ARE 210: Aug. 30, 2022 Section Notes

Joel Ferguson

1 Çinlar Prop. 1.6

Setup: A collection $\mathcal{C}$ of subsets of E is called a *p-system* if it is closed under intersection. A collection $\mathcal{D}$ of subsets of E is called a *d-system* if

1. $E \in \mathcal{D}$
2. $A,\ B \in \mathcal{D}$ and $B \subset A \Rightarrow A \setminus B \in \mathcal{D}$
3. $(A_n) \subset \mathcal{D}$ and $A_1 \subset A_2 \subset ...,\ \bigcup\limits_{n=1}^{\infty} = A \Rightarrow A \in \mathcal{D}$

If a collection of subsets of E, $\mathcal{E}$ is a p-system and a d-system on E then $\mathcal{E}$ is a σ-algebra on E.

Solution:
Strategy: Ususally we try to prove that something is a σ-algebra by proving it satisfies the three criteria: it contains $\emptyset$, it's closed under compliment, and it is closed under countable intersection. Criteria 1 and 2 of a d-system basically gets us criteria 1 and 2 of a σ-algebra, so the only thing left is closure under countable intersection, for which we'll use closure under comliments and closure under intersection (p-system), together with criterion 3 of a d-system.

Step 1: Prove Closure Under Compliment
We want to show $A \in \mathcal{E} \Rightarrow A^c \in \mathcal{E}$. We have something that looks almost like closure under compliments in criterion 2 of d-systems. We can use it by noting that by criterion 1 of a d-system $E \in \mathcal{E}$ and $A^c = E \setminus A$, so we have $A \in \mathcal{E} \Rightarrow E \setminus A = A^c \in \mathcal{E}$.

Step 2: Prove Empty Set Membership
Now that we have closure under compliment, it's easy to show $\emptyset \in \mathcal{E}$, since $E^c = \emptyset$.

Step 3: Prove Closure Under Countable Intersection
Now we want to prove $(A_n) \in \mathcal{E} \Rightarrow \bigcup\limits_{n=1}^{\infty} A_n \in \mathcal{E}$. The closest thing we have is criterion 3 of d-systems, so let's think about how to use it. We want to be able to take (A_n) and make it an increasing sequence, so it would be helpful to be

able to take unions. Luckily, we've already proved $\mathcal{E}$ is closed under compliment and we know that it is closed under intersection because it is a p-system, so we have $A, B \in \mathcal{E} \Rightarrow (A^c \cap B^c)^c = A \cup B \in \mathcal{E}$. Now we just need to use this to invoke criterion 3 of a d-system. Let $B_1 = A_1^c,\ B_2 = B_1 \cup A_2^c,\ B_3 = B_2 \cup A_3^c, ...,$ then $B_n \in \mathcal{E}$ and $B_1 \subset B_2 \subset ..$ so $(\bigcup\limits_{n=1}^{n} B_n)^c = \bigcap\limits_{n=1}^{n} A_n \in \mathcal{E}$, which completes the proof.

2 PS1 Question 9

Setup: Consider two independent tosses of a coin and define events:

- A: "Heads on first toss"
- B: "Heads on second toss"
- C: "Exactly one head and one tail (in any order) in the two tosses"

Show that

- $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$
- $\mathbb{P}(C \cap B) = \mathbb{P}(C)\mathbb{P}(B)$
- $\mathbb{P}(A \cap C) = \mathbb{P}(A)\mathbb{P}(C)$

Are the events (A, B, C) mutually independent? What do you take away from this exercise?

Solution:
Strategy: We basically get the first thing for free by the assumption of independence. We can prove the other two by conditioning C on the other event and then calculating the probability. The last bit is to realize that $\mathbb{P}(A \cap B \cap C) = 0$.

Step 1: Note $A \perp B$
Since A and B Are independent by assumption, we know $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

Step 2: Condition C on B
To calculate $\mathbb{P}(C \cap B)$ let try to calculate $\mathbb{P}(C|B)$. We can note that $C = (A \cap B^c) \cup (A^c \cap B)$, so

$$\begin{aligned}
\mathbb{P}(C|B) =& \mathbb{P}(C \cap B)/\mathbb{P}(B) \\
&= \mathbb{P}\left(((A \cap B^c) \cup (A^c \cap B)) \cap B\right)/\mathbb{P}(B) \\
&= \mathbb{P}(A^c \cap B)/\mathbb{P}(B) \\
&= \mathbb{P}(A^c)\mathbb{P}(B)/\mathbb{P}(B) = \mathbb{P}(A^c) = 0.5
\end{aligned}$$

where the last line follows because $A \perp B \Rightarrow A^c \perp B$. Similarly, we have $\mathbb{P}(C|B^c) = \mathbb{P}(A) = 0.5$, so $C \perp B$ and thus $\mathbb{P}(C \cap B) = \mathbb{P}(C)\mathbb{P}(B)$. The case for $\mathbb{P}(A \cap C) = \mathbb{P}(A)\mathbb{P}(C)$ is analogous, just rename B and A.

Step 3: Probability of all three

The last step is to note that $\mathbb{P}(A \cap B \cap C) = 0$. Intuitively this makes sense, $A \cap B$ means we get two heads, which means C can't be true. But $\mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) = 1/8$. Apparently, pairwise independence does not imply joint independence.