2 Fourier Series

2.1 Introduction

A continuous function f of a single variable is periodic with period 2π if

$$f(x + 2\pi) = f(x) \tag{2.1}$$

holds for all $x \in \mathbb{R}$. It then follows that $f(x) = f(x + 2n\pi)$ for all $x \in \mathbb{R}$ and all $n \in \mathbb{Z}$. Hence it is sufficient know the function on the interval $[-\pi, \pi]$; it can be extended to all $x \in \mathbb{R}$ by using the property (2.1). On the other hand, using (2.1) every continuous function that is defined on the interval $[-\pi, \pi]$ can be extended to all of $\mathbb{R}$ as a periodic function with period 2π. As an example take the function $f(x) = x$ for $x \in [-\pi, \pi]$. The graph of its continuation as a periodic function is shown in Figure 2.1 One sees that although the

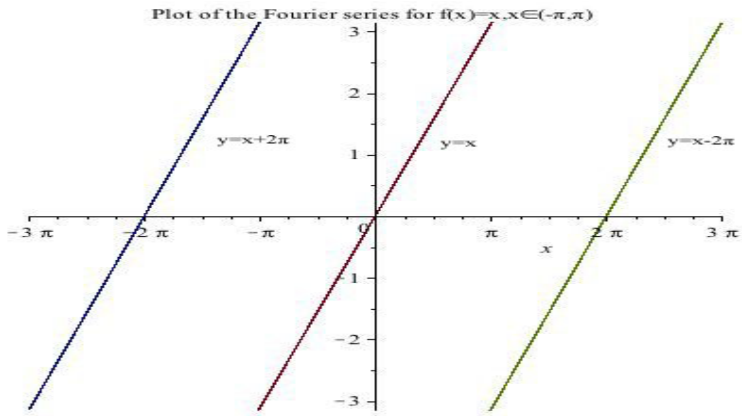


Figure 1: Continuation of the function $f(x)$ defined on $[-\pi, \pi]$

function is continuous on the interval $[-\pi, \pi]$, its extension to $\mathbb{R}$ has discontinuities at the points $(2n + 1)\pi$, $n \in \mathbb{Z}$. We will later see that this is a typical situation.

Other well known examples of periodic functions with period 2π are $\sin nx$ and $\cos nx$, where $n \in \mathbb{Z}$. Any (finite) linear combination

$$\sum_{n \in I} (a_n \cos nx + b_n \sin nx), \tag{2.2}$$

where $I \subset \mathbb{Z}$ is a finite subset, is also a periodic function. We take this example as a motivation for the following definition.

Definition 2.1. A series of the form

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{2.3}$$

is said to be a **trigonometric series**. On the set of points x where a trigonometric series converges, it defines a function f, whose value at x is the sum of the series for that value of x,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \tag{2.4}$$

Such a trigonometric series is said to be the **Fourier series** for f.

Remark 2.2. Since $\sin(-nx) = -\sin(nx)$ and $\cos(-nx) = \cos(nx)$, a trigonometric series does not require terms with negative n. The contribution with $n = 0$ is represented by the first term in the trigonometric series. The factor $\frac{1}{2}$ is conventional and turns out to be useful.

Proposition 2.3. *Assume that the Fourier series* (2.3) *converges at $x \in \mathbb{R}$. Then it converges at $x + 2\pi k$ for every $k \in \mathbb{Z}$ and, moreover, $f(x) = f(x + 2\pi k)$.*

Proof. Since each of the terms appearing in the Fourier series is periodic, with period 2π, the series converges for all x whenever it converges on $[-\pi, \pi]$. Hence, given $f(x)$ with $x \in [-\pi, \pi]$ defined by its Fourier series,

$$\begin{aligned} f(x + 2\pi) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n(x + 2\pi) + b_n \sin n(x + 2\pi)) \\ &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \\ &= f(x), \end{aligned} \tag{2.5}$$

so that the series defines a periodic function of x with period 2π. □

Now assume that the trigonometric series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{2.6}$$

converges on $[-\pi, \pi]$ and defines a function f,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \quad \text{for all } x \in [-\pi, \pi]. \tag{2.7}$$

Given the function f, how can one determine the coefficients a_n and b_n?

We first need the a useful quantity.

