PADERBORN UNIVERSITY
The University for the Information Society

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2nd Applied Project

Group Number X

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	Q1	Q2	Q3	Q4	Total
Points					

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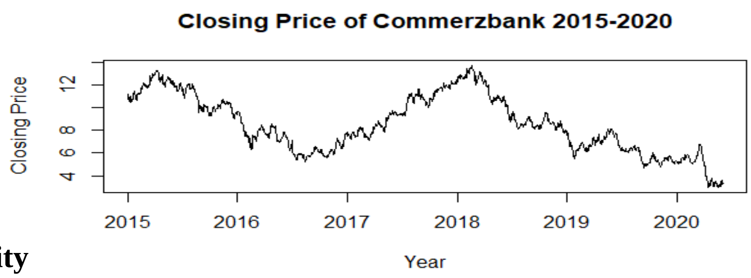
1- INTRODUCTION TO TIME-SERIES ANALYSIS

1-a) Summary

The Time Series is an ordered sequence of values of a variable obtained over a period, usually at regular intervals. For a given index set T, a time series is a collection of random variables $\{Xt,\ t \in T\}$, where t can be discrete or continuous. Xt can be discrete, continuous, or categorical, as well as univariate or multivariate.

To illustrate an example a Time Series we have used data of the Commerzbank AG which is one of the leading private and corporate customer banks in Germany. We have analyzed the evolution of the Closing Prices for the period of 2015-05.2020 Quelle: ***de.finance.yahoo.com.***



A- Stationarity

There are stationary (weakly and strongly Stationarity) and non-stationary Time Series.

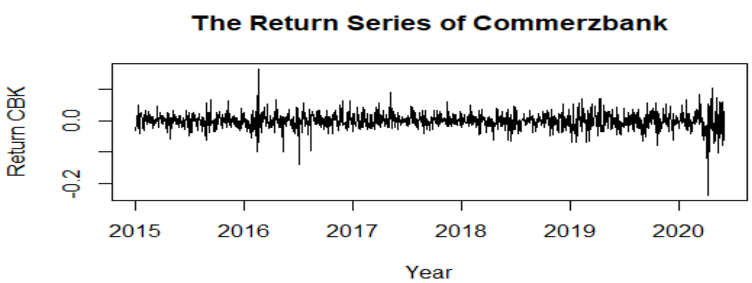
- A weakly stationary time series is when all statistical characteristics such as mean, variance, autocorrelation series are unchanged by shifts in time.

$$\mu(t) = \mathbb{E}x_t = \mu$$
$$\mathrm{Cov}(X_t, X_s) = \mathrm{Cov}(X_{t-s}, X_0)\forall t > s \in \mathbb{N}$$

That is, if the mean does not depend on time and the autocovariance between two elements only depends on the time between them and not the time of the first [1].

- A time series X_t is strongly stationary if the joint distribution of $(Xt1,...,X_{tm})$ is identical to that of $(X_{t1+k},...,Xt_{m+k})$ for all t, where m is an arbitrary positive integer and $(t_1,...,t_m)$ is a collection of m positive integers.

The figure below illustrates the return series of Commerzbank as a stationary Time Series,



B- Non-stationary time series

Non-stationary series is one whose statistical properties change over time. We referred to the figure evolution of the Closing Price of the Commerzbank which is a typical example of no-stationary Time Series.
To overcome this problem there are Operators which allow to make a non-stationary Time series into a weakly stationary temporary series such as:

- Backshift operator B: $BX_t = X_{t-1}$
- Differencing Operator $\Delta = 1 - B$: $\Delta X_t = X_t - X_{t-1}$
- Log-returns: $R_t = \Delta(\log(X_t)) = \log(X_t) - \log(X_{t-1})$

The theories of linear time series discussed include stationarity, dynamic dependence, autocorrelation function, modeling, and forecasting. The econometric models introduced in this part include autoregressive (AR) models, moving-average (MA) models, mixed autoregressive moving-average (ARMA) models.

I- Autoregressive-Process AR(p)

The autoregressive model is defined in terms of inhomogeneous difference equations. A time series X_t is said to be an autoregressive (AR(p)) if it can be written as

$$X_t = \beta_1 X_{t-1} + \beta_2 X_{t-2} + \cdots \ldots \ldots \ldots \ldots + \beta_p X_{t-p} + \varepsilon_t , \qquad \text{with } \beta_0 = 1$$

Where p is a nonnegative integer and $\mathcal{E}_t$ is white noise with zero mean and finite variance.

a) Stationary of AR(p)

The general AR(p) process is causal stationary only if the p roots of the characteristic equation of the AR polynomial lie outside the unit circle.

1- AR(1) Model; For Example a $AR(1) = \beta X_{t-1} + \varepsilon_t$ is causal stationary, when

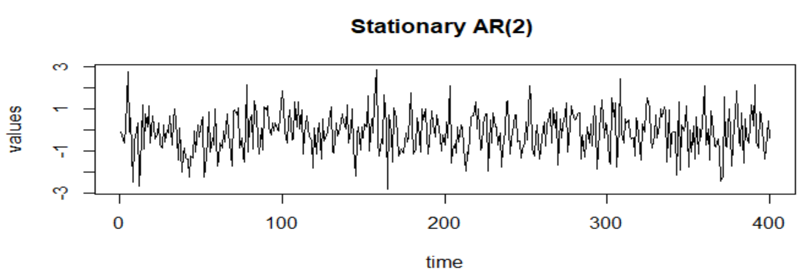
- $|z_1| = |1/\beta| > 1 \leftrightarrow |\beta| < 1$
- $X_t = \beta X_{t-1} + \varepsilon_t = \beta(\beta X_{t-2} + \varepsilon_{t-1}) + \varepsilon_t = ,,,,,, = \sum_{i=0}^{\infty} \alpha_i \, \varepsilon_{t-i}$

$$where \;\; \alpha_i = \beta^i \; with \;\; \alpha_0 = \beta^0 = 1 \; and \; we \; have \; \sum_{i=0}^{\infty} \alpha_i^2 = \sum_{i=0}^{\infty} \beta^{2i} = \frac{1}{1-\beta^2} < \infty, if \; |\beta| < 1$$

2- AR (2) Model: $AR(2) = \beta_1 X_{t-1} + \beta_2 X_{t-2} + \varepsilon_t$

For an AR (2) one uses rather the explicit conditions of stationarity which shows that an AR (2) = $ß_1 X_{t-1}$ + ß2X_{t-2} + $\mathcal{E}_t$ is stationary if the following conditions are respected. For all our example of AR(p) model, we estimated the parameters of an AR (2) model using 400 Observations of the Commerzbank.

