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A primal-dual interior-point algorithm for symmetric optimization based on a new method for finding search directions

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ABSTRACT
We introduce an interior-point method for symmetric optimization based on a new method for determining search directions. In order to accomplish this, we use a new equivalent algebraic transformation on the centring equation of the system which characterizes the central path. In this way, we obtain a new class of directions. We analyse a special case of this class, which leads to the new interior-point algorithm mentioned before. Another way to find the search directions is using barriers derived from kernel functions. We show that in our case the corresponding direction cannot be deduced from a usual kernel function. In spite of this fact, we prove the polynomial complexity of the proposed algorithm.

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1. Introduction

The projective potential reduction algorithm introduced by Karmarkar [1] had a great impact on the development of new methods for solving linear optimization (LO) problems. A very active domain of research has been emerged, namely the field of the interior-point methods (IPMs). The most important results related to these algorithms were presented in the monographs written by Roos et al. [2], Wright [3] and Ye [4].

Several IPMs have been extended to more general problems, like semidefinite optimization (SDO)(Boyd and Vandenberghe [5], Klerk [6], Li and Terlaky [7], Monteiro and Zhang [8], Salahi and Terlaky [9]), second-order cone optimization (SOCO) (Alizadeh and Goldfarb [10], Peng et al. [11], Wang and Bai [12]) and linear complementarity problems (LCPs) (Illés et al. [13–15], Kojima et al. [16], Lesaja and Roos [17], Potra and Sheng [18,19]). Recently, it turned out that the above-mentioned methods could be analysed in a unified way by considering the symmetric optimization (SO) problems. The SO problem is a convex programming problem, where the objective function is linear and the constraints are determined by the intersection of an affine subspace and a self-dual and homogenous cone, i.e. symmetric cone. It should be mentioned that there is a close relation between the symmetric cones and the Jordan algebras, as every symmetric cone can be realized as the cone of squares of Euclidean Jordan algebras. The approximate solution of these general problems using the IPMs is important, because methods based on pivoting cannot be applied in this context. Güler [20] proved that the symmetric cones coincide with the self-scaled cones studied earlier by Nesterov and Todd [21,22]. Faraut and Korányi [23] presented a more detailed analysis related to the theory of the Euclidean Jordan algebras and symmetric cones. Faybusovich [24,25] extended primal–dual

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IPMs for SDO to SO using Euclidean Jordan algebras. Sturm [26] studied the possibility of analysing different IPMs by considering similarity relations and extended Stein's theorem to symmetric cones. Schmieta and Alizadeh [27] analysed the relationship between polynomiality proofs of primal–dual IPMs for SO problems. Vieira [28] extended IPMs for LO based on kernel functions to SO and gave some properties of eigenvalues in Jordan algebras. Lesaja and Roos [29] considered a more general framework, namely the class of monotone LCPs over symmetric cones.

There are two main aspects of determining search directions in the case of the IPMs. One is based on barrier functions, the other one on an algebraic equivalent transformation. Nesterov and Nemirovskii [30] studied self-concordant barrier functions in order to introduce primal and dual IPMs for convex optimization. Bai et al. [31] introduced a new type of barrier function, where the corresponding kernel function has finite value in the origin. This type of barrier was generalized to SO by Vieira [32]. Moreover, Vieira [33] and Bai et al. [34] presented a comparative study of different kernel functions for LO. The other method for finding search directions was proposed by Darvay [35]. This approach is based on an algebraic equivalent transformation on the centring equation of the system which characterizes the central path. This can be achieved by applying componentwisely a continuously differentiable and invertible function on the above-mentioned centring equation. Every algebraic transformation yields a corresponding kernel function. Observe that if we consider the square root function proposed by Darvay [36] in order to transform the centring equation, then we obtain a finite barrier, too. This technique for determining search directions was extended to more general problems. Using this approach many results related LCPs appeared (Achache [37], Asadi and Mansouri [38], Kheirfam [39], Mansouri and Pirhaji [40]). Darvay's technique was also generalized to SO, SDO and SOCO by Wang and Bai [12,41,42]. Moreover, Wang [43] presented a generalization of this technique to monotone LCPs over symmetric cones.

Recently, Darvay and Takács [44] have introduced a new method for finding search directions using a new type of transformation on the centring equations. In this paper, we generalize this approach to SO. We achieve this by modifying the centring equation which can be written equivalently in the $v = v^2$ form, where v is the variance vector introduced in Section 4. After that, we apply a vector-valued function on this transformed equation. We obtain the new search directions using Newton's method and the Nesterov–Todd scaling. We make a connection between this new technique and the approach based on barrier functions. It should be mentioned that the corresponding kernel function is a positive-asymptotic kernel function which is introduced in the paper written by Darvay and Takács [45]. However, we prove that the new algorithm finds approximate solution in polynomial time.

The paper is organized as follows. In the next section, some main properties of the Euclidean Jordan algebras and symmetric cones are enumerated. Section 3 gives the SO problem and the system which defines the central path. Furthermore, in Section 4, the new method for determining search directions is proposed. In the following section, the new primal–dual algorithm for SO is introduced. Section 6 contains the analysis of the algorithm. In the last section, some important concluding remarks are presented.

2. Euclidean Jordan algebras and symmetric cones

In this section, some main properties of the Euclidean Jordan algebras and symmetric cones are presented in the works of Faraut and Korányi [23], Wang and Bai [41].

Let V be an n-dimensional vector space over $\mathbb{R}$ and let us consider the following bilinear map: $\circ : (x, y) \to x \circ y \in V$. Then, $(V, \circ)$ is named a Jordan algebra iff for all $x, y \in V$ the following statements hold: $x \circ y = y \circ x$ and $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$, where $x^2 = x \circ x$. We assume that there exists an identity element e such that $x \circ e = e \circ x = x, \forall x \in V$. A Jordan algebra is said to be Euclidean if there exists an associative inner product, i.e. $\langle x \circ y, z\rangle = \langle x, y \circ z\rangle$. Throughout the paper, we will assume that $(V, \circ)$ is an Euclidean Jordan algebra and we will denote it by V.

