Indicator-based Multi-objective Evolutionary Algorithms: A Comprehensive Survey

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For over 25 years, most multi-objective evolutionary algorithms (MOEAs) have adopted selection criteria based on Pareto dominance. However, the performance of Pareto-based MOEAs quickly degrades when solving multi-objective optimization problems (MOPs) having four or more objective functions (the so-called many-objective optimization problems), mainly because of the loss of selection pressure. Consequently, in recent years, MOEAs have been coupled with indicator-based selection mechanisms in furtherance of increasing the selection pressure so that they can properly solve many-objective optimization problems. Several research efforts have been conducted since 2003 regarding the design of the so-called indicator-based (IB) MOEAs. In this article, we present a comprehensive survey of IB-MOEAs for continuous search spaces since their origins up to the current state-of-the-art approaches. We propose a taxonomy that classifies IB-mechanisms into two main categories: (1) IB-Selection (which is divided into IB-Environmental Selection, IB-Density Estimation, and IB-Archiving) and (2) IB-Mating Selection. Each of these classes is discussed in detail in this article, emphasizing the advantages and drawbacks of the selection mechanisms. In the final part, we provide some possible paths for future research.

CCS Concepts: • **Theory of computation → Bio-inspired optimization**; • **Computing methodologies** → *Continuous space search*;

Additional Key Words and Phrases: Multi-objective optimization, quality indicators, indicator-based selection

ACM Reference format:
Jesús Guillermo Falcón-Cardona and Carlos A. Coello Coello. 2020. Indicator-based Multi-objective Evolutionary Algorithms: A Comprehensive Survey. *ACM Comput. Surv.* 53, 2, Article 29 (March 2020), 35 pages.
https://doi.org/10.1145/3376916

1 INTRODUCTION

Evolutionary multi-objective optimization (EMOO), originated in the mid-1980s, has been steadily growing since the late 1990s, focusing on the solution of problems that involve the simultaneous optimization of several, often conflicting, objective functions. Due to the conflict among the objectives, these multi-objective optimization problems (MOPs) do not have a single solution but a set of

The first author acknowledges support from CONACyT to pursue graduate studies in computer science at CINVESTAV-IPN. The second author gratefully acknowledges financial support from CONACyT Grant No. 2016-01-1920 (*Investigación en Fronteras de la Ciencia 2016*) and from a project from the 2018 SEP-Cinvestav Fund (Application No. 4).
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0360-0300/2020/03-ART29 $15.00
https://doi.org/10.1145/3376916

them that represent the best possible trade-offs among the objective functions. The solutions of an MOP conform the so-called *Pareto optimal set* (defined in decision variable space), and its image, in objective function space, is called *Pareto optimal front* ($\mathcal{PF}^*$). In the past 15 years, Multi-objective Evolutionary Algorithms (MOEAs) have become a popular choice for tackling complex MOPs [5, 22, 40, 109]. MOEAs are stochastic population-based metaheuristics that employ the principles of natural selection (i.e., the survival of the fittest individuals in a population) to drive a set of solutions toward the Pareto optimal front. Since it is, in general, not possible (as well as undesirable) to generate all the elements of the Pareto optimal set, MOEAs produce finite approximation sets of an MOP on a single algorithmic execution. Thus, the three main goals of an MOEA are: (1) to produce solutions that are as close as possible to $\mathcal{PF}^*$, (2) to generate evenly distributed solutions, and (3) to cover completely $\mathcal{PF}^*$.

For several years, most MOEAs have incorporated the concept of *Pareto dominance*[1] in their selection mechanisms [19]. Pareto-based MOEAs have shown a good performance when tackling MOPs with two and three objective functions [22, 115]. However, the selection pressure of Pareto-based MOEAs quickly dilutes when solving MOPs having four or more objective functions, i.e., the so-called many-objective optimization problems (MaOPs). This dilution is due to the rapid increase of the number of solutions preferred by the Pareto dominance relation, which eventually causes a Pareto-based selection mechanism to choose solutions at random [32]. In recent years, the design of methods that improve the performance of MOEAs on MaOPs has received much interest, since these problems are widespread in science and engineering applications. In general, the most popular methodologies to improve the performance of MOEAs on MaOPs are the following:

(1) *MOEAs using relaxed Pareto dominance relations:* In this case, the main idea is to employ alternative preference relations that relax the Pareto dominance relation [70]. Relaxed preference relations induce a finer grain order on the solutions belonging to MaOPs, which directly increases the selection pressure of MOEAs. Some examples of these preference relations are the following: the $(1-k)$-dominance relation proposed by Farina and Amato [32], the favour ranking proposed by Drechsler et al. [26], and the expansion relation that controls the dominance area of solutions [84].
(2) *Decomposition-based MOEAs:* This methodology aims to transform an MOP into multiple single-objective optimization problems (SOPs), using scalarizing functions such as the weighted Tchebycheff function. The distinctive feature of these MOEAs is that all the SOPs are solved in a single run, producing an entire approximation set. The most typical approach within this class is the MOEA based on Decomposition (MOEA/D) [94, 109].
(3) *Reference set-based MOEAs:* The Non-dominated Sorting Genetic Algorithm III (NSGA-III) [21] best represents this category. In this case, a reference set is constructed to guide the search process by measuring the quality of the population conforming it. According to Li et al. [61], the two main aspects related to this methodology are: (1) how to construct the reference set when no information about the Pareto optimal front is available, and (2) how to measure the quality of solutions using the reference set.
(4) *Indicator-based MOEAs:* These MOEAs employ quality indicators, which are functions that assess approximation sets, to define selection mechanisms. The underlying idea is to optimize the indicator value of the population throughout the evolutionary process. The S-Metric Selection Evolutionary Multi-objective Algorithm (SMS-EMOA) [5] is the most representative indicator-based MOEA (IB-MOEA). It employs the Hypervolume indicator

[1] A vector $\vec{u}$ Pareto dominates another vector $\vec{v}$ if the former is as good as the latter in every element and it is better in at least one of them.

(HV) [116] that measures the dominated volume of an approximation set, bounded by an anti-optimal point. SMS-EMOA imposes a total order among the solutions by calculating their contribution to the hypervolume indicator.

In this article, we focus our attention on IB-MOEAs whose popularity has increased since 2003 [5, 30, 40, 56, 58, 77, 114]. The backbone of IB-MOEAs is the use of quality indicators (QIs) as the basis of selection mechanisms. QIs are functions that assign a real value to one or more approximation sets, depending on certain quality aspects such as convergence and diversity of solutions. The origins of QIs can be traced back to the mid-1990 where some isolated efforts were undertaken to try to (numerically) assess the performance of MOEAs [29, 35, 60, 87]. However, the Ph.D. thesis of David Van Veldhuizen [98] can be considered as the cornerstone of QIs due to his comprehensive review of most of the QIs available at that time. In 2003, Zitzler et al. [117] provided the first theoretical analysis of QIs, using a mathematical framework to understand how QIs were related to a set of outperformance relations. In the past few years, Jiang et al. [55] and Liefooghe and Derbel [69] have conducted empirical studies aiming to determine the correlation between different QIs and their behavior when assessing a wide variety of Pareto front shapes. Additional reviews of QIs to evaluate the performance of MOEAs have been published by some researchers [28, 33, 37, 83, 89, 106, 107].

Since the late 1990s, several QIs have been proposed [67]. Van Veldhuizen [98] proposed different indicators, including the Generational Distance (GD) and Error Ratio (ER), among others. GD measures the average distance from the approximation set to a reference set, while ER reports the number of solutions of the approximation set that do not belong to $\mathcal{PF}^*$. Zitzler and Thiele [116] introduced the hypervolume indicator (HV) that rewards the convergence toward $\mathcal{PF}^*$ as well as the extent of solutions along the Pareto front. HV measures the size of the dominated space by an approximation set. Currently, HV is the only unary quality indicator that is known to be Pareto-compliant.[2] Hansen and Jaszkiewicz [38] proposed the R-family indicators ($R1$, $R2$, and $R3$) from a set of outperformance relations and some utility functions. Coello Coello and Cruz Cortés [18] proposed to measure the average Euclidean distances between the true Pareto front (or the reference set) and the approximation produced by an MOEA. Since this is exactly the opposite way in which GD operates, this indicator was called Inverted Generational Distance (IGD) and its use was reported for the first time in [20].[3] More recently, Ishibuchi et al. [48] proposed the Inverted Generational Distance plus (IGD$^+$) that determines the distance between an approximation set and a reference set, using a distance measure that adopts Pareto dominance. In spite of the plethora of QIs currently available, none of them can assess all the desired features of an approximation set, since each QI exhibits a specific preference, and, in this regard, Zitzler et al. claimed that it is necessary an infinite number of QI values to characterize the quality of an approximation set [117]. Hence, we should choose a QI depending on the type of conclusions we would like to draw.

IB-MOEAs transform the MOP into a single-objective optimization problem, i.e., the optimization of a given QI. To this end, IB-MOEAs solve at each iteration an indicator-based subset selection problem [4]. The origins of IB-MOEAs can be traced back to the work of Knowles and Corne [58] that proposed an HV-based external archive. By doing this, they maintained a convergent and well-distributed approximation to the Pareto front that maximized the HV value. However, the turning point in this area was the proposal of Zitzler et al. [113], called Indicator-based selection Evolutionary Algorithm (IBEA) where an individual's fitness value is calculated based on HV [111]

[2]A Pareto-compliant QI guarantees that one algorithm's indicator values are better than another in case the approximation sets of the former Pareto-dominates the other's.

[3]The original proposal of IGD was published in a journal but it appeared until 2005, whereas its first reported use was at a conference paper that was published in 2004.

or the ϵ^+ indicator [117]. It is worth emphasizing that over the years, HV has been widely used by MOEAs to improve their search properties due to its nice mathematical properties [3, 5, 17, 56, 110]. Nevertheless, the high computational cost involved in computing exact HV contributions when increasing the number of objectives, has limited the use of this QI. Consequently, other low-cost QIs have been proposed as selection mechanisms in spite of their theoretical limitations. Among these indicators, the most remarkable are $R2$ [12, 39], Δ_p [81, 85], the ϵ^+-indicator [113, 117] and IGD$^+$ [31, 48, 71]. It is worth emphasizing that the solutions found by an IB-MOEA, are strongly related to the preferences that the indicator expresses. Hence, by the No-Free Lunch Theorem [105], there is no IB-MOEA that can provide the best possible performance over all the possible classes of MOPs.

In spite of the numerous IB-MOEAs currently available, no comprehensive review of them has been published so far to the authors' best knowledge. Hence, this article aims to provide such an analysis of the currently available IB-MOEAs for continuous multi-objective optimization, using genetic algorithms as a search engine. Our review is based on a taxonomy (proposed here) that classifies approaches according to the usage that a QI has had within an MOEA. Furthermore, we also provide some possible future research directions in this area.

The remainder of this article is organized as follows. Section 2 introduces some basic concepts required to make of this a self-contained article. Section 3 briefly reviews some state-of-the-art QIs that have been commonly used by IB-MOEAs. Our proposed taxonomy and the description of IB-MOEAs, including a discussion of their main advantages and disadvantages, are presented in Section 4. Some potential future research trends related to IB-MOEAs are outlined in Section 5. Our conclusions are drawn in Section 6. The Supplementary Material is devoted to presenting a brief review of real-world applications tackled by IB-MOEAs.

2 BACKGROUND

According to Coello et al. [19], a multi-objective optimization problem[4] (MOP) is mathematically defined as

$$\min_{\vec{x}\in\mathbb{R}^n} \vec{F}(\vec{x}) := [f_1(\vec{x}), f_2(\vec{x}), \ldots, f_m(\vec{x})]^T, \tag{1}$$

subject to

$$g_i(\vec{x}) \le 0 \quad i = 1, 2, \ldots, q, \tag{2}$$

$$h_j(\vec{x}) = 0 \quad j = 1, 2, \ldots, p, \tag{3}$$

where $\vec{x} = (x_1, x_2, \ldots, x_n)^T$ is the n-dimensional vector of decision variables; $f_k : \mathbb{R}^n \to \mathbb{R}$, $k = 1, \ldots, m$ are the objective functions and $g_i, h_j : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, q, j = 1, \ldots, p$ are the constraint functions of the problem, which define that feasible region Ω.

Definition 1 (Pareto Dominance [19]). Given two vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$, $\vec{x}$ **Pareto dominates** $\vec{y}$ (denoted by $\vec{F}(\vec{x}) \prec \vec{F}(\vec{y})$) if $f_i(\vec{x}) \le f_i(\vec{y})$ for $i = 1, \ldots, m$ and there exists at least one index $j \in \{1, \ldots, m\}$ such that $f_j(\vec{x}) < f_j(\vec{y})$.

Definition 2 (Weak Pareto Dominance [19]). Given two vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$, $\vec{x}$ **weakly Pareto dominates** $\vec{y}$ (denoted by $\vec{F}(\vec{x}) \preceq \vec{F}(\vec{y})$) if $f_i(\vec{x}) \le f_i(\vec{y})$ for all $i = 1, \ldots, m$.

Definition 3 (Pareto Optimality [19]). A vector of decision variables $\vec{x}^* \in \Omega$ is **Pareto optimal** if there does not exist another $\vec{x} \in \Omega$ such that $\vec{F}(\vec{x}) \prec \vec{F}(\vec{x^*})$.