- $ß_1 + ß_2 < 1$
- $ß_2 - ß_1 < 1$
- $-1 < ß_2 < 1$



AR(2): X(t)= 0.0262X(t-1) + 0.0073X(t-2) + err(t)

Properties of AR Models: For effective use of AR models, it pays to study their basic properties. we can easily obtain the mean and the variance of this series, assuming that this AR(1) Model is weakly stationary then we have:

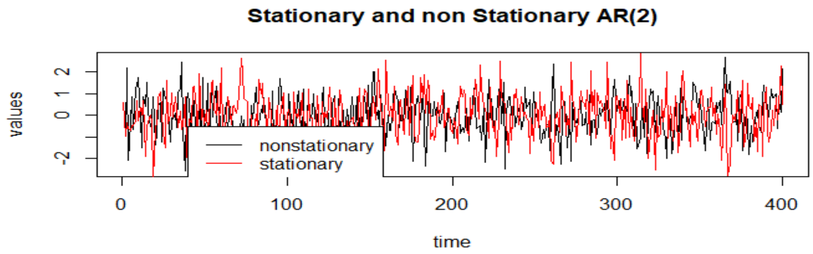
$E(X_t) = \mu$, $Var(X_t) = \gamma_0$, and $Cov(X_t, X_{t-j}) = \gamma_j$. Under the stationarity condition, $E(Xt) = E(X_{t-1}) = \mu$, $Var(Xt) = ßVar(Xt-1) + \sigma^2 = \sigma^2/(1-ß^2) = \gamma_0$.

Autocovariance/Autocorrelation function: In general $\gamma(k) = \sigma^2 \frac{\beta^{|k|}}{1-\beta^2}$ $and \; \rho(k) = \frac{\gamma(k)}{\gamma(0)} = \beta^{|k|}$

• $\rho(k)$ is positive for β>0 & with alternative sign for β<0.
• $|\rho(k)|$ decreases monotonically; tends to zero very quickly.
• These properties can be used in practice to check, whether an AR(1) model is suitable or not.

b) Non-stationary of an AR(p)

An AR(p) is non - stationary when it does not satisfy all the properties of stationarity.
Here by is an example of the combination of stationary and non-stationary AR(2) models



c) Parameter Estimation of AR(p)

The equation of autoregression is: $X(t) = \mu + \beta_1(X_{t-1} - \mu) + ... + \beta_p (X_{t-p} - \mu) + \varepsilon_t$, and altogether the model parameters are: $(\mu, \beta_1, ..., \beta_p, \sigma_\varepsilon)$ with $E(Xt) = \mu$. Suppose that the roots of the characteristic polynomial $P(z) = 1-(\beta_1 z + ... + \beta_p z^p)$ and if lay within the unit circle and so, there exists a stationary solution X(t) of the equation.

- The autoregression resembles a regression but the dependent variable X(t) and the independent (explanatory) variables X(t−1),...,X(t−p) are obtained from the same observation X(t), so the rows: the cases are interdependent, correlated.
- This does not change the estimation of the coefficients but does change the properties, the good ness, the distribution, invalidating the usual variances, tests, confidence bounds.
- The method of least squares is applicable as in regression.[3]

$$Q(\mu, \beta_1, \dots \dots \beta_p) = \sum_{t=p+1}^{N} \varepsilon_t^2 = \sum_{t=p+1}^{N} [X_t - \mu - \beta_1(X_{t-1} - \mu) - \cdots \dots \dots . . -\beta_p(X_{t-p} - \mu)]^2 \rightarrow min$$

For a AR(1) the estimated parameter is defined: $\hat{\beta} = \dfrac{\sum_{i=1}^{n-1}(X_i - \bar{X})(X_{i+1} - \bar{X})}{\sum_{i=1}^{n}(X_i - \bar{X})^2}$

d) Selection Rule

The penalty for each parameter used is 2 for AIC and ln (T) for BIC. Thus, com-pared with AIC, BIC tends to select a lower AR model when the sample size is moderate or large. To use AIC to select an AR model in practice, one computes AIC(ℓ) for ℓ =0,,,,p, where p is a prespecified positive integer and selects the order k that has the minimum AIC value. The same rule applies to BIC.[2]

II- Moving Average Process MA(q)

{Xt} is a moving average process of order q if $X_t = \alpha_1 \varepsilon_{t-1} + \alpha_2 \varepsilon_{t-2} + \dots \dots \dots \dots + \alpha_q \varepsilon_{t-q} + \varepsilon_t$

Stationarity and inversibility

Moving-average models are always causal and weakly stationary because they are finite linear combinations of a white noise sequence for which the first two moments are time invariant.

An MA (q) stationary Process is said to be invertible, if it can be writing as an (infinite order) AR Model

$$\varepsilon_t = \sum_{i=0}^{\infty} \beta_i X_{t-1} \; with \sum_{i=0}^{\infty} |\beta_i| < \infty$$

An MA (2) Process is invertible if the following conditions are met:

- $\alpha_1 + \alpha_2 > -1$
- $\alpha_1 - \alpha_2 < 1$
- $-1 < \alpha_2 < 1$

III- Autoregressive-moving average ARMA(p,q) Process

An even more general class of models may be obtained by combining AR and MA processes. A time series $\{X_t\}$ is said to be a mixed autoregressive-moving average process of order (p,q) (an ARMA (p,q) process) if it contains *p* AR terms and *q* MA terms. Then,

$$X_t = \beta_1 X_{t-1} + \beta_2 X_{t-2} + \cdots \dots \dots \dots \dots + \beta_p X_{t-p} + \alpha_1 \varepsilon_{t-1} +,,,,,,,,, + \alpha_q \varepsilon_{t-q} + \varepsilon_t$$

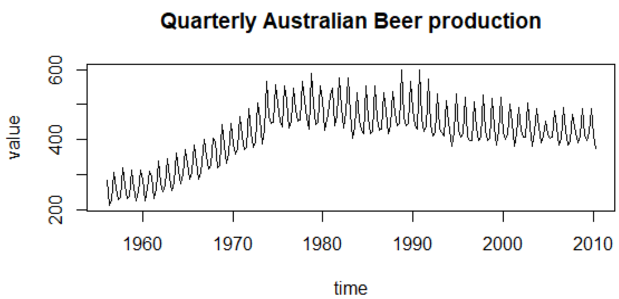
$$\emptyset(B) X_t = \varphi(B) \varepsilon_t , \qquad \emptyset(z) = 1 - \beta_1 z - \cdots - \beta_p z^p , \quad \varphi(z) = 1 + \alpha_1 z \dots . . + \alpha_p z^p$$

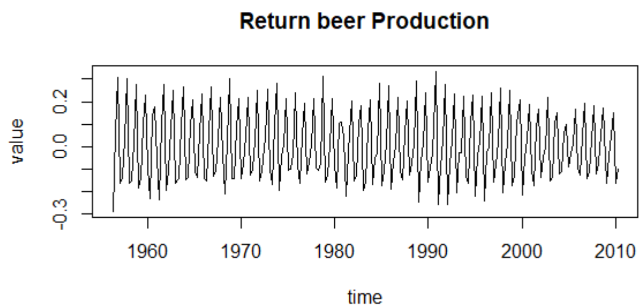
Stationarity and inversibility

- An ARMA is causal stationary if $|z_i| > 1$ for $\emptyset(z)$
- An ARMA is causal stationary if $|z_i| > 1$ for $\varphi(z)$
- An ARMA is causal stationary and inversibility if $|z_i| > 1$ for $\emptyset(z)$ and $\varphi(z)$

1-b) The Data we used describe the total quarterly beer production in Australia (in megalitres) from 1956: Q1 to 2010: Q2 with 218 Observations from Australian Bureau of Statistics. Cat. 8301.0.55.001.

