IGD$^+$-EMOA: A Multi-Objective Evolutionary Algorithm based on IGD$^+$

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Abstract*—In recent years, the design of selection mechanisms based on performance indicators has become a very popular trend in the development of new Multi-Objective Evolutionary Algorithms (MOEAs). The main motivation has been the well-known limitations of Pareto-based MOEAs when dealing with problems having four or more objectives (the so-called *many-objective problems*). The most commonly adopted indicator has been the hypervolume, mainly because of its nice mathematical properties (e.g., it is the only unary indicator which is known to be Pareto compliant). However, the hypervolume has a well-known disadvantage: its exact computation is very costly in high dimensionality, making it prohibitive for many-objective problems (this cost normally becomes unaffordable for problems with more than 5 objectives). Recently, a variation of the well-known inverse generational distance (IGD) was introduced. This indicator, which is called IGD$^+$ was shown to be weakly Pareto compliant, and presents some evident advantages with respect to the original IGD. Here, we propose an indicator-based MOEA, which adopts IGD$^+$. The proposed approach adopts a novel technique for building the reference set, which is used to assess the quality of the solutions obtained during the search. Our preliminary results indicate that our proposed approach is able to solve many-objective problems in an effective and efficient manner, being able to obtain solutions of a similar quality to those obtained by SMS-EMOA and MOEA/D, but at a much lower computational cost than required by the computation of exact hypervolume contributions (as adopted in SMS-EMOA).

I. Introduction

A wide variety of real-world problems have several (often conflicting) objectives which need to be optimized at the same time. They are generically called multiobjective optimization problems (MOPs) and their solution involves finding the best possible trade-offs among all the objectives. This set of trade-offs, when defined in decision variable space, is known as the *Pareto optimal set*. The image of the Pareto optimal set is called the Pareto front (PF).

Multi-Objective Evolutionary Algorithms (MOEAs) have been developed during the last 30 years, from which the last 15 have had a very intense activity [1], [2]. For several years, MOEAs adopted selection mechanisms based on Pareto optimality. However, recent studies have shown that Pareto-based multi-objective evolutionary algorithms do not perform properly when dealing with problems having more than three objectives (the so-called *many-objective optimization* problems) [3]. For this reason, some researchers have investigated the development of new selection schemes. One of the current research trends in this area is to optimize a quality assessment indicator that provides a good ordering among sets that represent Pareto approximations. A number of performance indicators have been proposed, from which the hypervolume is, with no doubt, the most popular so far, mainly because of its nice mathematical properties (it's the only unary indicator which is known to be Pareto compliant [4], [5], [6]). However, the main drawback of hypervolume-based MOEAs is the high computational cost associated with the computation of the exact hypervolume contributions, which becomes unaffordable in many-objective optimization problems.

A possible way to deal with this limitation is to adopt a different indicator to select solutions in a MOEA. Here, we propose a selection scheme based on the inverted generational distance (IGD) indicator, which was been recently modified by Ishibuchi [7], [8] to make it weakly Pareto compliant. This new version, called IGD$^+$ has a very low computational cost, even in high dimensional problems. However, its main drawback is that it requires a reference set to compute the indicator value. Here, we propose a technique to construct such a reference set and we show that the resulting MOEA has a competitive performance with respect to two state-of-the-art MOEAs (SMS-EMOA and MOEA/D), even when dealing with problems having a high number of objectives, while keeping a very affordable computational cost.

The remainder of this paper is organized as follows. Section II provides some basic concepts related to multi-objective optimization. Section III shows the most relevant previous related work. Our proposed approach is described in Section IV. The experimental results are presented in Section V-D, including the methodology and a short discussion of our main findings. Finally, conclusions and some possible paths for future research are provided in Section VI.

II. Basic Concepts

We are interested in solving problems of the type:

$$\text{minimize} \quad \vec{f}(\vec{x}) := [f_1(\vec{x}), f_2(\vec{x}), \ldots, f_m(\vec{x})]^T \qquad (1)$$

subject to:

$$g_i(\vec{x}) \leq 0, \quad i = 1, 2, \ldots, p \qquad (2)$$

$$h_j(\vec{x}) = 0, \quad j = 1, 2, \ldots, q \tag{3}$$

where $\vec{x} = [x_1, x_2, \ldots, x_n]$ is the vector of decision variables, $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$ are the objective functions and $g_i, h_j : \mathbb{R}^n \to \mathbb{R}$, $i = 0, \ldots, p$, $j = 1, \ldots, q$ are the constraint functions of the problem. Next, we introduce some definitions that will be used in the paper.

Definition 1: Let $\vec{x}, \vec{y} \in \mathbb{R}^m$, we say that $\vec{x}$ "dominates" $\vec{y}$ (denoted by $\vec{x} \prec \vec{y}$), if and only if: i) $x_i \leq y_i$ for all $i \in \{1, \ldots, m\}$ and ii) $x_j < y_j$ for at least one $j \in \{1, \ldots, m\}$.

Definition 2: We say that a vector of decision variables $\vec{x} \in \mathcal{X} \subset \mathbb{R}^n$ is "nondominated" with respect to $\mathcal{X}$, if there does not exist another $\vec{x}' \in \mathcal{X}$ such that $\vec{f}(\vec{x}') \prec \vec{f}(\vec{x})$.

Definition 3: We say that a vector of decision variables $\vec{x} \in \mathcal{F} \subset \mathbb{R}^n$ (where $\mathcal{F}$ is the feasible region) is "Pareto-optimal" if it is nondominated with respect to $\mathcal{F}$.