Definition 2.4. The **Kronecker delta** is defined by

$$\delta_{mn} = \begin{cases} 1, & \text{if } m = n \\ 0, & \text{if } m \neq n \end{cases} \quad \text{for all } m, n \in \mathbb{Z}. \tag{2.8}$$

With this we we get the following result.

Proposition 2.5. *For all $m, n \in \mathbb{Z}$,*

$$\begin{aligned}
\int_{-\pi}^{\pi} \sin mx \sin nx \, \mathrm{d}x &= \pi\delta_{mn}, \\
\int_{-\pi}^{\pi} \cos mx \cos nx \, \mathrm{d}x &= \pi\delta_{mn}, \\
\int_{-\pi}^{\pi} \sin mx \cos nx \, \mathrm{d}x &= 0.
\end{aligned} \tag{2.9}$$

Proof.

$$\begin{aligned}
\int_{-\pi}^{\pi} \sin mx \sin nx \, \mathrm{d}x &= \frac{1}{2}\int_{-\pi}^{\pi} \big(\cos(n-m)x - \cos(n+m)x\big)\, \mathrm{d}x \\
&= \frac{1}{2}\begin{cases} \left[x - \frac{\sin(n+m)x}{n+m}\right]_{-\pi}^{\pi} & \text{if } m = n \\ \left[\frac{\sin(n-m)x}{n-m} - \frac{\sin(n+m)x}{n+m}\right]_{-\pi}^{\pi} & \text{if } m \neq n \end{cases} \\
&= \begin{cases} \pi & \text{if } m = n \\ 0 & \text{if } m \neq n\ . \end{cases} \\
&= \pi\delta_{mn}.
\end{aligned} \tag{2.10}$$

The other cases are similar. □

These integrals can be used to find the coefficients a_n and b_n in the Fourier series of a function.

Proposition 2.6. *Assume that the trigonometric series*

$$f(x) = \frac{1}{2}a_0 + \sum_{m=1}^{\infty}(a_m \cos mx + b_m \sin mx) \tag{2.11}$$

converges for all $x \in [-\pi, \pi]$. Then the **Euler-Fourier formulae**

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, \mathrm{d}x \quad n = 0, 1, 2, \ldots, \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, \mathrm{d}x \quad n = 1, 2, 3, \ldots,
\end{aligned} \tag{2.12}$$

hold.

We remark that the coefficients in (2.12) are usually called **Euler-Fourier coefficients**.

Proof. We use (2.11), multiply with $\sin nx$ and integrate over the interval $[-\pi,\pi]$ with the help of Proposition 2.5,

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x &= \frac{1}{2}a_0\int_{-\pi}^{\pi}\sin nx\,\mathrm{d}x + \int_{-\pi}^{\pi}\sum_{m=1}^{\infty}(a_m\cos mx + b_m\sin mx)\sin nx\,\mathrm{d}x \\
&= \frac{1}{2}a_0\Big[-\frac{\cos nx}{n}\Big]_{-\pi}^{\pi} \\
&\quad + \sum_{m=1}^{\infty}a_m\int_{-\pi}^{\pi}\cos mx\sin nx\,\mathrm{d}x + \sum_{m=1}^{\infty}b_m\int_{-\pi}^{\pi}\sin mx\sin nx\,\mathrm{d}x \\
&= \sum_{m=1}^{\infty}b_m\pi\delta_{mn} = \pi b_n.
\end{aligned} \tag{2.13}$$

Similarly, for $n \neq 0$,

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x &= \frac{1}{2}a_0\int_{-\pi}^{\pi}\sin nx\,\mathrm{d}x + \int_{-\pi}^{\pi}\sum_{m=1}^{\infty}(a_m\cos mx + b_m\sin mx)\cos nx\,\mathrm{d}x \\
&= \frac{1}{2}a_0\Big[-\frac{\cos nx}{n}\Big]_{-\pi}^{\pi} \\
&\quad + \sum_{m=1}^{\infty}a_m\int_{-\pi}^{\pi}\cos mx\cos nx\,\mathrm{d}x + \sum_{m=1}^{\infty}b_m\int_{-\pi}^{\pi}\sin mx\cos nx\,\mathrm{d}x \\
&= \sum_{m=1}^{\infty}a_m\pi\delta_{mn} = \pi a_n
\end{aligned} \tag{2.14}$$

