1 Introduction to Sturm-Liouville systems

An equation involving a real-valued function $y(x)$ of a real variable x as well as its n lowest derivatives, $y'(x), y''(x), \dots y^{(n)}(x)$, is said to be an n^{th}-order ordinary differential equation. It can be written in the form

$$y^{(n)} = F(y^{(n-1)}, \dots, y', y, x). \tag{1.1}$$

Its general solution will contain n arbitrary constants. To determine values of these arbitrary constants, we need n conditions. Often, those conditions would be specified at one value x_0 of the independent variable, e.g., in the form $y(x_0) = a_0,\ y'(x_0) = a_1,\ \dots, y^{(n-1)}(x_0) = a_{n-1}$, where $a_0, \dots, a_{n-1}$ are real constants. Such a so-called **initial value problem** gives rise to a unique solution.

Example 1.1. The simplest first-order ordinary differential equation is

$$\frac{\mathrm{d}y}{\mathrm{d}x}(x) = y(x), \tag{1.2}$$

i.e., $n = 1$ and $F(y, x) = y$. Its general solution is of the form $y(x) = A\,\mathrm{e}^{x}$ where A is an arbitrary constant. By imposing the condition

$$y(0) = a_0, \tag{1.3}$$

at $x_0 = 0$, where a_0 is a specific constant, the value of the arbitrary constant A is fixed, $A = a_0$.

But what happens if we are thinking about conditions appropriate to plucking a guitar string, say? In such a case, the conditions would be specified for different values of the independent variable, a so-called *separated end-point problem*. Does such a problem have a solution and, if so, is it unique?

We will explore this idea for a second-order differential equation of the type

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}(x) + \lambda y(x) = 0, \tag{1.4}$$

where λ is a real parameter, and the function $y(x)$ is defined on an interval $[a, b]$. The general solution has two arbitrary constants that will be fixed by demanding one condition on $y(x)$ and/or $y'(x)$ at $x = a$ and another condition at $x = b$, i.e., at the boundaries of the interval $[a, b]$. A differential equation subject to such conditions is said to be a **boundary value problem**.

In order to specify the general solution of the equation (1.4) we need to distinguish the three cases where λ is negative, zero or positive:

(a) Letting $\lambda = -\alpha^2 < 0$, $\alpha \in \mathbb{R} \setminus \{0\}$ the equation (1.4) takes the form

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \alpha^2 y = 0, \tag{1.5}$$

and leads to the general solution

$$y_-(x) = A\cosh\alpha x + B\sinh\alpha x, \tag{1.6}$$

where A and B are arbitrary constants. We note the we can restrict to $\alpha > 0$ as $\pm\alpha$ lead to the same λ. Moreover, changing fromα to $-\alpha$ would change the sign of the sinh, but this can be absorbed in the arbitrary constant B.

(b) Letting $\lambda = 0$, the equation (1.4) becomes

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0. \tag{1.7}$$

Integrating twice, the general solution is

$$y_0(x) = cx + d, \tag{1.8}$$

for arbitrary constants c and d.

(c) Letting $\lambda = \beta^2 > 0$, $\beta \in \mathbb{R}\setminus\{0\}$, when the equation (1.4) is

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \beta^2 y = 0, \tag{1.9}$$

Thie general solution is

$$y_+(x) = E\sin\beta x + F\cos\beta x, \tag{1.10}$$

where E and F are arbitrary constants. For the same reason as in (a) we can restrict to $\beta > 0$.

In each of the three cases the boundary conditions will fix the possible values of the constants. Exploring this idea further, we will consider four similar, but slightly different examples.

Remark 1.2. As it will turn out, the equation and boundary conditions that we are going to impose are satisfied by the so-called **trivial solution** $y(x) \equiv 0$, which is not very interesting. So, throughout this chapter, our interest will focus on the existence of any non-trivial solutions.

