1 Introduction to Sturm-Liouville systems

An equation involving a real-valued function $y(x)$ of a real variable x as well as its n lowest derivatives, $y'(x), y''(x), \dots y^{(n)}(x)$, is said to be an n^{th}-order ordinary differential equation. It can be written in the form

$$y^{(n)} = F(y^{(n-1)}, \dots, y', y, x). \tag{1.1}$$

Its general solution will contain n arbitrary constants. To determine values of these arbitrary constants, we need n conditions. Often, those conditions would be specified at one value x_0 of the independent variable, e.g., in the form $y(x_0) = a_0,\ y'(x_0) = a_1,\ \dots, y^{(n-1)}(x_0) = a_{n-1}$, where $a_0, \dots, a_{n-1}$ are real constants. Such a so-called **initial value problem** gives rise to a unique solution.

Example 1.1. The simplest first-order ordinary differential equation is

$$\frac{\mathrm{d}y}{\mathrm{d}x}(x) = y(x), \tag{1.2}$$

i.e., $n = 1$ and $F(y, x) = y$. Its general solution is of the form $y(x) = A\,\mathrm{e}^x$ where A is an arbitrary constant. By imposing the condition

$$y(0) = a_0, \tag{1.3}$$

at $x_0 = 0$, where a_0 is a specific constant, the value of the arbitrary constant A is fixed, $A = a_0$.

But what happens if we are thinking about conditions appropriate to plucking a guitar string, say? In such a case, the conditions would be specified for different values of the independent variable, a so-called *separated end-point problem.* Does such a problem have a solution and, if so, is it unique?

We will explore this idea for a second-order differential equation of the type

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}(x) + \lambda y(x) = 0, \tag{1.4}$$

where λ is a real parameter, and the function $y(x)$ is defined on an interval $[a, b]$. The general solution has two arbitrary constants that will be fixed by demanding one condition on $y(x)$ and/or $y'(x)$ at $x = a$ and another condition at $x = b$, i.e., at the boundaries of the interval $[a, b]$. A differential equation subject to such conditions is said to be a **boundary value problem**.

In order to specify the general solution of the equation (1.4) we need to distinguish the three cases where λ is negative, zero or positive:

(a) Letting $\lambda = -\alpha^2 < 0$, $\alpha \in \mathbb{R} \setminus \{0\}$ the equation (1.4) takes the form

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \alpha^2 y = 0, \tag{1.5}$$

and leads to the general solution

$$y_-(x) = A\cosh \alpha x + B\sinh \alpha x, \tag{1.6}$$

where A and B are arbitrary constants. We note the we can restrict to $\alpha > 0$ as $\pm\alpha$ lead to the same λ. Moreover, changing fromα to $-\alpha$ would change the sign of the sinh, but this can be absorbed in the arbitrary constant B.

(b) Letting $\lambda = 0$, the equation (1.4) becomes

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 0. \tag{1.7}$$

Integrating twice, the general solution is

$$y_0(x) = cx + d, \tag{1.8}$$

for arbitrary constants c and d.

(c) Letting $\lambda = \beta^2 > 0$, $\beta \in \mathbb{R} \setminus \{0\}$, when the equation (1.4) is

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \beta^2 y = 0, \tag{1.9}$$

Thie general solution is

$$y_+(x) = E\sin \beta x + F\cos \beta x, \tag{1.10}$$

where E and F are arbitrary constants. For the same reason as in (a) we can restrict to $\beta > 0$.

In each of the three cases the boundary conditions will fix the possible values of the constants. Exploring this idea further, we will consider four similar, but slightly different examples.

Remark 1.2. As it will turn out, the equation and boundary conditions that we are going to impose are satisfied by the so-called **trivial solution** $y(x) \equiv 0$, which is not very interesting. So, throughout this chapter, our interest will focus on the existence of any non-trivial solutions.

Example 1.3. Consider the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}(x) + \lambda y(x) = 0\ , \quad x \in [0, \pi], \tag{1.11}$$

subject to the boundary conditions $y(0) = 0$ and $y(\pi) = 0$. Hence, in this case $a = 0$ and $b = \pi$, and the boundary conditions are imposed on $y(x)$, but not on $y'(x)$.

