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Thresholding and fast iso-contour extraction with fuzzy arithmetic

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Abstract

This paper introduces a fast technique to extract, from noisy images, regions whose pixels fall, with a degree of uncertainty, in a prescribed range of values. The method first builds from a digital image a smaller image whose pixels are assigned fuzzy numbers to describe local characteristic of the image and their uncertainty due to noise. This fuzzy picture is subsequently interrogated with a Marching-Cube-like algorithm. It is possible to visualize at once the results obtained using different presumption levels. This simple technique provides the user with cues to the amount of uncertainty of the found results. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Thresholding and its immediate generalization, window slicing (see Jain, 1989, pp. 407), are simple, yet powerful, tools for many vision algorithms. This paper presents a fast, simple and robust to noise method for window slicing. The new technique makes use of fuzzy arithmetic principles. The method can be implemented in hardware and provides a fast contouring algorithm that can greatly help some common diagnostic practices like ultrasonography.

Thresholding and window slicing are efficient and reliable segmentation techniques, especially when the object to extract is characterized by a local feature of the pixels. When the focus is on the pixels whose value is *right on* the threshold value one obtains algorithms for the determination of *iso-regions* or *iso-contours*. Window slicing with a band that marks the transition between two objects may be a fast way to find boundary locations. In applications that are not accuracy-critical, window slicing is frequently a viable alternative to more precise, but more costly algorithms. Thresholding segmentation is used in the treatment of remotely sensed IR images, scanned printed documents, X-ray images, sonar generated images, etc.

The effectiveness of a threshold based segmentation technique depends on the domain of application and, more crucially, on the right choice of an appropriate threshold value. A large amount of literature has been devoted to find automated criteria for the selection of threshold values that

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obey to some optimality principle. Seminal in this field has been the work of Otsu (1979). Other contributions are Pun (1981), Kittler and Illingworth (1986), O'Gorman (1994), Shanbhag (1994) and Liu and Srihari (1997). A comparison of several of the known techniques for the threshold selection can be found in (Glasbey, 1993).

When the data (pixels or voxels) have a smaller sampling resolution than the resolution required in the visualization a popular approach to windowing slicing is the Marching Cube Algorithm (MCA) introduced by Lorensen and Cline (1987). This technique makes use of a set of local rules to locate the polygons that approximate the iso-surface. Note that if the windowing band is not reduced to a single value the *iso-region* is a set of 3D polyhedra. Many variations of the Marching Cube Algorithm are known, but all seem to have two major sources of difficulties: noise sensitivity and topological ambiguity.

Even a small amount of spurious or erroneous data can confuse the windowing process destroying the topological integrity of the extracted region. Several heuristic rules have been proposed to resolve topological ambiguities in applying the MCA, but none of them seem to avoid all the possible problems (Wilhelms and van Gelder, 1990; Karron et al., 1992; Montani et al., 1993). A common cure to noisy data is to apply a smoothing filter like, for example, the non-linear median filter whose robustness to noise is rooted in the analogous property of the statistical median indicator. Median filtering, however, is relatively time-consuming and its use in practical applications is limited.

Another problem with MCA is that it provides only a *hard* classification; points are classified as in the inside of the approximating polyhedral surface or outside of it and any information about the uncertainty coming from the noise in the original image is lost. The user, hence, has no clue about the validity of the results.

The technique described in this paper for bidimensional data tries to address the three above mentioned issues noise reduction, topological soundness and uncertainty estimation, using fuzzy arithmetic. Its generalization to the 3D case is straightforward but the visualizations of the results are a challenge for scientific visualization and will not be considered here.

In our algorithm noise and uncertainty are naturally merged, in a controlled way, into a fuzzy model of the original image. The interrogation of this fuzzy model, done in a Marching-Cube-like fashion, provides in output regions where, with a given level of presumption, the pixel values fall in the required range. The regions relative to lower presumption levels naturally contain the regions relative to higher presumption levels. To color these regions with a slowly changing LUT provides visual information suggestive of the validity of the windowing process.

This short paper is organized as follows: Section 2 reviews some basics of fuzzy arithmetic and describes the two fundamental steps of the proposed technique, i.e., the fuzzification procedure of the original image and its interrogation. Section 3 reports experimental results and compares them with similar results obtained with a non-fuzzy approach. The paper concludes with a summary and with some notes about further related researches. Preliminary results obtained with the algorithm have been reported at the WSCG'98 Conference, held in Pilsen (Czech Republic).

2. Fuzzy digital images

2.1. Fuzzy arithmetic

In this subsection we review some basic concepts of fuzzy arithmetic. Definitions and results not explicitly mentioned here can be found in (Zimmermann, 1991; Kauffman and Gupta, 1991; Anile et al., 1995).

A *fuzzy real number F* is an interval $[a,b]$ of the real line together with a *membership function*, $m(t)$ from the set of the real numbers to the unit interval $[0,1]$ such that:

(i) $m(t) = 0$ for t in $R \backslash]a,b[$;

(ii) There is at least a point c in $[a,b]$ such that $m(c) = 1$.