Definition 4: The Pareto Optimal Set $\mathcal{P}^*$ is defined as:
$$\mathcal{P}^* = \{\vec{x} \in \mathcal{F} | \vec{x} \text{ is Pareto-optimal}\}$$
Definition 5: The "Pareto Front" $\mathcal{PF}^*$ is defined as follows:
$$\mathcal{PF}^* = \{\vec{f}(\vec{x}) \in \mathbb{R}^m | \vec{x} \in \mathcal{P}^*\}$$

III. Related Work

Previously, we said that the performance indicator which has been most commonly used for the selection mechanism of a MOEA, is the hypervolume [4], [9]. This indicator encapsulates in a single unary value a measure of the spread of the solutions along the Pareto front, as well as the distance of the approximation set to the true Pareto optimal front. One of the best approaches which use a hypervolume-based selection mechanism is the *S Metric Selection-Evolutionary Multi-Objective Optimization Algorithm* (SMS-EMOA) [10]. However, the high cost of the hypervolume limits the practical use of SMS-EMOA, making it unaffordable for problems having more than 5 objectives. The high computational cost associated with the computation of exact hypervolume contributions has motivated the development of alternative approaches. One of them is to estimate the ranking of a set of individuals that would be induced by the hypervolume indicator. This approach avoids computing the exact hypervolume value and it is called *Hypervolume Estimation algorithm for multi-objective optimization* (HypE)[11]. Although this is a very promising approach, comparative studies have shown that the performance of HypE is not competitive with respect to the use of other indicator-based MOEAs (see for example [12]).

Recently, a new performance indicator, called Δ_p, was proposed in [13]. Δ_p is considered as an "averaged Hausdorff distance" between the approximate set and the true Pareto Front. This indicator is based on two well-known performance indicators: Generational Distance (GD) [13] and Inverted Generational Distance (IGD) [14]. In spite of the fact that Δ_p is a pseudo-metric which simultaneously evaluates proximity to the Pareto front and spread of solutions along it, it is not Pareto compliant. Currently, there are already MOEAs whose selection mechanism is based on Δ_p. For example, Δ_p-EMOA [15], which is inspired on SMS-EMOA and incorporates a novel mechanism for building the reference set that is based on linearizing the nondominated (piecewise linear) front of the current population. This approach is, however, designed to solve only bi-objective problems. An extension of this approach was introduced in [16] for solving three-objective problems, but its generalization to any number of objectives is not trivial. Another approach based on this indicator is the *Reference Indicator-Based Evolutionary Multi-Objective Algorithm* (RIB-EMOA) [17], which builds a reference set by using a family of curves and incorporates a selection mechanism based on the exclusive contribution of a solution. This algorithm can solve many-objective optimization problems. However, its algorithm for approximating the reference set is not able to build a generalized curve. Finally, there is another approach based on this indicator, called DDE, which uses differential evolution as its search engine [18]. Although promising for many-objective optimization, this approach has some limitations related to the use of Δ_p (e.g., it doesn't work when dealing with multi-frontal problems).

More recently, the use of the $R2$ indicator [6] has also given rise to several MOEAs (see for example [12], [19], [20]). Although effective and suitable for many-objective optimization, these approaches require the generation of a set of weights, analogously to decomposition-based MOEAs, such as MOEA/D.

IV. Our Proposed Approach

Before describing our proposed approach, it is important to provide first more details about IGD$^+$. According to [7], the IGD$^+$ indicator can be viewed as follows:

$$IGD^+(\mathcal{A}, \mathcal{Z}) = \frac{1}{|\mathcal{Z}|} \left(\sum_{j=1}^{|\mathcal{Z}|} d_j^+(\vec{z}, \vec{a})^p \right)^{1/p} \tag{4}$$

where $\vec{a} \in \mathcal{A} \subset \mathbb{R}^m$, $\vec{z} \in \mathcal{Z} \subset \mathbb{R}^m$, A is the Pareto set approximation and Z is the reference set. $d^+(\vec{a}, \vec{z})$ is defined as:

$$d^+(\vec{z}, \vec{a}) = \sqrt{(\max\{a_1 - z_i, 0\})^2, \ldots, (\max\{a_m - z_m, 0\})^2}. \tag{5}$$

Therefore, a low IGD^+ value means that the set $\mathcal{A}$ has a better approximation to the real $\mathcal{PF}$ if we consider the reference set as $\mathcal{PF}$.

A. General Framework

Our approach starts with a population $\mathcal{P}_t$ which contains N randomly generated individuals. A new offspring is created by choosing two different parents from $\mathcal{P}$. The parents are recombined using Simulated Binary Crossover (SBX) and the resulting offspring are mutated (in this case, using Polynomial-Based Mutation [21]). This process is repeated until having λ offspring. The second step is to combine the parents and the offspring population to form the so-called Q set. The new population at generation $t+1$ is generated using an IGD^+-based selection mechanism. Next, we will provide more details of our proposed approach.

B. Selection Mechanism

Since we intend to use IGD^+ in the selection mechanism of a MOEA, we propose to transform this selection mechanism into a linear assignment problem (LAP), which is solved using Munkres' assignment algorithm [22]. This algorithm can obtain the best assignment in $\mathcal{O}(n^3)$, where n is the number of elements of the problem. Formally, the LAP can be expressed as:

Given two sets, $A = \{a_1, \ldots, a_n\}$ and $T = t_1, \ldots, t_n$ with the same cardinality, and a cost function $C : A \times T \to \mathbb{R}$ and having $\Phi : A \to T$ as the set of all bijections between A and T. So, the LAP can be formulated as follows:

$$\min_{\phi \in \Phi} \sum_{a \in A} C(a, \Phi(a)) \tag{6}$$

Normally, the cost is also presented as a squared matrix C, where each element $Ci, j = C(a_i, t_j)$ represents the relationship between a_i and t_j.

