of these methods is to bias the level of exploration in successive generations by controlling the diversity of the survivors [7]. Since premature convergence is one of the most common drawbacks in the application of EAs, modifications are usually performed with the aim of slowing down the convergence. One of the most popular proposals belonging to this group is the *crowding* method which is based on the principle that offspring should replace similar individuals from the previous generation [12]. Several replacement strategies that do not rely on crowding have also been devised. In some methods, diversity is considered as an objective. For instance, in the hybrid genetic search with adaptive diversity control (HGSADC) [13], individuals are sorted by their contribution to diversity and by their original cost. Then, the rankings of the individuals are used in the fitness assignment phase. A more recent proposal [7] incorporates a penalty approach to alter gradually the amount of diversity maintained in the population. Particularly, initial phases preserve a larger amount of diversity than the final phases of the optimization. This last method has inspired the design of the novel proposal put forth in this paper for multi-objective optimization.

It is important to remark that in the case of multi-objective optimization, few works related to the maintenance of diversity in the variable space have been developed. The following section reviews some of the most important MOEAs and introduces some of the works that consider the maintenance of diversity in the variable space.

B. Multi-objective Evolutionary Algorithms

In recent decades, several MOEAs have been proposed. While the purpose of most of them is to provide a well-spread set of solutions close to the Pareto front, several ways of facing this purpose have been devised. Therefore, several taxonomies have been proposed with the aim of better classifying the different schemes [14]. Particularly, a MOEA can be designed based on Pareto dominance, indicators and/or decomposition [15]. Since none of the groups has a remarkable superiority over the others, in this work all of them are taken into account to validate our proposal. This section introduces the three types of schemes and some of the most popular approaches belonging to each category. Then, one MOEA of each category is selected to carry out the validation of VSD-MOEA.

The dominance-based category includes those schemes where the Pareto dominance relation is used to guide the design of some of its components such as the fitness assignment, parent selection and replacement phase. The dominance relation does not inherently promotes the preservation of diversity in the objective space, therefore additional techniques such as objective-space density estimators are usually integrated with the aim of obtaining a proper spread and convergence to the Pareto front.

In order to assess the performance of MOEAs, several quality indicators have been devised. In the indicator-based MOEAs, the use of the Pareto dominance relation is substituted by some quality indicators to guide the decisions performed by the MOEA. An advantage of this kind of algorithms is that the indicators usually take into account both the quality and diversity in objective space, so incorporating additional mechanisms to promote diversity in the objective space is not required. Among the different indicators, hypervolume is a widely accepted Pareto-compliance quality indicator. The Indicator-Based Evolutionary Algorithm (IBEA) [16] was the first method belonging to this category. A more recent one is the R2-Indicator-Based Evolutionary Multi-objective Algorithm (R2-EMOA) [17], which has reported a quite promising performance in MOPs. Its most important feature is the use of the R2 indicator.

Finally, decomposition-based MOEAs [18] are based on transforming the MOP into a set of single-objective optimization problems that are tackled simultaneously. This transformation can be performed in several ways, e.g. with a linear weighted sum or with a weighted Tchebycheff function. Given a set of weights to establish different single-objective functions, the MOEA searches for a single high-quality solution for each of them. The weight vectors should be selected with the aim of obtaining a well-spread set of solutions [1]. The Multi-objective Evolutionary Algorithm Based on Decomposition (MOEA/D) [19] is the most popular decomposition-based MOEA. Its main principles include problem decomposition, weighted aggregation of objectives and mating restrictions through the use of neighborhoods.

It is important to stand out that none of the most popular algorithms in the multi-objective field introduce special mechanisms to promote diversity in variable space. However, some efforts have been dedicated to this principle. A popular approach to promote the diversity in the decision space is the application of fitness sharing [20] in a similar way than in single-objective optimization. Although, fitness sharing might be used to promote diversity both in objective and decision variable space, most popular variants consider only distances in the objective space. Another MOEA designed to promote diversity in both the decision and the objective space is the Genetic Diversity Evolutionary Algorithm (GDEA) [21]. In this case, each individual is assigned with a diversity-based objective which is calculated as the Euclidean distance in the genotype space to the remaining individuals in the population. Then, a ranking that considers both the original objectives and the diversity objective is used to sort individuals. Another somewhat popular approach is to calculate distances between candidate solutions by taking into account both the objective and variable space [22], [23] with the aim of promoting diversity in both spaces. A different proposal combines the use of two selection operators [24]. The first one promotes diversity and quality in the objective space whereas the second one promotes diversity in the decision space. A different approach was to modify the hypervolume to integrate the decision space diversity in a single metric [25]. In this approach, the proposed metric is used to guide the selection in the MOEA. Finally, some indirect mechanisms that might affect the diversity have also been taken into account. Probably, the most popular one is the use of mating restrictions [26], [18].

Attending to the results of previous described approaches, it is clear that taking into account the decision space diversity in the design phase might bring benefits to decision makers because the final solutions obtained by these methods present a larger decision space diversity than the ones obtained by traditional approaches [22], [27]. Thus, while clear improvements are obtained when taking into account metrics related to the Pareto set, the benefits in terms of the obtained Pareto front are not so clear. We claim that one of the reasons of this behavior might be that the diversity in the variable space is considered in the whole optimization process. However, in a similar way that in the single objective domain, reducing the importance granted to the diversity in the decision space as the generations progress [5] might be really important to attain better approximations of the Pareto front. Currently, no MOEA considers this idea, so this principle has guided the design of our novel MOEA.

III. PROPOSAL

This section is devoted to fully describe our novel proposal. The novelty of VSD-MOEA appears in the replacement phase, which incorporates the use of variable space diversity and a novel objective-space density estimator. The main principle behind the design of the novel replacement is to use the stopping criterion and elapsed generations with the aim of gradually moving from exploration to exploitation during the search process. Note that this principle might be incorporated in any of the three categories of MOEAs. In this paper, our decision was to incorporate it in a dominance-based approach and in the field of multi-objective problems which involves two and three objectives. Thus, some of our design decisions might not be suitable for dealing with many-objective optimization problems.

The general framework of VSD-MOEA is quite standard. Algorithm 1 shows the pseudocode of VSD-MOEA. The parent selection is performed with binary tournament based on the dominance raking with ties broken randomly. The variation stage is based on applying the well-known Simulated Binary Crossover (SBX) and polynomial mutation [28], [29]. Thus, the contribution appears in the replacement phase. The rest

Algorithm 1 Main procedure of VSD-MOEA

```
Initialization: Generate an initial population P_0 with N individuals.
Evaluation: Evaluate all individuals in the population.
Assign t = 0
while (not stopping criterion) do
    Mating selection: Fill the mating pool by performing binary
    tournament selection on P_t, based on the non-dominated ranks
    (ties are broken randomly).
    Variation: Apply SBX crossover and Polynomial mutation to
    the mating pool to create a child population Q_t.
    Evaluation: Evaluate all individuals in Q_t.
    Survivor selecction: Generate P_{t+1} by applying the replacement
    scheme described in Algorithm 2, using P_t and Q_t as input.
    t = t + 1
```

of this section is devoted to describe the replacement phase, including the novel objective-space density estimator.