$m(t)$ can be a general function, however, for efficiency and simplicity, only here, triangular fuzzy numbers are used. A *triangular fuzzy number* $F = ([a,b], m(t))$ is a fuzzy number such that there

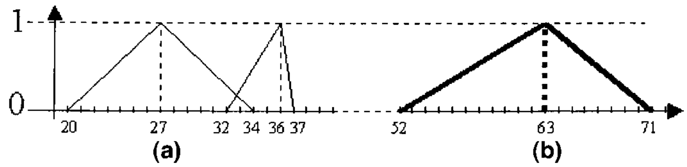

Fig. 1. (a) Triangular fuzzy numbers with bases [20, 34] and [32, 37] and vertexes 27 and 36. (b) The sum of the two fuzzy numbers of Fig. 1(a).

is only one point c in $[a, b]$ such that $m(c) = 1$ and the function $m(t)$ is linear and monotonically increasing from a to c and linear and monotonically decreasing from c to b. Examples of two triangular fuzzy numbers are diagrammed in Fig. 1(a).

Given a real number s in $[0, 1]$ the interval F_s is the subinterval $[l_s, h_s]$ of $[a, b]$ such that for every t in $[l_s, h_s]$, $m(t)$ is greater or equal to s. F_s is said *s-cut* of the number F.

A fuzzy number F can be equivalently assigned as a pair $([a, b], m(t))$ or as a family of intervals F_s with s in $[0, 1]$ such that if $s > t$, F_s is contained in F_t. In practice only a finite set of presumption levels is used and a common way to specify and operate with fuzzy numbers is to assign an ordered finite collection of nested intervals. Making use of s-cuts, it is possible to introduce arithmetic operations over fuzzy numbers. If $\cdot$ is a binary operation over real numbers the interval $[a, b] \cdot [c, d]$ is the interval defined as $[\min(t \cdot s),\ \max(t \cdot s)]$ with t in $[a, b]$ and s in $[c, d]$.

If $\cdot$ is a binary operation over the reals $F \cdot G$ is the fuzzy number described by the collection of s-cuts $F_s \cdot G_s$ with s in $[0, 1]$. If $\cdot$ is a linear operation triangular fuzzy numbers generate a new triangular fuzzy number. This is not necessarily true if $\cdot$ is non-linear. Fig. 1(b) shows the sum of two fuzzy numbers.

Linear interpolation between two fuzzy numbers F and G, $L(F, G, t)$, is defined over the unit real interval and takes value over the set of the fuzzy numbers. More precisely, in terms of s-cuts:

$$L_s(F, G, t) = F_s(1t) + G_s t.$$

In some sense the formula above describes parametrically a *fuzzy segment* or a linear *fuzzy function*. In Fig. 2(a)–(c) some s-cuts of the fuzzy segment interpolating the triangular numbers F and G with bases $[0, 6]$ and $[6, 9]$ and vertices 5 and 7, respectively, are diagrammed. In Fig. 2(d) they are represented simultaneously with different gray levels.

We propose to extend the window slicing procedure to the case of fuzzy segment as follows. Given a presumption level s and a range $[a, b]$, the set:

$$H_{[a,b],s} = \{t \text{ in}[0, 1] \text{ such that } L_s(F_s, G_s, t) \text{ and } [a, b] \text{ have a non-empty intersection}\}$$

is the set where the fuzzy function $L(F, G, t)$ assumes, with presumption s, values in $[a, b]$.

The fuzzy subset $H_{[a,b]}$ of the unit interval, defined with the s-cuts $H_{[a,b],s}$ is the answer to the windowing query of $L(F, G, t)$ with respect to the range $[a, b]$. Pictorially the process is described in Fig. 3.

2.2. Fuzzyfication of digital pictures

The first step in the proposed method is to reduce the data of the original image. In particular, we derive for overlapping square regions of the image a fuzzy number that provides a statistical descriptive summary of the pixel population in the region. A more expressive statistical description would include the complete histogram of the region, but such a model would be very hard to maintain and interrogate. Fuzzy numbers, on the other hand, naturally provide a compact, simple yet powerful way to organize this information.

A similar approach has been used to construct spline surfaces incorporating uncertainty of geographical data in (Anile et al., 1999; Gallo et al., 1999).

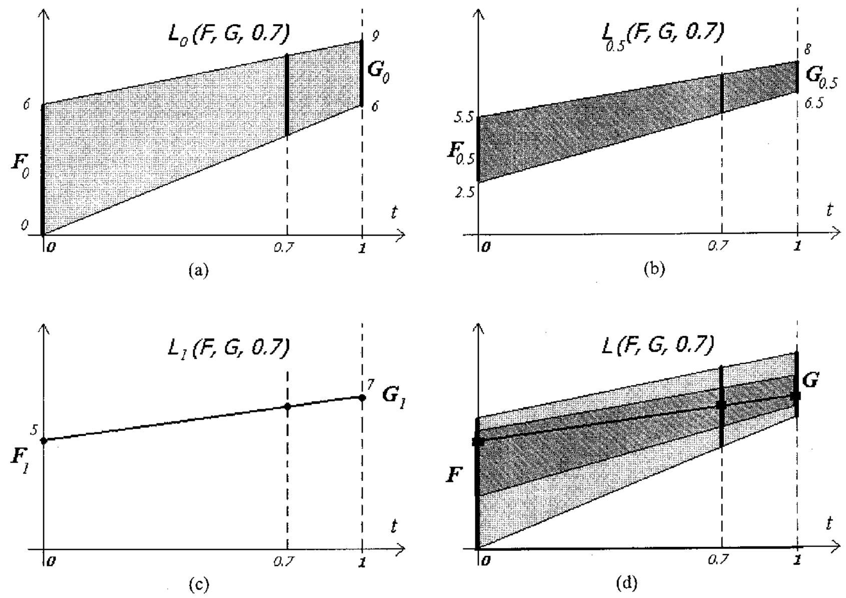

Fig. 2. Linear interpolation between two fuzzy numbers: (a) shows the interval interpolation between 0-level sets of the fuzzy numbers F and G; (b) and (c) show the same for 0.5-level and 1-level sets, respectively; (d) overlays together the results of (a)–(c).

