FORMULAS FOR THE COMPUTATION OF THE WEIGHTED L_2 DISCREPANCY

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ABSTRACT. For a d-dimensional quasi-Monte Carlo method with n points, calculating the weighted L_2 discrepancy using a formula obtained directly from its definition requires $O(2^{d-1}dn^2)$ operations. Here we give an alternative formula requiring $O(dn^2)$ operations. We also present the first numerical calculations of the weighted L_2 discrepancy. These results give supporting evidence for the 'limiting discrepancy'.

1. INTRODUCTION

Quasi-Monte Carlo methods may be used to approximate the multidimensional integral over $[0,1]^d$ given by

$$I_d(f) = \int_{[0,1]^d} f(\mathbf{t})\, \mathrm{d}\mathbf{t}.$$

In these methods, $I_d(f)$ is approximated by an equal-weight quadrature rule of the form

$$Q_d(f) = \frac{1}{n}\sum_{i=1}^{n} f(\mathbf{t}_i),$$

where the $\mathbf{t}_1, \ldots, \mathbf{t}_n$ are some sample of n points belonging to $[0,1]^d$.

In general, points $\mathbf{t}_1, \ldots, \mathbf{t}_n$ which are uniformly distributed over $[0,1]^d$ tend to give good approximations to the integral $I_d(f)$. Thus it is important to use a point set whose distribution is not too far away from the ideal uniform distribution. One quantity for measuring how close a set of points is to being uniformly distributed is the classical L_2 discrepancy (for example, see [4], [10], and [11]). This is defined in terms of the local discrepancy

$$g(\mathbf{t}) = \frac{1}{n}\Upsilon([0,t_1) \times \cdots \times [0,t_d)) - t_1 t_2 \cdots t_d, \tag{1.1}$$

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where $\mathbf{t} = (t_1, \ldots, t_d)$ and $\Upsilon([0, t_1) \times \cdots \times [0, t_d))$ is the number of points of Q_d that lie in the region $[0, t_1) \times \cdots \times [0, t_d)$. The definition of the classical L_2 discrepancy is then

$$\hat{D}(Q_d) := \left(\int_{[0,1]^d} g^2(\mathbf{t})\, \mathrm{d}\mathbf{t} \right)^{1/2}.$$

It is possible to derive the following explicit formula (see [11]) for $\hat{D}^2(Q_d)$:

$$\hat{D}^2(Q_d) = \left(\frac{1}{3}\right)^d - \frac{2}{n} \sum_{i=1}^{n} \prod_{j=1}^{d} \left(\frac{1}{2} - \frac{t_{i,j}^2}{2} \right) + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=1}^{n} \prod_{j=1}^{d} \left[1 - \max\left(t_{i,j}, t_{k,j}\right)\right], \tag{1.2}$$

where $\mathbf{t}_i = (t_{i,1}, \ldots, t_{i,d})$. In order to calculate this quantity it is clear that most of the work involved is in the double sum on the right-hand side of (1.2). It then follows that a straightforward computation will require $O(dn^2)$ operations. If d is not too large, then the recursive technique of Heinrich in [2] may be used to reduce the number of operations to $O(dn(\log n)^d)$.

As pointed out by Hickernell in [3], $\hat{D}(Q_d)$ does not appear explicitly in a quadrature error formula. For this reason, Hickernell suggested it would be preferable to use a certain variant of the L_2 discrepancy. This variant incorporates the classical L_2 discrepancy of the projections of the quadrature points onto the lower-dimensional faces of $[0,1]^d$ and appears in the well-known Koksma-Hlawka inequality which is discussed further below. This variant of the L_2 discrepancy is a special case of the weighted L_2 discrepancy to be considered here. This weighted L_2 discrepancy was first introduced by Sloan and Woźniakowski in [9].