Example 1.3. Consider the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}(x) + \lambda y(x) = 0\ , \quad x \in [0,\pi], \tag{1.11}$$

subject to the boundary conditions $y(0) = 0$ and $y(\pi) = 0$. Hence, in this case $a = 0$ and $b = \pi$, and the boundary conditions are imposed on $y(x)$, but not on $y'(x)$.

We now go through the three cases (a)–(c) from above:

(a) Letting $\lambda = -\alpha^2 < 0$, $\alpha > 0$, we impose the two boundary conditions on the general solution (1.6). From the first boundary condition, $y(0) = 0$, we get $A = 0$, giving $y(x) = B \sinh \alpha x$. From the second condition, $B \sinh \alpha\pi = 0$ and, since $\alpha \neq 0$, i.e., $\sinh \alpha\pi \neq 0$, we obtain $B = 0$. Combining these gives $y(x) \equiv 0$, the trivial solution. So there is no non-trivial solution for any strictly negative value of λ.

(b) Letting $\lambda = 0$, the boundary condition $y(0) = 0$ imposed on (1.8) implies $d = 0$, giving $y(x) = cx$. From the second condition, $c\pi = 0$ we obtain $c = 0$ and hence, once again, we only have the trivial solution $y(x) \equiv 0$ and no non-trivial solution for $\lambda = 0$.

(c) Letting $\lambda = \beta^2 > 0$, $\beta > 0$, we impose the boundary conditions on the general solution (1.10). From the first boundary condition, $y(0) = 0$, we get $F = 0$, giving $y(x) = E \sin \beta x$, and from the second condition, $E \sin \beta\pi = 0$. Hence, either $E = 0$, giving the trivial solution, or $\sin \beta\pi = 0$. But $\sin n\pi = 0$ when $n \in \mathbb{Z}$. Hence, we have a non-trivial solution for $\beta = n \in \mathbb{N}$ (as $\beta > 0$), giving $\lambda_n = n^2$ and $y_n(x) = E_n \sin nx$, adding a suffix $n \in \mathbb{N}$ to both the value of λ for which we have a non-trivial solution and to that solution itself.

Summarising our findings in this example, we have no solution when either λ is negative or zero, and we have countably infinitely many solutions when λ is positive.

Remark 1.4. We have met a similar situation in MT1820: the matrix equation

$$A\boldsymbol{x} = \lambda \boldsymbol{x} \tag{1.12}$$

only has a non-trivial solution for certain values of λ, the so-called eigenvalues, with $\boldsymbol{x} \neq \mathbf{0}$ the corresponding eigenvectors. Writing our differential equation (1.4) in the more suggestive form

$$-\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \lambda y, \tag{1.13}$$

the values of λ for which there is a non-trivial solution are referred to as the **eigenvalues** and the corresponding non-trivial solutions are referred to as the **eigenfunctions**. We shall meet a rigorous definition of these terms in Chapter 5.

What changes occur, if we modify the boundary conditions?

Example 1.5. Consider the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \lambda y = 0, \quad x \in [0, \pi], \tag{1.14}$$

subject to the boundary conditions $y(0) + y'(0) = 0$ and $y(\pi) = 0$. The only change from Example 1.3 is to the first boundary condition, which now involves $y(0)$ and $y'(0)$.

We now again go through the three cases (a)–(c) from above:

(a) Letting $\lambda = -\alpha^2 < 0$, $\alpha > 0$, the new boundary conditions are imposed on the general solution (1.6). For the first boundary condition we need $y'(x) = \alpha(A \sinh \alpha x + B \cosh \alpha x)$ so that the first condition gives