A. Replacement Phase of VSD-MOEA

The replacement phase of EAs is in charge of deciding in each generation which are the survivors among the members of the previous population and offspring. The novel replacement promotes a gradual movement from exploration to exploitation, which has been a quite beneficial principle in the design of single-objective optimizers [5]. Particularly, the replacement phase operates as follows. First, the members of the previous population and offspring are joined in a multi-set with $2 \times N$ individuals. Then, an iterative process that selects an additional individual at each iteration is used to pick up the N survivors. In order to take into account the diversity in the decision space, the Distance to Closest Survivor (DCS) of each individual is calculated at each iteration. Thus, the DCS of an individual I is calculated as $\min_{s \in S} Distance(I, s)$, where S is the multi-set containing the currently selected survivors. Normalized Euclidean distances are considered, so in order to calculate distances between any two individual A and B, Eq. (2) is applied. In the first iteration, the S multi-set is empty, so the DCS of each individual is infinity.

$$Distance(A, B) = \left(\frac{1}{n} \sum_{i=1}^{n} \left(\frac{A_i - B_i}{x_i^{(U)} - x_i^{(L)}} \right)^2 \right)^{1/2} \tag{2}$$

Note that individuals with larger DCS values are those that contribute more significantly to promote exploration. In order to avoid an excessive decrease of the exploration degree, individuals with a DCS value lower than a threshold value are penalized and they can only be selected if non-penalized individuals are missing.Then, among the non-penalized individuals, an objective-space density estimator is used to select the additional survivor of the iteration. In our case, the novel density estimator described in the next subsection is used.

In order to better understand the penalty method, it can be visualized in the following way. After selecting each survivor, a hypersphere centered in such a candidate solution — in the variable space — is created. Then, all the individuals that are inside a hypersphere are penalized and the objective-space estimator takes into account only the survivors and non-penalized individuals. This is illustrated in Fig. 1, which represents a state where three individuals have been selected to survive and an additional survivor must be picked up. The left side shows individuals in the variable space. Current survivors are marked with a red border and each one of them is surrounded by a blue dash circle with radius D_t. In this situation, the penalized individuals are the number 4, 5, and 6. In the objective space — right side — penalized individuals are shown with gray background, indicating that the objective-space density estimator does not take them into account.

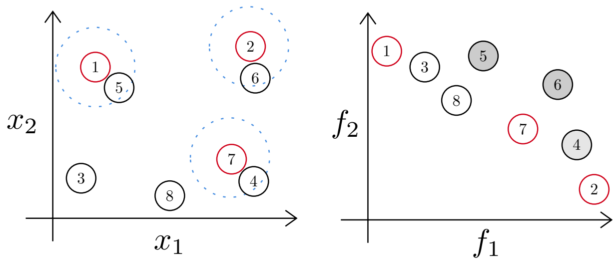

Fig. 1. Penalty Method of the Replacement Phase - The left side represents the variables space and the right side the objective space

Since penalizing with a large threshold value — radius of the hyperspheres — induces a large degree of exploration, it makes sense to reduce this value during the optimization process. This is precisely one of the keys of our proposal. The sizes of the hyperspheres are modified dynamically by taking into account the stopping criterion and elapsed generations. Particularly, the radius is decreased in a linear way starting from an initial distance. This means that in the initial phases exploration is promoted. However, as the size of the radius decreases only very close individuals are penalized, meaning that more exploitation is allowed. Note that this method requires a parameter which is the initial radius of the hyperspheres which is denoted as D_I. Setting this parameter with a large value might provoke the penalization of a lot of individuals, thus non-useful diversity might be maintained. However, too small values might not prevent fast convergence and therefore the approach might behave as a traditional non-diversity based MOEA. The robustness of the proposal with respect to this additional parameter is studied in our experimental validation.

Algorithm 2 fully describes the replacement phase of VSD-MOEA. First, the population of the previous generation (P_t) and the offspring (Q_t) are joined in R_t (line 3). The multiset R_t contains, at each iteration, the remaining non-penalized individuals that might be selected to survive. The population of survivors (P_{t+1}) and the set containing the penalized individuals are initialized to the empty set (lines 4 and 5). Then, the threshold value (D_t) that is used to penalize too close individuals is calculated (line 6). Note that D_I denotes the initial threshold value, $G_{Elapsed}$ is the amount of generations that have been evolved, and G_{End} is the stopping criterion, i.e. the number of generations that are to be evolved in the execution of VSD-MOEA. The linear decrease is calculated so that after the 50% of the generations, the D_t value is lower than 0, meaning that no penalties are performed. This means that in the first 50% of the generations, more exploration than

Algorithm 2 Replacement Phase of VSD-MOEA

1: Input: P_t (Population of current generation), Q_t (Offspring of current Generation)
2: Output: P_{t+1}
3: $R_t = P_t \cup Q_t$
4: $P_{t+1} = \emptyset$
5: $Penalized = \emptyset$
6: $D_t = D_I - D_I * \frac{G_{Elapsed}}{0.9 * G_{End}}$
7: **while** $|P_{t+1}| \leq$ N **do**
8: Compute DCS of individuals in R_t with P_{t+1} used as reference set
9: Move to $Penalized$ the individuals in R_t with $DCS < D_t$
10: **if** R_t is empty **then**
11: Compute DCS of individuals in $Penalized$ with P_{t+1} used as reference set
12: Move to R_t the individual in $Penalized$ with largest DCS
13: Identify the first front (F) in $R_t \cup P_{t+1}$ with an individual $I \in R_t$
14: Use the novel density estimator (Algorithm 3) to select a new survivor from F and move it to P_{t+1}
15: **return** P_{t+1}

in traditional MOEAs is induced.

Then, an iterative process that selects an individual at each iteration is executed until the survivors set contains N individuals (line 7). The iterative process works as follows. First, the DCS value of each remaining non-penalized individual is calculated (line 8). Then, those individuals with a DCS value lower than D_t are moved to the set of penalized individuals (line 9). If all the remaining individuals are penalized (line 10), it means that the amount of exploration is lower than the desired one. Thus, the individual with the largest DCS value is recovered, i.e. moved to the non-penalized individuals set (lines 11 and 12) and consequently it survives. Finally, the objective space is taken into account. Specifically, candidate non-penalized individuals and current survivors are joined. Then, the well-known non-dominated sorting procedure [30] is executed with such a set, stopping as soon as a front with a candidate individual is found, i.e. with an individual of R_t (line 13). Then, taking the identified front as input, a novel objective-space density estimator is used to select the next survivor (line 14). The specific way in which the diversity in the objective space is measured is described in the next section.

B. A Novel Density Estimator for the Objective Space

Since the dominance definition is not related to the preservation of diversity in the objective space, dominance-based MOEAs usually incorporate objective-space density estimators to promote the survival of diverse individuals. As it was previously described, our density estimator selects a new survivor from the front identified in line 14 of Algorithm 2. This front contains at least one individual belonging to R_t and it might also contain some elements of P_{t+1}. The aim behind the selection of the next survivor is to pick up an individual of the input front that contributes significantly in terms of objective-space quality and diversity.

