

# Mathematical Modeling on the Great Plague of London

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## Epidemic Description

This is a very well-known pandemic in history because the lack of knowledge in medicine in the 17<sup>th</sup> century led to a high mortality rate and a low recovery rate. The first plague epidemic reached England in 1348 and was named the Black Death. After this, there were repeated incidents occurring throughout England leading to the Great Plague of London which we will focus on. The Great Plague also known as the Bubonic Plague hit London in 1665 caused by the bite of an infected flea often carried on rodents like rats. City records indicate that some 68,596 people died during the epidemic, though the actual number of deaths is suspected to have exceeded 100,000 out of a total population estimated at 460,000. (Encyclopædia 2020). The symptoms involved the victim's skin turning black in patches, compulsive vomiting, swollen tongues, and agonizing headaches.

## Block diagram

The model we have decided to use for this disease is an NSIRS model using 4 blocks starting at N which represents the overall population leading to S which represents the susceptible population leading to I which represents the infectious population leading to R which represents the recovered population that loops back to the S block. Each block is linked through a rate to which we have given suitable labels,  $\nu$  represents the population growth rate from N to S,  $\beta$  represents the population flow from S to I,  $\gamma$  represents the population flow from I to R and lastly  $\alpha$  represents the population flow from R back to S completing the loop. These connections are accompanied by rates of mortality stemming down from each of the SIR blocks represented by  $\mu$ . All parameters are positive as we cannot have a negative population or negative growth(decrease in growth would be a decimal)

## Compartmental Model

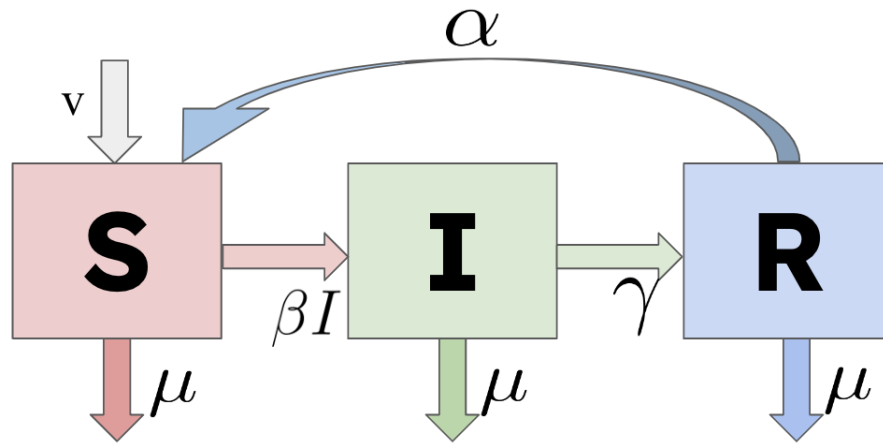


Figure 1. SIRS Model

## Differential equation setup

Next, we constructed 3 differential equations

$$\frac{dS}{dt} = VN - S\beta I - S\mu + R\alpha$$

$$\frac{dI}{dt} = S\beta I - I\mu + I\alpha$$

$$\frac{dR}{dt} = I\gamma - R\mu - R\alpha$$

$$N = S + I + R$$

## Disease Free Equilibrium

Step 1) replacing left sides with 0

$$0 = VN - S\beta I - S\mu + R\alpha$$

$$0 = S\beta I - I\mu + I\alpha$$

$$0 = I\gamma - R\mu - R\alpha$$

For DFE, I has to be 0.

After solving this system of equations with  $I = 0$ , we get the (S, I, R) value below

$$(S, I, R) = \left( \frac{VN}{\mu}, 0, 0 \right)$$

### Endemic Equilibrium

For EE, I should not be equal to 0.

After solving this system of equations with  $I \neq 0$ , we get the (S, I, R) value below

$$(S, I, R) = \left( \frac{\gamma + \mu}{\beta}, \frac{VN}{\gamma + \mu} + \frac{I\gamma\alpha}{(\gamma + \mu)(\mu + \alpha)} - \frac{\mu}{\beta}, \frac{I\gamma}{\mu + \alpha} \right)$$

note: we tried to complete a stability analysis for Endemic Equilibrium however it became too taxing after working out the determinant to find the eigenvalues. Resulted in a big jumble of letters with no obvious solution to analyse.

### Stability of Disease-Free Equilibrium

J =

$$\begin{bmatrix} -\beta I - \mu & -S\beta & \alpha \\ \beta I & S\beta - \gamma - \mu & 0 \\ 0 & \gamma & -\mu - \alpha \end{bmatrix}$$

J|DFE =

$$\begin{bmatrix} -\mu & vN\beta/\mu & \alpha \\ 0 & vN\beta/\mu - \gamma - \mu & 0 \\ 0 & \gamma & -\mu - \alpha \end{bmatrix}$$

$$\det(J - \lambda I) =$$

$$\begin{vmatrix} -\mu - \lambda & vN\beta/\mu & \alpha \\ 0 & vN\beta/\mu - \gamma - \mu - \lambda & 0 \\ 0 & \gamma & -\mu - \alpha - \lambda \end{vmatrix}$$

$$\lambda_{1,2,3} = \left\{ -\mu, \frac{vN\beta}{\mu} - \gamma - \mu, -\mu - \alpha \right\} \quad (\text{Notes: All parameters} > 0)$$

$$\frac{vN\beta}{\mu} - \gamma - \mu < 0$$

$$\frac{vN\beta}{\mu} < \gamma + \mu$$

$$\frac{vN\beta}{\mu(\gamma + \mu)} < 1$$

$$\frac{v}{\mu} r_0 < 1$$

$$\frac{v}{\mu} < \frac{1}{r_0}$$

$$\frac{\mu}{v} > r_0 \quad \text{but } r_0 > 1 \quad \text{and} \quad r_0 = \frac{N\beta}{(\gamma + \mu)}$$

$$\frac{\mu}{v} > 1$$

Mortality rate is greater than population growth rate for the disease to happen

$$\mu - \alpha < 0$$

$$-\mu < \alpha$$

$$\mu > -\alpha$$

Always True

## Conclusion

To conclude, we discovered in our modelling that there are three eigenvalues related to the subsequent Jacobian matrix. To check the stability of these results we set each one to be less than 0, with a bit of arranging solutions were found. Both  $-\mu < 0$  and  $-\mu - \alpha < 0$  are both true for any values as both parameters must be greater than 0.  $\nu N B / \mu - \gamma < 0$  resulted in the final equation  $\mu / \nu > 1$  using this we can gather that the mortality rate must be greater than the population growth rate for the disease to be modelled like a pandemic (with the  $r_0 > 1$ ). This relates to the Great plague as the  $\nu$  would be small at the time with nobody moving to London while the pandemic was occurring and the birth rate in London was drastically cut for the same reason. Relating to the data collected, the overall population was cut by approximately 15% this supports our theory that  $\mu / \nu > 1$  with a higher  $\mu$  value with all the deaths. Long-term behaviour depends on these two parameters and if population growth rate becomes greater than mortality rate then  $r_0 > 1$  is no longer supported and consequently can not be modelled as a pandemic.

### **Reference**

Great Plague of London. (2020). In Encyclopædia Britannica Online. Encyclopædia Britannica Inc.