$$A + \alpha B = 0. \tag{1.15}$$

The second boundary condition gives

$$A \cosh \alpha\pi + B \sinh \alpha\pi = 0. \tag{1.16}$$

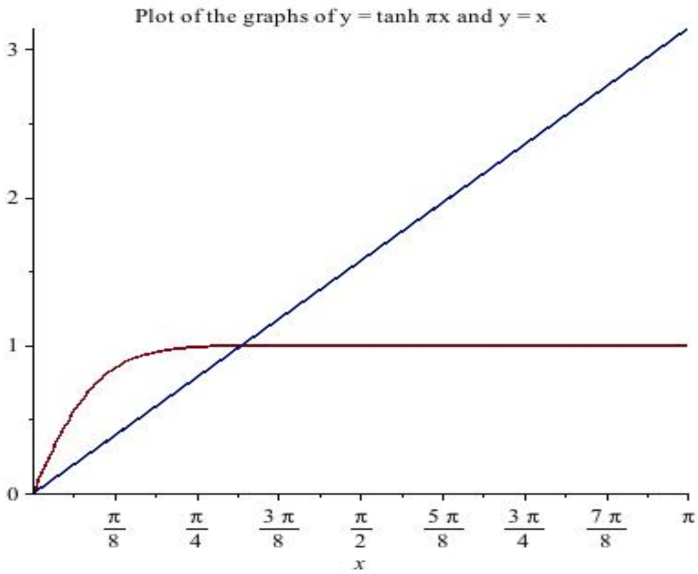
These are two homogeneous linear equations for the two unknowns A and B. These have non-trivial solutions only if

$$\det \begin{pmatrix} 1 & \alpha \\ \cosh \alpha\pi & \sinh \alpha\pi \end{pmatrix} = \sinh \alpha\pi - \alpha \cosh \alpha\pi = 0. \tag{1.17}$$

Since $\cosh \alpha\pi \neq 0$, this gives

$$\tanh \alpha\pi = \alpha, \tag{1.18}$$

which is a transcendental equation for α. In order to find its solution(s), plot the graphs of $y = \alpha$ and $y = \tanh \alpha\pi$, with $\alpha > 0$.



The two graphs have an intersection at the origin (not strictly positive) and at $\alpha = \alpha^* > 0$ say, giving us the eigenvalue $\lambda_{\alpha^*} = -\alpha^{*2}$, with the corresponding eigenfunction

$$y_{\alpha^*}(x) = B\big(-\alpha^* \cosh \alpha^* x + \sinh \alpha^* x\big). \tag{1.19}$$

W.l.o.g., set $B = 1$, because any multiple of $y_{\alpha^*}(x)$ will also satisfy the differential equation and the boundary conditions. Alternatively, using $\alpha^* = \tanh \alpha^*\pi$, we have the following form for the eigenfunction,

$$y_{\alpha^*}(x) = -\sinh \alpha^*\pi \cosh \alpha^* x + \cosh \alpha^*\pi \sinh \alpha^* x = \sinh \alpha^*(x - \pi). \tag{1.20}$$

(b) When $\lambda = 0$, the first boundary condition imposed on the general solution (1.8) gives $d + c = 0$, whilst the second one yields $c\pi + d = 0$. These simultaneous linear equations are satisfied only if $c = d = 0$, giving $y(x) \equiv 0$, i.e. the trivial solution. Hence, there is no non-trivial solution so that $\lambda = 0$ is not an eigenvalue.

(c) Letting $\lambda = \beta^2 > 0$, $\beta > 0$, the boundary conditions must be imposed on the general solution (1.10). For the first boundary condition we need $y'(x) = \beta(E\cos\beta x - F\cos\beta x)$. It then gives