In order to define the weighted L_2 discrepancy of [9], let $\mathfrak{u}$ be any subset of $\mathcal{D} := \{1, 2, \ldots, d-1, d\}$ and denote its cardinality by $|\mathfrak{u}|$. For the vector $\mathbf{t} \in [0,1]^d$, let $\mathbf{t}_{\mathfrak{u}}$ denote the vector from $[0,1]^{|\mathfrak{u}|}$ containing the components of $\mathbf{t}$ whose indices belong to $\mathfrak{u}$. By $(\mathbf{t}_{\mathfrak{u}}, \mathbf{1})$ we mean the vector from $[0,1]^d$ whose j-th component is t_j if $j \in \mathfrak{u}$ and 1 if $j \notin \mathfrak{u}$. It follows from (1.1) that the local discrepancy at the point $(\mathbf{t}_{\mathfrak{u}}, \mathbf{1})$ given by

$$g(\mathbf{t}_{\mathfrak{u}}, \mathbf{1}) = \frac{1}{n} \Upsilon\bigl(\prod_{j \in \mathfrak{u}} [0, t_j)\bigr) - \prod_{j \in \mathfrak{u}} t_j. \tag{1.3}$$

Now let $\boldsymbol{\gamma} = \{\gamma_j\}$ be a sequence of weights such that

$$\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_j \geq \cdots \geq 0,$$

with the sequence normalized by taking $\gamma_1 = 1$. For any nonempty subset $\mathfrak{u}$ of $\mathcal{D}$, we define

$$\boldsymbol{\gamma}_{\mathfrak{u}} := \prod_{j \in \mathfrak{u}} \gamma_j.$$

It is then shown in [9] that the absolute error $|I_d(f) - Q_d(f)|$ is bounded by

$$|I_d(f) - Q_d(f)| \leq D_{\boldsymbol{\gamma}}(Q_d) \|f\|, \tag{1.4}$$

where

$$D_{\boldsymbol{\gamma}}(Q_d) = \left(\sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} \boldsymbol{\gamma}_{\mathfrak{u}} \int_{[0,1]^{|\mathfrak{u}|}} g^2(\mathbf{t}_{\mathfrak{u}}, \mathbf{1})\, \mathrm{d}\mathbf{t}_{\mathfrak{u}} \right)^{1/2} \tag{1.5}$$

and

$$\|f\| = \left(\sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} \boldsymbol{\gamma}_{\mathfrak{u}}^{-1} \int_{[0,1]^{|\mathfrak{u}|}} \left| \frac{\partial^{|\mathfrak{u}|}}{\partial \mathbf{t}_{\mathfrak{u}}} f(\mathbf{t}_{\mathfrak{u}}, \mathbf{1}) \right|^2 \mathrm{d}\mathbf{t}_{\mathfrak{u}} \right)^{1/2}.$$

The L_2 version of the Koksma-Hlawka inequality [12] is recovered when $\gamma_j = 1$, $1 \leq j \leq d$. Following [9], we call the quantity $D_{\boldsymbol{\gamma}}(Q_d)$ the weighted L_2 discrepancy. This weighted L_2 discrepancy allows the consideration of functions whose behavior with respect to successive variables is diminishing. It also allows the consideration of questions of tractability of quasi-Monte Carlo methods. Further details may be found in [9].

We shall show in the next section that

$$\begin{aligned} D_{\boldsymbol{\gamma}}^2(Q_d) = \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} \Bigg[& \prod_{j \in \mathfrak{u}} \frac{\gamma_j}{3} - \frac{2}{n} \sum_{i=1}^{n} \prod_{j \in \mathfrak{u}} \frac{\gamma_j}{2} \left[1 - t_{i,j}^2\right] \\ & + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=1}^{n} \prod_{j \in \mathfrak{u}} \gamma_j \left[1 - \max\left(t_{i,j}, t_{k,j}\right)\right] \Bigg]. \end{aligned} \tag{1.6}$$

Thus the weighted L_2 discrepancy involves a sum over all nonempty subsets of the set $\mathcal{D} = \{1, 2, \ldots, d\}$. Since there are $\binom{d}{|\mathfrak{u}|}$ subsets of $\mathcal{D}$ having $|\mathfrak{u}|$ elements and the product under the double summation has $|\mathfrak{u}|$ terms, we see that calculation of $D_{\boldsymbol{\gamma}}^2(Q_d)$ by using (1.6) requires $O(\kappa n^2)$ operations, where

$$\kappa = \binom{d}{1} \times 1 + \binom{d}{2} \times 2 + \cdots + \binom{d}{d} \times d = \sum_{m=1}^{d} \binom{d}{m} m = 2^{d-1} d,$$

with the last step following from [1, equation 0.154]. For d large, $2^{d-1} d n^2$ is much greater than dn^2, which is a reason why the classical L_2 discrepancy given by (1.2) has previously been used in calculations rather than the quantity $D_{\boldsymbol{\gamma}}(Q_d)$. This latter quantity is more useful as it appears in the bound given in (1.4), of which the L_2 version of the Koksma-Hlawka inequality is a special case.

