ARE 210: Dec. 1, 2021 Section Notes

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PS 5 Q2a

Setup: Suppose we observe $\{X_i\}_{i=1}^n$ an i.i.d. sample from a $\mathcal{N}(\mu, 1)$ distribution where μ is unknown. Let t be a known scalar and define the parameter of interest $\theta \equiv P(X_1 \le t)$. Find the UMVUE and MLE of θ and discuss consistency.

Solution:
Strategy: To find the UMVUE, we'll want to find an unbiased estimator and a complete and sufficient statistic so we can apply the Lehmann-Scheffe theorem. Deriving the UMVUE explicitly requires some pretty tedious calculation, so I would guess it's unlikely you encounter something like this (in terms of the functional forms) on an exam. The MLE of μ is the mean, which should be known from the handout. We can then use the invariance property of MLE to derive the MLE for θ. Showing consistency of these estimators is then a matter of taking limits.

Step 1: Find an unbiased estimator

To apply the Lehmann-Scheffe theorem (which states that an estimator based only on a complete and sufficient statistic is the unique best unbiased estimator of its mean) and obtain a UMVUE, we want to first find an unbiased estimator of θ. We'll then find a complete and sufficient statistic and take the expectation of our unbiased estimator conditional on this statistic to obtain the UMVUE. Note that any unbiased statistic will do, since we know UMVUEs are unique. A simple unbiased estimator is

$$\hat{\theta}_1 = \mathbb{I}(X_1 \le t)$$

.

This is unbiased since $\mathbb{E}[X_1 \le t] = P(X_1 \le t)$.

Step 2: Find a complete and sufficient statistic

Since the normal distribution is an exponential family distribution, we know the sample mean is a complete statistic. This is because we can set $h(x) = (2\pi)^{-1/2}\exp(-(1/2)x^2)$, $\eta(\mu) = n\mu$, $T(X) = X/N$, $B(\mu) = \mu^2/2$. So $\sum T(X_i) = (1/n)\sum X_i$ is a complete statistic.

To show it's sufficient, we use the factorization theorem. We'll now call $T(\mathbf{X}) = (1/n)\sum X_i$. Our $g(T(\mathbf{X}), \mu)$ will be

$$g(T(\mathbf{X}), \mu) = \exp(\eta(\mu)T(\mathbf{X}) - B(\mu))$$

and our $h(\mathbf{X}) = \prod h(X_i)$. since $h(\mathbf{X})g(T(\mathbf{X}), \mu)$ is the likelihood, we know $T(\mathbf{X})$ is sufficient.

Step 3: Apply Lehmann-Scheffe to find the UMVUE

By the Lehmann-Scheffe theorem, we know

$$\hat{\theta}_{UMVUE} = \mathbb{E}[\hat{\theta}_1 | \bar{\mathbf{X}}]$$

Since this will be unbiased by the law of iterated expectation and is a function only of a complete and sufficient statistic. We can find an explicit form for the estimator by noting that this is the same as the conditional probability and applying Bayes' Rule

$$\hat{\theta}_{UMVUE} = \frac{P(X_1 \le t, \bar{\mathbf{X}} = \bar{\mathbf{x}})}{\bar{\mathbf{X}} = \bar{\mathbf{x}}}$$

To find the numerator, we need to first note that $(X_1, \bar{\mathbf{X}})$ is a multivariate normal random vector, since both components are linear combinations of independent normally distributed random variables. It's easy to show that both components have mean μ, we know $Var(X_1) = 1$, and $Var(\bar{\mathbf{X}}) = (1/n^2)\sum Var(X_i) = 1/n$. To find the covariance note

$$\begin{aligned}
Cov(\bar{\mathbf{X}}, X_1) &= \mathbb{E}[\bar{\mathbf{X}}X_1] - \mathbb{E}[\bar{\mathbf{X}}]\mathbb{E}[X_1] \\
&= \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[X_i X_1] - \mu^2 \\
&= \frac{1}{n}(\mathbb{E}[X_1^2] - \mu^2) + \frac{1}{n}\sum_{i=2}^{n}\mathbb{E}[X_i X_1] - \frac{n-1}{n}\mu^2 \\
&= \frac{1}{n} + \frac{1}{n}\sum_{i=2}^{n}\mu^2 - \frac{n-1}{n}\mu^2 = \frac{1}{n}
\end{aligned}$$