$$F + \beta E = 0, \tag{1.21}$$

whilst the second boundary condition requires

$$E\sin\beta\pi + F\cos\beta\pi = 0. \tag{1.22}$$

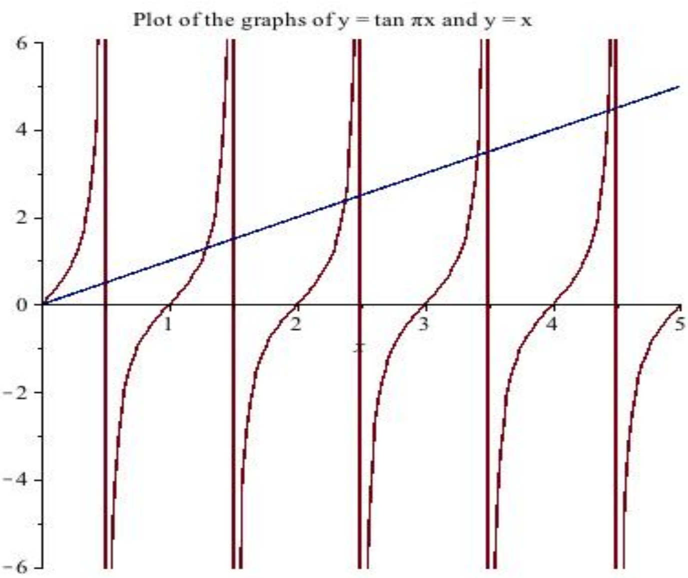
This pair of simultaneous homogeneous linear equations in E and F has non–trivial solution only if

$$\det\begin{pmatrix} \beta & 1 \\ \sin\beta\pi & \cos\beta\pi \end{pmatrix} = \beta\cos\beta\pi - \sin\beta\pi = 0. \tag{1.23}$$

Noting that here $\cos\beta\pi = 0$ would imply $\sin\beta\pi = 0$, giving a contradiction, we may assume that $\cos\beta\pi \neq 0$. This yields the transcendental equation

$$\tan\beta\pi = \beta. \tag{1.24}$$

Now plot the graphs of $y = \beta$ and $y = \tan\beta\pi$ for $\beta > 0$:



The graphs intersect at the origin (not strictly positive), and there exists one intersection in each of the intervals $(n, (n + \frac{1}{2}))$, $n \in \mathbb{N}$. Hence, there is a one-to-one correspondence between the points of intersection and the natural numbers and, therefore, there is a countably infinite number of intersections and hence a countably infinite number of positive eigenvalues. Letting β_n be the intersection point lying in the interval $(n, (n + \frac{1}{2}))$, the eigenfunction corresponding to $\lambda_n = \beta_n^2$ is

$$y_n(x) = \sin\beta_n\pi\cos\beta_n x - \cos\pi\beta_n\sin\beta_n x \equiv \sin\beta_n(\pi - x), \tag{1.25}$$

using an addition theorem.

Modifying the boundary conditions has simply yielded a similar structure to the first example. This time, we do have one negative eigenvalue but retain the countably infinite set of positive eigenvalues. However, whereas the eigenvalues in Example 1.3 are explicit, $\lambda_n = n^2$, they are not known explicitly in Example 1.5 since β_n is a solution of a transcendental equation.

So, exploring further, what happens if we change the interval of definition of the differential equation?

Example 1.6. Consider the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \lambda y = 0, \quad x \in [-\pi, \pi], \tag{1.26}$$

subject to the boundary conditions $y(-\pi) = 0$ and $y(\pi) = 0$.

As previously, we distinguish the three cases:

(a) Letting $\lambda = -\alpha^2 < 0$, $\alpha > 0$, imposing the first boundary condition on the general solution (1.6) gives

$$\begin{aligned} 0 &= A\cosh\alpha(-\pi) + B\sinh\alpha(-\pi) \\ &= A\cosh\alpha\pi - B\sinh\alpha\pi, \end{aligned} \tag{1.27}$$

whilst the second condition gives

$$0 = A\cosh\alpha\pi + B\sinh\alpha\pi. \tag{1.28}$$

Hence, adding and subtracting,

$$A\cosh\alpha\pi = 0 = B\sinh\alpha\pi. \tag{1.29}$$

As $\cosh\alpha\pi \neq 0$ and $\sinh\alpha\pi \neq 0$ (since $\alpha \neq 0$) we conclude $A = 0 = B$. Hence we are left with the trivial solution $y(x) \equiv 0$, so that there is no negative eigenvalue.