Algorithm 3 describes the selection of the next survivor. First, similarly to most state-of-the-art algorithms, an action

Algorithm 3 Density estimator

1: Input: P_{t+1} (Survivors), R_t (Candidates), F (Current front)
2: Output: $I \in R_t$
3: **for** $k \in$ number of objectives **do**
4: Select the best individual $I \in F$ of k according to Eq. 3.
5: **if** $I \in R_t$ **then**
6: **return** I
7: $FS = P_{t+1} \cap F$
8: $MaxID = 0$
9: **for** $Ic \in R_t$ **do**
10: $Improvement = \min_{s \in FS} ID(Ic, s)$
11: **if** $Improvement > MaxID$ **then**
12: $MaxID = Improvement$
13: $I = Ic$
14: **return** I

to promote the selection of boundary solutions is executed. Note that selecting the best solution for each objective might provoke some drawbacks related to accepting small improvement in an objective at the cost of important worsening in other objectives [31]. To solve this issue augmented functions can be applied, which has been the alternative used in this paper. Particularly, iteratively, for each objective k the candidate solution that minimizes the Augmented Weighted Function (AWF) given in Eq. 3 is calculated (lines 3 to 6). If such an individual belongs to R_t, i.e., it has not been selected yet as a survivor, the next survivor is such an individual and the process finalizes (line 6). Note that, augmented functions usually take into account weight vectors with the aim of dealing with objectives that present very different scales. Since benchmarks that have similar scales in each objective have been used in this paper, there was no need to apply such weight vectors.

$$AWF_k(\vec{x}) = f_k(\vec{x}) + 10^{-4} \times \sum_{j=1}^{M} f_j(\vec{x}) \tag{3}$$

In cases where the individuals that optimize each AWF_K function are already in P_{t+1}, a contribution to objective-space diversity is calculated by taking into account the current survivors of the front (lines 8 to 14). Particularly, the "Improvement Distance" (ID) defined for the indicator IGD+ [32] is used. The ID of an individual A with respect to an individual B is calculated by taking into account only the objectives where A is better. Specifically, Eq. (4) is used.

$$ID(A,B) = \left(\sum_{i=1}^{M} (max(0, B_i - A_i))^2 \right)^{1/2} \tag{4}$$

Denoting as FS the selected members of the input front that are already in P_{t+1} (line 7), the contribution of each member which belongs to the input front and that has not been selected (R_t) is calculated as $\min_{s \in FS} ID(I, s)$. Then, the individual with the higher contribution to the objective space is selected as the next survivor (lines 11 to 13).

IV. Experimental Validation

In this section the experimental validation is carried out, showing that controlling the diversity in the variable space

TABLE I
PARAMETERIZATION OF EACH MOEA

Algorithm	Configuration
MOEA/D	Max. updates by sub-problem (η_r) = 2, tour selection = 10, neighbor size = 10, period utility updating = 30 generations, probability local selection (δ) = 0.9,
VSD-MOEA	$D_I = 0.4$
R2-EMOA	$\rho = 1$, offspring by iteration = 1

is a way to improve further some of the results obtained by the state-of-art-MOEAs. The analyses carried out in this section are carefully designed to get a clear understanding of the particularities of VSD-MOEA. First, several technical specifications of the implemented agorithms, and test problems are described. Thereafter, one of the most meaningful comparisons between VSD-MOEA and state-of-the-art algorithms is presented. In addition, to have a broad perception of the VSD-MOEA three experiments are driven. Such analyses are designed to test the scalability in the decision variable space, the performance with different stopping criteria, and the behaviour with different initial thresholds which belong to the VSD-MOEA. Particularly, the second one is probed with several execution times as stopping criterion, i.e. from short-term to long-term executions.

This work is validated through some of the most popular bechmarks which are widely applied in the multi-objective field. Such problems are the WFG [33], DTLZ [34], and UF [35]. Furthermore, the experimental validation includes three well-known state-of-the-art-MOEAs and VSD-MOEA. The MOEAs that are taken into account are NSGA-II [36], MOEA/D [37], and R2-EMOA [38], which can be classified as based-dominance, based-decomposition, and based-indicator respectively. Particularly, the MOEA/D implementation belongs to the first place in the "Congress on Evolutionary Computation 2009" (CEC) [39]. Given that all the considered algorithms are stochastic, each execution was repeated 35 times with different seeds. The common configuration in all the executions was the following: the stopping criterion was set to $250,000$ generations, the population size was fixed to 100, the WFG test problems were configured with two and three objectives. This set of problems were set with 24 parameters, where 20 of them correspond to distance parameters and 4 to position parameters. In relation with the DTLZ test instances, the number of decision variables was set to $n = M + r - 1$, where $r = \{5, 10, 20\}$ for DTLZ1, DTLZ2 to DTLZ6 and DTLZ7 respectively. The UF benchmark is conformed by seven problems with two objectives (UF1-7) and three problems with three objectives (UF8-10), all of them were set with 30 decision variables. The operators employed on all the MOEAs were the Simulatied Binary Crossover (SBX), and polynomial mutation [28], [29]. Especifically, the crossover was set with the probability and distribution index values of 0.9 and 2 respectively. Similarly, the mutation probability and distribution index were fixed to $1/n$ and 50 respectively. In addition, the extra-parameterization of each algorithm is shown in Table I.

In addition, the weight-vector based algorithms (MOEA/D and R2-EMOA) are configured with the Tchebycheff approach as utility function. According to their implementations each algorithm takes into account different quantity and distribution of weight vectors. Particularly, since that R2-EMOA can be configured with a different number of weight vectors than the population size, it was applied with 501 and 496 weight vectors for two and three objectives respectively. In contrast, MOEA/D requires the same number of weight vectors than the population size, therefore the weight vectors were generated with the uniform design (UD) and the good lattice point method (GLP) [40], [41].

The experimental analyses were carried out taking into account the hypervolume indicator (HV). The HV metric measures the objective space dominated by the approximated solutions given a reference point, so the solutions dominated by the reference point were not considered. Particularly, the reference point is chosen to be a vector whose values are sightly larger (ten percent) than the nadir point as is suggested in [42]. Similarly that in [43], and to have a fair comparison the normalized HV is estimated. Specifically, the HV reported is computed as the ratio between the HV reached by a set of solutions and the HV of a set of points that belong to the Pareto Front. In this way, the more approximated to the unity this metric is, the more converged are the solutions to the Pareto Front.

In order to statistically compare the HV results, a similar guideline than the proposed in [44] was used. First a Shapiro-Wilk test was performed to check whatever or not the values of the results followed a Gaussian distribution. If, so, the Levene test was used to check for the homogeneity of the variances. If samples had equal variance, an ANOVA test was done; if not, a Welch test was performed. For non-Gaussian distributions, the non-parametric Kruskal-Wallis test was used to test whether samples are drawn from the same distribution. An algorithm X is said to win algorithm Y when the differences between them are statistically significant, if the mean and median obtained by X are higher than the mean and median achieved by Y.