In the next section we obtain an alternative formula for the weighted L_2 discrepancy, similar to the one given in (1.2). This means that it may be calculated in $O(dn^2)$ operations rather than the $O(2^{d-1}dn^2)$ required if (1.6) was used. In particular, when $\gamma_j = 1$, $1 \leq j \leq d$, the formula becomes

$$D_{\mathbf{1}}^2(Q_d) = \left(\frac{4}{3}\right)^d - \frac{2}{n} \sum_{i=1}^{n} \prod_{j=1}^{d} \left(\frac{3}{2} - \frac{t_{i,j}^2}{2} \right) + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=1}^{n} \prod_{j=1}^{d} \left[2 - \max\left(t_{i,j}, t_{k,j}\right)\right]. \tag{1.7}$$

This formula was originally obtained by Hickernell in [3] via the theory of reproducing kernels. Though this theory is powerful, the derivation of this particular result is not at all obvious. Here we make use of a simple lemma to prove the general result.

In Section 3, we present some results from the first numerical calculations of the weighted L_2 discrepancy. These calculations were made using three different choices of the sequence $\boldsymbol{\gamma} = \{\gamma_j\}$. These results give evidence for the 'limiting discrepancy' proposed in [9].

2. Discrepancy formulas

The local discrepancy $g(\mathbf{t}_{\mathfrak{u}}, \mathbf{1})$ given in (1.3) may be written as

$$g(\mathbf{t}_{\mathfrak{u}}, \mathbf{1}) = \frac{1}{n} \sum_{i=1}^{n} \prod_{j \in \mathfrak{u}} \mathcal{I}_{t_{i,j} < t_j} - \prod_{j \in \mathfrak{u}} t_j,$$

where $\mathcal{I}_{t_{i,j} < t_j}$ is the indicator function satisfying

$$\mathcal{I}_{t_{i,j} < t_j} = \begin{cases} 1, & \text{if } t_{i,j} < t_j, \\ 0, & \text{if } t_{i,j} \geq t_j. \end{cases}$$

Hence

$$g^2(\mathbf{t}_{\mathfrak{u}}, \mathbf{1}) = \prod_{j \in \mathfrak{u}} t_j^2 - \frac{2}{n} \sum_{i=1}^{n} \prod_{j \in \mathfrak{u}} t_j \mathcal{I}_{t_{i,j} < t_j} + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=1}^{n} \prod_{j \in \mathfrak{u}} \mathcal{I}_{t_{i,j} < t_j} \mathcal{I}_{t_{k,j} < t_j}.$$

Since

$$\int_0^1 t_j^2 \, \mathrm{d}t_j = \frac{1}{3}, \qquad \int_0^1 t_j \mathcal{I}_{t_{i,j} < t_j} \, \mathrm{d}t_j = \int_{t_{i,j}}^1 t_j \, \mathrm{d}t_j = \frac{1}{2} \left[1 - t_{i,j}^2 \right],$$

and

$$\int_0^1 \mathcal{I}_{t_{i,j} < t_j} \mathcal{I}_{t_{k,j} < t_j} \, \mathrm{d}t_j = \int_{\max(t_{i,j}, t_{k,j})}^1 1 \, \mathrm{d}t_j = 1 - \max\left(t_{i,j}, t_{k,j}\right),$$

it then follows from the expression for $D_{\boldsymbol{\gamma}}(Q_d)$ given in (1.5) that