(b) When $\lambda = 0$, and the general solution is (1.8), the first boundary condition gives $-c\pi + d = 0$, whilst the second one yields $c\pi + d = 0$. These simultaneous linear equations are satisfied only if $c = d = 0$, giving $y(x) \equiv 0$, i.e., we have the trivial solution. Hence there is no non-trivial solution so that $\lambda = 0$ is not an eigenvalue.

(c) Letting $\lambda = \beta^2 > 0$, $\beta > 0$, the general solution is (1.10). From the first boundary condition we obtain

$$\begin{aligned} 0 &= E \sin \beta(-\pi) + F \cos \beta(-\pi) \\ &= -E \sin \beta\pi + F \cos \beta\pi, \end{aligned} \tag{1.30}$$

whilst the second condition gives

$$0 = E \sin \beta\pi + F \cos \beta\pi. \tag{1.31}$$

This pair of simultaneous homogeneous linear equations in E and F has non–trivial solution only if

$$\det \begin{pmatrix} -\sin \beta\pi & \cos \beta\pi \\ \sin \beta\pi & \cos \beta\pi \end{pmatrix} = -2 \sin \beta\pi \cos \beta\pi = -\sin 2\beta\pi = 0. \tag{1.32}$$

Thus we conclude that

$$\beta = \frac{n}{2}, \quad n \in \mathbb{Z}. \tag{1.33}$$

Letting $n = 2k$, $k \in \mathbb{N}$, we find $\cos \beta\pi = \cos k\pi = \pm 1$ and $\sin \beta\pi = \sin k\pi = 0$ so that the first boundary condition forces $F = 0$. Hence we have eigenvalues $\lambda_k = k^2$, $k \in \mathbb{N}$, with corresponding eigenfunctions $y_k(x) = \sin kx$. However, letting $n = 2k-1$, $k \in \mathbb{N}$, we find $\cos \beta\pi = \cos(k - \frac{1}{2})\pi = 0$ and $\sin \beta\pi = \sin(k - \frac{1}{2})\pi = \pm 1$, so that the second boundary condition forces $E = 0$. Hence we have eigenvalues $\lambda_{k-\frac{1}{2}} = (k - \frac{1}{2})^2$, $k \in \mathbb{N}$, with corresponding eigenfunctions $y_{k-\frac{1}{2}}(x) = \cos(k - \frac{1}{2})x$.

Modifying the interval of definition, to one symmetrical about the origin, has allowed another sequence of eigenvalues and eigenfunctions. We now have a set of odd eigenfunctions (when n is odd) and a set of even ones (when n is even). We retain from Example 1.3 though the fact that there are no negative or zero eigenvalues, and that there are countably infinite positive eigenvalues.

In the final example of this group, we retain this symmetrical interval of definition but modify the boundary conditions. In the first three examples, each boundary condition involved just one end-point. Now, each condition involves both end-points: we still have so-called separated end-point conditions but of a different type.

Example 1.7. Consider the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \lambda y = 0, \quad x \in [-\pi, \pi], \tag{1.34}$$

subject to the boundary conditions $y(-\pi) = y(\pi)$ and $y'(-\pi) = y'(\pi)$.

Again going through the three cases:

(a) Letting $\lambda = -\alpha^2 < 0$, $\alpha > 0$, the general solution is (1.6). The first boundary condition gives

$$\begin{aligned} A \cosh \alpha\pi + B \sinh \alpha\pi &= A \cosh \alpha(-\pi) + B \sinh \alpha(-\pi) \\ &= A \cosh \alpha\pi - B \sinh \alpha\pi, \end{aligned} \tag{1.35}$$

hence

$$2B \sinh \alpha\pi = 0. \tag{1.36}$$

As $\alpha \neq 0$ means $\sinh \alpha\pi \neq 0$ we conclude that $B = 0$, i.e., $y(x) = A \cosh \alpha x$. Hence the second boundary condition gives