In Tables II and III are shown the normalized hypervolume with two and three objectives respectively. In this empirical results the VSD-MOEA achieved the best general mean and the lowest variability, therefore it is quite stable to different runs. According to Table II the dominance-based algorithms improved the remaining MOEAs. However, the general mean of NSGA-II (second best algorithm with 0.886) is quite similar to MOEA/D (0.881) and R2-EMOA (0.882), in contrast VSD-MOEA achieved the highest HV value (0.951). In particular, the dominance-based algorihms gave the best values in the kind of problems with disconnected Pareto geometries (WFG2, UF5 and UF6). Differently, taking into account three objectives, NSGA-II was placed as the worst algorithm (0.785), perhaps provoked by its density estimator, which has some drawbacks in the most difficult problems with three objectives (e.g. multi-frontal problems). The second place is attained by R2-EMOA (0.855), which is still quite low compared with VSD-MOEA (0.916).

Given that the general mean can be unsteady to atypical measurements, i.e. high variability, in Tables IV and V are shown the statistical tests with two and three objectives respectively. In addition, those tables incorporate the difference between the mean of each algorithm and the best mean achieved for each problem, which is tagged with “Diff”. A representative value of each column is shown in the last row which is conformed by the sum of the enterely column. The algorithm that attained more wins in the pairwise comparison is the VSD-MOEA with 48 and 52 wins in two and three objectives respectively. Besides that VSD-MOEA lost in few pairwise comparisons, in such problems this algorithm is close enough to the best results, this can be seen through the small “Diff” values attained. In fact, VSD-MOEA achieved 0.060 and 0.027 in contrast to the two second bests algorithms that are NSGA-II (1.542) and R2-EMOA (1.172) for two and three objectives respectively. Especifically, the worst “Diff” value achieved by the VSD-MOEA is with the WFG6 test-problem (0.045 and 0.024), which is uni-modal and non-separable. Nevertheless, this problem was better approximated with $D_I = 0.1$ whose means were of 0.913 and 0.868 for two and three objectives respectively. Indicating that this problem is improved inducing less diversity at the initial stages of the execution.

A. Decision Variable Scalability Analysis

In order, to study the scalability of the decision variables, and following the same configuration, the algorithms were test with 50, 100, and 250 variables. Particularly, since that long-term executions (250,000 generations) are time-consuming, this analysis is carried out taking into account middle-term executions (25,000 generations). In Figures 2 and 3 are shown the normalized mean of the HV results for two and three objectives respectively. Especially, increasing the number of variables provoked a degradation to different levels on the performance of each algorithm. Nevertheless, the based-dominance algorithms are the best with problems of two objectives. In contrast, taking into account problems of three objectives, the weight-vector based algorithms showed to be more stable than the dominance-based algorithms. However, VSD-MOEA is still the best algorithm in both two and three objectives. In problems of three objectives, the VSD-MOEA performance seems to be more affected increasing the number of variables in comparison to the weight-vector based algorithms. This might by caused for several reasons, perhaps two of the most important are the stopping criterion which might not be enough to attain an adequatly convergence and the distance metric taken into consideration for the magnament of diversity. The latter is popularly known as *The Curse of Dimensionality* [45], [46], meaning that under certain broad conditions, as dimensionality is increased, the distance to the nearest neighbour tends to be the same to the farthest neighbour. In other words, the contrast in distances to different data points becomes no-existent.

In order, to have a better understanding of the algorithms and following the guideline of middle-term executions, the diversity in the decision variable space with some WFG problems is calculated. Particularly, these problems divide the decision variables in two kinds of parameters: the distance parameters and the position parameters. A parameter x_i is

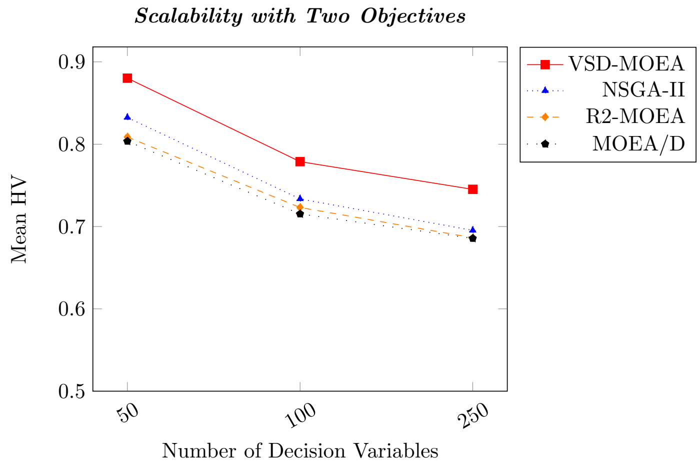

Fig. 2. Mean of the HV (35 runs) considering two objectives.

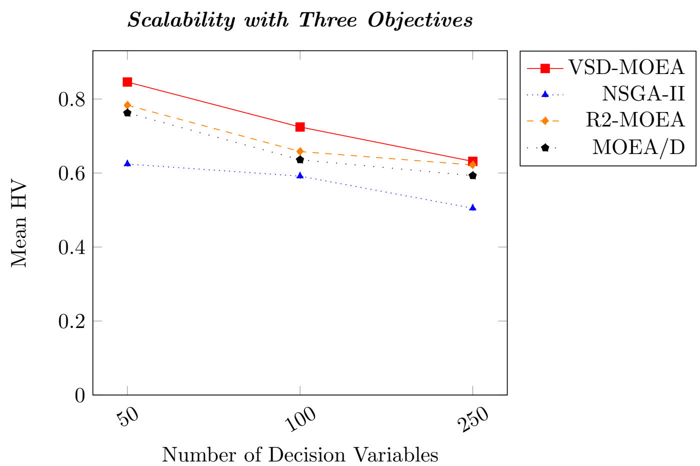

Fig. 3. Mean of the HV (35 runs) considering three objectives.

a distance parameter when for all parameter vectors $\vec{F}(\mathbf{x})$, modifying x_i in $\vec{F}(\mathbf{x})$ results in a parameter vector that dominates $\vec{F}(\mathbf{x})$, is equivalent to $\vec{F}(\mathbf{x})$, or is dominated by $\vec{F}(\mathbf{x})$. However, if x_i is a position parameter, modifying x_i in $\vec{F}(\mathbf{x})$ always results in a vector that is incomparable or equivalent to $\vec{F}(\mathbf{x})$ [47].