[4] Without loss of generality, we will assume only unconstrained minimization problems. To transform a minimization problem into a maximization one, we can use: $\max f = -\min(-f)$.

Based on the previous definitions, the **Pareto Optimal Set** $\mathcal{P}^*$ is defined as the set of all Pareto optimal decision vectors for a given MOP, i.e., $\mathcal{P}^* = \{\vec{x}^* \in \Omega \mid \vec{x} \text{ is Pareto optimal}\}$ [19]. The image of $\mathcal{P}^*$ in objective space is called the **Pareto Optimal Front** and it is defined as follows: $\mathcal{PF}^* = \{\vec{F}(\vec{x}^*) \in \mathbb{R}^m \mid \vec{x}^* \in \mathcal{P}^*\}$.

Definition 4 (Reference set [69]). We denote a reference set $\mathcal{Z}$ as a finite subset of solutions from the Pareto optimal front, i.e., $\mathcal{Z} \subseteq \mathcal{PF}^*$.

Definition 5 (Set Dominance [117]). Given two sets A and B ($A \neq B$) of m-dimensional points, we say that A dominates B (denoted as $A \preceq B$) if for every $\vec{b} \in B$ there exists at least one $\vec{a} \in A$ such that $\vec{a} \preceq \vec{b}$ [78].

Definition 6 (Ideal Objective Vector [78]). The *Ideal Objective Vector* ($\vec{z}^* \in \mathbb{R}^m$) is constructed using the minimum of each of the objective functions, considered separately. Each ith-component of the ideal vector is defined as $z_i^* = \min_{\vec{x}} f_i(\vec{x})$.

Definition 7 (Nadir Objective Vector [78]). The *Nadir Objective Vector* ($\vec{z}^{nad} \in \mathbb{R}^m$) is constructed using the worst values of $\mathcal{PF}^*$. Each ith-component is defined as $z_i^{nad} = \max_{\vec{x} \in \mathcal{P}^*} f_i(\vec{x})$.

To formally define a QI, we first describe the outcome of a multi-objective optimizer as a set of incomparable solutions [38].

Definition 8 (Approximation Set [117]). Let $\mathcal{A} \subseteq \Psi$ be a set of m-dimensional objective vectors. $\mathcal{A}$ is called an **approximation set** or **approximate Pareto front** if any element of $\mathcal{A}$ does not weakly dominate any other vector in $\mathcal{A}$. The set of all approximation sets is denoted as Ψ.

Definition 9 (Quality Indicator [117]). "A k-ary quality indicator I is a function $I : \Psi^k \rightarrow \mathbb{R}$, which assigns to each vector $(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k)$ of k approximate Pareto fronts a real value $I(\mathcal{A}_1, \ldots, \mathcal{A}_k)$".

Definition 10 (Pareto compliance [19]). "A QI denoted by $I : \Psi \rightarrow \mathbb{R}$ is Pareto compliant if for all $\mathcal{A}, \mathcal{B} \in \Psi : \mathcal{A} \preceq \mathcal{B} \Rightarrow I(\mathcal{A}) > I(\mathcal{B})$, assuming that greater indicator values correspond to higher quality (otherwise, $\mathcal{A} \preceq \mathcal{B} \Rightarrow I(\mathcal{A}) < I(\mathcal{B})$)."

3 QUALITY INDICATORS

Quality indicators are set functions that simultaneously assign a real value to k approximation sets. However, this definition is not enough to describe QIs, because they possess several features. Overall, Knowles and Corne [57], Zitzler et al. [112], and Jiang et al. [55] have distinguished their following characteristics: cardinality, performance criteria, Pareto compliance, scalability, scaling invariance, knowledge, and parameter dependence as well as computational complexity. The cardinality of a QI is the number k of approximation sets that can be simultaneously assessed. Performance criteria are related to what the QI measures: capacity,[5] convergence, diversity (divided into distribution and spread of solutions), or convergence-diversity. Pareto compliance is directly related to convergence QIs. The sensitivity of a QI to the different units and scales of the objective functions determines its scaling invariance. If the computation of the indicator requires knowledge of the MOP being solved, then it is knowledge-dependent. Similarly, a QI is parameter-dependent if it needs user-supplied parameters. A critical aspect in practice is its runtime complexity. Additionally, it is crucial that indicators can properly deal with solution sets to MOPs having a different

[5] According to Jiang et al. [55] "capacity QIs quantify the number or ratio of non-dominated solutions in the approximation set that conforms to the predefined requirements."

number of objectives as current MOEAs generate approximation sets in high-dimensional objective spaces, i.e., QIs should be scalable.

In this section, we mathematically define the indicators: HV, $R2$, GD, IGD, IGD$^+$, Δ_p, and ϵ^+, and we discuss their main properties. Additionally, we introduced the Shift-based Density Estimation method (SDE) [66]. These QIs have mostly promoted the development of IB-MOEAs. In all cases, let $\mathcal{A}$ be an approximation set, $\mathcal{Z}$ be a reference set, and m be the dimension of the objective space.

Definition 11 (Hypervolume [116]). Let Λ denote the Lebesgue measure in $\mathbb{R}^m$, then HV is defined as follows:

$$HV(\mathcal{A}, \vec{z}_{ref}) = \Lambda\left(\bigcup_{\vec{a}\in\mathcal{A}} \{\vec{x} \mid \vec{a} \prec \vec{x} \prec \vec{z}_{ref}\}\right), \tag{4}$$

where $\vec{z}_{ref} \in \mathbb{R}^m$ is a reference point that should be dominated by all points in $\mathcal{A}$.

HV is a unary QI that simultaneously assesses convergence and maximum spread of an approximation set of any dimension. To this end, HV considers the volume of the objective space dominated by $\mathcal{A}$. Currently, it is the only unary QI, which is known to be Pareto compliant. However, it has two main drawbacks. First, its computational cost increases super-polynomially with the number of objectives [104]. The other issue is related to the fact that it requires a reference point that bounds the dominated volume. Auger et al. [2] examined that the optimal μ-distributions preferred by HV depend on the choice of the reference point. A recent empirical study, proposed by Ishibuchi et al. [46], has supported the above-mentioned result, showing that HV's preferences on triangular and inverted triangular Pareto fronts, change based on the choice of $\vec{z}_{ref}$. Consequently, $\vec{z}_{ref}$ should be selected according to the Pareto front shape.

Definition 12 (Unary $R2$ indicator [12]). The unary $R2$ indicator is defined as follows:

$$R2(\mathcal{A}, W) = -\frac{1}{|W|}\sum_{\vec{w}\in W} \max_{\vec{a}\in\mathcal{A}}\{u_{\vec{w}}(a)\}, \tag{5}$$

where W is a set of m-dimensional weight vectors and $u_{\vec{w}} : \mathbb{R}^m \mapsto \mathbb{R}$ is a scalarizing function, parameterized by $\vec{w} \in W$, that assigns a real value to each solution vector.

Hansen and Jaszkiewicz initially proposed $R2$ as part of the R-family of indicators [38]. However, it was Brockhoff et al. [12] who proposed the discrete version that is currently employed. Given a set of weight vectors W and an utility function, $R2$ measures the average optimum utility values produced by $\mathcal{A}$. Consequently, $R2$ is a unary indicator for assessing convergence and distribution. Unlike HV, $R2$ is a weakly Pareto-compliant indicator whose complexity is $\Theta(m|W| \cdot |\mathcal{A}|)$. In spite of its low computational cost, $R2$ has some drawbacks. When using the Simplex-Lattice-Design method [109] to generate W, the cardinality of this set is a combinatorial number $N = C_{m-1}^{H+m-1}$, where $H \in \mathbb{N}$ is a user-supplied parameter, and m is the number of objective functions. Additionally, the preferences of $R2$ strongly depend on the choice of the utility functions, and most of these functions require a reference point that it is usually the ideal vector [78, 79].

Definition 13 (Generational Distance [100]). GD evaluates the average distance from each $\vec{a} \in \mathcal{A}$ to its closest reference point $\vec{z} \in \mathcal{Z}$. It is defined as follows:

$$\mathrm{GD}(\mathcal{A}, \mathcal{Z}) = \frac{1}{|\mathcal{A}|}\left(\sum_{\vec{a}\in\mathcal{A}} d(\vec{a}, \mathcal{Z})^p\right)^{1/p}, \tag{6}$$

where $p > 0$ is a user-defined parameter (usually set to $p = 2$) and d is the Euclidean distance from $\vec{a} \in \mathcal{A}$ to its nearest member of $\mathcal{Z}$:

$$d(\vec{a}, \mathcal{Z}) = \min_{\vec{z} \in \mathcal{Z}} \sqrt{\sum_{i=1}^{m} (a_i - z_i)^2}. \tag{7}$$

Definition 14 (Inverted Generational Distance [18]). In contrast to GD, "IGD measures the average distance from each reference point to its nearest solution in $\mathcal{A}$" as follows:

$$\mathrm{IGD}(\mathcal{A}, \mathcal{Z}) = GD(\mathcal{Z}, \mathcal{A}) = \frac{1}{|\mathcal{Z}|} \left(\sum_{\vec{z} \in \mathcal{Z}} d(\vec{z}, \mathcal{A})^p \right)^{1/p}, \tag{8}$$

where $p > 0$ is a parameter usually set to $p = 2$.

Definition 15 (Inverted Generational Distance plus [48]). The IGD^+, for minimization, is defined as follows:

$$\mathrm{IGD}^+(\mathcal{A}, \mathcal{Z}) = \frac{1}{|\mathcal{Z}|} \sum_{\vec{z} \in \mathcal{Z}} \min_{\vec{a} \in \mathcal{A}} d^+(\vec{a}, \vec{z}), \tag{9}$$

where $d^+(\vec{a}, \vec{z}) = \sqrt{\sum_{k=1}^{m} (\max\{a_k - z_k, 0\})^2}$.

GD was proposed by Van Veldhuizen and Lamont [100] and it estimates how far are the elements in $\mathcal{A}$ from those in $\mathcal{Z}$, i.e., it exclusively measures the convergence of the approximation set. Since GD is non-Pareto-compliant, in some cases, it produces misleading results when comparing MOEAs [48, 117]. Additionally, GD is sensitive to the size of the approximation set [6]. For example, an important problem takes place when $\mathcal{A}$ has very few points, but they all are clustered together. To overcome this issue, Coello Coello and Cruz Cortés [18] proposed IGD, which unlike GD, measures the average distance from $\mathcal{Z}$ to $\mathcal{A}$. Unfortunately, IGD is also a non-Pareto-compliant QI. In furtherance of improving the mathematical properties of IGD, Ishibuchi et al. proposed IGD^+ [48] that is a variant of it that adopts Pareto dominance in the Euclidean distance. Due to this modification, IGD^+ is weakly Pareto-compliant. Bezerra et al. [6] broadly discuss the differences between IGD and IGD^+. However, an important issue that GD, IGD, and IGD^+ share is how to construct $\mathcal{Z}$ when no information about $\mathcal{PF}^*$ is available [47]. The computational cost of these indicators is $\theta(m|\mathcal{Z}| \cdot |\mathcal{A}|)$.

Definition 16 (Δ_p indicator [85]). For a given $p > 0$, Δ_p is defined as follows:

$$\Delta_p(\mathcal{A}, \mathcal{Z}) = \max \left\{ GD_p(\mathcal{A}, \mathcal{Z}), IGD_p(\mathcal{A}, \mathcal{Z}) \right\}. \tag{10}$$

Δ_p is composed of two indicators: GD_p and IGD_p, which are slight modifications of GD and IGD, respectively. These are defined as follows:

Definition 17 (GD_p indicator [85]).

$$\mathrm{GD}_p(\mathcal{A}, \mathcal{Z}) = \left(\frac{1}{|\mathcal{A}|} \sum_{\vec{a} \in \mathcal{A}} d(\vec{a}, \mathcal{Z})^p \right)^{1/p}. \tag{11}$$

Definition 18 (IGD_p indicator [85]).

$$\mathrm{IGD}_p(\mathcal{A}, \mathcal{Z}) = GD_p(\mathcal{Z}, \mathcal{A}) = \left(\frac{1}{|\mathcal{Z}|} \sum_{\vec{z} \in \mathcal{Z}} d(\vec{z}, \mathcal{A})^p \right)^{1/p}. \tag{12}$$

Although the Hausdorff distance is a metric in the mathematical sense on the set of compact subsets of $\mathbb{R}^m$, it tends to penalize outlier solutions that are commonly generated by MOEAs. Consequently, Schütze et al. introduced the Δ_p indicator that measures the averaged Hausdorff distance of the approximation set to the reference set [85]. Δ_p is based on the indicators GD_p and IGD_p, which are slight modifications of the original GD and IGD indicators that aim to reduce the penalization of outliers. It is worth noting that such averaging of the distances leads to violations of the triangle inequality, and hence, Δ_p is not a metric. Additionally, Δ_p is a non-Pareto-compliant QI, and it assesses convergence and distribution simultaneously. Similar to GD, IGD, and IGD^+, this indicator requires a reference set, which is difficult to construct without information of the Pareto optimal front.

