Ordinary Differential Equations

Seperable - trivial solution

First order linear - integrating factor method

e.g. $\frac{dy}{dx} + \frac{1}{x} y = x^2$

(if coefficient of $\frac{dy}{dx} \neq 1$, divide through by this coefficient)

$p(x) = \frac{1}{x}$.

$\Rightarrow \mu(x) = \exp\left[\int \frac{1}{x}\, dx\right] = \exp\left[\ln x + c\right]$

$= C_2 x$ for constant C_2.

Multiply eqn. by x

$\Rightarrow x\left(\frac{dy}{dx} + \frac{1}{x} y\right) = x^3$

$\Rightarrow \frac{d}{dx}(xy) = x^3$

$\Rightarrow xy = \frac{x^4}{4} + C_1$

Bernoulli - "almost first order linear"

e.g. $2\frac{dy}{dx} + \frac{4}{x}y = y^3x^5$

$\Rightarrow \frac{2}{y^3}\frac{dy}{dx} + \frac{4}{x}\frac{y}{y^3} = x^5$

$\Rightarrow 2y^{-3}\frac{dy}{dx} + \frac{4}{x}y^{-2} = x^5$

Let $v(x) = y^{-2}(x) \Rightarrow \frac{dv}{dx} = -2y^{-3}\frac{dy}{dx}$

$\Rightarrow \frac{dv}{dx} - \frac{4}{x}v = -x^5$

This is first order linear.

Homogeneous - n^{th} order linear with "constant" term $= 0$

$$2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 2y = 0$$

Let $y = e^{mx}$

$$(2m^2 + 5m + 2)e^{mx} = 0$$

$\Rightarrow 2m^2 + 5m + 2 = 0 \quad \because e^{mx} > 0 \quad \forall x$

$\Rightarrow (2m+1)(m+2) = 0$

$\Rightarrow m_1 = -\frac{1}{2}, \quad m_2 = -2$

General solution

$$y(x) = c_1 e^{-\frac{1}{2}x} + c_2 e^{-2x}$$

(constants c_1, c_2) PTO

(for complex solutions, remember $e^{i\theta} = \cos\theta + i\sin\theta$)

i Real roots:

$$y_c = c_1 e^{m_1 x} + c_2 e^{m_1 x}$$

ii Repeated root:

$$y_c = c_1 x e^{mx} + c_2 e^{mx} = (c_1 x + c_2)(e^{mx})$$

Complex roots

$$y_c = e^{\alpha x}\left[c_1 \cos \beta x + c_2 \sin \beta x\right]$$

$$(m = \alpha \pm i\beta)$$

Note: solutions i and ii generalise intuitively to n^{th} order DEs.

Non-homogeneous - e.g. $2\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 6y = 3e^{2x}$

Find solution $y_c(x)$ to complimentary equation:

$$2\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 6y = \cancel{3e^{2x}}\ 0$$

$$\vdots$$

$$y_c = c_1 e^{3x} + c_2 e^{-x}$$

Particular integral

Trial functions

RHS	particular integral trial sol.
constant	c
linear	$ax + b$
quadratic	$ax^2 + bx + c$
exponential (e^{px})	ke^{px}
$\sin(px)$ / $\cos(px)$	$a\cos(px) + b\sin(px)$

Continuing example from above...

$RHS = 3e^{2x}$

$\Rightarrow$ Let $y_p = ke^{2x}$

$\Rightarrow \frac{dy}{dx} = 2ke^{2x}$

and $\frac{d^2y}{dx^2} = 4ke^{2x}$

$\Rightarrow 2(4ke^{2x}) - 4(2ke^{2x}) - 6(ke^{2x}) = 3e^{2x}$

$\Rightarrow k = -\frac{1}{2} \Rightarrow y_p(x) = -\frac{1}{2}e^{2x}$

General solution:

$y(x) = y_c(x) + y_p(x)$

$= c_1 e^{3x} + c_2 e^{-x} - \frac{1}{2}e^{2x}$