Particularly, the selected problems were used to show that either premature convergence or stagnation appears in the set of distance parameters. Consequently, the operators involved lose its exploratory strength. We select the WFG1-WFG7 problems because their distance parameters values associated to Pareto optimal solutions have exactly the same values. This values is shown as follows:

$$x_{i=k+1:n} = 2i \times 0.35 \tag{5}$$

In order to analyze the diversity, the average Euclidean distance among individuals (ADI) is calculated, i.e. the mean value of all pairwise distances among individuals in the population is reported. In Figures 4 and 5 are shown the evolution of diverstiy with two and three objectives. Concretely, each figure shows the diversity maintained in the position parameters and the distance parameters. The diversity is calculated with 50, 100, and 250 variables for each algorithm. Mainly, the state-

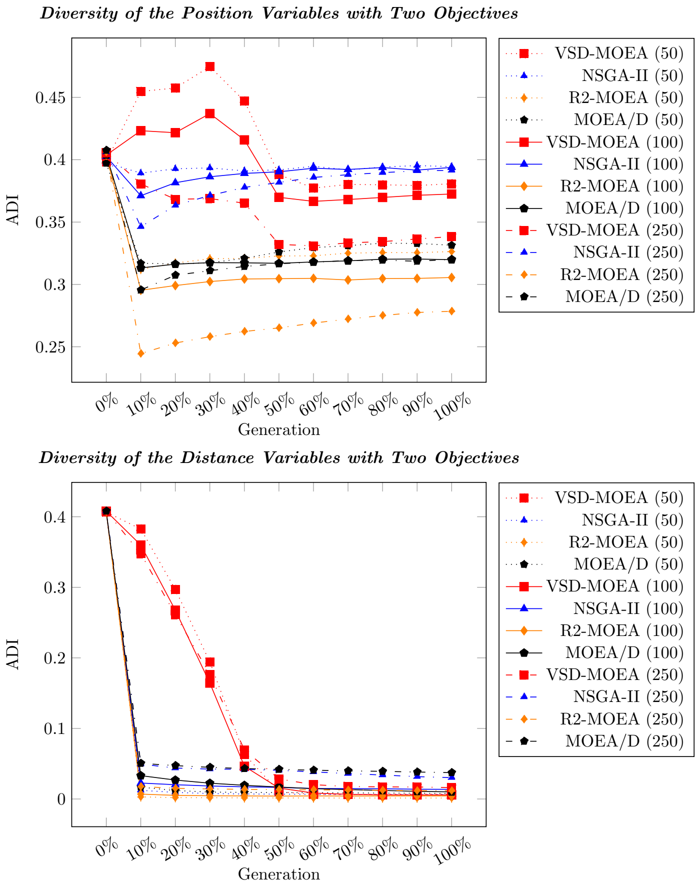

Fig. 4. Evolution of average distance individuals of the problems WFG1-WFG7 with two objectives.

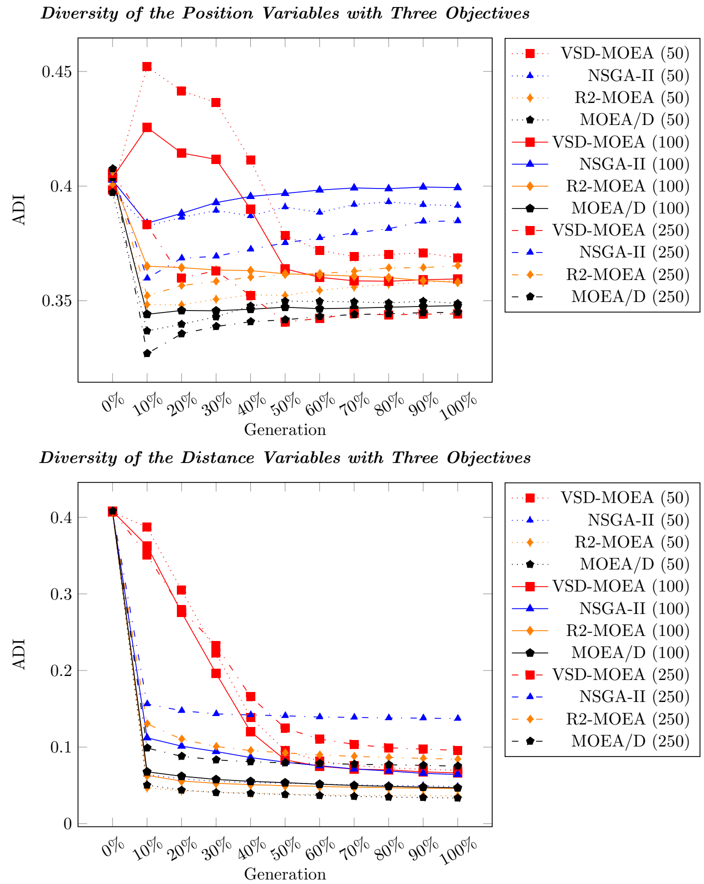

Fig. 5. Evolution of average distance individuals of the problems WFG1-WFG7 with three objectives.

of-the-art algorithms converged to promising regions in the distance parameters after the 10% of the total time-execution. However, their diversity in position parameters increases as the time elapses, meaning that the remaining 90% is devoted to promote diverse solutions in the objective space. In spite this, it seems that the algorithms converge on the distance parameters to promising regions, even though those regions could not be the optimal. In contrast, VSD-MOEA keeps diverse solutions in both kind of parameters until the 50% of the time elapsed, this strategy allows to explore more adequatly the promising regions and might avoid stagnation in several stages.

In relation with the state-of-the-art algorithms, the diversity maintained in problems of three objectives was larger than with problems of two objectives, this might occurs since that the quantity of prominent solutions increases meaninfully between two and three objectives. Therefore, there is present an implicit relation of diversity between the number of objectives and the decision variables space, in this situation three objectives keeps implictly more diversity than two objectives. However, this implicit diversity maintained is not enough to avoid stagnation in several problems. Thus, the consideration of long-term executions in the state-of-the-art algorithms could not provide meaningfully improvements. In contrast, VSD-MOEA could improve even more with long-term executions, since that it keeps diversity at several stages. In general, as the number of variables is increased, the convergence in the distance parameters is even more slow, therefore to have quality solutions should be required a greater time-execution, especifically in the diversity-based approaches.

B. Performance of MOEAs Through the Execution

This section is devoted to test the performance of the algorithms with several stopping criteria, i.e. maximum number of generations. As it is previously discussed, state-of-the-art algorithms might suffer of premature convergence or stagnation. Therefore, the solutions could converge to promising –but not optimal– regions at the first stages of the execution, wasting a meaningful quantity of generations. To deal with this issue, VSD-MOEA maintains diversity at different stages as long as the stopping criterion is reached. Accordingly, to have a better understanding of the performance obtained in each algorithm several stopping criteria have been probed. In those runs the mean of normalized HV of all the problems for two and three objectives are calculated. Mainly, three ranges of stopping criterion were explored. Each range was split in ten intervals, such ranges considered were $[250, 2500]$, $[2500, 25000]$ and $[25000, 250000]$ that are named as short-term, middle-term and long-term executions respectively. In Figures 6 and 7 are showed the mean HV values attained with each MOEA with two and three objectives respectively. Each figure is conformed by three graphics which correspond to short-term, middle-term and long-term. Those figures demostrate that

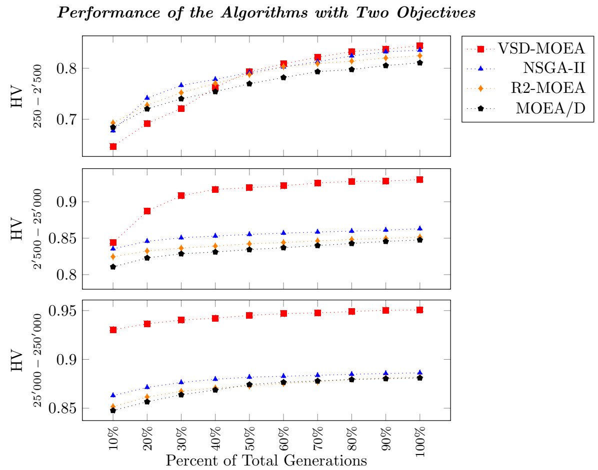

Fig. 6. Performance of the MOEAs considering three ranges of stopping criterion. The configurations take place with short-term (first row), middle-term (second row) and long-term (third row) executions.