- ***State the model and show that this model is stationary. Plot of quarterly beer production in Australia and the return***





To find the best model we compared all the models according to the AIC, then we chose the third model (with p = 0, ….,2 and q = 0,……..,2) that has the smallest AIC. Considering that the first model corresponds to p = 0, thus the third model corresponds to p = 2 and q = 2.

p \ q	0	1	2
0	-183.7442	-300.0075	-301.2934
1	-188.8034	-298.0385	-306.5586
2	-335.9858	-470.4594	**-597.6298**

According to AIC the best Model is **ARMA(2,0,2)**

arima(x = Return.beer, order = c(2, 0, 2))					
Coefficients	**ar1**	**ar2**	**ma1**	**ma2**	**intercept**
	-0.0267	-0.8974	-1.0551	0.6951	0.0024
s.e.	0.0312	0.0310	0.0634	0.0396	0.0013
sigma^2 estimated as 0.00344: log likelihood = 304.81, aic = -597.63					

$$\text{Model } X_t = -0.0267X_{t-1} - 0.8974X_{t-2} - 1.0551\varepsilon_{t-1} + 0.6951\varepsilon_{t-2} + \varepsilon_t$$

- ***Check for invertibility and stationary of the best model and plot***

$$\beta_1 = -0.0267, \qquad \beta_2 = -0.8974, \qquad \alpha_1 = -1.0551, \qquad \alpha_2 = 0.6951$$

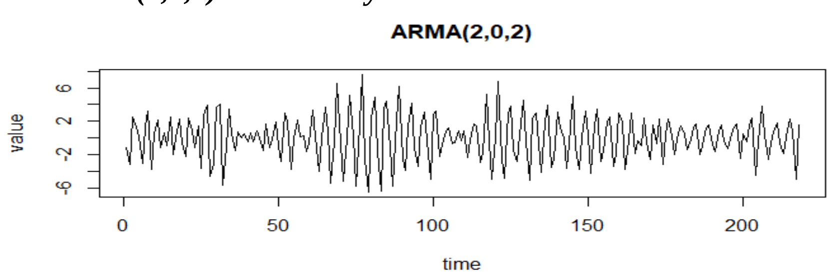
$\alpha_1 + \alpha_2 = -1.0551 + 0.6951 = -0.36 > -1$ and $\alpha_1 - \alpha_2 = -1.7502 < 1$ *and* $-1 < \alpha_2 = 0.6951 < 1$

$\rightarrow$ **MA (2) is invertible**

$\beta_1 + \beta_2 = -0.0267 - 0.8974 = -0.9241 < 1$ and $\beta_2 - \beta_1 = -0.8707 < 1$ *and* $-1 < \beta_2 = -0.8974 < 1$

$\rightarrow$ **AR (2) is stationary**

Then the best Model ARMA(2,0,2) is stationary and invertible



2- FINANCIAL TIME SERIES

2.a) Summary

- **ARCH - Model**

The Autoregressive Conditional Heteroscedastic (ARCH) was mad by Engle in 1982 (Nobel Prize in Econ 2003). It is a time-series statistical model used to analyze effects left unexplained by econometric models. It is related to the AR with a conditionally normal with zero mean ($E[Y_t] = 0$) and the conditional variance h_t ($var[Y_t|\mathcal{F}_{t-1}] = h_t$) with these properties: $Y_t|\mathcal{F}_{t-1} \sim \mathrm{N}(0, h_t)$,
$h_t = \alpha_0 + \alpha_1 Y_{t-1}^2 + \ldots + \alpha_p Y_{1-p}^2$. $\alpha_0 > 0,\ \alpha_p \geq 0, i = 1, \ldots, p, \mathcal{F}_{t-1}$ past information. $h_t > 0$ is the conditional variance of Y_t. We must notice that the h_t here in the ARCH model does not depend on the past period h_{t-1}.

- **GARCH - Model**

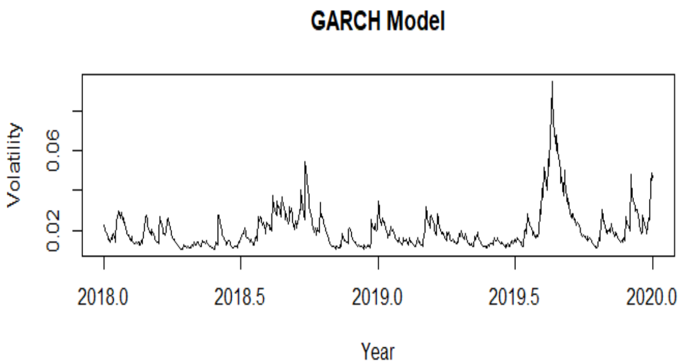
Generalized Autoregressive Conditional Heteroscedastic (GARCH) was mad by Bollerslev in 1986. It is a statistical model which estimate the volatility of stock returns. It is related to the ARMA and is used when the variance of the error term is not constant.
$Y_t|\mathcal{F}_{t-1} \sim \mathrm{N}(0, h_t)$, $h_t = \alpha_0 + \alpha_1 Y_{t-1}^2 + \ldots + \alpha_p Y_{1-p}^2 + \beta_1 h_{t-1} + \ldots + \beta_q h_{t-q}$.
$\alpha_0 > 0, \alpha_i \geq 0, i = 1, \ldots, p, \beta_j \geq 0, j = 1, \ldots, q, \mathcal{F}_{t-1}$ the same as before. Here h_t depends on $h_{t-1}, \ldots, h_{t-q}$.
According to the basic properties of the ARCH/GARCH defined as follow: $E[Y_t|\mathcal{F}_{t-1}] = 0$, by definition $\Longrightarrow$ $E[Y_t] = 0$. $cov(Y_t|\mathcal{F}_{t-1}, Y_{t+k}|\mathcal{F}_{t+k-1} = 0$, for $k > 0 \Longrightarrow \gamma(k) = cov(Y_t, Y_{t+1}) = 0$, Y_t by the GARCH process are uncorrelated, but not independent, because $cov(Y_t^2,\ Y + Y_{t+k}^2)\ \neq 0$. Conditionally normal, but non-normal. The ARCH/GARCH models can also be written like: $Y_t|\mathcal{F}_{t-1} \sim \eta_t\sqrt{h_t})$, η_t iid N(0,1), $\sqrt{h_t}$ is the conditional SD.

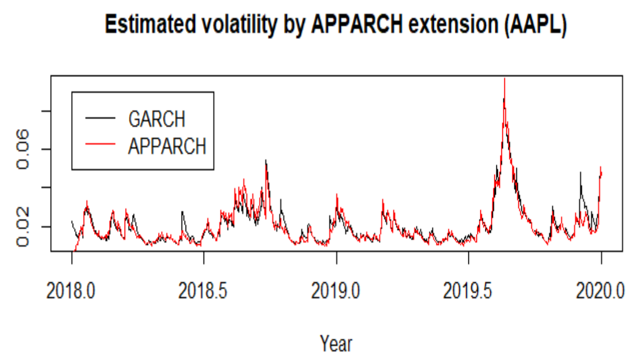
Standard GARCH models assume that positive and negative error terms have a symmetric effect on the volatility. But in practice this assumption is frequently violated, by stock returns, in that the volatility increases more after negative shocks (bad news) than positive shocks (good news). To solve these disadvantages, we can use GARCH Extensions such as: GARCH-t, GARCH with skewed innovation, asymmetric power ARCH (APARCH), exponential GARCH (EGARCH), ARMA-GARCH.