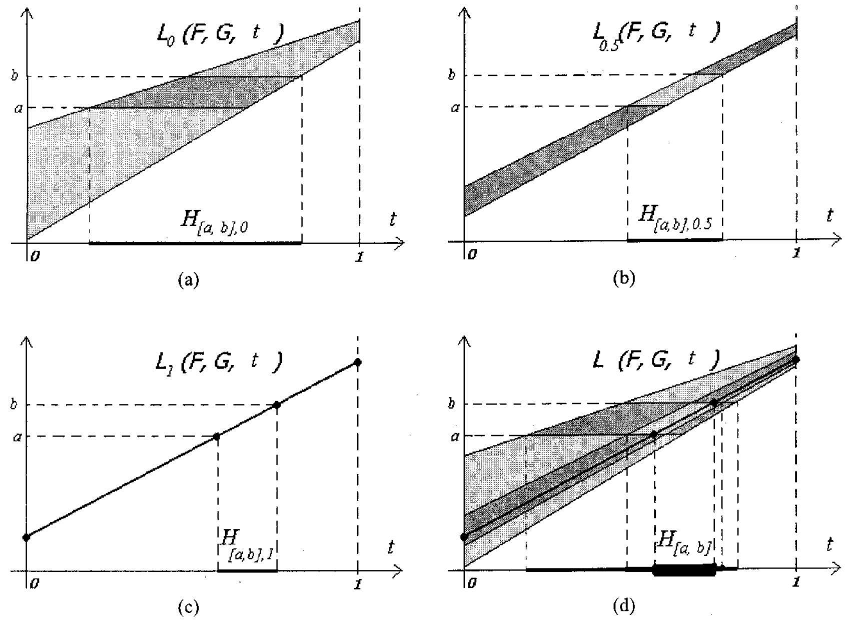

Fig. 3. Window slicing procedure extended to the case of a fuzzy segment: (a) the 0-level set of the interpolation between fuzzy numbers F and G is interrogated with respect to the interval $[a, b]$; (b) the same relatively to the 0.5-level set; (c) the same relatively to the 1-level; (d) overlays the results of (a)–(c) together.

Let $\boldsymbol{h}$ be a positive integer. Let $\boldsymbol{M} \times \boldsymbol{N}$ be the dimension in pixels of the original picture. The fuzzy version of the picture is a lattice of $M/h \times N/h$ fuzzy numbers. To each point (i,j) of the lattice, $i = 1, \dots, M/h$, $j = 1, \dots, N/h$, we assign a triangular fuzzy number obtained as follows:

1. Consider the square sub-region of the original picture of side $2\boldsymbol{h}$ pixels, whose center is the pixel (ih, jh).
2. Compute the median (MED) the first quartile (FQ) and the third quartile (TQ) of the values of the $4\boldsymbol{h}^2$ pixels in the square.
3. Assign to the point (i,j) the fuzzy triangular number whose basis is the *interval* $[FQ, TQ]$ and with vertex MED.

This simple procedure provides a statistical summary of the original picture and uses indicators that are well known for their robustness to noise and outliers, (see Hogg and Craig, 1978). Robustness is, here, a key issue: the proposed methodology is suitable to the interrogation of noisy pictures where spurious gray values may frequently appear. A filtering mechanism to minimize the effects of such spurious values is hence needed.

Another possible choice to approximate the local histogram with a fuzzy number could be to use as a basis the range spanned between minimum and maximum gray value in a region. This choice is sensible to outliers and once the fuzzy model is interrogated, it leads to very wide intervals.

More sophisticated choices in the shape of the fuzzy number (i.e., more complex linear splines instead of a triangle) are possible. We learned from experiments that very little is gained from the choice of a more complex shape for the fuzzy number: the increase in complexity and storage requirements in this case seems to be not fully justifiable.

The fuzzy data in the lattice realize a form of data reduction of the original data set, maintaining, at the same time, minimal information about the statistical variability. It is, of course, possible that different statistical distributions of pixels provide similar fuzzy summaries. Since quartile and median operators are not linear it is hard to predict their relationships with the successive moments of the gray values distribution in a region: pathological examples may be found (see Fig. 4). These pictures suggest the hypothesis that there is certain degree of correlation between the fuzzy summary and the perceived light value in a region. Further investigation and experiments are needed to test this hypothesis. A weak evidence in support of this idea comes, indirectly, from the performances of our technique: when the fuzzy model is interrogated the results have always been sufficiently coherent with approximate evaluation made by a human observer. Fig. 4 shows that histograms have the same representative weakness with respect to spatial distribution. To minimize such problems we enhance spatial coherence of the new data overlapping the $2\boldsymbol{h} \times 2\boldsymbol{h}$ squares. On the other hand fuzzy summaries require less storage and computational resources than histograms.

Notice that the state-of-the-art methods in the analysis of spatial data, as reported in (Kruse and Meyer, 1987; Cressie, 1996; Viertl, 1996; Deutsche and Journel, 1998), ask generally for more information than the simple summaries used here. This is achieved with a greater computational effort. We found experimentally that the proposed fuzzy technique balances well the complexity issues with the precision required for many applications in computer vision.

2.3. Interrogation of the fuzzy data set

In this section we show how to interrogate the fuzzy summary obtained in the previous section.

The kind of query considered here is a pair $([a,b], s)$. The first component of the query is a real interval: we are looking for regions of the image whose pixels fall, with same uncertainty, in the range $[a,b]$. The second component of the query, s, is a real number in $[0,1]$ and represents the presumption level associated with the query.