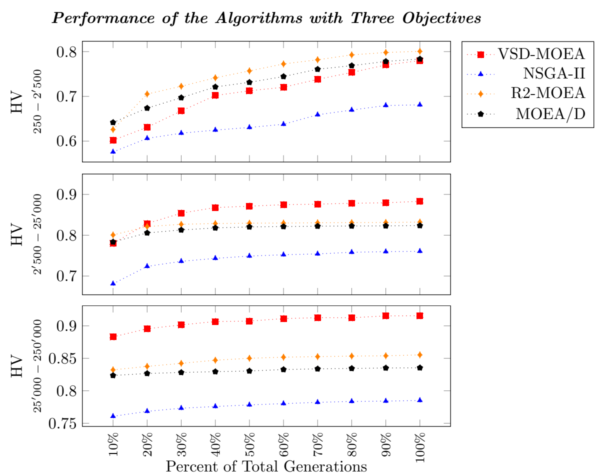

Fig. 7. Performance of the MOEAs considering three ranges of stopping criterion. The configurations take place with short-term (first row), middle-term (second row) and long-term (third row) executions.

VSD-MOEA attained competitive results considering short-term executions. Even more, as the maximum number of generations is increased its performance is improved. In fact after $2,500$ generations VSD-MOEA has a notorious improvement respect to the state-of-the-art-MOEAs in both two and three objectives (second row of Figures 6 and 7). In problems of two objectives state-of-the-art algorithms attained quite similar HV values in long-term executions ($250,000$ generations). The weight-vector based algorithms had a similar behaviour variating the stoping criterion. Particularly, R2-EMOA improved state-of-the-art algorithms with problems of three objectives (Figure 7). Moreover, this algorithm showed a lightly constant improvement increasing the number of generations. However, VSD-MOEA attained the best results in long-term executions and could improve even more increasing the time-execution.

C. Analysis of the Initial Threshold Value

Avoidance of premature convergence and stagnation in long-term executions is a dificult task mainly in multi-objective problems since that –depending of the problem– the objective space implicitly involves a relation of diversity in the decision variable space. For this reason, a potential strategy to mitigate those drawbacks is to lead the algorithm through different diversity stages and taking into account the stoping criterion to stimulate an appropiate balance between exploration and exploitation. Nevertheless, the latter strategy, which is incorpored in VSD-MOEA, requires an initial parameter, which represents the initial diversity induced in the algorithm, better known as the initial threshold value (D_I). Accordingly, this initial threshold value is decreasing as the time elapses. In order, to have a better insight about the effect raised of inducing several initial diversity stages, VSD-MOEA is tested with several initial threshold values and in long-term executions ($250,000$ generations). Particularly, this parameter is computed as a fraction of the main normalized diagonal, which belongs to the unitary hyper-cube, the portions considered are $D_I = \{0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$. In Figure 8 is shown the general normalized mean of the HV for each configuration with two an three objectives. Specifically, setting $D_I = 0.0$ implies that the diversity is not promoted resulting in a similar behaviour as the dominance-based algorithm. In spite of this setting, VSD-MOEA attained better HV values with all the settings than the state-of-the-art algorithms, in fact the lowest values were attained without inducing diversity, such values were 0.912 and 0.893 for two and three objectives respectively. Especially, the maximum benefit achieved of inducing diversity is found with an initial threshold value of $DI = 0.4$ with two and three objectives. In addition, the behaviour of setting several initial threshold value is quite similar with two and three objectives. Therefore, inducing diversity and taking into account the elapses generations can be useful mainly for middle-term and long-term executions.

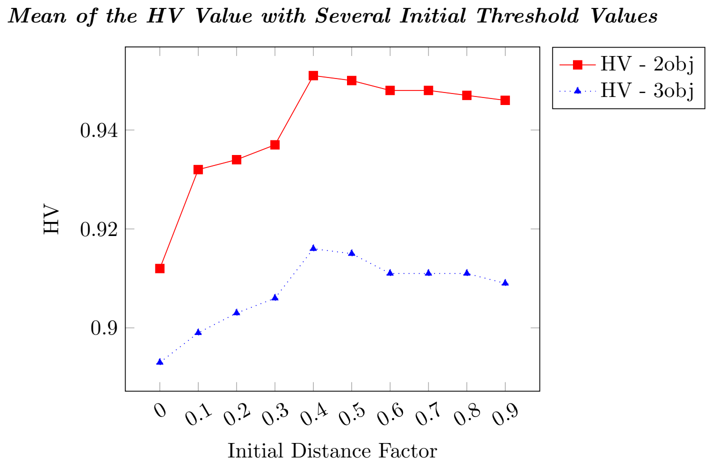

Fig. 8. Mean of HV values taking into account all instances with several initial threshold values.

V. Conclusion

The evolutionary algorithms have been one of the most popular approaches to deal with complex optimization problems. The performance of multi-objective algorithms is measured directly in the objective space, therefore the majority of algorithms incorporate sophisticated mechanisms to attain