- **Asymmetric Power ARCH (APARCH, Ding, Granger, Engle, 1993)**

This model can well express the Fat tails, Excess kurtosis, and leverage Effects. Generally, this extension shows the Asymmetry of the effect of negative and positive returns, formulate of conditional variance different for positive or negative returns, and the power (δ) of the absolut returns. The general structure is as follows: $Y_t \sim APARCH(p, q)$, $Y_t = \eta_t \sigma_t, \sigma_t^\delta = \alpha_0 + \sum_{i=1}^p \alpha_i\ (|Y_i| - \gamma_i Y_{t-i})^\delta + \sum_{j=1}^q \beta_j \sigma_{t-j}^\delta$. This formula must satisfy the following conditions: $\sigma_t > 0$: the conditional standard deviation:
$0 < \delta \leq 2$: power index and $0 \leq \gamma_i < 1$: the leverage effect (higher volatility for negative past returns and lower volatility for positive past returns). In special case of APARCH, GARCH with $\delta = 2, \gamma_i = 0$, APARCH model is NGARCH model. When δ = 1, $\gamma_i = 0$, APRCH model is TS-GARCH model, when δ = 2, $0 \leq \gamma_i < 1$ APARCH model is TGARCH model. For our example we used the Apple data from 11.09.2018 to 11.09.2020 to plot the volatility of a fitted GARCH model in a single plot and plot the volatility the APPARCH extension. Here by is an example of the best APPARCH model:

Coef	mu	omega	alpha1	gamma1	beta1	delta
Values	0.00192	0.00036	0.1722	0.4453	0.8224	1.239





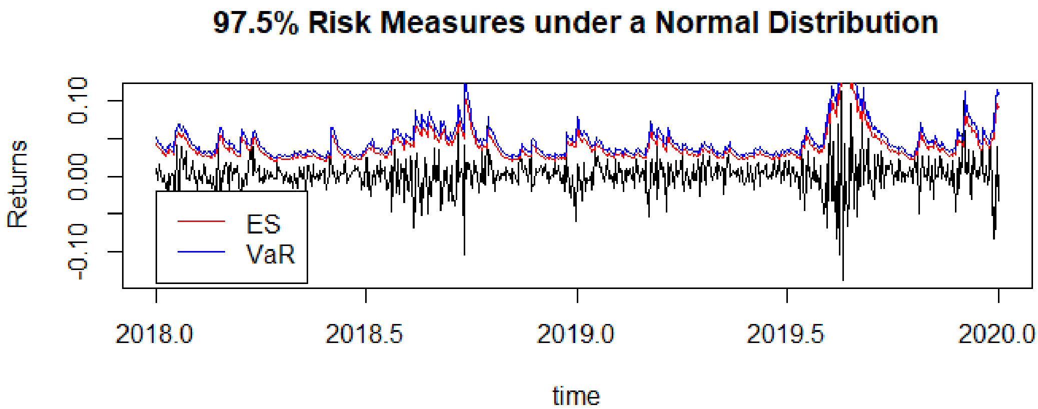
The estimated APPARCH (1,1) is given by: $Y_t = 1.9 \cdot 10^{-3} + \varepsilon_t \sigma_t, \varepsilon_t \sim N(0,1), \widehat{\delta} = 1.239$, $\sigma_t^{\widehat{\delta}} = 3.6 \cdot 10^{-4} + 0.1722(|Y_{t-1}| - 0.4453 \cdot Y_{t-1})^{\widehat{\delta}} + 0.8224\sigma_t^{\widehat{\delta}}$

- Exponential GARCH (EGARCH, Nelson, 1991): This model shows some differences in the volatility (measured by σ_t^2) from the standard model and is defined as: $log\sigma_t^2 = \omega_t + \sum_{k=1}^{\infty} \beta_k g(Z_{t-k})$, where ω_t, β_k are deterministic coefficients and $g(Z_t) = \theta Z_t + \gamma(|Z_t| - E|Z_t|)$. It can be directly seen that: $E[g(Z_t)] = 0$
- GARCH-t: This model is a GARCH with a symmetric conditional t-distribution after standardization and is defined as: $Y_t \sim GARCH(p,q), Y_t = \eta_t \sqrt{h_t}, h_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i Y_{t-i}^2 + \sum_{j=1}^{q} \beta_j h_{t-1}$.
- Component-GARCH (CGARCH, Lee, Engle, 2003) which can also be estimated using *rugarch* package.
- ARMA-GARCH is an ARMA(p, q) with GARCH innovations and is defined as: $Y_t = \sum_{i=1}^{p} \varphi_i Y_{t-p} + \sum_{j=1}^{q} \psi_j \varepsilon_{t-j} + \varepsilon_t, \varepsilon_t \sim GARCH(s,t), \varepsilon_t = \zeta_t \sqrt{h_t},\ h_t = \alpha_0 + \sum_{i=1}^{k} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{s} \beta_j h_{t-j}$

Market risk is caused by exposure to uncertainly in the market price of an investment portfolio. Any financial institution which holds a portfolio of financial assets Is exposed to this kind of risk and consequently should implement risk measurement such as Value at Risk (VaR) and Expected Shortfall (ES), and management methods in order to optimize the manner in which risk is taken. They are other source of risk like Liquidity Risk, Credit Risk or Operational Risk which are possible.

- Value at Risk (VaR): VaR modeling determines the potential for loss in the entity being assessed and the probability of occurrence for the defined loss. It is defined as: $VaR_{\alpha =} q_\alpha(F_L) = F_L^{\leftarrow}(\alpha)$, or $VaR_\alpha = \inf\{l \, \epsilon \, R : P(L > l) \leq 1 - \alpha$. Theoretically this is the α-quantile of the loss-distribution
- Expected Shortfall: The ES gives information about frequency and size of large losses and is define as follows: $ES_\alpha = E(L \mid L > VaR_\alpha)$,

For a real realized volatility with finite mean of the quantiles, ES is the average of all VaRs for $p \geq \alpha, \alpha \, \epsilon \, (0,1)$ with $ES_\alpha = \frac{1}{1-\alpha} \int_\alpha^1 VaR_u(L)du$, and for any given α, ES > VaR. Here by is an example of a VaR and ES on the AAPL data.



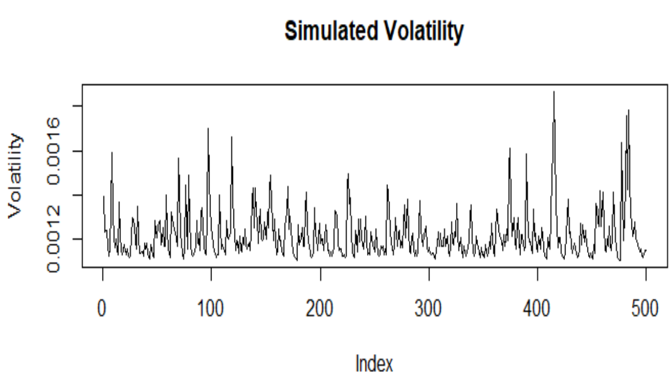
High-frequency financial are observations on financial variables taken daily or at a finer time scale and are often irregularly spaced over time, and we are talking about the equivalence (EHF) when we have five, ten, or one day data. The ultra-high-frequency (UHF) are tick-by-tick data i.e. the time interval between two transactions does not following each other.