When $n = 0$,

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x &= \frac{1}{2}a_0\int_{-\pi}^{\pi}\mathrm{d}x + \int_{-\pi}^{\pi}\sum_{m=1}^{\infty}(a_m\cos mx + b_m\sin mx)\,\mathrm{d}x \\
&= \frac{1}{2}a_0\,2\pi + \sum_{m=1}^{\infty}a_m\int_{-\pi}^{\pi}\cos mx\,\mathrm{d}x + \sum_{m=1}^{\infty}b_m\int_{-\pi}^{\pi}\sin mx\,\mathrm{d}x \\
&= \pi a_0.
\end{aligned} \tag{2.15}$$

□

2.2 Fourier Theorems

So far, we considered trigonometric series with given Euler-Fourier coefficients. If the series converges, it defines a periodic function from which the Euler-Fourier coefficients can be recovered with the help of the Euler-Fourier formulae. Now we we look at the following problem: a periodic function f (or simply a function on the in terval $[-\pi,\pi]$) is given and we apply the Euler-Fourier formulae to obtain coefficients a_n and b_n. Using these coefficients we can set up the trigonometric series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \tag{2.16}$$

We may then ask:

- Does the series converge?
- Is it the Fourier series of the original function f?

A theorem giving conditions under which a trigonometric series converges to the function used in the definition of its coefficients is said to be a **Fourier Theorem**.

Remark 2.7. Examples exist to show that the trigonometric series corresponding to a function f may not converge to $f(x)$, or may not even converge at all.

The Fourier Theorem which we shall use gives sufficient conditions for the convergence of a Fourier series; the conditions are by no means necessary. Nor are they the most general ones guaranteeing convergence. But they do include ideas from the example shown in Figure 2.1, which motivates the following.

Definition 2.8. A function f defined on an interval $[a, b]$ is said to be **piecewise continuous**, if the interval can be subdivided by a finite number of points,

$$a = x_0 < x_1 < \ldots < x_n = b, \tag{2.17}$$

such that

(i) f is continuous on each open subinterval (x_{k-1}, x_k), $k = 1 \ldots, n$,

(ii) f has a finite limit as the end-points of each subinterval are approached from within the subinterval.

Remark 2.9. We use the following notation:

$$\begin{aligned} f(x_k^+) &= \lim_{x \searrow x_k} f(x), \\ f(x_k^-) &= \lim_{x \nearrow x_k} f(x). \end{aligned} \tag{2.18}$$

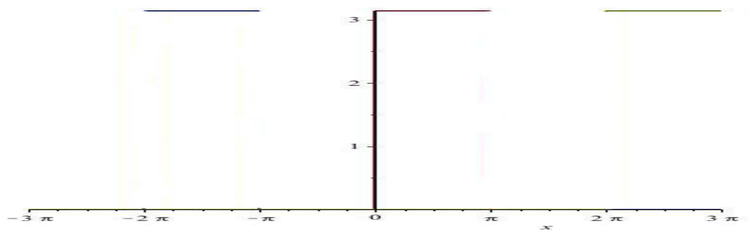
(Here the first limit is the limit $x \to x_k$ taken with the condition $x > x_k$, whereas the second limit is taken with the condition $x < x_k$.)

Note that the definition of a piecewise continuous function does **not** require that $f(x_k^+) = f(x_k^-)$, but only that these limits exist for all $k = 0, \ldots, n$. This means that a piecewise continuous function are allowed to have finite jumps at the points x_k.

Example 2.10. Determine the Fourier series for the function f defined as

$$f(x) = \begin{cases} 0, & \text{if } -\pi < x < 0 \\ \pi, & \text{if } 0 < x < \pi \end{cases}, \tag{2.19}$$

and $f(x + 2\pi) = f(x)$ for all x. We do not specify a value for $f(x_k)$ at the points of discontinuity $x_k = k\pi$, $k \in \mathbb{Z}$. The only condition we impose is that such a value shall be finite.