$$
\begin{aligned}
D_{\boldsymbol{\gamma}}^2(Q_d) &= \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} \boldsymbol{\gamma}_{\mathfrak{u}} \left[\left(\frac{1}{3}\right)^{|\mathfrak{u}|} - \frac{2}{n} \sum_{i=1}^{n} \prod_{j \in \mathfrak{u}} \frac{1}{2} [1 - t_{i,j}^2] \right. \\
&\qquad \left. + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=1}^{n} \prod_{j \in \mathfrak{u}} [1 - \max(t_{i,j}, t_{k,j})] \right] \\
&= \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} \left[\prod_{j \in \mathfrak{u}} \frac{\gamma_j}{3} - \frac{2}{n} \sum_{i=1}^{n} \prod_{j \in \mathfrak{u}} \frac{\gamma_j}{2} [1 - t_{i,j}^2] \right. \\
&\qquad \left. + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=1}^{n} \prod_{j \in \mathfrak{u}} \gamma_j [1 - \max(t_{i,j}, t_{k,j})] \right].
\end{aligned}
\tag{2.1}
$$

As pointed out in the previous section, calculation of $D_{\boldsymbol{\gamma}}^2(Q_d)$ by using the above formula requires $O(2^{d-1} d n^2)$ operations. An alternative formula for $D_{\boldsymbol{\gamma}}^2(Q_d)$ which is more suited to computations is given in the following theorem.

Theorem 2.1. *We have*

$$
\begin{aligned}
D_{\boldsymbol{\gamma}}^2(Q_d) &= \prod_{j=1}^{d} \left(1 + \frac{\gamma_j}{3}\right) - \frac{2}{n} \sum_{i=1}^{n} \prod_{j=1}^{d} \left(1 + \frac{\gamma_j}{2} [1 - t_{i,j}^2]\right) \\
&\quad + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=1}^{n} \prod_{j=1}^{d} \left(1 + \gamma_j [1 - \max(t_{i,j}, t_{k,j})]\right).
\end{aligned}
\tag{2.2}
$$

The proof of this result is based on the following lemma.

Lemma 2.2. *For numbers $a_1, \ldots, a_d$ and $\mathcal{D} = \{1, 2, \ldots, d\}$, we have*

$$\prod_{j=1}^{d} (1 + a_j) = 1 + \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} \prod_{j \in \mathfrak{u}} a_j.$$

Proof. The result is nothing more than an expansion of a product and presumably is well-known. However, we give a short proof for completeness. A longer proof by induction is also possible.

Expansion of the product $\prod_{j=1}^{d} (1 + a_j)$ yields 2^d terms. Each of these terms is made up of a product of i of the a_j and $(d - i)$ 1's, that is, each of the terms consists of a product of i of the a_j with i running from $i = 0$ (the term in this case is just 1) to $i = d$. All these terms are distinct and for each i there are $\binom{d}{i}$ such terms. It is then clear that the 2^d terms in the expansion are the same as the 2^d terms in

$$1 + \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} \prod_{j \in \mathfrak{u}} a_j,$$

thus completing the proof. □

We remark that Lemma 2.2 has other applications. For example, when $\gamma_j = 1$, $1 \le j \le d$, the lemma may be used to prove that the square of the modified L_2 discrepancy proposed in [6] may be written as

$$D_{\mathbf{1}}^2(Q_d) - \left(\frac{5}{4}\right)^d + \frac{2}{n}\sum_{i=1}^{n}\prod_{j=1}^{d}\left(\frac{3}{2} - \frac{t_{i,j}}{2}\right) - \frac{1}{n^2}\sum_{i=1}^{n}\sum_{k=1}^{n}\prod_{j=1}^{d}\left[2 - t_{i,j} - t_{k,j} + t_{i,j}t_{k,j}\right],$$

where $D_{\mathbf{1}}^2(Q_d)$ is given by (1.7).

We now proceed to prove Theorem 2.1. By using Lemma 2.2, the three terms on the right-hand side of (2.2) become

$$\prod_{j=1}^{d}\left(1 + \frac{\gamma_j}{3}\right) = 1 + \sum_{\emptyset \ne \mathfrak{u} \subseteq \mathcal{D}} \prod_{j \in \mathfrak{u}} \frac{\gamma_j}{3}, \tag{2.3}$$