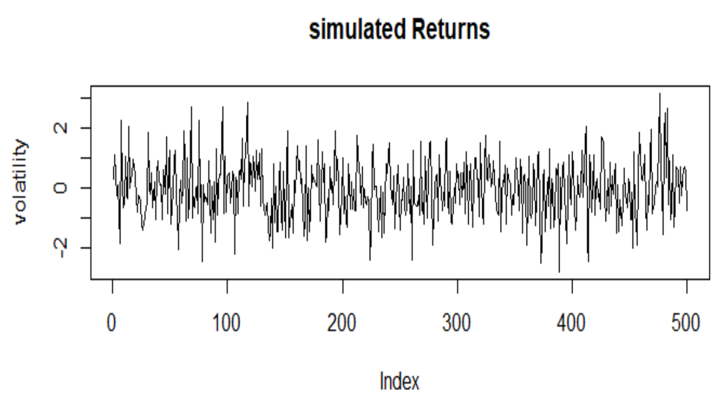
- Realized volatility is a model-free new volatility approach based on high-frequency financial data and is defined as follow:

$$RV_1 = \sum_{i=1}^{N} y_{t,i}^2, RV_1 \text{ is unbiased estimate to } var(x_t), if\ E[y_{t,i}] = \quad and\ E[y_{t,i}, y_{s,j}] = 0.$$

- Realized covariance and realized correlation: Financial returns are usually correlated and the realized covariance on one day t is defined as follow: $RCov_t^{AB} = \sum_{i=1}^{N_t} y_{t,i}^A\, y_{t,i}^B$. And the realized correlations $RCov_t^{AB} = \frac{RCov^{AB}}{\sqrt{RV_t^A}\sqrt{RV_t^B}}$.

2.b) Plot of the reproducible volatility and the estimated returns





After fitted the GARCH (p,q) with the BIC we obtain the GARCH (1,1) as best model because it has the smallest value of BIC (BIC = 2.838389). But as we can see on the above plot, the GARCH (1,1) model is certainly suitable for estimating the volatility, but it can be improved by using GARCH extensions.

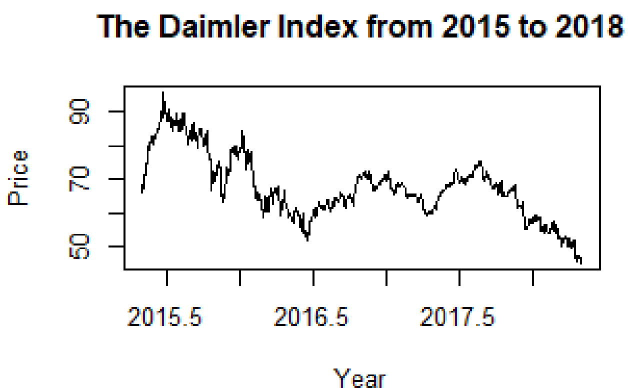
PROBLEM 3: RISK MANAGEMENT

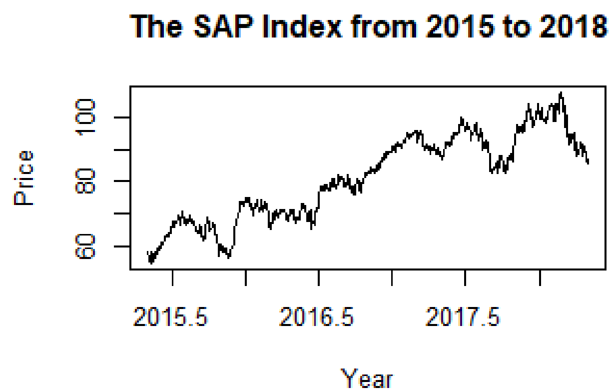
➢ List and description of the companies

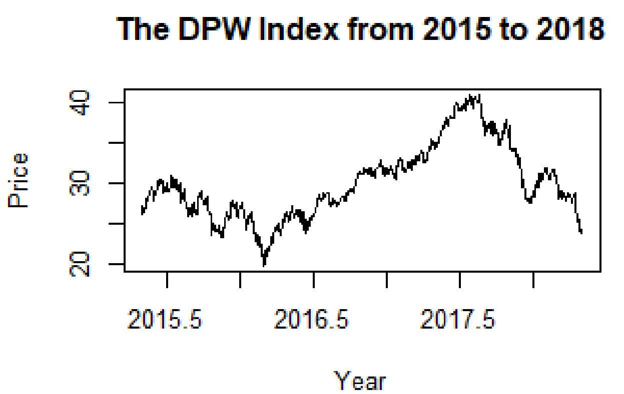
Companies	Branch	Country
SAP SE (SAP.DE)	DAX Companies	Germany
DAIMLER AG (DAI.DE)	DAX Companies	Germany
Deutsche Post AG(DPW.DE)	DAX Companies	Germany
Deutsche Telekom AG(DTE.DE)	DAX Companies	Germany

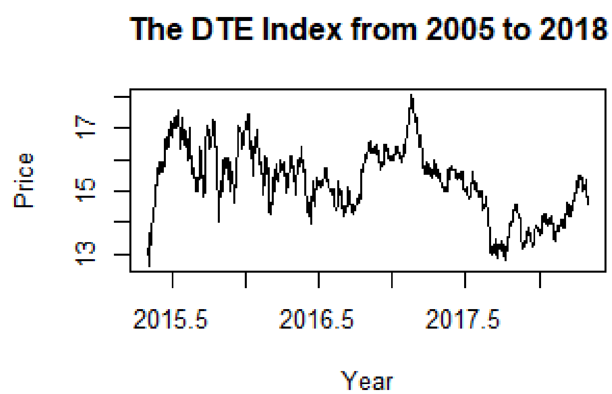
The data of each company goes from 02.01.2015 to 28.12.2018 working on the financial market (or 4 years)
<u>Quelle</u> : *https://finance.yahoo.com/*

3-1) Display the stock price time-series





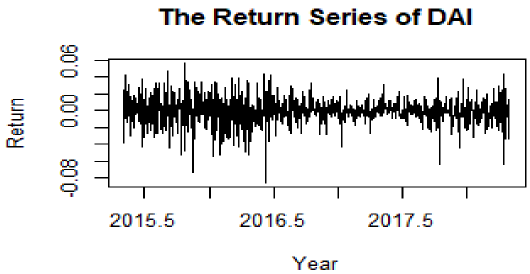


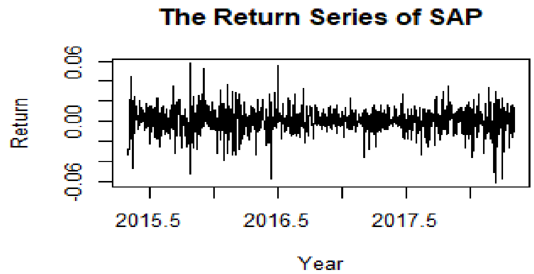


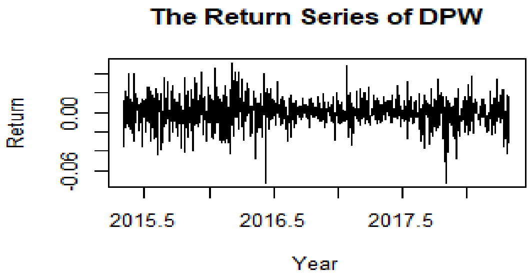
negative trend in stock prices from 2015 to 2018. SAP and the Deutsche Post show a positive trend in stock prices from 2015 to 2017 before experiencing a short downwards trend.