So $\Sigma = \begin{pmatrix} 1 & 1/n \\ 1/n & 1/n \end{pmatrix}$. To use the multivariate normal density, we need $\det(\Sigma) = (1/n) - (1/n^2) = (n-1)/n^2$ and $\Sigma^{-1} = \begin{pmatrix} n/(n-1) & -n/(n-1) \\ -n/(n-1) & n^2/(n-1) \end{pmatrix}$. We can now write[1]

[1] After solving for $(x_1 - \mu, \bar{\mathbf{x}} - \mu)\Sigma^{-1}(x_1 - \mu, \bar{\mathbf{x}} - \mu)^T = (x_1 - \mu)^2 - 2(x_1 - \mu)(\bar{\mathbf{x}} - \mu) + n(\bar{\mathbf{x}} - \mu)^2]$.

$$
\begin{aligned}
P(X_1 \le t, \bar{\mathbf{X}} = \bar{\mathbf{x}}) &= \int_{-\infty}^{t} \frac{1}{2\pi} \frac{n}{\sqrt{n-1}} exp(-\frac{1}{2}\frac{n}{n-1}[(x_1-\mu)^2 - 2(x_1-\mu)(\bar{\mathbf{x}}-\mu) + n(\bar{\mathbf{x}}-\mu)^2])dx_1 \\
P(\bar{\mathbf{X}} = \bar{\mathbf{x}}) &= \sqrt{\frac{n}{2\pi}} exp(-\frac{n}{2}(\bar{\mathbf{x}}-\mu)^2) \\
\hat{\theta}_{UMVUE} &= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{n}{n-1}} \exp(-\frac{1}{2}[\frac{n}{n-1}((x_1-\mu)^2 - 2(x_1-\mu)(\bar{\mathbf{x}}-\mu) + n(\bar{\mathbf{x}}-\mu)^2) - n(\bar{\mathbf{x}}-\mu)^2])dx_1 \\
&= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{n}{n-1}} \exp(-\frac{1}{2}[\frac{n}{n-1}((x_1-\mu)^2 - 2(x_1-\mu)(\bar{\mathbf{x}}-\mu) + n(\bar{\mathbf{x}}-\mu)^2 - (n-1)(\bar{\mathbf{x}}-\mu)^2) \\
&= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{n}{n-1}} \exp(-\frac{1}{2}[\frac{n}{n-1}((x_1-\mu)^2 - 2(x_1-\mu)(\bar{\mathbf{x}}-\mu) + (\bar{\mathbf{x}}-\mu)^2)])dx_1 \\
&= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{n}{n-1}} \exp(-\frac{1}{2}[\frac{n}{n-1}(x_1-\bar{\mathbf{x}})^2])dx_1
\end{aligned}
$$

Noting that the integrand is the pdf of a $\mathcal{N}(\bar{\mathbf{x}}, n/(n-1))$ random variable, we can say $\hat{\theta}_{UMVUE} = \Phi(\sqrt{n/(n-1)}(t - \bar{\mathbf{X}}))$.

Step 4: Use the invariance property to find $\hat{\theta}_{MLE}$

We have seen many times that $\hat{\mu}_{MLE} = \bar{\mathbf{X}}$. We can use the invariance principle if we are able to find a mapping from μ to θ. We can rewrite $\theta = \Phi(t-\mu)$, so we know $\hat{\theta}_{MLE} = \Phi(t - \bar{\mathbf{X}})$.

Step 5: Show consistency

The key to showing consistency of both of these estimators is the continuous mapping theorem, since $\Phi(.)$ is continuous. For $\hat{\theta}_{MLE}$, note that by the WLLN, $\bar{\mathbf{X}} \xrightarrow[n\to\infty]{p} \mu$, so by the CMT, $\hat{\theta}_{MLE} \xrightarrow[n\to\infty]{p} \theta$. For $\hat{\theta}_{UMVUE}$, first note $\sqrt{n/(n-1)} \xrightarrow[n\to\infty]{p} 1$, so by Slutsky's lemma, $\sqrt{n/(n-1)}(t-\bar{\mathbf{X}}) \xrightarrow[n\to\infty]{p} (t-\mu)$ and the same result as for $\hat{\theta}_{MLE}$ goes through.

