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AN ADAPTIVE SCALARIZATION METHOD IN MULTIOBJECTIVE OPTIMIZATION*

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Abstract. This paper presents a new method for the numerical solution of nonlinear multiobjective optimization problems with an arbitrary partial ordering in the objective space induced by a closed pointed convex cone. This algorithm is based on the well-known scalarization approach by Pascoletti and Serafini and adaptively controls the scalarization parameters using new sensitivity results. The computed image points give a nearly equidistant approximation of the whole Pareto surface. The effectiveness of this new method is demonstrated with various test problems and an applied problem from medicine.

Key words. multicriteria optimization, vector optimization, approximation, sensitivity, scalarization approaches

AMS subject classifications. 90C29, 90C31, 90C59

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1. Introduction.

The optimization problems arising nowadays in application areas like engineering, economics, or the sciences are often multiobjective; i.e., several competing objective functions have to be minimized all at once. Those optimization problems have in general not only one best solution but the solution set is very large.

In the last decades the main focus was on finding one minimal solution, e.g., by interactive methods, whereas objective numerical calculations alternate with subjective decisions done by a so-called decision maker (d.m.). Based on much better computer performance it is now possible to represent the whole efficient set. Having an approximation of the whole efficient set available, the d.m. gets a useful insight in the problem structure. Especially in engineering tasks it is interesting to have all design alternatives available [28]. So in this paper we aim to generate an approximation of the efficient set as it is done in many other works, e.g., in [9, 18, 19, 24, 42, 48].

However, the information provided by this approximation depends mainly on the quality of the approximation. Many points are related to a high numerical effort and to too many points which have to be interpreted by the d.m. A sparse approximation neglects large parts of the efficient set. According to different quality criteria as discussed in [47], an approximation is good in the sense of a stinted but representative presentation if the approximation points are evenly spread with equal distances over the whole image of the solution set (see also [19]). Thus our aim is to generate equidistant points in the value space. For this we use a parameter dependent scalarization approach by Pascoletti and Serafini [45] and control the choice of the parameters adaptively.

A common concept for minimality in the multiobjective context is Edgeworth–Pareto- (EP-) minimality based on the natural ordering defined by the ordering cone $\boldsymbol{R}^m_+ := \{x \in \boldsymbol{R}^m \mid x_i \geq 0,\ i = 1, \ldots, m\}$. Using arbitrary partial orderings minimality is defined similarly (see, e.g., [4, 29, 31, 49, 58]). Allowing this, preference structures can be mapped which cannot be formulated explicitly as an objective function; see [55,

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Example 4.1]. In decision theory and in economics, arbitrary partial orderings are a well-known tool to model the relative importance of several criteria or to incorporate groups of decision makers, as promoted, for instance, by Wiecek [56]. For example, cones being a superset of the positive orthant can be defined by allowable trade-offs between the objectives or by grouping objectives according to their importance. This provides a more useful representation of the decision makers' preferences than the standard cone because the set of efficient points is reduced by undesired solutions.

For example, in [26, 27] convex polyhedral cones are used for modeling the preferences of a d.m. based on trade-off information facilitating multicriteria decision making. In portfolio optimization [2] polyhedral cones as well as nonfinitely generated cones are considered. Besides, orderings, other than the natural ordering, are important in [20] where a scalar bilevel optimization problem is reformulated as a multiobjective problem. There a nonconvex cone that is the union of two convex cones is used. In [13] a multiobjective optimization problem w.r.t. a cone $K = \boldsymbol{R}^m_+ \times \{0_n\}$ is considered for solving multiobjective bilevel optimization problems. Helbig [24] constructs various cones as a tool for finding EP-minimal points; see also [37, 51]. In addition to that, Wu [57] considers convex cones for a solution concept in fuzzy multiobjective optimization. Hence, multiobjective optimization problems w.r.t. arbitrary partial orderings are essential in decision making and are further an important tool in other areas. Therefore we develop our results w.r.t. more general partial orderings defined by closed pointed convex cones.

In the remainder we proceed as follows: in section 2 we recall the basic concepts in multiobjective optimization. In section 3 we discuss the well-known scalarization approach by Pascoletti and Serafini and we give some properties of this approach. We choose this scalarization because it is very general in the sense that many other scalarizations can be seen as a special case of it; see section 7 and [15]. In section 4 we present our main sensitivity theorem on which we base our new adaptive method in section 5. In section 6 this is applied to some test problems and to a problem in intensity modulated radiotherapy in medicine. We conclude with some remarks on the presented scalarization approach and on the transferability of the given procedure to other scalarization approaches in section 7.

2. Basic notations and concepts. We consider multiobjective optimization problems formally defined by

$$\begin{array}{c} \min_K f(x) = (f_1(x), \dots, f_m(x))^\top \\ \text{subject to the constraint} \\ x \in \Omega \subset \boldsymbol{R}^n. \end{array} \tag{2.1}$$

Here, K represents the considered partial ordering defined later.

We assume the following:

Assumption 1. Let C be a closed convex cone in $\boldsymbol{R}^p$, $\hat{S} \subset \boldsymbol{R}^n$ a nonempty open subset, and $S \subset \hat{S}$ closed and convex. Let the functions $f\colon \hat{S} \to \boldsymbol{R}^m$, $g\colon \hat{S} \to \boldsymbol{R}^p$, and $h\colon \hat{S} \to \boldsymbol{R}^q$ ($m, n \in \boldsymbol{N}$, $p, q \in \boldsymbol{N}_0$, $m \geq 2$) be continuously differentiable on $\hat{S}$. Let the set $\Omega \subset \boldsymbol{R}^n$, given by $\Omega = \{x \in S \mid g(x) \in C,\ h(x) = 0_q\}$, be compact.

A *convex cone* $C \subset \boldsymbol{R}^p$ is a subset of $\boldsymbol{R}^p$ with the property $\lambda(x + y) \in C$ for all $\lambda \geq 0,\ x, y \in C$. For defining minimality we need a partial ordering "$\leq$" in the image space $\boldsymbol{R}^m$. Here we mean by a partial ordering a binary relation which is reflexive, transitive, and compatible with addition and with nonnegative scalar multiplication. Any partial ordering $\leq$ defines a convex cone K by $K := \{x \in \boldsymbol{R}^m \mid 0_m \leq x\}$ and any convex cone $K \subset \boldsymbol{R}^m$, then called *ordering cone*, defines a partial ordering by

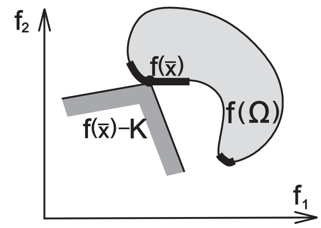

Fig. 2.1. *Illustration of Definition* 2.1 *for* $m = 2$. *The thick part of the boundary of* $f(\Omega)$ *denotes the set* $\mathcal{E}(f(\Omega), K)$.

$\leq_K := \{(x, y) \in \boldsymbol{R}^m \times \boldsymbol{R}^m \mid y - x \in K\}$. For example, the natural ordering is defined by the cone $\boldsymbol{R}^m_+$. The partial ordering is antisymmetric if the related ordering cone is *pointed*, i.e., $K \cap (-K) = \{0_m\}$. Here we consider only partial orderings defined by closed pointed convex cones.

Assumption 2. Let Assumption 1 hold. In addition let $K \subset \boldsymbol{R}^m$ be a closed pointed convex cone.

Minimality is then defined by (see, among others, [29, 31, 49, 58]):

Definition 2.1. *Let K be a closed pointed convex cone. A point $\bar{x} \in \Omega$ is called K-minimal of* (2.1) *if $(f(\bar{x}) - K) \cap f(\Omega) = \{f(\bar{x})\}$. Additionally for int$(K) \neq \emptyset$ a point $\bar{x} \in \Omega$ is called weakly K-minimal of* (2.1) *if $(f(\bar{x}) - \text{int}(K)) \cap f(\Omega) = \emptyset$.*

For an illustration of this definition see Figure 2.1.

We denote the set of all K-minimal points by $\mathcal{M}(f(\Omega), K)$ and the set of all weakly K-minimal points by $\mathcal{M}_w(f(\Omega), K)$. The set $\mathcal{E}(f(\Omega), K) := \{f(x) \in \boldsymbol{R}^m \mid x \in \mathcal{M}(f(\Omega), K)\}$ is called *efficient set* (see Figure 2.1) and the set $\mathcal{E}_w(f(\Omega), K) := \{f(x) \in \boldsymbol{R}^m \mid x \in \mathcal{M}_w(f(\Omega), K)\}$ *weakly efficient set.* For $K = \boldsymbol{R}^m_+$ the K-minimal points are denoted as *Edgeworth–Pareto (EP)-minimal* points, too.

Later on we need that in the bicriteria case ($m = 2$) every ordering cone is finitely generated. This is stated in the following lemma. The proof is omitted here and the interested reader is referred to [13]. For the definition of a finitely generated cone (or polyhedral cone) see [46, Definition 2.17, 2.18].

Lemma 2.2. *Let $K \subset \boldsymbol{R}^2$ be a closed pointed convex cone with $K \neq \{0_2\}$. Then K is polyhedral and there is either a $k \in \boldsymbol{R}^2 \setminus \{0_2\}$ with $K = \{\lambda k \mid \lambda \geq 0\}$ or there are $l^1, l^2 \in \boldsymbol{R}^2 \setminus \{0_2\}$, l^1, l^2 linearly independent, and $\tilde{l}^1, \tilde{l}^2 \in \boldsymbol{R}^2 \setminus \{0_2\}$, $\tilde{l}^1, \tilde{l}^2$ linearly independent, with*

(2.2)
$$K = \left\{y \in \boldsymbol{R}^2 \mid l^{1\top} y \geq 0,\ l^{2\top} y \geq 0\right\} = \left\{y \in \boldsymbol{R}^2 \mid y = \lambda^1 \tilde{l}^1 + \lambda^2 \tilde{l}^2,\ \lambda^1, \lambda^2 \geq 0\right\}.$$

In general, Lemma 2.2 is not true for $m \geq 3$. This is illustrated by the ice-cream cone $\{(x_1, x_2, x_3) \in \boldsymbol{R}^3 \mid x_1 \geq \sqrt{x_2^2 + x_3^2}\}$ which is not polyhedral.

3. Scalarization approach. For determining minimal solutions of the multiobjective optimization problem (2.1) a common approach is the scalarization of the problem. We examine the scalarization problem by Pascoletti and Serafini [45] named (SP(a, r)) which is defined by

$$\text{(SP}(a,r)) \qquad \begin{array}{c} \min t \\ \text{subject to the constraints} \\ a + t\, r - f(x) \in K, \\ x \in \Omega, \quad t \in \boldsymbol{R} \end{array}$$

for parameters $a, r \in \boldsymbol{R}^m$. Properties of this scalarization approach can be found in [13, 45]. Here we concentrate on some major results only.

THEOREM 3.1.

(a) *Let $\bar{x}$ be a K-minimal point of* (2.1)*, then $(0, \bar{x})$ is a global minimal solution of* $(\mathrm{SP}(a,r))$ *with $a = f(\bar{x})$, $r \in K \setminus \{0_m\}$.*

(b) *Let $(\bar{t}, \bar{x})$ be a global minimal solution of* $(\mathrm{SP}(a,r))$, *then $\bar{x}$ is a weakly K-minimal solution of* (2.1).

Theorem 3.1 is given for global minimal solutions, but the statements can easily be adapted to local minimal solutions (for a continuous function f) too; see [13, 45]. Theorem 3.1 (a) yields that any K-minimal solution of the multiobjective optimization problem can be found by solving the scalar problem $(\mathrm{SP}(a,r))$ for appropriate parameters, even for nonconvex problems. This is not the case for any scalarization approach as, e.g., the weighted sum method [59] shows. In the following Theorems 3.2, 3.3, and 3.5 we even see that it is sufficient to consider only variations of the parameter a in (a subset of) a hyperplane for a fixed chosen parameter $r \in K \setminus \{0_m\}$. Then it is still possible that any minimal solution of the multiobjective optimization can be recovered.

THEOREM 3.2. *Let $\bar{x} \in \mathcal{M}(f(\Omega), K)$ and $r \in K$ be given. We define a hyperplane H by $H = \{y \in \boldsymbol{R}^m \mid b^\top y = \beta\}$ with $b \in \boldsymbol{R}^m \setminus \{0_m\}$, $b^\top r \neq 0$, $\beta \in \boldsymbol{R}$. Then there is a parameter $a \in H$ and some $\bar{t} \in \boldsymbol{R}$ so that $(\bar{t}, \bar{x})$ is a minimal solution of* $(\mathrm{SP}(a,r))$.

