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An approximate strong KKT condition for multiobjective optimization

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Abstract
In this paper, we introduce a sequential approximate strong Karush–Kuhn–Tucker (ASKKT) condition for a multiobjective optimization problem with inequality constraints. We show that each local efficient solution satisfies the ASKKT condition, but weakly efficient solutions may not satisfy it. Subsequently, we use a so-called cone-continuity regularity (CCR) condition to guarantee that the limit of an ASKKT sequence converges to an SKKT point. Finally, under the appropriate assumptions, we show that the ASKKT condition is also a sufficient condition of properly efficient points for convex multiobjective optimization problems.

Keywords Multiobjective optimization · Strong KKT conditions · Approximate KKT conditions · Regularity conditions · Properly efficient solutions

Mathematics Subject Classification 90C29 · 90C46

1 Introduction

A necessary optimality condition plays a crucial role in the development of algorithms to solve nonlinear optimization problems. There is no doubt that the most popular necessary optimality condition is Karush–Kuhn–Tucker (KKT) conditions, which are extensively studied both for scalar problems and for vector/multiobjective problems. When a suitable constrain qualification is satisfied, we can obtain that a KKT condition holds.

There are several types of KKT conditions in the case of multiobjective optimization (see, e.g., Bigi and Pappalardo 1999; Maciel et al. 2009). One of them is a strong KKT

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(SKKT) condition, which demands that all Lagrange multipliers associated with the objective functions are positive. That is to say, each Lagrange multiplier associated with the objective functions has a role in such a necessary condition of optimality. However, as realized by Maeda (1994) that the SKKT may not be satisfied at an efficient point even when the linear independence constrain qualification holds. To guarantee the fulfillment of SKKT, some additional regularity conditions involving the objectives need to be assumed (Maeda 1994; Maciel et al. 2009).

A sequential approximate KKT (AKKT) condition is another kind of necessary optimality conditions. Andreani et al. (2011) proved that every local minimum point of a smooth constrained optimization problem satisfies the AKKT condition. It seems that the AKKT is better understood in some practical algorithms than the KKT. In fact, as far as we know, the sequential AKKT condition has been widely used to define the stopping criteria of many practical constrained optimization algorithms, for instance, the SQP method of Qi and Wei (2000), the interior-point method of Chen and Goldfarb (2006), and the augmented Lagrangian algorithms (Birgin and Martínez 2014). On the other hand, to guarantee that AKKT sequences generated by practical algorithms converge to KKT points, strict constraint qualifications have been involved such as constant positive linear dependence condition (Qi and Wei 2000), constant positive generator condition (Andreani et al. 2012) and cone-continuity property (Andreani et al. 2016).

Recently, Giorgi et al. (2016) extended the AKKT condition to smooth multiobjective optimization and proved that such approximate optimality condition is a necessary one of weakly efficient solutions without any constraint qualification. They also proved that a limit point of an AKKT sequence is also a KKT point if a quasi-normality constraint qualification holds. However, in many practical multiobjective problems, one pays more attention on investigating an efficient solution, which has better properties than weakly efficient solution, and the SKKT condition.

In the current paper, we study a smooth multiobjective optimization with inequality constraints and introduce an approximate strong KKT (ASKKT) condition. We show that each local efficient point satisfies such optimality condition and the limit point of an ASKKT sequence is an SKKT point, provided a cone-continuity regularity (CCR) condition holds. Moreover, under suitable hypotheses, we prove that the ASKKT condition is sufficient for a properly efficient solution of convex multiobjective optimization.

The rest of this paper is organized as follows. Section 2 contains some basic definitions and notations needed throughout this work. In Sect. 3, we introduce a new sequential AKKT condition. Section 4 is devoted to analyzing the consequence properties of such AKKT methods. In Sect. 5, a sufficient AKKT sequential optimality condition is achieved for proper efficiency. Finally, we show a numerical example in Sect. 6.

2 Preliminaries

Recall that, for a set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, the sequential Painlevé–Kuratowski outer limit of F as $x \to \bar{x}$ is denoted by

$$\operatorname*{Lim\,sup}_{x\to\bar{x}} F(x) := \left\{ y \in \mathbb{R}^m \mid \exists x^k \to x, \exists y^k \to y \text{ with } y^k \in F(x^k), \forall k \in \mathbb{N} \right\}$$

and the inner limit by

$$\operatorname*{Lim\,inf}_{x\to\bar{x}} F(x) := \left\{ y \in \mathbb{R}^m \mid \forall x^k \to x, \exists y^k \to y \text{ with } y^k \in F(x^k), \forall k \in \mathbb{N} \right\}.$$

We say that the set-valued mapping F is outer semicontinuous at $\bar{x}$ if

$$\operatorname*{Lim\,sup}_{x\to\bar{x}} F(x) \subset F(\bar{x}),$$

but inner semicontinuous at $\bar{x}$ if

$$F(\bar{x}) \subset \operatorname*{Lim\,inf}_{x\to\bar{x}} F(x).$$

If the set-valued mapping F is both outer semicontinuous and inner semicontinuous at $\bar{x}$, we say that F is continuous at $\bar{x}$.

Given a nonempty set $\Omega \subset \mathbb{R}^n$, the polar cone of Ω is denoted as:

$$\Omega^* := \left\{ v \in \mathbb{R}^n \mid \langle v, x\rangle \leq 0, \forall x \in \Omega \right\}.$$

The Bouligand–Severi tangent cone to Ω at $\bar{x} \in \Omega$ is defined by taking the sequential Painlevé–Kuratowski upper limit as:

$$T_\Omega(\bar{x}) := \operatorname*{Lim\,sup}_{t\downarrow 0} \frac{\Omega - \bar{x}}{t} = \left\{ w \in \mathbb{R}^n \mid \exists t_k \downarrow 0, w^k \to w \text{ with } \bar{x} + t_k w^k \in \Omega, \forall k \in \mathbb{N} \right\}.$$

We denote $a_+ := \max\{0, a\}$ and $a_+^2 := (a_+)^2$ for $a \in \mathbb{R}$. $\|\cdot\|$ denotes the Euclidean distance. The nonnegative orthant in $\mathbb{R}^n$ is signified by $\mathbb{R}^n_+$. Formally, given two vectors $y, z \in \mathbb{R}^n$, by $y \leq z$, we mean $y_i \leq z_i$ for $i = 1, \ldots, n$; by $y < z$, we mean $y_i < z_i$ for $i = 1, \ldots, n$. In this paper, we consider the optimization problem of the form

$$\min_{\mathbb{R}^p_+} f(x) := (f_1(x), \ldots, f_p(x)) \text{ subject to } g_j(x) \leq 0, \; j = 1, \ldots, m, \tag{1}$$

where $f_i (i = 1, \ldots, p)$ and $g_j (j = 1, \ldots, m) : \mathbb{R}^n \to \mathbb{R}$ are always assumed to be continuously differentiable unless otherwise indicated. Set $I := \{1, \ldots, p\}$ and $J := \{1, \ldots, m\}$. We denote the feasible set of problem (1) by $X := \{x \in \mathbb{R}^n \mid g_j(x) \leq 0, j \in J\}$. The set of active indexes at a point $x \in X$ is given by $A(x) := \{j \in J \mid g_j(x) = 0\}$.