$$\begin{aligned} \alpha A \sinh \alpha\pi &= \alpha A \sinh \alpha(-\pi) \\ &= -A\alpha \sinh \alpha\pi, \end{aligned} \tag{1.37}$$

or

$$2A\alpha \sinh \alpha\pi = 0. \tag{1.38}$$

Again, $\alpha \neq 0$ means $\sinh \alpha\pi \neq 0$ which implies $A = 0$, i.e., $y(x) \equiv 0$ is the trivial solution, so that there is no negative eigenvalue.

(b) When $\lambda = 0$ and the general solution is (1.8), the first condition gives $-c\pi + d = c\pi + d$, i.e., $c = 0$, whence, $y(x) = d$. The second condition is trivially satisfied now that $y'(x) = 0$. We therefore have the non-trivial solution $y(x) = d$. Here, $\lambda = 0$ is an eigenvalue, with corresponding eigenfunction $y_0(x) = 1$, w.l.o.g.

(c) Letting $\lambda = \beta^2 > 0$, $\beta > 0$, the general solution is (1.10). Here the first boundary condition gives

$$\begin{aligned} E \sin \beta\pi + F \cos \beta\pi &= E \sin \beta(-\pi) + F \cos \beta(-\pi) \\ &= -E \sin \beta\pi + F \cos \beta\pi, \end{aligned} \tag{1.39}$$

i.e., $2E \sin \beta\pi = 0$. Hence, either $E = 0$ or $\sin \beta\pi = 0$, which is satisfied only if $\beta = n \in \mathbb{N}$. For the second condition we need $y'(x) = \beta(E \cos \beta x - F \sin \beta x)$, and find

$$\begin{aligned} \beta(E \cos \beta\pi - F \sin \beta\pi) &= \beta(E \cos \beta(-\pi) - F \sin \beta(-\pi)) \\ &= \beta(E \cos \beta\pi + F \sin \beta\pi), \end{aligned} \tag{1.40}$$

or $2F \sin \beta\pi = 0$. Hence, either $F = 0$ or $\sin \beta\pi = 0$, which is satisfied only if $\beta = n \in \mathbb{N}$. The choice $E = 0 = F$ gives the trivial solution so that, for non-trivial solution, we must take $\beta = n \in \mathbb{N}$, giving the eigenvalues $\lambda_n = n^2, n \in \mathbb{N}$. With this choice, we may take the corresponding E_n and/or F_n to be non-zero to give the corresponding eigenfunctions. Hence, for this system, we have eigenvalues $\lambda_n = n^2, n \in \mathbb{N}_0$, with corresponding eigenfunctions

$$y_n(x) = \left\{ \begin{array}{cc} 1 & \text{if } n = 0 \\ \left\{ \begin{array}{c} \sin nx \\ \cos nx \end{array} \right\} & \text{if } n \neq 0 \end{array} \right. . \tag{1.41}$$

The bracketing on the eigenfunctions for $n \neq 0$ indicates that a non-trivial linear combination of the functions shown is also an eigenfunction. We retain the infinite set of strictly positive eigenvalues but there are now two linearly independent eigenfunctions corresponding to each eigenvalue.

Remark 1.8. In Example 1.7 we found that there are the two eigenfunctions $\sin nx$ and $\cos nx$ corresponding to the same eigenvalue $\lambda_n = n^2$ for every $n \in \mathbb{N}$. Following Remark 1.4, this corresponds to the case of a matrix A with two linearly independent eigenvectors $\boldsymbol{x}_n$ and $\tilde{\boldsymbol{x}}_n$ that correspond to the same eigenvalue λ_n . In that context one says that the two linearly independent eigenvectors span the two-dimensional eigenspace corresponding to the eigenvalue λ_n, and that λ_n has multiplicity two. We carry over this terminology to the case of boundary value problems.