Proof. We set $\bar{t} = (b^\top f(\bar{x}) - \beta)/(b^\top r)$ and $a = f(\bar{x}) - \bar{t}\, r$. Then $a \in H$ and $(\bar{t}, \bar{x})$ is feasible for $(\mathrm{SP}(a,r))$. We assume $(\bar{t}, \bar{x})$ is not a minimal solution of $(\mathrm{SP}(a,r))$. Then there are $t' \in \boldsymbol{R}$, $t' < \bar{t}$, $x' \in \Omega$, and $k' \in K$ so that $a + t' r - f(x') = k'$. Using the definition of the parameter a, this results in $f(\bar{x}) = f(x') + k' + (\bar{t} - t')r$. Since K is a pointed convex cone, $r \in K \setminus \{0_m\}$ and $\bar{t} - t' > 0$ we have $f(\bar{x}) \in f(x') + K \setminus \{0_m\}$ for $x' \in \Omega$ which is a contradiction to $\bar{x}$ K-minimal. ∎

We can restrict the parameter set even further. First we consider the bicriteria case ($m = 2$). Then, according to Lemma 2.2, every closed pointed convex cone is finitely generated. We suppose the cone K is given by (2.2). (The more trivial case $K = \{\lambda k \mid \lambda \geq 0\}$ for some $k \in \boldsymbol{R}^2 \setminus \{0_2\}$ can be handled similarly, but we will not consider this less interesting case here. For more details see [13, pp. 73f]). Next we solve the scalar optimization problems

$$\min_{x \in \Omega} l^{i\top} f(x), \qquad i = 1, 2, \tag{3.1}$$

with minimal solutions $\bar{x}^i$, $i = 1, 2$. Then the points $\bar{x}^i$, $i = 1, 2$, are weakly K-minimal and we can easily show that for every K-minimal point x of (2.1) we have

$$l^{1\top} f\left(\bar{x}^1\right) \leq l^{1\top} f(x) \leq l^{1\top} f\left(\bar{x}^2\right) \quad \text{and} \quad l^{2\top} f\left(\bar{x}^2\right) \leq l^{2\top} f(x) \leq l^{2\top} f\left(\bar{x}^1\right). \tag{3.2}$$

Using (3.2), $l^{1\top} f(\bar{x}^1) = l^{1\top} f(\bar{x}^2)$ implies $l^{1\top} f(x) = l^{1\top} f(\bar{x}^2)$ and, consequently, $f(x) \in f(\bar{x}^2) + K$ for all $x \in \mathcal{M}(f(\Omega), K)$ resulting in $\mathcal{E}(f(\Omega), K) = \{f(\bar{x}^2)\}$. The same, $l^{2\top} f(\bar{x}^2) = l^{2\top} f(\bar{x}^1)$, leads to $\mathcal{E}(f(\Omega), K) = \{f(\bar{x}^1)\}$. Thus, assuming the efficient set does not consist of one point only, we have

$$l^{1\top} f\left(\bar{x}^1\right) < l^{1\top} f\left(\bar{x}^2\right) \quad \text{and} \quad l^{2\top} f\left(\bar{x}^2\right) < l^{2\top} f\left(\bar{x}^1\right). \tag{3.3}$$

If we define the points $\bar{a}^i$, $i = 1, 2$, by a projection of the points $f(\bar{x}^i)$, $i = 1, 2$, in direction $r \in K$ on the hyperplane H, then we get

$$\bar{a}^i := f\left(\bar{x}^i\right) - \bar{t}^i\, r \in H \quad \text{with} \quad \bar{t}^i := \frac{b^\top f\left(\bar{x}^i\right) - \beta}{b^\top r}, \qquad i = 1, 2. \tag{3.4}$$

In the next theorem we see that we can restrict ourselves to the set $H^a := \{y \in \boldsymbol{R}^2 \mid y = \lambda \bar{a}^1 + (1-\lambda)\bar{a}^2,\ \lambda \in [0,1]\}$ for choosing the parameter a.

Theorem 3.3. *We consider the multiobjective optimization problem* (2.1) *with* $m = 2$ *and* K *as in* (2.2). *Further let* $\bar{a}^i,\ i = 1,2$, *be defined as in* (3.4) *with* $\bar{x}^i$, $i = 1,2$, *minimal solutions of* (3.1) *and assume* $\bar{x} \in \mathcal{M}(f(\Omega), K)$. *Then there is a parameter* $a \in H^a \subset H$ *and some* $\bar{t} \in \boldsymbol{R}$ *so that* $(\bar{t}, \bar{x})$ *is a minimal solution of* $(\mathrm{SP}(a,r))$.

Proof. Notice that $\bar{a}^1, \bar{a}^2 \in H$ and, hence, $H^a \subset H$. According to Theorem 3.2 we have for any $\bar{x} \in \mathcal{M}(f(\Omega), K)$ a parameter $a \in H$ and some $\bar{t} \in \boldsymbol{R}$ so that $(\bar{t}, \bar{x})$ is a minimal solution of $(\mathrm{SP}(a,r))$. This is achieved by $\bar{t} = (b^\top f(\bar{x}) - \beta)/(b^\top r)$ and $a = f(\bar{x}) - \bar{t}\, r$. Hence, it suffices to show $a = \lambda \bar{a}^1 + (1-\lambda)\bar{a}^2$ for some $\lambda \in [0,1]$. Using the definitions of a, $\bar{a}^1$, and $\bar{a}^2$, this equation can be written as

$$f(\bar{x}) - \bar{t}\, r = \lambda \left(f\left(\bar{x}^1\right) - \bar{t}^1\, r \right) + (1-\lambda) \left(f\left(\bar{x}^2\right) - \bar{t}^2\, r \right). \tag{3.5}$$

If the efficient set consists of one point only (and then this point is $f(\bar{x}^1)$ or $f(\bar{x}^2)$) (3.5) is fulfilled for $\lambda = 1$ or $\lambda = 0$. Otherwise, the strict inequalities (3.3) hold. Equation (3.5) can further be written as

$$f(\bar{x}) = \lambda\, f\left(\bar{x}^1\right) + (1-\lambda)\, f\left(\bar{x}^2\right) + \left(\bar{t} - \lambda\, \bar{t}^1 - (1-\lambda)\, \bar{t}^2\right) r, \tag{3.6}$$

and we differentiate the following two cases: $(\bar{t} - \lambda\, \bar{t}^1 - (1-\lambda)\, \bar{t}^2) = \frac{1}{b^\top r}(b^\top(f(\bar{x}) - \lambda\, f(\bar{x}^1) - (1-\lambda)\, f(\bar{x}^2)) \geq 0$ and $(\bar{t} - \lambda\, \bar{t}^1 - (1-\lambda)\, \bar{t}^2) < 0$.

For $\bar{t} - \lambda\, \bar{t}^1 - (1-\lambda)\, \bar{t}^2 \geq 0$ we suppose (3.6) is satisfied for $\lambda < 0$. Applying the linear map l^1 on (3.6) results, together with $r \in K$ and (3.3), in the following:

$$\begin{aligned} l^{1\top} f(\bar{x}) &= \lambda\, l^{1\top} f\left(\bar{x}^1\right) + (1-\lambda)\, l^{1\top} f\left(\bar{x}^2\right) + \underbrace{\left(\bar{t} - \lambda\, \bar{t}^1 - (1-\lambda)\, \bar{t}^2\right)}_{\geq 0} \underbrace{l^{1\top} r}_{\geq 0} \\ &\geq \underbrace{\lambda}_{<0} \underbrace{l^{1\top} f\left(\bar{x}^1\right)}_{< l^{1\top} f(\bar{x}^2)} + (1-\lambda) l^{1\top} f\left(\bar{x}^2\right) \\ &> \lambda\, l^{1\top} f\left(\bar{x}^2\right) + (1-\lambda)\, l^{1\top} f\left(\bar{x}^2\right) = l^{1\top} f\left(\bar{x}^2\right), \end{aligned}$$

which is a contradiction to (3.2).

Instead, assuming (3.6) is satisfied for $\lambda > 1$, we get by applying l^2 on (3.6) together with (3.3)

$$l^{2\top} f(\bar{x}) \quad \geq \quad \lambda\, l^{2\top} f\left(\bar{x}^1\right) + \underbrace{(1-\lambda)}_{<0}\, \underbrace{l^{2\top} f\left(\bar{x}^2\right)}_{< l^{2\top} f(\bar{x}^1)} > l^{2\top} f\left(\bar{x}^1\right)$$

in contradiction to (3.2). Hence, we have $\lambda \in [0,1]$ for $\bar{t} - \lambda\, \bar{t}^1 - (1-\lambda)\, \bar{t}^2 \geq 0$.

In the same way we show $\lambda \in [0,1]$ for $\bar{t} - \lambda\, \bar{t}^1 - (1-\lambda)\, \bar{t}^2 < 0$, too, and the assertion of the theorem is proven. ∎

A generalization to the case $m \geq 3$ is not possible because then a cone need not be finitely generated as we have seen. Even if the cone is finitely generated, even if it is the positive orthant, the previous results are not true in general for more than two objectives as the following example shows.

Example 3.4. We consider the function $f\colon \boldsymbol{R}^3 \to \boldsymbol{R}^3$ with $f(x) = x$ for all $x \in \boldsymbol{R}^3$ and the set $\Omega = \{x \in \boldsymbol{R}^3 \mid x_1^2 + x_2^2 + x_3^2 \leq 1\}$ representing the unit ball in $\boldsymbol{R}^3$. Let the ordering cone be the natural ordering cone $K = \boldsymbol{R}^3_+$ finitely generated by $l^1 = (1,0,0)^\top$, $l^2 = (0,1,0)^\top$, and $l^3 = (0,0,1)^\top$. The multiobjective optimization problem $\min_{x\in\Omega} f(x)$ has the minimal solution set $\mathcal{M}(f(\Omega),\boldsymbol{R}^3_+) = \{x \in \boldsymbol{R}^3 \mid x_1^2+x_2^2+x_3^2 = 1,\ x_i \le 0,\ i=1,2,3\}$. By solving the scalar optimization problems $\min_{x\in\Omega} l^{i\top} f(x),\ i=1,2,3$ (compare (3.1)), we get the minimal solutions $\bar{x}^1 = (-1,0,0)^\top,\ \bar{x}^2 = (0,-1,0)^\top$, and $\bar{x}^3 = (0,0,-1)^\top$. If we consider now the hyperplane $H = \{y \in \boldsymbol{R}^3 \mid (-1,-1,-1)\cdot y = 1\}$ we have $f(\bar{x}^i) = \bar{x}^i \in H$ for $i=1,2,3$.

Choosing $r=(1,1,1)^\top$ and determining the points $\bar{a}^i \in H$, $i=1,2,3$, as in (3.4) results in $\bar{a}^1 = (-1,0,0)^\top,\ \bar{a}^2 = (0,-1,0)^\top,\ \bar{a}^3 = (0,0,-1)^\top$ and, hence, in the set H^a consisting of all convex combinations of the points $\bar{a}^i,\ i=1,2,3$. In this example the point $\bar{x} = (-1/\sqrt{2},-1/\sqrt{2},0)^\top$ is EP-minimal, but there is no parameter $\bar{a}\in H^a$ such that $\bar{x}$ is a minimal solution of (SP$(\bar{a},r)$). For $\bar{a} = -1/(3\sqrt{2})\cdot(1+\sqrt{2},1+\sqrt{2},\sqrt{2}-2)^\top$ and $\bar{t} = (1-\sqrt{2})/3$ the point $(\bar{t},\bar{x})$ is a minimal solution of (SP$(\bar{a},r)$) but it is $\bar{a}\notin H^a$.