Definition 2.1 The (weak) KKT condition for problem (1) is said to hold at a feasible point $x^0 \in X$ if there exist vectors $\lambda \in \mathbb{R}^p_+$ and $\mu \in \mathbb{R}^m_+$ such that

$$\text{(K1)} \sum_{i=1}^{p} \lambda_i \nabla f_i(x^0) + \sum_{j\in A(x^0)} \mu_j \nabla g_j(x^0) = 0,$$

$$\text{(K2)}\ \lambda \neq 0,$$

$$\text{(K3)}\ g_j(x^0) < 0 \Rightarrow \mu_j = 0, \; j = 1, \ldots, m.$$

We say that a point $x^0 \in X$ is an efficient (resp. weakly efficient) solution of problem (1) if there is no other $x \in X$ such that $f(x) \leq f(x^0)$, $f(x) \neq f(x^0)$ (resp. $f(x) < f(x^0)$). We say that x^0 is a local efficient (resp. local weakly efficient) solution to problem (1) if x^0 is an efficient (resp. weakly efficient) solution in $V(x^0) \cap X$, where $V(x^0)$ is some neighborhood of x^0.

Andreani et al. (2011) proved an interesting result that every local minimum point of a smooth constrained optimization problem fulfills an approximate KKT condition. This optimality condition gives a good property for local minima without constraint qualifications, and it can be used to define the stopping criteria of many practical nonlinear programming algorithms (Qi and Wei 2000; Birgin and Martínez 2014; Chen and Goldfarb 2006). Recently, Giorgi et al. (2016) extended this approximate KKT condition to a smooth multiobjective optimization problem.

Definition 2.2 The approximate KKT condition (AKKT) for problem (1) is said to hold at a feasible point $x^0 \in X$ (x^0 is called an AKKT point) if there exist sequences $\{x^k\} \subset \mathbb{R}^n$ ($\{x^k\}$ is called an AKKT sequence), $\{\lambda^k\} \subset \mathbb{R}^p_+$ and $\{\mu^k\} \subset \mathbb{R}^m_+$ such that

$$\text{(A0) } x^k \to x^0,$$

$$\text{(A1) } \sum_{i=1}^{p} \lambda_i^k \nabla f_i(x^k) + \sum_{j=1}^{m} \mu_j^k \nabla g_j(x^k) \to 0,$$

$$\text{(A2) } \sum_{i=1}^{p} \lambda_i^k = 1,$$

$$\text{(A3) } g_j(x^0) < 0 \;\Rightarrow\; \mu_j^k = 0 \text{ for sufficiently large} k,\; j = 1, \ldots, m.$$

It was proved by Giorgi et al. (2016) that every local weakly efficient point must be an AKKT point. Let us observe that the point sequence $\{x^k\}$ is not required to be feasible and the multiplier sequence $\{\lambda^k\}$ is always non-null and bounded. Actually, we can further assume that the multiplier sequence $\{\lambda^k\}$ satisfies the following condition:

$$\sum_{i=1}^{p} \lambda_i^k = 1, \quad \text{and} \quad \lambda_i^k > 0,\; i = 1, \ldots, p. \tag{2}$$

Indeed, for any sequence $\{(x^k, \lambda^k, \mu^k)\}$ satisfying AKKT condition $(A0)$–$(A3)$, there is always a sequence $\varepsilon_k \downarrow 0$ such that

$$\sum_{i=1}^{p} (\lambda_i^k + \varepsilon_k) \nabla f_i(x^k) + \sum_{j=1}^{m} \mu_j^k \nabla g_j(x^k) \to 0$$

(choosing sufficiently small ε_k if necessary). By setting

$$\eta_k := \sum_{i=1}^{p} (\lambda_i^k + \varepsilon_k), \quad \hat{\lambda}_i^k := \frac{\lambda_i^k + \varepsilon_k}{\eta_k},\; i = 1, \ldots, p, \quad \hat{\mu}_j^k := \frac{\mu_j^k}{\eta_k},\; j = 1, \ldots, m,$$

the sequence $\{(x^k, \hat{\lambda}^k, \hat{\mu}^k)\}$ will verify (A0)–(A1), (A3) and (2).

3 An approximate strong KKT condition

Maeda (1994) introduced a strong KKT condition for differentiable multiobjective problems. This optimality condition demands that Lagrange multipliers associated with the objective functions all are positive in KKT condition. In view of this, it is a natural way for us to introduce an approximate strong KKT condition.

Definition 3.1 An approximate strong KKT condition (ASKKT) for problem (1) is said to hold at a point $x^0 \in X$ (x^0 is called an ASKKT point) if there exist sequences $\{x^k\} \subset \mathbb{R}^n$ ($\{x^k\}$ is called an ASKKT sequence), $\{\lambda^k\} \subset \mathbb{R}^p_+$ and $\{\mu^k\} \subset \mathbb{R}^m_+$ such that

$$\text{(A0)}\ x^k \to x^0,$$

$$\text{(A1)}\ \sum_{i=1}^{p} \lambda_i^k \nabla f_i(x^k) + \sum_{j=1}^{m} \mu_j^k \nabla g_j(x^k) \to 0,$$

$$\text{(A2}'\text{)}\ \lambda_i^k \geq 1,\ i = 1, \ldots, p,$$

$$\text{(A3)}\ g_j(x^0) < 0\ \Rightarrow\ \mu_j^k = 0 \text{ for sufficiently large } k,\ \ j = 1, \ldots, m.$$

It is obvious from (A2′) that λ_i^k does not require an upper bound in the definition of ASKKT. Thus, the existence of the limitation of the multipliers associated with the different objective functions is not necessary for verifying a feasible point to be an ASKKT point. To explain this assertion, we provide a simple example.

Example 3.1 Consider problem (1) with the following data:

$$\begin{aligned} f_1(x_1, x_2) &= \frac{1}{2}(x_1)^2 + x_2,\ \ f_2(x_1, x_2) \\ &= -\frac{1}{2}(x_2)^2 + x_1,\ \ g(x_1, x_2) = -x_1,\ \ x^0 = (0, 0). \end{aligned}$$

If we take $x^k := (\frac{1}{\sqrt{k}}, \frac{1}{\sqrt{k}})$, $\lambda^k := (\sqrt{k}, k)$ and $\mu^k = k + 1$, then we get

$$\lambda_1^k \nabla f_1(x^k) + \lambda_2^k \nabla f_2(x^k) + \mu^k \nabla g(x^k) = (0, 0).$$

Thus, x^0 is an ASKKT point. However, the limitation of $\{\lambda^k\}$ does not exist.

Note that an ASKKT point is exactly a limit point of an ASKKT sequence. Obviously the ASKKT condition collapses to the AKKT condition of Andreani et al. (2011) in the case of scalar optimization problems. In general, the ASKKT implies the AKKT, but the converse is not true. The following example illustrates this case.