A linear assignment problem can be created in terms of a MOP, by using the m-dimensional objective vectors which represent individuals from the population and reference set. So, a cost matrix is created using the modified distance d^+ between each element in the reference set and all objective vectors in the population. This transformation aims to find the best relationship between them. As evidenced in [23], the solution of this LAP allows convergence to the true Pareto front and, at the same time, produces a good distribution of solutions along the Pareto front.

We need to normalize the objective vectors of the current population, in order to handle objectives having different units. This normalization can be expressed as:

$$f'_i = \frac{f_i}{u_i} \tag{7}$$

where $\vec{u} \in \mathcal{R}^m$ and its i^{th}-element is defined as $u_i = \max\limits_{j=1,\ldots,\mu+\lambda} f_i(\vec{x}_j)$, $i = 1, \ldots, m$. The second step is to compute the C cost matrix and we can then express each element of the C cost matrix as follows:

$$C_{i,j} = d^+(a_i, z_j), \quad i = 1, \ldots, n, \quad j = 1, \ldots, n. \tag{8}$$

The solution to our assignment problem is found by identifying the combination of values in C resulting in the smallest sum. This solution corresponds to the best relationship of the current points of the population with respect to a reference set.

C. Approximating the Reference Set

In most multi-objective optimization problems, the geometrical shape of the true PF is unknown. However, we can approximate certain types of PFs (i.e., at least those having a smooth convex or concave surface) using superspheres. A γ-supersphere is a type of curve and it is defined as follows:

$$\{(y_1, \ldots, y_m) \in \mathbb{R}^m_+ | \quad y_1^\gamma + \cdots + y_m^\gamma = 1\} \tag{9}$$

where $\gamma \in \mathbb{R}_+$ is an arbitrary and fixed value. We only consider the "positive" parts of the γ-superspheres. According to [24], we can view the positive parts of the γ-superspheres as concave if $\gamma > 1$ or as convex if $0 < \gamma < 1$.

Clearly, we can see that a set of weight vectors satisfies equation (9) when $\gamma = 1$, since a weight vector is defined as:

Definition 6: Let $\vec{w} = [w_1, \ldots, w_m] \in \mathcal{R}^m$. We say that $\vec{w}$ is a weight vector if $\sum_{j=1}^m w_j = 1$ and $w_j \geq 0$.

In order to build the reference set, we assume that we have a set of weight vectors which is used to construct the reference set. We need to find the γ-value which will be used to transform the weights set into the reference set.

Clearly, in order to find the γ-value, equation (9) would be a root-finding problem and we can say that the γ-value needs to satisfy:

$$y_1^\gamma + \cdots + y_m^\gamma - 1 = 0 \tag{10}$$

For solving equation (10), we propose to use Newton's method for approximating the γ-value. Now, we can see that the next approximation to the root is defined as:

$$\gamma_{k+1} = \gamma_k - \frac{(\sum_{j=1}^m y_j^{\gamma_k}) - 1}{\sum_{j=1}^m y_j^{\gamma_k} \log(y_j)} \tag{11}$$

Then, the computation of the reference set is described according to the following description.

Let $\mathcal{Q}$ be the current set which was created combining the parent and offspring population. Thus, the reference set is created by Algorithm 1.

In the first part of the algorithm, we find the non-dominated points which will be used as a reference for building the curve (these points establish the non-dominated region). After that, we search the best relationship between each weighted vector $\vec{w}$ and the non-dominated points. For this reason, we calculate the perpendicular distance between both sets. In order to construct the reference surface, we project the nearest non-dominated point to a specific weight vector $\vec{w}$. Once this is done, we can search the γ-value using Newton's method, which is described by equation (11). Finally, the reference point is computed using the γ-value. We can see that this process is repeated for all weight vectors of set $\mathcal{Z}$.

It is worth noting that this approach uses a predefined set of weights in order to ensure diversity. We adopted Das and Dennis' approach which places points on a $(m-1)$-dimensional hyperplane [25]. The total number of vectors is represented by the combinatorial number C_{m-1}^{H+m-1}, where H is the number of divisions of the objective space.

Input: A current set $\mathcal{Q} \subset \mathbb{R}^m$ and a set of weighted vectors $W \subset \mathbb{R}^m$, where m is the number of objectives

Output: The reference set $\mathcal{Z}$ which is the best approximation of the set Q

Find the nondominated points from $\mathcal{Q}$ and save to $\mathcal{Q}'$
for each $\vec{p} \in \mathcal{Q}'$ **do**
  for each $\vec{w} \in \mathcal{W}$ **do**
    Compute $d^{\perp}(\vec{p}, \vec{w}) = \| \vec{p} - \vec{w}^T \vec{s}\vec{w} / \| \vec{w} \|^2 \|$
  end for
  Assign $r(w) = \underset{\vec{p} \in \mathcal{Q}'}{\operatorname{argmin}} \quad d^{\perp}(\vec{p}, \vec{w})$
end for
$j \leftarrow 0$
for each $\vec{w} \in \mathcal{W}$ **do**
  $stepsize \leftarrow \vec{p}_{r(\vec{w})} \cdot \vec{w} / \| \vec{w} \|$
  $\vec{y} \leftarrow stepsize * \vec{w}$
  Approximate the γ value using equation (10)
  Compute supersphere point as $z_{j,k} \leftarrow w_{j,k}^{\gamma}$ for all $j = 1, \ldots, m$
  $j \leftarrow j + 1$
end for

Algorithm 1: Computation of the reference set which is based on supersphere curves

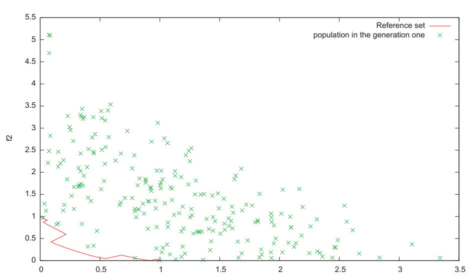

Fig. 1. Approximation of the reference set for DTLZ1

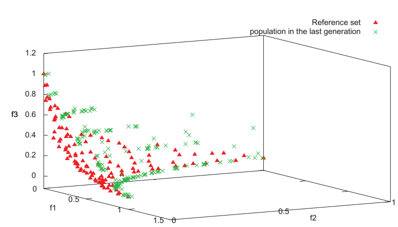

Fig. 2. Approximation of the reference set for WFG2.