Restricted to the interval $[a, b] = [-\pi, \pi]$, this function is piecewise continuous when we introduce the subdivision

$$a = x_0 = -\pi < x_1 = 0 < x_2 = b = \pi. \tag{2.20}$$

Calculating the Euler-Fourier coefficients,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{\pi}\int_{0}^{\pi} \pi\,\mathrm{d}x = \pi. \tag{2.21}$$

Similarly, for $n \in \mathbb{N}$,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x = \frac{1}{\pi}\int_{0}^{\pi} \pi\cos nx\,\mathrm{d}x \\ &= \left[\frac{\sin nx}{n}\right]_0^{\pi} = 0, \end{aligned} \tag{2.22}$$

and for $n \in \mathbb{N}_0$,

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x = \frac{1}{\pi}\int_{0}^{\pi} \pi\sin nx\,\mathrm{d}x \\ &= \left[\frac{-\cos nx}{n}\right]_0^{\pi} = \frac{1-\cos n\pi}{n} = \frac{1-(-1)^n}{n} \\ &= \begin{cases} 0, & \text{if } n = 2k\,,\ k \in \mathbb{N} \\ \frac{2}{2k-1}, & \text{if } n = 2k-1\,,\ k \in \mathbb{N}. \end{cases} \end{aligned} \tag{2.23}$$

Hence, the trigonometric series associated with f is

$$\frac{\pi}{2} + 2\sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1}. \tag{2.24}$$

We may now ask whether this trigonometric series converges and, if so, whether it converges to the function (2.19). In order to answer this question we need a Fourier Theorem.

Theorem 2.11 (A Fourier Theorem)**.** *Let f be a periodic function with period. 2π. Assume that, when restricted to the interval $[-\pi, \pi]$, the function f and its derivative f' are piecewise continuous on the interval $[-\pi, \pi]$. Let a_n and b_n be given by the Euler-Fourier formulae for f. Then, for $x \in (-\pi, \pi)$, the trigonometric series*

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{2.25}$$

converges to

(i) *$f(x)$, if f is continuous at x,*

(ii) *$\frac{f(x_+)+f(x_-)}{2}$ if f is discontinuous at x,*

(iii) *$\frac{f(-\pi_+)+f(\pi_-)}{2}$ at $x = \pm\pi$.*

We note that at a point of discontinuity, x, where the function jumps from $f(x_-)$ to $f(x_+)$, the trigonometric series converges to the mid-point of the jump. At the end-points $\pm\pi$ of the interval the series converges to the mid-point of the end-values $f(-\pi_+)$ and $f(\pi_-)$.

We can apply this theorem to Example 2.10 as we already identified the function f to be piecewise continuous, and its derivative, $f'(x) = 0$, is also piecewise continuous with the same subdivision of the interval. Hence, the series converges to 0 for $x \in (-\pi, 0)$ and to π for $x \in (-\pi, 0)$. At the origin, it converges to

$$\frac{f(0_+) + f(0_-)}{2} = \frac{\pi + 0}{2} = \frac{\pi}{2}, \tag{2.26}$$

and at $x = \pm\pi$ it converges to

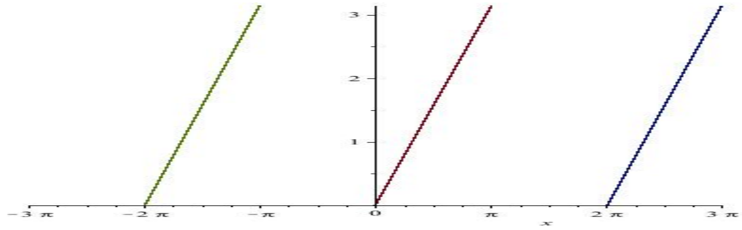
$$\frac{f(-\pi_+) + f(\pi_-)}{2} = \frac{0 + \pi}{2} = \frac{\pi}{2}. \tag{2.27}$$

Above we did not specify a value for f at these points. here we see that the most convenient value to assign to $f(x)$ at the points of discontinuity $x_k = k\pi$, $k \in \mathbb{Z}$, is $\frac{\pi}{2}$; then the series converges to the value of f at these points. If $f(x)$ were assigned any other value at the points of discontinuity, we would have an example of the Fourier series converging to a finite value which is not the value of the function at these points.

Example 2.12. Determine the Fourier series for the function f defined as

$$f(x) = \begin{cases} 0, & \text{if } -\pi < x \leq 0 \\ x, & \text{if } 0 \leq x < \pi \end{cases}, \tag{2.28}$$

with $f(x + 2\pi) = f(x)$ for all x; sum the resulting series.



