ARE 210: Oct. 20, 2021 Section Notes

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1 Midterm Q3

Setup: Let (X, Y) be i.i.d. exponential random variables with parameter 1, i.e. with marginal CDF $F(z) = 1 - \exp(-z)$. a) Find the joint density of $U = X + Y$ and $V = \frac{X}{X+Y}$ b) Find the distribution of V.

Solution:
Strategy: We want to find the joint density of U and V. Since X and Y are independent, it's straightforward to find their joint distribution. The main challenge of this problem is finding a inverse transformation $g^{-1}(U, V) = (X, Y)$. This approach is more appropriate because one of our variables of interest, V, is not a linear combination of random variables, so we can't use any of the tricks we know for moment generating functions and characteristic functions. Even conditional on being able to work with the (multivariable) MGF/CF, we'd still have to figure out the distribution it corresponds to, which is generally only easy if the joint distribution is a widely used one. Once we have the joint distribution of U, V we can integrate out U to find the marginal distribution of V.

Step 1: Find $g^{-1}(.,.)$
We want to find a function $g^{-1}(U, V) = (X, Y)$. The first step here is to try to write either X or Y as a function of U, V. Since we have a lone X in the numerator of V, that seems like a good thing to try to isolate, and we can do by multiplying $UV = X$.

Now we need to find Y. Given that we know we can write X as UV and $U = X + Y$, we can write $Y = U - UV$. So our inverse mapping is $g^{-1}(U, V) = (UV, U - UV)$.

Step 2: Find $|J(g^{-1})|$
Now that we have $g^{-1}(.,.)$, we can find the Jacobian and its determinant. The Jacobian is computed as

$$J(g^{-1}) = \begin{bmatrix} \frac{dX}{dU} & \frac{dX}{dV} \\ \frac{dY}{dU} & \frac{dY}{dV} \end{bmatrix} = \begin{bmatrix} V & U \\ 1 - V & -U \end{bmatrix}$$

And it's determinant is

$$|J(g^{-1})| = |-VU - U(1 - V)| = |-U| = U$$

Step 3: Plug in to PDF Transformation
Using the multivariate PDF transformation formula, we know

$$f_{UV}(u,v) = f_{XY}(f^{-1}(u,v))|J(g^{-1})| = e^{-uv}e^{-(u-uv)}u = ue^{-u}$$

The last step is to find the domain. Since $X, Y \in \mathbb{R}_+$ we have $U \in \mathbb{R}_+$. And since $X < \leq X + Y$, we have $V \in [0,1]$. We can incorporate these into the PDF as

$$f_{UV}(u,v) = e^{-uv}e^{-(u-uv)}U = ue^{-u}\mathbb{I}(u \geq 0)\mathbb{I}(v \in [0,1])$$

This completes part a).

Step 4: Integrate out U to find the distribution of V
We know that we can integrate out U from the joint distribution of UV to find the marginal distribution of V. This is a relatively straightforward calculation (using integration by parts), we only need to be mindful of the indicators.

$$\begin{aligned} f_V(v) &= \int_{-\infty}^{\infty} f_{UV}(u,v)du \\ &= \int_{-\infty}^{\infty} Ue^{-u}\mathbb{I}(u \geq 0)\mathbb{I}(v \in [0,1])du \\ &= \int_{0}^{\infty} ue^{-u}\mathbb{I}(v \in [0,1])du = -e^{-u}(1+u)\mathbb{I}(v \in [0,1])\Big|_0^{\infty} = \mathbb{I}(v \in [0,1]) \end{aligned}$$

So $V \sim U[0,1]$.

2 Midterm Q6a

Setup: Suppose that $X_1, X_2, \cdots, X_k, \cdots$ are independent and identically distributed random variables with $X_i \sim U[-1,1]$ (i.e. uniformly distributed over the interval $[-1,1]$). Define

$$Y_n \equiv \frac{\sum_{k=1}^{n} X_k}{\sum_{k=1}^{n} X_k^2 + \sum_{k=1}^{n} X_k^3}$$

Show $Y_n \xrightarrow[n\to\infty]{p} \mu$ and find μ

Solution:
Strategy: Generally, with problems where we want to show convergence of somekind and we have many elements of the random variable of interest that look like sample averages, we want to rewrite the variable in terms of sample averages and then use Slutsky's Lemma, CMT, Delta Method, etc to get the

result. In this problem, Y is a ratio of sample averages and we know what they converge to via the WLLN.