For an element $x \in V$, let $L(x) : V \to V$ be a linear operator such that for every $y \in V$, $x \circ y = L(x)y$. In particular, $L(x)e = x$ and $L(x)x = x^2$. For each $x \in V$, we define

$$P(x) := 2L(x)^2 - L(x^2),$$

where $L(x)^2 := L(x)L(x)$. The map $P(x)$ is called the quadratic representation of V.

For a fixed $x \in V$ let r be the smallest integer such that the set $\{e, x, \dots, x^r\}$ is linearly dependent. Then, r is called the degree of the element x and it is denoted as $deg(x)$. The rank of V, denoted as $rank(V)$, is the largest $deg(x)$, for all $x \in V$. Throughout the paper, we will assume that V is an Euclidean Jordan algebra with rank r.

An element $c \in V$ is said to be indempotent iff $c \neq 0$ and $c^2 = c$. The idempotent elements c_1 and c_2 are orthogonal if $c_1 \circ c_2 = 0$. An idempotent element c is said to be primitive if it is nonzero and cannot be expressed by sum of two other nonzero idempotents. A set of primitive idempotents $\{c_1, \dots, c_r\}$ is called Jordan frame iff for all primitive idempotents c_i the following hold: $c_i \circ c_j = 0, i \neq j$ and $\sum_{i=1}^{r} c_i = e$.

Theorem 2.1 (Spectral decomposition, Theorem III.1.2 in Faraut and Korányi [23]): *Let $x \in V$. Then, there exists a Jordan frame $\{c_1, \dots, c_r\}$ and the real numbers $\lambda_1(x), \dots, \lambda_r(x)$ such that*

$$x = \sum_{i=1}^{r} \lambda_i(x) c_i. \tag{1}$$

The numbers $\lambda_i(x)$ are the eigenvalues of x.

Using the eigenvalues the trace of an element $x \in V$ can be defined as $tr(x) = \sum_{i=1}^{r} \lambda_i(x)$.

This theorem gives the opportunity to extend the definition of any real-valued univariate continuous function to elements of Euclidean Jordan algebras using eigenvalues. We assume that φ is a real-valued univariate function differentiable on the interval $(\kappa^2, +\infty)$ such that $2t\varphi'(t^2) - \varphi'(t) > 0, \forall t > \kappa^2$, where $0 < \kappa < 1$. Let us introduce the vector-valued function using the function φ. Let $x \in V$ be with the spectral decomposition given in (1). The vector-valued function φ is defined in the following way:

$$\varphi(x) := \varphi(\lambda_1(x))c_1 + \dots + \varphi(\lambda_r(x))c_r. \tag{2}$$

Similarly, the vector-valued function φ' is defined in the following way:

$$\varphi'(x) := \varphi'(\lambda_1(x))c_1 + \dots + \varphi'(\lambda_r(x))c_r. \tag{3}$$

Let us introduce the inner product:

$$\langle x, s \rangle := tr(x \circ s), \tag{4}$$

where $x, s \in V$. The Frobenius norm, $\| \cdot \|_F$, which is induced by the trace inner product given in (4), is defined by

$$\|x\|_F := \sqrt{\langle x, x \rangle}. \tag{5}$$

Moreover,

$$|\lambda_{\max}(x)| \leq \|x\|_F \quad \text{and} \quad |\lambda_{\min}(x)| \leq \|x\|_F, \tag{6}$$

where $\lambda_{\max}(x)$ and $\lambda_{\min}(x)$ denote the largest and the smallest eigenvalues of x.

A convex cone is said to be symmetric if it is self-dual and homogeneous. We define the cone of squares

$$K := \{x^2 : x \in V\}.$$

The following lemma gives the scaling introduced by Nesterov and Todd, which was first studied for SO by Faybusovich.

Lemma 2.2 (NT-scaling, Lemma 3.2 in Faybusovich [25]): *Let $x, s \in \operatorname{int} K$. Then, there exists a unique $w \in \operatorname{int} K$ such that*

$$x = P(w)s.$$

Moreover,

$$w = P(x)^{\frac{1}{2}} \left(P(x)^{\frac{1}{2}} s\right)^{-\frac{1}{2}} \left[= P(s)^{-\frac{1}{2}} \left(P(s)^{\frac{1}{2}} x\right)^{\frac{1}{2}}\right].$$

The point w is called the scaling point of x and s.

In the following section, we present the SO problem.

3. The SO problem

Let us introduce the following notations:

$$x \succeq_K 0 \quad \Leftrightarrow \quad x \in K \qquad \text{and} \qquad x \succ_K 0 \quad \Leftrightarrow \quad x \in \operatorname{int} K.$$

Beside this,

$$x \succeq_K s \quad \Leftrightarrow \quad x - s \succeq_K 0 \qquad \text{and} \qquad x \succ_K s \quad \Leftrightarrow \quad x - s \succ_K 0.$$

Let us consider the following primal problem:

$$min \ \{\langle c, x\rangle : Ax = b, x \succeq_K 0\} \tag{SOP}$$

and its dual problem:

$$\max \left\{b^T y : A^T y + s = c, s \succeq_K 0\right\} \tag{SOD}$$

where c and the rows of A belong to V, $b, y \in \mathbb{R}^m$ and we assume that $rank(A) = m$. We assume that the *interior-point condition* (IPC) holds for the primal and dual problems, i.e. there exists (x^0, y^0, s^0) so that:

$$\begin{aligned} Ax^0 &= b, \qquad x^0 \succ_K 0, \\ A^T y^0 + s^0 &= c, \qquad s^0 \succ_K 0. \end{aligned} \tag{IPC}$$

The optimal solution of the primal–dual pair can be given by the following system of equations:

$$\begin{aligned} Ax &= b, \qquad x \succeq_K 0, \\ A^T y + s &= c, \qquad s \succeq_K 0, \\ x \circ s &= 0. \end{aligned} \tag{7}$$

The IPMs replace the complementary condition by a parameterized equation. Hence, we obtain the system which defines the central path:

$$\begin{aligned} Ax &= b, \qquad x \succeq_K 0, \\ A^T y + s &= c, \qquad s \succeq_K 0, \\ x \circ s &= \mu e, \end{aligned} \tag{8}$$

where $\mu > 0$.