Definition 19 (Unary ϵ^+ indicator [117]). Mathematically, it is defined as follows:

$$\epsilon^+(\mathcal{A}, \mathcal{Z}) = \max_{\vec{z} \in \mathcal{Z}} \min_{\vec{a} \in \mathcal{A}} \max_{1 \leq i \leq m} \{z_i - a_i\}. \tag{13}$$

Zitzler et al. [117] introduced the unary ϵ-indicator to measure the minimum distance that an approximation set needs to be translated in each dimension to weakly Pareto dominate a reference set. Consequently, ϵ^+ exclusively assesses the convergence of a Pareto front approximation. It is worth emphasizing that ϵ^+ is a weakly Pareto-compliant QI. Although it is a parameterless QI, a reference set is required for its computation. Additionally, ϵ^+ is not very sensitive to small changes of the solutions in $\mathcal{A}$.

Li et al. [66] introduced in 2014 the SDE that is a general density estimation methodology. Given $\vec{a} \in \mathcal{A}$, a general density estimator D is a function as follows:

$$D(\vec{a}, \mathcal{A}) = D\left(\left\{\text{dist}(\vec{a}, \vec{b})\right\}_{\vec{b} \in \mathcal{A} \setminus \{\vec{a}\}}\right), \tag{14}$$

where $\text{dist}(\vec{a}, \vec{b})$ is a function, usually a distance function, that determines the similarity between solutions $\vec{a}$ and $\vec{b}$. D measures the similarity between $\vec{a}$ and each element in $\mathcal{A} \setminus \{\vec{a}\}$. It is worth noting that the implementation of D depends on the MOEA employed. Regarding the issues of Pareto-based MOEAs when tackling MaOPs, Li et al. proposed a modification to Equation (14) to improve the performance of a density estimator. They proposed to shift the location of other individuals in $\mathcal{A} \setminus \{\vec{a}\}$ according to the convergence comparison between them and $\vec{a}$, giving rise to the SDE methodology.

Definition 20 (Shift-based Density Estimator [66]). Given the density estimator of Equation (14), the SDE methodology is defined as follows:

$$D'(\vec{a}, \mathcal{A}) = D\left(\left\{\text{dist}(\vec{a}, S(\vec{b}, \vec{a}))\right\}_{\vec{b} \in \mathcal{A} \setminus \{\vec{a}\}}\right), \tag{15}$$

where S is the shift function

$$\vec{v} = S(\vec{b}, \vec{a}) = \left\{\begin{array}{ll} a_j, & \text{if } b_j < a_j \\ b_j, & \text{otherwise} \end{array}\right\}_{j \in \{1, \ldots, m\}}. \tag{16}$$

4 INDICATOR-BASED MOEAS

Multi-objective Evolutionary Algorithms aim to determine a set of solutions that satisfy specific optimality properties. The Pareto dominance relation [19] has represented the most general optimality notion. However, the degree of freedom to determine what is an optimal solution is still considerable. Consequently, QIs integrated into MOEAs (i.e., the so-called indicator-based MOEAs) introduce additional information that describes the preference of the user and improves the notion of optimality imposed by the Pareto dominance relation [90, 113]. The underlying idea of

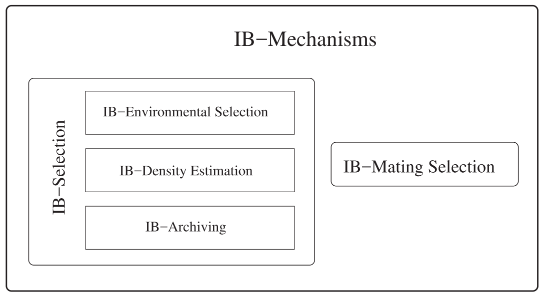

Fig. 1. IB-Mechanisms are divided into two main categories: (1) IB-Selection and (2) IB-Mating Selection. Furthermore, the former category is classified in three groups: IB-Environmental Selection, IB-Density Estimation, and IB-Archiving.

ALGORITHM 1: MOEA general framework

1 Generate initial population $\mathcal{P}$ of size μ;
2 Randomly initialize $\mathcal{A}$;
3 **while** *stopping criterion is not fulfilled* **do**
4     $M \leftarrow$ Select μ parents from $\mathcal{P}$ or $\mathcal{A}$;
5     $O \leftarrow$Generate a set of λ offspring solutions based on M, using variation operators;
6     $Q \leftarrow \mathcal{P} \cup O$;
7   Update archive $\mathcal{A}$ using Q;
8     $S \leftarrow \text{select}(Q, \mathcal{A}, \mu)$;
9     $\mathcal{P} \leftarrow S$;
10 **end**
11 **return** $\mathcal{P}$ or $\mathcal{A}$

IB-MOEAs is to optimize an indicator value through the evolutionary process [90]. Since 2003, IB-MOEAs have been developed to overcome the drawbacks of Pareto-based MOEAs, with a particular emphasis on diversity issues and the dilution of selection pressure when tackling MaOPs [61, 101]. In this section, we review the development of IB-MOEAs from the earliest proposals to the current state-of-the-art approaches.

4.1 Our Proposed Taxonomy

Currently, several indicator-based mechanisms (IB-Mechanisms) have been designed and integrated into MOEAs. However, there is no clear distinction among them. Consequently, we introduce a taxonomy (see Figure 1) to classify IB-Mechanisms based on a comprehensive review of the state-of-the-art approaches. It is worth emphasizing that, to the authors' best knowledge, this is the first taxonomy ever proposed to organize the mechanisms of IB-MOEAs.

Let us start the discussion of our proposed taxonomy for IB-Mechanisms by analyzing the general framework of an MOEA, shown in Algorithm 1. First, the main population $\mathcal{P}$ and the archive $\mathcal{A}$ (also known as the external population) are initialized. Lines 3 to 9 show the main loop of an MOEA. At each iteration, mating selection is performed to select μ parent solutions from either $\mathcal{P}$ or $\mathcal{A}$. Variation operators (e.g., crossover and mutation) further explore this set M of parent solutions to create the set O of λ offspring solutions. Then, the archive $\mathcal{A}$ is updated using selection rules that are executed on the joint population $Q = \mathcal{P} \cup O$ in line 7. The next step (line 8) involves solving or approximating a subset selection problem, since it is necessary to choose the best μ solutions, according to a certain criterion from Q to shape the set S. Finally, S replaces the main population $\mathcal{P}$. In line 11, an MOEA returns either $\mathcal{P}$ or $\mathcal{A}$, since both of them contain Pareto front approximations. This decision is based on the following: (1) if $\mathcal{P}$ is returned, then the MOEA employed $\mathcal{A}$ to maintain diversity during the evolutionary process, and (2) if $\mathcal{A}$ is the outcome, then the main population was utilized as an exploration and exploitation tool to construct $\mathcal{A}$.

Our taxonomy, shown in Figure 1, is based on the primary mechanisms that are present in Algorithm 1, i.e., mating selection and selection schemes applied on the main population and the archive. It is worth emphasizing that our taxonomy aims to classify the indicator-based mechanisms that MOEAs have adopted, rather than classifying IB-MOEAs themselves. IB-Mechanisms are divided into two categories: (1) IB-Mating Selection, and (2) IB-Selection. On the one hand, IB-Mating Selection aims to select μ parents to create λ promising offspring solutions. These IB-Mating Selection methods can be variations, for instance, of the commonly employed tournament selection, roulette wheel selection, or proportional selection [7, 19] or new proposals, but applying QIs as the core of the methods. On the other hand, IB-Selection comprises three classes: (1) IB-Environmental Selection (IB-ES), (2) IB-Density Estimation (IB-DE), and (3) IB-Archiving (IB-AR). The main difference between IB-Mating Selection and IB-Selection is that the latter aims to solve or approximate the Indicator-based Subset Selection Problem (IBSSP) [4] that is defined as follows:

Definition 21 (Indicator-based Subset Selection Problem [4]). Let $Z \subseteq \mathbb{R}^m$ be a space of m-dimensional vectors, and let I be a unary quality indicator. Without loss of generality, we assume that this indicator is to be maximized. Given $R \subseteq Z$, a reference set of cardinality N, and $k \leq N$ a positive subset size, the IBSSP consists in determining the set $S \subseteq R$ of maximal quality:

$$\arg\max_{\substack{S \subseteq R \\ |S|=k}} I(S). \tag{17}$$

In other words, IBSSP aims to select a subset of $k \leq N$ solutions that optimizes the indicator value. The solution space size of IBSSP is $\binom{N}{k}$. Apart from the inherent complexity of computing indicator values, this makes of IBSSP a challenging combinatorial optimization problem. The next question to solve is why IB-Selection is divided into three categories, since, at first sight, they seem to overlap.

On the one hand, IB-Archiving corresponds to selection rules that update the external population $\mathcal{A}$. The first proposed IB-MOEA implements an IB-AR scheme based on the hypervolume indicator [58, 59]. In this regard, many MOEAs have an external population whose purpose is to maintain an approximation to the Pareto front, i.e., the global non-dominated solutions generated by the MOEA [19]. Each time a new point is considered for addition to the archive, it must be analyzed for non-dominance concerning the points currently stored. Hence, at the end of the evolutionary process, the archive will only include solutions that are non-dominated for all the objective vectors that have been generated by the MOEA (i.e., the global non-dominated solutions).

On the other hand, IB-ES and IB-DE are exclusively applied to the main population $\mathcal{P}$. These two mechanisms are involved in line 8 of Algorithm 1. However, there is no clear distinction between IB-ES and IB-DE. Environmental selection dictates which solutions should survive at each generation. Thus, this mechanism has particular importance, because convergence is closely related to it. Thiele [90] has pointed out that IB-ES can be designed using two algorithmic methodologies: hierarchical selection and selection based on fitness assignment. Hierarchical selection imposes a partial order among the solutions in $\mathcal{P}$ just as the non-dominated sorting algorithm [22] does, but it needs a refinement using a density estimator. On the contrary, fitness assignment imposes a total order among the solutions, which implies that a density estimator is not required. As we previously mentioned, density estimators refine the partial order imposed by hierarchical environmental selection mechanisms. In this regard, they perform as the second selection criterion to reduce the size of the joint population of parents and offspring, aiming simultaneously to enhance diversity. Although they have different purposes, an IB-DE can perform as an IB-ES under certain conditions. For instance, when the non-dominated sorting algorithm is coupled with an IB-DE [5, 13, 31], the latter will eventually replace the former, since it tends to create a single rank of solutions when tackling MaOPs, i.e., it loses selection pressure. Hence, only when an IB-DE is working with an environmental selection mechanism that loses selection pressure, it will perform as the primary selection mechanism. Otherwise, an IB-DE will perform as the second selection criterion.

In the following sections, we will review several IB-Mechanisms that have been proposed until the end of 2018. The review will be done following our taxonomy. For each proposal, we will focus on discussing their properties, advantages, and drawbacks.

4.2 IB-Selection

The first IB-MOEA was proposed by Knowles et al. [59] in 2003. This algorithm employed an archiving update rule based on the hypervolume indicator. Since then, a plethora of proposals have been published in the specialized literature. In this section, we present a review of the most important MOEAs that use an IB-Selection mechanism. Tables 1, 2, and 3 summarize MOEAs adopting IB-ES, IB-DE, and IB-AR methods, respectively.

4.2.1 IB-Environmental Selection.

HV. Zitzler and Künzli [114] proposed the Indicator-based Evolutionary Algorithm (IBEA) whose general framework is shown in Algorithm 2. The underlying idea of IBEA is to provide a general framework for $(\mu + \lambda)$ environmental selection based on arbitrary binary indicators that are integrated into a fitness function (see line 5). This fitness function measures the loss in quality when each $\vec{x} \in Q$ is removed. According to the authors, the exponential function is employed to amplify the influence of dominating solutions members over dominated ones. This fitness function also requires a fitness scaling factor κ that depends on the indicator being used and the MOP. The IB-ES, described in lines 6 to 10, is a greedy algorithm that deletes at each iteration the solution having the minimal fitness value while $|Q| > \mu$. To illustrate the effectiveness of IBEA, its authors proposed the IBEA_{HV} that uses a binary hypervolume indicator. IBEA_{HV} was compared to NSGA-II [22] and SPEA2 [115] on the 100-item 0/1 knapsack problem and the low-dimensional continuous MOPs: ZDT6, DTLZ2, DTLZ6, and Kursawe (KUR) [19]. IBEA_{HV} outperformed NSGA-II in almost all MOPs. Interestingly enough, however, the bi-objective KUR problem, which has a disconnected and a concave Pareto front, was the most challenging problem for IBEA_{HV}.

An important disadvantage of IBEA is related to the κ parameter, which is dependent on the MOP and the indicator employed. Regarding this drawback, IBEA2 [54] was proposed to adaptively adjust parameter κ. This adaptive mechanism looks for the best κ parameter using the Nelder-Mead method where the objective function is defined as a similarity measure between two sets: one produced by the IB-ES of IBEA_{HV} and another one using a density estimator based on HV [56]. IBEA2 was tested on the benchmarks ZDT, DTLZ, and WFG for two to five objective functions, comparing its performance to NSGA-II, SPEA2, MOEA/D, and IBEA_{HV} in terms of HV. Based on the analysis of the experimental results, IBEA2 is a very good optimizer for MOPs whose Pareto fronts are linear, concave, and degenerated. However, it has low performance when tackling MOPs having disconnected, mixed, and multifrontal Pareto fronts. Additionally, the results show that due to the use of the adaptive mechanism, IBEA2 is better than IBEA_{HV} in most of the test problems adopted.