The query is processed with an approach that is reminiscent of the MCA. This is a reasonable choice because the fuzzy image has a lower resolution than the original picture and local linearity can be safely assumed, at least for small values of h. The interrogation algorithm considers, one at the time, the unit rectangles of the lattice of fuzzy data.

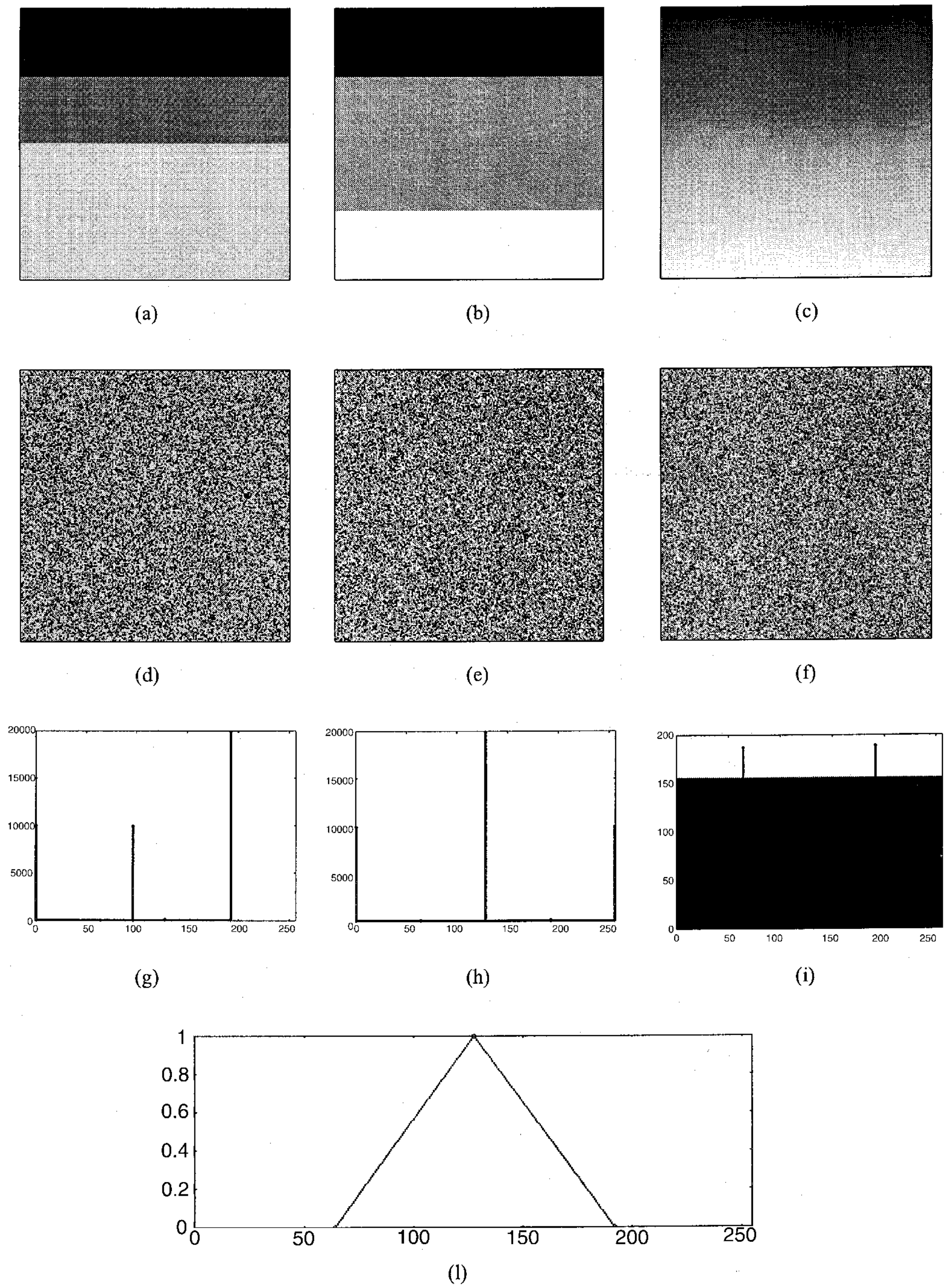

Fig. 4. The images in (a), (b) and (c) have the same histograms than, respectively, (d), (e) and (f); histograms are shown below each pair in (g), (h) and (i). All of these pictures are summarized, in the proposed approach with the same fuzzy number shown in (l). Observe that there is very little perceptual difference between (d), (e) and (f).

- Let Upper Left (UL) be the fuzzy number associated with the vertex (i,j) of the rectangle.
- Let Upper Right (UR) be the fuzzy number associated with the vertex $(i+1,j)$ of the rectangle.
- Let Lower Left (LL) be the fuzzy number associated with the vertex $(i,j+1)$ of the rectangle.
- Let Lower Right (LR) be the fuzzy number associated with the vertex $(i+1,j+1)$ of the rectangle.

The linear interpolations $L(\mathrm{UL}, \mathrm{UR}, t)$, $L(\mathrm{UR}, \mathrm{LR}, t)$, $L(\mathrm{LR}, \mathrm{LL}, t)$ and $L(\mathrm{LL}, \mathrm{UL}, t)$ are queried, at the presumption level s, with the interval $[a,b]$ as shown in Section 2.1. As a result one obtains at most four sub-segments of the edges joining the points (i,j), $(i+1,j)$, $(i+1,j+1)$, $(i,j+1)$ of the lattice. The answer to the original query, inside the rectangle, is obtained as the convex hull of such sub-segments. The set-theoretic union of these convex hulls over all the rectangles in the lattice provides the output of the query (see Fig. 5). It is straightforward to observe that if $s_1 < s_2$ the convex hull obtained as above for the presumption level s_1 contains the corresponding convex hull for the presumption level s_2.

