Mathematical Programming 1 (1971) 239–266. North-Holland Publishing Company

PROBLEMS AND METHODS WITH MULTIPLE OBJECTIVE FUNCTIONS *

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Received 15 October 1970
Revised manuscript received 6 April 1971

Let A be a set of feasible alternatives or decisions, and suppose n different indices, measures, or objectives are associated with each possible decision of A. How can a "best" feasible decision be made? What methods can be used or experimented with to reach some decision?

The purpose of this paper is to attempt a synthesis of the main approaches to this problem which have been studied to date. Four different classes of approaches are distinguished: (1) aggregation of multiple objective functions into a unique function defining a complete preference order; (2) progressive definition of preference together with exploration of the feasible set; (3) definition of a partial order stronger than the product of the n complete orders associated with the n objective functions; and (4) maximum reduction of uncertainty and incomparability.

0. Introduction

0.1 When a decision-maker (DM henceforth) has to choose an action among several alternative actions, he usually wants to select the "best" one. When a mathematician builds a model for the purpose of helping the DM, he generally assumes that no problem arises regarding the DM's preferences: he introduces an a priori single objective function f whose value $f(a)$ for a given feasible action a expresses the utility of a for the DM. That is a very workable manner to define a complete order on the set A of feasible actions (which is also supposed clearly defined). In this schema, the only problem for the mathematician is to exhibit extreme points of A.

0.2 In many real problems (choice of investments for a firm or a

* This paper was presented at the 7th Mathematical Programming Symposium 1970, The Hague, The Netherlands.

community; selection, assignment or remuneration policies; product planning, scheduling or sequencing problems ..) comparison between actions must be made on the basis of multiple heterogeneous and complex consequences (dealing with cash-flow, shares, market conditions, future investment possibilities, quality, comfort, security, growth, welfare, ..). Thus it is not easy to summarize for each action a in A the essentials of all its consequences ($C(a)$ henceforth) by the means of a single numeraire. But it might be possible to associate with each $C(a)$ (that is to say with each a) a sequence of n numbers:

$$\gamma_1(a), \gamma_2(a), ..., \gamma_n(n)$$

which, for all practical purposes, sufficiently summarizes all the relevant information in $C(a)$. These numbers $\gamma_i(a)$ can be interpreted as the index, or score of $C(a)$ on the i^{th} criterion or attribute. In this model, the unique objective function of the previous model is replaced by n * objective functions and the complete order on A is changed into a partial order.

0.3. Where is now the problem for the theoretician who wants to bring some help to the DM in his choice? What methods can he use or experiment with? The purpose of this paper is precisely to attempt a synthesis of the main approaches which have been studied in this context. Though the interest for it is rather new, the literature is already copious (see the selected bibliography at the end). Considering this literature I think that four kinds of approaches in particular have to be distinguished:

1. aggregation of multiple objective functions in a single function defining a complete preference order;
2. progressive definition of preference together with exploration of the feasible set;
3. definition of a partial order stronger than the product of the n complete orders associated with the n objective functions;
4. maximum reduction of uncertainty and incomparability.

Though these approaches do not include all the studies referenced, I

* In this paper we will suppose $n \geqslant 3$. For $n = 2$, the best general approach is to compute numerically and to graph values of all efficient actions (see paragraph 2.1) and then ask the DM to select the best point on the tradeoff-curve obtained. However, obtaining such a curve may be difficult. The easiest cases are those in which efficient actions can be enumerated, or when parametric programming techniques are applicable (see in particular Gass and Saaty [27], Geoffrion [29], Grigoriadis [33]).

will limit this paper to these four kinds of approaches. I will present the general philosophy of each and then discuss its conceptual basis and practical limits, together with a short presentation of some particular methods or results.

1. Aggregation of multiple objective functions in a unique function defining preferences

1.1. There exist many cases in which it is relatively easy, for each criterion or attribute considered separately from the others, to evaluate on a common dimension (usually money or other conventional unit) each index or score available. If these evaluations may be regarded as additive values, it is possible to substitute for the n objective functions, a value function of the following type:

$$v_1[\gamma_1(a)] + \ldots + v_n[\gamma_n(a)] \; . \tag{1.1}$$

It is with reference to such hypothesises that the current use of a weighted sum:

$$p_1 \cdot \gamma_1(a) + \ldots + p_n \cdot \gamma_n(a) \tag{1.2}$$

is justified. In [45] J.R. Miller discusses conditions and techniques to associate a "worth" value to a vector of attribute scores having the form (1.1) with:

$$v_i[\gamma_i(a)] = p_i \cdot w_i[\gamma_i(a)]$$

where the weights p_i are constrained to be positive and to sum to unity, and where the w_i-functions are constrained to lie in the interval from 0 to 1.

Without dwelling on this kind of approach, I want to draw attention on to what G. Boldur proposed in [9] to make it operational, especially in linear programming with multiple objective functions *. Denote by $\bar{a}_i$ and $\underline{a}_i$ two elements of A giving to γ_i its maximum and minimum, respectively, on A. G. Boldur suggests trying to estimate utilities which can be associated with these extreme positions, say $\bar{u}_i$ and $\underline{u}_i$ respectively. If a linear interpolation is realistic we can define preference with:

* A may be for instance a convex polyhedron in R^m and the γ_i are n linear functions of coordinates of a.

$$\sum_{i=1}^{n} [\alpha_i \cdot \gamma_i(a) + \beta_i] \tag{1.3}$$

where α_i and β_i are solutions of:

$$\alpha_i \cdot \gamma_i(\bar{a}_i) + \beta_i = \bar{u}_i$$

$$\alpha_i \cdot \gamma_i(\underline{a}_i) + \beta_i = \underline{u}_i \quad .$$

1.2. Additive expressions such as (1.1) are not the only value functions where one can think to aggregate multiple objective functions to define a complete preference order. I mention these (such as multiplicative expressions, see for instance H. Terry [64]), only in passing, as they are basically similar though the functional forms differ. The difficulty with them generally is a greater complexity in the computation of the optimum, but as we will show with the following case, there may exist tricks to stay in linear programming if for others aspects of the problem we are in this framework.