If the IPC holds, then for a fixed $\mu > 0$ system (8) has unique solution, which is called the *μ-centre* or *analytic centre* ([24,46]). If μ tends to zero, then the central path converges to the optimal solution of the problem.

4. New search directions using an equivalent algebraic transformation

In this section, we will modify the method introduced by Darvay [35] and generalized by Wang and Bai [41] in order to get new search directions. Let us consider the φ vector-valued function, which is induced by the real-valued univariate function φ. Using this, system (8) can be written in the following way:

$$
\begin{aligned}
Ax &= b, \qquad x \succeq_K 0,\\
A^T y + s &= c, \qquad s \succeq_K 0,\\
\varphi\left(\frac{x \circ s}{\mu}\right) &= \varphi(e).
\end{aligned}
\tag{9}
$$

Lemma 4.1 (Lemma 28 in Schmieta and Alizadeh [27]): *Let $u \in \operatorname{int} K$. Then,*

$$x \circ s = \mu e \quad \Leftrightarrow \quad P(u)x \circ P(u)^{-1}s = \mu e.$$

Let $u = w^{-\frac{1}{2}}$, where w is the NT-scaling point of x and s. Let us introduce the following notation:

$$v := \frac{P(w)^{-\frac{1}{2}}x}{\sqrt{\mu}} \left[= \frac{P(w)^{\frac{1}{2}}s}{\sqrt{\mu}} \right]. \tag{10}$$

Using this, we have $\frac{x \circ s}{\mu} = v \circ v$.

We introduce new search directions starting from a simple equivalence. Using that $v^2 = \sum_{i=1}^{r} \lambda_i (v)^2 c_i$ and $e = \sum_{i=1}^{r} c_i$, we obtain

$$
\begin{aligned}
v^2 &= e \Leftrightarrow \sum_{i=1}^{r} \lambda_i(v)^2 c_i = \sum_{i=1}^{r} c_i \Leftrightarrow \sum_{i=1}^{r} (\lambda_i(v)^2 - 1)c_i = 0\\
&\Rightarrow \lambda_i(v)^2 = 1, \forall i = 1, \dots, r.
\end{aligned}
\tag{11}
$$

Using that $x \succ_K 0$ and $s \succ_K 0$, we obtain $v \succ_K 0$, which means that $\lambda_{\min}(v) > 0$. Using this, (11) implies

$$\lambda_i(v) = 1, \forall i = 1, \dots, r \Leftrightarrow v = e. \tag{12}$$

Beside this,

$$v = e \Rightarrow v \circ v = v \circ e = v. \tag{13}$$

Using (11)–(13), we obtain

$$\frac{x \circ s}{\mu} = e \Leftrightarrow v \circ v = e \Leftrightarrow v = e \Leftrightarrow v \circ v = v. \tag{14}$$

Using (14) system (9) can be transformed in the following way:

$$
\begin{aligned}
Ax &= b, \qquad x \succeq_K 0,\\
A^T y + s &= c, \qquad s \succeq_K 0,\\
\varphi\left(v \circ v\right) &= \varphi(v).
\end{aligned}
\tag{15}
$$

Let g be the vector-valued function induced by the real-valued univariate function $g(t)=\varphi(t)-\varphi\left(\sqrt{t}\right), \forall t>\kappa^2$. Suppose that the function φ is chosen in a way such that $g'(t)>0,\ \forall t>\kappa^2$. Using this, the third equation of system (15) can be written in the following form: $g(v\circ v)=g(e)$. We will use Newton's method in order to define search directions. For the strictly feasible $x\in \operatorname{int} K$ and $s\in \operatorname{int} K$ we want to find the search directions $(\Delta x, \Delta y, \Delta s)$ so that $\lambda_{\min}\left(\frac{(x+\Delta x)\circ(s+\Delta s)}{\mu}\right)>\kappa^2$ and

$$\begin{aligned} A\Delta x &= 0, \\ A^T\Delta y+\Delta s &= 0, \\ g\left(\frac{(x+\Delta x)\circ(s+\Delta s)}{\mu}\right) &= g(e). \end{aligned} \tag{16}$$

Let us introduce the following notations:

$$\bar{A}:=\frac{1}{\sqrt{\mu}}AP(w)^{\frac{1}{2}},\quad d_x:=\frac{P(w)^{-\frac{1}{2}}\Delta x}{\sqrt{\mu}},\quad d_s:=\frac{P(w)^{\frac{1}{2}}\Delta s}{\sqrt{\mu}}. \tag{17}$$

Using (10) and (17) and Lemma 4.1 after some reductions, we obtain

$$\begin{aligned} \bar{A}d_x &= 0, \\ \bar{A}^T\Delta y+d_s &= 0, \\ g\left((v+d_x)\circ(v+d_s)\right) &= g(e), \end{aligned} \tag{18}$$

Neglecting the term $d_x\circ d_s$ from the third equation of system (18), we have

$$g\left(v\circ v+v\circ(d_x+d_s)\right)=g(e). \tag{19}$$

Using that $v\circ v+v\circ(d_x+d_s)=e$, from Corollary 2.3 given by Wang and Bai [41], we obtain that the equation (19) is equivalent to

$$g(v\circ v)+g'(v\circ v)\circ v\circ(d_x+d_s)=0. \tag{20}$$

After some reductions we obtain the scaled system

$$\begin{aligned} \bar{A}d_x &= 0, \\ \bar{A}^T\Delta y+d_s &= 0, \\ d_x+d_s &= p_v, \end{aligned} \tag{21}$$

where

$$p_v=[2\varphi(v)-2\varphi(v\circ v)]\circ[2v\circ\varphi'(v\circ v)-\varphi'(v)]^{-1}.$$

Observe that the condition $g'(t)>0, \forall t>\kappa^2$ yields that $2v\circ\varphi'(v\circ v)-\varphi'(v)$ is invertible. For different functions φ, we obtain different search directions that lead to new algorithms. In the next part of the paper, we analyse the algorithm obtained by considering a concrete function φ.