This leads to a simple way to visualize at once the answers obtained from several queries of the kind $([a,b],s_1),([a,b],s_2),\ldots,([a,b],s_k)$ with $s_1 < s_2 < \cdots < s_k$.

We assign a sequence of gray values $(g_1,g_2,\ldots,g_k)$ to the sequence $(s_1,s_2,\ldots,s_k)$ with the property: $g_i > g_j$ whenever $s_i > s_j$.

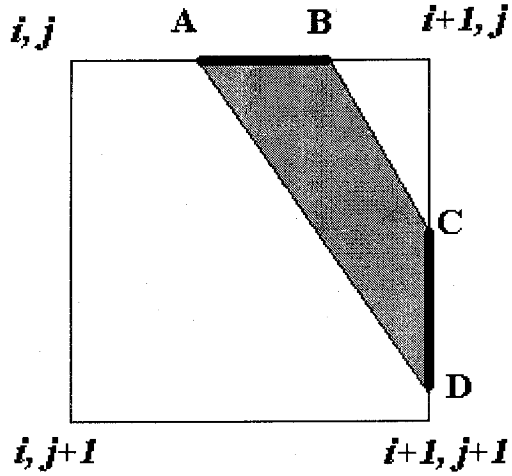

Fig. 5. The thick segments AB and CD, mark the intervals resulting from the intersections of the linear interval functions obtained interpolating the intervals UL_s and UR_s, UR_s and LR_s, respectively with the query interval $[a,b]$; the shaded region is the convex hull of the two segments.

The visualization proceeds filling with gray value g_1 the region obtained as answer to the query $([a,b],s_1)$ and successively filling the regions obtained as answer to queries $([a,b],s_2),\ldots,([a,b],s_k)$ using, respectively, gray values $g_2,\ldots,g_k$.

The process relative to three presumption levels is summarized in Figs. 6 and 7(c). Fig. 8(c) shows the results of the interrogation of two medical images.

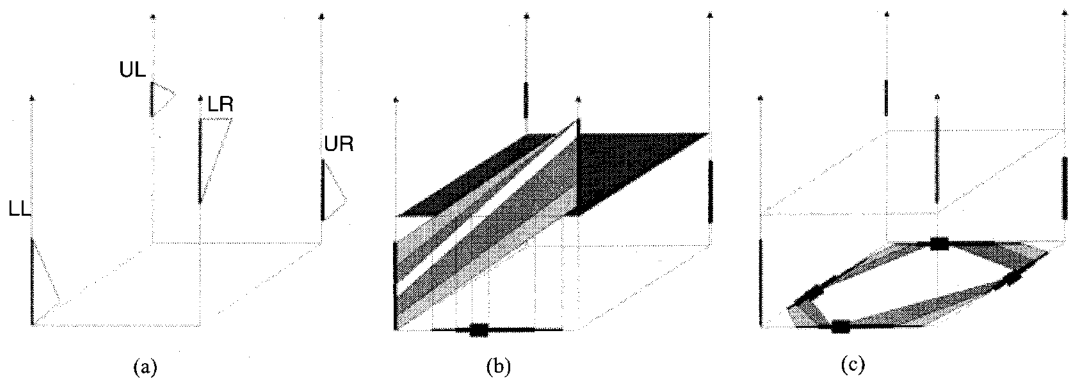

Fig. 6. (a) An example of a rectangle of the lattice. (b) and (c) Results of some interrogations of the rectangle in (a).