Problem	Reference points
DTLZ1	$(1, 1, 1, \ldots, 1)$
DTLZ2-6	$(2, 2, 2, \ldots, 2)$

TABLE I
REFERENCE POINTS USED FOR THE HYPERVOLUME INDICATOR.

V. EXPERIMENTAL RESULTS

We compare the performance of our proposed algorithm with respect to that of two state-of-the-art MOEAs. The first is the *S Metric Selection-Evolutionary Multiobjective Optimization Algorithm* (SMS-EMOA) [10]. SMS-EMOA is a steady state evolutionary algorithm in which each newly created solution is ranked and a solution is removed from the worst ranked front in order to keep constant the population size. The solution that contributes the least to the hypervolume of the worst ranked front is then discarded (see [10] for details). We use here a version that incorporates the algorithm proposed in [11] for estimating the hypervolume using Monte Carlo sampling, instead of the exact hypervolume calculations adopted in the original implementation of SMS-EMOA. The second approach adopted for our comparative study is the multi-objective evolutionary algorithm based on decomposition (MOEA/D) [26], which transforms a multi-objective problem into several single-objective optimization problems which are simultaneously optimized.

A. Test problems

For our comparative study, we adopted two benchmarks: the Deb-Thiele-Laumanns-Zitzler (DTLZ) test suite [27] and the Walking-Fish-Group (WFG) test suite [28]. These problems include aspects such as separability and multifrontality which make them more difficult to solve.

B. Methodology

For our comparative study, we decided to adopt the hypervolume indicator, which assesses both convergence and maximum spread along the Pareto front. Mathematically, if Λ denotes the Lebesgue measure, the hypervolumen can be described as follows:

$$I_H(\mathcal{A}, \vec{y_{ref}}) = \Lambda \left(\bigcup_{\vec{y} \in \mathcal{A}} \{\vec{x} | \quad \vec{y} \prec \vec{x} \prec \vec{y_{ref}}\} \right) \tag{12}$$

where $\mathcal{A}$ is the approximation of the true Pareto front and $\vec{y_{ref}} \in \mathbb{R}^k$ denotes de reference point. To compute I_H, we used the reference points shown in Table I.

Additionally, we also compared the running time of each MOEA, which was measured in seconds.

C. Parameterization

In the DTLZ test suite, the total number of decision variables is given by $n = m + k - 1$, where m is the number of objectives and k was set to 5 for DTLZ1 and to 10 for DTLZ2-6. The number of decision variables in WFG was set to 24, and the position-related parameter was set to $m - 1$.

Objectives	H	Population size
2	119	120
3	14	120
4	7	120
5	5	126
6	4	126
7	3	84
8	3	120

TABLE II
PARAMETERIZATION VALUES

The parameters of each MOEA used in our study were chosen in such a way that we could do a fair comparison among them. The distribution indexes for the SBX and polynomial-based mutation operators [21], used by our approach and SMS-EMOA, were set as: $\eta_c = 20$ and $\eta_m = 20$, respectively. The crossover probability was set to $p_c = 0.9$ and the mutation probability was set to $p_m = 1/L$, where L is the number of decision variables. Otherwise, the number of samples was set to 100,000. The total number of function evaluations was set in such a way that it did not exceed 60,000.

In MOEA/D and our proposed approach, the number of weight vectors was set to the same value as the population size. The population size N is dependent on H which specifies the number of divisions in objective space. H was set in such a way that N took a value not greater than 130. MOEA/D used the Tchebycheff approach with a neighborhood size of 20. The main characteristics of the hardware used for the experiments are the following: An Intel Core i7-3930k CPU running at 2.30 GHz, with 8GB of RAM.

D. Discussion of Results

Table III provides the average hypervolume over the 30 independent executions of each compared MOEA for each instance of the DTLZ test suite. The best results are presented in **boldface**. We used Wilcoxon's rank sum for the statistical assessment of our results. It is clear that the winner in this experimental study is our proposed approach, since it was able to outperform SMS-EMOA-HYPE in all problems. We can observe that the null hypothesis can be rejected in all cases (the medians of two results are distinct), which means that the differences obtained are statistically significant.

Table IV shows the comparison of results with respect to MOEA/D. As can be observed, our approach was able to outperform MOEA/D in twenty-four cases and in a few more, both approaches obtained similar results. The null hypothesis cannot be rejected in only three cases (DTLZ2 and DTLZ6 with 2, 6 and 7 dimensions, respectively). In the other cases, the differences obtained are statistically significant. For DTLZ5, MOEA/D performs better than our proposed approach in all the instances of this problem. The reason is probably that the true Pareto front of this problem is linear, which makes the approximations produced by our approach to converge to a single region.

Table V indicates that MOEA/D has the lowest running times. However our proposed approach was able to solve the problems in a reasonably low running time (if we consider the running time of SMS-EMOA, which would be significantly higher if exact hypervolume contributions had been computed). Overall, our proposed approach produced results that are very competitive, while requiring a reasonably low computational cost.