5. New primal–dual interior-point algorithm

In this section, we analyse the case of $\varphi:\left(\frac{1}{\sqrt{2}},\infty\right)\to\mathbb{R},\ \varphi(t)=t^2$, which gives a new search direction. In this case

$$p_v=(v-v^3)\circ(2v^2-e)^{-1}. \tag{22}$$

We assume that $\lambda_{\min}(v) > \frac{1}{\sqrt{2}}$. Hence, the inverse of $2v^2 - e$ exists.
In order to analyse the algorithm, it is necessary to give a measure to the central path:

$$\delta(v) = \delta(x, s; \mu) := \frac{\|p_v\|_F}{2} = \frac{1}{2}\left\|(v - v^3) \circ (2v^2 - e)^{-1}\right\|_F. \tag{23}$$

From the first two equations of system (21), we obtain

$$\langle d_x, d_s \rangle = \langle d_s, d_x \rangle = 0. \tag{24}$$

Let us introduce the following notation:

$$q_v = d_x - d_s. \tag{25}$$

Then,

$$d_x = \frac{p_v + q_v}{2}, \quad d_s = \frac{p_v - q_v}{2}, \quad d_x \circ d_s = \frac{p_v \circ p_v - q_v \circ q_v}{4}. \tag{26}$$

From (24), we get

$$\|p_v\|_F = \|q_v\|_F = 2\delta(v). \tag{27}$$

Furthermore, we consider the other method for determining search directions based on kernel functions. For this, let us present the notion of positive-asymptotic kernel function and it's corresponding barrier defined by Darvay and Takács [45].

Definition 1 (Definition 5.1 in Darvay and Takács [45]): Let $0 \leq \kappa < 1$ and $D = (\kappa, +\infty)$ be an open interval. Moreover, suppose that $\psi : D \rightarrow [0, +\infty)$ is a twice differentiable function. We call this function a *κ-asymptotic kernel function* if

(i) $\psi'(1) = \psi(1) = 0$;
(ii) $\psi''(t) > 0$, for all $t > \kappa$;
(iii) $\psi(t)$ is e-convex on D;
(iv) $\lim_{t\to\kappa} \psi(t) = \lim_{t\to+\infty} \psi(t) = +\infty$.

Definition 2 (Definition 5.2 in Darvay and Takács [45]): A function is a *positive-asymptotic kernel function* iff it is κ-asymptotic and $0 < \kappa < 1$.

The barrier function associated with this type of function is called *positive-asymptotic barrier function*.

Observe that the third equation of (21) is equivalent to

$$d_x + d_s = -\nabla\Psi(v),$$

where $\Psi(v) = \sum_{i=1}^{n} \psi(v_i)$, and $\psi(t)$ is the corresponding positive-asymptotic kernel function. In the case of the function $\varphi(t) = t^2$, we obtain the following kernel function:

$$\psi(t) = \frac{t^2 - 1}{4} - \frac{\log(2t^2 - 1)}{8}. \tag{28}$$

Note that this function is a positive-asymptotic kernel function, where $\kappa = \frac{1}{\sqrt{2}}$.

Primal-dual interior-point algorithm for SO

Let $\epsilon > 0$ *be the accuracy parameter,* $0 < \theta < 1$ *the update parameter (default* $\theta = \frac{1}{14\sqrt{r}}$*) and* τ *the proximity parameter (default* $\tau = \frac{1}{8}$*). Assume that for* (x^0, y^0, s^0) *the IPC holds and* $\mu^0 = \frac{\langle x^0, s^0 \rangle}{r}$*. Furthermore, suppose that* $\delta(x^0, s^0; \mu^0) < \tau$ *and* $\lambda_{min}\left(\frac{x^0 \circ s^0}{\mu}\right) > \frac{1}{2}$*.*

begin
  $x := x^0$; $y := y^0$; $s := s^0$; $\mu := \mu^0$;
  while $\langle x, s \rangle \geq \epsilon$ **do begin**
    calculate $(\Delta x, \Delta y, \Delta s)$ *from system (21) using (17);*
    $x := x + \Delta x$;
    $y := y + \Delta y$;
    $s := s + \Delta s$;
    $\mu := (1 - \theta)\mu$;
  end
end.

Figure 1. Primal–dual interior-point algorithm for SO.

One of the most general classes of kernel functions studied in the literature is the class of eligible kernel functions introduced by Bai et al. [34] for LO. This class was extended to more general problems, too [29,47].

A coercive kernel function is eligible, if satisfies the following conditions:

$$t\psi''(t) + \psi'(t) > 0, \quad t < 1, \tag{29}$$

$$\psi'''(t) < 0, \tag{30}$$

$$2\psi''(t)^2 - \psi'(t)\psi'''(t) > 0, \quad t < 1, \tag{31}$$

$$\psi''(t)\psi(\beta t) - \beta\psi'(t)\psi''(\beta t) > 0, \quad t > 1, \beta > 1. \tag{32}$$

Note that the kernel function given in (28) does not belong to the class of eligible kernel functions, because it is not defined on the whole $(0, \infty)$ interval. Moreover, it does not satisfy the condition (32), even if we consider the $\left(\frac{1}{\sqrt{2}}, \infty\right)$ interval as the domain of the function.

The algorithm is given in Figure 1.

In the next section, we study the complexity of the algorithm.

6. Analysis of the algorithm

Firstly, observe that using (10) and (17) we can write

$$x_+ = x + \Delta x = \sqrt{\mu} P(w)^{\frac{1}{2}} (v + d_x) \tag{33}$$

and

$$s_+ = s + \Delta s = \sqrt{\mu} P(w)^{-\frac{1}{2}} (v + d_s). \tag{34}$$

The first lemma proves the strict feasibility of the full-Newton step.