I will now consider the case in which we may define for each criterion or attribute an acceptable level L *: we strongly prefer not to fall short of this level, and we hope to exceed it. I will also suppose (see for example, assignment models studied by A. Charnes, W.W. Cooper, R.J. Niehaus and A. Stedry in [15]) that scales (used for scores or indexes) are defined in such a way that a deviation from the acceptable level L_i, of a given magnitude D ($D = \gamma_i(a) - L_i$), has exactly the same importance on each criterion. With this hypothesis it may seem reasonable to compare actions on the basis of:

$$\text{Min } [\gamma_1(a) - L_1, \ldots, \gamma_n(a) - L_n] \tag{1.4}$$

where the higher the minimum, the better the action. As is well known, one may find the maximum of (1.4) by writing:

maximize z

subject to:

$$\gamma_i(a) - L_i - z \geqq 0 \quad \text{for all } i$$

so that the problem remains linear if it was in all other aspects.

* Without loss of generality we will suppose in the whole paper that for each i, the higher is γ_i the better it is.

Keeping the same notations, L_i's can also be thought of as goals or subgoals – that is to say best suitable levels instead of acceptable levels – as is done in what is now currently named *goal programming* (see mainly A. Charnes and W.W. Cooper [13], Y. Ijiri [35] and E. Johnsen [37]). Value function (1.4) has then to be replaced by a weighted sum of absolute deviations $|\gamma_i(a) - L_i|$ with positive weights c_i^+ when the deviation is positive and c_i^- when it is negative. The minimum of such a quantity may be easily obtained by linear programming (if all other constraints are linear) in writing:

$$\text{minimize} \sum_{i=1}^{n} (c_i^+ \cdot y_i^+ + c_i^- \cdot y_i^-) \tag{1.5}$$

subject to:

$$y_i^- \geqq 0\,, \quad y_i^+ \geqq 0\,, \quad \text{for all } i$$

$$\gamma_i(a) - L_i + y_i^- - y_i^+ = 0 \quad \text{for all } i\,.$$

Finally I want to mention the possibility of choosing as value function an ordered set of functions intervening in a lexicographical way. For example, in an assignment problem, we may consider function (1.4) as the major criterion, function (1.5) as a secondary criterion to decide between ties resulting from (1.4), and the sum of positive deviations as a third criterion to decide between ties resulting from both (1.4) and (1.5).

1.3. Whatever be the unique function ((1.1) to (1.5) or others) which is substituted for the n objective functions, a major problem is to be sure of having a good definition of DM's preferences according to the real problem. To judge if this is so, we may consider the *indifference map* associated with the value function. It is always useful to verify that the combinations of scores, obtained as solutions of the equation derived by setting the value function equal to a constant, may realistically be regarded as equivalent.

Everyone who attempts to give a good representation of preferences with a value (or utility) function defined on n-tuples = $(s_1, \ldots, s_n)$ * must always recall conditions under which additive forms (cf. (1.1),

* s_i = any admissible score for the ith criterion or attribute.

(1.2), (1.3)) are admissible. Two cases have to be considered according to the non probabilistic or probabilistic character of the outcomes.

In non probabilistic choice situations, the value function is only required to satisfy:

$$v(s'_1, \ldots, s'_n) \geqq v(s''_1, \ldots, s''_n)$$

when and only when the n-tuple $(s'_1, \ldots, s'_n)$ is preferred or indifferent to $(s''_1, \ldots, s''_n)$. It is clear that the additive form:

$$v(s_1, \ldots, s_n) = v_1(s_1) + \ldots + v_n(s_n) \tag{1.6}$$

that implies, for all i and j components, holding all the other components fixed, the *substitution rate* between the ith and jth components * does not depend on the values of components other than i and j. G. Debreu [18] has proved that, for $n \geqslant 3$ and under continuity conditions, this independence substitution rate condition is sufficient to imply the additive form (1.6). Although such a condition seems little restrictive, in practice, it is not rare to have substitution rates affected by values of non directly concerned components. When the worst of the values influences preferences (see an extreme case in paragraph 1.2), the concept of substitution rate itself loses the major part of its interest.

Consider now probabilistic choice situations. To avoid any confusion, we will speak in this case of utility functions (denoted by u) instead of value functions. It is then usually required that the expected utility number be an appropriate index to maximize. Here, the utility function not only has to reflect the ordinal preference (as previously) but also the utility of a lottery L giving q_i chance at n-tuple $s^{(i)}$ ($i = 1, \ldots, r$). The utility of L must be:

$$u(L) = q_1 \cdot u(\underline{s}^{(1)}) + \ldots + q_r \cdot u(\underline{s}^{(r)}) .$$

The existence of an additive form for u in the probabilistic case requires a stronger hypothesis than in the non probabilistic case. The difference is essentially expressed by the following question. Consider

* Recall that this substitution rate is defined by the answer to the following question: if we change s_i to s'_i in a given n-tuple, by how much must we change s_j so that the modified n-tuple is indifferent to the original n-tuple?

four scores s'_i, s'_j, s''_i, s''_j ($s'_i \leqq s''_i$, $s'_j \leqq s''_j$). Suppose a lottery where the probabilities of s'_i and s''_i are fifty-fifty, and the probabilities of s'_j and s''_j are also fifty-fifty (every other score being invariant). Is the lottery with extreme pairings (s'_i, s'_j) and (s''_i, s''_j) indifferent to the lottery with intermediate pairings (s'_i, s''_j) and (s''_i, s'_j)? (see fig. 1).
The additive form requires that the answer be *yes*. It is obvious that in practice the juxtaposition of bad values may for instance be a dangerous disadvantage and a lottery with intermediate pairings may be preferred.