Figures 3, 4, 5 and 6 present a graphical representation of the approximations of the Pareto front obtained by our proposed approach in some of the WFG test problems adopted with 24 variables and 3 objectives. These plots correspond to the mean hypervolume value from 30 independent executions. It is interesting to note that our proposed approach is able to properly converge to the true Pareto front of WFG2 which is disconnected. This confirms that the use of a selection mechanism based on IGD$^+$ is an effective way to solve many-objective problems.

VI. CONCLUSIONS AND FUTURE WORK

We have proposed a new indicator-based approach for solving many-objective problems. The core idea of our proposed algorithm is to adopt the IGD$^+$ performance indicator in the selection mechanism of a MOEA. Our proposal includes a new method for constructing the reference set which is based on Newton's Method using superspheres. Our preliminary results indicate that our proposed approach is very competitive with respect to two state-of-the-art MOEAs (SMS-EMOA and MOEA/D), which require a relatively low computational cost (lower than that required by SMS-EMOA).

As part of our future work, we are interested in studying the sensitivity of our proposed approach to the reference set. We are also interested in exploring alternative techniques for improving the construction of the reference sets, such that our approach can properly deal with linear Pareto fronts. Finally, we are interested in incorporating a local search mechanism into our proposed MOEA.

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Objectives	IGD$^+$-EMOA (I_H)	SMS-EMOA (I_H)	$P(H)$
m	DTLZ1		
2	0.873783426	0.603475637	0.0000(1)
3	**0.97402027**	0.630213232	0.0000(1)
4	**0.994388397**	0.661586784	0.0000(1)
5	**0.9919105**	0.768368948	0.0000(1)
6	**0.892359232**	0.776399218	0.0000(1)
7	**0.854507595**	0.738417486	0.0000(1)
8	**0.861695367**	0.814606014	0.0000(1)
DTLZ2			
2	3.210821317	1.863197179	0.0000(1)
3	**7.421812488**	3.375262182	0.0000(1)
4	**15.56741135**	6.747488035	0.0000(1)
5	**31.66763799**	14.90159836	0.0000(1)
6	**58.07564274**	33.83189984	0.0000(1)
7	**116.7077942**	65.20591399	0.0000(1)
8	201.1809779	150.6662452	0.0000(1)
DTLZ3			
2	**3.204155341**	1.88237071	0.0000(1)
3	**7.355071834**	6.411731511	0.0000(1)
4	**15.53694896**	14.12060072	0.0000(1)
5	**31.63537503**	30.64233556	0.0000(1)
6	56.53710167	**59.64630155**	0.0000(1)
7	**96.20407137**	60.7350761	0.0000(1)
8	**198.890958**	129.4343786	0.0000(1)
DTLZ4			
2	**3.210795477**	2.083725111	0.0000(1)
3	**7.082647895**	3.866538974	0.0000(1)
4	**15.20291716**	7.715636261	0.0000(1)
5	**29.27224775**	17.69125542	0.0000(1)
6	**59.24047887**	39.52867707	0.0000(1)
7	**113.5313178**	74.07283972	0.0000(1)
8	**245.9811967**	168.8209989	0.0000(1)
DTLZ5			
2	**3.210822115**	1.859887057	0.0000(1)
3	**4.042831297**	3.575537122	0.0000(1)
4	**8.003414255**	6.12118542	0.0000(1)
5	**16.00248339**	11.30170488	0.0000(1)
6	**31.99996981**	21.59984771	0.0000(1)
7	**63.99998659**	39.53980236	0.0000(1)
8	**127.999845**	80.08300234	0.0000(1)
DTLZ6			
2	**3.106126387**	1.714714707	0.0000(1)
3	**5.66835877**	3.019997564	0.0000(1)
4	**7.437133923**	5.724252402	0.0000(1)
5	**14.82343849**	10.70214298	0.0000(1)
6	**29.76776111**	20.11681978	0.0000(1)
7	**58.53573762**	36.44267101	0.0000(1)
8	**117.596708**	75.84189126	0.0000(1)

TABLE III

RESULTS OBTAINED IN THE DTLZ TEST PROBLEMS BY SMS-EMOA(HYPE) AND OUR PROPOSED IGD$^+$-EMOA, USING THE HYPERVOLUME INDICATOR (I_H). THE THIRD COLUMN SHOWS THE RESULTS OF THE STATISTICAL ANALYSIS APPLIED TO OUR EXPERIMENTS USING WILCOXON'S RANK SUM, WHERE P IS THE PROBABILITY OF OBSERVING THE GIVEN RESULT (THE NULL HYPOTESHIS IS TRUE). IF THE P-VALUE IS SMALL, THE DATA INDICATE THAT THE NULL HYPOTHESIS CAN BE REJECTED AT THE 5% LEVEL AND WE CAN CONCLUDE THAT THE TWO RESULTS ARE DISTINCT ($H = 1$) AND THEIR DIFFERENCE IS STATISTICALLY SIGNIFICANT.