We now go through the three cases (a)–(c) from above:

(a) Letting $\lambda = -\alpha^2 < 0$, $\alpha > 0$, we impose the two boundary conditions on the general solution (1.6). From the first boundary condition, $y(0) = 0$, we get $A = 0$, giving $y(x) = B \sinh \alpha x$. From the second condition, $B \sinh \alpha\pi = 0$ and, since $\alpha \neq 0$, i.e., $\sinh \alpha\pi \neq 0$, we obtain $B = 0$. Combining these gives $y(x) \equiv 0$, the trivial solution. So there is no non-trivial solution for any strictly negative value of λ.

(b) Letting $\lambda = 0$, the boundary condition $y(0) = 0$ imposed on (1.8) implies $d = 0$, giving $y(x) = cx$. From the second condition, $c\pi = 0$ we obtain $c = 0$ and hence, once again, we only have the trivial solution $y(x) \equiv 0$ and no non-trivial solution for $\lambda = 0$.

(c) Letting $\lambda = \beta^2 > 0$, $\beta > 0$, we impose the boundary conditions on the general solution (1.10). From the first boundary condition, $y(0) = 0$, we get $F = 0$, giving $y(x) = E \sin \beta x$, and from the second condition, $E \sin \beta\pi = 0$. Hence, either $E = 0$, giving the trivial solution, or $\sin \beta\pi = 0$. But $\sin n\pi = 0$ when $n \in \mathbb{Z}$. Hence, we have a non-trivial solution for $\beta = n \in \mathbb{N}$ (as $\beta > 0$), giving $\lambda_n = n^2$ and $y_n(x) = E_n \sin nx$, adding a suffix $n \in \mathbb{N}$ to both the value of λ for which we have a non-trivial solution and to that solution itself.

Summarising our findings in this example, we have no solution when either λ is negative or zero, and we have countably infinitely many solutions when λ is positive.

Remark 1.4. We have met a similar situation in MT1820: the matrix equation

$$A\boldsymbol{x} = \lambda \boldsymbol{x} \tag{1.12}$$

only has a non-trivial solution for certain values of λ, the so-called eigenvalues, with $\boldsymbol{x} \neq \mathbf{0}$ the corresponding eigenvectors. Writing our differential equation (1.4) in the more suggestive form

$$-\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \lambda y, \tag{1.13}$$

the values of λ for which there is a non-trivial solution are referred to as the **eigenvalues** and the corresponding non-trivial solutions are referred to as the **eigenfunctions**. We shall meet a rigorous definition of these terms in Chapter 5.

What changes occur, if we modify the boundary conditions?

Example 1.5. Consider the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \lambda y = 0, \quad x \in [0, \pi], \tag{1.14}$$

subject to the boundary conditions $y(0) + y'(0) = 0$ and $y(\pi) = 0$. The only change from Example 1.3 is to the first boundary condition, which now involves $y(0)$ and $y'(0)$.

We now again go through the three cases (a)–(c) from above:

(a) Letting $\lambda = -\alpha^2 < 0$, $\alpha > 0$, the new boundary conditions are imposed on the general solution (1.6). For the first boundary condition we need $y'(x) = \alpha(A\sinh\alpha x + B\cosh\alpha x)$ so that the first condition gives