Lemma 6.1: *If* $\delta := \delta(x, s; \mu) < 1$ *and* $\lambda_{\min}(v) > \frac{1}{\sqrt{2}}$*, then* $x_+ \succ_K 0$ *and* $s_+ \succ_K 0$.

***Proof*:** Let $0 \leq \alpha \leq 1$ and let us define

$$v_x(\alpha) := v + \alpha d_x \quad \text{and} \quad v_s(\alpha) := v + \alpha d_s.$$

Then,

$$\begin{aligned} v_x(\alpha) \circ v_s(\alpha) &= (v + \alpha d_x) \circ (v + \alpha d_s) = v \circ v + \alpha v \circ (d_x + d_s) + \alpha^2 (d_x \circ d_s) \\ &= (1-\alpha) v^2 + \alpha (v^2 + v \circ p_v) + \alpha^2 \left(\frac{p_v \circ p_v - q_v \circ q_v}{4} \right). \end{aligned} \tag{35}$$

We know that

$$\begin{aligned} v \circ v + v \circ p_v &= v \circ v + (v^2 - v^4) \circ (2v^2 - e)^{-1} \\ &= v \circ v \circ (2v^2 - e) \circ (2v^2 - e)^{-1} + (v^2 - v^4) \circ (2v^2 - e)^{-1} \\ &= (2v^4 - v^2 + v^2 - v^4) \circ (2v^2 - e)^{-1} \\ &= v^4 \circ (2v^2 - e)^{-1}. \end{aligned} \tag{36}$$

From $(v^2 - e)^2 \succeq_K 0$, it follows that $v^4 \succeq_K 2v^2 - e$ and

$$v^4 \circ (2v^2 - e)^{-1} \succeq_K e. \tag{37}$$

Using (35), after some reductions we get

$$\begin{aligned} v_x(\alpha) \circ v_s(\alpha) = (1-\alpha) v^2 + \alpha \left(v^2 + v \circ p_v - e + \frac{p_v^2}{4} \right) \\ + \alpha \left[e - \left((1-\alpha) \frac{p_v^2}{4} + \alpha \frac{q_v^2}{4} \right) \right]. \end{aligned} \tag{38}$$

From (37) and $\frac{p_v^2}{4} \in K$ we obtain

$$v^2 + v \circ p_v - e + \frac{p_v^2}{4} \in K. \tag{39}$$

Moreover, using Lemma 2.12 proposed by Gu et al. [48], we obtain:

$$\begin{aligned} \left\| (1-\alpha) \frac{p_v \circ p_v}{4} + \alpha \frac{q_v \circ q_v}{4} \right\|_F &\leq (1-\alpha) \left\| \frac{p_v \circ p_v}{4} \right\|_F + \alpha \left\| \frac{q_v \circ q_v}{4} \right\|_F \\ &\leq (1-\alpha) \frac{\|p_v\|_F^2}{4} + \alpha \frac{\|q_v\|_F^2}{4} = \delta^2 < 1. \end{aligned} \tag{40}$$

From Lemma 2.6 given by Darvay and Takács [45] we have

$$e - (1-\alpha) \frac{p_v \circ p_v}{4} - \alpha \frac{q_v \circ q_v}{4} \in \operatorname{int} K. \tag{41}$$

Moreover, using (39) and (41), we get:

$$v_x(\alpha) \circ v_s(\alpha) \in \operatorname{int} K.$$

Substituting $\alpha = 1$ in Lemma 4.1 proposed by Wang and Bai [41], we obtain:

$$v_x(1) = v + d_x \in \operatorname{int} K \quad \text{and} \quad v_s(1) = v + d_s \in \operatorname{int} K. \tag{42}$$

Using that $P(w)^{\frac{1}{2}}$ and $P(w)^{-\frac{1}{2}}$ are automorphisms of $\operatorname{int} K$, we deduce that the following statements are equivalent:

$$x_+ \in \operatorname{int} K \quad \Leftrightarrow \quad v + d_x \in \operatorname{int} K,$$

and

$$s_+ \in \operatorname{int} K \quad \Leftrightarrow \quad v + d_s \in \operatorname{int} K.$$

Using these and (42) we get

$$x_+ \succ_K 0 \quad \text{and} \quad s_+ \succ_K 0,$$

which proves the lemma. □

Let us introduce the following notation:

$$v_+ := \frac{P(w_+)^{-\frac{1}{2}} x_+}{\sqrt{\mu}} \left[= \frac{P(w_+)^{\frac{1}{2}} s_+}{\sqrt{\mu}} \right], \tag{43}$$

where w_+ is the scaling point of x_+ and s_+.

The following lemma is a technical one which will be used in the next part of the analysis.

Lemma 6.2 (Lemma 7.4 in Darvay and Takács [45]): *Let $f : D \to \mathbb{R}_+^*$ be a decreasing function, where $D = [d, +\infty), d > 0$. Furthermore, let us consider the vector $v \in K$ such that $\lambda_{\min}(v) > d$ and let $\eta > 0$. Then,*

$$\|f(v) \circ (\eta^2 e - v \circ v)\|_F \leq f(\lambda_{\min}(v)) \cdot \|\eta^2 e - v \circ v\|_F \leq f(d) \cdot \|\eta^2 e - v \circ v\|_F.$$

In the following lemma, we prove the quadratic convergence of the full-Newton step.