Objectives	IGD$^+$-EMOA (I_H)	MOEA/D (I_H)	$P(H)$
m	DTLZ1		
2	0.873783426	**0.873862461**	0.0000(1)
3	**0.97402027**	0.968914097	0.0000(1)
4	**0.994388397**	0.970400938	0.0000(1)
5	**0.9919105**	0.724927498	0.0000(1)
6	**0.892359232**	0.762830249	0.0001(1)
7	**0.854507595**	0.656643222	0.0000(1)
8	**0.861695367**	0.402984134	0.0000(1)
DTLZ2			
2	3.210821317	**3.210869248**	0.0000(1)
3	**7.421812488**	7.382922569	0.0000(1)
4	**15.56741135**	13.31884553	0.0000(1)
5	**31.66763799**	27.15582568	0.0000(1)
6	**58.07564274**	53.83549288	0.0000(1)
7	**116.7077942**	115.8005853	0.2890(0)
8	201.1809779	**216.0379868**	0.0000(1)
DTLZ3			
2	3.204155341	**3.206666982**	0.0009(1)
3	7.355071834	**7.374166479**	0.0215(1)
4	**15.53694896**	12.91659236	0.0000(1)
5	**31.63537503**	24.2992914	0.0000(1)
6	**56.53710167**	48.35142688	0.0000(1)
7	96.20407137	**100.9289111**	0.0000(1)
8	198.890958	**224.3638975**	0.0000(1)
DTLZ4			
2	**3.210795477**	2.395550932	0.0001(1)
3	**7.082647895**	6.233954506	0.0003(1)
4	**15.20291716**	11.7039561	0.0000(1)
5	**29.27224775**	22.23443593	0.0000(1)
6	**59.24047887**	47.35243703	0.0000(1)
7	**113.5313178**	93.66821328	0.0000(1)
8	**245.9811967**	184.5468339	0.0000(1)
DTLZ5			
2	3.210822115	**3.210869361**	0.0000(1)
3	4.042831297	**6.091452678**	0.0000(1)
4	8.003414255	**10.80302487**	0.0000(1)
5	16.00248339	**16.15695638**	0.0037(1)
6	31.99996981	**37.8234936**	0.0000(1)
7	63.99998659	**78.10923993**	0.0000(1)
8	127.999845	**150.5444428**	0.0000(1)
DTLZ6			
2	**3.106126387**	3.056665166	0.0000(1)
3	5.66835877	**5.801128621**	0.5792(0)
4	7.437133923	**8.877262691**	0.0000(1)
5	**14.82343849**	11.95437872	0.0000(1)
6	**29.76776111**	28.53823221	0.5793(0)
7	58.53573762	**71.77293638**	0.0000(1)
8	117.596708	**131.4029039**	0.0000(1)

TABLE IV

RESULTS OBTAINED IN THE DTLZ TEST PROBLEMS BY MOEA/D AND OUR PROPOSED IGD$^+$-EMOA, USING THE HYPERVOLUME INDICATOR (I_H). THE THIRD COLUMN SHOWS THE RESULTS OF THE STATISTICAL ANALYSIS APPLIED TO OUR EXPERIMENTS USING WILCOXON'S RANK SUM, WHERE P IS THE PROBABILITY OF OBSERVING THE GIVEN RESULT (THE NULL HYPOTHESIS IS TRUE). IF THE P-VALUE IS LARGE, THE DATA DO NOT GIVE ANY REASON TO REJECT THE NULL HYPOTHESIS AND WE CAN CONCLUDE THAT THE TWO RESULTS ARE THE SAME ($H = 0$). OTHERWISE, IF P-VALUE IS SMALL, THE NULL HYPOTHESIS CAN BE REJECTED AT THE 5% LEVEL AND THE TWO RESULTS ARE DISTINCT ($H = 1$) AND THEIR DIFFERENCE IS STATISTICALLY SIGNIFICANT.

Objectives	IGD$^+$-EMOA	MOEA/D	SMS-EMOA
m		DTLZ1	
2	5.638792	**0.451672**	929.482702
3	10.521783	**0.529422**	3.958571
4	13.902771	**0.245426**	2111.873245
5	28.46273	**0.435324**	2014.102227
6	32.162026	**0.566348**	2082.745213
7	19.710173	**0.27731**	3510.322044
8	46.412152	**0.067746**	5038.534303
		DTLZ2	
2	29.264559	**0.618055**	1259.578953
3	42.169365	**0.694392**	2200.86988
4	54.027164	**0.32018**	3689.959608
5	90.588769	**0.575118**	4539.503577
6	107.13428	**0.86347**	5563.185747
7	12.719386	**0.337427**	5290.054592
8	27.656435	**0.085446**	6934.750096
		DTLZ3	
2	8.252379	**0.573409**	800.213934
3	49.478735	**0.662682**	999.843201
4	23.923694	**0.295604**	1185.619497
5	48.132021	**0.533946**	1052.808177
6	74.052142	**0.733004**	1505.031887
7	15.886005	**0.337056**	3180.209343
8	27.307364	**0.076317**	4069.487323
		DTLZ4	
2	7.620927	**0.49825**	1353.118432
3	10.11606	**0.698973**	2337.602668
4	13.871761	**0.265944**	3948.222166
5	22.35422	**0.55728**	4213.046424
6	25.576331	**0.78043**	5512.432487
7	15.039888	**0.315422**	5241.15327
8	31.035888	**0.070369**	6897.864537
		DTLZ5	
2	7.983771	**0.600385**	1238.279526
3	26.447281	**0.714718**	1977.824246
4	19.755846	**0.242651**	4047.525445
5	22.538165	**0.315002**	3862.21531
6	21.331467	**0.334229**	6723.838925
7	72.28353	**0.347957**	6149.572591
8	23.338065	**0.397939**	8314.476698
		DTLZ6	
2	7.171413	**0.578733**	2007.65159
3	33.712275	**0.623824**	2526.082216
4	44.556186	**0.200934**	3039.004372
5	52.202535	**0.472714**	3307.124979
6	56.476568	**0.349478**	3985.982441
7	43.149346	**0.296292**	4645.460144
8	99.324617	**0.384934**	7241.438203

TABLE V
HERE, WE SHOW THE COMPUTATIONAL TIME (MEASURED IN SECONDS) REQUIRED BY EACH EXECUTION OF THE MOEAS COMPARED. ALL ALGORITHMS WERE COMPILED USING THE GNU C COMPILER AND THEY WERE EXECUTED ON THE SAME COMPUTER.