Table 1. IB-Environmental Selection Mechanisms

Indicator	Algorithm	Algorithmic approach	Method	Problems	# Objectives	Year	Ref.
HV	IBEA_{HV}	Greedy	Fitness assignment	ZDT, KUR, DTLZ	2, 3	2004	[113]
	Iterative-IBEA	Greedy	Fitness assignment	500-item 0/1 Knapsack problem	2–4	2007	[50]
	iSMS-EMOA	Greedy	Worst contribution	DTLZ & WFG	3–6	2013	[73]
	DIVA	Greedy	Fitness assignment	WFG	2	2014	[95]
	IBEA2	Greedy	Fitness assignment	ZDT, DTLZ, WFG	2–5	2016	[54]
	mIBEA	Greedy	Fitness assignment	DTLZ	3	2017	[68]
	aviSMS-EMOA	Greedy	Worst contribution	DTLZ & WFG	3–6	2017	[76]
$R2$	$R2$-MOGA	Hierarchical	Worst contribution	ZDT, DTLZ, WFG	2–10	2013	[24]
	$R2$-MODE	Hierarchical	Worst contribution	ZDT, DTLZ, WFG	2–10	2013	[24]
	$R2$-IBEA	Greedy	Fitness assignment	ZDT & DTLZ	2–5	2013	[96]
	MOMBI	Hierarchical	Scalarizing function optimization	DTLZ & WFG	2–8	2013	[39]
	MOMBI-II	Hierarchical	Scalarizing function optimization	DTLZ & WFG	3–10	2015	[40]
	TS-R2EA	Hierarchical	R2 contribution and fitness assignment	DTLZ & WFG	3–15	2018	[63]
	R2-MOEA/D	Decomposition	Worst contribution	DTLZ & WFG	3–15	2018	[64]
IGD^+	IGD^+-EMOA	Linear Assignment Problem	Kuhn-Munkres algorithm	DTLZ & WFG	2–8	2015	[71]
	IGD^+-EMOA II	Linear Assignment Problem	Kuhn-Munkres algorithm	DTLZ, WFG, MAF, Viennet	2–8	2018	[72]
GD	GD-MOEA	Greedy	Minimize distance to $\mathcal{PF}^*$	DTLZ, WFG	3–6	2015	[74]
	GDE-MOEA	Greedy	Minimize distance to $\mathcal{PF}^*$ and ϵ dominance	DTLZ, WFG	3–6	2015	[75]
ϵ^+	IBEA_{ϵ^+}	Greedy	Fitness assignment	ZDT6, KUR, DTLZ2, DTLZ6	2–3	2004	[113]
	AGE	Greedy	Worst contribution	DTLZ	2–20	2011	[11]
	AGE-II	Greedy	Worst contribution	DTLZ, WFG	2–20	2013	[102]
	Two_Arch2	Greedy	Fitness assignment	DTLZ, WFG	2–20	2014	[103]
Δ_p	Δ_p-DDE	Greedy	Worst contribution	ZDT, DTLZ	2–10	2012	[81]
	Δ_p-MOEA	Greedy Hierarchical	Minimize point-to-set distance	WFG	3–6	2016	[77]

The algorithmic structure of the IB-ES is termed as "Algorithmic approach," while "Method" indicates what kind of information is employed to select the solutions. For each approach, we show the test problems and the number of objective functions on which the IB-MOEA was tested on.

Another improvement of the IBEA framework is the modified IBEA (mIBEA), proposed by Li et al. [68]. Its main motivation was to improve the bad distribution of points generated by IBEA_{HV}. To this aim, the authors hybridized the IBEA framework with the non-dominated sorting algorithm to exclude dominated solutions at each generation. Due to this modification, the scaling of the objective functions scores is no longer affected by dominated solutions, which are far away from the best non-dominated solutions. The uniformity and convergence analysis of solutions showed that mIBEA performs better than IBEA_{HV}. However, the authors did not show an exhaustive performance analysis with respect to other state-of-the-art MOEAs.

Table 2. IB-Density Estimators

Indicator	Algorithm	Method	Problems	# Objectives	Year	Ref.
HV	ESP	Worst contribution	ZDT	2	2003	[43]
	SIBEA	Worst contribution	ZDT	2	2007	[110]
	SMS-EMOA	Worst contribution	ZDT & DTLZ	2, 3	2007	[5]
	MO-CMA-ES	Worst contribution	ZDT & FON	2	2007	[44]
	SMS-EMOA-Apr	HV contribution using ASF-based approximation	DTLZ	3, 6	2010	[52]
	HypE	HV approximation	DTLZ, WFG knapsack problem	2, 3, 5, 7, 10, 25, 50	2011	[3]
	FV-MOEA	Worst contribution	ZDT, DTLZ, WFG	2–5	2015	[56]
	I-SIBEA	Worst contribution	ZDT & DTLZ	2, 3	2015	[17]
R2	R2-EMOA	Worst contribution	ZDT, DTLZ	2	2015	[13]
IGD^+	IGD^+-MaOEA	Worst contribution	DTLZ & $DTLZ^{-1}$	3–7	2018	[31]
IGD	MyO-DEMR	Worst contribution	DTLZ	2, 3, 5, 8, 10, 15, 20	2013	[23]
	MOEA/IGD-NS	Worst contribution	ZDT & DTLZ	2–3	2016	[92]
	MaOEA/IGD	Linear Assignment Problem	DTLZ & WFG	8, 15, 20	2018	[88]
	AR-MOEA	Worst contribution	DTLZ, $DTLZ^{-1}$, WFG, MAF	3, 5, 10	2018	[91]
Δ_p	RIB-EMOA	Worst contribution	DTLZ	3–10	2014	[108]

The "Method" determines how the solutions are selected. For each IB-MOEA, it is shown in which problems it has been tested on as well as the number of objectives adopted in each case.

Table 3. IB-Archiving Methods

Indicator	Algorithm	Method	Problems	# Objectives	Year	Ref.
HV	LAHC	Worst contribution	General point sequences	2, 3	2003	[59]
	ϵ-MOPSO$^{UD}_{MRV}$	Worst contribution	ZDT & DTLZ	2, 3	2009	[53]
Δ_p	Δ_p-EMOA	Worst contribution	DTLZ, ZDT spheres model, DENT	2	2011	[36]
	Δ_p-M-EMOA	Worst contribution	DTLZ & Viennet	3	2012	[93]
	Δ_p-T-EMOA	Worst contribution	DTLZ & Viennet	3	2013	[82]
	SMS-DPPSA	Worst contribution	DTLZ	4	2013	[25]
	PS-EMOA	Worst contribution	DTLZ	4	2013	[25]

The "Method" determines how the solutions are selected. For each IB-MOEA, it is shown in which problems it has been tested on as well as the number of objectives of the MOPs adopted.

In 2007, Ishibuchi et al. [50] proposed the Iterative-IBEA[6] that produces at each execution a single solution to maximize the HV value. In other words, it iteratively constructs the final solution set S. At execution k, Iterative-IBEA uses $\text{HV}(S^{(k-1)} \cup \{\vec{u}\})$ as its fitness function (where $S^{(k-1)}$ is the solution set generated in the previous execution, thus, $|S^{(k-1)}| = k-1$). The IB-ES computes the fitness values of the $(\mu + \lambda)$ solutions and deletes the worst λ individuals. If we want a Pareto front approximation of size N, then we have to execute Iterative-IBEA the same number of times, which implies a high-computational cost due to the use of the hypervolume

[6]Although this MOEA is called IBEA, it does not follow the IBEA framework of Algorithm 2.

indicator. This IB-MOEA was only compared to NSGA-II on the 500-item 0/1 knapsack problem for two, three, and four objective functions.

Ulrich et al. [95] proposed the Diversity Integrating Hypervolume-based Search Algorithm (DIVA) that combines a decision space diversity measure and HV into one single set measure, where the trade-off between the two measures is tunable. DIVA employs a greedy environmental selection that aims to remove the worst contributing solutions to HV to obtain the best μ solutions out of a set of $(\mu + \lambda)$ individuals. DIVA was tested on the WFG test suite for two objective functions. Experimental results showed that DIVA significantly improves diversity compared to HypE [3] (see Section 4.2.2) due to the integration of the diversity measure into HV. However, its main drawback is related to the high computational cost associated with both HV and the diversity measure adopted.

Since the main limitation of the above-mentioned proposals is the expensive calculation of HV, Menchaca-Mendez and Coello Coello [73] proposed an IB-ES that exploits the locality property of HV.[7] This approach, called iSMS-EMOA, generates at each generation one offspring solution that must compete, regarding its HV contribution, with its nearest neighbor in objective space and $r \geq 1$ randomly chosen solutions from the population (they set $r = 1$). Consequently, it only needs to compute $r + 2$ HV contributions instead of handling the whole population. iSMS-EMOA was compared to SMS-EMOA and HypE on the DTLZ test suite for three to six objective functions, regarding HV. The stopping criterion of all MOEAs was 50,000 function evaluations or a maximum of four hours of running time (since SMS-EMOA computes the exact HV, it is very time-consuming for many-objective problems). For three and four objective functions, iSMS-EMOA presented a competitive performance compared to SMS-EMOA. However, for five and six objective functions, iSMS-EMOA completely outperformed the adopted MOEAs, since they ran out of running time. Hence, iSMS-EMOA can be considered as a promising approach for solving MaOPs due to its less expensive IB-ES while still relying on the nice mathematical properties of HV.

Finally, in 2017, the same authors proposed an improvement of iSMS-EMOA. denoted as approximate version of the improved SMS-EMOA (aviSMS-EMOA) [76]. The idea of aviSMS-EMOA is to combine the selection scheme of iSMS-EMOA with a recently proposed mechanism to approximate the hypervolume contributions with a minimal error [9]. Due to the exploitation of the locality property and the mechanism to approximate HV contributions, aviSMS-EMOA is able to balance the quality of the outcome set and the running time required to obtain it. In its experimental results, considering the DTLZ and WFG test suites, aviSMS-EMOA was able to outperform both iSMS-EMOA and HypE. However, SMS-EMOA obtained better results, since it employs the exact calculation of the hypervolume, but if the trade-off between computational cost and quality is taken into account, then aviSMS-EMOA is a very competitive alternative to the use of SMS-EMOA.

R2. In 2013, Phan and Suzuki [96] proposed R2-IBEA following the scheme of IBEA (see Algorithm 2) but adopting a binary version of $R2$. Additionally, the authors suggested two new mechanisms. First, a hypervolume-based weight vector generation approach that uniformly distributes the weight vectors required by $R2$, aiming to maximize HV. Additionally, they proposed an adaptive reference point adjustment mechanism that aids R2-IBEA to reduce the bias of the $R2$ indicator to prefer the knee of the Pareto front therefore promoting the generation of uniform solutions. The performance of R2-IBEA was examined on MOPs from the ZDT and DTLZ test suites (using three and five objectives), using the HV, GD, IGD, and ϵ^+ indicators and it was compared with Pareto-based, indicator-based, and decomposition-based MOEAs. Although R2-IBEA had remarkable results in almost all problems in terms of convergence and diversity, it is worth

[7] For two dimensions, “given three consecutive points on the Pareto front, moving the middle point will only affect the HV contribution that is dedicated to this point, but the joint HV contribution remains fixed” [2].

ALGORITHM 2: IBEA general framework

Input: Fitness scaling factor κ
Output: Pareto front approximation
Randomly initialize population $\mathcal{P}$ of size μ;
while *stopping criterion is not fulfilled* **do**
  Create the set O of λ offspring solutions;
  $Q \leftarrow \mathcal{P} \cup O$;
  fitness$(\vec{x}) = \sum_{\vec{y} \in Q \setminus \{\vec{x}\}} -e^{-I(\{\vec{y}\}, \{\vec{x}\})/\kappa}, \forall \vec{x} \in Q$;
  while $|Q| > \mu$ **do**
    $\vec{x}^{\min} \leftarrow arg\, min_{\vec{x} \in Q}$fitness$(\vec{x})$;
    $Q \leftarrow Q \setminus \{\vec{x}^{\min}\}$;
    Update fitness values of all individuals in Q;
  end
end
return $\mathcal{P}$

emphasizing its poor performance on multifrontal MOPs, i.e., ZDT4 and DTLZ3 for two and three objective functions, respectively. For five-dimensional problems, R2-IBEA obtained the best HV value in problems DTLZ1, DTLZ3, DTLZ4, and DTLZ7. However, the authors did not test their proposal on degenerated problems (DTLZ5 and DTLZ6).

In the same year, Díaz-Manríquez et al. [24] proposed a hierarchical IB-ES similar to the non-dominated sorting scheme of NSGA-II. However, instead of computing ranks of non-dominated solutions, they suggested the creation of ranks of contributing solutions to the $R2$ indicator. The first layer contains solutions that do contribute to $R2$; then, these solutions are temporarily removed and a new layer is generated. This process continues until there are no more solutions left. The authors embedded this IB-ES into two search engines: a genetic algorithm and a differential evolution algorithm, giving rise to the R2-Multi-objective Genetic Algorithm (R2-MOGA) and the R2-Multi-objective Differential Evolution (R2-MODE). Both proposals were compared to NSGA-II, MOEA/D and SMS-EMOA on the ZDT, DTLZ, and WFG test suites using two and three objective functions. According to their HV results, R2-MOGA and R2-MODE had a poor performance on these MOPs. Additionally, the authors experimented on MaOPs, comparing R2-MOGA and R2-MODE to SMS-EMOA and HypE using the HV indicator. They employed the DTLZ1-DTLZ4 test instances with 4 to 10 objective functions. These results indicated that their two proposals were competitive with respect to SMS-EMOA and HypE regarding HV. Additionally, the proposed approaches outperformed the HV-based MOEAs adopted in the comparison in terms of computational time.