Step 1: Rewrite Y as sample averages

As mentioned in the strategy, we want to try to rewrite Y_n in terms of sample averages. In this case, we can multiply and divide by n^{-1} to make all of the sums into averages.

$$Y_n = \frac{\sum\limits_{k=1}^{n} X_k}{\sum\limits_{k=1}^{n} X_k^2 + \sum\limits_{k=1}^{n} X_k^3} = \frac{n^{-1}\sum\limits_{k=1}^{n} X_k}{n^{-1}(\sum\limits_{k=1}^{n} X_k^2 + \sum\limits_{k=1}^{n} X_k^3)} = \frac{n^{-1}\sum\limits_{k=1}^{n} X_k}{n^{-1}\sum\limits_{k=1}^{n} X_k^2 + n^{-1}\sum\limits_{k=1}^{n} X_k^3}$$

Step 2: Apply the WLLN

Now that we have a bunch of sample averages, we can apply the WLLN. Note that since the X_is are iid, so are the functions of them. Thus, we have

$$\begin{aligned} n^{-1}\sum_{k=1}^{n} X_k &\xrightarrow[n\to\infty]{p} \mathbb{E}[X_1] \\ n^{-1}\sum_{k=1}^{n} X_k^2 &\xrightarrow[n\to\infty]{p} \mathbb{E}[X_1^2] \\ n^{-1}\sum_{k=1}^{n} X_k^3 &\xrightarrow[n\to\infty]{p} \mathbb{E}[X_1^3] \end{aligned}$$

Step 3: Find these expectations

Now that we know that all of these components of Y converge in probability to something, we need to find their values. Since $X_1 \sim U[-1,1]$, we have $f_{X_1}(x) = 1/2\mathbb{I}(x \in [-1,1])$. So we have

$$\begin{aligned} \mathbb{E}[X_1] &= \int_{-1}^{1} \frac{x}{2}dx = \frac{x^2}{4}\Big|_{-1}^{1} = 0 \\ \mathbb{E}[X_1^2] &= \int_{-1}^{1} \frac{x^2}{2}dx = \frac{x^3}{6}\Big|_{-1}^{1} = \frac{1}{3} \\ \mathbb{E}[X_1^3] &= \int_{-1}^{1} \frac{x^3}{2}dx = \frac{x^4}{8}\Big|_{-1}^{1} = 0 \end{aligned}$$

Step 4: Apply the Continuous Mapping Theorem

We now have $(n^{-1}\sum\limits_{k=1}^{n} X_k, n^{-1}\sum\limits_{k=1}^{n} X_k^2, n^{-1}\sum\limits_{k=1}^{n} X_k^3) \xrightarrow[n\to\infty]{p} (0, 1/3, 0)$. The final step is to apply the CMT to the function $g(a,b,c) = a/(b+c)$

$$g((n^{-1}\sum_{k=1}^{n} X_k, n^{-1}\sum_{k=1}^{n} X_k^2, n^{-1}) = Y_n \xrightarrow[n\to\infty]{p} \frac{0}{1/3+0} = 0$$

so $Y_n \xrightarrow[n\to\infty]{p} 0$.

3 Midterm Q6b

Setup: Show that Y_n, suitably normalized, converges in distribution as $n \to \infty$

Solution:
Strategy: Now we want to show convergence in distribution. Since we know Y_n is a transformation of a bunch of sample means, we want to use the CLT to make one of them converge in distribution and then use Slutsky's Lemma to find what Y_n converges to. Since we know the numerator goes to 0 in probability, we want to make that converge in distribution so we don't end up with a degenerate distribution. Similarly, we know the denominator converges in probability (and thus distribution) to 1/3, so we won't end up dividing by 0.