Similar considerations have been done in [9, pp. 635f] in connection with the normal boundary intersection method. Nevertheless we want to restrain the set H from which we choose the parameters a for the case of more than two objectives. For this aim we project the set $f(\Omega)$ in the direction r into the set H and determine the set $\tilde{H} := \{y \in H \mid y + t\,r = f(x),\ t\in\boldsymbol{R},\ x\in\Omega\} \subset H$. Of course we would get a stricter limitation by projecting the set $\mathcal{E}(f(\Omega),K)$ instead of $f(\Omega)$, but in general the efficient set is not known in advance. Because the set $\tilde{H}\subset H$ has usually an irregular shape, which complicates a methodic procedure, we embed the set $\tilde{H}$ in the image of an $(m-1)$-dimensional hyperplane under a linear transformation $H^0 \subset \boldsymbol{R}^m$, which we attempt to choose minimally. For doing this we first determine $m-1$ orthogonal vectors $v^1,\dots,v^{m-1}$ spanning the hyperplane H with $\tilde{H}\subset H$. Hence, we have

$$H = \left\{ y\in\boldsymbol{R}^m \mid y = \sum_{i=1}^{m-1} s_i v^i,\ s\in\boldsymbol{R}^{m-1} \right\}. \tag{3.7}$$

Next we solve the following $2(m-1)$ scalar optimization problems

$$\begin{array}{ccc} \min s_j & & \min -s_j \\ \text{subject to the constraints} & & \text{subject to the constraints} \\ \displaystyle\sum_{i=1}^{m-1} s_i v^i + t\,r = f(x), & \text{and} & \displaystyle\sum_{i=1}^{m-1} s_i v^i + t\,r = f(x), \\ t\in\boldsymbol{R},\ x\in\Omega,\ s\in\boldsymbol{R}^{m-1}, & & t\in\boldsymbol{R},\ x\in\Omega,\ s\in\boldsymbol{R}^{m-1} \end{array} \tag{3.8}$$

for $j\in\{1,\dots,m-1\}$ with minimal solutions $(t^{\min,j},x^{\min,j},s^{\min,j})$ and minimal values $s_j^{\min,j}$ and $(t^{\max,j},x^{\max,j},s^{\max,j})$ and minimal values $-s_j^{\max,j}$, respectively. These optimization problems are generally nonconvex even if the related multiobjective optimization problem is convex. However, note that it suffices to provide lower bounds for the optimal function values for the following results. Then we obtain the set H^0 by $H^0 := \{y\in\boldsymbol{R}^m \mid y = \sum_{i=1}^{m-1} s_i v^i,\ s_i \in [s_i^{\min,i}, s_i^{\max,i}],\ i=1,\dots,m-1\}$. The set H^0 includes the set $\tilde{H}$ and is calculated numerically as small as possible.

THEOREM 3.5. *Let $\bar{x}\in\mathcal{M}(f(\Omega),K)$. Then there is a parameter $\bar{a}\in H^0$ and some $\bar{t}\in\boldsymbol{R}$ so that $(\bar{t},\bar{x})$ is a minimal solution of* (SP$(\bar{a},r)$).

Proof. According to the proof of Theorem 3.2 we have for $\bar{t} = (b^\top f(\bar{x})-\beta)/(b^\top r)$ and $\bar{a} = f(\bar{x}) - \bar{t}\,r$ that $(\bar{t},\bar{x})$ is a minimal solution of (SP$(\bar{a},r)$) with $\bar{a}\in H$. Because $H^0\subset H$ it suffices to show $\bar{a}\in H^0$. Because $\bar{a}\in H$ there is, according to the representation in (3.7), a vector $\bar{s}\in\boldsymbol{R}^{m-1}$ with $\bar{a} = \sum_{i=1}^{m-1} \bar{s}_i v^i$. Because of $\bar{a}+\bar{t}\,r =$

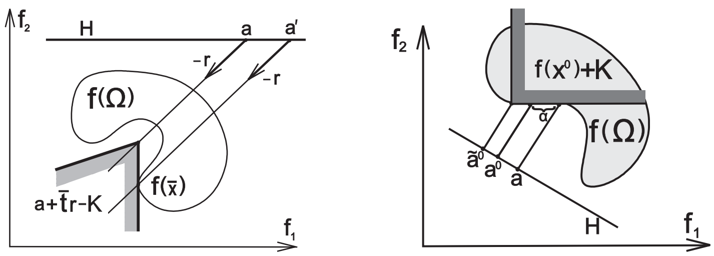

FIG. 3.1. (a) *Visualization of Theorem* 3.6. (b) *Choosing the new parameter in the case* $a^0 + t^0\, r - f(x^0) \neq 0_2$ *and* $\tilde{a}^0_1 < a^0_1$.

$f(\bar{x})$, the point $(\bar{t}, \bar{x}, \bar{s})$ is feasible for the optimization problems (3.8) and thus we have $s_i^{\min,i} \leq \bar{s}_i \leq s_i^{\max,i}$ for $i = 1, \dots, m-1$. Hence, it follows $\bar{a} \in H^0$. $\square$

Thus, it is sufficient to consider only parameters $a \in H^0$ for a fixed $r \in K \setminus \{0_m\}$ to find all K-minimal solutions of the multiobjective optimization problem (2.1).

For our considerations in the next section it will be important that in the minimal solution $(\bar{t}, \bar{x})$ the constraint $a + t\, r - f(x) \in K$ is active; i.e., $a + \bar{t}\, r - f(\bar{x}) = 0_m$. If this is not the case for the choice of a parameter a, we can easily define a new parameter a' for which this property is satisfied (see Figure 3.1(a)).

THEOREM 3.6. *Let the hyperplane* $H = \{y \in \boldsymbol{R}^m \mid b^\top y = \beta\}$ *with* $b \in \boldsymbol{R}^m \setminus \{0_m\}$, $\beta \in \boldsymbol{R}$ *be given. Suppose* $(\bar{t}, \bar{x})$ *is a minimal solution of* $(\mathrm{SP}(a, r))$ *for* $a \in H$, $r \in K$, $b^\top r \neq 0$. *Then there is a* $\bar{k} \in K$ *with* $a + \bar{t}\, r - f(\bar{x}) = \bar{k}$. *However, then there is a point* $a' \in H$ *and some* $t' \in \boldsymbol{R}$ *so that* $(t', \bar{x})$ *is also a minimal solution of* $(\mathrm{SP}(a', r))$ *with* $a' + t'\, r - f(\bar{x}) = 0_m$.

Proof. We set $t' := (b^\top f(\bar{x}) - \beta)/(b^\top r)$ and $a' := a + (\bar{t} - t')\, r - \bar{k} = f(\bar{x}) - t'\, r$. Then $a' \in H$ and $a' + t'\, r - f(\bar{x}) = 0_m$. Next we show that $(t', \bar{x})$ is a minimal solution of $(\mathrm{SP}(a', r))$. Otherwise there is a feasible point $(\hat{t}, \hat{x}) \in \boldsymbol{R} \times \Omega$ with $\hat{t} < t'$ and there is a $\hat{k} \in K$ with $a' + \hat{t}\, r - f(\hat{x}) = \hat{k}$. Together with the definition of a' this leads to $a + (\bar{t} - t' + \hat{t})\, r - f(\hat{x}) = \hat{k} + \bar{k} \in K$. However, then $(\bar{t} - t' + \hat{t}, \hat{x})$ is feasible for $(\mathrm{SP}(a, r))$ with $\bar{t} - t' + \hat{t} < \bar{t}$ in contradiction to $(\bar{t}, \bar{x})$ a minimal solution of $(\mathrm{SP}(a, r))$. $\square$

It can also be shown that there is no K-minimal point $\tilde{x}$ for which $\tilde{a} = a + \lambda\,(a' - a)$ for some $\lambda \in]0, 1[$ with $\tilde{a} := f(\tilde{x}) - \tilde{t}\, r$ and $\tilde{t} := (b^\top f(\tilde{x}) - \beta)/(b^\top r)$ [13, Theorem 4.2.12]. Thus the section on the hyperplane H between the parameters a and a' can be neglected. According to Theorem 3.6 we have for the parameter a' not only $a' \in H$ but also $a' + t'\, r = f(\bar{x})$. Hence, for the set H^0 as in Theorem 3.5, it holds $a' \in H^0$.

Under the Assumption 2 it can be shown that if a minimal solution $(\bar{t}, \bar{x})$ of $(\mathrm{SP}(a, r))$ has Lagrange multipliers $(\mu, \nu, \xi) \in K^* \times C^* \times \boldsymbol{R}^q$, then these are also Lagrange multipliers to the point $(t', \bar{x})$ for $(\mathrm{SP}(a', r))$ under the transformation of Theorem 3.6. For the definition of the Lagrange function and the Lagrange multipliers see, for instance, [32, p. 115]. Here, $\mu \in K^*$ is the Lagrange multiplier to the constraint $a + t\, r - f(x) \in K$ (and thus also to $a' + t\, r - f(x) \in K$), $\nu \in C^*$ corresponds to the constraint $g(x) \in C$, and $\xi \in \boldsymbol{R}^q$ to the equality constraint $h(x) = 0_q$.

4. Sensitivity results. The main point of our adaptive parameter control for the parameter dependent scalarization approach $(\mathrm{SP}(a, r))$ is a sensitivity theorem based on theorems by Alt [3, Theorem 5.3, 6.1]. For two reasons we apply these theorems on a modified version of problem (SP(a,r)) called ($\overline{\mathrm{SP}}(a,r)$):

$$(\overline{\mathrm{SP}}(a,r)) \qquad \begin{array}{c} \min t \\ \text{subject to the constraints} \\ a + t\,r - f(x) = 0_m, \\ t \in \boldsymbol{R},\ x \in \Omega \end{array}$$

with the constraint set $\overline{\Sigma}(a,r) := \{(t,x) \in \boldsymbol{R} \times \boldsymbol{R}^n \mid a + t\,r - f(x) = 0_m,\ x \in \Omega\}$.

First, by getting sensitivity results for the minimal value $t = t(a,r)$ of the problem ($\overline{\mathrm{SP}}(a,r)$) we can at once conclude how a variation of the parameters influences the generated points $f(x(a,r))$ by using the equality constraint $a + t(a,r)\,r - f(x(a,r)) = 0_m$. Second, for applying the sensitivity theorems by Alt directly to the problem (SP(a,r)), for the Hessian of the Lagrange function $\mathcal{L}(t,x,\mu,\nu,\xi,a,r) = t - \mu^\top(a + t\,r - f(x)) - \nu^\top g(x) - \xi^\top h(x)$ w.r.t. the variables (t,x)

$$\nabla^2_{(t,x)}\mathcal{L}(t,x,\mu,\nu,\xi,a,r) = \begin{pmatrix} 0 & 0 \\ 0 & W(x,\mu,\nu,\xi) \end{pmatrix}$$

with $W(x,\mu,\nu,\xi) = \sum_{i=1}^m \mu_i \nabla^2 f_i(x) - \sum_{j=1}^p \nu_j \nabla^2 g_j(x) - \sum_{k=1}^q \xi_k \nabla^2 h_k(x)$ the assumption

$$\left(t, x^\top\right)\ \nabla^2_{(t,x)}\mathcal{L}(t,x,\mu,\nu,\xi,a,r) \begin{pmatrix} t \\ x \end{pmatrix} \geq \alpha \left\| \begin{pmatrix} t \\ x \end{pmatrix} \right\|^2$$

has to be satisfied for some constant $\alpha > 0$ for all $(t,x) \in \boldsymbol{R}^{n+1}$ with $\nabla h(x)x = 0_q$. This is always contradicted by the points $(t,x) = (t,0_n)$ with $t \neq 0$.

Problem ($\overline{\mathrm{SP}}(a,r)$) is in general nonconvex, even if the original multiobjective optimization problem is convex. However, note that the problem ($\overline{\mathrm{SP}}(a,r)$) is never actually solved. We solve only the problems (SP(a,r)) (which are convex if the original multiobjective optimization problem is convex) and we use the problem ($\overline{\mathrm{SP}}(a,r)$) only for approximating the minimal value and the points $f(x(a,r))$ of the problem (SP(a,r)).

We obtain the connection between the problems (SP(a,r)) and ($\overline{\mathrm{SP}}(a,r)$) by Theorem 3.6: if we solve the problem (SP(a,r)) with a minimal solution $(\bar{t},\bar{x})$ and with $a + \bar{t}\,r - f(\bar{x}) = \bar{k} \in K$, then there always exists a parameter a' and some t' so that $(t',\bar{x})$ is minimal for (SP(a',r)) with $a' + t'\,r - f(\bar{x}) = 0_m$ and then $(t',\bar{x})$ is a minimal solution of ($\overline{\mathrm{SP}}(a',r)$) too. We examine now the dependence of the minimal values of the problem ($\overline{\mathrm{SP}}(a',r)$) on the parameter a' (in Theorem 4.2) and from that we conclude on the (approximated) dependence of the minimal value of the problem (SP(a',r)) on the parameter a' (in section 5.1). Before we come to the main sensitivity result we need the following assumption and the following lemma.

