ARE 210: Oct. 12, 2022 Section Notes

Joel Ferguson

1 2020 Midterm Q4

Setup: $\{X_i\}_{i=1}^n$ are iid $U[0,1]$. Define $Z_n = \left(\prod_{i=1}^n X_i\right)^{1/n}$
a) Show $Z_n \xrightarrow[n\to\infty]{p} \exp(-1)$ (Hint: $\int_0^1 \log(x)dx = -1$)
b) Let $V = Var(\log(X_i))$. Derive the limiting distribution of $\sqrt{n}(Z_n - \exp(-1))$.

Solution:
Strategy: It might be possible to derive the distribution of Z_n explicitly, but we should first be thinking about using the WLLN and CLT. You can see this a little bit in the hints: we're given the value of $\mathbb{E}[\log(x)]$ and we're told to use V in place of $Var(\log(X_i))$, so it seems like we should be taking logs somewhere. Fortunately, $\log(Z_n)$ is a sample average, so we derive its asymptotics pretty easily. Once we have the asymptotics of $\log(Z_n)$, we just need to find ways to transform it back into what we care about: the CMT in the case of part **a** and the delta method in the case of part **b**.

Step 1: Take the log of Z_n
As noted in the strategy, there seem to be a couple of things pointing us to take the log of Z_n. Even without the hint, it should be relatively straightforward to see that taking the log will give us a sample average, which is much easier to analyze than this geometric average.

$$\log(Z_n) = \frac{1}{n}\sum_{i=1}^n \log(X_i)$$

Step 2: Find $\mathbb{E}[\log(Z_n)]$

We're now almost set to invoke the WLLN and the CLT, we just need to find $\mathbb{E}[\log(Z_n)]$. Luckily, this is provided in the hint to part **a**

$$\mathbb{E}[\log(Z_n)] = int_0^1 \log(x)dx = -1$$

So by the WLLN $\log(Z_n) \xrightarrow[n\to\infty]{p} -1$ and by the CLT $\sqrt{n}(\log(Z_n) - (-1)) \xrightarrow[n\to\infty]{d} N(0, V)$.

Step 3: Apply CMT Since we're looking to prove that $Z_n \xrightarrow[n\to\infty]{p} \exp(-1)$, we need to manipulate the result we have from the WLLN: $\log(Z_n) \xrightarrow[n\to\infty]{p} -1$. Since we essentially just want to exponentiate both sides of this result, the CMT justifies doing this.

$$\log(Z_n) \xrightarrow[n\to\infty]{p} -1 \Rightarrow Z_n \xrightarrow[n\to\infty]{p} \exp(-1)$$

Step 4: Apply the Delta Method Now we're trying to get the limiting distribution of $\sqrt{n}(Z_n - \exp(-1))$. Since we are trying to exponentiate the two terms in the result we have, $\sqrt{n}(\log(Z_n) - (-1)) \xrightarrow[n\to\infty]{d} N(0, V)$, we need to apply the delta method with $g(.) = \exp(.)$

$$\begin{aligned}\sqrt{n}(\log(Z_n) - (-1)) \xrightarrow[n\to\infty]{d} N(0, V) &\Rightarrow \sqrt{n}(Z_n - \exp(-1)) \xrightarrow[n\to\infty]{d} \exp'(-1)N(0, V)\\ &\Rightarrow \sqrt{n}(Z_n - \exp(-1)) \xrightarrow[n\to\infty]{d} N(0, \exp(-2)V)\end{aligned}$$

2 2021 Midterm Q6

Setup: $\{X_i\}_{i=1}^n$ are iid $U[-1, 1]$. Define

$$Y_n = \frac{\sum_{i=1}^n X_i}{\sum\limits_{i=1}^n X_i^2 + \sum_{i=1}^n X_i^3}$$

(Hint: $\int_{-1}^1 x^c dx = \frac{1-(-1)^{c+1}}{c+1}$)

a) Show $Y_n \xrightarrow[n\to\infty]{p} \mu$ and find μ

b) Show that Y_n suitably normalized converges in distribution

c) Define $Z_n = \left(\frac{1}{n}\sum\limits_{i=1}^n X_i\right)\left(\frac{1}{\sqrt{n}}\sum\limits_{i=1}^n X_i^2\right)$ and state whether the following are true or false:

1. $\frac{1}{n}\sum\limits_{i=1}^n X_i \xrightarrow[n\to\infty]{p} 0$
2. $\frac{1}{\sqrt{n}}\sum\limits_{i=1}^n X_i^2 \xrightarrow[n\to\infty]{d} N(\mathbb{E}[X_1^2], Var(X_1^2))$
3. $Z_n \xrightarrow[n\to\infty]{p} 0$

d) Define $W_n = \max\{X_i : i = 1, 2, ..., n\}$. Derive the limiting distribution of

$$Q_n = \frac{\frac{1}{\sqrt{n}}\sum\limits_{i=1}^n X_i}{W_n}$$

Solution:

Strategy: With so many terms that look like sample averages just missing the $1/n$ we should immediately be thinking WLLN and CLT. It turns out that multiplying and dividing by $1/n$ will be all we need to invoke them. Part **a** will just be an application of the WLLN and Slutsky's lemma. For part **b** we'll need to choose whether to have the numerator or denominator converge in distribution, which will depend on what each of them converge to in probability. Part **c** is a little tricky: it's making sure we know that we have to recenter sample averages to invoke the CLT. For part **d** we'll need to explicitly derive the distribution of W_n, since it doesn't have any sample averages. Intuitively, we should see that it converges in probability to 1 since that is the maximum value X_i can take.