TABLE II
STATISTICS HV WITH TWO OBJECTIVES

	MOEA/D				NSGA-II				R2-MOEA				VSD-MOEA			
	Min	**Max**	**Mean**	**Std**	**Min**	**Max**	**Mean**	**Std**	**Min**	**Max**	**Mean**	**Std**	**Min**	**Max**	**Mean**	**Std**
WFG1	0.984	0.993	0.992	0.002	0.987	0.993	0.992	0.002	0.946	0.994	0.988	0.012	0.984	0.994	0.992	0.003
WFG2	0.965	0.996	0.967	0.007	0.966	0.998	0.974	0.014	0.965	0.966	0.966	0.000	0.998	0.998	0.998	0.000
WFG3	0.992	0.992	0.992	0.000	0.987	0.988	0.987	0.000	0.991	0.992	0.991	0.000	0.992	0.992	0.992	0.000
WFG4	0.988	0.988	0.988	0.000	0.983	0.987	0.985	0.001	0.991	0.991	0.991	0.000	0.990	0.990	0.990	0.000
WFG5	0.876	0.893	0.882	0.005	0.884	0.899	0.890	0.002	0.886	0.895	0.891	0.003	0.911	0.946	0.926	0.008
WFG6	0.879	0.940	0.914	0.016	0.894	0.942	0.913	0.012	0.875	0.942	0.912	0.015	0.858	0.885	0.869	0.006
WFG7	0.988	0.988	0.988	0.000	0.983	0.987	0.984	0.001	0.991	0.991	0.991	0.000	0.990	0.990	0.990	0.000
WFG8	0.800	0.822	0.811	0.006	0.771	0.801	0.789	0.006	0.803	0.824	0.815	0.005	0.830	0.955	0.947	0.020
WFG9	0.795	0.972	0.883	0.082	0.793	0.966	0.832	0.070	0.797	0.976	0.884	0.079	0.964	0.975	0.970	0.003
DTLZ1	0.993	0.993	0.993	0.000	0.990	0.992	0.991	0.000	0.992	0.992	0.992	0.000	0.992	0.992	0.992	0.000
DTLZ2	0.989	0.989	0.989	0.000	0.986	0.988	0.987	0.000	0.991	0.992	0.992	0.000	0.990	0.990	0.990	0.000
DTLZ3	0.989	0.989	0.989	0.000	0.987	0.989	0.989	0.001	0.991	0.992	0.992	0.000	0.990	0.990	0.990	0.000
DTLZ4	0.259	0.989	0.781	0.330	0.259	0.988	0.863	0.274	0.259	0.992	0.657	0.365	0.990	0.990	0.990	0.000
DTLZ5	0.989	0.989	0.989	0.000	0.986	0.988	0.987	0.000	0.991	0.992	0.992	0.000	0.990	0.990	0.990	0.000
DTLZ6	0.448	0.910	0.700	0.105	0.138	0.511	0.322	0.075	0.510	0.922	0.691	0.107	0.990	0.990	0.990	0.000
DTLZ7	0.996	0.996	0.996	0.000	0.996	0.997	0.996	0.000	0.997	0.997	0.997	0.000	0.996	0.996	0.996	0.000
UF1	0.991	0.993	0.992	0.000	0.986	0.989	0.988	0.000	0.978	0.994	0.990	0.005	0.992	0.995	0.994	0.000
UF2	0.987	0.993	0.991	0.002	0.980	0.983	0.981	0.001	0.984	0.991	0.988	0.002	0.986	0.992	0.989	0.002
UF3	0.481	0.674	0.597	0.043	0.678	0.871	0.784	0.048	0.531	0.704	0.589	0.041	0.805	0.909	0.867	0.025
UF4	0.881	0.917	0.908	0.006	0.875	0.910	0.889	0.008	0.923	0.935	0.929	0.003	0.920	0.930	0.925	0.002
UF5	0.035	0.792	0.484	0.165	0.256	0.766	0.641	0.104	0.123	0.792	0.566	0.192	0.586	0.762	0.658	0.043
UF6	0.255	0.711	0.447	0.114	0.235	0.801	0.635	0.120	0.349	0.767	0.568	0.113	0.668	0.922	0.827	0.080
UF7	0.987	0.991	0.990	0.001	0.980	0.983	0.981	0.001	0.557	0.991	0.910	0.150	0.975	0.991	0.988	0.003
Mean	0.806	0.935	0.881	0.038	0.808	0.927	0.886	0.032	0.801	0.940	0.882	0.048	0.930	0.964	0.951	0.008

TABLE III
STATISTICS HV WITH THREE OBJECTIVES

	MOEA/D				NSGA-II				R2-MOEA				VSD-MOEA			
	Min	**Max**	**Mean**	**Std**	**Min**	**Max**	**Mean**	**Std**	**Min**	**Max**	**Mean**	**Std**	**Min**	**Max**	**Mean**	**Std**
WFG1	0.958	0.969	0.966	0.002	0.925	0.945	0.935	0.005	0.968	0.979	0.975	0.002	0.978	0.984	0.982	0.001
WFG2	0.973	0.978	0.976	0.001	0.959	0.974	0.968	0.004	0.962	0.963	0.963	0.000	0.988	0.991	0.989	0.001
WFG3	0.992	0.992	0.992	0.000	0.976	0.988	0.985	0.002	0.991	0.992	0.992	0.000	0.989	0.989	0.989	0.000
WFG4	0.864	0.865	0.865	0.000	0.854	0.883	0.868	0.007	0.903	0.905	0.904	0.000	0.918	0.920	0.919	0.000
WFG5	0.795	0.804	0.797	0.002	0.806	0.836	0.821	0.008	0.843	0.853	0.848	0.002	0.842	0.861	0.853	0.005
WFG6	0.777	0.832	0.809	0.013	0.788	0.836	0.815	0.011	0.847	0.875	0.857	0.007	0.823	0.848	0.834	0.007
WFG7	0.864	0.865	0.865	0.000	0.858	0.889	0.875	0.008	0.901	0.905	0.904	0.001	0.919	0.920	0.919	0.000
WFG8	0.778	0.785	0.782	0.002	0.697	0.730	0.716	0.008	0.816	0.821	0.819	0.001	0.877	0.912	0.902	0.009
WFG9	0.726	0.851	0.819	0.039	0.720	0.833	0.746	0.027	0.773	0.895	0.872	0.038	0.763	0.881	0.872	0.019
DTLZ1	0.950	0.950	0.950	0.000	0.935	0.950	0.943	0.004	0.939	0.943	0.941	0.001	0.962	0.966	0.964	0.001
DTLZ2	0.899	0.899	0.899	0.000	0.871	0.901	0.886	0.007	0.913	0.916	0.915	0.001	0.928	0.930	0.930	0.000
DTLZ3	0.899	0.899	0.899	0.000	0.876	0.901	0.890	0.006	0.914	0.916	0.915	0.000	0.928	0.931	0.930	0.000
DTLZ4	0.151	0.899	0.813	0.238	0.871	0.904	0.888	0.007	0.151	0.916	0.675	0.298	0.929	0.932	0.930	0.001
DTLZ5	0.978	0.978	0.978	0.000	0.982	0.984	0.983	0.001	0.985	0.986	0.986	0.000	0.986	0.986	0.986	0.000
DTLZ6	0.310	0.889	0.591	0.142	0.183	0.382	0.243	0.056	0.400	0.946	0.672	0.143	0.986	0.986	0.986	0.000
DTLZ7	0.914	0.914	0.914	0.000	0.907	0.935	0.924	0.006	0.837	0.893	0.860	0.014	0.963	0.966	0.964	0.001
UF8	0.151	0.830	0.773	0.107	0.324	0.646	0.463	0.069	0.578	0.917	0.898	0.057	0.876	0.926	0.909	0.010
UF9	0.753	0.916	0.846	0.067	0.368	0.782	0.728	0.096	0.778	0.954	0.844	0.079	0.901	0.973	0.946	0.021
UF10	0.145	0.555	0.341	0.162	0.060	0.391	0.242	0.067	0.143	0.578	0.413	0.166	0.410	0.723	0.591	0.101
Mean	0.730	0.877	0.835	0.041	0.735	0.826	0.785	0.021	0.771	0.903	0.855	0.043	0.893	0.928	0.916	0.009

well spread and converged solutions among the Pareto front. Nevertheless, the multi-objective problems are build with some difficulties which mislead several algorithms and drive to premature convergence or stagnation. Mainly, in difficult problems and with long-term executions stagnation could be present wasting important computational resources. In evolutionary algorithms, the diversity of the decision variable space has showed a critical role to avoid premature convergence or stagnation. Particularly, stagnation could converge in sub-optimal regions provoking that the operators lose its exploratiory strength. For this reason, a potential strategy to mitigate those drawbacks dwell in leading the search process through different diversity stages, which change as the stopping criterion is reached.