All these questions about additive value and additive utility are discussed by H. Raiffa in a very clear and comprehensive paper [50] in which he is "concerned mainly with the association of utility numbers to consequences when these consequences are complicated stimuli". He writes further: "this assessment of utility values is far from easy but there are many tricks of the trade that can help systematize one's thought process and in some circumstances crude evaluations might suffice to resolve the action problem; but if these crude evaluations do not suffice, then at least we will know how to refine the evaluations". Additive, but also non additive cases are considered, and particular attention is devoted to the cases in which it is recommended to introduce a hierarchical structure of attributes. Concerning this matter I want to cite again Raiffa: "Also one can elect to decompose the problem down to a very fine partition of detail in *qualitative* terms but introduce *quantitative* evaluations only at a much broader level of aggregation. As an extreme case of this, one might choose to elaborate the detailed goal fabric of a problem but refuse to assign utility numbers at any level of disaggregation below the super-goal of "the good life"."

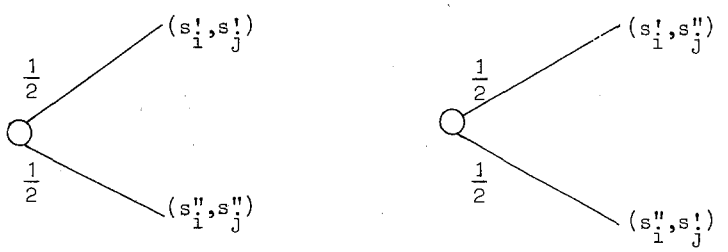

Fig. 1. Are these two lotteries indifferent?

2. Progressive definition of preferences together with exploration of the feasible set

2.1. In practice, as Raiffa said, it is "far from easy" to make explicit preferences between every pair of n-tuples in building a value or utility function. Even with the tricks and techniques proposed, many problems will present either too much work to have a satisfactory reflection of real preferences of the DM, or a too arbitrary representation of those preferences in a very crude function.

In fact, several aspects of the work required to build the value function are superfluous for finding the best action.

First, in building a value function, we make explicit preferences between every pair of n-tuples and yet we generally do not need to know how to compare two n-tuples, one of which is unrealistic in the sense that there does not exist any action in A having this n-tuple as consequence vector. Furthermore we only need to work on the comparison between two *efficient actions*: by efficient action we mean an action a such that there is no other action in A having a higher score for at least one attribute, and the same score for other attributes (Pareto optimum). Consider for instance the attribute which seems a priori to have the major importance for determining the choice; its contribution to a general value function must be examined carefully in any case. It may nevertheless happen that such work is inappropriate because scores obtained using this attribute with all efficient actions are nearly the same or sufficiently exceed an acceptable level, that the discriminatory power of this attribute appears a posteriori completely negligible. On the other hand, we may conceive of another attribute which a priori seems secondary, but if it leads to a large range of scores having very bad values for some efficient actions, we are required to consider this attribute with great attention. I would like to add that the DM's answers to questions on his preferences will be more exact if they are asked in terms of comparison between two real, efficient actions.

But, it is often difficult to formalize the set of efficient actions and to work with it (see for instance J. Philip [49]). With these remarks, we are led to define preferences together with an exploration of the feasible set for seeking efficient solutions. Such a pragmatic approach may appear as non-scientific, accustomed as we are to solve a problem only after it has been completely and clearly formulated. Here we begin to "solve" the problem when the nature of the "best choice" we are

looking for is still fuzzy, and exploit the work done step by step, in progressing to a definition of the unknown. I see nothing irrational or clumsy in this kind of approach (cf. B. Roy [52]). On the contrary, I believe it is very close to what is effectively done by the DM himself.

2.2. Within the linear programming framework, two approaches have been undertaken recently and independently by J. Saska [59] in Czechoslovakia, and R. Benayoun and J. Tergny [5] in France. Both works introduce a concept of "best compromise" to replace that of best solution (in the sense of optimum). Also both begin by finding extreme solutions for each linear objective function considered independently from the others. The approaches differ in the way towards a best compromise. Saska describes two elementary procedures to obtain such a best compromise. Benayoun and Tergny in a general approach named POP (Progressive Orientation Procedure) propose a more highly developed procedure. The method is illustrated by an application to manpower management. In [7] they study three cases according to the information one has concerning the relative importance of the n objective functions: (1) it is quantified, (2) it is known but not quantifiable and (3) it is completely implicit. They describe a very interesting algorithm named STEM in which an automatic mechanism allows the integration step by step of the DM's answers to very simple questions for guiding the exploration of the feasible set and defining the best compromise.

In STEM each iteration is made up of a calculation phase and a decision-making phase. The calculation consists of the construction of a Pay-Off Table and the calculation of a feasible compromise solution which will be proposed to the decision-maker in the decision phase. The pay-off table contains in the jth row values of the γ_i's for one of the actions which maximize γ_j on A. The diagonal of this table contains the optimum values and represents an "ideal" solution which, in general, is not in A (they only consider this case). The compromise solution calculated is the feasible solution with value nearest (in the minimax sense) to that of the ideal solution. In this calculation, "weights" are introduced to define the relative importance of the distances to the ideal solution. They depend on what we know in advance about the relative importance of the objectives, but in all the three cases mentioned above, their determination is influenced by the values of the objectives in the pay-off table. In the decision phase the ideal solution and the compromise solution are shown to the decision-maker. Com-

paring them, he decides if the compromise solution is satisfactory; if it is, the compromise is the solution required and the procedure terminates. Otherwise he must accept a relaxation of a criterion to improve the values given by the others. He then indicates that criterion and the maximum amount of relaxation he can accept, with help of a sensitivity analysis giving the behaviour of the different objective functions in the neighbourhood of the last compromise solution. Then a return is made to the calculation phase. According to this information, new weights are determined and a new compromise solution is proposed. The algorithm produces the solution in less than n iterations.

Another approach of this second kind is that proposed by A.M. Geoffrion in [31] here again within the mathematical programming framework. He considers the case in which A is a convex set of R^q defined by a family of differentiable constraints on the q components of a; the γ_i's are assumed to be differentiable concave functions of these components. Moreover, he supposes the existence for the DM of an implicit ordinal preference function:

$$V(\gamma_1 \ldots,\gamma_n)$$

V being assumed to be concave, increasing and differentiable, but obviously determined up to an arbitrary increasing differentiable transformation. Although this preference function remains implicit, the DM must be able to give some specific kinds of information about it. In the procedure, some new information intervene at the beginning of each iteration to determine the direction of the exploration, and at the end to fix a new starting point for the next iteration.