Lemma 6.3: *Let $\delta = \delta(x, s; \mu) < \frac{1}{8}$ and $\lambda_{\min}(v) > \frac{1}{\sqrt{2}}$. Then, $\lambda_{\min}(v_+) > \frac{1}{\sqrt{2}}$ and*

$$\delta(x_+, s_+; \mu) \leq \frac{5\delta^2 \sqrt{1 - \delta^2}}{1 - 2\delta^2}.$$

Furthermore,

$$\delta(x_+, s_+; \mu) < 6\delta^2.$$

Proof: Substituting $\alpha = 1$ in (35), we get

$$(v + d_x) \circ (v + d_s) = v^2 + v \circ p_v - e + \frac{p_v^2}{4} + e - \frac{q_v^2}{4}. \tag{44}$$

From (36) and (37), we have

$$v^2 + v \circ p_v \succeq_K e. \tag{45}$$

Using (39) and (44), we get

$$(v + d_x) \circ (v + d_s) \succeq_K e - \frac{q_v^2}{4}. \tag{46}$$

Using Lemma 2.12 given by Gu et al. [48], Lemma 14 proposed by Schmieta and Alizadeh [27], (6) and (39), we get the following:

$$\begin{aligned} \lambda_{\min}((v + d_x) \circ (v + d_s)) &\geq \lambda_{\min}\left(v^2 + v \circ p_v - e + \frac{p_v^2}{4}\right) + \lambda_{\min}\left(e - \frac{q_v^2}{4}\right) \\ &\geq 1 - \frac{\|q_v^2\|_F}{4} \geq 1 - \frac{\|q_v\|_F^2}{4} = 1 - \delta^2. \end{aligned}$$

From Proposition 5.9.3 given by Vieira [28] and Theorem 4 introduced by Sturm [26] it follows that

$$\begin{aligned}\lambda_{\min}(v_+) &= \lambda_{\min}\left(\left(P(v+d_x)^{\frac{1}{2}}(v+d_s)\right)^{\frac{1}{2}}\right)\\ &\geq \left(\lambda_{\min}\left((v+d_x)\circ(v+d_s)\right)\right)^{\frac{1}{2}} \geq \sqrt{1-\delta^2}. \qquad (47)\end{aligned}$$

From $\delta < \frac{1}{8}$ it follows that $\sqrt{1-\delta^2} > \frac{\sqrt{63}}{8}$. Using this and (47), we obtain that $\lambda_{\min}(v_+) > \frac{1}{\sqrt{2}}$. We write $\delta(v_+) = \delta(x_+, s_+; \mu)$ in the following way:

$$\begin{aligned}\delta(v_+) &= \frac{1}{2}\left\|(v_+ - v_+^3)\circ(2v_+^2 - e)^{-1}\right\|_F\\ &= \frac{1}{2}\left\|v_+\circ(2v_+^2-e)^{-1}\circ(e-v_+^2)\right\|, \qquad (48)\end{aligned}$$

where $(2v_+^2 - e)^{-1}$ is well defined, because $\lambda_{\min}(2v_+^2 - e) > 0$, which means that $2v_+^2 - e$ is invertible. Let us introduce the function $f : \left(\frac{1}{\sqrt{2}}, +\infty\right) \to \mathbb{R}, f(t) = \frac{t}{2t^2-1} > 0, \forall t > \frac{1}{\sqrt{2}}$. From $f'(t) < 0$ it follows that f is a decreasing function. Using Lemma 6.2, (47) and (48), we obtain the following:

$$\delta(x_+, s_+; \mu) \leq \frac{1}{2} f\left(\sqrt{1-\delta^2}\right)\|e - v_+\circ v_+\|_F. \qquad (49)$$

We know that $v^2 + v\circ p_v = v^4 \circ (2v^2 - e)^{-1}$, which implies

$$\begin{aligned}e + (v^2-e)^2\circ(2v^2-e)^{-1} &= \left[(v^2-e)^2 + 2v^2 - e\right]\circ(2v^2-e)^{-1}\\ &= v^4\circ(2v^2-e)^{-1} = v^2 + v\circ p_v. \qquad (50)\end{aligned}$$

Using (22), (44) and (50), we have

$$\begin{aligned}(v+d_x)\circ(v+d_s) &= e + (v^2-e)^2\circ(2v^2-e)^{-1}\\ &\quad + \frac{v^2\circ\left(e-v^2\right)^2\circ(2v^2-e)^{-2}}{4} - \frac{q_v^2}{4}\\ &= e + (v^2-e)^2\circ(2v^2-e)^{-1}\\ &\quad \circ\left[e + \frac{v^2\circ(2v^2-e)^{-1}}{4}\right] - \frac{q_v^2}{4}\\ &= e + \frac{(v^2-e)^2\circ(9v^2-4e)\circ(2v^2-e)^{-2}}{4} - \frac{q_v^2}{4}. \qquad (51)\end{aligned}$$

From (22) and (51)

$$\begin{aligned}e - (v+d_x)\circ(v+d_s) &= \frac{q_v^2}{4} - \frac{(v^2-e)^2\circ(9v^2-4e)\circ(2v^2-e)^{-2}}{4}\\ &= \frac{q_v^2}{4} - \frac{v^2\circ\left(e-v^2\right)^2\circ(2v^2-e)^{-2}}{4}\\ &\quad \circ v^{-2}\circ(9v^2-4e)\\ &= \frac{q_v^2}{4} - (9v^2-4e)\circ v^{-2}\circ\frac{p_v^2}{4}. \qquad (52)\end{aligned}$$

From $4e \succ_K 0$, we have $9v^2 - 4e \prec_K 9v^2$. Using this, (27), (52), Lemma 2.12 proposed by Gu et al. [48], Lemma 30 of Schmieta and Alizadeh [27] and Proposition 5.9.3 given by Vieira [28] we can write

$$
\begin{aligned}
\|e - v_+ \circ v_+\|_F &= \|e - P(v + d_x)^{\frac{1}{2}}(v + d_s)\|_F \le \|(v + d_x) \circ (v + d_s) - e\|_F \\
&\le \left\| \frac{q_v^2}{4} \right\|_F + \left\| (9v^2 - 4e) \circ v^{-2} \circ \frac{p_v^2}{4} \right\|_F \\
&\le \frac{\|q_v\|_F^2}{4} + 9\frac{\|p_v\|_F^2}{4} = 10\delta^2. \qquad (53)
\end{aligned}
$$

From (49) and (53) we obtain

$$
\delta(x_+, s_+; \mu) \le \frac{5\delta^2\sqrt{1-\delta^2}}{1 - 2\delta^2},
$$

which proves the first statement of the lemma. We will transform this in order to get the second statement of the lemma.