When restricted to the interval $[a, b] = [-\pi, \pi]$, this function is continuous. Its derivative is the function in Example 2.10, which is piecewise continuous on the same interval with the subdivision (2.20). When restricted to the interval $[-\pi, \pi]$, the function (2.28), therefore, satisfies the assumptions of the Fourier Theorem.

Evaluating the Euler-Fourier coefficients

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{\pi}\int_{0}^{\pi} x\,\mathrm{d}x = \frac{\pi}{2}. \tag{2.29}$$

Similarly, for $n \in \mathbb{N}$,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x = \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,\mathrm{d}x \\ &= \frac{1}{n\pi}\big[x\sin nx\big]_0^{\pi} - \frac{1}{n\pi}\int_0^{\pi}\sin nx\,\mathrm{d}x \\ &= \left[\frac{\cos nx}{n^2\pi}\right]_0^{\pi} = \frac{1}{n^2\pi}[(-1)^n - 1] \\ &= \begin{cases} 0, & \text{if } n = 2k\,,\ k \in \mathbb{N} \\ -\frac{2}{(2k-1)^2\pi}, & \text{if } n = 2k-1\,,\ k \in \mathbb{N}, \end{cases} \end{aligned} \tag{2.30}$$

and

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x = \frac{1}{\pi}\int_{0}^{\pi} x\sin nx\,\mathrm{d}x \\ &= \left[\frac{-x\cos nx}{n\pi}\right]_0^{\pi} + \int_0^{\pi}\cos nx\,\mathrm{d}x \\ &= -\frac{\cos n\pi}{n} = \frac{(-1)^{n+1}}{n}. \end{aligned} \tag{2.31}$$

Hence, the trigonometric series generated from the function f is

$$\frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{\pi n^2}\cos nx + \frac{(-1)^{n+1}}{n}\sin nx\right). \tag{2.32}$$

Since we observed that f defined in (2.28) satisfies the conditions of the Fourier Theorem, the trigonometric series is the Fourier series for f and converges to the value x, $x \in [0, \pi)$ and to the value 0, $x \in (-\pi, 0]$. Note that the function is continuous at $x = 0$. At $x = \pm\pi$, the series converges to

$$\frac{f(-\pi_+) + f(\pi_-)}{2} = \frac{\pi}{2}. \tag{2.33}$$

2.3 Evaluation of some infinite sums

Evaluating the series at particular values will generate formulae that can be used to sum certain specific series. E.g., evaluating (2.32) at $x = 0$, a point of continuity, gives

$$0 = f(0) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{1}{n^2 \pi}[(-1)^n - 1]. \tag{2.34}$$

In other words,

$$-\frac{\pi}{4} = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2}[(-1)^n - 1] \tag{2.35}$$

This can be rearranged to give

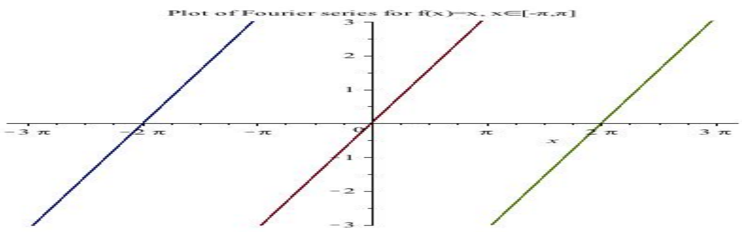
$$\frac{\pi^2}{8} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots \tag{2.36}$$

At $x = \pi$, a point of discontinuity, an application of the Fourier Theorem assigns the value $\frac{\pi}{2}$ (see (2.33)) to the series

$$\frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{1}{n^2 \pi}[(-1)^n - 1] \cos n\pi = \frac{\pi}{4} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}[(-1)^n - 1]. \tag{2.37}$$

This brings us back to (2.35), and finally to (2.36).

Example 2.13. Determine the Fourier series for the function $f(x) = x$, $x \in (-\pi, \pi)$, with $f(x + 2\pi) = f(x)$ for all x.