$$A + \alpha B = 0. \tag{1.15}$$

The second boundary condition gives

$$A\cosh\alpha\pi + B\sinh\alpha\pi = 0. \tag{1.16}$$

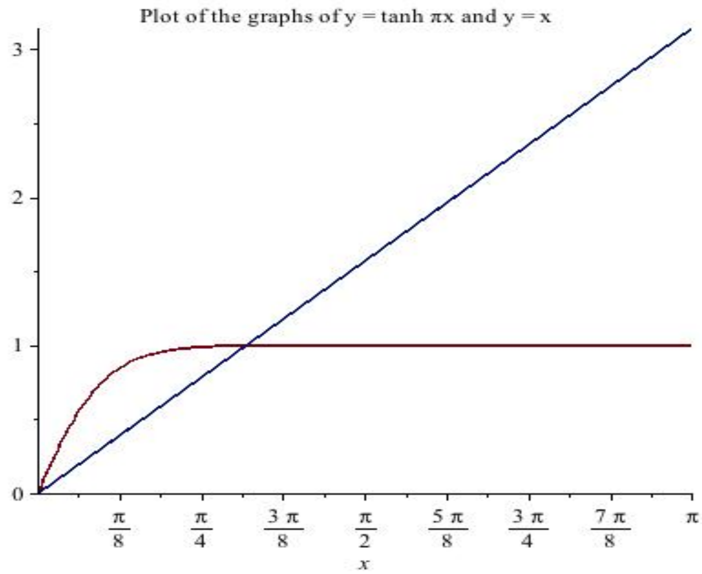
These are two homogeneous linear equations for the two unknowns A and B. These have non-trivial solutions only if

$$\det\begin{pmatrix} 1 & \alpha \\ \cosh\alpha\pi & \sinh\alpha\pi \end{pmatrix} = \sinh\alpha\pi - \alpha\cosh\alpha\pi = 0. \tag{1.17}$$

Since $\cosh\alpha\pi \neq 0$, this gives

$$\tanh\alpha\pi = \alpha, \tag{1.18}$$

which is a transcendental equation for α. In order to find its solution(s), plot the graphs of $y = \alpha$ and $y = \tanh\alpha\pi$, with $\alpha > 0$.



The two graphs have an intersection at the origin (not strictly positive) and at $\alpha = \alpha^* > 0$ say, giving us the eigenvalue $\lambda_{\alpha^*} = -\alpha^{*2}$, with the corresponding eigenfunction

$$y_{\alpha^*}(x) = B\big(-\alpha^*\cosh\alpha^* x + \sinh\alpha^* x\big). \tag{1.19}$$

W.l.o.g., set $B = 1$, because any multiple of $y_{\alpha^*}(x)$ will also satisfy the differential equation and the boundary conditions. Alternatively, using $\alpha^* = \tanh\alpha^*\pi$, we have the following form for the eigenfunction,

$$y_{\alpha^*}(x) = -\sinh\alpha^*\pi\cosh\alpha^* x + \cosh\alpha^*\pi\sinh\alpha^* x = \sinh\alpha^*(x - \pi). \tag{1.20}$$

(b) When $\lambda = 0$, the first boundary condition imposed on the general solution (1.8) gives $d + c = 0$, whilst the second one yields $c\pi + d = 0$. These simultaneous linear equations are satisfied only if $c = d = 0$, giving $y(x) \equiv 0$, i.e. the trivial solution. Hence, there is no non-trivial solution so that $\lambda = 0$ is not an eigenvalue.

(c) Letting $\lambda = \beta^2 > 0$, $\beta > 0$, the boundary conditions must be imposed on the general solution (1.10). For the first boundary condition we need $y'(x) = \beta(E\cos\beta x - F\cos\beta x)$. It then gives

$$F + \beta E = 0, \tag{1.21}$$

whilst the second boundary condition requires

$$E\sin\beta\pi + F\cos\beta\pi = 0. \tag{1.22}$$

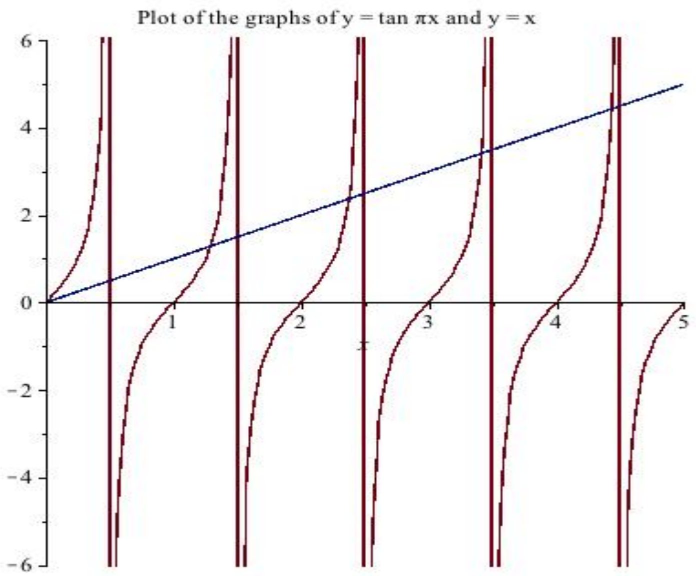
This pair of simultaneous homogeneous linear equations in E and F has non–trivial solution only if