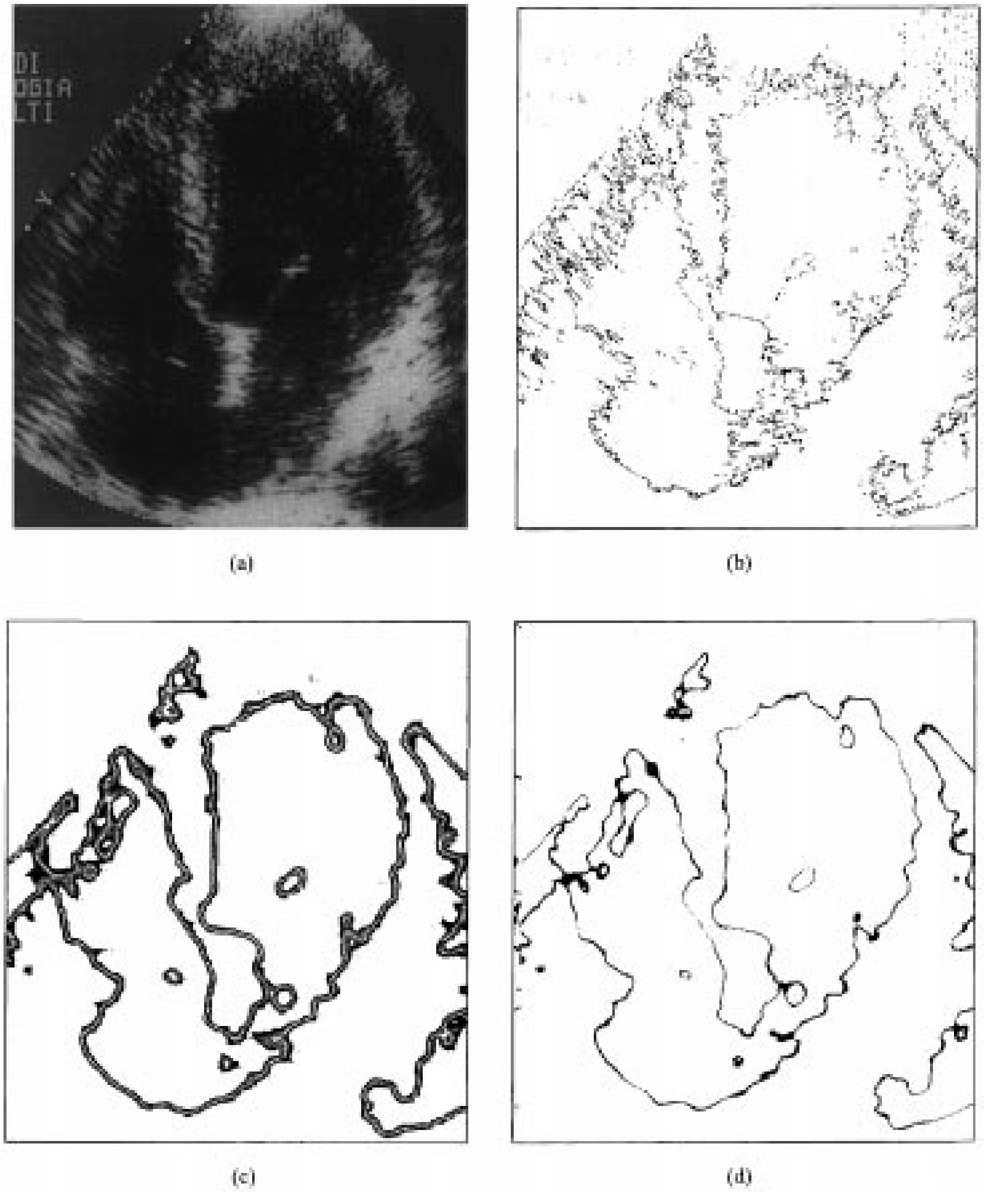

Fig. 7. (a) The original picture. (b) The results obtained with traditional window slicing using the interval [36, 40]. (c) The result obtained with the new technique, at presumption levels $s = 0,\ 0.5$ and 0.9 and $h = 15$ relatively to the same interval. (d) The result obtained with the traditional window slicing, in the same interval, after median smoothing.

3. Experimental results

The proposed algorithm has been implemented both in C+ and in Java on Pentium machines and has been tested on a set of pictures including natural scenes, noisy ultrasonography images, CAT, synthetic images and scanned text with uniform noise.

To validate the quality of the results obtained with the proposed methodology we first compared