The first kind of information deals with marginal substitution rates * between the ith criterion and a fixed reference criterion, say the nth (choosen so as to allow easy comparisons). The author makes the following strong hypothesis: these $n - 1$ marginal substitution rates can be specified by the DM at any given point a in A **. These values define the tangential hyperplane to the indifference surface passing through the point considered and consequently the direction of gradient. Classical methods in nonlinear programming (see Zoutendijk

* See note paragraph 1.3. We specify marginal to underline that this substitution rate may depend on all the values of the γ_i's at the point considered.

** We will see in paragraph 4.2 another use of these marginal substitution rates but in a weaker hypothesis (cf. note paragraph 4.2).

[66]) may then be applied to determine a feasible direction for increasing V. Denote by θ a scalar variable used to fix the current point $a(\theta)$ on the feasible interval of the chosen direction. It is now easy to compute and graph $\gamma_i[a(\theta)]$ as a function of θ, on the interval of feasibility, for each i. The curves obtained (see fig. 2) can be superimposed on a common abscissa. The iteration ends by requiring the DM to select the most preferred value of θ, that is to say the value of the feasibility interval which maximizes:

$$V[\gamma_1(a(\theta)),\ldots,\gamma_n(a(\theta))],$$

(that is the second kind of information needed about this implicit function). The value selected defines a new point from which a new feasible direction will be derived (if possible) and so on.

In the same paper [31] Geoffrion studies a generalization of this "vector maximum problem" to the case in which each γ_i function "is now assumed to be the optimal value of an optimization problem parametrized by a". It is what he calls "vector maximal decomposition program". He gives an illustration of this class of problem "in terms of the management of a decentralized organization composed of n semi-autonomous operating divisions, each of which may be pursuing a different goal within the charter of the organization as a whole. Then a would be interpreted as the vector of decision variables under the control of the top administrator of the organization (e.g. resource allocations to each division)".

2.3. As useful as convexity and differentiability properties are for the development of the above procedures, I think that, according to the general philosophy of paragraph 2.1, methods based on a progressive

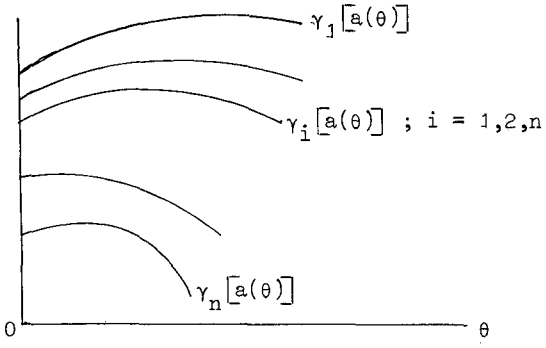

Fig. 2.

definition of preferences together with an exploration of the feasible set can be conceived within other contexts (outside of continuous mathematical programming). I want only mention here some directions which probably have not been enough investigated in this direction.

First, in discrete mathematical programming with multiple objective functions, branch and bound procedures (particularly in the general framework of P.S.E.S. and P.S.E.P. as in B. Roy [56] ch. 10) might allow building suitable algorithms.

Second, it could be interesting, for instance, to develop such methods for the shortest route problem in a graph when, instead of having a unique number attached to each arc to define its length, we have a vector.

Finally, cases in which A is defined by a simple enumeration of its elements (new products, R. and D. projects, ... see for instance P. Buffet, J.C. Gremy, M. Marc, B. Sussmann [12]) with a vector index attached to each, might also be studied in a similar perspective. Some of the results obtained by M. Barbut in [4] and Jaquet-Lagreze in [36] could perhaps be combined with the DM's answers to successive comparisons of appropriate actions, to build progressively a complete order on A.

3. Definition of a partial order stronger than the product of the n complete orders associated with the n objective functions

3.1. Denote by S_i the set of possible scores or indexes for the ith criterion or attribute; each S_i is completely ordered by preferences: we may always suppose that $s' \epsilon S_i$ will be preferred to $s'' \epsilon S_i$ if and only if $s' > s''$. The product $S_1 \times \ldots \times S_n$ is partially ordered by the classical product of the n complete orders. This partial order (*dominance order* henceforth) is usually very poor. On the other hand approaches in sections 1 and 2 suppose the existence of the richest order we would conceive of: I refer here to the complete order associated with a value or utility function. When we are confronted with a real problem, we must ask ourselves the following question: is it necessary and realistic to suppose the existence of such a complete order and is it possible to develop it into a sufficiently good approximation?

We must consider cases in which the DM may be willing and able to arrive at preference decisions only for some pairs of vectors, while for

the others he may be unwilling or unable to arrive at a decision. He may consider that the data are too rough or that validating the decision would require a too expensive study. Concerning experimental work on utility theory, Luce and Raiffa mention in [43] the possibility of intransitivities "when a subject forces choices between inherently incomparable alternatives". To give an example, consider a first action a' for which the $\gamma_i(a')$'s are all acceptable medium values, and a second action a'' such that for all criteria except the first one a'' is slightly preferred or indifferent to a', but $\gamma_1(a'')$ is very bad in comparison to $\gamma_1(a')$. May such an important negative deviation be compensated or not by the family of small positive deviations? This may be a very difficult question for the DM and he may prefer to admit an incomparability.

R.J. Aumann studied in [1] (revised in [2]) what happens with utility theory when we drop out the completeness axiom*. He shows that we still get a utility function u which satisfies the expected utility hypothesis (cf. paragraph 1.3), and still "represents" the preference order but now in a weaker sense. As before, if a' is preferred to a'', then $u(a') > u(a'')$, but the reverse implication is no longer true (since real numbers are completely ordered). Indeed, this utility function "retains many of the useful properties of the Von Neumann-Morgenstern utility. For example, we can solve maximization problems with it: maximization of our utility over a given constraint set will always lead to a maximal element of the constraint set; conversely, for every maximal element a there is a utility whose maximization leads to a." Nevertheless, the major difficulty is still the construction of such utility functions, and here the complexity seems greater than in the classical case with the completeness axiom (see 1.3).