$$\det\begin{pmatrix} \beta & 1 \\ \sin\beta\pi & \cos\beta\pi \end{pmatrix} = \beta\cos\beta\pi - \sin\beta\pi = 0. \tag{1.23}$$

Noting that here $\cos\beta\pi = 0$ would imply $\sin\beta\pi = 0$, giving a contradiction, we may assume that $\cos\beta\pi \neq 0$. This yields the transcendental equation

$$\tan\beta\pi = \beta. \tag{1.24}$$

Now plot the graphs of $y = \beta$ and $y = \tan\beta\pi$ for $\beta > 0$:



The graphs intersect at the origin (not strictly positive), and there exists one intersection in each of the intervals $(n, (n + \frac{1}{2}))$, $n \in \mathbb{N}$. Hence, there is a one-to-one correspondence between the points of intersection and the natural numbers and, therefore, there is a countably infinite number of intersections and hence a countably infinite number of positive eigenvalues. Letting β_n be the intersection point lying in the interval $(n, (n+\frac{1}{2}))$, the eigenfunction corresponding to $\lambda_n = \beta_n^2$ is

$$y_n(x) = \sin\beta_n\pi\cos\beta_n x - \cos\pi\beta_n\sin\beta_n x \equiv \sin\beta_n(\pi - x), \qquad (1.25)$$

using an addition theorem.

Modifying the boundary conditions has simply yielded a similar structure to the first example. This time, we do have one negative eigenvalue but retain the countably infinite set of positive eigenvalues. However, whereas the eigenvalues in Example 1.3 are explicit, $\lambda_n = n^2$, they are not known explicitly in Example 1.5 since β_n is a solution of a transcendental equation.

So, exploring further, what happens if we change the interval of definition of the differential equation?

Example 1.6. Consider the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \lambda y = 0, \quad x \in [-\pi, \pi], \qquad (1.26)$$

subject to the boundary conditions $y(-\pi) = 0$ and $y(\pi) = 0$.

As previously, we distinguish the three cases:

(a) Letting $\lambda = -\alpha^2 < 0$, $\alpha > 0$, imposing the first boundary condition on the general solution (1.6) gives

$$\begin{aligned} 0 &= A\cosh\alpha(-\pi) + B\sinh\alpha(-\pi) \\ &= A\cosh\alpha\pi - B\sinh\alpha\pi, \end{aligned} \qquad (1.27)$$

whilst the second condition gives

$$0 = A\cosh\alpha\pi + B\sinh\alpha\pi. \qquad (1.28)$$

Hence, adding and subtracting,

$$A\cosh\alpha\pi = 0 = B\sinh\alpha\pi. \qquad (1.29)$$

As $\cosh\alpha\pi \neq 0$ and $\sinh\alpha\pi \neq 0$ (since $\alpha \neq 0$) we conclude $A = 0 = B$. Hence we are left with the trivial solution $y(x) \equiv 0$, so that there is no negative eigenvalue.

(b) When $\lambda = 0$, and the general solution is (1.8), the first boundary condition gives $-c\pi + d = 0$, whilst the second one yields $c\pi + d = 0$. These simultaneous linear equations are satisfied only if $c = d = 0$, giving $y(x) \equiv 0$, i.e., we have the trivial solution. Hence there is no non-trivial solution so that $\lambda = 0$ is not an eigenvalue.