Example 3.2 (AKKT does not imply ASKKT) Consider problem (1) with the following data:

$$f_1(x_1, x_2) = x_1, \quad f_2(x_1, x_2) = x_2, \quad g(x_1, x_2) = -x_1, \quad x^0 = (0, 0).$$

On one hand, AKKT is verified at x^0 since

$$\nabla f_1(x^0) + 0 \cdot \nabla f_2(x^0) + \nabla g(x^0) = (0, 0).$$

On the other hand, ASKKT does not hold at x^0. Consider arbitrary sequences $\{x^k\} \subset \mathbb{R}^2$, with $x^k \to x^0$, $\{\lambda^k\} \subset \mathbb{R}^2$ and $\{\mu^k\} \subset \mathbb{R}$ such that $\lambda_1^k \geq 1$, $\lambda_2^k \geq 1$ and $\mu^k \geq 0$. One has

$$\lambda_1^k \nabla f_1(x^k) + \lambda_2^k \nabla f_2(x^k) + \mu^k \nabla g(x^k) = (\lambda_1^k - \mu^k, \lambda_2^k) \not\to (0, 0).$$

Thus, condition $(A1)$ is not valid.

Now, we prove that each local efficient solution fulfills an ASKKT necessary optimality condition.

Theorem 3.1 *If x^0 is a local efficient point of problem (1), then x^0 satisfies the ASKKT condition.*

Proof Let x^0 be a local efficient point. By Theorem 1 in Wendell and Lee (1977), we know that x^0 is local minimal of the scalarization problem

$$\min \sum_{i=1}^{p} f_i(x) \quad \text{s.t. } x \in X,\ f_i(x) \leq f_i(x^0), \quad i = 1, \ldots, p.$$

That means that there is $\delta > 0$ such that

$$\sum_{i=1}^{p} f_i(x^0) \leq \sum_{i=1}^{p} f_i(x) \text{ for } x \in X,\ f_i(x) \leq f_i(x^0), \quad i = 1, \ldots, p,\ \|x - x^0\| \leq \delta.$$

Thus, x^0 is an unique global minimum point of the problem

$$\min \sum_{i=1}^{p} f_i(x) + \frac{1}{2}\|x - x^0\|^2 \quad \text{s.t. } x \in X,\ f_i(x) \leq f_i(x^0),\ i = 1, \ldots, p,\ \|x - x^0\| \leq \delta.$$

Let $\{\rho_k\}$ be a sequence of positive scalars with $\rho_k \to +\infty$. For every $k \in \mathbb{N}$, we define

$$\psi_k(x) := \sum_{i=1}^{p} f_i(x) + \frac{1}{2}\|x - x^0\|^2 + \frac{\rho_k}{2}\left(\sum_{i=1}^{p}(f_i(x) - f_i(x^0))_+^2 + \sum_{j=1}^{m} g_j(x)_+^2\right),$$

and let x^k be a global minimum point of

$$\min\ \psi_k(x) \quad \text{s.t. } \|x - x^0\| \leq \delta. \tag{3}$$

Observe that $\{x^k\}$ is well defined, as the function $\psi_k(x)$ is continuous and the constraint set $B(x^0, \delta) = \{x \in \mathbb{R}^n | \|x - x^0\| \leq \delta\}$ is nonempty and compact. Further, as x^0 is feasible to problem (3), we arrive at

$$\sum_{i=1}^{p} f_i(x^k) + \frac{1}{2}\|x^k - x^0\|^2 + \frac{\rho_k}{2}\left(\sum_{i=1}^{p}(f_i(x^k) - f_i(x^0))_+^2 + \sum_{j=1}^{m} g_j(x^k)_+^2\right)$$
$$\leq \sum_{i=1}^{p} f_i(x^0). \tag{4}$$

By the convergence theory of external penalty methods (Fiacco and McCormick 1968), we conclude that $x^k \rightarrow x^0$. Therefore, for sufficiently large k, x^k is an interior point of the closed ball $B(x^0, \delta)$, we have by the necessary optimality condition that $\nabla\psi_k(x^k) = 0$, that is,

$$\sum_{i=1}^{p} \nabla f_i(x^k) + (x^k - x^0) + \sum_{i=1}^{p} \rho_k(f_i(x^k) - f_i(x^0))_+\nabla f_i(x^k)$$
$$+ \sum_{j=1}^{m} \rho_k g_j(x^k)_+\nabla g_j(x^k) = 0.$$

By setting $\lambda_i^k := 1 + \rho_k(f_i(x^k) - f_i(x^0))_+$ and $\mu_j^k := \rho_k g_j(x^k)_+$, $(A0)$, $(A1)$ and $(A2')$ are verified. Moreover, from the continuity of g_j, $(A3)$ is also satisfied. □

It is worthwhile to mention that, from the proof of Theorem 3.1, the following complementary condition also is satisfied:

$$\sum_{i=1}^{p} \left|\lambda_i^k \left(f_i(x^k) - f_i(x^0)\right)\right| + \sum_{i=1}^{m} \left|\mu_j^k g_j(x^k)\right| \rightarrow 0. \tag{5}$$

Indeed, due to the fact that $x^k \rightarrow x^0$, by the continuity of f_i, from (4) we arrive at

$$\sum_{i=1}^{p} \rho_k \left(f_i(x^k) - f_i(x^0)\right)_+^2 + \sum_{j=1}^{m} \rho_k g_j(x^k)_+^2 \rightarrow 0.$$

Thus, by replacing $\rho_k(f_i(x^k) - f_i(x^0))_+$ with $\lambda_i^k - 1$, and $\rho_k g_j(x^k)_+$ with μ_j^k, we see

$$\sum_{i=1}^{p} \left|(\lambda_i^k - 1)(f_i(x^k) - f_i(x^0))\right| + \sum_{i=1}^{m} \left|\mu_j^k g_j(x^k)\right| \rightarrow 0.$$

This implies (5) from the continuity of f_i.

Clearly, the complementary condition (5) is equivalent to saying that

$$\lambda_i^k(f_i(x^k) - f_i(x^0)) \to 0,\ i = 1, \dots, p, \quad \mu_j^k g_j(x^k) \to 0,\ j = 1, \dots, m.$$

Note that such a condition involves an complementarity for the Lagrange multipliers associated with the objective functions, and it can be used to strengthen the stopping criteria of algorithms with the validity of ASKKT. It should also be noted that, in the case of scalar optimization, the ASKKT condition satisfying condition (5) collapses to the complementary approximate KKT condition that appeared in Andreani et al. (2010).

The following example illustrates Theorem 3.1 and the case mentioned above.