$$
\begin{aligned}
\delta(x_+, s_+; \mu) &\le \frac{5\delta^2\sqrt{1-\delta^2}}{1-2\delta^2} = \frac{5\delta^2(\sqrt{2}\sqrt{1-\delta^2} - 1)}{\sqrt{2}(1-2\delta^2)} + \frac{5\delta^2}{\sqrt{2}(1-2\delta^2)} \\
&= \frac{5\delta^2}{\sqrt{2}(1+\sqrt{2}\sqrt{1-\delta^2})} + \frac{5\delta^2}{\sqrt{2}(1-2\delta^2)}. \qquad (54)
\end{aligned}
$$

From $\delta < \frac{1}{8}$ we have $\delta^2 < \frac{1}{64}$ and $\sqrt{1-\delta^2} > \frac{\sqrt{63}}{8}$. This yields

$$
\sqrt{2}(1 + \sqrt{2}\sqrt{1-\delta^2}) > \sqrt{2} + 2\frac{\sqrt{63}}{8} = \frac{8\sqrt{2} + 2\sqrt{63}}{8}. \qquad (55)
$$

Beside this,

$$
\sqrt{2}(1 - 2\delta^2) > \frac{31\sqrt{2}}{32}. \qquad (56)
$$

From (54)–(56), it follows that

$$
\delta(v_+) < 5\delta^2 \left(\frac{8}{8\sqrt{2} + 2\sqrt{63}} + \frac{32}{31\sqrt{2}} \right) < 6\delta^2.
$$

Thus,

$$
\delta(x_+, s_+; \mu) < 6\delta^2,
$$

which proves the last statement of the lemma. □

The following lemma gives an upper bound for the duality gap after a full-Newton step.

Lemma 6.4: *We assume that we obtained x_+ and s_+ after a full-Newton step. Then,*

$$
\langle x_+, s_+ \rangle \le \mu(r + 8\delta^2).
$$

Furthermore, if $\delta < \frac{1}{8}$ and $r \ge 1$, then

$$
\langle x_+, s_+ \rangle < \frac{9}{8}\mu r.
$$

***Proof*:** Using (4), (24), (33), (34) and (36) we get

$$\begin{aligned}\langle x_+, s_+\rangle &= tr(x_+ \circ s_+) = \mu\, tr((v+d_x)\circ(v+d_s))\\ &= \mu tr(v\circ v + v\circ p_v + d_x \circ d_s)\\ &= \mu\langle e, v\circ v + v\circ p_v\rangle + \mu\langle e, d_x\circ d_s\rangle\\ &= \mu\langle e, v^4\circ(2v^2-e)^{-1}\rangle + \mu\langle d_x, d_s\rangle\\ &= \mu\langle e, v^4\circ(2v^2-e)^{-1}\rangle. \end{aligned}\tag{57}$$

From $4v^{-2} \succeq_K 0$, (22) and (36), we have

$$\begin{aligned} v^2 + v\circ p_v &= v^4\circ(2v^2-e)^{-1} = e + (v^2-e)^2\circ(2v^2-e)^{-1}\\ &= e + 4(2v^2-e)\circ v^{-2}\circ \frac{v^2\circ\left(e-v^2\right)^2\circ(2v^2-e)^{-2}}{4}\\ &= e + 4(2v^2-e)\circ v^{-2}\circ\frac{p_v^2}{4} = e + (8e-4v^{-2})\circ\frac{p_v^2}{4}\\ &\preceq_K e + 8\frac{p_v^2}{4}. \end{aligned}\tag{58}$$

From (57), (58) and Lemma 2.5 of Darvay and Takács [45], it follows that

$$\langle x_+, s_+\rangle = \mu\langle e, v^4\circ(2v^2-e)^{-1}\rangle \le \mu\langle e,e\rangle + \mu\langle e, 8\frac{p_v^2}{4}\rangle = \mu(r+8\delta^2), \tag{59}$$

which proves the first statement of the lemma. If $\delta < \frac{1}{8}$ and $r \ge 1$, then $1 \le \sqrt{r}$, which yields $\delta^2 < \frac{1}{64}r$. Using this and (59), we obtain

$$\langle x_+, s_+\rangle < \frac{9}{8}\mu r,$$

which proves the second part of the lemma. □

In the following lemma, we analyse the effect of the full-Newton step on the proximity measure after reducing the value of μ.

Lemma 6.5: *Let $\delta = \delta(x,s;\mu) < \frac{1}{8}$ and $\mu_+ = (1-\theta)\mu$, where $0<\theta<1$. Let us define $\eta = \sqrt{1-\theta}$ and $v_\sharp = \frac{1}{\eta}v_+$. Then,*

$$\delta(x_+, s_+; \mu_+) < \frac{\sqrt{1-\delta^2}(\theta\sqrt{r}+10\delta^2)}{2\sqrt{1-\theta}(1-2\delta^2)}.$$

Moreover, if $\theta = \frac{1}{14\sqrt{r}}$ and $r \ge 1$, then $\delta(x_+, s_+; \mu_+) < \frac{1}{8}$.

***Proof*:** From $v_\sharp = \frac{1}{\eta}v_+$, it follows that

$$\begin{aligned}(v_\sharp - v_\sharp^3)\circ(2v_\sharp^2-e)^{-1} &= \left(\frac{1}{\eta}v_+ - \frac{1}{\eta^3}v_+^3\right)\circ\left(\frac{2}{\eta^2}v_+^2 - e\right)^{-1}\\ &= \frac{1}{\eta^3}v_+\circ(\eta^2 e - v_+^2)\circ\left(\frac{1}{\eta^2}(2v_+^2-\eta^2 e)\right)^{-1}\\ &= \frac{1}{\eta}v_+\circ(\eta^2 e - v_+^2)\circ\left(2v_+^2-\eta^2 e\right)^{-1}. \end{aligned}\tag{60}$$

Let us consider the function $h(t) = \frac{t}{2t^2-\eta^2}$, where $t > \frac{\eta}{\sqrt{2}}$. Then,

$$\left(v_\sharp - v_\sharp^3\right) \circ \left(2v_\sharp^2 - e\right)^{-1} = \frac{1}{\eta} h(v_+) \circ (\eta^2 e - v_+^2).$$

