ARE 210: Oct. 27, 2021 Section Notes

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Final 2020: Q1f-i

Setup: Suppose that $\{X_i\}_{i=1}^n$ are a sequence of iid Poisson random variables with $\mathbb{P}(X_i = k) = \frac{\exp(-\theta)\theta^k}{k!}$ for $\theta > 0$.

f) Comput the MLE of θ and denote it $\hat{\theta}$.
g) What does $\hat{\theta}$ converge to in probability?
h) Derive the limiting distrubtion of (a suitably normalized version of) $\hat{\theta}$
i) Define the estimator $\hat{\delta} = \exp(-\hat{\theta})$. Is $\hat{\delta}$ the MLE of $\exp(-\theta)$? Justify your answer.
Solution:
Strategy: For part f we'll simply write down the sample likelihood and find the θ that maximizes it Part g will follow from a law of large numbers. Similarly, part h will follow from a central limit theorem. For part i, rather than deriving the sample likelihood for this new estimand we'll invoke the invariance property of MLE to say $\hat{\delta}$ is indeed the MLE of $\exp(-\theta)$.

Step 1: Write down sample likelihood
Since $\mathbb{P}(X_i = k) = \frac{\exp(-\theta)\theta^k}{k!}$ and $\{X_i\}_{i=1}^\infty$ are independent we have

$$\begin{aligned}
\mathcal{L}(\{x_i\}_{i=1}^\infty; \theta) &= \prod_{i=1}^n \frac{\exp(-\theta)\theta^{x_i}}{x_i!} \\
&= \frac{\exp(-n\theta)\theta^{\sum\limits_{i=1}^n x_i}}{\prod\limits_{i=1}^n x_i!}
\end{aligned}$$

Step 2: Optimize

Taking the log, then derivative of both sides and settting equal to zero yields

$$\begin{aligned}
\log(\mathcal{L}(\{x_i\}_{i=1}^{\infty};\theta)) &= -n\theta + \log(\theta)\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} log(x_i!\,) \\
\frac{\partial \log(\mathcal{L}(\{x_i\}_{i=1}^{\infty};\hat{\theta}))}{\partial\hat{\theta}} &= -n + \frac{1}{\hat{\theta}}\sum_{i=1}^{n} x_i = 0 \\
&\Rightarrow \frac{1}{\hat{\theta}}\sum_{i=1}^{n} x_i = n \\
&\Rightarrow \frac{1}{n}\sum_{i=1}^{n} x_i = \hat{\theta}
\end{aligned}$$

So the MLE is the sample mean.

Step 3: WLLN To find the probability limit of $\hat{\theta}$, we simply need to apply the WLLN. Since $\mathbb{E}[X_i] = \theta$, we have $\hat{\theta} \xrightarrow[n\to\infty]{p} \theta$

Step 4: CLT To find the limiting distribution, we'll want to invoke the CLT. Since $\hat{\theta} = \frac{1}{n}\sum\limits_{i=1}^{n} x_i$, we have

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow[n\to\infty]{d} N(0, \theta)$$

Where $Var(X_i) = \theta$ follows from the Poisson distribution.

Step 5: Invariance Property
Part i follows directly from the invariance property of MLE. To see why, note that $\log(\mathcal{L}(\{x_i\}_{i=1}^{\infty}; -\log(\delta)) = \log(\mathcal{L}(\{x_i\}_{i=1}^{\infty};\theta)$. So $\hat{\theta}$ is the MLE of $-\log(\delta))$ and since $-\log(.)$ is a one-to-one function it must be that $\hat{\delta}$ is the MLE of δ.