Assumption 3. Let Assumption 2 hold. In addition let the functions f, g, and h be twice continuously differentiable on $\hat{S}$.

LEMMA 4.1. *Let Assumption* 3 *hold. Let (t^0,x^0) be a local minimal solution of ($\overline{\mathrm{SP}}(a^0,r^0)$) with Lagrange multipliers $(\mu^0,\nu^0,\xi^0) \in \boldsymbol{R}^m \times C^* \times \boldsymbol{R}^q$. Assume there exists some constant $\tilde{\alpha} > 0$ such that for the matrix $W(x^0,\mu^0,\nu^0,\xi^0) = \mu^{0\top}\nabla^2 f(x^0) - \nu^{0\top}\nabla^2 g(x^0) - \xi^{0\top}\nabla^2 h(x^0)$ we have*

$$x^\top W\left(x^0,\mu^0,\nu^0,\xi^0\right) x \geq \tilde{\alpha}\,\|x\|^2 \tag{4.1}$$

for all $x \in \{x \in \boldsymbol{R}^n \mid \nabla h(x^0)x = 0_q,\ \nabla f(x^0)x = r^0\,t \ \text{ for a } t \in \boldsymbol{R}\}$. Then there exists *some constant* $\alpha > 0$ *such that for the Lagrange function* $\overline{\mathcal{L}}$ *to* $(\overline{\mathrm{SP}}(a,r))$ *we have*

$$(4.2)\qquad \left(t, x^{\top}\right) \nabla^2_{(t,x)} \overline{\mathcal{L}}\left(t^0, x^0, \mu^0, \nu^0, \xi^0, a^0, r^0\right) \begin{pmatrix} t \\ x \end{pmatrix} \geq \alpha \left\| \begin{pmatrix} t \\ x \end{pmatrix} \right\|^2$$

for all $(t,x) \in \{(t,x) \in \boldsymbol{R} \times \boldsymbol{R}^n \mid \nabla h(x^0)x = 0_q,\ \nabla f(x^0)x = r^0\, t\}$.

Proof. Because (t^0, x^0) is a local minimal solution of $(\overline{\mathrm{SP}}(a^0, r^0))$ with Lagrange multipliers (μ^0, ν^0, ξ^0) we have for the associated Lagrange function

$$(4.3)\qquad \nabla_{(t,x)} \overline{\mathcal{L}}\left(t^0, x^0, \mu^0, \nu^0, \xi^0, a^0, r^0\right)^{\top} \begin{pmatrix} t - t^0 \\ x - x^0 \end{pmatrix} \geq 0 \text{ for all } t \in \boldsymbol{R},\ x \in S.$$

With $\frac{\partial \overline{\mathcal{L}}(t^0, x^0, \mu^0, \nu^0, \xi^0, a^0, r^0)}{\partial t} = 1 - \mu^{0\top} r^0$ and because (4.3) has to be fulfilled for all $t \in \boldsymbol{R}$ we have

$$(4.4)\qquad \mu^{0\top} r^0 = 1$$

and, therefore, $\mu^0 \neq 0_m,\ r^0 \neq 0_m$. Because in $\boldsymbol{R}^n$ and $\boldsymbol{R}^m$, respectively, all norms are equivalent, there exist positive constants $M^l, M^u \in \boldsymbol{R}_+$ and $\tilde{M}^l, \tilde{M}^u \in \boldsymbol{R}_+$, respectively, with $M^l \|x\|_2 \leq \|x\| \leq M^u \|x\|_2$ and

$$\tilde{M}^l \left\| \begin{pmatrix} t \\ x \end{pmatrix} \right\|_2 \leq \left\| \begin{pmatrix} t \\ x \end{pmatrix} \right\| \leq \tilde{M}^u \left\| \begin{pmatrix} t \\ x \end{pmatrix} \right\|_2$$

for all $(t,x) \in \boldsymbol{R} \times \boldsymbol{R}^n$. For all $(t,x) \in \boldsymbol{R} \times \boldsymbol{R}^n$ with $\nabla f(x^0)x = r^0\, t$ we have together with (4.4) the equation $\mu^{0\top} \nabla f(x^0)x = t$ and then we get the estimation $|t|^2 = |\mu^{0\top} \nabla f(x^0)x|^2 \leq \|\mu^0\|_2^2\, \|\nabla f(x^0)\|_2^2\, \|x\|_2^2$. If we set now

$$\alpha := \frac{\tilde{\alpha}\ \left(M^l\right)^2}{\left(\tilde{M}^u\right)^2\ (1 + \|\mu^0\|_2^2\, \|\nabla f\,(x^0)\,\|_2^2)} > 0,$$

we conclude from (4.1) for all $(t,x) \in \{(t,x) \in \boldsymbol{R} \times \boldsymbol{R}^n \mid \nabla h(x^0)x = 0_q,\ \nabla f(x^0)x = r^0\, t\}$

$$\begin{aligned} x^{\top} W\left(x^0, \mu^0, \nu^0, \xi^0\right)\, x &\geq \tilde{\alpha}\, \|x\|^2 \geq \tilde{a}\, \left(M^l\right)^2 \|x\|_2^2 \\ &= \alpha \left(\tilde{M}^u\right)^2\ \left(1 + \|\mu^0\|_2^2\, \|\nabla f(x^0)\|_2^2\right)\, \|x\|_2^2 \\ &\geq \alpha \left(\tilde{M}^u\right)^2\ \left(\|x\|_2^2 + |t|^2\right) \\ &= \alpha \left(\tilde{M}^u\right)^2 \left\| \begin{pmatrix} t \\ x \end{pmatrix} \right\|_2^2 \geq \alpha \left\| \begin{pmatrix} t \\ x \end{pmatrix} \right\|^2. \end{aligned}$$

With

$$\nabla^2_{(t,x)} \overline{\mathcal{L}}\left(t^0, x^0, \mu^0, \nu^0, \xi^0, a^0, r^0\right) = \begin{pmatrix} 0 & 0 \\ 0 & W\left(x^0, \mu^0, \nu^0, \xi^0\right) \end{pmatrix}$$

the assertion is proven. ∎

The condition (4.2) for all (t,x) of the given set is called strict second-order sufficient condition. If this condition is fulfilled for a regular point, then this is sufficient for strict local minimality of the considered point [41, Theorem 5.2].

THEOREM 4.2. *Let Assumption* 3 *and the assumptions of Lemma* 4.1 *hold. We consider the parametric optimization problem* $(\overline{\mathrm{SP}}(a,r))$ *with the constraint set* $\overline{\Sigma}(a,r)$ *starting with a reference problem* $(\overline{\mathrm{SP}}(a^0,r^0))$ *with a local minimal solution* (t^0,x^0) *and with Lagrange multipliers* $(\mu^0,\nu^0,\xi^0)\in \boldsymbol{R}^m\times C^*\times \boldsymbol{R}^q$.

(i) *Suppose the point* (t^0,x^0) *is regular for the set* $\overline{\Sigma}(a^0,r^0)$, *i.e., we have*

$$0_{m+p+q}\in int\left\{\left(\begin{array}{rcl} r^0\,\left(t-t^0\right) & - & \nabla f\left(x^0\right)\left(x-x^0\right)\\ g\left(x^0\right) & + & \nabla g\left(x^0\right)\left(x-x^0\right)-c\\ & & \nabla h\left(x^0\right)\left(x-x^0\right)\end{array}\right)\ \middle|\ \begin{array}{c} c\in C,\\ x\in S,\\ t\in \boldsymbol{R}\end{array}\right\}.$$

(ii) *Assume there exists some* $\zeta>0$ *such that the following holds for all* p^1, $p^2\in \zeta\tilde{B}$ *(with* $\tilde{B}$ *the closed unit ball in* $\boldsymbol{R}^{1+n+m+p+q}$*) with* $p^i=(t^{*i},x^{*i},u^i,v^i,w^i)$, $i=1,2$*: if* (t^1,x^1) *and* (t^2,x^2)*, respectively, are solutions of the quadratic optimization problem*

$$\begin{gathered}\min J\left(t,x,p^i\right)\\ \textit{subject to the constraints}\\ r^0\,\left(t-t^0\right)-\nabla f\left(x^0\right)\left(x-x^0\right)-u^i=0_m,\\ g\left(x^0\right)+\nabla g\left(x^0\right)\left(x-x^0\right)-v^i\in C,\\ \nabla h\left(x^0\right)\left(x-x^0\right)-w^i=0_q,\\ t\in\boldsymbol{R},\ x\in S,\end{gathered}$$

$(i=1,2)$ *with* $J(t,x,p^i):=\frac{1}{2}(x-x^0)^\top W(x^0,\mu^0,\nu^0,\xi^0)\,(x-x^0)+(t-t^0)-t^{*i}\,(t-t^0)-(x^{*i})^\top(x-x^0)$, *then the Lagrange multipliers* $(\mu_q^i,\nu_q^i,\xi_q^i)$ *to the solutions* (t^i,x^i), $i=1,2$, *are uniquely determined and*

$$\left\|\left(\mu_q^1,\nu_q^1,\xi_q^1\right)-\left(\mu_q^2,\nu_q^2,\xi_q^2\right)\right\|\le c_M\left(\left\|\left(t^1,x^1\right)-\left(t^2,x^2\right)\right\|+\left\|p^1-p^2\right\|\right)$$

with some constant c_M.

Then there exists some $\delta>0$ *and a neighborhood* $N(a^0,r^0)$ *of* (a^0,r^0) *so that the local minimal value function* $\overline{\tau}^\delta(a,r):=\inf\{t\mid (t,x)\in\overline{\Sigma}(a,r)\cap B_\delta(t^0,x^0)\}$ *is differentiable on* $N(a^0,r^0)$ *with the derivative*

$$\nabla_{(a,r)}\overline{\tau}^\delta(a,r)=\nabla_{(a,r)}\overline{\mathcal{L}}\left(\bar{t}(a,r),\bar{x}(a,r),\mu(a,r),\nu(a,r),\xi(a,r),a,r\right).$$

Here $(\bar{t}(a,r),\bar{x}(a,r))$ *denotes the strict local minimal solution of* $(\overline{\mathrm{SP}}(a,r))$ *for* $(a,r)\in N(a^0,r^0)$ *with the unique Lagrange multipliers* $(\mu(a,r),\nu(a,r),\xi(a,r))$. *In addition to that the mapping* ϕ: $N(a^0,r^0)\to B_\delta(t^0,x^0)\times B_\delta(\mu^0,\nu^0,\xi^0)$ *defined by* $\phi(a,r)=(\bar{t}(a,r),\bar{x}(a,r),\mu(a,r),\nu(a,r),\xi(a,r))$ *is Lipschitzian on* $N(a^0,r^0)$.

Proof. By using Lemma 4.1 it can easily be shown that all premises for applying the Theorems 5.3 and 6.1 by Alt [3] are met. ∎

Remark 4.3. The condition (ii) of the preceding theorem is always satisfied if we have only equality constraints [3, Theorem 7.1] or, in the case of the natural ordering $C=\boldsymbol{R}^n_+$, if the gradients of the active constraints are linearly independent; compare [16, Theorem 2.1] and [35, Theorem 2].