2018 Final Exam Q1 remaining sub-parts

Setup: Suppose $\{X_i\}_{i=1}^{\infty}$ is an iid sample with the density of X_i given by

$$
p(x, \theta, \nu) = \frac{\theta\nu^\theta}{x^{\theta+1}}\mathbb{I}(x \ge \nu)
$$

where $\nu > 0$ and $\theta > 0$.

Results we already have:

- **a)** Not exponential family.

- **b)** $(X_{(1)}, \prod_{i=1}^{n} X_i)$ is sufficient for ν, θ
- **c)** $\hat{\nu}_1 = \frac{\theta-1}{\theta}\frac{1}{n}\sum_{i=1}^{n} X_i$, $\mathbb{E}[X_i] = \frac{\theta\nu}{\theta-1}$.
- **d)** $\hat{\nu}_2 = \mathbb{E}[\hat{\nu}_1 | X_{(1)}]$ has smaller variance than $\hat{\nu}_1$
- **g)** $\hat{\nu}_{MLE} = X_{(1)}$
- **h)** $F_{\hat{\nu}_{MLE}}(x) = 1 - \left(\frac{\nu}{x}\right)^{n\theta}$ for $x \geq \nu$
- **n)** $\hat{\theta}_{MLE} = \frac{1}{\frac{1}{n}\sum_{i=1}^{n} \log\left(\frac{X_i}{X_{(1)}}\right)}$

Questions

- **e)** Derive the (appropriately normalized) limiting distribution of $\hat{\nu}_1$ for the case where $\theta = 3$. Derive the asymptotic variance explicitly.
- **f)** How does your answer to **e** change if $\theta = 2$?
- **j)** Suppose that θ is known. Construct an exact level-α test of H: $\nu = 1$ against K: $\nu > 1$. Derive the critical value of the test explicitly. Denote the critical value by c^* and write down the test $\delta = \mathbb{I}(T(\mathbf{X}) > c^*)$ for some statistic $T(\mathbf{X})$.
- **k)** Derive the power function $\mathbb{P}_\nu(T > c^*)$ for the test proposed in part **j**.
- **l)** Propose an appropriate normalization and derive the limiting distribution of $\hat{\nu}_{MLE}$.
- **m)** Show that $\hat{\nu}_{MLE}$ is consistent for ν. Hint: $a_n X_n = O_p(1)$ and $a_n \to \infty$ implies $X_n = o_p(1)$.
- **o)** Show that $\hat{\theta}_{MLE}$ is consistent for θ. You may use the fact that $\mathbb{E}_{\nu,\theta}[\log(X_i)] = \log(\nu) + (1/\theta)$
- **p)** Derive the limiting distribution of $\frac{1}{n}\sum_{i=1}^{n} \log\left(\frac{X_i}{X_{(1)}}\right)$.
- **q)** Use the delta method to derive the asymptotic distribution of $\hat{\theta}_{MLE}$. In particular, derive a form for the asymptotic variance.
- **r)** Construct an approximate level-α test for H: $\theta = 3$ against the alternative K: $\theta \neq 3$. Outline a test procedure and be explicit about the sense in which the test is *approximately* level-α.

Solution:

Strategy:

For **e**, since the estimator is a scalar multiple of a sample mean, we will be able to use the Lindberg-Levy CLT to derive the limiting distribution. In calculating the variance, we will find that it doesn't exist for $\theta = 2$, giving us the answer to **f**. For part **j** we'll use the distribution derived in **h** to find a value c^* such that $P(X_(1) \geq c^*) = \alpha$ under the null hypothesis that $\nu = 1$. We'll work in cases for part **k**, since the probability of rejecting the null will be 1 for large enough values of ν. To show consistency we'll simply take the limit in the case where the probability of rejection isn't already 1. For part **l** we'll need to use the fact (given as a hint) that $\lim_{n\to\infty}\left(\frac{a}{a+t/n}\right)^{nb} = exp(-\frac{tb}{a})$. Essentially, we'll try to find a transformation of $X_{(1)}$ such that it's distribution contains that expression. We'll combine the expression we get with the fact that $n \to \infty$ to show that $\hat{\nu}_{MLE}$ is consistent in part **m**. We'll combine this with the hint to part **o** via the continuous mapping theorem to get its solution. For part **p**, we'll rewrite the expression as a sum of two things we know the distribution of, one will be a sample mean with it's expected value subtracted off and the other will be $\hat{\nu}_{MLE} - \nu$, which will give us a tractable expression. We'll use this result to derive an estimator of $1/\theta$ and then apply the delta method to derive the asymptotic distribution $\hat{\theta}_{MLE}$. We'll develop our approximate level-α test for part **r** by normalizing $\hat{\theta}_{MLE}$ using the expression for the variance derived in **q**.