Hernández Gómez and Coello Coello [39] proposed the $R2$-ranking algorithm, which is similar to the one previously described. However, instead of creating ranks of $R2$-contributing solutions, this approach identifies solutions that optimize the set of utility functions involved in the definition of $R2$. This selection mechanism was implemented in the Many-objective Metaheuristic based on the $R2$ Indicator (MOMBI). The authors adopted the DTLZ and WFG test suites in their experiments, assessing the performance of MOMBI, MOEA/D, and SMS-EMOA in terms of HV using MOPs with from two to eight objective functions. From the results, it is evident that MOMBI outperformed the other MOEAs on the multifrontal MOPs and those having disconnected and mixed Pareto front shapes, such as DTLZ7 and WFG1, respectively. However, MOMBI was outperformed by the other MOEAs on degenerated MOPs (DTLZ5, DTLZ6, and WFG3) and concave MOPs (DTLZ2 and WFG4-WFG9). Regarding concave MOPs, the distribution of points generated by MOMBI was strongly biased toward the knee of the Pareto front due to the adoption of the weighted Tchebycheff utility function (WTCH). Additionally, a running time analysis showed that MOMBI was less computationally expensive than SMS-EMOA and that it was slightly more costly than MOEA/D.

In furtherance of solving the distribution bias of MOMBI, the same authors proposed MOMBI-II [40] that uses the achievement scalarizing function (ASF) instead of WTCH, where the former promotes a more uniform distribution of points. Additionally, MOMBI-II employs a mechanism to statistically estimate the nadir point that is needed to normalize the population. MOMBI-II was tested on one linear MOP (DTLZ1) and five concave MOPs (DTLZ2-DTLZ4, WFG6, and WFG7) using three and five objective functions and it was compared to numerous MOEAs, using the indicators HV and Δ_p. The results show that due to the use of ASF, MOMBI-II produces evenly distributed solutions in the considered MOPs, outperforming all of the adopted MOEAs. However, the authors did not test MOMBI-II in MOPs having complicated Pareto fronts such as DTLZ5, DTLZ6, WFG3, nor in MOPs having degenerated or disconnected Pareto fronts such as DTLZ7 and WFG2.

In recent years, some studies have emphasized that MOEAs using a set of convex weight vectors,[8] using, for example, the Simplex Lattice Design method [109], may lose diversity when solving MaOPs [21, 40]. This is the case of all the above mentioned $R2$-based MOEAs (except for R2-IBEA). Consequently, Li et al. [63] have enhanced the diversity management of the $R2$ environmental selection mechanisms by taking advantage of the Reference Vector guided Evolutionary Algorithm (RVEA) [15]. This selection mechanism first divides the population into contributing and non-contributing solutions to the $R2$ indicator, keeping the contributing ones. If the next population is not complete, then the remaining solutions are selected from the non-contributing ones, enhancing diversity with the RVEA mechanism that clusters the population into different subregions where a weight vector defines each subregion. Then, the idea is to avoid deleting solutions from isolated subregions, i.e., to remove solutions from the most crowded subregions. The proposed approach, called Two-Stage $R2$ Evolutionary Algorithm (TS-R2EA), was mainly compared to MOMBI-II and RVEA on the DTLZ and WFG test suites for 3 to 15 objective functions, using the HV and IGD$^+$ indicators. Statistically, TS-R2EA had a better performance on MaOPs with more than 10 objective functions and on multifrontal problems such as DTLZ1. However, it did not show good results when tackling test problems with irregular Pareto fronts, namely, WFG1, WFG2, and WFG3, which have mixed, disconnected, and degenerated Pareto front geometries.

In 2018, Li et al. [64] proposed an environmental selection mechanism that combines the $R2$ indicator with the decomposition strategy of MOEA/D [109]. The motivation of this algorithm (called R2-MOEA/D) is that in some cases a simple $R2$ selection strategy may lose the diversity of solutions in objective space. Based on the set of weight vectors required by $R2$, the authors defined subspaces where the solutions are assigned depending on their proximity to each weight vector. Instead of using the scalarizing values for each solution as in MOEA/D, R2-MOEA/D assigns to each solution its contribution to $R2$. The underlying idea of this hybrid selection scheme is to avoid deleting solutions from isolated subregions even if the pure $R2$ selection mechanism aims to do it. In this way, a better diversity of solutions is promoted. To the authors' best knowledge this is the first approach that combines an IB-Mechanism with a decomposition strategy. Compared to MOMBI-II, R2-MOEA/D produces better results regarding the Δ_p indicator that assesses convergence and diversity simultaneously. However, due to the use of convex weight vectors, it is likely that the performance of R2-MOEA/D strongly depends on specific Pareto front shapes [49].

[8] A vector $\vec{w}$ is a convex weight vector if and only if $\sum_{i=1}^{m} w_i = 1$ and $\forall i \in \{1, \ldots, m\} w_i \geq 0$.

***IGD*$^+$.** Manoatl and Coello Coello introduced the IGD$^+$-EMOA, which is the only algorithm currently available (to the authors' best knowledge) that uses an environmental selection based on IGD$^+$ [71]. It transforms the selection process into a Linear Assignment Problem (LAP) such that the best relationship between the reference set and the population is found from the modified Euclidean distance of IGD$^+$. The authors proposed to use the Kuhn-Munkres' algorithm to solve the LAP. The reference set is a crucial aspect of IGD$^+$-EMOA. Manoatl and Coello Coello proposed the construction of this set using γ-superspheres [27] to approximate convex, linear, or concave geometries of the Pareto front. A γ-supersphere is a type of curve defined as: $\{(y_1, \ldots, y_m) \in \mathbb{R}^m_+ \mid y_1^\gamma + \cdots + y_m^\gamma = 1\}$, where $\gamma \in \mathbb{R}^m_+$ is a parameter that controls the geometry of the curve. The authors proposed to find the value of γ by solving a root-finding problem using Newton's method. However, due to the limitation of geometries that can be approximated using this strategy, IGD$^+$-EMOA cannot properly solve MOPs having degenerated and disconnected Pareto front shapes. IGD$^+$-EMOA II [72], proposed by the same authors, aims to solve the problem of the previous version with difficult Pareto front shapes. Its main contribution is a new method to generate the reference set. This mechanism employs an external archive of non-dominated solutions from which certain ones are selected to be part of the reference set based on their contribution to the hypercube (a concept closely related to the hypervolume). Regarding the experimental results, it is confirmed that the HV-based strategy to build the reference set allows IGD$^+$-EMOA II to solve MOPs having complex Pareto front shapes, for example, the Viennet problems [99] and the MAF test suite [16], making it a very versatile multi-objective optimizer.

GD. Menchaca and Coello Coello [74] proposed the Generational Distance-based Multi-objective Evolutionary Algorithm (GD-MOEA) that employs an environmental selection scheme based on GD. However, since GD only promotes convergence leaving aside diversity, the authors introduced a mechanism based on Euclidean distances that compensates for this deficiency. At each iteration, the population is divided into non-dominated and dominated solutions. In the case that we have less than μ non-dominated solutions, the remaining solutions are selected from the dominated ones using a GD-based environmental selection mechanism in which the non-dominated individuals represent the reference set. The IB-DE aims to choose the nearest dominated solutions to the reference set, diversifying the set by analyzing its nearest neighbors. If the number of non-dominated solutions is greater than μ, then the diversity mechanism is applied. GD-MOEA was compared to MOEA/D and HypE on the DTLZ and WFG test suites using from three to six objective functions, and adopting the hypervolume indicator. The results reported by the authors showed that HypE outperformed GD-MOEA in 75% of the test problems and the latter outperformed MOEA/D in 67% of the MOPs. It is worth noting that GD-MOEA had difficulties to solve multifrontal MOPs such as DTLZ1 and DTLZ3. We can only distinguish a good performance of GD-MOEA on problems DTLZ7 and WFG7 for high-dimensional objective spaces. Regarding running time, GD-MOEA is 168 times faster than HypE and 1.46 times slower than MOEA/D.

Menchaca et al. [75] found that the diversity mechanism of GD-MOEA did not sufficiently increase the selection pressure when tackling MaOPs. Thus, they replaced the old diversity mechanism with one based on the ϵ-dominance[9] whose aim is to uniformly distribute solutions among the hypercubes produced by this dominance relation. The new algorithm, called GDE-MOEA, controls the ϵ value that divides objective function space. To validate the performance of GDE-MOEA, the authors employed the same experimental setup of GD-MOEA and this algorithm was also included. From the results, it is clear that GDE-MOEA is a bad option for MOPs similar to DTLZ1 and DTLZ2, since in all cases it was outperformed by the adopted MOEAs. However, with respect

[9]Given $\vec{u}, \vec{v} \in \mathbb{R}^m$ and $\epsilon > 0$, $\vec{u}$ is said to ϵ-dominate $\vec{v}$ (denoted by $\vec{u} \prec_\epsilon \vec{v}$) if for all $i = 1, \ldots, m$ $u_i \leq (v_i + \epsilon)$ and $\exists j \in \{1, \ldots, m\} : u_i < (v_i + \epsilon)$.

to HV, GDE-MOEA was an outstanding optimizer for disconnected problems such as DTLZ7 and WFG2 in which it consistently obtained the best results. In general, GDE-MOEA had better results than GD-MOEA and MOEA/D. Unlike GD-MOEA that was outperformed by HypE, GDE-MOEA had a competitive performance with respect to this HV-based MOEA.

ϵ^+. When Zitzler and Künzli proposed the IBEA framework (see Algorithm 2), they also showed the effectiveness of the binary ϵ^+ indicator in its selection mechanism, giving rise to IBEA$_{\epsilon^+}$ [114]. The experimental scenario was the same as that of IBEA$_{\text{HV}}$'s. However, IBEA$_{\epsilon^+}$ performed significantly better on the problem DTLZ6, which is a degenerated MOP that other MOEAs cannot properly solve because of the generation of numerous weakly dominated solutions.

The Approximation-Guided Evolutionary Multi-objective Optimizer (AGE) [11] employs a greedy environmental selection mechanism that reduces the population size based on the worst contribution to the ϵ^+ indicator.[10] An external archive that stores non-dominated solutions is employed as the reference set. AGE does not need additional parameters, unlike IBEA$_{\epsilon^+}$. Furthermore, the authors proposed a fast method to reduce the number of calculations related to the greedy selection mechanism. AGE was comprehensively tested on four MOPs of the DTLZ test suite, varying the number of objective functions from 2 to 20. Experimental results showed that AGE does not perform well on MOPs having two and three objective functions and it is competitive in MaOPs with respect to SMS-EMOA, IBEA$_{\text{HV}}$, and NSGA-II.

Wagner and Neumann proposed AGE-II [102] that tackles the two main limitations of AGE. First, AGE adopts an unbounded external archive whose update rule is solely based on Pareto dominance, slowing down its execution time. In contrast, AGE-II uses a bounded external archive adopting ϵ dominance in the update rule. However, this introduces the need for a parameter ϵ_{grid} for the ϵ dominance. To improve performance in MOPs with two and three objective functions, AGE-II proposes a crowding distance-based parent selection that aims to maintain a diverse genetic material for the variation operators. Experimental results based on HV showed that AGE-II outperforms its predecessor, AGE, for MOPs in low- and high-dimensional objective space. Additionally, under time constraints, AGE-II is also able to be competitive or even get better results than SMS-EMOA and IBEA$_{\text{HV}}$. AGE-II presented remarkable results for multifrontal problems, i.e., DTLZ1 and DTLZ3.

The Two Archive algorithm 2 (Two_Arch2) [103] is a hybrid MOEA that uses two subpopulations, one dedicated to maintain convergence and the other to preserve diversity. Two_Arch2 was especially designed to tackle MaOPs. To this purpose, the convergence subpopulation is updated based on the ϵ^+ indicator, using the scheme of IBEA$_{\epsilon^+}$. The current solutions of the convergence subpopulation and the non-dominated solutions from the newly created offspring are merged into a single temporary population Q. If Q exceeds its maximum allowable size, then a greedy selection is performed by calculating the fitness values of all solutions, using the equation on Line 5 of Algorithm 2 with $\kappa = 0.05$. The solution with the lowest fitness value is removed and, then, all fitness values are updated. This process continues until Q reaches its maximum size. The other subpopultion of Two_Arch2 aims to maintain diversity by using an update rule based on a $L_{1/m}$ norm, where m is the number of objective functions. Both subpopulation interact to produce a Pareto front approximation with both convergence and diversity properties. Two_Arch2 was tested on MOPs from the DTLZ and WFG benchmarks, varying the number of objective functions from 2 to 20. The performance of Two_Arch2 was compared with respect to IBEA$_{\epsilon^+}$ and AGE-II, outperforming both algorithm in low- and high-dimensional MOPs.

Δ_p. In 2012, Rodríguez Villalobos and Coello Coello [81] proposed to use the Δ_p indicator in a selection mechanism coupled to differential evolution, giving rise to the Δ_p-DDE algorithm.

[10] In the original paper, the authors denoted the ϵ^+ indicator as the α indicator.