Example 3.3 Consider problem (1) with the following data:

$$f_1(x_1, x_2) = 2x_1 - (x_2)^2, \quad f_2(x_1, x_2) = -x_2, \quad g(x_1, x_2) = (x_2)^2 - x_1.$$

The point $x^0 = (0, 0)$ is an efficient solution as can be checked. By Theorem 3.1, we can find a sequence $\{(x^k, \lambda^k, \mu^k)\}$ satisfying conditions (A0)–(A1), (A2′), (A3) and (5). First, for the convenience we begin with the point sequence $x^k = (0, \frac{1}{k})$, $k \in \mathbb{N}$. Observe that all points x^k are nonfeasible. Second, to arrive at

$$\lambda_1^k \nabla f_1(x^k) + \lambda_2^k \nabla f_2(x^k) + \mu^k \nabla g(x^k) = \left(2\lambda_1^k - \mu^k, \frac{2(\mu^k - \lambda_1^k)}{k} - \lambda_2^k\right) \to (0, 0)$$

for $\lambda_1^k, \lambda_2^k \geq 1$ and $\mu^k \geq 0$, we can choose

$$\lambda_2^k = 1,\ \lambda_1^k = \frac{\mu^k}{2} = \max\left\{\frac{k}{2}, 1\right\}.$$

(A3) is satisfied, as $g(x^0) = 0$. Finally, condition (5) is also verified since for $k \geq 2$

$$\left|\lambda_i^k\left(f_i(x^k) - f_i(x^0)\right)\right| + \left|\lambda_i^k\left(f_i(x^k) - f_i(x^0)\right)\right| + \left|\mu^k g(x^k)\right| = \frac{5}{2k} \to 0.$$

To close this section, we provide an example to demonstrate that a local weakly efficient solution may not fulfill the ASKKT condition.

Example 3.4 (Weak efficiency does not imply ASKKT) Consider Example 3.2. The point $x^0 = (0, 0)$ is a weakly efficient solution but not a (local) efficient solution as can be checked. But at $x^0 = (0, 0)$, the ASKKT condition does not hold.

4 Global convergence under regularity conditions

In scalar optimization, the AKKT condition has been proved to be a genuine necessary condition of optimality. However, to make it become a sufficient condition, some assumptions have to be considered. For example, even in the convex optimization, the AKKT condition requiring an extra complementary condition could imply the optimality. See Andreani et al. (2010, 2011), Haeser and Schuverdt (2011) for more discussion about the relations between AKKT and optimality. Therefore, for a general nonlinear optimization problem, the limit point of an AKKT sequence usually is only a feasible point. For this reason, the most popular numerical algorithms focusing on whether the limit points generated by the AKKT sequences are KKT points. For example, the AKKT sequence generated by the sequential quadratic programming algorithm of Qi and Wei (2000) converges to a KKT point if a CPLD holds, while paper (Andreani et al. 2016) is devoted to finding the weakest constraint qualification, called cone-continuity property, instead of CPLD.

Given $x^0 \in X$, we denote the linearized cone to X at x^0 by

$$\mathcal{L}_X(x^0) := \left\{ d \in \mathbb{R}^n \mid \langle \nabla g_j(x^0), d \rangle \leq 0,\ \forall j \in A(x^0) \right\},$$

and define

$$\mathcal{K}(x) := \left\{ \sum_{j \in A(x^0)} \mu_j \nabla g_j(x) \,\middle|\, \mu_j \in \mathbb{R}_+ \right\}.$$

It is clear to see that $\mathcal{K}(x)$ is a closed convex cone and coincides with the polar cone of $\mathcal{L}_X(x^0)$ at x^0. Observe that the set-valued mapping $\mathcal{K} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is inner semicontinuous at x^0, that is,

$$\mathcal{K}(x^0) \subset \operatorname*{Lim\,inf}_{x \to x^0} \mathcal{K}(x),$$

due to the continuity of the gradients and the definition of $\mathcal{K}(x)$. For this reason, in view of the continuity for set-valued mappings the definition given below is reasonable.

Definition 4.1 We say that $x^0 \in X$ satisfies the cone-continuity property (CCP) if

$$\operatorname*{Lim\,sup}_{x \to x^0} \mathcal{K}(x) \subset \mathcal{K}(x^0).$$

In Theorem 3.2 of Andreani et al. (2016), the CCP is proved to be the weakest strict constraint qualification in smooth scalar optimization problems. In other words, the fulfillment of CCP ensures that limits of AKKT sequences converge to KKT points for every continuously differentiable objective function.

We note that for a non-null vector $\lambda^0 \in \mathbb{R}^p_+$ and an objective function $f_0(x)$ scalarized from problem (1) as:

$$f_0(x) := \sum_{i=1}^{p} \lambda_i^0 f_i(x),$$

in view of Theorem 3.2 of Andreani et al. (2016), a limit point of an AKKT sequence, where the CCP holds, is a KKT point for the scalarization problem. With this, as a sequence $\{\lambda^k\} \subset \mathbb{R}^p_+$ satisfying condition $(A2)$ has at least one non-null accumulation point, it turns out that the verification of CCP is also sufficient for problem (1) to guarantee that limits of AKKT sequences are (weak) KKT points, as the following theorem shows.

Theorem 4.1 *If $x^0 \in X$ is a limit point of an AKKT sequence $\{x^k\}$ of problem (1) and verifies the CCP, then x^0 is a KKT point.*

Proof By definition, we can assume, without loss of generality, there exist sequences $\{\lambda^k\} \subset \mathbb{R}^p_+$ and $\{\mu^k\} \subset \mathbb{R}^m_+$ such that $\sum_{i=1}^p \lambda_i^k = 1$, $\mu_j^k = 0$ for $j \notin A(x^0)$ and

$$w^k := \sum_{i=1}^p \lambda_i^k \nabla f_i(x^k) + \sum_{j \in A(x^0)} \mu_j^k \nabla g_j(x^k) \to 0. \tag{6}$$

Since the positive sequence $\{\lambda^k\}$ is bounded with $\sum_{i=1}^p \lambda_i^k = 1$, there exists a subsequence $\{\lambda^{k,l}\}$ such that $\lambda^{k,l} \to \lambda^0$ satisfying $\lambda^0 \in \mathbb{R}^p_+$ and $\sum_{i=1}^p \lambda_i^0 = 1$. It follows from (6)

$$w^{k,l} - \sum_{i=1}^p \lambda_i^{k,l} \nabla f_i(x^{k,l}) = \sum_{j \in A(x^0)} \mu_j^{k,l} \nabla g_j(x^{k,l}) \in \mathcal{K}(x^{k,l}). \tag{7}$$

Thus, taking limits in (7) when l tends to infinity, using the continuity of the gradients, from (6) we get

$$-\sum_{i=1}^p \lambda_i^0 \nabla f_i(x^0) \in \operatorname*{Lim\,sup}_{l \to \infty} \mathcal{K}(x^{k,l}) \subset \operatorname*{Lim\,sup}_{x \to x^0} \mathcal{K}(x) \subset \mathcal{K}(x^0),$$

where the last inclusion is due to the fulfillment of the CCR. This means that there is vector $\mu^0 \in \mathbb{R}^m_+$ such that

$$\sum_{i=1}^p \lambda_i^0 \nabla f_i(x^0) + \sum_{j \in A(x^0)} \mu_j^0 \nabla g_j(x^0) = 0.$$

This completes the proof. □

Definition 4.2 The strong KKT condition (SKKT) for problem (1) is said to hold at a point $x^0 \in X$ if there exist vectors $\lambda \in \mathbb{R}^p$ and $\mu \in \mathbb{R}^m$ such that

$$\text{(S1)} \sum_{i=1}^{p} \lambda_i \nabla f_i(x^0) + \sum_{j \in A(x^0)} \mu_j \nabla g_j(x^0) = 0,$$
$$\text{(S2)}\ \lambda > 0, \quad \mu \geq 0,$$
$$\text{(S3)}\ g_j(x^0) < 0 \ \Rightarrow \ \mu_j = 0,\ j = 1, \ldots, m.$$

Unfortunately, the fulfillment of CCP is not enough to ensure that the limits of ASKKT sequences are SKKT points.