Step 1: Part e

From part **c** we know $\hat{\nu}_1 = \frac{\theta-1}{\theta}\frac{1}{n}\sum_{i=1}^{n} X_i$ and $\mathbb{E}[X_i] = \frac{\theta\nu}{\theta-1}$. To apply the Lindberg-Levy CLT, we'll need to subtract $\frac{\theta-1}{\theta}\frac{\theta\nu}{\theta-1}$ from $\hat{\nu}_1$ and then multiply the difference by $\sqrt{n}$.

$$\begin{aligned}\sqrt{n}(\hat{\nu}_1 - \frac{\theta-1}{\theta}\frac{\theta\nu}{\theta-1}) &= \frac{\theta-1}{\theta}\sqrt{n}(\frac{1}{n}\sum_{i=1}^{n} X_i - \nu) \\ &\xrightarrow[n\to\infty]{d} \frac{\theta-1}{\theta}\mathcal{N}(0, Var(X_i))\end{aligned}$$

We're given that $\theta = 3$, so the last piece we need is the variance of X_i, which boils down to finding it's second (uncentered) moment.

$$\begin{aligned}\mathbb{E}[X_i^2] &= \int_{-\infty}^{\infty} x^2 \frac{\theta\nu^\theta}{x^{\theta+1}}\mathbb{I}(x > \nu)dx \\ &= \int_{\nu}^{\infty} \frac{\theta\nu^\theta}{x^{\theta-1}}dx \\ &= \frac{\theta\nu^\theta}{\theta-2}x^{-(\theta-2)}|_{\nu}^{\infty} = \frac{\theta\nu^2}{\theta-2}\end{aligned}$$

This gives us $Var(X_i) = \frac{\theta\nu^2}{\theta-2} - \frac{\theta^2\nu^2}{(\theta-1)^2} = \frac{\theta\nu^2(\theta-1)^2-\theta^2\nu^2(\theta-2)}{(\theta-2)(\theta-1)^2} = \frac{\theta\nu^2}{(\theta-2)(\theta-1)^2}$. Plugging in $\theta = 3$, we get $\sqrt{n}(\hat{nu}_1 - \nu) \xrightarrow[n\to\infty]{d} \mathcal{N}(0, \nu^2/3)$.

Step 2: Part f

In the case where $\theta = 2$, the variance of X_i is not finite. As a result, we can't apply the CLT or derive the asymptotic distribution.

Step 3: Part j

The statistic we'll base our decision rule on is $X_{(1)} = \hat{\nu}_{MLE}$. From parth **h** we have $F_{\hat{\nu}_{MLE}}(x) = 1 - \left(\frac{\nu}{x}\right)^{n\theta}$ for $x \geq \nu$. Under the null hypothesis of $\nu = 1$, we thus have

$$\mathbb{P}(X_{(1)} > c) = \mathbb{E}[\mathbb{I}(X_{(1)} > c)] = c^{-n\theta}$$

To find a level-α test, we want to a value c^* such that under the null hypothesis, $\mathbb{E}[\mathbb{I}(X_{(1)} > c^*)] = \alpha$. Solving $c^{*-n\theta} = \alpha$ for c^* yields $c^* = \alpha^{-1/n\theta}$.