The selection method used in this case is a greedy algorithm that removes the worst contributing solutions. Since Δ_p is based on GD_p and IGD_p, the authors decided to give more importance to the latter, because it rewards both convergence and distribution while the former is specifically focused on convergence. Hence, when the individual Δ_p contributions of all solutions are computed, the first criterion to check is the IGD_p contribution. It is worth noting that unlike other IB-MOEAs using a greedy strategy, Δ_p-DDE only needs to compute all contributions once and then it identifies the best μ solutions. Consequently, this reduces its computational cost. Regarding the reference set, it is constructed by fitting the current non-dominated points of the population into a frame formed by the approximations to the ideal and nadir points, uniformly distributing the points using a distance measure. Regarding HV results on the ZDT and DTLZ test suites, Δ_p-DDE produced poor solution sets for discontinuous problems (such as DTLZ7 and ZDT3) and the degenerated DTLZ5 and DTLZ6 problems. However, it had outstanding results when tackling multifrontal problems such as DTLZ1 and DTLZ3. Unfortunately, Δ_p-DDE was not exhaustively tested on MaOPs, and the authors only considered the many-objective version of DTLZ2 in which SMS-EMOA obtained better results than Δ_p-DDE.

In a further paper, Menchaca et al. designed the Δ_p-selection mechanism that was integrated into the Δ_p-MOEA [77]. Unlike Δ_p-DDE, which prefers to use IGD_p for selection, Δ_p-MOEA switches between IGD_p and GD_p to select solutions, depending on which of these two QIs best assesses the population. The GD_p-selection is the same as GDE-MOEA. The IGD_p-based environmental selection is similar to the non-dominated sorting mechanism of NSGA-II but considering the proximity of the solutions to the reference set. Concerning the reference set, it is constructed using the non-dominated solutions and ϵ dominance to assign each solution to a certain hypercube. Δ_p-MOEA was compared to MOEA/D and HypE on the WFG test suite using from three to six objective functions, adopting HV and Δ_p as QIs. Δ_p-MOEA outperforms both MOEAs, only showing poor results for WFG1 whose Pareto front is mixed (i.e., it combines convex and concave shapes). Additionally, Δ_p-MOEA performs better on discontinuous MOPs such as WFG2 where Δ_p-DDE fails. It is worth noting that even Δ_p-MOEA has better results than GD-MOEA and GDE-MOEA.

Discussion. Due to its strong mathematical properties, the hypervolume seems to be the best option to design environmental selection schemes. It has been proved that the maximization of the HV is related to finding the Pareto optimal set [34]. However, its high computational cost is a critical drawback. From Table 1, we observe that none of the HV-ES performs a hierarchical selection; as a matter of fact, all of them are greedy algorithms. The lack of HV-based hierarchical mechanisms is related to the complexity of the hypervolume subset selection problem (HSSP), which is NP-hard [10]. The solution of the HSSP would allow us to find a subset of solutions that maximizes the HV value. However, there is no polynomial-time algorithm able to solve the HSSP unless P = NP. Due to the computational limitations involved, greedy strategies have to be implemented. Nevertheless, Bradstreet et al. [8] claim that a greedy strategy does not always produce the desired results. From the HV-selection mechanisms presented before, the one implemented in iSMS-EMOA and aviSMS-EMOA should be strongly considered. The key aspect of both IB-MOEAs is to exploit the locality property of HV to reduce the number of times the HV-contribution is computed. These algorithms considerably reduce the computational time with respect to the other proposals without sacrificing the quality of the generated approximation in a significant way.

Other indicators have been employed to avoid the drawbacks of the HV. The $R2$ indicator is a remarkable option because of its weak Pareto compliance. MOMBI-II and TS-R2EA are perhaps the best $R2$-based algorithms currently available. However, since all $R2$-based MOEAs need to be supplied a set of convex weight vectors, that forms an $(m-1)$-simplex, to define the utility functions and to maintain diversity. This requirement is indeed its main drawback. As the dimensionality of the objective space increases, the number N of weight vectors increases in a combinatorial fashion, i.e., $N = C_{m-1}^{h+m-1}$, where h is an integer parameter [12, 65]. Maintaining a relatively small N in MaOPs affects the number of intermediate vectors in the simplex. A two-layer set of weight vectors has been used to overtake this last issue [21]. Another important problem related to the set of weight vectors is a possible overspecialization on Pareto fronts that are strongly coupled with these vectors [49]. Regarding ϵ^+, Δ_p, GD, and IGD$^+$, their performance is strongly related to the way in which the reference set is built [47]. Each approach proposes a different way to define the reference set. However, from the experimental results, the method to construct the reference set of IGD$^+$-EMOA II seems the best option currently available, since it is based on a HV-based method, which is less computationally expensive. Its usage allows IGD$^+$-EMOA II to solve problems having complex Pareto front shapes, which directly tackles the problem stated by Ishibuchi et al. [49].

4.2.2 IB-Density Estimation.

HV. In 2007, Beume et al. introduced the S-Metric Selection Evolutionary Multiobjective Optimization Algorithm (SMS-EMOA) [5], which is a steady-state[11] version of NSGA-II, in which the density estimator originally based on crowding distance is replaced by one that removes the least HV-contributing solution. Algorithm 3 provides the general framework of SMS-EMOA. At each generation, the union set $\mathcal{P} \cup \{\vec{a}_{\text{new}}\}$ (where $\vec{a}_{\text{new}}$ is the newly created solution and $\mathcal{P}$ is the population) is categorized in layers or ranks $\{R_1, R_2, \ldots, R_k\}$ using the Pareto dominance relation (see line 5). If R_k has more than one solution, then we need to delete in line 8 the solution $\vec{q}_{\text{worst}}$ that has the least HV contribution (this is the HV-based density estimator). In case R_k has one solution, this one is eliminated, since it is the worst solution regarding Pareto dominance. Due to the mathematical properties of HV, SMS-EMOA can theoretically solve any MOP, producing convergent and uniformly distributed solutions along the Pareto front (although it prefers the Pareto front knee, since solutions in this part of the front have a more substantial HV contribution). An essential aspect of SMS-EMOA is that it can solve MOPs whose Pareto front is degenerated such as DTLZ5, DTLZ6, and WFG3, which are very difficult for other MOEAs, including IB-MOEAs. However, it has some drawbacks. First, when solving MaOPs the number of ranks tends to one, forcing SMS-EMOA to calculate the HV contributions of the whole population, which implies an additional computational cost to the already expensive calculation of HV in high-dimensional objective spaces. Recently, it has been empirically shown that the reference point employed by HV is dependent on the MOP and its geometry [45, 46]. This has a severe impact on the performance of HV-based MOEAs.

Although SMS-EMOA is the most remarkable HV-based MOEA, the underlying idea of its density estimator had been previously presented by Huband et al. in 2003 in their Evolution Strategy with Probabilistic Mutation (ESP) algorithm [43]. The only difference between SMS-EMOA and ESP is that the latter is a $(\mu + \lambda)$-Evolution Strategy (ES), which implies that the HV-based density estimator has to compute much more HV contributions than the HV-based density estimator of SMS-EMOA. This is a critical aspect that avoids the use of ESP and that is why all HV-based proposals employ a steady-state scheme. ESP was compared to a variety of other MOEAs, outperforming them on the ZDT test suite due to the use of its HV-based density estimator.

Igel et al. [44] proposed the Multi-objective Covariance Matrix Adaptation Evolution Strategy (MO-CMA-ES) that uses two types of density estimators: one adopting crowding distance (as NSGA-II) and another one, which is the same as that adopted in SMS-EMOA and ESP. A steady-state MO-CMA-ES was tested on the ZDT test suite and the Fonseca problem (FON) [19], obtaining similar results to SMS-EMOA.

However, Zitzler et al. [110] defined the weighted HV indicator, which allows the user to incorporate preferences through a set of weight vectors (as $R2$-based MOEAs do), preserving the Pareto

[11] It implements a $(\mu + 1)$-selection scheme.

ALGORITHM 3: SMS-EMOA general framework

Output: Pareto front approximation
Randomly initialize population $\mathcal{P}$ of size μ;
while *stopping criterion is not fulfilled* **do**
  Create a new offspring solution $\vec{a}_{\text{new}}$;
  $Q \leftarrow \mathcal{P} \cup \{\vec{a}_{\text{new}}\}$;
  $\{R_1, \ldots, R_k\} \leftarrow$ non-dominated sorting(Q);
  if $|R_k| > 1$ **then**
    $z_i^{\max} \leftarrow \max_{\vec{q} \in Q} f_i(\vec{q}), \forall i = 1, \ldots, m$;
    $\vec{q}_{\text{worst}} \leftarrow arg\,min_{\vec{q} \in R_k} HV(Q, \vec{z}^{\max}) - HV(Q \setminus \{\vec{q}\}, \vec{z}^{\max})$;
  else
    $\vec{q}_{\text{worst}}$ is equal to the sole solution in R_k;
  end
  $\mathcal{P} \leftarrow Q \setminus \{\vec{q}_{\text{worst}}\}$;
end
return $\mathcal{P}$

compliance of the original HV indicator. A density estimator based on this weighted HV was integrated into the NSGA-II framework that performs a $(\mu + \lambda)$-selection scheme, giving rise to the Simple Indicator-based Evolutionary Algorithm (SIBEA). Due to the incorporation of preferences, SIBEA can focus the search on a specific region of the objective space and it can circumvent the bias toward the knee of the Pareto front that is related to traditional HV-based mechanisms. However, SIBEA is restricted to bi-objective MOPs due to the mathematical definition of the weighted HV that makes it computationally expensive. Additionally, due to the $(\mu + \lambda)$-selection scheme, the execution of SIBEA is very time-consuming even for bi-objective MOPs.

Due to the high computational cost of the hypervolume, some authors have proposed its approximation. Bader and Zitzler [3] proposed the Hypervolume Estimation Algorithm for Multiobjective Optimization (HypE). HypE uses Monte Carlo sampling with the aim of approximating the HV contributions. The authors stated that "the main idea is that the actual indicator values are not important, but rather the rankings of solutions induced by the hypervolume indicator" [3]. HypE is a $(\mu + \lambda)$-EA that works under the same framework of NSGA-II. It is worth noting that HypE uses the exact HV contribution for two and three dimensions of the objective space and, in case of MaOPs, it employs the approximation. Additionally, the quality of the solution set is sensitive to the number of samples that the Monte Carlo method adopts. Although the quality of the solutions is a bit lower than those of SMS-EMOA, the computational cost of HypE in MaOPs is considerably lower. HypE was compared to NSGA-II, SPEA2, and IBEA_{HV} on the DTLZ and WFG test suites using 2, 3, 5, 10, 25, and 50 objective functions. According to the hypervolume results, HypE performs better than the adopted MOEAs, which implies that the approximation of the ranks produced by HV allows an MOEA to get good results.

Ishibuchi et al. [52] proposed an HV approximation approach based on the use of achievement scalarizing functions (ASFs) [79]. They decided to use ASFs because: (1) they have shown to be effective when solving MaOPs, and (2) the accuracy and computational load can be adjusted through the number of weight vectors employed. The idea of the approximation is to measure the distance from the reference vector employed by HV to the solution set using the ASFs. This approximation method was incorporated into SMS-EMOA, giving rise to SMS-EMOA-Apr. The authors claim that this new approach drastically decreased the runtime of SMS-EMOA. Furthermore, they observed that the use of an approximation to HV does not severely deteriorate the quality of the solutions produced. However, SMS-EMOA-Apr was only tested in DTLZ1 and DTLZ2, leaving aside MOPs with interesting properties. It is worth noting that the accuracy of this approximation method strongly depends on the number of weight vectors. As the dimensionality of the objective space increases, it would be necessary to provide even more weight vectors than those commonly employed by $R2$-based MOEAs, which will increase the runtime of the algorithm.

In 2015, Jiang et al. [56] presented the Simple and Fast Hypervolume Indicator-based MOEA (FV-MOEA). The main contribution of this work is a new method to update the exact HV contributions of different solutions based on the locality property of HV. The IB-density estimator using this fast HV calculation was incorporated into the NSGA-II framework. Experimental results confirmed the superiority of FV-MOEA regarding HV as well as its lower cost with respect to SMS-EMOA and IBEA_{HV}. The distribution of points is similar to other HV-based MOEAs. A possible drawback of FV-MOEA is that the locality property is well-defined for the two-objective case, while it is not entirely stated for $m > 2$ objectives.

Finally, the Interactive Simple Indicator-based Evolutionary Algorithm (I-SIBEA), introduced by Chugh et al. [17], is an interactive MOEA [78], i.e., it asks the decision maker (DM) to provide preference information throughout the evolutionary process. I-SIBEA uses a density estimator based on the weighted HV to reduce the population size, and it collects data from the decision maker, related to preferred and non-preferred solutions. Since I-SIBEA is an interactive MOEA, it asks for preference information through the search process. Due to this interaction, the solution process of I-SIBEA could be complicated, especially when the decision maker does not have a clear idea of his preferences. However, focusing the search on some areas of the objective space could be beneficial when solving MaOPs because of the reduction of the search space. Additionally, the DM can decide how many times s(he) wants to interact with the algorithm and how many solutions s(he) wants to compare while interacting. Therefore, the DM does not need to compare more solutions than s(he) is able to consider at a time. Unfortunately, the comparative study only included ZDT4, DTLZ1, and DTLZ2 problems using two and three objective functions. These MOPs are not very challenging for most MOEAs.