Calculating the Euler-Fourier formulae we find that

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x \, \mathrm{d}x = 0, \tag{2.38}$$

and for $n \in \mathbb{N}$,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos nx \, \mathrm{d}x = 0; \tag{2.39}$$

in both cases as we are integrating an odd function over a symmetric interval. Furthermore, for $n \in \mathbb{N}$,

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,\mathrm{d}x = \frac{1}{\pi}\int_{0}^{\pi} x\sin nx\,\mathrm{d}x + \frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\,\mathrm{d}x \\ &= \frac{2}{\pi}\int_{0}^{\pi} x\sin nx\,\mathrm{d}x = \frac{2}{\pi}\Big[-\frac{x}{n}\cos nx\Big]_0^{\pi} + \frac{2}{n\pi}\int_0^{\pi}\cos nx\,\mathrm{d}x \\ &= -\frac{2}{n}\cos n\pi + 0 = -\frac{2}{n}(-1)^n. \end{aligned} \tag{2.40}$$

The resulting trigonometric series is

$$2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx. \tag{2.41}$$

Since f and f' are both continuous on $(-\pi,\pi)$, the Fourier Theorem applies and implies that the series converges at all points of the interval to the value x, and at the endpoints $\pm\pi$ to

$$\frac{f(-\pi_+)+f(\pi_-)}{2} = \frac{-\pi+\pi}{2} = 0. \tag{2.42}$$

Remark 2.14. Considering Examples 2.12 and 2.13, we have two distinct series, (2.32) and (2.41), both of which converge to x for $x \in (0,\pi)$ but take distinct values on $(-\pi, 0)$.

Evaluating the series (2.41) at $x = \frac{\pi}{2}$, where the function f is continuous and takes the value $\frac{\pi}{2}$, we first note that

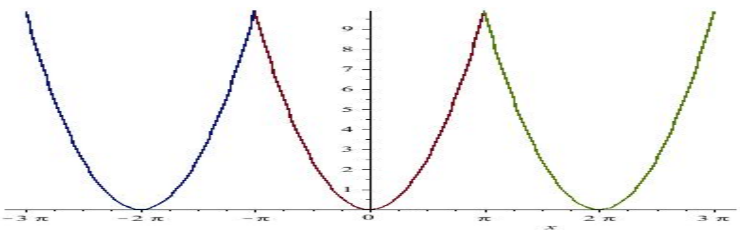
$$\sin n\frac{\pi}{2} = \begin{cases} \sin k\pi = 0, & \text{if } n = 2k, \quad k \in \mathbb{N} \\ \sin(2k-1)\frac{\pi}{2} = (-1)^{k+1}, & \text{if } n = 2k-1, \quad k \in \mathbb{N} \end{cases}. \tag{2.43}$$

Hence,

$$\frac{\pi}{4} = \sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{2k-1}. \tag{2.44}$$

This is another example of a particular infinite series that can be evaluated with the current method. And one final example before we explore some of the relationships between Fourier series, what we may and may not do with them.

Example 2.15. Determine the Fourier series for the function $f(x) = x^2$, $x \in (-\pi,\pi)$, with $f(x+2\pi) = f(x)$ for all x.



The Euler-Fourier formulae give the following coefficients:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \,\mathrm{d}x = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{2\pi^2}{3}, \tag{2.45}$$

and, for $n \in \mathbb{N}$,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx \,\mathrm{d}x = \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos nx \,\mathrm{d}x \\ &= \frac{2}{n\pi}\left[x^2 \sin nx\right]_0^{\pi} - \frac{4}{n\pi}\int_0^{\pi} x \sin nx \,\mathrm{d}x \\ &= -\frac{4}{n\pi}\left[-\frac{\pi}{n}\cos nx\right]_0^{\pi} = \frac{4}{n^2}(-1)^n. \end{aligned} \tag{2.46}$$

Moreover, for $n \in \mathbb{N}$,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx \,\mathrm{d}x = 0, \tag{2.47}$$

since an odd function is integrated over a symmetric interval. The resulting trigonometric series is

$$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx. \tag{2.48}$$

Since both f and f' are continuous over $(-\pi, \pi)$, the Fourier Theorem applies and implies that the series converges to x^2 on this interval and to $\frac{f(-\pi_+)+f(\pi_-)}{2} = \pi^2$ at $x = \pm\pi$.

Since $x = 0$ is a point of continuity the series converges to $f(0) = 0$, i.e.,

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos 0, \tag{2.49}$$

which can be rewritten as

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}. \tag{2.50}$$

Evaluating at $x = \pi$, however, we have

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos n\pi, \tag{2.51}$$

or

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}. \tag{2.52}$$