Step 4: Part k

Using our result from part **h** again, we have

$$\mathbb{P}_\nu(X_{(1)} > c^*) = 1 - F_{X_{(1)}}(c^*) = 1 - (1 - \nu^{n\theta}\alpha)\mathbb{I}(\alpha^{-1/n\theta} \geq \nu)$$

To determine whether the test is consistent we'll start by holding n and ν fixed. Note that for $\alpha^{-1/n\theta} < \nu$ we have $\mathbb{P}_\nu(X_{(1)} > c^*) = 1$, so we only have to consider the case where $\alpha^{-1/n\theta} \geq \nu$, in which case

$$\mathbb{P}_\nu(X_{(1)} > c^*) = \nu^{n\theta}\alpha$$

Since $\theta > 0$, we know $\alpha^{-1/n\theta} \to 0$. Finally, since $\nu > 0$, there's some value of N such that $\alpha^{-1/n\theta} < \nu$ for $n > N$. So the test is consistent.

Step 5: Part l

Again, we'll use $F_{X_{(1)}}(x) = 1 - \left(\frac{\nu}{x}\right)^{n\theta}$ for $x \geq \nu$ along with the hint that $\lim_{n\to\infty}\left(\frac{a}{a+t/n}\right)^{nb} = exp(-\frac{tb}{a})$. The hint suggests that we'd like to replace the x in the expression for $F_{X_{(1)}}(x)$ with $\nu + x/n$, which we can do by evaluating it at $\nu + x/n$ rather than x. Since $F_{X_{(1)}}(x) = \mathbb{P}(X_{(1)} \leq x)$,

$$F_{X_{(1)}}(\nu + x/n) = \mathbb{P}(X_{(1)} \leq \nu + x/n) = \mathbb{P}((X_{(1)} - \nu)n \leq x) = 1 - \left(\frac{\nu}{\nu + x/n}\right)^{n\theta}$$

We can apply the hint to this expression to get $\lim F_{(X_{(1)}-\nu)n}(x) = 1 - exp(-x\theta/\nu)$. Thus, $(X_{(1)} - \nu)n \xrightarrow[n\to\infty]{d} exponential(\theta/\nu)$.

Step 6: Part m

As we just showed, $n(\hat{\nu}_{MLE} - \nu) = O_p(1)$, since it converges in distribution. We also know that $n \to \infty$, so by the hint $\hat{\nu}_{MLE} - \nu = o_p(1)$ which implies $\hat{\nu}_{MLE} \xrightarrow[n\to\infty]{p} \nu$.

Step 7: Part o From part **n** we have $\hat{\theta}_{MLE} = \frac{1}{\frac{1}{n}\sum\limits_{i=1}^{n} \log\left(\frac{X_i}{X_{(1)}}\right)}$. We can rewrite this as

$$\hat{\theta}_{MLE} = [\frac{1}{n}\sum_{i=1}^{n} \log(X_i) - \log(X_{(1)})]^{-1}$$

By the WLLN, we know $\frac{1}{n}\sum\limits_{i=1}^{n} \log(X_i) \xrightarrow[n\to\infty]{p} \mathbb{E}[\log(X_i)] = \log(\nu)+(1/\theta)$. We also just showed that $X_{(1)} \xrightarrow[n\to\infty]{p} \nu$. So by the Continuous Mapping Theorem, we have

$$\hat{\theta}_{MLE} \xrightarrow[n\to\infty]{p} [\log(\nu) + \frac{1}{\theta} - \log(\nu)]^{-1} = \theta$$

So $\hat{\theta}_{MLE}$ is consistent for θ.

Step 8: Part p

First we'll start by rewriting $\frac{1}{n}\sum\limits_{i=1}^{n} \log\left(\frac{X_i}{X_{(1)}}\right)$ as a sum

$$\frac{1}{n}\sum_{i=1}^{n} \log\left(\frac{X_i}{X_{(1)}}\right) = \frac{1}{n}\sum_{i=1}^{n} \log(X_i) - \log(X_{(1)})$$

This looks like we'll want to multiply it by $\sqrt{n}$ and apply the CLT at least once. To do so, we'll have to add and subtract a term within the sum that is the mean of one of the random variables. We know that we're trying to derive the limiting distribution for $\hat{\theta}_{MLE}$, so that should tell us that we'll want to use $\mathbb{E}[\log(X_i)] = \log(\nu) + (1/\theta)$. Another hint that that's the one we want to plug in is that we haven't found $\mathbb{E}[\log(X_{(1)}]$. Adding and subtracting $\log(\nu) + (1/\theta)$ yields

$$\frac{1}{n}\sum_{i=1}^{n} \log(X_i) - \log(X_{(1)}) = \frac{1}{n}\sum_{i=1}^{n} \left[(\log(X_i) - (\log(\nu) + 1/\theta)) - \left(\log(X_{(1)}) - (\log(\nu) + 1/\theta)\right)\right]$$