R2. Brockhoff et al. [13] proposed the $R2$-EMOA that substitutes the HV-based density estimator of SMS-EMOA by one based on the $R2$ indicator. $R2$-EMOA obtained promising results because of the weakly Pareto compliance of the $R2$ indicator and the even distribution of points promoted by the required set of convex weight vectors. Another remarkable feature of $R2$-EMOA is that it is less computationally expensive than SMS-EMOA, allowing it to solve MaOPs at an affordable time. However, the cardinality of the set of weight vectors increases in a combinatorial fashion as the number of objective functions does. Moreover, the performance strongly depends on the utility function that is employed. Each utility function allows $R2$-EMOA to find solutions in different regions of the objective space [79]. The authors performed an empirical analysis of their greedy heuristic strategy, i.e., the R2-based density estimator could produce the μ-optimal distributions associated with the R2 indicator. To this aim, they employed some ZDT and DTLZ problems. The experimental results showed that R2-EMOA produces approximation sets similar to the μ-optimal distributions. However, no comprehensive comparative study is available yet.

***IGD⁺*.** Motivated by the overspecialization of MOEAs using convex weight vectors (as search directions, reference sets or as part of a quality indicator) on certain benchmark problems, Falcón-Cardona and Coello Coello have proposed an MOEA that employs the IGD^+ indicator as its density estimator [31]. Based on an empirical analysis, they found out that an IGD^+-based search could produce similar Pareto front approximations to those of SMS-EMOA. Hence, they proposed the IGD^+ Many-objective Evolutionary Algorithm (IGD^+-MaOEA) that replaces the HV density estimator of SMS-EMOA in Algorithm 3 by the IGD^+ contributions of all solutions in the last dominated rank. Additionally, they introduced a method to reduce the computational cost of computing the

IGD^+ contributions of the entire population. The authors compared IGD^+-MaOEA with respect to IGD^+-EMOA, NSGA-III, MOEA/D, and SMS-EMOA on the DTLZ and $DTLZ^{-1}$ [49] test suites using from three to seven objective functions. The $DTLZ^{-1}$ test suite is a slight modification of the original DTLZ test suite, where all the objective functions of the MOPs are multiplied by -1, producing in some cases MOPs whose Pareto front is not correlated to the simplex that a set of convex weight vectors forms. On the basis of the HV results, IGD^+-MaOEA is very competitive with respect to the adopted MOEAs in the DTLZ test suite and it outperforms all the other MOEAs when tackling the $DTLZ^{-1}$ test problems. On the basis of these results, the authors claimed that IGD^+-MaOEA is a more general many-objective optimizer, since its performance does not depend on the Pareto front shapes.

IGD. In 2013, Denysiuk et al. [23] introduced the Many-objective Differential Evolution with Mutation Restriction (MyO-DEMR) based on the NSGA-II framework. MyO-DEMR replaces the density estimator of NSGA-II by one based on the individual contributions to the IGD indicator. The reference set $\mathcal{Z}$ is equal to the hyperplane that dominates the set of solutions in the rank R_j that makes the population size to exceed μ solutions. After calculating the individual contribution of each solution in R_j, the contributions are sorted in descending order, and the elements that make the population equal to μ are kept. This strategy differs from the one employed by $R2$-EMOA where each time a solution is removed, all the contributions must be recomputed. Hence, this saves computational time. MyO-DEMR showed competitive results with respect to state-of-the-art algorithms, producing well-covered and well-distributed solutions for problems with up to 20 objectives. However, MyO-DEMR was not tested on MOPs having degenerated and discontinuous Pareto fronts. It is worth noting that MyO-DEMR presented remarkable results for multifrontal MOPs such as DTLZ1 and DTLZ3, although the authors mentioned that this behavior was due to the restriction strategy contained in the differential evolution search engine.

Tian et al. [92] proposed an enhanced IGD indicator, called IGD with noncontributing solution detection (IGD-NS). This indicator defines a value on the basis of all noncontributing solutions, which penalizes the original IGD value. Noncontributing solutions are the ones which are avoided to be the nearest neighbors of any point in the required reference set. The authors proposed the MOEA/IGD-NS that employs Pareto dominance as its environmental selection mechanism and a density estimator that removes the worst IGD-NS contributing solutions, i.e., it follows the NSGA-II scheme. MOEA/IGD-NS has an external archive that stores non-dominated solutions, improving their diversity with a scheme similar to that of RVEA, i.e., it clusters solutions in different subregions [15]. This external archive is employed as the reference set of IGD. Evidently, the identification of noncontributing solutions increases the computational cost of MOEA/IGD-NS in comparison to MyO-DEMR. Experimental results on low-dimensional instances of the ZDT and DTLZ test suites showed that MOEA/IGD-NS produces evenly distributed solutions. Nevertheless, the authors claimed that MOEA/IGD-NS could not solve MaOPs, because it is complicated to maintain an external archive with both a good convergence and a good diversity. Hence, the performance of the approach is not scalable.

In 2018, Sun et al. [88] introduced the IGD Indicator-based Many-objective Evolutionary Algorithm (MaOEA/IGD). This IB-MOEA first constructs an ideal version of the Pareto front on the basis of the $(m-1)$-dimensional hyperplance that is shaped by the set of approximated extreme points. This hyperplane is employed to classify the population in three groups based on the Pareto dominance relation: (1) rank R_1 contains points that dominate the hyperplane, (2) rank R_2 has points mutually non-dominated with the hyperplane, and (3) points dominated by the hyperplane belong to rank R_3. Unlike the non-dominated sorting procedure that creates different ranks of solutions, the previous classification only generates three ranks of solutions that are added to the next population whenever the population size is not exceeded. It is worth noting that depending on the rank, a distance between a point and the reference set (the hyperplane) is measured: a negative Euclidean distance, a modified Euclidean distance of IGD$^+$, and an Euclidean distance for $R1$, $R2$, and R_3, respectively. Starting from the rank that exceeds the population, the remaining solutions are selected by transforming the selection problem into a Linear Assignment Problem just as Manoatl and Coello Coello proposed in IGD$^+$-EMOA [71]. The authors were interested in showing the effectiveness of MaOEA/IGD in MaOPs. Thus, they compared it with NSGA-III, MOEA/D, HypE and RVEA on the DTLZ and WFG test suites using 8, 15, and 20 objective functions, adopting the hypervolume indicator. Their experimental results showed that MaOEA/IGD performs better than the other MOEAs on MaOPs with 8 and 20 objective functions and its performance is significantly better for MaOPs DTLZ1, DTLZ7, WFG1, and WFG3, which covers a wide range of Pareto front shapes. However, an important aspect to consider is that its performance mainly depends on the proper construction of the approximated hyperplane. The method to approximate the extreme points that shape the hyperplane is very time-consuming, since it involves m additional approximation sets and it may sometimes fail to produce good points.

As in the case of IGD$^+$-MaOEA whose primary motivation is the overspecialization of MOEAs using convex weight vectors on specific benchmark problems, Tian et al. presented the Adaptive Reference Set-based MOEA (AR-MOEA) [91] as an alternative to deal with a wider variety of Pareto front shapes. AR-MOEA is an improvement of MOEA/IGD-NS, since the latter showed to have a poor performance on MOPs having different Pareto front geometries. Although the indicator IGD-NS solves some problems of IGD when selecting solutions, the former tends to choose non-contributing solutions that are clustered together and too near to a reference point. Consequently, the density estimator based on IGD-NS of AR-MOEA also includes an adaptive reference set that approximates the inner geometry of the Pareto front, using an approach similar to that of RVEA [15]. Hence, AR-MOEA takes advantage of both the adaptive reference set and the IGD-NS-based selection. AR-MOEA follows the framework of NSGA-II, i.e., it uses Pareto dominance as its main selection criterion and, additionally, the IGD-NS density estimator is adopted to delete the worst-contributing solution from the last rank of solutions. AR-MOEA was tested on several test problems that cover a wide range of Pareto front shapes. They used the DTLZ, DTLZ^{-1}, WFG, and MAF test suites adopting 3, 5, and 10 objective functions. Their experimental results showed that AR-MOEA is a more versatile optimizer, since its performance does not depend on the Pareto front shape. For three objective functions, AR-MOEA could rank first in terms of HV values only on DTLZ4, WFG3, and MAF4. In case of MaOPs, its performance was competitive with respect to NSGA-III, RVEA, and MOMBI-II.

Δ_p. The Reference Indicator-based Evolutionary Multi-objective Algorithm (RIB-EMOA) proposed by Zapotecas et al. in 2014 [108] is a steady-state MOEA that adopts Δ_p as its second selection criterion. RIB-EMOA follows the SMS-EMOA framework although it tries to save some computations of Δ_p contributions. Regarding the required reference set, the authors proposed to generalize its construction using Lamé superspheres such that specific Pareto front geometries are approximated, namely, linear, concave and convex shapes. Experimental results showed that RIB-EMOA has some difficulties when solving multifrontal MOPs, namely, DTLZ1 and DTLZ3 as well as MOPs having degenerated Pareto fronts, e.g., DTLZ5 and DTLZ6. Nevertheless, RIB-EMOA presents better results than IGD$^+$-EMOA (see Section 4.2.1), which uses a similar strategy to build the reference set, in MOPs having degenerated and discontinuous Pareto fronts.

Discussion. IB-Density estimators are mechanisms that aim to reduce the population size of MOEAs once the main selection criterion has been applied. Unlike mechanisms such as crowding distance or fitness sharing, the advantages of IB-DEs become clear when solving MaOPs, because the use of quality indicators increases the selection pressure once the Pareto dominance relation cannot properly rank solutions in several layers. It is worth emphasizing that most of the IB-MOEAs in Table 2 are based on the NSGA-II framework (a number of them adopt a steady-state version).

The main issue with HV-based MOEAs is the high computational cost associated with computing exact HV contributions. However, as stated by Ishibuchi et al. [52], the use of an approximation to the HV does not severely deteriorate the quality of the solutions and allows a significant reduction in the computational cost required. However, the approximation of HV is currently an open research area. A remarkable approach within this class is FV-MOEA, which calculates exact HV contributions using an efficient algorithm based on the locality property of HV. FV-MOEA is indeed capable of producing solutions of the same quality as SMS-EMOA but at a considerably lower computational cost. However, scalability is an issue in this case, since for high-dimensional objective spaces the locality property is difficult to interpret.

Regarding the other approaches, they produce competitive results although the Pareto compliance property is not held in the worst case. $R2$-EMOA produces competitive results but it is necessary to analyze its performance in high-dimensional objective spaces. Its main drawbacks are the need for a set of convex weight vectors whose cardinality increases with the number of objective functions as well as a possible overspecialization on certain Pareto fronts due to the method adopted to generate the weight vectors. The performance of RIB-EMOA, MyO-DEMR, and MOEA/IGD-NS is strongly dependent on the construction of the reference set. Hence, it would be useful to analyze the impact that the reference sets have on the performance of these approaches, with the aim of designing an adaptive method that could generalize their use. Two interesting MOEAs within this class are AR-MOEA and IGD$^+$-MaOEA. The motivation of these two IB-MOEAs is to avoid the overspecialization of MOEAs on certain benchmark, i.e., they have traced the path to design more general many-objective optimizers whose performance does not depend on some particular Pareto front shapes. Both algorithms have been tested on different benchmarks that cover a wide range of Pareto front geometries. In the future, it would be interesting to compare both algorithms to determine their advantages and drawbacks.

4.2.3 IB-Archiving.

HV. To the authors' best knowledge, the Lebesgue Archiving Hillclimber (LAHC) was the first IB-Mechanism ever proposed [59]. The operation of the archiver $\mathcal{A}$ is very simple. If we try to add a solution $\vec{u}_{\text{new}}$ to the archive and the archive is full, then the solution from $\mathcal{A} \cup \{\vec{u}_{\text{new}}\}$ having the least contribution to HV, is deleted. This is similar to the operation of an HV-based steady-state MOEA, such as SMS-EMOA (see Algorithm 3), although LAHC operates on an external archive. LAHC was tested on general point sequences that intend to approximate different Pareto front geometries. However, the experimental analysis did not include the implementation of LAHC in an MOEA. In spite of this, it is very likely that some MOEA using LAHC behaves in a similar way to SMS-EMOA, since LAHC operates under a similar principle.

Jiang and Cai [53] introduced the ϵ-MOPSO$^{UD}_{MRV}$ that uses an archive acceptance rule called Minimum Reduced Hypervolume (MRV) that combines HV and ϵ-dominance. The solutions in the archive are placed on the hyperboxes created by the ϵ-dominance mechanism. Considering a hyperbox with more than one solution inside, the update rule will delete the solution having the lowest contribution to the HV. By doing this, convergence and diversity of the archive is promoted. Experimental results showed that the combination of ϵ-dominance and HV in the archive improves both convergence and diversity of the Pareto front approximations. However, ϵ-MOPSO$^{UD}_{MRV}$ was only tested on the ZDT and DTLZ test suites using two and three objective functions.

$\boldsymbol{\Delta_p}$**.** Gerstl et al. [36] introduced the first MOEA based on the Δ_p indicator. This MOEA, called Δ_p-EMOA, is a modified version of the SMS-EMOA (see Section 4.2.2), which is restricted to two-dimensional objective spaces. Δ_p-EMOA adds a Δ_p-based archive to the SMS-EMOA to improve diversity by compensating the distribution bias of the SMS-EMOA's HV-based density estimator. The reference set for Δ_p is constructed from the current population produced by SMS-EMOA, which is linearly interpolated, aiming to distribute solutions evenly. Each time a new solution is considered to be added to the archive, the Δ_p contributions of all solutions, including the new one, are calculated, and the one with the worst value is removed. Unfortunately, the authors did not show an exhaustive performance analysis. However, a clear disadvantage of Δ_p-EMOA is its reference set construction method that relies on linear interpolation of the main population, which makes the MOEA unable to solve MOPs with more than three objective functions.