Step 1: Multiply and divide by $1/n$
Since we're pretty sure we'll want to use the WLLN and CLT, we need to get sample averages. It turns out they're actually in Y_n already, we just have to multiply and divide by $1/n$. If you don't see this from the start, you could begin by figuring out how to transform Y_n to get the sample average in the numerator and how to transform it to get the sample averages in the denominator.

$$Y_n = \frac{1/n}{1/n} \frac{\sum_{i=1}^n X_i}{\sum\limits_{i=1}^n X_i^2 + \sum_{i=1}^n X_i^3} = \frac{\frac{1}{n}\sum_{i=1}^n X_i}{\frac{1}{n}\sum\limits_{i=1}^n X_i^2 + \frac{1}{n}\sum_{i=1}^n X_i^3}$$

Step 2: Get expectations and apply WLLN+Slutsky's Now to figure out what Y_n converges to we need to apply the WLLN to each of its component sample averages. Using the hint we have

$$\begin{aligned}
\mathbb{E}[X_1] &= 0 \\
\mathbb{E}[X_1^2] &= 1/3 \\
\mathbb{E}[X_1^3] &= 0
\end{aligned}$$

So by Slutsky's Lemma we have $Y_n \xrightarrow[n\to\infty]{p} \frac{0}{1/3+0} = 0$.

Step 3: Apply CLT to numerator, then CMT We know we want to apply the CLT, but to which sample average? We essentially have the choice of the numerator or both terms in the denominator. It should be clear though, that we can't use the denominator because the numerator converges in probability to 0, so our limiting distribution would be degenerate. So we need to multiply by $\sqrt{n}$ and subtract off $\mathbb{E}[X_1]/\frac{1}{n}\sum\limits_{i=1}^n X_i^2 + \frac{1}{n}\sum_{i=1}^n X_i^3$. Luckily, the term we need to subtract is 0, so we have

$$\begin{aligned}
\sqrt{n}Y_n &= \frac{\frac{1}{\sqrt{n}}\sum_{i=1}^n X_i}{\frac{1}{n}\sum\limits_{i=1}^n X_i^2 + \frac{1}{n}\sum_{i=1}^n X_i^3} \\
&\xrightarrow[n\to\infty]{d} \frac{1}{1/3}N(0,1/3) \\
&= N(0,3)
\end{aligned}$$

Since $Var(X_1) = \mathbb{E}[X_1^2] - \mathbb{E}[X_1] = \mathbb{E}[X_1^2] = 1/3$.

Step 4: Consider $\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i^2$

We know $\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow[n\to\infty]{p} 0$ by the WLLN, so statement 1 in part **c** is true. The next part we need to consider is $\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i^2$. We know from the CLT that $\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \frac{1}{3}\right) \xrightarrow[n\to\infty]{d} N(0, Var(X_1^2)$, but this doesn't imply statement 2 is true. Note that $\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i^2 = \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \frac{1}{3}\right) + \frac{\sqrt{n}}{3}$. While the first term on the right-hand-side converges in distribution, $\frac{\sqrt{n}}{3}$ diverges, so $\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i^2$ does not converge in distribution and statement 2 is false.

Step 5: Rearrange Z_n to find its limiting distribution

We've seen that $\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i^2$ diverges, so we can't analyze the asymptotics of Z_n as it's written. From the previous part, we do know that $\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i \xrightarrow[n\to\infty]{d} N(0, 1/3)$ and $\frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow[n\to\infty]{p} 1/3$, so we can re-write

$$
\begin{aligned}
Z_n &= \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i^2\right) \\
&= \left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) \\
&\xrightarrow[n\to\infty]{d} \frac{1}{3}N(0, \frac{1}{3}) \\
&= N(0, \frac{1}{27})
\end{aligned}
$$

so statement 3 is also false.

Step 6: Derive the distribution of W_n Since W_n has no sample averages we need to explicitly derive its distribution to make headway. We can start by writing the definition of its distribution and using the fact that the X_is are iid.

$$
\begin{aligned}
\mathbb{P}(W_n \leq w) &= \mathbb{P}(X_1 \leq w, X_2 \leq w, ..., X_n \leq w) \\
&= \prod_{i=1}^{n} \mathbb{P}(\mathbb{X}_{\beth} \leq \gtrapprox) \\
&= P(X_1 \leq w)^n \\
&= \frac{w+1}{2}^n 1\{w \in [-1, 1]\} + 1\{w > 1\}
\end{aligned}
$$

Step 7: Take the limit of $F_{W_n}(w)$ As mentioned in the strategy, our intuition should be that $W_n \xrightarrow[n\to\infty]{p} 1$. We can prove this explicitly by taking the limit of $F_{W_n}(w)$

$$\lim_{n\to\infty} \mathbb{P}(W_n \leq w) = 1\{w \geq 1\}$$

since for $w \in [-1, 1)$ we have $\frac{w+1}{2} < 1$, so $\frac{w+1}{2}^n \to 0$. Since W_n only takes values in $[-1, 1]$, that means $W_n \xrightarrow[n\to\infty]{p} 1$.

Step 7: Apply CLT and Slutsky's From previous parts we know $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \xrightarrow[n\to\infty]{d} N(0, 1/3)$. So by Slutsky's, we have

$$\begin{aligned} Q_n &\xrightarrow[n\to\infty]{d} \frac{1}{1} N(0, 1/3) \\ &= N(0, 1/3) \end{aligned}$$