Mainly, in this work two contributions applied in the multi-objective field are described. First, is presented a mechanism to simultaneoulsy manage the diversity in the variable space and the objective space. This strategy is incorporated in the replacement phase, specifically the diversity maintained in the decision variable space is decremented according the elapsed time and the stopping criterion. Second, a novel density estimator of the objective space, which is based in the IGD+ indicator, is suggested. In the experimental validation carried out, is shown that our proposal not only improves the state-of-the-art algorithms in long-term executions, also offers a competitive performance in short-term executions. In addition,

TABLE IV
STATISTICAL TESTS OF HV WITH TWO OBJECTIVES

	MOEA/D				NSGA-II				R2-MOEA				VSD-MOEA			
	↑	↓	↔	**Diff**	↑	↓	↔	**Diff**	↑	↓	↔	**Diff**	↑	↓	↔	**Diff**
WFG1	1	0	2	0.000	0	2	1	0.000	0	0	3	0.005	1	0	2	0.000
WFG2	1	2	0	0.032	2	1	0	0.024	0	3	0	0.033	3	0	0	0.000
WFG3	2	1	0	0.001	0	3	0	0.005	1	2	0	0.001	3	0	0	0.000
WFG4	1	2	0	0.003	0	3	0	0.006	3	0	0	0.000	2	1	0	0.001
WFG5	0	3	0	0.044	1	1	1	0.036	1	1	1	0.035	3	0	0	0.000
WFG6	1	0	2	0.000	1	0	2	0.001	1	0	2	0.002	0	3	0	0.045
WFG7	1	2	0	0.003	0	3	0	0.007	3	0	0	0.000	2	1	0	0.001
WFG8	1	2	0	0.136	0	3	0	0.158	2	1	0	0.133	3	0	0	0.000
WFG9	1	1	1	0.087	0	3	0	0.138	1	1	1	0.086	3	0	0	0.000
DTLZ1	3	0	0	0.000	0	3	0	0.002	2	1	0	0.001	1	2	0	0.001
DTLZ2	1	2	0	0.002	0	3	0	0.004	3	0	0	0.000	2	1	0	0.001
DTLZ3	1	2	0	0.002	0	3	0	0.003	3	0	0	0.000	2	1	0	0.001
DTLZ4	0	2	1	0.209	1	1	1	0.128	0	0	3	0.334	2	0	1	0.000
DTLZ5	1	2	0	0.002	0	3	0	0.004	3	0	0	0.000	2	1	0	0.001
DTLZ6	1	1	1	0.291	0	3	0	0.668	1	1	1	0.299	3	0	0	0.000
DTLZ7	0	3	0	0.001	2	1	0	0.001	3	0	0	0.000	1	2	0	0.001
UF1	1	1	1	0.002	0	3	0	0.006	1	1	1	0.004	3	0	0	0.000
UF2	3	0	0	0.000	0	3	0	0.010	1	1	1	0.003	1	1	1	0.002
UF3	0	2	1	0.270	2	1	0	0.084	0	2	1	0.279	3	0	0	0.000
UF4	1	2	0	0.020	0	3	0	0.040	3	0	0	0.000	2	1	0	0.003
UF5	0	3	0	0.175	1	0	2	0.018	1	0	2	0.092	1	0	2	0.000
UF6	0	3	0	0.380	2	1	0	0.192	1	2	0	0.258	3	0	0	0.000
UF7	2	0	1	0.000	1	2	0	0.009	0	3	0	0.079	2	0	1	0.001
Total	23	36	10	1.661	13	49	7	1.542	34	19	16	1.643	48	14	7	0.060

TABLE V
STATISTICAL TESTS OF HV WITH THREE OBJECTIVES

	MOEA/D				NSGA-II				R2-MOEA				VSD-MOEA			
	↑	↓	↔	**Diff**	↑	↓	↔	**Diff**	↑	↓	↔	**Diff**	↑	↓	↔	**Diff**
WFG1	1	2	0	0.017	0	3	0	0.047	2	1	0	0.007	3	0	0	0.000
WFG2	2	1	0	0.014	1	2	0	0.022	0	3	0	0.027	3	0	0	0.000
WFG3	3	0	0	0.000	0	3	0	0.008	2	1	0	0.001	1	2	0	0.004
WFG4	0	3	0	0.055	1	2	0	0.052	2	1	0	0.015	3	0	0	0.000
WFG5	0	3	0	0.056	1	2	0	0.032	2	1	0	0.005	3	0	0	0.000
WFG6	0	2	1	0.048	0	2	1	0.043	3	0	0	0.000	2	1	0	0.024
WFG7	0	3	0	0.055	1	2	0	0.044	2	1	0	0.015	3	0	0	0.000
WFG8	1	2	0	0.121	0	3	0	0.187	2	1	0	0.084	3	0	0	0.000
WFG9	1	2	0	0.053	0	3	0	0.127	2	1	0	0.001	3	0	0	0.000
DTLZ1	2	1	0	0.014	1	2	0	0.022	0	3	0	0.024	3	0	0	0.000
DTLZ2	1	2	0	0.031	0	3	0	0.044	2	1	0	0.015	3	0	0	0.000
DTLZ3	1	2	0	0.031	0	3	0	0.040	2	1	0	0.015	3	0	0	0.000
DTLZ4	0	2	1	0.117	1	1	1	0.042	0	1	2	0.255	3	0	0	0.000
DTLZ5	0	3	0	0.007	1	2	0	0.003	2	0	1	0.000	2	0	1	0.000
DTLZ6	1	2	0	0.395	0	3	0	0.743	2	1	0	0.314	3	0	0	0.000
DTLZ7	1	2	0	0.050	2	1	0	0.041	0	3	0	0.104	3	0	0	0.000
UF8	1	2	0	0.136	0	3	0	0.445	2	0	1	0.011	2	0	1	0.000
UF9	1	1	1	0.100	0	3	0	0.218	1	1	1	0.102	3	0	0	0.000
UF10	0	2	1	0.250	0	2	1	0.349	2	1	0	0.178	3	0	0	0.000
Total	16	37	4	1.551	9	45	3	2.507	30	22	5	1.172	52	3	2	0.027

some scalability experiments in the decision variable space are carried out with middle-term executions, results indicate the superiority and stability measured with the hypervolume indicator. In the same line, the evolution of diversity in the variable space with some specific problems is analysed. Showing that the problems of two objectives maintain implicitly less diveristy in decision variable space than the problems of three objectives.

In the future, we plan to develop an adaptive scheme for the initial threshold value, as well the development of alternatives in short-term executions to provide improved solutions. In order, to attain even better results, these strategies are going to be incorporated in a multi-objective memetic algorithm. Finally, the integration of the replacement phase in a different multi-objective paradigm (indicator or decomposition) seems to be promising to give an emphasis of the diversity implications in the multi-objective field.

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