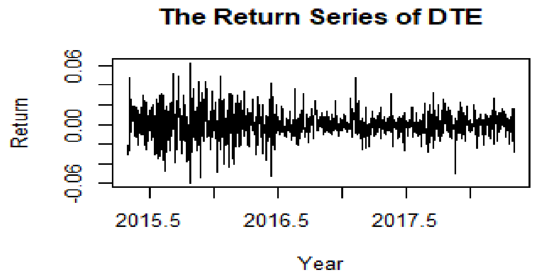
- The stock prices time-series for all companies are not stable and consequently not stationary.

3-2) Compute and display the returns for each of the time-series.









- the return-series for all companies are stable and consequently stationary
- Report the GARCH and APARCH of the DAIMLER

GARCH (1,1)

mu	omega	alpha1	beta1
-1.411926e-04	8.022072e-07	3.820307e-02	9.594849e-01

$$Y_t = -1.412 * 10^{-4} + \varepsilon_t\sqrt{h_t} \quad ,\varepsilon_t \sim (0,1)$$
$$h_t = 8.022 * 10^{-7} + 0.0382{Y^2}_{t-1} + 0.959h_{t-1}$$

GARCH (1,2)

mu	omega	alpha1	beta1	beta2
-1.3162e-04	9.265857e-07	4.788289e-02	6.7702e-01	2.7e-01

$$Y_t = -1.316 * 10^{-4} + \varepsilon_t\sqrt{h_t} \quad ,\varepsilon_t \sim (0,1)$$
$$h_t = 9.27 * 10^{-7} + 0.048{Y^2}_{t-1} + 0.677h_{t-1} + 0.27h_{t-2}$$

GARCH (2,1)

mu	omega	alpha1	alpha2	beta1
-1.325e-04	7.818505e-07	3.781202e-02	1.e-08	9.6e-01

$$Y_t = -1.325 * 10^{-4} + \varepsilon_t\sqrt{h_t} \quad ,\varepsilon_t \sim (0,1)$$
$$h_t = 7.82 * 10^{-7} + 0.0378{Y^2}_{t-1} + 1.e\text{-}08{Y^2}_{t-2} + 0.96h_{t-1}$$

GARCH (2,2)

mu	omega	alpha1	alpha2	beta1	beta2
-1.316e-04	9.27e-07	4.79e-02	1.e-08	6.77e-01	2.7e-01

$$Y_t = -1.316 * 10^{-4} + \varepsilon_t\sqrt{h_t} \quad ,\varepsilon_t \sim (0,1)$$
$$h_t = 9.27 * 10^{-7} + 0.048{Y^2}_{t-1} + 1.e\text{-}08{Y^2}_{t-2} + 0.677h_{t-1} + 0.27h_{t-2}$$

APARCH (1,1)

mu	omega	alpha1	gamma1	beta1	delta
-0.000168472	0.000849	0.0323	0.4304	0.9695	0.45516

$$Y_t = -1.685 * 10^{-4} + \varepsilon_t \sigma_t \quad, \varepsilon_t \sim (0,1), \hat{\delta} = 0.45516$$
$$\sigma^{\hat{\delta}}{}_t = 8.49 * 10^{-4} + 0.0323(|Y_{t-1}| - 0.4304 * Y_{t-1})^{\hat{\delta}} + 0.9695\sigma^{\hat{\delta}}{}_{t-1}$$

APARCH (1,2)

mu	omega	alpha1	gamma1	beta1	beta2	delta
-0.0002	0.0014	0.0463	0.382	0.55	0.406	0.417

$$Y_t = -2 * 10^{-4} + \varepsilon_t \sigma_t \quad, \varepsilon_t \sim (0,1), \hat{\delta} = 0.417$$
$$\sigma^{\hat{\delta}}{}_t = 1.4 * 10^{-3} + 0.0463(|Y_{t-1}| - 0.382 * Y_{t-1})^{\hat{\delta}} + 0.55\sigma^{\hat{\delta}}{}_{t-1} + 0.406\sigma^{\hat{\delta}}{}_{t-2}$$

APARCH (2,1)

mu	omega	alpha1	alpha2	gamma1	gamma2	beta1	delta
-2e-4	9.3e-4	0.0325	1e-8	0.39	-0.4373	0.969	0.4378

$$Y_t = -2 * 10^{-4} + \varepsilon_t \sigma_t \quad, \varepsilon_t \sim (0,1), \hat{\delta} = 0.4378$$
$$\sigma^{\hat{\delta}}{}_t = 9.3 * 10^{-4} + 0.0325(|Y_{t-1}| - 0.39 * Y_{t-1})^{\hat{\delta}} + 1 * 10^{-8}(|Y_{t-2}| + 0.4373 * Y_{t-2})^{\hat{\delta}} + 0.969\sigma^{\hat{\delta}}{}_{t-1}$$

APARCH (2,2)

mu	omega	alpha1	alpha2	gamma1	gamma2	beta1	beta2	delta
-2e-4	0.0014	0.0462	1e-8	0.381	0.0067	0.555	0.4	0.421

$$Y_t = -2 * 10^{-4} + \varepsilon_t \sigma_t \quad, \varepsilon_t \sim (0,1), \hat{\delta} = 0.421$$
$$\sigma^{\hat{\delta}}{}_t = 1.4 * 10^{-3} + 0.0462(|Y_{t-1}| - 0.381 * Y_{t-1})^{\hat{\delta}} + 1 * 10^{-8}(|Y_{t-2}| - 0.0067 * Y_{t-2})^{\hat{\delta}} + 0.555\sigma^{\hat{\delta}}{}_{t-1} + 0.4\sigma^{\hat{\delta}}{}_{t-2}$$

- **GARCH (p,q) and APARCH (p,q)**

The task was to fit 4 GARCH and APARCH models for p, q=1, 2 of the time series. To build these models we used the GarchFit formula out of the “fGarch” package. To select the best GARCH and APARCH model we created the corresponding BIC’s of each GARCH and APARCH model. The best GARCH and APARCH model is the model that has the smallest BIC

1- GARCH (p,q)

Daimler AG(DAI): The best GARCH is GARCH (1,1)

mu	omega	alpha1	beta1
-1.412e-04	8.022e-07	3.8203e-02	9.595e-01

$$Y_t = -1.412 * 10^{-4} + \varepsilon_t \sqrt{h_t} \quad, \varepsilon_t \sim (0,1)$$
$$h_t = 8.022 * 10^{-7} + 0.0382 Y^2{}_{t-1} + 0.959 h_{t-1}$$

SAP SE (SAP): The best GARCH is GARCH (1,1)

mu	omega	alpha1	beta1
6.763e-04	3.343e-06	6.259e-02	9.1771e-01

$$Y_t = 6.763 * 10^{-4} + \varepsilon_t \sqrt{h_t} \quad, \varepsilon_t \sim (0,1)$$
$$h_t = 3.343 * 10^{-6} + 0.0626 Y^2{}_{t-1} + 0.9177 h_{t-1}$$