Example 4.1 (ASKKT does not imply SKKT under CCP) Consider the problem (1) with the following data:

$$f_1(x_1, x_2) = x_1 - (x_2)^2,\ f_2(x_1, x_2) = -2x_2,\ f_3(x_1, x_2) = -(x_2)^2,\ g(x_1, x_2) = -x_1.$$

Let us study properties of the point $x^0 = (0, 0)$. We take sequences

$$x^k = \left(0, -\frac{1}{k}\right), \quad \lambda_1^k = \lambda_2^k = \mu^k = 1, \quad \lambda_3^k = k.$$

Thereby, one has

$$\lambda_1^k \nabla f_1(x^k) + \lambda_2^k \nabla f_2(x^k) + \lambda_3^k \nabla f_3(x^k) + \mu^k \nabla g(x^k) = \left(0, \frac{2}{k}\right) \to (0, 0).$$

Thus, x^0 is an ASKKT point. Meanwhile, it is clear to verify that the CCP holds at x^0. On the other hand, suppose that the multipliers $\lambda \in \mathbb{R}^3_+$ and $\mu \in \mathbb{R}_+$ satisfy

$$\lambda_1 \nabla f_1(x^0) + \lambda_2 \nabla f_2(x^0) + \lambda_3 \nabla f_3(x^0) + \mu \nabla g(x^0) = (0, 0),$$

we get that λ_2 must be 0. Thus, x^0 is not an SKKT point.

So, a natural question is what conditions are able to ensure that the limits of ASKKT sequences are SKKT points. In the following, in order to obtain a multiplier rule with $\lambda > 0$ for an ASKKT point, we introduce a called cone-continuity regularity condition.

Given $x^0 \in X$, we denote the cone of critical directions at x^0 by

$$\mathcal{C}(x^0) := \left\{ d \in \mathbb{R}^n \mid \langle \nabla f_i(x^0), d \rangle \leq 0,\ \forall i \in I,\ \langle \nabla g_j(x^0), d \rangle \leq 0,\ \forall j \in A(x^0) \right\},$$

and we define

$$\mathcal{D}(x) := \left\{ \sum_{i=1}^{p} \lambda_i \nabla f_i(x) + \sum_{j \in A(x^0)} \mu_j \nabla g_j(x) \,\middle|\, \lambda_i, \mu_j \geq 0 \right\} \text{ for all } x \in \mathbb{R}^n.$$

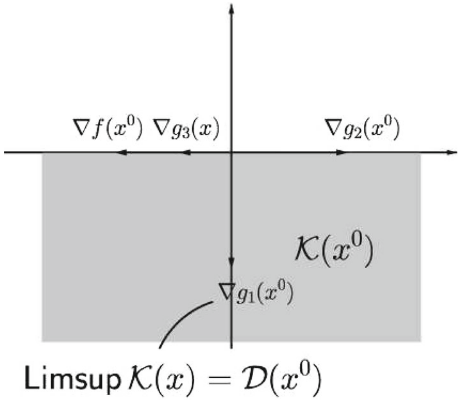


Fig. 1 Example 4.2

By definition, $\mathcal{D}(x)$ is a closed cone containing the cone $\mathcal{K}(x)$ for $x \in \mathbb{R}^n$ and coincides with $\left[\mathcal{C}(x^0)\right]^*$ at x^0. Similarly to $\mathcal{K}(x)$, the set-valued mapping $\mathcal{D} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is inner semicontinuous at x^0.

Definition 4.3 We say that $x^0 \in X$ satisfies the cone-continuity regularity (CCR) if

$$\operatorname*{Lim\,sup}_{x \to x^0} \mathcal{D}(x) \subset \mathcal{D}(x^0).$$

Compared with the CCP, the CCR involves the objective functions. It seems that CCR is just an analogy of CCP, treating objectives as inequality constraints, but there are no implications between such two conditions, in fact. To see these, we provide two simple examples as follows.

Example 4.2 (CCR does not imply CCP) In $\mathbb{R}^2$. Consider the scalar problem:

$$\min\ -x_1 \quad \text{s.t.}\ -x_1 \le 0,\ x_2 \le 0,\ -(x_1)^3 \le 0.$$

At the point $x^0 = (0, 0)$, the CCR holds but the CCP does not hold. See Fig. 1.

Example 4.3 (CCP does not imply CCR) In $\mathbb{R}^2$. Consider the scalar problem:

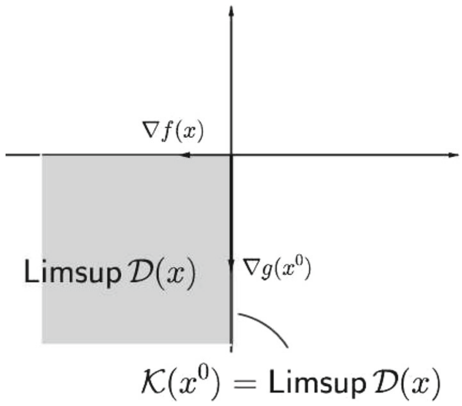
$$\min\ -(x_1)^3 \quad \text{s.t.}\ -x_2 \le 0.$$

At the point $x^0 = (0, 0)$, the CCP holds but the CCR does not hold. See Fig. 2.

Note that the CCR is neither stronger nor weaker than the CCP even in scalar optimization problems. In Example 4.1, we have explained that an ASKKT point may not be an SKKT point even if the CCP holds. However, when the CCR holds, we can get that the result holds. In order to achieve this, we first state an equivalent characterization for an SKKT point.