$$\begin{aligned} -\frac{2}{n}\sum_{i=1}^{n}\prod_{j=1}^{d}\left(1 + \frac{\gamma_j}{2}\left[1 - t_{i,j}^2\right]\right) &= -\frac{2}{n}\sum_{i=1}^{n}\left[1 + \sum_{\emptyset \ne \mathfrak{u} \subseteq \mathcal{D}} \prod_{j \in \mathfrak{u}} \frac{\gamma_j}{2}\left[1 - t_{i,j}^2\right]\right] \\ &= -2 - \sum_{\emptyset \ne \mathfrak{u} \subseteq \mathcal{D}}\left(\frac{2}{n}\sum_{i=1}^{n}\prod_{j \in \mathfrak{u}} \frac{\gamma_j}{2}\left[1 - t_{i,j}^2\right]\right), \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} &\frac{1}{n^2}\sum_{i=1}^{n}\sum_{k=1}^{n}\prod_{j=1}^{d}\left(1 + \gamma_j\left[1 - \max\left(t_{i,j}, t_{k,j}\right)\right]\right) \\ &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{k=1}^{n}\left[1 + \sum_{\emptyset \ne \mathfrak{u} \subseteq \mathcal{D}} \prod_{j \in \mathfrak{u}} \gamma_j\left[1 - \max\left(t_{i,j}, t_{k,j}\right)\right]\right] \\ &= 1 + \sum_{\emptyset \ne \mathfrak{u} \subseteq \mathcal{D}}\left(\frac{1}{n^2}\sum_{i=1}^{n}\sum_{k=1}^{n}\prod_{j \in \mathfrak{u}} \gamma_j\left[1 - \max\left(t_{i,j}, t_{k,j}\right)\right]\right). \end{aligned} \tag{2.5}$$

By adding (2.3), (2.4), and (2.5) together, we see that the formulas for $D_{\gamma}^2(Q_d)$ given in (2.1) and (2.2) are equivalent.

We remark that for computational efficiency, it is better to rewrite (2.2) in the form

$$D_{\gamma}^2(Q_d) = \prod_{j=1}^{d}\left(1+\frac{\gamma_j}{3}\right) - \frac{2}{n}\sum_{i=1}^{n}\prod_{j=1}^{d}\left(1+\frac{\gamma_j}{2}\left[1-t_{i,j}^2\right]\right) + \frac{1}{n^2}\sum_{i=1}^{n}\prod_{j=1}^{d}\left(1+\gamma_j\left[1-t_{i,j}\right]\right) + \frac{2}{n^2}\sum_{i=1}^{n-1}\sum_{k=i+1}^{n}\prod_{j=1}^{d}\left(1+\gamma_j\left[1-\max\left(t_{i,j},t_{k,j}\right)\right]\right).$$

3. Numerical results

In [9] it is proved that if n is fixed, then the limit

$$\lim_{d\to\infty} D_{\boldsymbol{\gamma}}(Q_d)$$

exists if and only if the series

$$\sum_{j=1}^{\infty}\gamma_j$$

is convergent. In this result, the implicit assumption is that if $d < d'$, then for each quadrature point $\mathbf{t}_i$ of Q_d, there is a quadrature point $\mathbf{t}'_{i'}$ of $Q_{d'}$ such that the components of $\mathbf{t}'_{i'}$ include the d components of Q_d.

We now present some numerical results which show this limiting discrepancy. We fix $n = 5003$ and present results for the three different choices of the sequence $\boldsymbol{\gamma}$ given by

$$\text{(i) } \gamma_j = 1, \quad \text{(ii) } \gamma_j = \frac{1}{j}, \quad \text{(iii) } \gamma_j = \frac{1}{2^{j-1}}. \tag{3.1}$$

The results in Tables 1 to 3 are obtained by choosing the points $\mathbf{t}_i$ in Q_d randomly from $[0,1]^d$ with d running from 5 to 100 in steps of 5. These points were generated using the random number generator found in [7] with starting seed 123456. The quadrature points $\mathbf{t}_i$ for $d = 5$ were generated first. For $d = 10$, an extra five random components in $[0,1]$ were added to the end of each $\mathbf{t}_i$ used for $d = 5$ to give quadrature points with ten components. In turn, for $d = 15$, an extra five random components in $[0,1]$ were added to the end of each quadrature point used for $d = 10$ to give quadrature points with 15 components etc. This procedure was adopted to ensure that the implicit assumption mentioned above was satisfied. The computer program was restarted for each choice of $\boldsymbol{\gamma}$, that is, the quadrature rules Q_d used to produce the results in each of Tables 1 to 3 were the same.

