MT280: LINEAR ALGEBRA

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1. Revision from MT182 (with a few extras thrown in)

1.1. **Determinants.** Let $A = (a_{ij})$ be an $n \times n$ matrix, with entries in a field F.

The **determinant** of A, written $\det(A)$, was defined in MT182 as a horrible sum over all $n!$ permutations of $\{1, \ldots, n\}$, with each term in the sum being a product of n matrix entries (including one from each row and one from each column) multiplied by the sign of the permutation.

For any i and j (each between 1 and n), we can delete row i and column j to form an $(n-1) \times (n-1)$ matrix, $\tilde{A}_{ij}$. These matrices are sometimes called the **minors** of A.

(More generally, deleting any subset of the rows and the same number of columns to leave a square matrix produces a minor. If the same numbered rows and columns are deleted, then the minor is called principal. But the only minors we shall need are the $(n-1) \times (n-1)$ minors, so we simply call these minors.)

The **cofactors** of A, c_{ij} are signed determinants of the minors:

$$c_{ij} = (-1)^{i+j} \det(\tilde{A}_{ij})\,.$$

We saw in MT182 that

$$\det(A) = \sum_{j=1}^{n} c_{ij} a_{ij}$$

(expansion along a row; we also met expansion down a column). Note that this is n formulas for $\det(A)$, as i can take any value between 1 and n.

This formula for $\det(A)$ is sometimes called the **Laplace expansion** of $\det(A)$. In 'the book', expansion along the first row is taken as the *definition* of $\det(A)$, and other properties are derived from this definition.

A comment about computing determinants in practice: the formula from the sum over all permutations of n objects has $n!$ terms. Expanding along a row (or column) is conceptually nice, in that it reduces the problem to a bunch of smaller problems, but it is not generally computationally efficient: the computation is 'reduced' to computing n determinants of $(n-1) \times (n-1)$ matrices, and iterating this reduction eventually leads to computing $n!$ products again (no surprise!). There is a more efficient way: using row reduction to transform the matrix to triangular form. The determinant of the reduced matrix is the product of the diagonal entries. Each elementary row operation changes the determinant in a simple, predictable way. Doing this in a straightforward way the number of steps (multiplications, additions, etc., ignoring the sizes of the entries) grows with n^3 (bounded by some multiple of n^3), which for large n is much better than $n!$ steps.

One very important property which we met was that for two $n \times n$ matrices A and B we have

$$\det(AB) = \det(A)\det(B)\,.$$

We saw that a matrix A is **invertible** (there exits B such that $AB = I$ (remember I?)) if and only if $\det(A) \neq 0$, and gave a (horrible) formula for the inverse. We will recall other properties if we need them.

1.2. **Vector spaces.** We recall several key definitions from MT182.

A **vector space** V over a field F (the **scalars**) is a set of vectors with two operations (addition of vectors, and scalar multiplication)

$$\begin{array}{rcl} V \times V & \to & V \\ (\mathbf{v}_1, \mathbf{v}_2) & \mapsto & \mathbf{v}_1 + \mathbf{v}_2 \end{array}$$

$$\begin{array}{rcl} F \times V & \to & V \\ (\lambda, \mathbf{v}) & \mapsto & \lambda\mathbf{v} \end{array}$$

satisfying lots of axioms (listed in the MT182 notes), namely (for all vectors and scalars as relevant)

- $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
- $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$
- there exists $\mathbf{0} \in V$ such that $\mathbf{0} + \mathbf{x} = \mathbf{x} + \mathbf{0} = \mathbf{x}$
- for each $\mathbf{x}$ in V there exists $\mathbf{y}$ in V such that $\mathbf{x} + \mathbf{y} = \mathbf{0}$ (so far these axioms say that V is an abelian group under the operation of addition)
- $1\mathbf{x} = \mathbf{x}$
- $(ab)\mathbf{x} = a(b\mathbf{x})$
- $a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}$
- $(a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$

When writing things by hand, I shall underline vectors. Using bold face (or underlining) is not a logical necessity, and not all books do it, and I might not always remember to do it, but it does help to keep clear what is a vector and what is not.

A non-empty subset U of a vector space V is called a **subspace** of V if it forms a vector space using the same operations as those in V. It follows that (i) $\mathbf{0} \in U$; (ii) if $\mathbf{u}_1$, $\mathbf{u}_2 \in U$, then $\mathbf{u}_1 + \mathbf{u}_2 \in U$; (iii) if $\mathbf{u} \in U$ and $a \in F$, then $a\mathbf{u} \in U$. We saw that these three properties are equivalent to the definition of being a subspace (and provide a useful test).

Some standard examples of vector spaces:

(i) F^n, the set of all n-tuples $(a_1, \ldots, a_n)$, with each a_i in F, with addition and scalar multiplication done componentwise

(ii) $M_{m\times n}(F)$, the set of all $m \times n$ matrices with entries from F, with addition and scalar multiplication done componentwise

(iii) the set of all polynomials with coefficients in F, with the obvious rules for addition and scalar multiplication

(iv) the subspace of (iii) where we pick r and then restrict to polynomials of degree at most r

We met the delicate subject of **linear independence**. A list of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ is said to be **linearly independent** if and only if the only way to write $\mathbf{0} = a_1\mathbf{v}_1 + \cdots + a_r\mathbf{v}_r$ is to take $a_1 = \cdots = a_r = 0$. (And recall that anything of the shape $a_1\mathbf{v}_1 + \cdots + a_r\mathbf{v}_r$ is a **linear combination** of $\mathbf{v}_1, \ldots, \mathbf{v}_r$.)

The **span** of a set of vectors S, written $\mathrm{span}(S)$, is the set of all (finite) linear combinations $a_1\mathbf{v}_1 + \cdots + a_r\mathbf{v}_r$ with $\mathbf{v}_1, \ldots, \mathbf{v}_r \in S$. (If S is empty, then we include $\mathbf{0}$ in its span, by convention.) For any subset $S \subseteq V$, $\mathrm{span}(S)$ is a subspace of V. Note that $\mathrm{span}(\emptyset) = \{\mathbf{0}\}$.

The set S is a **spanning set** if $\mathrm{span}(S) = V$.

If $\mathrm{span}(S) = V$ and also S is linearly independent, then we say that S is a **basis** for V. (Some good will is needed here in switching between lists and sets.)

If V has a finite spanning set, then we say that V is **finite-dimensional**. We saw that every finite-dimensional vector space has a finite basis, and that the number of elements in such a basis is the same for all bases: this number is the **dimension** of V, written $\dim(V)$. If V is not finite-dimensional, then we write $\dim(V) = \infty$.

For the above four examples of vectors spaces the dimensions are (i) n, (ii) mn, (iii) ∞, (iv) $r + 1$.