Lemma 4.1 *Let $x^0 \in X$ be a feasible point. Then, x^0 satisfies the SKKT condition of problem* (1) *if and only if $-\nabla f_i(x^0) \in \mathcal{D}(x^0)$ for $i = 1, \ldots, p$.*

Fig. 2 Example 4.3



Proof Suppose that there exist vectors $\lambda \in \mathbb{R}^p$, with $\lambda > 0$, and $\mu \in \mathbb{R}^m_+$ such that

$$\sum_{i=1}^{p} \lambda_i \nabla f_i(x^0) + \sum_{j\in A(x^0)} \mu_j \nabla g_j(x^0) = 0.$$

It follows that, for each $i \in I$,

$$-\nabla f_i(x^0) = \sum_{l\in I\setminus\{i\}} \lambda_l \nabla f_l(x^0) + \sum_{j\in A(x^0)} \mu_j \nabla g_j(x^0) \in \mathcal{D}(x^0).$$

To prove the converse, let $-\nabla f_i(x^0) \in \mathcal{D}(x^0)$ for $i = 1, \ldots, p$. Then, for each $i \in I$, there exist vectors $\lambda^i \in \mathbb{R}^p_+$ and $\mu^i \in \mathbb{R}^m_+$ such that

$$-\nabla f_i(x^0) = \sum_{l=1}^{p} \lambda_l^i \nabla f_l(x^0) + \sum_{j\in A(x^0)} \mu_j^i \nabla g_j(x^0).$$

That is,

$$\nabla f_i(x^0) + \sum_{l=1}^{p} \lambda_l^i \nabla f_l(x^0) + \sum_{j\in A(x^0)} \mu_j^i \nabla g_j(x^0) = 0, \quad i = 1, \ldots, p.$$

If we now add up the p equations, and put

$$\hat{\lambda}_l := 1 + \sum_{i=1}^{p} \lambda_l^i,\ l = 1, \ldots, p, \quad \hat{\mu}_j := \sum_{i=1}^{m} \mu_j^i, \quad j = 1, \ldots, m,$$

then we have that $\hat{\lambda} > 0$ and $\hat{\mu} \geq 0$ satisfying

$$\sum_{l=1}^{p} \hat{\lambda}_l \nabla f_l(x^0) + \sum_{j \in A(x^0)} \hat{\mu}_j \nabla g_j(x^0) = 0.$$

The proof is complete. □

Now, we state our main result.

Theorem 4.2 *If $x^0 \in X$ is a limit point of an ASKKT sequence $\{x^k\}$ of problem (1) and verifies the CCR, then x^0 is an SKKT point.*

Proof Let x^0 be an ASKKT point. To show that the SKKT condition holds at x^0, in view of Lemma 4.1, it suffices to prove $-\nabla f_i(x^0) \in \mathcal{D}(x^0)$ for $i = 1, \ldots, p$. Since x^0 fulfills the ASKKT condition, there exist sequences $\{x^k\} \subset \mathbb{R}^n$ and $\{(\lambda^k, \mu^k)\} \subset \mathbb{R}^p_+ \times \mathbb{R}^m_+$ such that $\lambda^k_i \geq 1,\ i = 1, \ldots, p$, $\mu^k_j = 0$ for $j \notin A(x^0)$ and

$$w^k := \sum_{i=1}^{p} \lambda^k_i \nabla f_i(x^k) + \sum_{j \in A(x^0)} \mu^k_j \nabla g_j(x^k) \to 0. \tag{8}$$

It follows that, for each $i \in I$,

$$w^k - \nabla f_i(x^k) = (\lambda^k_i - 1)\nabla f_i(x^k) + \sum_{l \in I\setminus\{i\}} \lambda^k_l \nabla f_l(x^k) + \sum_{j \in A(x^0)} \mu^k_j \nabla g_j(x^k).$$

Thus, for such i,

$$w^k - \nabla f_i(x^k) \in \mathcal{D}(x^k). \tag{9}$$

Taking limits in (9), using the continuity of the gradients of f_i, from (8) we arrive at

$$-\nabla f_i(x^0) \in \operatorname*{Lim\,sup}_{k\to\infty} \mathcal{D}(x^k) \subset \operatorname*{Lim\,sup}_{x\to x^0} \mathcal{D}(x) \subset \mathcal{D}(x^0),$$

as the CCR holds at x^0. □

Clearly, as a local efficient point must be an ASKKT point, Theorem 4.2 has the following immediate corollary.

Corollary 4.1 *If $x^0 \in X$ is a local efficient solution of problem (1) and verifies the CCR, then x^0 is an SKKT point.*

By virtue of Theorem 4.4 in Andreani et al. (2016), it is easy to prove that the CCR implies the Abadie regularity condition (Abadie RC), which appears in Maeda (1994). It holds at a feasible point $x^0 \in Q$ if $\mathcal{C}(x^0) \subset T_Q(x^0)$ holds, where

$$Q := \left\{x \in X \mid f_i(x) \leq f_i(x^0),\ i = 1, \ldots, p\right\}.$$

It was proved in Maeda (1994) that the SKKT condition is satisfied at an efficient solution when the Abadie RC holds. Naturally, an interesting question will arise: Whether do the limit of an ASKKT sequence converge to an SKKT point under the Abadie RC? Unfortunately, the following example answers the question negatively. In other words, the CCR cannot be relaxed to the weaker Abadie regularity condition in Theorem 4.2.

Example 4.4 (ASKKT does not imply SKKT under Abadie RC) Consider Example 4.1. The Abadie RC is verified at x^0 since (after some calculations)

$$Q = \left\{(x_1, x_2)|x_2 \geq 0, 0 \leq x_1 \leq (x_2)^2\right\}, \quad \mathcal{C}(x^0) = T_Q(x^0) = \{0\} \times \mathbb{R}_+.$$

However, we can see that at the ASKKT point x^0 the SKKT condition does not hold. In fact, the fulfillment of the CCR at x^0 has been violated.

The following theorem demonstrates that when the CCR holds, the AKKT condition also implies the KKT condition.

Theorem 4.3 *If $x^0 \in X$ is a limit point of an AKKT sequence $\{x^k\}$ of problem (1) and verifies the CCR, then, x^0 is a KKT point.*

Proof It follows the same reasoning of the proof of Theorem 4.1 that

$$-\sum_{i=1}^{p} \lambda_i^0 \nabla f_i(x^0) \in \mathcal{D}(x^0),$$

which validates the KKT condition. □

However, the fulfillment of CCR is not enough for the limit points of AKKT sequences to be SKKT points.

Example 4.5 (AKKT does not imply SKKT under CCR) Consider Example 3.2. The point $x^0 = (0, 0)$ is an AKKT point and satisfies the CCR, as $\mathcal{D}(x) = \mathcal{D}(x^0) = \mathbb{R} \times \mathbb{R}_+$ for all $x \in \mathbb{R}^2$. However, x^0 does not satisfy the SKKT condition.

Corollary 4.2 (Scalar Problems) *Let us consider problem (1) with $p = 1$. If $x^0 \in X$ is a limit point of an AKKT sequence $\{x^k\}$ and verifies the CCR, then x^0 is a KKT point.*

Since it is known in Andreani et al. (2011) that each local minimum point fulfills the condition of AKKT, Corollary 4.2 exactly expresses that the CCR ensures the validity of KKT at a local minimum point. In fact, the CCR is different from the classical Guignard CQ (GCQ). The following example shows that Corollary 4.2 is still applicable when GCQ fails in scalar optimization problems.