LEMMA 4.4. *Let the assumptions of Theorem* 4.2 *be satisfied with* $S=\boldsymbol{R}^n$. *Then there is some* $\delta>0$ *and a neighborhood* $N(a^0,r^0)$ *of* (a^0,r^0) *so that for all* $(a,r)\in N(a^0,r^0)$ *the derivatives of the local minimal value function are given by*

$$\begin{aligned} &\nabla_a\overline{\tau}^\delta(a,r)=-\mu(a,r)-\nabla_a\nu(a,r)^\top g(\bar{x}(a,r))\\ \textit{and}\quad &\nabla_r\overline{\tau}^\delta(a,r)=-\bar{t}(a,r)\mu(a,r)-\nabla_r\nu(a,r)^\top g(\bar{x}(a,r)).\end{aligned}$$

Proof. According to Theorem 4.2 there is a neighborhood $N(a^0,r^0)$ of (a^0,r^0) such that for all $(a,r)\in N(a^0,r^0)$ there is a strict minimal solution $(\bar t(a,r),\bar x(a,r))$ with unique Lagrange multipliers $(\mu(a,r),\nu(a,r),\xi(a,r))$. Because of $S=\boldsymbol{R}^n$ we have for the derivative of the Lagrangian $\nabla_{(t,x)}\overline{\mathcal{L}}(\bar t(a,r),\bar x(a,r),\mu(a,r),\nu(a,r),\xi(a,r),a,r)=0_{n+1}$. Then it follows

$$
\begin{aligned}
0_m &= \nabla_a \begin{pmatrix} \bar t(a,r)\\ \bar x(a,r)\end{pmatrix}^{\top} \nabla_{(t,x)}\overline{\mathcal{L}}(\bar t(a,r),\bar x(a,r),\mu(a,r),\nu(a,r),\xi(a,r),a,r)\\
&= \nabla_a \bar t(a,r) - \sum_{i=1}^{m}\mu_i(a,r)\Big(\nabla_a\bar t(a,r)\,r_i - \nabla_a\bar x(a,r)^{\top}\nabla_x f_i(\bar x(a,r))\Big) \qquad (4.5)\\
&\quad - \sum_{j=1}^{p}\nu_j(a,r)\nabla_a\bar x(a,r)^{\top}\nabla_x g_j(\bar x(a,r)) - \sum_{k=1}^{q}\xi_k(a,r)\nabla_a\bar x(a,r)^{\top}\nabla_x h_k(\bar x(a,r)).
\end{aligned}
$$

According to Theorem 4.2 there exists some $\delta>0$ so that the derivative of the local minimal value function is given by $\nabla_{(a,r)}\overline{\tau}^{\delta}(a,r)=\nabla_{(a,r)}\overline{\mathcal{L}}(\bar t(a,r),\bar x(a,r),\ \mu(a,r),$ $\nu(a,r),\ \xi(a,r),a,r)$. Applying standard rules of differentiation and together with (4.5) we conclude

$$
\begin{aligned}
\nabla_a\overline{\tau}^{\delta}(a,r) &= \nabla_a\bar t(a,r) - \sum_{i=1}^{m}\mu_i(a,r)\Big(e_i + \nabla_a\bar t(a,r) r_i - \ \nabla_a\bar x(a,r)^{\top}\nabla_x f_i(\bar x(a,r))\Big)\\
&\qquad - \sum_{i=1}^{m}\nabla_a\mu_i(a,r)\,\underbrace{(a_i+\bar t(a,r)\,r_i - f_i(\bar x(a,r)))}_{=0}\\
&\qquad - \sum_{j=1}^{p}\nu_j(a,r)\nabla_a\bar x(a,r)^{\top}\nabla_x g_j(\bar x(a,r)) - \sum_{j=1}^{p}\nabla_a\nu_j(a,r) g_j(\bar x(a,r))\\
&\qquad - \sum_{k=1}^{q}\xi_k(a,r)\nabla_a\bar x(a,r)^{\top}\nabla_x h_k(\bar x(a,r)) - \sum_{k=1}^{q}\nabla_a\xi_k(a,r)\,\underbrace{h_k(\bar x(a,r))}_{=0}\\
&= -\mu(a,r) - \nabla_a\nu(a,r)^{\top} g(\bar x(a,r)).
\end{aligned}
$$

The same for $\nabla_r\overline{\tau}^{\delta}(a,r)$. ∎

We can use that inactive constraints remain inactive for small parameter changes and then in the case $C=\boldsymbol{R}^p_+$ we conclude (using the arguments in [17, Theorem 3.2.2, Proof of Theorem 3.4.1]) $\nabla_{(a,r)}\nu(a^0,r^0)^{\top} g(\bar x(a^0,r^0)) = 0_{2m}$. This results in the following.

COROLLARY 4.5. *Under the assumptions of Lemma* 4.4 *and with* $C=\boldsymbol{R}^p_+$ *it follows*

$$
\nabla_{(a,r)}\overline{\tau}^{\delta}\left(a^0,r^0\right) = -\begin{pmatrix}\mu^0\\ t^0\,\mu^0\end{pmatrix}.
$$

Hence, we get in this special case the derivative information via the Lagrange multipliers without additional effort just by solving the problems $(\mathrm{SP}(a,r))$. Otherwise, the derivative of the local minimal value function, being equivalent to the derivative of the Lagrange function, has to be approximated. Under some special additional assumptions as $C=\boldsymbol{R}^p_+,\ K=\boldsymbol{R}^m_+,\ \hat S = S=\boldsymbol{R}^n$, and nondegeneracy the second-order information $\nabla_a^2\tau^{\delta}(a^0,r^0) = -\nabla_a\mu(a^0,r^0)$ and $\nabla_r^2\tau^{\delta}(a^0,r^0) = t^0\mu^0(\mu^0)^{\top} - t^0\nabla_r\mu(a^0,r^0)$ is available [13, Theorem 3.2.4], too.

5. Parameter control and algorithm. In the literature several quality criteria have been discussed for approximations of the efficient set (see [7, 12, 39, 47, 54], and others). Most of them have been developed for evaluating evolutionary algorithms, as, e.g., the quality criteria of measuring the distance of the approximation set to the efficient set. As our approximation points are determined by solving the problems ($\mathrm{SP}(a,r)$) they are at least weakly K-minimal. Here we suppose that a numerical solver is at our disposal which allows us to find global minimal solutions of the considered scalar optimization problems. However, generally numerical methods generate only approximations of a minimal solution if not even only local minimal solutions. Then the distance to the efficient set depends on the numerical solvers used and not on the adaptive parameter control in which we are interested here. Therefore, quality criteria as the distance to the efficient set are not considered in this context.

The most interesting criteria, in the case of scalarization approaches, are the three proposed by Sayin [47] called coverage error, uniformity, and cardinality. In our opinion and with respect to these targets, an approximation possesses a high quality in the sense of a concise but representative approximation if it consists of almost equidistant approximation points.

We want to use the sensitivity results from section 4 to reach the aim of an equidistant approximation, at least locally. We first apply these results for developing a method for determining the parameters a such that we can control the distance between the generated approximation points. This will be used in section 5.2 for locally refining coarse approximations of the efficient set with equidistant points. In section 5.3 we specialize our results to the bicriteria case ($m = 2$), because for two objective functions we do not have to determine a coarse approximation first which is then refined. Instead we can adaptively control the parameter a from the beginning to generate an equidistant approximation of the whole efficient set.

5.1. Parameter control. We start by assuming that we have already solved a so-called reference problem ($\mathrm{SP}(a^0,r)$) with minimal solution (t^0,x^0) with Lagrange multipliers (μ^0,ν^0,ξ^0) and with $a^0+t^0\,r-f(x^0)=0_m$. (Otherwise, for $a^0+t^0\,r-f(x^0)=k\neq 0_m$, we can apply Theorem 3.6 and determine a scalar t' and a parameter a' with $a'+t'\,r-f(x^0)=0_m$. Then we take the problem ($\mathrm{SP}(a',r)$) as reference problem.) Then (t^0,x^0) is a minimal solution of the modified problem ($\overline{\mathrm{SP}}(a^0,r)$). We further assume that the derivative $\nabla_a\overline{\tau}^\delta(a^0,r)$ of the local minimal value function is known as a consequence of Theorem 4.2.

We will concentrate on a variation of the parameter a. We use the derivative information for a first-order Taylor series approximation (assuming this is possible) of the minimal value t of problem ($\overline{\mathrm{SP}}(a,r)$) depending on the parameter a given by $\bar{t}(a,r)\approx t^0+\nabla_a\overline{\tau}^\delta(a^0,r)^\top(a-a^0)$. Using the equality constraint $f(\bar{x}(a,r))=a+\bar{t}(a,r)\,r$ of problem ($\overline{\mathrm{SP}}(a,r)$) we get, together with $f(x^0)=a^0+t^0\,r$,

$$
\begin{aligned}
f(\bar{x}(a,r)) &\approx a^0+\left(a-a^0\right)+\left(t^0+\nabla_a\overline{\tau}^\delta\left(a^0,r\right)^\top\left(a-a^0\right)\right)\,r\\
&= f\left(x^0\right)+\left(a-a^0\right)+\left(\nabla_a\overline{\tau}^\delta\left(a^0,r\right)^\top\left(a-a^0\right)\right)\,r.
\end{aligned}
$$

We now use $(\bar{t}(a,r),\bar{x}(a,r))$ (the minimal solution of ($\overline{\mathrm{SP}}(a,r)$)) as an approximation of the minimal solution $(t(a,r),x(a,r))$ of ($\mathrm{SP}(a,r)$). We have at least $t(a,r)\leq\bar{t}(a,r)$ and $(\bar{t}(a,r),\bar{x}(a,r))$ feasible for ($\mathrm{SP}(a,r)$). Hence, we get the following local approximation of the generated weakly efficient points of (2.1) depending on the parameter a:

$$f(x(a,r)) \approx f\left(x^0\right) + \left(a - a^0\right) + \left(\nabla_a \overline{\tau}^\delta \left(a^0, r\right)^\top \left(a - a^0\right)\right) \ r. \tag{5.1}$$

Our goal is to compute equidistant approximation points and for a predefined distance of $\alpha > 0$ we want to find a new parameter a^1 such that

$$\left\| f\left(x\left(a^0, r\right)\right) - f\left(x\left(a^1, r\right)\right)\right\| = \alpha \tag{5.2}$$

with $f(x(a^0, r)) = f(x^0)$, i.e., such that the new approximation point has a distance of α to the former. In Theorem 3.2 we have seen that it is sufficient to consider parameters a in a hyperplane H. Assuming $a^0 \in H = \{y \in \boldsymbol{R}^m \mid b^\top y = \beta\}$, we choose a direction $v \in \boldsymbol{R}^m$ with $b^\top v = 0$ such that $a^0 + s\,v \in H$ for all $s \in \boldsymbol{R}$. Because we want $a^1 \in H$ we set $a^1 = a^0 + s^1\,v$, $s^1 \in \boldsymbol{R}$, and together with (5.2) and (5.1) this results in

$$\begin{aligned}
\alpha &= \left\| f\left(x\left(a^0, r\right)\right) - f\left(x\left(a^1, r\right)\right)\right\| \\
&\approx \left\| f\left(x^0\right) - \left(f\left(x^0\right) + s^1\,v + s^1 \left(\nabla_a \overline{\tau}^\delta \left(a^0, r\right)^\top v\right) r\right)\right\| \\
&= \left|s^1\right| \left\| v + \left(\nabla_a \overline{\tau}^\delta \left(a^0, r\right)^\top v\right) r\right\|.
\end{aligned}$$

Hence, we choose

$$s^1 := \frac{\alpha}{\left\| v + \left(\nabla_a \overline{\tau}^\delta \left(a^0, r\right)^\top v\right) r\right\|}. \tag{5.3}$$

For the new parameter $a^1 := a^0 + s^1\,v$ (or $a^1 := a^0 - s^1\,v$) we now solve the problem $(\mathrm{SP}(a^1, r))$ with minimal solution (t^1, x^1), and if the quality of our approximations have been good this results in $\|f(x^0) - f(x^1)\| \approx \alpha$.

5.2. General multiobjective case. For the general multiobjective case ($m \geq 2$) we have seen in Theorem 3.5 that it is sufficient to vary the parameter a in the subset H^0 for being able to detect all K-minimal points of the multiobjective optimization problem. We use this information for determining in a first step a coarse approximation of the efficient set. Then, in a second step, this approximation is refined locally with (almost) equidistant points.

The coarse approximation is done using equidistant parameters only. The d.m. can, for instance, define the desired number of points N^i in each direction v^i, $i = 1, \ldots, m-1$, spanning the hyperplane H (see (3.7)). With a distance of $L_i := (s_i^{\max,i} - s_i^{\min,i})/N^i$ this leads to $\prod_{i=1}^{m-1} N^i$ equidistant discretization points. For any of these parameters $a \in H^0$ and for $r \in K$ constant we solve the scalar optimization problem $(\mathrm{SP}(a, r))$ with minimal solution (t^a, x^a) (if one exists) and Lagrange multiplier μ^a to the constraint $a + t\,r - f(x) \in K$.