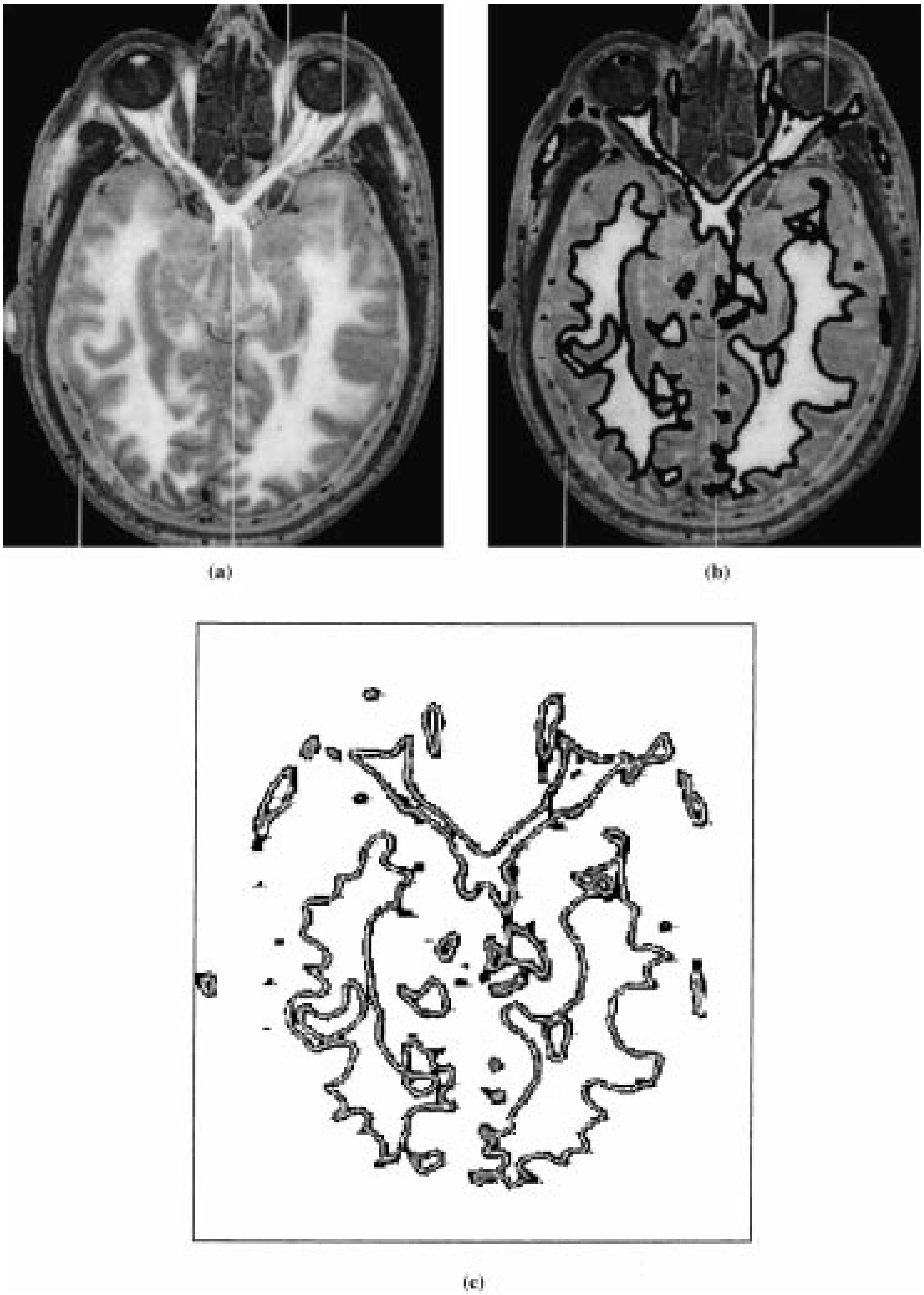

Fig. 8. A brain section from the Visible Human data set (a) has been fuzzy thresholded to separate white and gray matter (b). The result of interrogation at several presumption levels $(0, 0.5, 0.9)$ has been visualized simultaneously (c). The picture (c) has been enlarged for visibility purposes.

them with the results obtained using a traditional window slicing technique.

The differences between the results obtained with the two techniques can be visually compared looking at Fig. 7(a)–(c). It is evident that the results obtained with the fuzzy technique are more coherent and less affected by noise. On the other hand we think that to compare a simple traditional window slicing with our technique is not totally fair.

This happens because our technique implicitly filters out high frequencies in a new linear fashion. For this reason we have also compared our results with the results obtained with a traditional window slicing technique after the original image has been filtered with a median filter over a rectangle with size $h \times h$ (Fig. 7 (d)).

To get a more quantitative evaluation of the performance of the new method we have classified the pixels in the original image into four classes:

- *IN–IN*: This class contains all the pixels that are in the *inside* of the selected window both following the traditional median + windowing technique and the proposed approach;
- *OUT–OUT*: This class contains all the pixels that are in the *outside* of the selected window both in the traditional median + windowing technique and the proposed approach;
- *IN–OUT*: This class contains all the pixels that are in the *inside* of the selected window in the traditional median + windowing technique, but are in the *outside* of the window obtained with the proposed approach;
- *OUT–IN*: This class contains all the pixels that are in the *outside* of the selected window in the traditional median + windowing technique, but are in the *inside* of the window obtained with the proposed approach.

Of course the performance depends for the fuzzy technique on the presumption level associated with every query. We have computed the percentage of pixels falling in the four classes above averaged over 100 different pictures relatively to presumption levels $s = 0$, 0.5 and 0.9. In all cases we choose the same window slice both in the fuzzy and in the traditional method. The results are summarized in Table 1. Observe that although the median

Table 1
Percentages of the four classes IN–IN, IN–OUT, OUT–IN and OUT–OUT described in the text in the case when presumption level $s = 0$, 0.5, 0.9 and values $h = 5$, 15, 25 have been chosen[a]

s	$h = 5$				$h = 15$				$h = 25$			
	IN–IN (%)	IN–OUT (%)	OUT–IN (%)	OUT–OUT (%)	IN–IN (%)	IN–OUT (%)	OUT–IN (%)	OUT–OUT (%)	IN–IN (%)	IN–OUT (%)	OUT–IN (%)	OUT–OUT (%)
0	11.2	11.7	4.4	72.7	14.3	11.5	9.7	64.5	16.7	10.7	15.9	56.7
0.5	7.0	15.9	2.1	75.0	8.6	17.2	4.3	69.9	10.6	16.8	8.4	64.2
0.9	2.4	20.5	0.8	76.3	2.5	23.3	1.2	73.0	3.3	24.1	3.7	68.9

[a] The results are averaged over 100 images of different nature.

+ traditional thresholding results are not guaranteed to be more precise than the results obtained with our technique, we think that the data reported in Table 1 provide a strong argument for the validity of the fuzzy approach.

In conclusion the results obtained with a traditional and with the proposed approach are of comparable quality, with richer information provided by the proposed algorithm thanks to the visualization of several presumption levels at once.

Although traditional window slicing is faster than the proposed approach there is a relevant difference in efficiency between the proposed approach and the median + window slicing approach. This difference increases as the parameter h increases, see Table 2.

Table 2
Average processing times over 100 images of 500 × 500 pixels

Method	Time (s)
Traditional window slicing	1.03
Fuzzy window slicing ($h = 5$)	12.34
Fuzzy window slicing ($h = 7$)	10.53
Median + Traditional window slicing ($h = 5$)	19.82
Median + Traditional window slicing ($h = 7$)	23.12

The fuzzy approach, finally, provides very good visual clues to the quality of the segmentation obtained. The analysis is hence greatly simplified as Figs. 9–11 dramatically show.