Example 4.6 In $\mathbb{R}^2$. Consider the scalar problem:

$$\min\ x_2 \quad \text{s.t.} -x_1 \leq 0,\ -x_2 \leq 0,\ -(x_1)^3 + (x_2)^2 \leq 0.$$

The point $x^0 = (0, 0)$ is a local minimum point as can be checked. At x^0, the CCR holds but the GCQ does not hold, while the KKT condition is satisfied.

5 Sufficient conditions for proper efficiencies

This section is devoted to sufficient optimality conditions of an ASKKT point to be a properly efficient point for convex multiobjective optimization problems.

In the literature, there are many notions of proper efficiency, as those introduced by Geoffrion, Borwein, Benson and Henig. Since problem (1) considered in this section is convex, all these concepts are equivalent each other in terms of linear scalarization (see, e.g., Ehrgott 2005), so that we recall only Geoffrion's definition: a point $x^0 \in X$ is said to be properly efficient if it is efficient and there exists a scalar $\rho > 0$ such that, for all $i \in I$ and $x \in X$ satisfying $f_i(x^0) > f_i(x)$, there exists $l \in I$ such that $f_l(x^0) < f_l(x)$ and

$$\frac{f_i(x^0) - f_i(x)}{f_l(x) - f_l(x^0)} \leq \rho. \tag{10}$$

In our main results of this section, we will use the following general hypotheses:

(H1) $f_i(i = 1, \ldots, p)$ and $g_j(j = 1, \ldots, m)$ are convex;
(H2) x^0 is a feasible point satisfying the ASKKT condition with sequences $\{x^k\}$, $\{\lambda^k\}$, $\{\mu^k\}$;
(H3) the multiplier sequence $\{\lambda^k\}$ is bounded;
(H4) the ASKKT sequence $\{x^k\}$ and the multipliers $\{\mu^k\}$ satisfy

$$\sum_{i=1}^{m} \mu_j^k g_j(x^k) \to 0. \tag{11}$$

For any $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^p$, and $\mu \in \mathbb{R}^m$, we define the generalized Lagrangian function of problem (1) by

$$L(x, \lambda, \mu) := \sum_{i=1}^{p} \lambda_i f_i(x) + \sum_{j=1}^{m} \mu_j g_j(x).$$

Now, we already state the following sufficient optimality result.

Theorem 5.1 *Assume that* (H1)–(H4) *hold. Then, x^0 is a properly efficient solution of problem* (1).

Proof Let $x \in X$ be an arbitrary feasible point differed from x^0. As f_i, g_j are differentiable and convex, so is $L(\cdot, \lambda^k, \mu^k)$ for every $k \in \mathbb{N}$. Thus, it results

$$L\left(x, \lambda^k, \mu^k\right) \geq L\left(x^k, \lambda^k, \mu^k\right) + \left\langle \nabla_x L(x^k, \lambda^k, \mu^k), x - x^k \right\rangle$$

for the ASKKT sequence $\{x^k\}$. Furthermore, we have

$$\sum_{i=1}^{p} \lambda_i^k f_i(x) \geq L\left(x, \lambda^k, \mu^k\right) \geq L\left(x^k, \lambda^k, \mu^k\right) + \left\langle \nabla_x L(x^k, \lambda^k, \mu^k), x - x^k \right\rangle, \tag{12}$$

where the validation of the first inequality is due to $\mu_j^k \geq 0$ and $g_j(x) \leq 0$. Since the sequence $\{\lambda^k\}$ is bounded, there is a subsequence $\{\lambda^{k,l}\}$ and a positive vector λ^0 such that $\lambda^{k,l} \to \lambda^0$ as $l \to \infty$. From (12), we get

$$\sum_{i=1}^{p} \lambda_i^{k,l} f_i(x) \geq \sum_{i=1}^{p} \lambda_i^{k,l} f_i(x^{k,l}) + \sum_{j=1}^{m} \mu_j^{k,l} g_j(x^{k,l}) + \left\langle \nabla_x L\left(x^{k,l}, \lambda^{k,l}, \mu^{k,l}\right), x - x^{k,l} \right\rangle.$$

Taking the limit when l goes to infinity, using the continuity of f_i, from (A0)–(A1) and (11) we deduce that

$$\sum_{i=1}^{p} \lambda_i^0 f_i(x) \geq \sum_{i=1}^{p} \lambda_i^0 f_i(x^0). \tag{13}$$

Thus, by Theorem 1 in Geoffrion (1968), combining with (13) and the fact $\lambda^0 > 0$ gives us that x^0 is a properly efficient solution. □

It is worth mentioning that (H4) can be implied by the complementary condition (5), and it can be replaced by a sign condition (SGN)

$$\mu_j^k g_j(x^k) \geq 0, \quad \text{for all } j = 1, \ldots, m$$

to conform with the statement of Theorem 5.1. Note also that the SGN has been used in Theorem 3.2 of Giorgi et al. (2016) for sufficient conditions of weakly efficient solutions of convex multiobjective problems.

The next result may be found in Ehrgott (2005, Theorem 3.27).

Corollary 5.1 *Assume that (H1) holds. Suppose that $x^0 \in X$ is an SKKT point. Then, x^0 is a properly efficient solution of problem* (1).

Proof If x^0 is an SKKT point, then (H2) is naturally verified, and there exist vectors $\lambda^0 \in \mathbb{R}^p$ and $\mu^0 \in \mathbb{R}^m$ such that (S1)–(S3) hold. By taking $x^k = x^0$, $\lambda^k = \lambda^0$ and $\mu^k = \mu^0$ for all k, (H3) and (H4) are valid. Hence, the conclusion follows from Theorem 5.1. □

Since we know that the SKKT condition does not need to be satisfied at an ASKKT point, as mentioned in Example 4.1, it allows us to show that Corollary 5.1 may fail to work even if Theorem 5.1 is applicable.

Example 5.1 Consider the problem (1) with the following data:

$$f_1(x_1, x_2) = x_1 + (x_2)^2, \quad f_2(x_1, x_2) = (x_1)^2, \quad g(x_1, x_2) = (x_1)^2.$$

Let us note that it is a convex problem. We consider $x^0 = (0, 0)$ and

$$x^k = \left(-\frac{1}{k}, 0\right), \quad \lambda^k = (1, 1), \ \mu^k = \frac{k}{2}, \quad \text{for each } k \in \mathbb{N}.$$

Then, one has

$$\lambda_1^k \nabla f_1(x^k) + \lambda_2^k \nabla f_2(x^k) + \mu^k \nabla g(x^k) = \left(-\frac{2}{k}, 0\right) \to (0, 0).$$

Thus, x^0 satisfies the ASKKT. Observe also that $\{\lambda^k\}$ is bounded and

$$\mu^k g(x^k) = \frac{1}{2k} \to 0.$$

Hence, Theorem 5.1 is applicable and then x^0 is verified to be a properly efficient point. On the other hand, if we assume $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2_+$ and $\mu \geq 0$ such that

$$\lambda_1 \nabla f_1(x^0) + \lambda_2 \nabla f_2(x^0) + \mu \nabla g(x^0) = (0, 0),$$

then we get that λ_1 has to be zero. Thus, x^0 does not fulfill the SKKT, so that Corollary 5.1 fails to work in this context.