Based on this coarse approximation the d.m. gets a first overview over the efficient set and can now choose which areas or points are of special interest for doing a refinement now using the results of section 5.1. Let $f(x^0)$ be such a chosen approximation point (with $a^0 + t^0\,r - f(x^0) = 0_m$) and assume a refinement with $\bar{n} \in \boldsymbol{N}$ additional points in every direction should be done. Thus, we search for parameters a with $\|f(x(a)) - f(x^0)\| = \alpha$ for a distance $\alpha > 0$. As the hyperplane H is spanned by the $m-1$ vectors v^k, $k=1,\dots,m-1$, we set for the new parameters $a=a^0+s\cdot v^k$, $k\in\{1,\dots,m-1\}$, for some $s\in\boldsymbol{R}$. This leads, according to (5.3), to

$$s^k := \frac{\alpha}{\left\| v+\left(\nabla_a\overline{\tau}^\delta\left(a^0,r\right)^\top v^k\right) r\right\|}.$$

Hence, we get the $2(m-1)$ new parameters $a:=a^0\pm s^k\,v^k$, $k\in\{1,\dots,m-1\}$.

We extend this to a consideration of all $(2\bar{n}+1)^2-1$ parameters a with $a=a^0+\sum_{k=1}^{m-1} l_k\,s^k\,v^k$ for $l_k\in\{-\bar{n},\dots,\bar{n}\}\subset\boldsymbol{Z}$, $k=1,\dots,m-1$, $l:=(l_1,\dots,l_{m-1})\neq 0_{m-1}$, for getting $\bar{n}$ new parameters in every direction. Solving $(\mathrm{SP}(a,r))$ for all these parameters results in a refined approximation with locally equidistant points (around $f(x^0)$).

For the calculation of the values s^k we need the derivative of the local minimal value function which is given in Theorem 4.2. For that, the derivative of the Lagrange function has to be approximated. In the case of $S=\mathbb{R}^n$ this is reduced to an approximation of the derivative of the function ν w.r.t. a. According to Corollary 4.5, in the case of $C=\mathbb{R}^p_+$ the derivative is even immediately given by $\nabla_a\overline{\tau}^\delta(a^0,r)=-\mu^0$. Choosing additionally $K=\mathbb{R}^n_+$ and the hyperplane $H=\{y\in\boldsymbol{R}^m\mid y_m=0\}$, i.e., $b=e_m$, with e_m the mth unit vector in $\boldsymbol{R}^m$, $\beta=0$, and $r=e_m$, then solving the problem $(\mathrm{SP}(a,r))$ for parameters $a=(a_1,\dots,a_{m-1},0)\in H$ is equivalent to solve the problem

$$\begin{gathered}\min f_m(x)\\ \text{subject to the constraints}\\ f_i(x)\le a_i,\qquad i=1,\dots,m-1,\\ x\in\Omega.\end{gathered}$$

This problem is well known as ε-constraint scalarization [21]. Choosing for v^i the unit vectors the problems (3.8) then reduce to $\min_{x\in\Omega} f_i(x)$ and $\max_{x\in\Omega} f_i(x)$ for $i=1,\dots,m-1$, and thus the calculation of the set H^0 (Step 1) is facilitated. Also, the points a', t' to a minimal solution $\bar{x}$ as in Theorem 3.6 are just given by $t'=f_m(\bar{x})$ and $a'=(f_1(\bar{x}),\dots,f_{m-1}(\bar{x}),0)$. Hence, we assume for the given algorithm:

Assumption 4. *Let Assumption* 3 *hold with $S=\boldsymbol{R}^n$, $K=\mathbb{R}^m_+$, and $C=\mathbb{R}^p_+$. To any choice of parameters (a,r) for which we consider the optimization problem* $(\mathrm{SP}(a,r))$ *or* $(\overline{\mathrm{SP}}(a,r))$ *let there exist a minimal solution $(\bar{t},\bar{x})$ with Lagrange multipliers $(\bar{\mu},\bar{\nu},\bar{\xi})\in\boldsymbol{R}^m\times\boldsymbol{R}^p_+\times\boldsymbol{R}^q$ and let the assumptions of Theorem* 4.2 *in $(\bar{t},\bar{x})$ be satisfied.*

This simplifies the algorithm considerably and allows a short representation. For multiobjective optimization problems with arbitrary ordering cones, the ε-constraint reformulation instead of $(\mathrm{SP}(a,r))$ is generally not possible and the determination of the set H^0 is more laborious. Also the calculation of the derivatives $\nabla_a\overline{\tau}^\delta(a,r)$ is more costly if C is not the natural ordering and if S does not equal the whole space (see Lemma 4.4). As the formulation of the general algorithm goes straightforward we restrain here to this common special case.

ALGORITHM 5.1 (Algorithm for an adaptive parameter control).

Input: Set $r=e_m$, $b=e_m$, $\beta=0$. Choose desired number of discretization points N^i in direction $v^i=e_i$ for $i=1,\dots,m-1$.

Step 1: Solve problem $\min_{x\in\Omega} f_i(x)$ with minimal solution $x^{\min,i}$ and minimal value $f(x^{\min,i})=:a_i^{\min}$ for $i=1,\dots,m-1$, and problem $\max_{x\in\Omega} f_i(x)$ with maximal solution $x^{\max,i}$ and maximal value $f(x^{\max,i})=:a_i^{\max}$ for $i=1,\dots,m-1$.

Step 2: Set $L_i := (a_i^{\max} - a_i^{\min})/N^i$ for $i = 1, \ldots, m-1$ and solve $(\mathrm{SP}(a, r))$ for all $a \in E$ with $E := \{a = (a_1, \ldots, a_{m-1}, 0) \in \boldsymbol{R}^m \mid a_i = a_i^{\min} + L_i/2 + l_i \cdot L_i \;\text{ for } l_i = 0, \ldots, N^i - 1,\; i = 1, \ldots, m-1\}$ with minimal solution x^a and Lagrange multiplier μ^a to the constraint $a + t\,r - f(x) \in \boldsymbol{R}_+^m$. Determine the set $A := \{x^a \mid x^a$ minimal solution of (SP(a,r)) for $a \in E\}$.

Step 3: Determine the set $D := \{f(x) \mid x \in A\}$ and set $l = 0$.

Input: Choose $y \in D$ with $y = f(x^a)$ and associated Lagrange multiplier μ^a and set $a = (f_1(x^a), \ldots, f_{m-1}(x^a), 0)$. If y is a sufficient good solution, then stop. Otherwise, if additional points in the neighborhood of y are desired, then define a distance $\alpha \in \boldsymbol{R},\ \alpha > 0$, and the number $n^l \in \boldsymbol{N}$ of additional points for each direction and go to step 4.

Step 4: For all $\bar{\imath} = (i_1, \ldots, i_{m-1}) \in \{(i_1, \ldots, i_{m-1}) \in \boldsymbol{Z}^{m-1} \setminus \{(0, \ldots, 0)\} \,\big|\, i_j = -n^l, \ldots, n^l$, for $j = 1, \ldots, m-1\}$ set

$$a^{\bar{\imath}} := a + \sum_{j=1}^{m-1} i_j \cdot \frac{\alpha}{\sqrt{1 + (\mu_j^a)^2}} \cdot e_j$$

and solve $(\mathrm{SP}(a^{\bar{\imath}}, r))$. If there exists a solution $x^{\bar{\imath}}$ with Lagrange multiplier $\mu^{\bar{\imath}}$ set $A := A \cup \{x^{\bar{\imath}}\}$. Set $l := l + 1$ and go to Step 3.

Output: The set D is an approximation of the set of weakly efficient points.

Note that some of the problems considered in Step 2 and Step 4 may be infeasible. Thus in general not all parameters result in approximation points of the efficient set; see also test problem 4 in section 6.1. In [5, 23] conditions are given under which there exist minimal solutions of the problems $(\mathrm{SP}(a, r))$. In detail, we have that for $K = \boldsymbol{R}_+^m$ and a nonempty efficient set there always exists a minimal solution for $a \in \boldsymbol{R}^m$, $r \in \mathrm{int}(\boldsymbol{R}_+^m)$. However, here we have chosen $r = e_m \notin \mathrm{int}(\boldsymbol{R}_+^m)$ for simplicity.

For solving the scalar optimization problems in Steps 1, 2, and 4, an appropriate numerical method has to be used as, e.g., the SQP method. However, using just a local solver can lead to only local minimal solutions of the scalar problems and thus to only locally weakly EP-minimal points of (2.1). As a starting point for a numerical method for solving problem $(\mathrm{SP}(a^{\bar{\imath}}, r))$ in Step 4 the point $(f(x^a), x^a)$ can be used.

In Steps 2 and 3 a coarse approximation of the efficient set is calculated and in Step 4 around the special chosen points the refinement is done. Based on this algorithm it is possible to generate local equidistant approximations. With the coarse approximation in Step 2 it is ensured that all parts of the efficient set are covered and that the d.m. gets a survey of the efficient set. Then the method changes to an interactive part where the d.m. has to choose the areas in which a refinement is done.

5.3. Biobjective case. Now we come to the bicriteria case, i.e., $m = 2$. Of course the general Algorithm 5.1 presented in the previous section can be applied for $m = 2$, too. However, in the biobjective case we can use some special properties which do not hold generally for $m \geq 3$. This allows us not only to refine a coarse approximation locally but to determine equidistant approximations of the whole efficient set.

For $m = 2$ we can restrict the parameter set to a line segment H^a (Theorem 3.3). On this line segment we can easily define a total ordering, for instance increasing order w.r.t. the first coordinate. Then, points in the set H^a that are neighbors to each other are neighbors w.r.t. this order, too. This is no longer possible for the set H^0

for $m \geq 3$. As we have to know which points are neighbors to a point already found for using sensitivity information and for controlling the distance between the points, we start with a coarse approximation in the general case (section 5.2). However, since we are considering the biobjective case, we can use the special structure of H^a for adaptively determining the parameter a from the beginning without a previous coarse approximation. The result is an (almost) equidistant approximation of the efficient curve.

In the following we choose the parameters a in increasing order w.r.t. the first coordinate i.e., $a_1^0 \leq a_1^1 \leq \cdots \leq a_1^l \leq a_1^{l+1} \leq \cdots$ assuming we have $b_2 \neq 0$ for the hyperplane H. We choose $v \in \boldsymbol{R}^2$ with $b^\top v = 0$ such that we have $a^0 + s\,v \in H$ for $s \in \boldsymbol{R}$ and for $a^0 \in H$. Further, let $v_1 > 0$. We assume again we have already solved a reference problem (SP(a^0, r)) with a minimal solution (t^0, x^0). For $a^0 + t^0\,r - f(x^0) = 0_2$ we choose the next parameter a^1 by $a^1 = a^0 + s^1\,v$ with $s^1 > 0$ as in (5.3). Then, we have $a_1^1 > a_1^0$. For the case $a^0 + t^0\,r - f(x^0) = k^0 \neq 0_2$ we calculate $\tilde{a}^0$ as described in the proof of Theorem 3.6 by $\tilde{a}^0 = f(x^0) - \tilde{t}^0\,r$ and $\tilde{t}^0 = (b^\top f(x^0) - \beta)/(b^\top r)$. Then $\tilde{a}^0 + \tilde{t}^0\,r - f(x^0) = 0_2$.

In the case of $\tilde{a}_1^0 \geq a_1^0$ we set $a^1 := \tilde{a}^0 + s^1\,v$ (i.e., $a_1^1 > a_1^0$). For the ordering cone $K = \boldsymbol{R}^2_+$ we can show by an easy calculation using the fact that $a^0 + t^0\,r - f(x^0) = k^0$ with $k^0 \in \partial \boldsymbol{R}^2_+ \setminus \{0_2\}$ (see [45], but it is also a direct conclusion of the proof of Theorem 3.1 (b)) that $\tilde{a}_1^0 \geq a_1^0$ if and only if $\left(k_1^0 = 0,\ k_2^0 > 0 \text{ and } \frac{r_1 b_2}{b^\top r} > 0\right)$ or $(k_1^0 > 0,\ k_2^0 = 0 \text{ and } \frac{r_2 b_2}{b^\top r} < 0)$.