Deutsche Post AG(DPW): The best GARCH is GARCH (1,1)

mu	omega	alpha1	beta1
3.8204e-04	1.6044e-06	3.6301e-02	9.562e-01

$$Y_t = 3.82 * 10^{-4} + \varepsilon_t \sqrt{h_t} \quad, \varepsilon_t \sim (0,1)$$
$$h_t = 1.604 * 10^{-6} + 0.0363 Y^2{}_{t-1} + 0.9562 h_{t-1}$$

Deutsche Telekom AG(DTE): The best GARCH is GARCH (1,1)

mu	omega	alpha1	beta1
1.532e-04	1.347e-06	3.467e-02	9.568e-01

$$Y_t = 1.532 * 10^{-4} + \varepsilon_t\sqrt{h_t} \quad , \varepsilon_t \sim (0,1)$$
$$h_t = 1.347 * 10^{-6} + 0.0347{Y^2}_{t-1} + 0.957h_{t-1}$$

2- APARCH

Daimler AG(DAI): The best APARCH is APARCH (1,1)

mu	omega	alpha1	gamma1	beta1	delta
-0.0001685	0.000849	0.0323	0.4304	0.9695	0.45516

$$Y_t = -1.685 * 10^{-4} + \varepsilon_t\sigma_t \quad , \varepsilon_t \sim (0,1), \hat{\delta} = 0.45516$$
$$\sigma^{\hat{\delta}}{}_t = 8.49 * 10^{-4} + 0.0323(|Y_{t-1}| - 0.4304 * Y_{t-1})^{\hat{\delta}} + 0.9695\sigma^{\hat{\delta}}{}_{t-1}$$

SAP SE(SAP): The best APARCH is APARCH (1,1)

mu	omega	alpha1	gamma1	beta1	delta
0.00086987	0.008964	0.05311	0.90692	0.93	0.280

$$Y_t = 8.7 * 10^{-4} + \varepsilon_t\sigma_t \quad , \varepsilon_t \sim (0,1), \hat{\delta} = 0.280$$
$$\sigma^{\hat{\delta}}{}_t = 8.964 * 10^{-3} + 0.0531(|Y_{t-1}| - 0.9069 * Y_{t-1})^{\hat{\delta}} + 0.93\sigma^{\hat{\delta}}{}_{t-1}$$

Deutsche Post AG(DPW): The best APARCH is APARCH (1,1)

mu	omega	alpha1	gamma1	beta1	delta
-1.03e-08	2.069e-04	0.0274	1	0.96915	0.93668

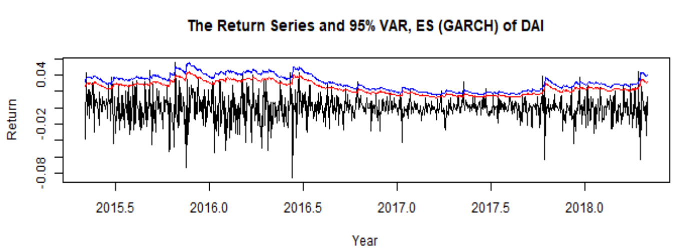
$$Y_t = -1.031 * 10^{-8} + \varepsilon_t\sigma_t \quad , \varepsilon_t \sim (0,1), \hat{\delta} = 0.93668$$
$$\sigma^{\hat{\delta}}{}_t = 2.069 * 10^{-4} + 0.0274(|Y_{t-1}| - 1 * Y_{t-1})^{\hat{\delta}} + 0.96915\sigma^{\hat{\delta}}{}_{t-1}$$

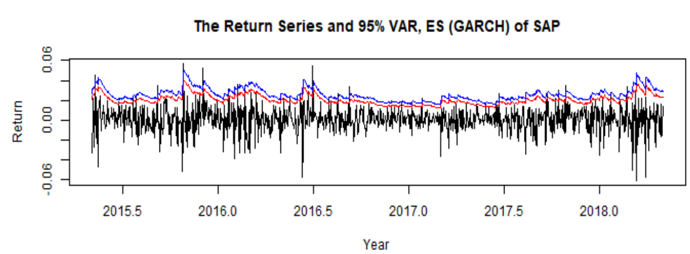
Deutsche Telekom AG(DTE): The best APARCH is APARCH (1,2)

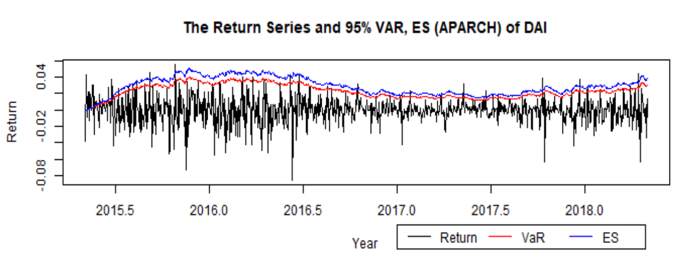
mu	omega	alpha1	gamma1	beta1	beta2	delta
8.7e-05	2.619e-03	7.356e-02	0.6995	8.166e-02	0.8468	0.4146

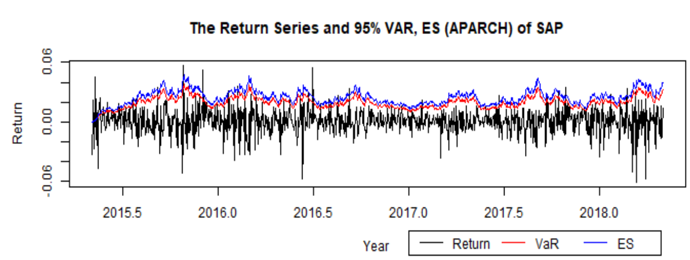
$$Y_t = 8.7 * 10^{-5} + \varepsilon_t\sigma_t \quad , \varepsilon_t \sim (0,1), \hat{\delta} = 0.4146$$
$$\sigma^{\hat{\delta}}{}_t = 2.619 * 10^{-3} + 0.07356(|Y_{t-1}| - 0.7 * Y_{t-1})^{\hat{\delta}} + 0.08166\sigma^{\hat{\delta}}{}_{t-1} + 0.8468\sigma^{\hat{\delta}}{}_{t-2}$$

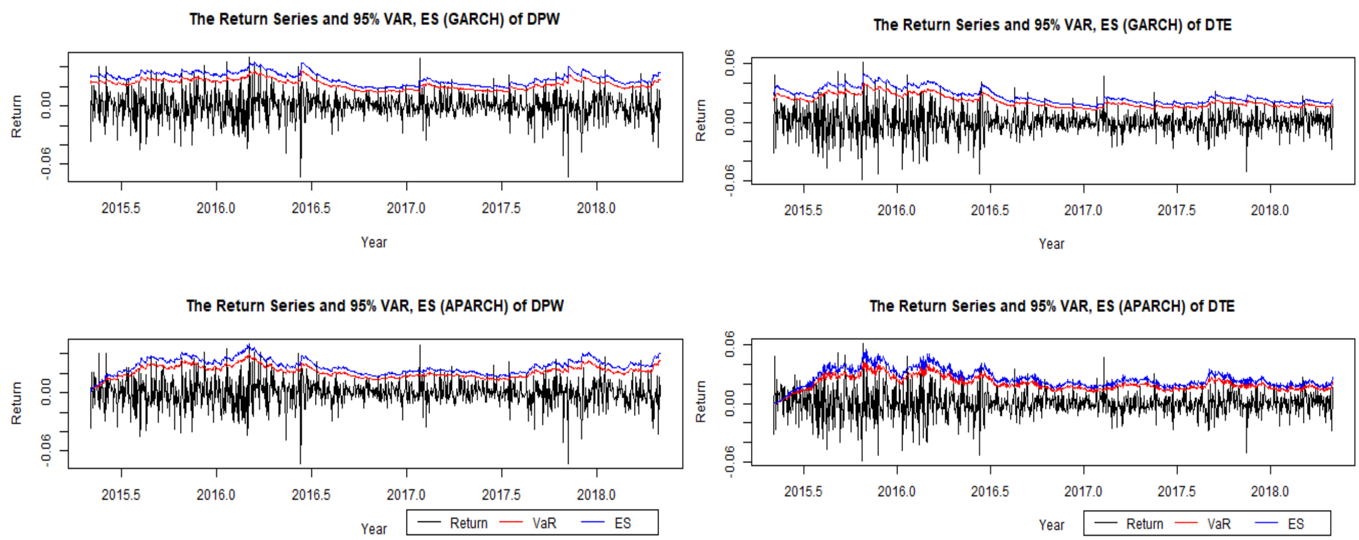
- **Plot Return and 95% VAR and ES for both models for the 4 time series**