(c) Letting $\lambda = \beta^2 > 0$, $\beta > 0$, the general solution is (1.10). From the first boundary condition we obtain

$$\begin{aligned} 0 &= E \sin \beta(-\pi) + F \cos \beta(-\pi) \\ &= -E \sin \beta\pi + F \cos \beta\pi, \end{aligned} \tag{1.30}$$

whilst the second condition gives

$$0 = E \sin \beta\pi + F \cos \beta\pi. \tag{1.31}$$

This pair of simultaneous homogeneous linear equations in E and F has non–trivial solution only if

$$\det \begin{pmatrix} -\sin \beta\pi & \cos \beta\pi \\ \sin \beta\pi & \cos \beta\pi \end{pmatrix} = -2 \sin \beta\pi \cos \beta\pi = -\sin 2\beta\pi = 0. \tag{1.32}$$

Thus we conclude that

$$\beta = \frac{n}{2}, \quad n \in \mathbb{Z}. \tag{1.33}$$

Letting $n = 2k$, $k \in \mathbb{N}$, we find $\cos \beta\pi = \cos k\pi = \pm 1$ and $\sin \beta\pi = \sin k\pi = 0$ so that the first boundary condition forces $F = 0$. Hence we have eigenvalues $\lambda_k = k^2$, $k \in \mathbb{N}$, with corresponding eigenfunctions $y_k(x) = \sin kx$. However, letting $n = 2k-1$, $k \in \mathbb{N}$, we find $\cos \beta\pi = \cos(k - \frac{1}{2})\pi = 0$ and $\sin \beta\pi = \sin(k - \frac{1}{2})\pi = \pm 1$, so that the second boundary condition forces $E = 0$. Hence we have eigenvalues $\lambda_{k-\frac{1}{2}} = (k - \frac{1}{2})^2$, $k \in \mathbb{N}$, with corresponding eigenfunctions $y_{k-\frac{1}{2}}(x) = \cos(k - \frac{1}{2})x$.

Modifying the interval of definition, to one symmetrical about the origin, has allowed another sequence of eigenvalues and eigenfunctions. We now have a set of odd eigenfunctions (when n is odd) and a set of even ones (when n is even). We retain from Example 1.3 though the fact that there are no negative or zero eigenvalues, and that there are countably infinite positive eigenvalues.

In the final example of this group, we retain this symmetrical interval of definition but modify the boundary conditions. In the first three examples, each boundary condition involved just one end-point. Now, each condition involves both end-points: we still have so-called separated end-point conditions but of a different type.

Example 1.7. Consider the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \lambda y = 0, \quad x \in [-\pi, \pi], \tag{1.34}$$

subject to the boundary conditions $y(-\pi) = y(\pi)$ and $y'(-\pi) = y'(\pi)$.

Again going through the three cases:

(a) Letting $\lambda = -\alpha^2 < 0$, $\alpha > 0$, the general solution is (1.6). The first boundary condition gives

$$\begin{aligned} A \cosh \alpha\pi + B \sinh \alpha\pi &= A \cosh \alpha(-\pi) + B \sinh \alpha(-\pi) \\ &= A \cosh \alpha\pi - B \sinh \alpha\pi, \end{aligned} \tag{1.35}$$

hence

$$2B \sinh \alpha\pi = 0. \tag{1.36}$$

As $\alpha \neq 0$ means $\sinh \alpha\pi \neq 0$ we conclude that $B = 0$, i.e., $y(x) = A \cosh \alpha x$. Hence the second boundary condition gives

$$\begin{aligned} \alpha A \sinh \alpha\pi &= \alpha A \sinh \alpha(-\pi) \\ &= -A\alpha \sinh \alpha\pi, \end{aligned} \tag{1.37}$$

or

$$2A\alpha \sinh \alpha\pi = 0. \tag{1.38}$$

Again, $\alpha \neq 0$ means $\sinh \alpha\pi \neq 0$ which implies $A = 0$, i.e., $y(x) \equiv 0$ is the trivial solution, so that there is no negative eigenvalue.