Step 1: Apply the CLT to $n^{-1}\sum_{k=1}^{n} X_k$

Since $n^{-1}\sum_{k=1}^{n} X_k$ is a sample average of iid random variables and as shown in the previous question $\mathbb{E}[n^{-1}\sum_{k=1}^{n} X_k] = 0$, we can apply the CLT without centering

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n} X_k \xrightarrow[n\to\infty]{d} N(0, \mathbb{V}(X_1))$$

Step 2: Find $\mathbb{V}(X_1)$

Since we know $\mathbb{E}[X_1] = 0$, and $\mathbb{E}[X_1^2] = 1/3$, we can combine these to find

$$\mathbb{V}(X_1) = \mathbb{E}[X_1^2] - \mathbb{E}[X_1]^2 = \frac{1}{3} - 0 = \frac{1}{3}$$

So we have $\frac{1}{\sqrt{n}}\sum_{k=1}^{n} X_k \xrightarrow[n\to\infty]{d} N(0, 1/3)$.

Step 3: Apply Slutsky's Lemma

We now know $\frac{1}{\sqrt{n}}\sum_{k=1}^{n} X_k \xrightarrow[n\to\infty]{d} N(0, 1/3)$ and $n^{-1}(\sum_{k=1}^{n} X_k^2 + \sum_{k=1}^{n} X_k^3) \xrightarrow[n\to\infty]{p} 1/3$. We now need to standardize Y_n to be a ratio of these terms and apply Slutsky's Lemma to derive the limiting distribution.

$$\frac{\frac{1}{\sqrt{n}}\sum_{k=1}^{n} X_k}{n^{-1}(\sum_{k=1}^{n} X_k^2 + \sum_{k=1}^{n} X_k^3)} = \sqrt{n}\frac{n^{-1}\sum_{k=1}^{n} X_k}{n^{-1}(\sum_{k=1}^{n} X_k^2 + \sum_{k=1}^{n} X_k^3)}$$
$$= \sqrt{n}Y_n \xrightarrow[n\to\infty]{d} 3N(0,1/3) = N(0,3)$$

4 Midterm Q6d

Setup: Define $W_n = \max_{1\leq k\leq n} X_k$. Derive the limiting distribution of

$$Q_n = \frac{\frac{1}{\sqrt{n}}\sum_{k=1}^{n} X_k}{W_n}$$

Solution:
Strategy: We already know the limiting distribution of the numerator of Q_n, so we should hope to find that W_n converges in probability to a constant so we can apply Slutsky's Lemma. Intuitively, W_n should converge in probability to the upper end of the domain of X_i, which is 1, because as $n \to \infty$ the max should get closer and closer to 1. Since this turns out to be the case, Q_n has the same limiting distribution as its numerator.

Step 1: Derive the CDF of W_n
Since the X_is are iid, we know that

$$\begin{aligned}
\mathbb{P}(W_n \leq w) &= \mathbb{P}(X_1 \leq w, X_2 \leq 2, \cdots, X_n \leq w) \\
&= \prod_{i=1}^{n} \mathbb{P}(X_i \leq w) \\
&= \mathbb{P}(X_1 \leq w)^n = \left(\frac{w+1}{2}\right)^n
\end{aligned}$$

Step 2: Show $W_n \xrightarrow[n\to\infty]{p} 1$ Now we want to show that our intuition that $W_n \xrightarrow[n\to\infty]{p}$ 1 was correct. We can write

$$\begin{aligned}
\mathbb{P}(|W_n - 1| < \varepsilon) &= \mathbb{P}(1 - W_n < \varepsilon) = \mathbb{P}(W_n > 1-\varepsilon) = 1 - \mathbb{P}(W_n < 1-\varepsilon) \\
&= 1 - \left(\frac{2-\varepsilon}{2}\right)^n = \left(\frac{\varepsilon}{2}\right)^n
\end{aligned}$$

Since $\left(\frac{\varepsilon}{2}\right)^n \xrightarrow{n\to\infty} 0$, we have $W_n \xrightarrow[n\to\infty]{p} 1$.

Step 3: Apply Slutsky's Lemma As shown above, $\frac{1}{\sqrt{n}} \sum_{k=1}^{n} X_k \xrightarrow[n\to\infty]{d} N(0, 1/3)$.
Since we've just shown $W_n \xrightarrow[n\to\infty]{p} 1$, we know $Q_n \xrightarrow[n\to\infty]{d} N(0, 1/3)$.