Tables 4 to 6 contain the results obtained when Q_d is a number-theoretic rule with quadrature points given by

$$\mathbf{t}_i = \left\{\frac{i\mathbf{z}}{n}\right\}, \quad 1 \le i \le n,$$

where $n = 5003$, the braces around a vector indicate that we take the fractional part of each component, and $\mathbf{z}$ is a well-chosen integer vector. Further details about these number-theoretic rules may be found in [5] or [8]. Here we take $\mathbf{z}$ to be of the form

$$\mathbf{z} = (1, \ell, \ell^2, \ldots, \ell^{d-1}), \quad 1 \le \ell \le n-1.$$

As explained in [8], there are theoretical reasons for using a vector $\mathbf{z}$ of this form. In the results presented here, ℓ was taken to be 780 as this value minimizes the quantity $D^2_{\mathbf{1}}(Q_5)$ given by (1.7) with $d = 5$.

In Table 5 a large jump in the value of $D_{\boldsymbol{\gamma}}(Q_d)$ from $d = 35$ to $d = 40$ is observed. However, we have no explanation for this behavior. Except for this observation, the results given in the tables are what one would expect from the theory. For the sequences (i) and (ii) given in (3.1), the corresponding series are not convergent and the results in Tables 1, 2, 4, and 5 indicate that the values of $D_{\boldsymbol{\gamma}}(Q_d)$ are not converging. On the other hand, the corresponding series for the sequence (iii) in (3.1) is convergent and one can see the rapid convergence of the values of $D_{\boldsymbol{\gamma}}(Q_d)$ in Tables 3 and 6.

In Table 4, we observe that for large s (say, $s \ge 60$), the values of $D_{\boldsymbol{\gamma}}(Q_d)$ are approximately $2^{s/2}/n$. A future work will show that this is not totally unexpected.

In Tables 7 to 10 we set $d = 20$ and present results which show how the weighted L_2 discrepancy varies with n. We give results for the sequences $\boldsymbol{\gamma}$ with $\gamma_j = 1$ and $\gamma_j = 1/2^{j-1}$. The results using random points are given in Tables 7 and 8. For the sequence $\gamma_j = 1$ in Table 7, we observe that the behavior of $D_{\boldsymbol{\gamma}}(Q_d)$ is approximately $O(n^{-1/2})$. This is consistent with the known result that the expected value of $D_{\boldsymbol{\gamma}}(Q_d)$ for random points is $O(n^{-1/2})$ (see [3]). The last two tables contain the results obtained using the number-theoretic rule described above.

TABLE 1. Values of $D_{\boldsymbol{\gamma}}(Q_d)$ using random points with $n = 5003$ and $\gamma_j = 1$

d	$100 \times D_{\boldsymbol{\gamma}}(Q_d)$	d	$100 \times D_{\boldsymbol{\gamma}}(Q_d)$
5	2.67699	55	9.82096E+04
10	9.31336	60	2.70500E+05
15	26.8156	65	7.46610E+05
20	79.4829	70	2.06965E+06
25	221.427	75	5.77043E+06
30	608.561	80	1.58571E+07
35	1680.61	85	4.27465E+07
40	4661.24	90	1.16601E+08
45	12854.8	95	3.23477E+08
50	35602.6	100	8.80015E+08

TABLE 2. Values of $D_{\boldsymbol{\gamma}}(Q_d)$ using random points with $n = 5003$ and $\gamma_j = 1/j$

d	$100 \times D_{\boldsymbol{\gamma}}(Q_d)$	d	$100 \times D_{\boldsymbol{\gamma}}(Q_d)$
5	1.09624	55	2.90431
10	1.57817	60	3.00775
15	1.82282	65	3.08877
20	2.04874	70	3.16541
25	2.22472	75	3.22537
30	2.35526	80	3.28967
35	2.50009	85	3.35646
40	2.63006	90	3.42245
45	2.73197	95	3.48425
50	2.82262	100	3.53833