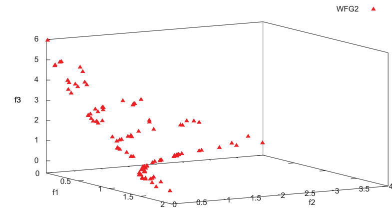

Fig. 3. Solutions obtained by IGD$^+$-EMOA for WFG2.

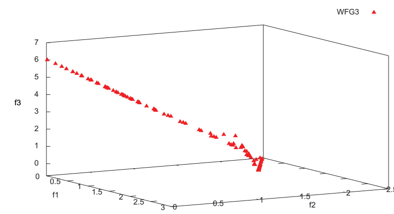

Fig. 4. Solutions obtained by IGD$^+$-EMOA for WFG3.

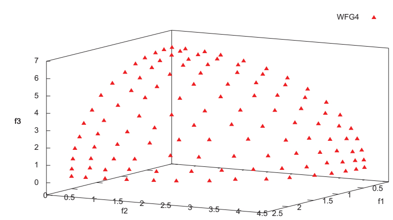

Fig. 5. Solutions obtained by IGD$^+$-EMOA for WFG4.

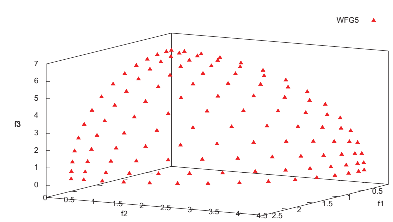

Fig. 6. Solutions obtained by IGD$^+$-EMOA for WFG5.

References

[1] C. A. Coello Coello, G. B. Lamont, and D. A. Van Veldhuizen, *Evolutionary Algorithms for Solving Multi-Objective Problems*, 2nd ed. New York: Springer, September 2007, iSBN 978-0-387-33254-3.

[2] K. Deb and D. Kalyanmoy, *Multi-Objective Optimization Using Evolutionary Algorithms*. New York, NY, USA: John Wiley & Sons, Inc., 2001.

[3] H. Ishibuchi, N. Tsukamoto, and Y. Nojima, "Evolutionary many-objective optimization: A short review," in *2008 Congress on Evolutionary Computation (CEC'2008)*. Hong Kong: IEEE Service Center, June 2008, pp. 2424–2431.

[4] E. Zitzler, L. Thiele, M. Laumanns, C. M. Fonseca, and V. G. da Fonseca, "Performance Assessment of Multiobjective Optimizers: An Analysis and Review," *IEEE Transactions on Evolutionary Computation*, vol. 7, no. 2, pp. 117–132, April 2003.

[5] K. Bringmann and T. Friedrich, "Approximating the Least Hypervolume Contributor: NP-Hard in General, But Fast in Practice," in *Evolutionary Multi-Criterion Optimization. 5th International Conference, EMO 2009*, M. Ehrgott, C. M. Fonseca, X. Gandibleux, J.-K. Hao, and M. Sevaux, Eds. Nantes, France: Springer. Lecture Notes in Computer Science Vol. 5467, April 2009, pp. 6–20.

[6] D. Brockhoff, T. Wagner, and H. Trautmann, "On the Properties of the $R2$ Indicator," in *2012 Genetic and Evolutionary Computation Conference (GECCO'2012)*. Philadelphia, USA: ACM Press, July 2012, pp. 465–472, iSBN: 978-1-4503-1177-9.

[7] H. Ishibuchi, H. Masuda, Y. Tanigaki, and Y. Nojima, "Modified Distance Calculation in Generational Distance and Inverted Generational Distance," in *Evolutionary Multi-Criterion Optimization, 8th International Conference, EMO 2015*, A. Gaspar-Cunha, C. H. Antunes, and C. Coello Coello, Eds. Guimarães, Portugal: Springer. Lecture Notes in Computer Science Vol. 9019, March 29 - April 1 2015, pp. 110–125.

[8] H. Ishibuchi, H. Masuda, and Y. Nojima, "A Study on Performance Evaluation Ability of a Modified Inverted Generational Distance Indicator," in *2015 Genetic and Evolutionary Computation Conference (GECCO 2015)*. Madrid, Spain: ACM Press, July 11-15 2015, pp. 695–702, iSBN 978-1-4503-3472-3.

[9] N. Beume, "S-Metric Calculation by Considering Dominated Hypervolume as Klee's Measure Problem," *Evolutionary Computation*, vol. 17, no. 4, pp. 477–492, Winter 2009.

[10] N. Beume, B. Naujoks, and M. Emmerich, "SMS-EMOA: Multiobjective selection based on dominated hypervolume," *European Journal of Operational Research*, vol. 181, no. 3, pp. 1653–1669, 16 September 2007.

[11] J. Bader and E. Zitzler, "HypE: An Algorithm for Fast Hypervolume-Based Many-Objective Optimization," *Evolutionary Computation*, vol. 19, no. 1, pp. 45–76, Spring, 2011.

[12] R. Hernández Gómez and C. A. Coello Coello, "MOMBI: A New Metaheuristic for Many-Objective Optimization Based on the $R2$ Indicator," in *2013 IEEE Congress on Evolutionary Computation (CEC'2013)*. Cancún, México: IEEE Press, 20-23 June 2013, pp. 2488–2495, iSBN 978-1-4799-0454-9.

[13] O. Schütze, X. Esquivel, A. Lara, and C. A. Coello Coello, "Using the Averaged Hausdorff Distance as a Performance Measure in Evolutionary Multiobjective Optimization," *IEEE Transactions on Evolutionary Computation*, vol. 16, no. 4, pp. 504–522, August 2012.