3.2. Since the dominance order is too poor and loses all information about comparisons when they are not unanimous for all criteria, it is natural (cf. B. Roy [56], 1st volume, pp. 185–186) to examine some way to build a richer binary relation taking into account this information. Such a binary relation, say R, must be understood as an attempt to synthetize the n orders associated with the n objective functions. Usually these orders will be in contradiction; because of some of the aspects discussed in 3.1, there is no reason (and it even may be dangerous) to force R to be transitive. In effect, for any couple (a', a'')

* See also Fishburn [23].

of actions, in relation to the two vectors of γ_i's and to the quality of these data, the question is: is it realistic to regard a' as almost surely better than a''? When the answer is *yes* (and only in this case) $a'Ra''$ holds. Then for instance the attributes (and general reasons) which allow one to decide a^1Ra^2 and those which allow a^2Ra^3 may be too distinct to justify a^1Ra^3. Such a relation R (outranking relation henceforth) must not be viewed as an exact reflection of all the DM's preferences, but only as the expression of the part of his preferences that can be well accounted for by means of the available data.

An outranking relation may be very useful to rough out a problem. First, it may be a basis for more profound comprehension and expression of the DM's preferences. Second, if cycles * may be regarded as expressing indifference, it gives the possibility to restrict the choice problem to a subset of A. This subset is defined (cf. [56] 1st volume, pp. 192–193) from the kernel K_C of the graph G_C defined as follows.Let:

- C be the equivalence relation defined by $a'Ca''$ if and only if there exists in R a cycle passing by a' and a''
- B denote the set of classes in the equivalence C
- R_C denote the relation defined on B and verified by the couple of classes (b', b'') if and only if there exist $a' \in b'$ and $a'' \in b''$ such that $a'Ra''$; when R is acyclic: $B = A$ and $R_C = R$
- G_C be the acyclic graph associated with the relation R_C **.

A subset $K_C \subset B$ is called a *kernel* of G_C if:

(i) $\forall b', b''$ in B: not $b'R_Cb''$;
(ii) $\forall b \in B - K_C$ there exists $k \in K_C$ such that kR_Cb.

It is well known that an acyclic graph has a kernel and only one, cf. [56]. For instance, the single kernel of the graph drawn in fig. 6 is formed of three nodes 2, 4 and 5.

Consider now the subset K of A formed of all the elements of A belonging to a class of K_C. By the definition of K, for each action a of $A - K$, there exists an action k in K such that kRa holds, and any two actions in K are incomparable by R or indifferent by C. If R is an order, K is reduced to the subset of actions which are not outranked in R_C, that is to say, if R was complete, K would be the first class of R_C. In

* Directed path beginning in a node and coming back to this node.

** By definition of such a graph there exists an arc from node a' to node a'' if and only if $a'R_Ca''$.

any non-trivial case, K is smaller than the subset of all efficient actions of A.

R. Benayoun, B. Roy and B. Sussmann in [6] proposed a particular method of this type named ELECTRE I (see also B. Roy [53]). I would like to introduce here some more general considerations for building an outranking relation R, which improve what have been done in ELECTRE I.

3.3. We will consider here cases in which $A, S_1,...,S_n$ are finite sets such as in the following example.

Suppose the DM to be a forty year old man having to choose a car for his family. The most important criterion for him is the price; immediately thereafter comes comfort; speed and beauty are two other slightly less important criteria he wants also to introduce. For each of these four criteria he defines scales (see fig. 3) on which he can easily and quickly evaluate each available car. Size of the car is also an important factor he has to consider, but he estimates more suitable its use as a constraint to define the set A_c of types of cars (among those offered by the market) he has to compare. Suppose he selects only the 7 types described in fig. 4. * According to the quality of the data, he may estimate that it will be too difficult and too long to assess a value function; he may also reject the second kind of approach as not appropriate. Then he may want to build an outranking relation, but how can he do so?

For all this paragraph, we will assume that A is composed of only efficient and non equivalent actions. Consider now a couple (a', a'') of feasible actions (corresponding in the example to two different types of cars of A_c still denoted by a' and a''). It leads to a trichotomic partition of the set $I = \{1, 2, ..., n\}$ of criteria:

$I^+(a', a'')$: set of criteria for which a' is preferred to a'',
$I^=(a', a'')$: set of criteria for which a' is indifferent to a'',
$I^-(a', a'')$: set of criteria for which a'' is preferred to a'.

Suppose that we may account for the relative importance of each of these three subsets of I, each of them being considered as defining a new criterion made of the union of its elements. Let:

* We may identify A and A_c: any feasible action a of A being defined as the purchase of a car belonging to a well specified type of A_c.

$$P^+(a', a''), \quad P^=(a', a''), \quad P^-(a', a''),$$

be three numbers (something like weights) representative of this relative importance of the three subsets. We have here a first kind of information to assess an outranking relation.

To proceed in this way we may imagine one (or several) condition of the form:

$$C[P^+(a', a''), P^=(a', a''), P^-(a', a'')] \geqq c \tag{3.1}$$

such that $a'Ra''$ cannot hold if the couple (a',a'') does not satisfy it. The left hand side of (3.1) must be conceived as an indicator of the more or less good "concordance" – we will say, the degree of concordance – between criteria in the sense a' prefered to a''; the right hand side appears as an acceptable level c that the degree of concordance must reach. In this *concordance condition, c* may be viewed as a parameter whose influences on R must be studied. In ELECTRE I (see [53]) the degree of concordance is merely defined as the sum $P^+ + P^=$. The resulting concordance condition (3.1) allows $a'Ra''$ together with $a''Ra'$, while this is impossible with the following *:

$$P^+(a', a'')/P^-(a', a'') \geqq c\,. \tag{3.2}$$

In ELECTRE II, cf [56 bis], the selected solution combines the above one referring to ELECTRE I and one condition of the (3.2) type.