Because of Theorem 4.2, Corollary 5.1 has the following immediate result.

Corollary 5.2 *Assume that (H1) and (H2) hold. Suppose that x^0 satisfies the CCR. Then, x^0 is a properly efficient solution of problem* (1).

6 Numerical results

In the earlier sections, we have discussed that, under appropriate hypotheses, a limit point of an ASKKT sequence might be an SKKT point or an efficient point. However, in practical applications, the first question that we should face is how to verify an ASKKT sequence. In Dutta et al. (2013), a KKT-proximity measure estimate scheme was suggested to identify that the limit of a sequence is an AKKT point. With this, we now extend this scheme to the situation of ASKKT sequence for dealing with our multiobjective problems.

Definition 6.1 Given a point $x \in \mathbb{R}^n$ and a scalar $\lambda_{\max} \in [1, +\infty]$, an SKKT-proximity measure of problem (1) is the optimal value of the following problem:

$$\min \quad \varepsilon \tag{14a}$$

$$\text{subject to} \quad \left\| \sum_{i=1}^{p} \lambda_i \nabla f_i(x) + \sum_{j=1}^{m} \mu_j \nabla g_j(x) \right\|^2 \leq \varepsilon, \tag{14b}$$

$$\sum_{j=1}^{m} |\mu_j g_j(x)| \leq \varepsilon, \quad \mu_j \geq 0, \quad j = 1, \ldots, m, \tag{14c}$$

$$1 \leq \lambda_i \leq \lambda_{\max}, \quad i = 1, \ldots, p. \tag{14d}$$

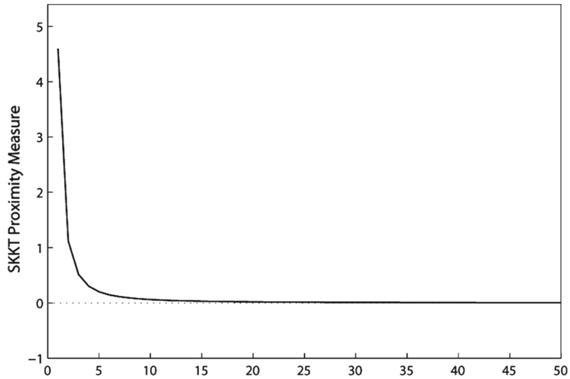


Fig. 3 SKKT-proximity measure for problem (15)

Note that the above problem is always solvable. Let $\{x^k\}$ be a given sequence. For every iterate x^k, we may denote the optimal solution of problem (14) by $(\varepsilon_k^*, \lambda_*^k, \mu_*^k)$. It is clear that if the value ε_k^* reduces to zero as $k \to \infty$, then any feasible limit point of $\{x^k\}$ will be an ASKKT point. In the following, we show an example to illustrate this idea.

We shall point out that the limit of a sequence of non-KKT points may be an ASKKT point even an SKKT point.

$$\min_{\mathbb{R}^2_+} f(x_1, x_2) = ((x_1)^2 + (x_2)^2, 0.3|x_1|^{3/2} - 2x_2) \tag{15a}$$

$$\text{subject to } g_1(x_1, x_2) := 3x_1 - x_2 + 1 \leq 0, \tag{15b}$$

$$g_2(x_1, x_2) := (x_1)^2 + (x_2 - 2)^2 - 1 \leq 0. \tag{15c}$$

Let us consider a sequence of points $x^k = (\frac{0.6}{k}, 1 + \frac{1.8}{k}),\ k = 1, 2, \ldots$, along the linear constraint boundary from (0.6, 2.8) to (0, 1). Clearly, the sequence $\{x^k\}$ converges to the point (0, 1), denoted by x^0. Next for every iterate x^k we will investigate the behavior of SKKT-proximity measure estimate scheme by solving problem (14), where we assign $\lambda_{\max}$ a finite value (= 10).

Figure 3 shows the first 50 SKKT-proximity measure estimate values. From the figure, we observe that the starting point x^1 has a large SKKT-proximity measure estimate value, but as x^1 is iterated to x^{15} the estimate value has a distinct reduction. Since x^k is not an SKKT point throughout, the SKKT-proximity measure estimate value will never exactly be zero. But, as points towards to point (0, 1); sequentially, the estimate value will reduce to zero. The behavior of the SKKT-proximity measure can be also analyzed from Fig. 4. Let us note that the limit of multiplier sequence $\{(\lambda_*^k, \mu_*^k)\}$ converges to a vector (1, 1, 0, 0), which happens to satisfy the SKKT condition at x^0. Thus, the estimate value ε_k^* must tend to zero, and the limit point x^0 of $\{x^k\}$ must be an ASKKT point as well. Besides, since problem (15) is convex, x^0 is verified to be properly efficient from Theorem 5.1 or Corollary 5.1.

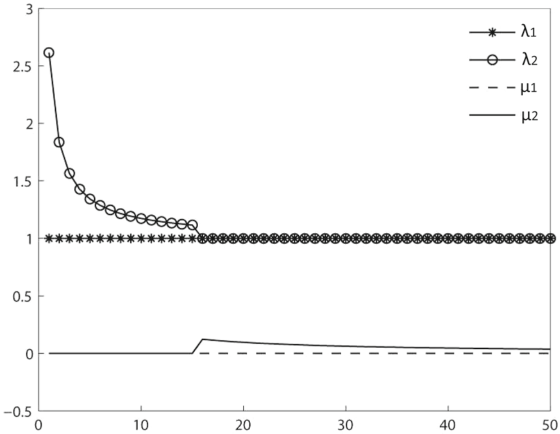


Fig. 4 Lagrange multipliers for problem (15)

7 Concluding remarks

In this work, a new sequential optimality condition, mentioned as the ASKKT condition, is introduced for smooth multiobjective optimization problems with inequality constraints. We prove that every local efficient point fulfills the ASKKT condition, as provided in Theorem 3.1. However, for weakly efficient points, the ASKKT optimality condition may not be verified as shown in Example 3.4. A regularity condition CCR has been suggested, which is used to guarantee that a limit point of ASKKT sequence is an SKKT point (Theorem 4.2). We achieve a sufficient ASKKT sequential optimality condition for properly efficiencies in convex optimization, as shown in Theorem 5.1.

We realize that, by means of a scalarization method, solving a smooth multiobjective optimization problem can be transformed to solving a smooth scalar problem, as stated in the proof Theorem 3.1. With this, another interesting research line is to study second-order sequential optimality conditions for local efficient points of smooth multiobjective optimization problems. See Andreani et al. (2017) where a second-order sequential optimality condition is introduced to analyze the global convergence of second-order algorithms.

On the other hand, practical algorithms should be invited for solving constrained multiobjective optimization problems in the future study. And we believe that approximate KKT conditions (Giorgi et al. 2016) and approximate SKKT conditions are useful sequential optimality conditions for generating or improving numerical algorithms of constrained multiobjective optimization problems.

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