[14] C. A. Coello Coello and M. Reyes Sierra, "A Study of the Parallelization of a Coevolutionary Multi-Objective Evolutionary Algorithm," in *Proceedings of the Third Mexican International Conference on Artificial Intelligence (MICAI'2004)*, R. Monroy, G. Arroyo-Figueroa, L. E. Sucar, and H. Sossa, Eds. Springer Verlag. Lecture Notes in Artificial Intelligence Vol. 2972, April 2004, pp. 688–697.

[15] K. Gerstl, G. Rudolph, O. Schütze, and H. Trautmann, "Finding Evenly Spaced Fronts for Multiobjective Control via Averaging Hausdorff-Measure," in *The 2011 8th International Conference on Electrical Engineering, Computer Science and Automatic Control (CCE'2011)*. Mérida, Yucatán, México: IEEE Press, October 2011, pp. 975–980.

[16] H. Trautmann, G. Rudolph, C. Dominguez-Medina, and O. Schütze, "Finding Evenly Spaced Pareto Fronts for Three-Objective Optimization Problems," in *EVOLVE - A Bridge between Probability, Set Oriented Numerics, and Evolutionary Computation II*, O. Schütze, C. A. Coello Coello, A.-A. Tantar, E. Tantar, P. Bouvry, P. Del Moral, and P. Legrand, Eds. Berlin, Germany: Springer, Advances in Intelligent Systems and Computing Vol. 175, 2012, pp. 89–105, iSBN 978-3-642-31519-0.

[17] S. Zapotecas Martínez, V. A. Sosa Hernández, H. Aguirre, K. Tanaka, and C. A. Coello Coello, "Using a Family of Curves to Approximate the Pareto Front of a Multi-Objective Optimization Problem," in *Parallel Problem Solving from Nature - PPSN XIII, 13th International Conference*, T. Bartz-Beielstein, J. Branke, B. Filipič, and J. Smith, Eds. Ljubljana, Slovenia: Springer. Lecture Notes in Computer Science Vol. 8672, September 13-17 2014, pp. 682–691.

[18] C. A. Rodríguez Villalobos and C. A. Coello Coello, "A New Multi-Objective Evolutionary Algorithm Based on a Performance Assessment Indicator," in *2012 Genetic and Evolutionary Computation Conference (GECCO'2012)*. Philadelphia, USA: ACM Press, July 2012, pp. 505–512, iSBN: 978-1-4503-1177-9.

[19] A. Díaz-Manríquez, G. Toscano-Pulido, C. A. Coello Coello, and R. Landa-Becerra, "A Ranking Method Based on the $R2$ Indicator for Many-Objective Optimization," in *2013 IEEE Congress on Evolutionary Computation (CEC'2013)*. Cancún, México: IEEE Press, 20-23 June 2013, pp. 1523–1530, iSBN 978-1-4799-0454-9.

[20] D. ung H. Phan and J. Suzuki, "R2-IBEA: R2 Indicator Based Evolutionary Algorithm for Multiobjective Optimization," in *2013 IEEE Congress on Evolutionary Computation (CEC'2013)*. Cancún, México: IEEE Press, 20-23 June 2013, pp. 1836–1845, iSBN 978-1-4799-0454-9.

[21] K. Deb, A. Pratap, S. Agarwal, and T. Meyarivan, "A Fast and Elitist Multiobjective Genetic Algorithm: NSGA–II," *IEEE Transactions on Evolutionary Computation*, vol. 6, no. 2, pp. 182–197, April 2002.

[22] H. W. Kuhn and B. Yaw, "The hungarian method for the assignment problem," *Naval Res. Logist. Quart*, pp. 83–97, 1955.

[23] J. A. M. Berenguer and C. A. Coello Coello, "Evolutionary Many-Objective Optimization Based on Kuhn-Munkres' Algorithm," in *Evolutionary Multi-Criterion Optimization, 8th International Conference, EMO 2015*, A. Gaspar-Cunha, C. H. Antunes, and C. Coello Coello, Eds. Guimarães, Portugal: Springer. Lecture Notes in Computer Science Vol. 9019, March 29 - April 1 2015, pp. 3–17.

[24] M. T. M. Emmerich and A. H. Deutz, "Test problems based on lamé superspheres," in *Proceedings of the 4th International Conference on Evolutionary Multi-criterion Optimization*, ser. EMO'07. Berlin, Heidelberg: Springer-Verlag, 2007, pp. 922–936. [Online]. Available: http://dl.acm.org/citation.cfm?id=1762545.1762622

[25] I. Das and J. E. Dennis, "Normal-Boundary Intersection: A new method for generating the pareto surface in nonlinear multicriteria optimization problems," *SIAM J. on Optimization*, vol. 8, no. 3, pp. 631–657, Mar. 1998. [Online]. Available: http://dx.doi.org/10.1137/S1052623496307510

[26] Q. Zhang and H. Li, "MOEA/D: A Multiobjective Evolutionary Algorithm Based on Decomposition," *IEEE Transactions on Evolutionary Computation*, vol. 11, no. 6, pp. 712–731, December 2007.

[27] K. Deb, L. Thiele, M. Laumanns, and E. Zitzler, "Scalable Test Problems for Evolutionary Multiobjective Optimization," in *Evolutionary Multiobjective Optimization. Theoretical Advances and Applications*, A. Abraham, L. Jain, and R. Goldberg, Eds. USA: Springer, 2005, pp. 105–145.

[28] S. Huband, P. Hingston, L. Barone, and L. While, "A Review of Multiobjective Test Problems and a Scalable Test Problem Toolkit," *IEEE Transactions on Evolutionary Computation*, vol. 10, no. 5, pp. 477–506, October 2006.