To scale up Δ_p-EMOA, Trautmann et al. proposed Δ_p-M-EMOA [93], which is able to solve three-objective problems. The mechanism to construct the reference set uses the Multi-Dimensional Scaling (MDS) method to perform a dimensionality reduction from m dimensions to two-objective spaces. The solution having the worst Δ_p contribution within a set of grid points inside the ϵ^+-convex hull of the population in MDS space is removed from the archive. The experimental study presented by the authors considered DTLZ1, DTLZ2, DTLZ3 and the Viennet problem with three objective functions and to assess performance, the Δ_p indicator was employed. From the experimental results, it is clear that Δ_p-M-EMOA produce Pareto front approximations with more uniform solutions than those generated by SMS-EMOA, NSGA-II, and MOEA/D. However, as in the case of the Δ_p-EMOA, the archiving strategy is computationally expensive. It is worth noting, however, that reducing the dimensionality always to two-dimensional objective spaces may lead to the loss of important information, especially when dealing with MaOPs.

Regarding Δ_p-M-EMOA, Rudolph et al. [82] claimed that the sequential generation of reference sets in Δ_p-M-EMOA is highly complex. Hence, they proposed the Δ_p-T-EMOA, which follows the direction of Δ_p-EMOA, but it can solve three-dimensional MOPs using a triangulation technique to generate the reference set. Using this technique, they circumvented the dimensionality reduction of Δ_p-M-EMOA. This archiving strategy is considerably faster than that of Δ_p-M-EMOA. An important issue of Δ_p-T-EMOA is the detection of the border of the Pareto front and its inability to solve MaOPs. Δ_p-T-EMOA was compared to Δ_p-M-EMOA, SMS-EMOA, MOEA/D, and NSGA-II using the same experimental setup as in Δ_p-M-EMOA. From the experimental results, it is clear that the Δ_p-based archive improves the uniformity of solutions, outperforming all the adopted MOEAs. Additionally, the archive update rule of Δ_p-T-EMOA is less computationally expensive than the one of Δ_p-M-EMOA.

Finally, Domínguez et al. [25] proposed two algorithms that can be seen as an improvement of Δ_p-EMOA, Δ_p-M-EMOA, and Δ_p-T-EMOA. For this purpose, the core idea is the use of the Part and Selection Algorithm (PSA). Unlike SMS-EMOA, SMS-EMOA with PSA (SMS-EMOA-DPPSA) adds a Δ_p-based external archive to store convergent and evenly spaced solutions that are not outliers. PSA processes the current population to generate a reference set of a given size. Roughly speaking, PSA is similar to a clustering algorithm, because it partitions the set of solutions according to a dissimilarity function aiming to group alike solutions. Using the reference set described above, the archiver removes the worst Δ_p contributing solutions until the desired population size is reached. As SMS-DPPSA is a variant of SMS-EMOA, it uses a density estimator based on the hypervolume. Hence, it is highly expensive when solving MaOPs. To circumvent this issue, PS-EMOA is introduced. PS-EMOA uses Pareto dominance as its main selection criterion and PSA to get samples of the current archive. The above algorithm is the online version of PS-EMOA. Domínguez et al. also proposed an offline version of PS-EMOA that first executes the online version to store all possible non-dominated solutions in the archive and, after that, the Δ_p archiver is run to prune it, optimizing the Δ_p value. When using DTLZ1, DTLZ2, and DTLZ3, both algorithms were able to produce

Table 4. IB-Mating Selection Mechanisms

Indicator	Algorithm	Method	Comparison	Year	Ref.
R2	R2-IBEA	Binary tournament	Comparison of binary $R2$ value	2013	[96]
IGD	MaOEA/IGD	Binary tournament	Comparison of rank and distance value	2018	[88]
	AR-MOEA	Binary tournament	Comparison of IGD-NS contribution	2018	[91]

"Method" is related to the basic algorithm on which the mechanisms are based on, and "Comparison" is related to the information employed to select solutions.

evenly distributed solutions due to the bias of Δ_p. Unfortunately, SMS-DPPSA and PS-EMOA were only well-suited for MOPs having a maximum of four objective functions.

Discussion. Unlike IB-ESs or IB-DEs, IB-ARs have received little attention from the EMOO community. It is worth noting that some important reasons for this is the computational overhead associated with the use of unbounded archives [58]. Also, and mainly for pragmatic reasons (e.g., to allow a fair comparison with other MOEAs), most approaches adopted bounded archives.

In spite of the nice mathematical properties of HV, its high computational cost has prevented its use in more mechanisms. The only two HV-archivers currently available employ the same mechanism; i.e., they remove the solution associated with the lowest HV-contribution. Therefore, there is no real improvement when considering these archivers. Regarding Δ_p-EMOA and its variants, the main drawback is that they are hard-wired to a specific dimensionality of the objective space, i.e., they are not MOEAs intended for general use. Furthermore, the construction of the reference set in all cases is highly complicated, and they do not offer a clear advantage over other IB-MOEAs that adopt a reference set. Clearly, there is a lot of room for improvement in IB-Archiving techniques.

4.3 IB-Mating Selection

IB-Mating Selection involves the identification of good parent solutions based on quality indicator values. This type of selection mechanisms does not aim to solve or approximate the indicator-based subset selection problem. Instead, this mechanism tries to produce promising offspring solutions to accelerate the evolutionary process. Unfortunately, the EMOO community has not tuned at all its gaze toward these mechanisms. To the authors' best knowledge, there are currently three MOEAs that use IB-Mating Selection. Such methods are summarized in Table 4. It is worth emphasizing that none of the authors has conducted experiments to determine the actual contribution in the search process of these IB-Mating Selection methods. In the following, we will briefly describe the functioning of them that are entirely based on a binary tournament selection scheme.

By adopting a binary tournament selection scheme [19], R2-IBEA iteratively fills its gene pool by selecting those solutions that have a higher value regarding the binary $R2$ indicator. Since the binary $R2$ indicator is weakly Pareto-compliant [96], R2-IBEA always ensures to select solutions that are better in terms of weak Pareto dominance, i.e., the solutions having non-zero contribution. In second place, we have MaOEA/IGD [88] that creates a binary tournament where the comparisons among solutions are made on the basis of their rank and their distance value to the hyperplane (the reference set for the IGD indicator). The first stage is to compare the ranks of solutions, where the lower the rank, the better. In case that both solutions have the same rank, their distance values (negative Euclidean distance, the modified Euclidean distance of IGD$^+$ or Euclidean distance) are compared and the one having the minimum value is chosen. If there is a tie, then a random solution is selected. This IGD-based mating selection scheme aims to choose solutions having a good degree of convergence. Finally, AR-MOEA [91] computes the IGD-NS contribution of those solutions that are selected to compete in the tournament. The one having the larger contribution (the authors called it fitness value) wins the competition and it is added to the gene pool. As in the case of MaOEA/IGD, this mating selection process aims to generate offspring solutions on the basis of the best solutions in terms of convergence.

5 FUTURE RESEARCH DIRECTIONS

In spite of the numerous IB-MOEAs that have been proposed, this is a research area with several potential topics for future research. Some of them are briefly described next:

5.1 Design of Multi-indicator-based MOEAs

Currently, IB-MOEAs are based on a single indicator that imposes a certain search bias originated by its own strengths and weaknesses. Hence, a possible research direction is to propose Multi-indicator-based MOEAs (MIB-MOEAs). The core idea of MIB-MOEAs would be to combine the properties of each indicator-based mechanism to obtain a better global search behavior. Phan and Suzuki [80] were apparently the first to propose a MIB-MOEA (called BIBEA) that boosts existing IB-selection operators, using the AdaBoost algorithm. The proposed multi-indicator selection scheme aims to select the potential parents for crossover. In a further work, Phan et al. [97] proposed BIBEA-P, which improves BIBEA's parent selection scheme by using an ensemble learning method. The authors also proposed a multi-indicator environmental selection mechanism. An issue of both proposals is that they require supervised offline training, using certain MOPs. Hence, apparently, they would not be able to solve all types of MOPs. Unfortunately, the experimental results did not show that the proposals outperformed state-of-the-art MOEAs. Unlike BIBEA and BIBEA-P, which are ensemble methods, the Stochastic Ranking-based Multi-indicator Algorithm (SRA) [62] is an MOEA that aims to balance the search biases of the indicators ϵ^+ and SDE. SRA is a steady-state MOEA that uses the stochastic ranking algorithm as its environmental selection mechanism to sort the population using the two considered indicators as its sorting criteria. After the sorting is done, the worst solution is deleted. The authors showed a comprehensive series of experiments using benchmark problems in low- and high-dimensional objective spaces, comparing their results with those produced by a wide variety of state-of-the-art MOEAs. However, Hernández Gómez and Coello Coello [41] proposed an MOEA, called MOMBI-III, that combines the convergence effect of an $R2$-selection mechanism and a density estimator based on the s-energy indicator for improving diversity. Additionally, the $R2$-selection mechanism employs a hyper-heuristic to select the most suitable utility function for the $R2$ indicator. Their experimental results showed that MOMBI-III outperforms several state-of-the-art MOEAs.

5.2 Use of Hyper-heuristics

Due to the No Free Lunch Theorem [105], IB-MOEAs cannot possibly have a good performance in all types of MOPs. With the aim of reducing the effect of the NFL, hyper-heuristics arise as a good option, because they find from among a pool of low-level heuristics, the one that is more suitable for a certain problem [14]. Hence, a hyper-heuristic could decide which is the most effective indicator-based mechanism depending on the MOP being tackled. To the author's best knowledge, Falcón-Cardona and Coello Coello [30] were the first to propose a hyper-heuristic that selects from a pool of density estimators based on the indicators $R2$, IGD$^+$, ϵ^+, and Δ_p, the most suitable choice for a given problem.

5.3 Parallel IB-MOEAs

We believe that it is possible to take advantage of parallelism in at least two ways. First, by designing parallel IB-mechanisms that reduce the computational cost of IB-MOEAs. In this regard, Hernández Gómez and Coello Coello [42] proposed a parallel version of SMS-EMOA based on the island model, where each island has a micro population. This version can substantially reduce the computational cost of SMS-EMOA without sacrificing the population's quality in a significant way, regarding HV. However, the interactions of subpopulations, using different IB-Mechanisms, in a parallel model (e.g., the island model) could produce new global search behaviors. To the author's best knowledge, no work in this direction has been reported yet.

5.4 IB-MOEAs' Theory

Currently, the understanding of QIs and IB-MOEAs is far from being complete. Regarding QIs, it is necessary to mathematically analyze them not only in an individual fashion but also when they are combined. An open research direction is to mathematically prove if there are other unary Pareto-compliant QIs besides the hypervolume. In case there are no more Pareto-compliant QIs, we have to turn our attention to the design of efficient algorithms for the exact hypervolume computation [104] or, at least, a good approximation of it that can be obtained at a relatively low computational cost [51, 86]. Another remarkable aspect is to study the preference incorporation in QIs as in the case of the weighted HV [1]. Regarding IB-MOEAs, it would be valuable to have a theoretical study that characterizes IB-Selection mechanisms with respect to their speed of convergence, distribution, and spread of solutions. It would also be interesting to propose new selection mechanisms based on existing or on new QIs and investigate their properties from a theoretical point of view. Finally, a theoretical aspect around IB-MOEAs that is worth analyzing is why mechanisms based on non-Pareto-compliant indicators can indeed produce good results, e.g., Δ_p-MOEA or MOEA/IGD-NS. It is clear that from the point of view of quality comparison of MOEAs, Pareto-compliance is required to avoid generating misleading results. However, based on the results of IB-MOEAs using non-Pareto-compliant indicators, convergence of an IB-MOEA is apparently not strictly related to the use of a Pareto-compliant QI. Hence, an interesting topic for future research is to determine if it is really necessary to adopt a Pareto-compliant QI as part of a MOEA to produce high-quality results. If not, then it is clearly important to know the reasons.

6 CONCLUSIONS

In this article, we have presented a comprehensive survey of different indicator-based (IB) MOEAs. Such algorithms transform a multi-objective optimization problem into the optimization of a quality indicator. IB-MOEAs have represented a viable way to solve many-objective optimization problems due to their increase of selection pressure in comparison to Pareto-based MOEAs. We have proposed a taxonomy to classify the IB-Mechanisms of IB-MOEAs currently available in the specialized literature. Our proposed taxonomy considers two main categories: (1) IB-Mating Selection, and (2) IB-Selection, where this last category is further divided into three classes: (a) IB-Environmental Selection, (b) IB-Density Estimation, and (c) IB-Archiving.

Based on our proposed taxonomy, we reviewed several state-of-the-art IB-MOEAs, emphasizing their main advantages and drawbacks. Moreover, we have outlined some future research paths that have not been broadly explored so far and that we believe that could produce significant advances in the design of MOEAs and in our understanding of quality indicators.

ACKNOWLEDGMENT

The second author is on Sabbatical leave from CINVESTAV-IPN, Department of Computer Science, Mexico City, México.

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Received February 2019; revised May 2019; accepted December 2019