In practice, when criteria overlap little (that is to say deal with sufficiently distinct aspects of the consequences) one may write:

$$P^\star(a', a'') = \sum_{i \in I^\star(a', a'')} p_i \quad \text{for} \quad \star \in \{+, =, -\} \tag{3.3}$$

where p_i is a weight (positive number) giving the importance of the ith criterion. In the example, according to what we said above, we must have:

$$p_1 > p_2 > p_3 = p_4\,. \tag{3.4}$$

Suppose for the DM that criteria 3 and 4 considered together have

* It can be accepted only when n is sufficiently small to avoid cases in which P^+ and P^- are both not equal to zero but very small compared to $P^=$.

slightly greater importance than the price:

$$p_3 + p_4 > p_1 \tag{3.5}$$

he can arrive at the following weights:

$$p_1 = 5\,, \quad p_2 = 4\,, \quad p_3 = 3\,, \quad p_4 = 3\,. \tag{3.6}$$

Values of the ratio P^+/P^- (cf. formula 3.2) are indicated (under conditions 3.2 and 3.6) in fig. 5 for each couple (a', a''), (with a' corresponding to the rows, and a'' to the columns) such that the ratio is greater than one.

If a given high degree of concordance (cf. formula 3.1) seems necessary to decide that a' outranks a'', it usually does not seem sufficient. Indeed we assumed that, we were not able to aggregate on the one hand, all the positive deviations $\gamma_i(a') - \gamma_i(a'')$ for $i \in I^+(a', a'')$ and on the other hand all the negative deviations $\gamma_i(a') - \gamma_i(a'')$ for $i \in I^-(a', a'')$, for seeing if the first one really compensates the second one. Nevertheless, we can establish additional conditions upon the negative deviations which have to be satisfied to accept that $a'Ra''$ holds when (3.1) is verified: among others, the following kind of rule may be used.

Suppose that, (3.1) being satisfied, for a given i in I^- (a', a''), $\gamma_i(a')$ is very bad compared to $\gamma_i(a'')$. Such a "discordance" may appear as too big to be compensated in all cases by positive deviations. As we agreed to avoid this much too difficult kind of discussion, the simplest procedure in such a case is to decide that a' cannot outrank a''. It is easy to formalize such a *discordance condition* by giving for each i in I a subset $D_i \subset S_i \times S_i$ such that, for any couple (a', a''):

$$\gamma_i(a') = s' \quad \text{and} \quad \gamma_i(a'') = s'' \quad \text{with} \quad (s', s'') \in D_i \tag{3.7}$$

$$\Rightarrow \; a'Ra'' \text{ cannot hold.}$$

In brief, $a'Ra''$ holds if and only if (3.1) is verified and (3.7) does not imply incomparability.

In the example the DM may let:

$$D_1 = \{(45, 25), (45, 30), (40, 25)\} \tag{3.8}$$

$$D_2 = \{(N, H)\} \quad D_3 = \emptyset \qquad D_4 = \emptyset\,.$$

Criteria	Scales	Codes
$i = 1$: Price	less than \$2 700	25
	from \$2 800 to \$3 200	30
	from \$3 300 to \$3 700	35
	from \$3 800 to \$4 200	40
	from \$4 300 to \$4 700	45
$i = 2$: Comfort	High	H
	Medium	M
	Low	L
$i = 3$: Speed	Fast	F
	Slow	S
$i = 4$: Beauty	Beautiful	B
	Acceptable	A

Fig. 3.

Types		Evaluations											
		γ_1	γ_2	γ_3	γ_4		①	②	③	④	⑤	⑥	⑦
	①	45	H	F	B	①	–	–	–	1.40	1.40	–	2.00
	②	40	H	S	B	②	1.67	–	1.33	–	–	–	1.20
	③	40	M	F	B	③	1.25	–	–	–	–	–	2.00
A	④	35	M	F	A	④	–	1.14	1.67	–	–	1.33	1.40
	⑤	35	M	S	B	⑤	–	1.25	1.67	–	–	1.33	1.40
	⑥	35	L	F	B	⑥	1.25	2.00	1.20	–	–	–	1.20
	⑦	25	L	S	A	⑦	–	–	–	–	–	–	–

Fig. 4.

$p_1 = 5$ $p_2 = 4$
$p_3 = 3$ $p_4 = 3$

Fig. 5.

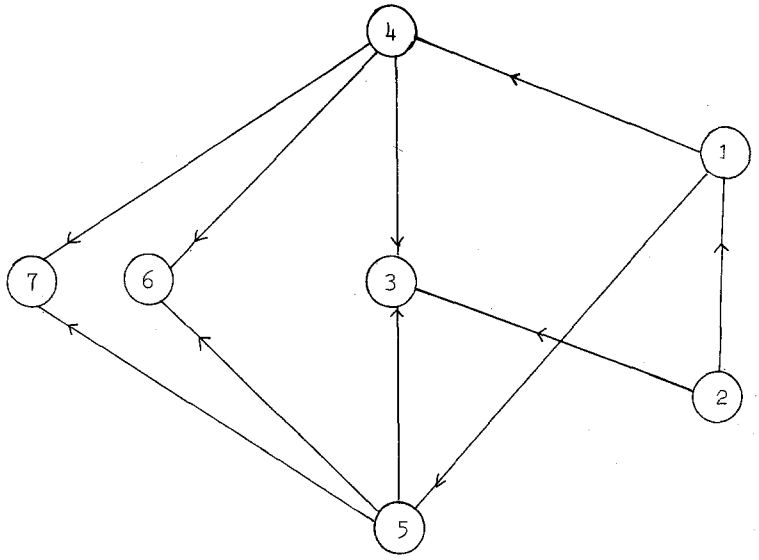

Fig. 6.

Is it possible to restrict ourselves to a discordance condition (3.7) for succeeding to define R? It obviously depends on the problem, but if it seems necessary to go deeper, two directions must be investigated. First, instead of only considering separately each criterion, we may also introduce subsets D for two criteria (or more). Second, in order to avoid that all deviations play the same role whatever be their magnitude, we may introduce a hierarchy by means of two three levels of preference on each criterion; by an appropriate extension of the definitions of sets I^+, I^- and D it is possible to establish stronger and more sophisticated conditions for defining R. *

Suppose now R to be well defined, and denote by G its associated graph. Fig. 6 shows the graph obtained in the car choice problem when R is defined by the table of concordance of fig. 5 with $c = 4/3$ and discordance condition (3.7) (3.8). Here G has no cycle and its unique kernel K (see end of paragraph 3.2) is formed of nodes 2, 4 and 5. It follows that types of cars associated with the four other nodes can be eliminated, and attention must be concentrated on those which are in the kernel. For progressing in comparison between them, one may use one of the approaches studied in the next section.