Since $\hat{\theta}_{MLE} \xrightarrow[n\to\infty]{p} \theta$, it's natural here to subtract off $1/\theta$. Another way see that is to note that $X_{(1)} \xrightarrow[n\to\infty]{p} \nu$, so we might use the CMT to conclude $\log(X_{(1)}) \xrightarrow[n\to\infty]{p} \log(\nu)$, which would require subracting $1/\theta$. Doing so yields

$$\frac{1}{n}\sum_{i=1}^{n}\log(X_i)-\log(X_{(1)})-\frac{1}{\theta}=\frac{1}{n}\sum_{i=1}^{n}\left[(\log(X_i)-(\log(\nu)+1/\theta))-\big(\log(X_{(1)})-(\log(\nu)+1/\theta)\big)\right]-\frac{1}{\theta}$$
$$=\frac{1}{n}\sum_{i=1}^{n}\left[(\log(X_i)-(\log(\nu)+1/\theta))-\big(\log(X_{(1)})-\log(\nu)\big)\right]$$

Finally, we'll multiply this by $\sqrt{n}$ so we can apply the CLT to the first part of the sum.

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[(\log(X_i)-(\log(\nu)+1/\theta))\right]=\frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n}\left[\log(X_i)\right]-\left(\log(\nu)+\frac{1}{\theta}\right)\right)$$
$$\xrightarrow[n\to\infty]{d}\mathcal{N}(0,Var(\log(X_i)))$$

The second part of the sum simplifies to $\frac{1}{\sqrt{n}}[\log(X_{(1)})-\log(\nu)]$. We can use a similar strategy to the one used in part **l** to show $n[\log(X_{(1)})-\log(\nu)]\xrightarrow[n\to\infty]{d}$ $exponential(\theta)$, so $n[\log(X_{(1)})-\log(\nu)]=O_p(1)$. We can rewrite $n[\log(X_{(1)})-\log(\nu)]=n^{3/2}\frac{1}{sqrtn}[\log(X_{(1)})-\log(\nu)]$, and since $n^{3/2}\to\infty$, we know from the hint to **m** that $\frac{1}{sqrtn}[\log(X_{(1)})-\log(\nu)]=o_p(1)$. Applying Slutsky's lemma to the two sums gives

$$\sqrt{n}[\frac{1}{n}\sum_{i=1}^{n}\log\left(\frac{X_i}{X_{(1)}}\right)-\frac{1}{\theta}]\xrightarrow[n\to\infty]{d}\mathcal{N}(0,Var(\log(X_i)))$$

Step 9: Part q

Define $T_n=\frac{1}{n}\sum\limits_{i=1}^{n}\log\left(\frac{X_i}{X_{(1)}}\right)$. We've just shown that

$$\sqrt{n}(T_n-1/\theta)\xrightarrow[n\to\infty]{d}\mathcal{N}(0,Var(\log(X_i)))$$

From part **n** we know $\hat{\theta}_{MLE}=1/T_n$, so we're interested in the asymptotic distribution of $\sqrt{n}(1/T_n-\theta)$. Letting $g(x)=1/x$, we know by the delta method that

$$\sqrt{n}(1/T_n-\theta)\xrightarrow[n\to\infty]{d}-\theta^{-2}\mathcal{N}(0,Var(\log(X_i)))$$
$$=\mathcal{N}(0,\theta^{-4}Var(\log(X_i)))$$

Step 10: Part r

Under the null hypothesis,

$$z=\frac{\sqrt{n}(1/T_n-\theta)}{\theta^{-2}\sqrt{Var(\log(X_i))}}\xrightarrow[n\to\infty]{d}\mathcal{N}(0,1)$$

So an approximately level-α test is to reject if $|z| > Z_{1-\alpha/2}$. This is only approximate because it relies on the limiting distribution, so it isn't necessarily size-α in finite samples.