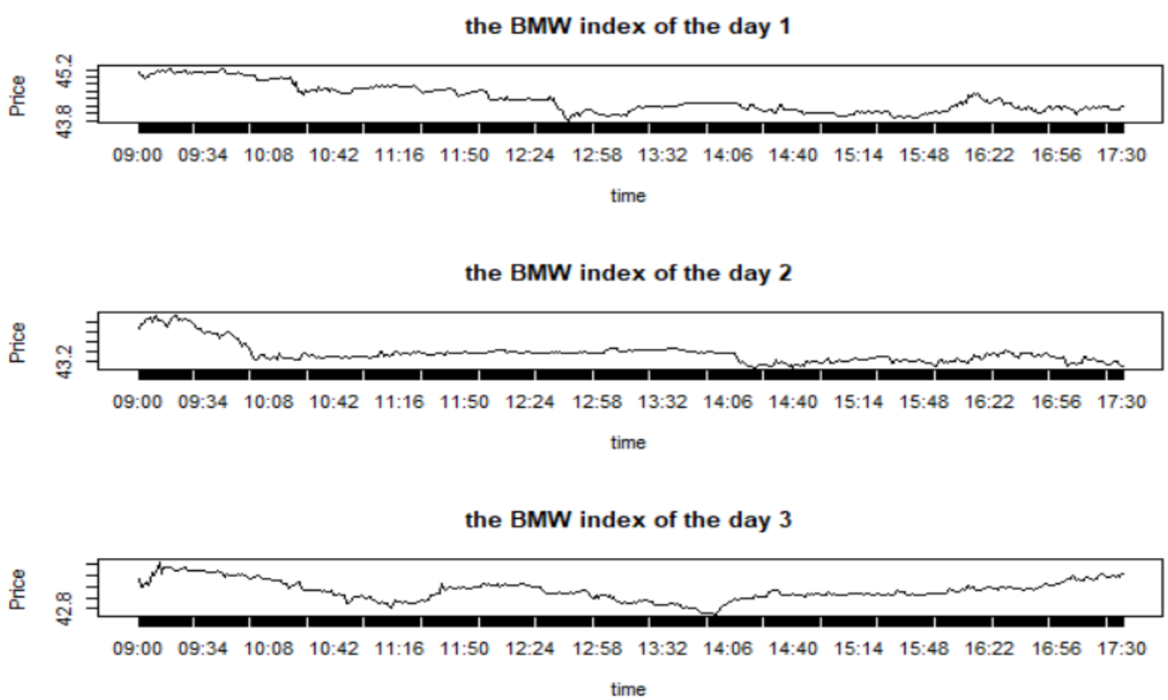




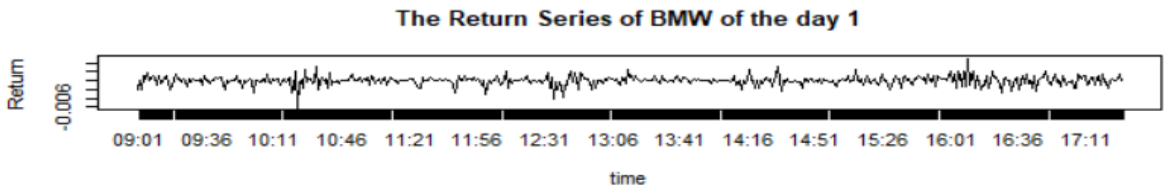


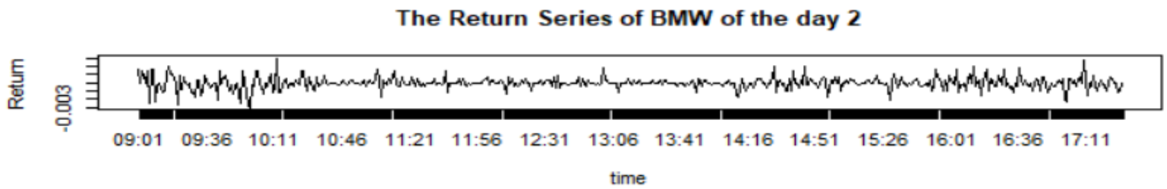
4 -**High-Frequency Financial Data**

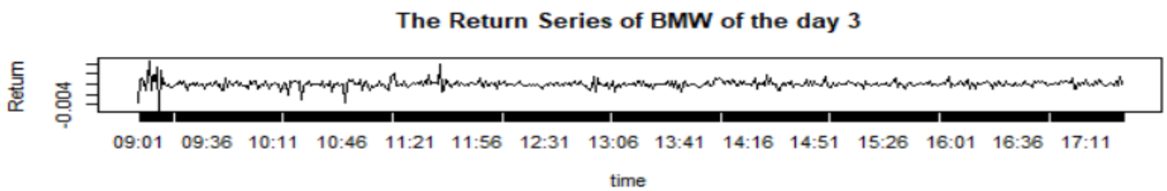
After loading the required packages into the R-file, we will work with the HF-BWM data: The BMW is a German company belonging to DAX. Its database contains data of the Close Prices on several days that from 9:00 to 17:30 in which I will extract 3 days. To have an overview on the development of the close prices, we plotted them in the following graphs:



As we can see, the BMW Close Prices show a decreasing course for the first two days. For the third day the BMW Close Prices show a decreasing course from 9:00 to approximately 14:00 and then increases until the closure of the financial stock exchange (17:30). The following graphs show the development of the log returns for three days. To calculate the realized volatility on each of three days we created the **GARCH (1,1)** model for the total time series of the tree days, out of the log returns of the tree days. After that we have determined the volatility of the three days using the @**sigma.t** of the created GARCH model. the table below shows the volatility of 3 days for the first 6 minutes.







Volatility_Day_1	**Volatility_Day_2**	**Volatility_Day_3**
0.001001853	0.0007262744	0.000718338
0.001215470	0.0008663754	0.001391455
0.001093694	0.0008128989	0.001277678
0.001273346	0.0008887307	0.001229151
0.001227607	0.0008866240	0.001134660
0.001107769	0.0008313313	0.001106262

SOURCES

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