* It is a third direction which has been adopted in ELECTRE II (cf [56 bis]).

4. Maximum reduction of uncertainty and incomparability

4.1. I now arrive at the last kind of approach I wanted to examine. It may be characterized by a systematic manner to reduce uncertainty and incomparability. As in the preceding kinds of approaches, we assume the existence on A of a preference order, complete or not, but insufficiently explicit to allow a decision as to the best choice. Suppose this order could be accounted for in the frame of a general formalism, in which some factors (like marginal substitution rates, subjective probabilities, ...) are not well known. Methods of this kind consist first, in a proposal of such a formalism, and second, in a systematic elaboration of limits for such factors of uncertainty or incomparability so that a best choice (or at least a small number of best efficient actions) might be exhibited in spite of the remaining uncertainty and incomparability.

Here, the objective is not to specify the preference order by means of a mathematical form (value or utility function) as in the first kind of approach. If, as in the second kind, we find some systematic effort towards a better knowledge of preferences, they are not integrated into an exploration of A, but they deal with specific information like ranges of parameters, signs of differences and so on. In that way, effects of uncertainty and incomparability can be systematically reduced without passing by a global definition of something like an outranking relation as in the third kind of approach.

4.2. There is a very good illustration of this fourth kind of approach that C. Maier-Rothe and M.F. Stankard Jr. propose in [44]. The authors consider the case in which $S_1 \times \ldots \times S_n$ is convex, so that a' and a'' being two feasible actions, they may introduce a hypothetical feasible action $a^\star$ whose score for the ith criterion is $\frac{1}{2}[\gamma_i(a') + \gamma_i(a'')]$. Then they suppose the existence of an unknown utility function u and they show that $u(a') - u(a'') \geqq 0$ if and only if:

$$\sum_{i=1}^{n} [\gamma_i(a') - \gamma_i(a'')] R_i \geqq 0 \qquad (4.1)$$

where R_i is the finite difference approximation to the marginal substitution rate of components γ_i and, say *, γ_n at the point $a^\star$:

* Contrary to what has been done by Geoffrion (see note paragraph 2.2), these marginal substitution rates are not assumed to be known here. It is supposed that only lower and upper bounds can be specified through answers to appropriate concrete questions, as we will explain further.

$$R_i = \left.\frac{\Delta u/\Delta\gamma_i}{\Delta u/\Delta\gamma_n}\right|_{a^\star} \qquad (R_i \geqq 0) \qquad i = 1, \ldots, n \,. \tag{4.2}$$

The question as to whether a' is preferred or not to a'' can be answered as soon as we have enough information on the R_i's to determine the sign of the left hand side of (4.1). Denote by $Z(R_1, \ldots, R_n)$ this linear expression of the R_i's, and by D the domain of possible vectors $\underline{R} = (R_1, \ldots, R_n)$. With this formalism, the problem is now to arrive at a definition of D such that:

$$\underset{\underline{R}\in D}{\text{Min}}\, Z(\underline{R}) \quad \text{and} \quad \underset{\underline{R}\in D}{\text{Max}}\, Z(\underline{R}) \quad \text{have the same sign} \tag{4.3}$$

Maier-Rothe and Stankard explained how this can easily be done by obtaining from the decision maker answers to a sequence of simple questions dealing with his preferences between vectors:

$$\begin{pmatrix} \gamma_1(a^\star) \\ \vdots \\ \gamma_i(a^\star) + \Delta\gamma_i \\ \vdots \\ \gamma_n(a^\star) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \gamma_1(a^\star) \\ \vdots \\ \gamma_i(a^\star) \\ \vdots \\ \gamma_n(a^\star) + \Delta\gamma_n \end{pmatrix} . \tag{4.4}$$

When his preference is for the left hand vector, we have, by definition of S_i:

$$R_i \geqq \Delta\gamma_n/\Delta\gamma_i \tag{4.5}$$

(then we have a lower bound better than zero) and when it is the reverse, we have an upper bound:

$$R_i \leqq \Delta\gamma_n/\Delta\gamma_i \,. \tag{4.6}$$

If lower bounds zero or (4.5), together with upper bounds (4.6) are too broad to satisfy condition (4.3), the authors explain how to gather additional efficient information by new comparisons (4.4).

I would mention here another approach having the same purpose of building a complete order on a finite set A, but with a different

hypothesis on the nature of scales S_1 ....,S_n which are here supposed to be finite sets. In this case $S_1 \times ... \times S_n$ is also a finite set and the complete order we are looking for may always be defined by the mean of a classification rank assigned to each n-tuple of scores. Discussions on marginal substitution rates can be replaced by discussions on comparative effects on the classification rank of increasing or decreasing a score to the nearest one for some pairs of attributes. An appropriate formalism dealing with such comparisons leads to rules expressed in terms of "déclassements comparés" (see an example of application in H. le Boulanger and B. Roy [40]). The difficulty is to know how to progress towards a family of such rules which, considered together are such that all systems of classification ranks compatible with it, appear very "similar" at least for the top of the classification. Some investigations of these problems are presented by O. Larichev in [39].

4.3. Consider now a risky choice problem in which consequences of any feasible action a may be influenced by some random events. Denote by:

$$E = \{e_1, ..., e_m\}$$

the set of all exclusive events we have to consider. m vectors are then necessary to describe consequences of each action a of A:

$$\gamma_1(a, e_j), ..., \gamma_n(a, e_j) \qquad j = 1, ..., m . \tag{4.7}$$

Suppose it is realistic to introduce, for each j, a utility function u_j so that the m vectors (4.7) could be summarized in m numeraires:

$$u_j(a) \qquad j = 1, ..., m . \tag{4.8}$$

As we explained in section 1, the estimations of the u_j's for all a in A may be a diffcult task, even if A is finite with r elements (as we will assume until the end of the paper).