TABLE 3. Values of $D_{\boldsymbol{\gamma}}(Q_d)$ using random points with $n = 5003$ and $\gamma_j = 1/2^{j-1}$

d	$100 \times D_{\boldsymbol{\gamma}}(Q_d)$	d	$100 \times D_{\boldsymbol{\gamma}}(Q_d)$
5	0.888778	55	0.944644
10	0.943660	60	0.944644
15	0.944603	65	0.944644
20	0.944643	70	0.944644
25	0.944644	75	0.944644
30	0.944644	80	0.944644
35	0.944644	85	0.944644
40	0.944644	90	0.944644
45	0.944644	95	0.944644
50	0.944644	100	0.944644

TABLE 4. Values of $D_{\boldsymbol{\gamma}}(Q_d)$ using a number-theoretic rule with $n = 5003$ and $\gamma_j = 1$

d	$100 \times D_{\boldsymbol{\gamma}}(Q_d)$	d	$100 \times D_{\boldsymbol{\gamma}}(Q_d)$
5	0.186375	55	3.79531E+06
10	2.28046	60	2.14638E+07
15	12.3828	65	1.21410E+08
20	48.4456	70	6.86786E+08
25	186.602	75	3.88503E+09
30	797.763	80	2.19771E+10
35	3951.36	85	1.24321E+11
40	21356.1	90	7.03265E+11
45	119166	95	3.97827E+12
50	671617	100	2.25045E+13

TABLE 5. Values of $D_{\boldsymbol{\gamma}}(Q_d)$ using a number-theoretic rule with $n = 5003$ and $\gamma_j = 1/j$

d	$100 \times D_{\boldsymbol{\gamma}}(Q_d)$	d	$100 \times D_{\boldsymbol{\gamma}}(Q_d)$
5	0.0523464	55	0.896692
10	0.147287	60	0.951598
15	0.228291	65	0.993283
20	0.302085	70	1.03058
25	0.359615	75	1.06742
30	0.429666	80	1.10284
35	0.487779	85	1.13431
40	0.693758	90	1.16581
45	0.776233	95	1.20453
50	0.841426	100	1.23472

TABLE 6. Values of $D_\gamma(Q_d)$ using a number-theoretic rule with $n = 5003$ and $\gamma_j = 1/2^{j-1}$

d	$100 \times D_\gamma(Q_d)$	d	$100 \times D_\gamma(Q_d)$
5	0.0376439	55	0.0476619
10	0.0474087	60	0.0476619
15	0.0476400	65	0.0476619
20	0.0476616	70	0.0476619
25	0.0476618	75	0.0476619
30	0.0476618	80	0.0476619
35	0.0476619	85	0.0476619
40	0.0476619	90	0.0476619
45	0.0476619	95	0.0476619
50	0.0476619	100	0.0476619

TABLE 7. Values of $D_\gamma(Q_d)$ using random points with $d = 20$ and $\gamma_j = 1$

n	$100 \times D_\gamma(Q_d)$	n	$100 \times D_\gamma(Q_d)$
619	222.867	20011	40.0411
1249	151.591	40009	27.0012
2503	106.006	80021	19.4693
5003	75.8419	160001	13.0357
10007	54.1609		

TABLE 8. Values of $D_\gamma(Q_d)$ using random points with $d = 20$ and $\gamma_j = 1/2^{j-1}$

n	$100 \times D_\gamma(Q_d)$	n	$100 \times D_\gamma(Q_d)$
619	2.17935	20011	0.622402
1249	1.91245	40009	0.415896
2503	1.45587	80021	0.273317
5003	1.03614	160001	0.144760
10007	0.777481		

TABLE 9. Values of $D_\gamma(Q_d)$ using a number-theoretic rule with $d = 20$ and $\gamma_j = 1$

n	$100 \times D_\gamma(Q_d)$	n	$100 \times D_\gamma(Q_d)$
619	264.224	20011	47.1519
1249	136.081	40009	19.2840
2503	93.7432	80021	8.86261
5003	48.4456	160001	7.02539
10007	40.7503		

TABLE 10. Values of $D_{\gamma}(Q_d)$ using a number-theoretic rule with $d = 20$ and $\gamma_j = 1/2^{j-1}$

n	$100 \times D_{\gamma}(Q_d)$	n	$100 \times D_{\gamma}(Q_d)$
619	0.506262	20011	0.0363918
1249	0.279360	40009	0.0186478
2503	0.192500	80021	0.0178260
5003	0.0476616	160001	0.0135685
10007	0.117992		

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