For the case $\tilde{a}_1^0 < a_1^0$ special considerations have to be made as it is not desirable to continue with the parameter $\tilde{a}^0$ instead of a^0 as we are looking for parameters with increasing first coordinate. In that case we still use the parameter a^0 for determining a^1. We can no longer assume $f(x(a,r)) = a + t(a,r)\,r$ as we have $f(x^0) = a^0 + t^0\,r - k^0$ with $k^0 \neq 0_2$. However, we can presume that the constraint $a + t\,r - f(x) \in K$ remains inactive and thus in view of $a^0 + t^0\,r = f(x^0) + k^0$ we set $a + t\,r = f(x^0) + k^0 + s\,k^0$ for some $s > 0$. For $s := \alpha/\|k^0\|$ we have a distance of $\alpha > 0$ between the points $a + t\,r$ and $a^0 + t^0\,r$, see Figure 3.1(b). Thus, we set the new parameter as

$$a^1 := f\left(x^0\right) + (1+s)\,k^0 - t\,r \tag{5.4}$$

with $s = \alpha/\|k^0\|$ and with some $t \in \boldsymbol{R}$. As we still demand $a \in H$ we choose $t = \frac{b^\top\left(f(x^0)+(1+s)\,k^0\right)-\beta}{b^\top r}$. Using the definition of $\tilde{a}^0$ we get $a^1 = \tilde{a}^0 + (1+s)(k^0 - \frac{b^\top k^0}{b^\top r} r)$. Because we have $b^\top a^1 = \beta$, the vector a^1 is actually an element of the hyperplane H. Again by an easy calculation, we can show that $a_1^1 \geq a_1^0$ for a_1 as in (5.4) and $s \geq 0$; i.e., the next parameter is chosen with increasing first coordinate.

By repeating the described steps for finding the next parameters $a^2, a^3, \ldots$ we can adaptively determine an almost equidistant approximation $f(x^0), f(x^1), \ldots$ of the efficient set of (2.1). However, it can happen that $\|f(x^l) - f(x^{l-1})\| \gg \alpha$ or $\|f(x^l) - f(x^{l-1})\| \ll \alpha$. This can be due to a strong varying curvature of the efficient set (see, e.g., [10, test problem CTP2]), especially if the distance α is not chosen appropriately small. It can also be due to gaps in the efficient set, i.e., nonconnected parts, see test problem 3 in section 6.1. In practice the efficient set of the examined multiobjective optimization problems is very often smooth (see, e.g., the application problem given in section 6.2 or in [6, 30, 36]). Nevertheless, if the distance between consecutive points is too large a refinement strategy as in Step 4 in Algorithm 5.1 can be applied subsequently. Too small distances can be eliminated afterwards just by reducing the approximation set.

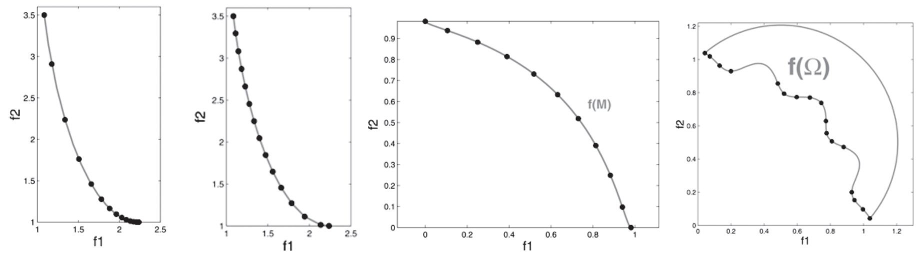

FIG. 6.1. (a) *Efficient set and approximation points of test problem* 1 *with weighted sum method and* (b) *with adaptive parameter control.* (c) *Efficient set and approximation points of test problem* 2. (d) *Image set and approximation points of test problem* 3.

6. Numerical results. In the following we apply the proposed methods on some test problems and on an application in intensity modulated radiotherapy.

6.1. Test problems. We start by considering some test problems which demonstrate the main properties of the proposed methods. It is shown that equidistant approximations are generated and that nonconvex problems as well as problems with a nonconnected efficient set or more than two objectives are covered.

Test problem 1: Comparison with the weighted sum method. The following test problem is chosen to show the advantage of the new method compared to the well-known habitual weighted sum method with which by a variation of the weights approximations of the efficient set can also be generated. For the bicriteria case the scalarization $\min_{x\in\Omega} w_1 f_1(x) + w_2 f_2(x)$ with weights $w_1, w_2 \in [0,1],\ w_1 + w_2 = 1$, is considered. Applying this scalarization to the test problem

$$\min_{\boldsymbol{R}^2_+} \begin{pmatrix} \sqrt{1+x_1^2} \\ x_1^2 - 4x_1 + x_2 + 5 \end{pmatrix}$$
subject to the constraints
$$x_1^2 - 4x_1 + x_2 + 5 \leq 3.5,$$
$$x_1 \geq 0,\ x_2 \geq 0,$$

choosing uniformly distributed weights leads to the approximation shown in Figure 6.1(a). This approximation has an uneven distribution and thus a low uniformity and a high coverage error. A much better result again with 15 points is gained by applying the procedure of section 5.3 with a hyperplane $H = \{y \in \boldsymbol{R}^2 \mid (1,1)y = 2.5\}$, $r = (1,0)^\top$, and a predefined distance of $\alpha = 0.2$ between the approximation points, see Figure 6.1(b).

Test problem 2: Nonconvex set. The following example by van Veldhuizen [54, p. 545], see also [11, 33], has a nonconvex image. Letting $n \in \boldsymbol{N}$ be a parameter the problem is defined as follows:

$$\min_{\boldsymbol{R}^2_+} \begin{pmatrix} 1 - \exp\left(-\sum_{i=1}^{n}\left(x_i - \frac{1}{\sqrt{n}}\right)^2\right) \\ 1 - \exp\left(-\sum_{i=1}^{n}\left(x_i + \frac{1}{\sqrt{n}}\right)^2\right) \end{pmatrix}$$
subject to the constraints
$$x_i \in [-4,4] \quad \text{for all } i = 1,\ldots,n.$$

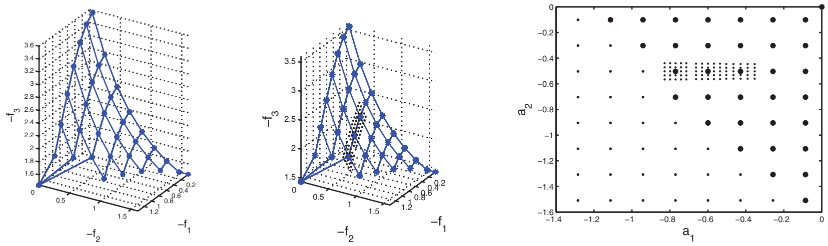

FIG. 6.2. *Test problem* 4: (a) *coarse approximation,* (b) *refined approximation, and* (c) *parameter set.*

We observe that by using the weighted sum method not all EP-minimal points can be found. For $n = 40$ we get with $r = (1,1)^\top$, $b = (1,0)^\top$, $\beta = 1.2$, $\alpha = 0.15$, and the procedure of section 5.3 the approximation shown in Figure 6.1(c). The connected line shows the efficient set of test problem 2 denoted as $f(M) := f(\mathcal{M}(f(\Omega), \boldsymbol{R}^2_+))$.

Test problem 3: Nonconnected efficient set. In the following problem by Tanaka [52] the image set $f(\Omega)$ is nonconvex, too, and additionally the efficient set is nonconnected (in a topological meaning):

$$\min_{\boldsymbol{R}^2_+} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$\text{subject to the constraints}$$
$$x_1^2 + x_2^2 - 1 - 0.1\cos\left(16\arctan\left(\frac{x_1}{x_2}\right)\right) \geq 0,$$
$$(x_1 - 0.5)^2 + (x_2 - 0.5)^2 \leq 0.5,$$
$$x_1, x_2 \in [0, \pi].$$

We have $f(\Omega) = \Omega$. By choosing $r = (1,2)^\top$, $b = (1,1)^\top$, $\beta = 0.5$, $\alpha = 0.08$, and with the procedure of section 5.3 we get the approximation of Figure 6.1(d). Here the difficulty is that by solving the scalar optimization problems (SP(a,r)) global solutions have to be found and thus, by using only a local method as the SQP method only local EP-minimal points instead of global solutions of (2.1) are guaranteed.

Test problem 4: Three objectives. This test problem with a nonconvex image set is a modified version of a problem in [38]. For the ordering cone $K = \boldsymbol{R}^3_+$ we consider

$$\min_{\boldsymbol{R}^3_+} \begin{pmatrix} -x_1 \\ -x_2 \\ -x_3^2 \end{pmatrix}$$
$$\text{subject to the constraints}$$
$$-\cos(x_1) - \exp(-x_2) + x_3 \leq 0,$$
$$0 \leq x_1 \leq \pi,\ x_2 \geq 0,\ x_3 \geq 1.2.$$

We use Algorithm 5.1 with $N^1 = N^2 = 8$ for a coarse approximation. The result after Steps 1–3 is given in Figure 6.2(a). Note that the negative of the objective function values is drawn and that the approximation points are connected with lines.

In the input step of the algorithm we choose the three points $y \in D$ for which $y_1 \leq -0.4$ and $-0.6 \leq y_2 \leq -0.4$ holds. We do a refinement with $n^1 = 2$ and $\alpha = 0.06$. This

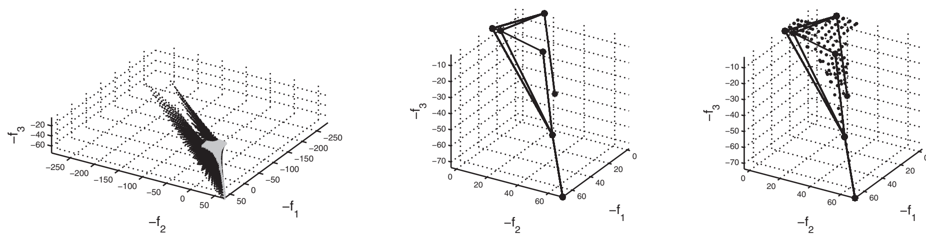

FIG. 6.3. *Test problem* 5: (a) *image set in dark color and efficient set in grey,* (b) *coarse approximation with the approximation points connected with lines, and* (c) *refined approximation.*

results in the refined approximation of the efficient set of Figure 6.2(b). In the course of the algorithm several scalar optimization problems $(\mathrm{SP}(a,r))$ to parameters $a = (a_1, a_2, 0)$ are solved. These parameters are plotted as points (a_1, a_2) in Figure 6.2(c) as dots. To the parameters represented by the smallest dots, no minimal solution of the problem $(\mathrm{SP}(a,r))$ exists. Around three parameters one can see the parameters of the refinement step (Step 4 of the algorithm), and it can well be seen that the distance between the refinement parameters differs depending on the sensitivity information delivered by the Lagrange multipliers (corresponding to the steepness of the efficient set).

Test problem 5: Comet problem. This problem [11, p. 9]

$$\min_{\boldsymbol{R}^3_+} \begin{pmatrix} (1+x_3)\left(x_1^3 x_2^2 - 10x_1 - 4x_2\right) \\ (1+x_3)\left(x_1^3 x_2^2 - 10x_1 + 4x_2\right) \\ 3(1+x_3)x_1^2 \end{pmatrix}$$

subject to the constraints

$$1 \le x_1 \le 3.5,\ -2 \le x_2 \le 2,\ 0 \le x_3 \le 1$$

with the set of K-minimal points $\mathcal{M}(f(\Omega), \boldsymbol{R}^3_+) = \{x \in \boldsymbol{R}^3 \mid 1 \le x_1 \le 3.5,\ -2 \le x_1^3 x_2 \le 2,\ x_3 = 0\}$ has its name because of the image of the efficient set with a short broad and a long small area (see Figure 6.3(a)). A first coarse approximation for $N^1 = N^2 = 12$ delivers only a few approximation points (Figure 6.3(b)), but by doing a refinement according to Step 4 of Algorithm 5.1 with $n^1 = 3$, $\alpha = 4$ for all points with no other point with a distance of less than 5 in the neighborhood, an approximation with a high quality is finally achieved (Figure 6.3(c)).

6.2. Application in IMRT. We have also examined a problem in intensity modulated radiotherapy (IMRT) in medical engineering. Here, a patient with, e.g., a prostate tumor has to be irradiated to destroy the tumor. An optimal irradiation plan which is represented by an optimal intensity profile $x \in \mathbb{R}^{400}$ to 400 separate controllable beamlets B_i, $i = 1, \ldots, 400$ has to be found. We assume the beam geometry to be fixed. The problem is that the healthy surrounding organs should be damaged as little as possible while in each cell of the tumor a minimal curative dose has to be reached [1, 8, 40, 43].