From $h'(t) < 0$, it follows that h is a strictly decreasing function, so we can use Lemma 6.2 and we get

$$\begin{aligned}
\delta(x_+, s_+; \mu_+) &= \frac{1}{2}\left\| (v_\sharp - v_\sharp^3) \circ (2v_\sharp^2 - e)^{-1} \right\|_F \\
&\le \frac{1}{2\eta} h(\sqrt{1-\delta^2}) \|\eta^2 e - v_+^2\|_F \\
&= \frac{\sqrt{1-\delta^2}}{2\sqrt{1-\theta}(1-2\delta^2+\theta)} \left\|(1-\theta)e - v_+^2\right\|_F \\
&< \frac{\sqrt{1-\delta^2}}{2\sqrt{1-\theta}(1-2\delta^2)} \left\|(1-\theta)e - v_+^2\right\|_F
\end{aligned} \tag{61}$$

Using (53) we have

$$\|(1-\theta)e - v_+^2\|_F \le \| - \theta e\|_F + \|e - v_+^2\|_F \le \theta\sqrt{r} + 10\delta^2,$$

thus

$$\delta(x_+, s_+; \mu_+) < \frac{\sqrt{1-\delta^2}(\theta\sqrt{r} + 10\delta^2)}{2\sqrt{1-\theta}(1-2\delta^2)},$$

which proves the first statement of the lemma. We will transform this in order to obtain the second statement of the lemma. We know that

$$\begin{aligned}
\frac{\sqrt{1-\delta^2}(\theta\sqrt{r}+10\delta^2)}{2\sqrt{1-\theta}(1-2\delta^2)} &= \frac{\theta\sqrt{r}+10\delta^2}{2\sqrt{1-\theta}} \frac{\sqrt{1-\delta^2}}{1-2\delta^2} \\
&= \frac{\theta\sqrt{r}+10\delta^2}{2\sqrt{1-\theta}} \left(\frac{\sqrt{2}\sqrt{1-\delta^2}-1}{\sqrt{2}(1-2\delta^2)} + \frac{1}{\sqrt{2}(1-2\delta^2)} \right) \\
&= \frac{\theta\sqrt{r}+10\delta^2}{2\sqrt{1-\theta}} \left(\frac{1}{2\sqrt{1-\delta^2}+\sqrt{2}} + \frac{1}{\sqrt{2}(1-2\delta^2)} \right).
\end{aligned} \tag{62}$$

From $\theta = \frac{1}{14\sqrt{r}}$ and $\delta^2 < \frac{1}{64}$, we have $\theta\sqrt{r} + 10\delta^2 < \frac{1}{14} + \frac{5}{32}$. Beside this, from $r \ge 1$ we get $\sqrt{1-\theta} \ge \sqrt{\frac{13}{14}}$. Using these, (55), (56) and (62) we obtain

$$\delta(x_+, s_+; \mu_+) < \frac{\frac{1}{14} + \frac{5}{32}}{2\sqrt{\frac{13}{14}}} \left(\frac{8}{8\sqrt{2} + 2\sqrt{63}} + \frac{32}{31\sqrt{2}} \right) < \frac{1}{8},$$

which proves the second statement of the lemma. □

The following lemma gives an upper bound for the number of iterations of the algorithm.

Lemma 6.6: *We assume that (x^0, s^0) is a strictly feasible pair, $\mu^0 = \frac{\langle x^0, s^0\rangle}{r}$ and $\delta(x^0, s^0; \mu^0) < \frac{1}{8}$. Moreover, let x^k and s^k be the vectors obtained after k iterations. Then, we have $\langle x^k, s^k\rangle < \epsilon$, if*

$$k \ge \left\lceil \frac{1}{\theta} \log \frac{9\langle x^0, s^0\rangle}{8\epsilon} \right\rceil.$$

Proof: From Lemma 6.4, it follows that

$$\langle x^k, s^k \rangle < \frac{9}{8} r\mu^k = \frac{9}{8} r(1-\theta)^k \mu^0 = \frac{9}{8}(1-\theta)^k \langle x^0, s^0 \rangle.$$

Then, we have $\langle x^k, s^k \rangle < \epsilon$, if

$$(1-\theta)^k \langle x^0, s^0 \rangle \leq \frac{8\epsilon}{9}.$$

Taking logarithms, we may write

$$k \log (1-\theta) + \log\langle x^0, s^0 \rangle \leq \log \frac{8\epsilon}{9}.$$

Using $-\log(1-\theta) \geq \theta$, we see that the inequality stands only if

$$k\theta \geq \log\langle x^0, s^0 \rangle - \log \frac{8\epsilon}{9} = \log \frac{9\langle x^0, s^0 \rangle}{8\epsilon}.$$

This proves the lemma. □

Theorem 6.7: *Let $\theta = \frac{1}{14\sqrt{r}}$. Then, the algorithm given in Figure 1 demands no more than*

$$14\sqrt{r} \log \frac{9\langle x^0, s^0 \rangle}{8\epsilon}$$

interior-point iterations. Furthermore, the resulting (x, s) pair satisfies $\langle x, s \rangle < \epsilon$.

Let $f, g : \mathbb{R} \to \mathbb{R}_+$ be two functions. Then, we say that $f(t) = O(g(t))$, if there exists a positive constant $\bar{c}$ and a real number t_0, for which the inequality $f(t) \leq \bar{c} g(t)$ stands, for all $t > t_0$.

Corollary 6.8: *If $x^0 = s^0 = e$, then the algorithm given in Figure 1 requires no more than*

$$O\left(\sqrt{r} \log \frac{r}{\epsilon}\right)$$

interior-point iterations.

7. Conclusion

In this paper, we introduced a new IPM for SO. We achieved this by proposing a new technique for determining search directions based on a new algebraic equivalent transformation. We considered a special case by applying the square function on the modified centring equation. We established a connection between this method and the approach based on kernel functions. In this way, we obtained that the corresponding function is a positive-asymptotic kernel function. Moreover, we proved that the new algorithm has the same complexity as the best-known IPMs for SO.

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