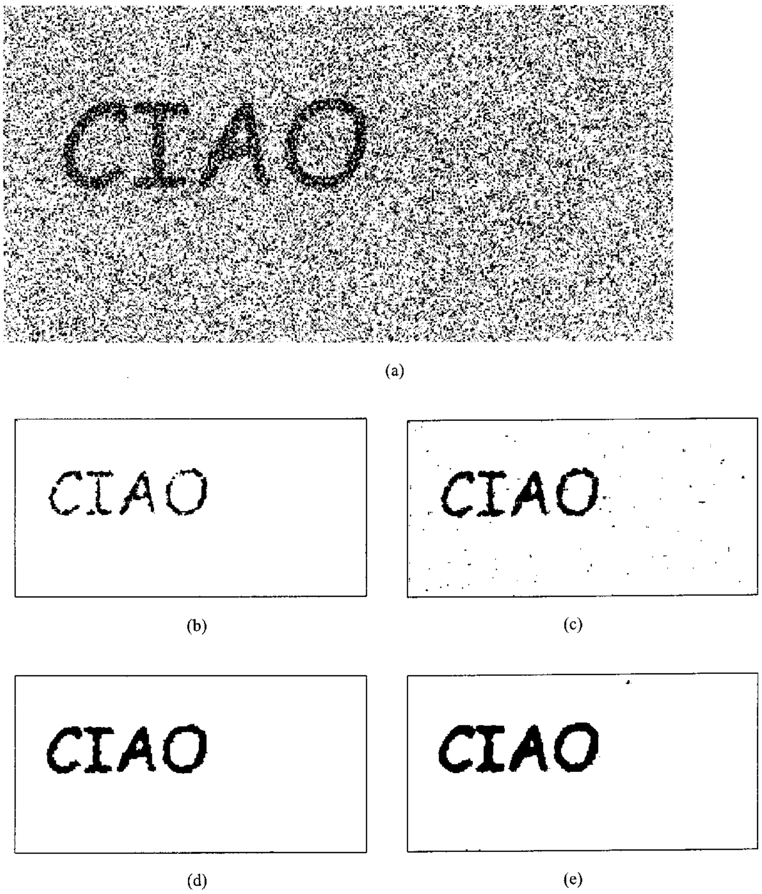

Fig. 9. A fragment of scanned text with added uniform noise (a) has been median filtered and thresholded with a narrow window (b) and with a wider window (c). The same picture has been fuzzy thresholded with grid sizes 5 × 5 and threshold 25 in (d) and grid sizes 7 × 7 with threshold 45 in (e).

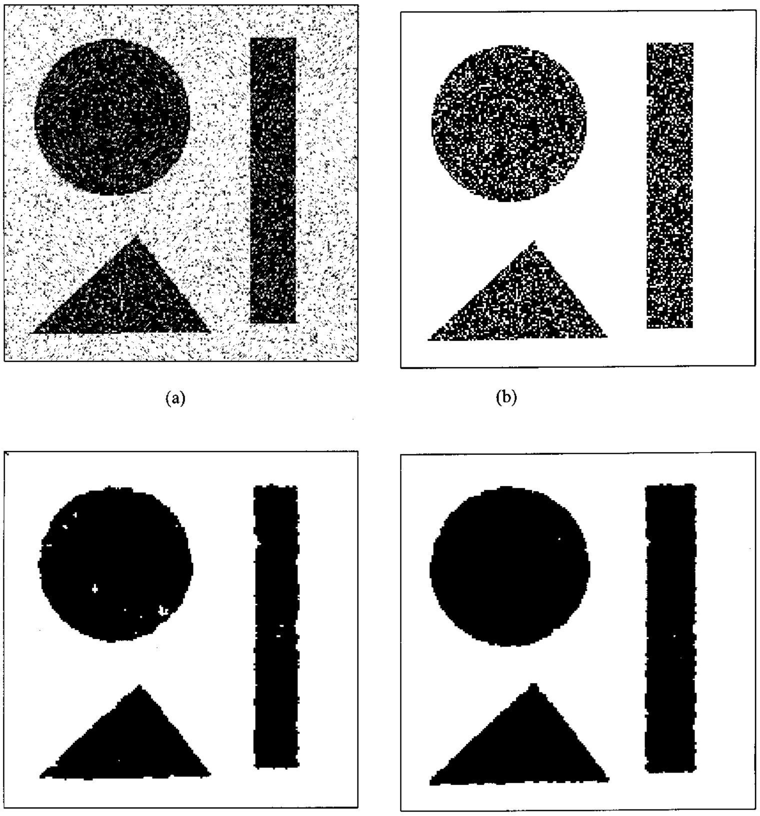

Fig. 10. (a) A synthetic image with additional uniform noise. (b) Result obtained by simple window slicing. (c) and (d) Results obtained with median + windowing and the proposed fuzzy approach, respectively.

4. Conclusions

We have described a new approach, based on fuzzy arithmetic, to pixel classification for digital images. The proposed method provides results of comparable or better quality than similar more traditional window techniques. Its relatively low computational cost makes the technique a good interactive tool for image analysis and enhancement. The algorithm provides qualitative information about the validity of the results obtained incorporating, in a simple and natural way, statistical information about data variability and uncertainty.

The new algorithm can be extended in a straightforward way to higher dimensions. Research to find a suitable graphical representation to visualize the results in this case is in progress. Moreover varying the parameter $\boldsymbol{h}$ different scales of the images may be analyzed: applications of the *fuzzy marching squares* technique to multi-scale analysis seem to be a promising direction to pursue.

Finally, experience has shown that good threshold selection from a human operator is much easier when using the proposed approach than when using traditional techniques. Research to automate optimal threshold selection for fuzzy windowing is in progress.

Fig. 11. The gradient magnitude picture (b) of the standard "Lena" (a) has been computed. (c) The results of median filter and threshold with a narrow window. (d) The same with a wider window. The same picture has been fuzzy thresholded with grid sizes 5×5 for (e) and 7×7 for (f) and a single threshold value 55.

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