Hence, this problem is a multiobjective optimization problem which has formerly been solved by just summing up the objective functions to one single scalar-valued objective using a weighted sum approach. Thereby, the difficulty is that the weights have no medical interpretation and that the physician has to find a good irradiation plan by a laborious trial and error process [40]. Using instead a multiobjective

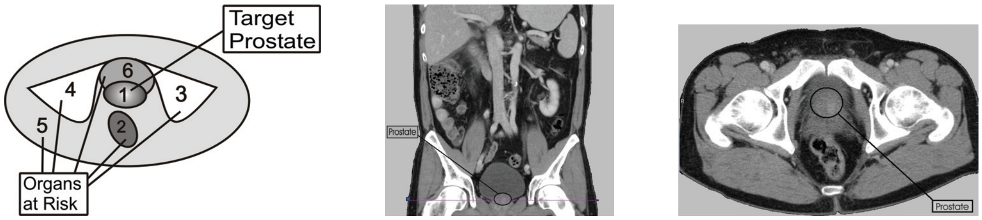

FIG. 6.4. (a) *Schematic axial body cut.* (b) *Coronar and axial CT-cut.*[1]

optimization approach with an approximation of the whole efficient set [22, 40, 43] simplifies this procedure and improves the results significantly and is, therefore, now actually applied. Thereby, a high quality approximation with equidistant points is demanded [40].

The number of objective functions depends on the number of different healthy tissues surrounding the tumor. In our example, the treatment planning for a prostate tumor, one is especially concerned about the bladder (V_6) and the rectum (V_2), while it has to be ensured that the doses in the remaining tissues as left (V_3) and right (V_4) hip bone and surrounding unspecified tissue (V_5) remain below an upper level (for the location of the organs see Figure 6.4(a)).

For evaluating and comparing the radiation stress in the several organs the concept of the equivalent uniform dose by Niemierko [44] based on p-norms is used. Therefore, the relevant part of the patient's body is mapped with the help of a computer tomography (CT), see Figure 6.4(b), and according to the thickness of the slices dissected in cubes, the so-called voxels. Then, using a clustering method [40, 50], whereas voxels with equal radiation exposure are collected, the high number of 435 501 voxels is reduced to 11 877 clusters $c_j,\ j = 1, \dots, 11\,877$, which are allocated to the seven volumes $V_0, \dots, V_6$ by a physician.

Volumes V_0 and V_1 describe the tumor (the so-called target-tissue) while V_1 is the boost-tissue, which is tumor tissue that has to be irradiated especially high. Thus, depending on the volume V_k, the number of voxels $N(V_k)$ in this organ, the number of voxels $N(c_j)$ in cluster c_j, and the dose limit U_k, the radiation stress is evaluated by

$$\mathrm{EUD}_k(x) := \frac{1}{U_k}\left(\frac{1}{N(V_k)}\sum_{\{j|c_j\in V_k\}} N(c_j)\cdot(P_j x)^{p_k}\right)^{\frac{1}{p_k}} - 1, \qquad k = 2, \dots, 6.$$

The vector P_j denotes the jth row of the matrix $P = (P_{ji})_{j=1,\dots,11\,877, i=1,\dots,400}$ which describes the emission by the beamlet B_i ($i = 1, \dots, 400$) in the cluster c_j ($j = 1, \dots, 11\,877$) at one radiation unit.[2] For the intensity profile $x \in \boldsymbol{R}^{400}$, $P_j x$ denotes the irradiation dose in the cluster c_j caused by the beamlets B_i, $i = 1, \dots, 400$. The parameter $p_k \in [1, \infty[$, which represents the physiology of the organ, is determined statistically and is given, like the other parameters, in Table 6.1.

The irradiation stress should remain below a critical value which results in the constraints $U_k(\mathrm{EUD}_k(x) + 1) \le Q_k,\ k = 2, \dots, 6$, which can be restated as

$$\sum_{\{j|c_j\in V_k\}} N(c_j)(P_j x)^{p_k} \le Q_k^{p_k}\, N(V_k), \quad k = 2, \dots, 6.$$

[1] By courtesy of Dr. R. Janka, Institute of Diagnostic Radiology, Univ. Erlangen-Nürnberg.
[2] The data are available on request by sending an email to the author.

TABLE 6.1
Critical values for the organs at risk.

	number of organ (k)	p_k	U_k	Q_k	$N(V_k)$
rectum	2	3.0	30	36	6 459
left hip-bone	3	2.0	35	42	3 749
right hip-bone	4	2.0	35	42	4 177
remaining tissue	5	1.1	25	35	400 291
bladder	6	3.0	35	42	4 901

TABLE 6.2
Critical values for the tumor tissues.

	number of organ (k)	L_k	δ_k	ε_k
target-tissue	0	67	0.11	0.11
boost-tissue	1	72	0.07	0.07

The radiation stress in the tumor tissues V_0 and V_1 is considered w.r.t. each single cluster, as it is important to destroy each single cancer cell. For homogeneity reasons this results in the constraints

$$\begin{aligned} & L_0(1-\varepsilon_0) \leq P_j x \leq L_0(1+\delta_0), && \forall j \text{ with } c_j \in V_0 \\ \text{and} \quad & L_1(1-\varepsilon_1) \leq P_j x \leq L_1(1+\delta_1), && \forall j \text{ with } c_j \in V_1, \end{aligned}$$

with constants $L_0,\ L_1,\ \varepsilon_0, \varepsilon_1, \delta_0$ and δ_1 given in Table 6.2. Volume V_0 consists of 8 593 clusters while V_1 has 302 clusters. Including nonnegativity constraints for the beamlet intensity, this results in the feasible set

$$\begin{aligned} \Omega = \{x \in \mathbb{R}^{400}_+ \mid \quad & U_k(\mathrm{EUD}_k(x)+1) \leq Q_k, && k = 2, \ldots, 6, \\ & L_0(1-\varepsilon_0) \leq P_j x \leq L_0(1+\delta_0), && \forall j \text{ with } c_j \in V_0, \\ & L_1(1-\varepsilon_1) \leq P_j x \leq L_1(1+\delta_1), && \forall j \text{ with } c_j \in V_1\} \end{aligned}$$

with 17 795 constraints.

The objectives are a minimization of the dose stress in the rectum (V_2) and in the bladder (V_6) as these two healthy organs always have the highest irradiation stress and a stress reduction for the rectum deteriorates the level for the bladder and vice versa. This leads to the biobjective optimization problem

$$\begin{gathered} \min_{\boldsymbol{R}^2_+} \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} = \begin{pmatrix} \mathrm{EUD}_6(x) \\ \mathrm{EUD}_2(x) \end{pmatrix} \\ \text{subject to the constraint} \\ x \in \Omega. \end{gathered}$$

We apply the procedure of section 5.3 with $r = (1,1)^\top$, $\alpha = 0.04$, and $H = \{y \in \boldsymbol{R}^2 \mid y_1 = 0\}$, and we get that only parameters $a \in H^a$ with $H^a = \{y \in \boldsymbol{R}^2 \mid y_1 = 0,\ y_2 = \lambda \cdot 0.1841 + (1-\lambda) \cdot (-0.2197),\ \lambda \in [0,1]\}$ have to be considered. The approximation given in Figure 6.5(a) with 10 approximation points (connected with lines) is generated. These points as well as the distances δ^i between consecutive approximation points are listed in Table 6.3.

Based on these results the physician can choose a treatment plan by weighting the damage to the bladder and the rectum against each other. Besides he can choose an interesting plan and refine around it by using the strategy as in Step 4 of Algorithm 5.1. Further he can choose a point y determined by interpolation between the

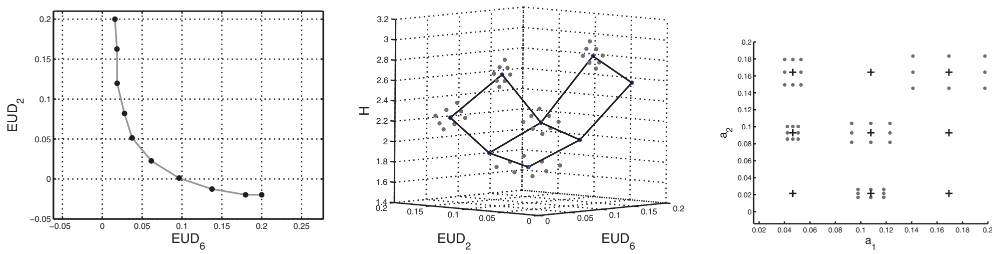

FIG. 6.5. *IMRT problem:* (a) *efficient set and approximation points of the bicriteria problem,* (b) *refined approximation of the tricriteria problem, and* (c) *parameter set of the tricriteria problem.*

TABLE 6.3
Approximation points and distances δ^i between them for $\alpha = 0.04$.

approximation point	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$
$\mathrm{EUD}_2\ (\bar{x}^i)$	0.2000	0.1625	0.1197	0.0819	0.0515
$\mathrm{EUD}_6\ (\bar{x}^i)$	0.0159	0.0184	0.0187	0.0278	0.0374
δ_i	–	0.0375	0.0429	0.0389	0.0319

approximation point	$i=6$	$i=7$	$i=8$	$i=9$	$i=10$
$\mathrm{EUD}_2\ (\bar{x}^i)$	0.0228	0.0012	−0.0126	−0.0197	−0.0197
$\mathrm{EUD}_6\ (\bar{x}^i)$	0.0615	0.0964	0.1376	0.1796	0.2000
δ_i	0.0375	0.0411	0.0434	0.0426	0.0204

approximation points and solve problem $(\mathrm{SP}(a,r))$ to the correspondent parameters, see [53], to get a new approximation point.

As it turned out that the treatment success depends also on the irradiation homogeneity, this objective can be added to the former two objective functions. Thereby the homogeneity of the irradiation is measured by

$$H(x) := \sqrt{\frac{\sum\limits_{\{j \mid c_j \in V_0\}} N(c_j)\,(P_j x - L_0)^2 + \sum\limits_{\{j \mid c_j \in V_1\}} N(c_j)\,(P_j x - L_1)^2}{N(V_0) + N(V_1)}}$$

with $N(V_0) = 13\,238$ and $N(V_1) = 2686$. This results in the multiobjective optimization problem

$$\min_{\boldsymbol{R}^3_+} \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{pmatrix} = \begin{pmatrix} \mathrm{EUD}_6(x) \\ \mathrm{EUD}_2(x) \\ H(x) \end{pmatrix}$$

subject to the constraint

$$x \in \Omega.$$

We have solved this problem using Algorithm 5.1 with $N^1 = N^2 = 3$. In Step 1 we get $a_1^{\min} = 0.0158$, $a_1^{\max} = 0.2000$, $a_2^{\min} = -0.0141$, and $a_2^{\max} = 0.2000$. This results in $L_1 = 0.0614$, $L_2 = 0.0714$ and thus in the parameter set $E := \{a \in \mathbb{R}^3 \mid a_1 \in \{0.0465,\ 0.1079,\ 0.1693\},\ a_2 \in \{0.0216,\ 0.0929,\ 0.1643\},\ a_3 = 0\}$. For solving the related scalar optimization problems we use the SQP procedure implemented in Matlab with 600 iterations and a restart after 150 iteration steps. We do not get a solution for the parameter $a = (0.0465, 0.0216, 0)$. We assume a physician chooses certain points and we do a refinement around these points with $n^1 = 1$ and $\alpha = 0.07$.

This results in the refined approximation shown in Figure 6.5(b). The determined parameters $a = (a_1, a_2, 0)$ according to Steps 2 and 4 are shown in Figure 6.5(c) as points (a_1, a_2).

If we choose, e.g., the approximation point $(0.0465, 0.1643, 0.2294)$ of the efficient set and calculate the distances between that point and the surrounding refinement points, we get the following 12 distances: 0.0697, 0.0801, 0.0795, 0.0814, 0.0880, 0.0679, 0.0736, 0.0624, 0.0663, 0.0687, 0.0640, and 0.0712 with a rounded average value of 0.0727.

A more detailed and more technical description of this problem can be found in [13, 14, 40].

7. Outlook. Here we have developed an adaptive parameter control for the Pascoletti–Serafini scalarization. We have chosen this scalarization because it is not only suitable also for finding K-minimal points with $K \neq \boldsymbol{R}^m_+$, but it is also a very general method. Many other scalarization approaches such as the weighted Chebyshev norm, the ε-constraint method (see p. 1707 and [14]), the Polak method [34], or the normal boundary intersection (NBI) method [9] can be seen as a special case of this method (see [15]), and thus the presented results can be applied there too.

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