Two cases have to be distinguished:

(α) the choice of some a in A may influence the chances for some of the random events;

(β) whatever be the choice in A, the chances for random events remain unchanged.

Investigations of the sign of the difference between the expected utilities of two actions under various conditions of "partial" measure-

ment of utilities and subjective probabilities for cases (α) and (β) have been made in some detail by Fishburn: see [20bis].

J. Vedder in [65] considers the case (α) and he shows that, in many cases (interesting illustrations to planning problems are given) a best choice can be exhibited with only elementary items of information on the u_j's. But he has to assume that the conditional probability of each event e_j in E, given the choice of an action a^k of A:

$$\Pr(e_j/a^k) = p_j^k$$

can be estimated with sufficiently good precision. * With this formalism, we can define the utility U of any feasible action by its expected utility:

$$U(a^k) = \sum_{j=1}^{m} p_j^k \cdot u_j(a^k) \qquad k = 1, \ldots, r\,. \tag{4.9}$$

As in Maier-Rothe and Stänkard's approach, the knowledge of the exact value of U is not necessary to determine the best choice: in effect the knowledge of the sign of:

$$U(a^k) - U(a^h) \tag{4.10}$$

suffices to eliminate one of the two actions (so if we always succeed to establish this sign for couples successively considered, a best choice may be obtained in exactly $r - 1$ comparisons). Vedder introduces "successively higher levels of measurement" of the u_j's to arrive at a decision on such signs. When he does not reach a best choice at a given level he goes to the next level.

The lower level is that of the dominance order (see paragraph 3.1) on the set of vectors (4.7) for all a in A. Regarding this order, a system of inequalities of the form:

$$u_j(a^k) \geqq u_g(a^h) \qquad \text{for some } j, g, k, h \tag{4.11}$$

may be introduced to delimit a first domain D_1 for the u_j's. A method is proposed to arrive, (if it is possible), within these limits, at a decision

* In fact, this assumption can be slightly relaxed because if we use, as we suggest further, linear programming, a sensitivity analysis can allow us to be satisfied with crude approximations.

on the sign of differences (4.10). The second level consists in defining a complete order on the u_j's. * It results in a new domain D_2, and signs which have not been decided with D_1 are reconsidered with D_2. The third level makes intervening comparisons between differences in utilities. An ultimate level is that of an appropriate estimation of utilities dealing with remaining incomparable pairs. With the same procedure for reducing incomparability, decisions on signs could also be obtained by minimizing and maximizing (4.10), which is a linear expression of the u_j's, these variables being constrained to verify successive systems of linear inequalities defining D_1, D_2, ... This would have the advantage among others indicated in note paragraph 4.3.

In connection with concrete questions about airbus realization, Fourgeaud, Lenclud and Sentis in [25] consider a particular problem belonging to case (β). Contrary to Vedder they assume that the utility numbers $u_j(a)$ can be estimated with a sufficiently good approximation, while one can introduce for each event $e_j \in E$ only a subjective probability p_j whose value is fuzzy. In this context, difficulty of comparisons is only due to uncertainty but p_j can also be viewed in a deterministic context as fuzzy weights entering in an additive utility formula $\Sigma\ p_j \cdot u_j(a)$ similar to expected utility. For instance, the p_j's may deal with relative weightings by customers, of different characteristics of a new product. Although we will continue to refer to the p_j's as subjective probabilities, what follows is pertinent for both probabilistic and non-probabilistic contexts.

The theory of games approach is the first one of which it is natural to think when we are confronted with to the above type of problem. The authors of [25] pointed out that, in many cases, the p_j's are not completely unknown, and they study the case in which a partial order on these numbers can be introduced. Let us denote by D_1 the domain of possible values for $p_1, ..., p_m$ defined by:

$$\sum_{j=1}^{m} p_j = 1\ , \quad p_j \geqq 0 \ \text{ for } j = 1, ..., m$$
$$p_j \geqq p_g \ \text{ for all couples } (j, g) \text{ comparable in the given partial order.} \tag{4.12}$$

* An intermediate level could be that of a transitive outranking relation (see paragraph 3.2).

Then, they define a best choice as an action a of A such that:

$$m(a) = \underset{(p_1, \ldots, p_m) \in D_1}{\text{minimum}} \sum_{j=1}^{m} p_j \cdot u_j(a) \tag{4.13}$$

be a maximum on A. It suffices to solve for each a in A the linear program (4.12) (4.13) to obtain those for which $m(a)$ is maximum. Sentis, Fourgeaud and Lenclud develop in [61] a particular algorithm to solve this special type of linear program.

I suggest extending this approach in the following way. Let a^* be an action which maximizes $m(a)$ on A, and denote by $p_j(a)$ the subjective probability values obtained in the computation of $m(a)$. For each a in A when $r = |A|$ is small, and only for those giving highest values to $m(a)$ when r is great, examine the m-tuple of probabilities $p_1(a),\ldots,p_m(a)$ to be sure it is really reasonable. If it is, for any a examined, then choose a^*. If it does not seem realistic for some a, introduce new constraints of the following type:

$$\begin{aligned} &p_j - p_g \geqq a_{jg}\,, \qquad p_j/p_g \geqq b_{jg}\,, \\ &p_j + p_g \geqq p_f \quad \text{or} \quad p_j + p_g \leqq p_f\,, \end{aligned} \tag{4.14}$$

*

so that each impossible m-tuple be eliminated from the new domain D_2 defined by (4.12) and (4.14). After a new computation of $m(a)$'s (D_1 being changed into D_2) in (4.13) a new examination of the $p_j(a)$'s obtained must be made until all the m-tuples entering in the definition of the $m(a)$'s seem reasonable; then the maximum criterion appears perfectly legitimate to decide on the best choice, regarding the remaining uncertainty on the p_j's. By such a procedure, we are systematically led to a maximum reduction of uncertainty as far as it influences the decision.

* See what is said at formula (3.5) for a better interpretation of this type of constraint.

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