(b) When $\lambda = 0$ and the general solution is (1.8), the first condition gives $-c\pi + d = c\pi + d$, i.e., $c = 0$, whence, $y(x) = d$. The second condition is trivially satisfied now that $y'(x) = 0$. We therefore have the non-trivial solution $y(x) = d$. Here, $\lambda = 0$ is an eigenvalue, with corresponding eigenfunction $y_0(x) = 1$, w.l.o.g.

(c) Letting $\lambda = \beta^2 > 0$, $\beta > 0$, the general solution is (1.10). Here the first boundary condition gives

$$\begin{aligned} E \sin \beta\pi + F \cos \beta\pi &= E \sin \beta(-\pi) + F \cos \beta(-\pi) \\ &= -E \sin \beta\pi + F \cos \beta\pi, \end{aligned} \tag{1.39}$$

i.e., $2E \sin \beta\pi = 0$. Hence, either $E = 0$ or $\sin \beta\pi = 0$, which is satisfied only if $\beta = n \in \mathbb{N}$. For the second condition we need $y'(x) = \beta(E \cos \beta x - F \sin \beta x)$, and find

$$\begin{aligned} \beta(E \cos \beta\pi - F \sin \beta\pi) &= \beta(E \cos \beta(-\pi) - F \sin \beta(-\pi)) \\ &= \beta(E \cos \beta\pi + F \sin \beta\pi), \end{aligned} \tag{1.40}$$

or $2F \sin \beta\pi = 0$. Hence, either $F = 0$ or $\sin \beta\pi = 0$, which is satisfied only if $\beta = n \in \mathbb{N}$. The choice $E = 0 = F$ gives the trivial solution so that, for non-trivial solution, we must take $\beta = n \in \mathbb{N}$, giving the eigenvalues $\lambda_n = n^2, n \in \mathbb{N}$. With this choice, we may take the corresponding E_n and/or F_n to be non-zero to give the corresponding eigenfunctions. Hence, for this system, we have eigenvalues $\lambda_n = n^2, n \in \mathbb{N}_0$, with corresponding eigenfunctions

$$y_n(x) = \left\{ \begin{array}{cc} 1 & \text{if } n = 0 \\ \left\{ \begin{array}{c} \sin nx \\ \cos nx \end{array} \right\} & \text{if } n \neq 0 \end{array} \right. . \tag{1.41}$$

The bracketing on the eigenfunctions for $n \neq 0$ indicates that a non-trivial linear combination of the functions shown is also an eigenfunction. We retain the infinite set of strictly positive eigenvalues but there are now two linearly independent eigenfunctions corresponding to each eigenvalue.

Remark 1.8. In Example 1.7 we found that there are the two eigenfunctions $\sin nx$ and $\cos nx$ corresponding to the same eigenvalue $\lambda_n = n^2$ for every $n \in \mathbb{N}$. Following Remark 1.4, this corresponds to the case of a matrix A with two linearly independent eigenvectors $\boldsymbol{x}_n$ and $\tilde{\boldsymbol{x}}_n$ that correspond to the same eigenvalue λ_n . In that context one says that the two linearly independent eigenvectors span the two-dimensional eigenspace corresponding to the eigenvalue λ_n, and that λ_n has multiplicity two. We carry over this terminology to the case of boundary value problems.

2 Fourier Series

2.1 Introduction

A continuous function f of a single variable is periodic with period 2π if

$$f(x + 2\pi) = f(x) \tag{2.1}$$

holds for all $x \in \mathbb{R}$. It then follows that $f(x) = f(x + 2n\pi)$ for all $x \in \mathbb{R}$ and all $n \in \mathbb{Z}$. Hence it is sufficient know the function on the interval $[-\pi, \pi]$; it can be extended to all $x \in \mathbb{R}$ by using the property (2.1). On the other hand, using (2.1) every continuous function that is defined on the interval $[-\pi, \pi]$ can be extended to all of $\mathbb{R}$ as a periodic function with period 2π. As an example take the function $f(x) = x$ for $x \in [-\pi, \pi]$. The graph of its continuation as a periodic function is shown in Figure 2.1 One sees that although the

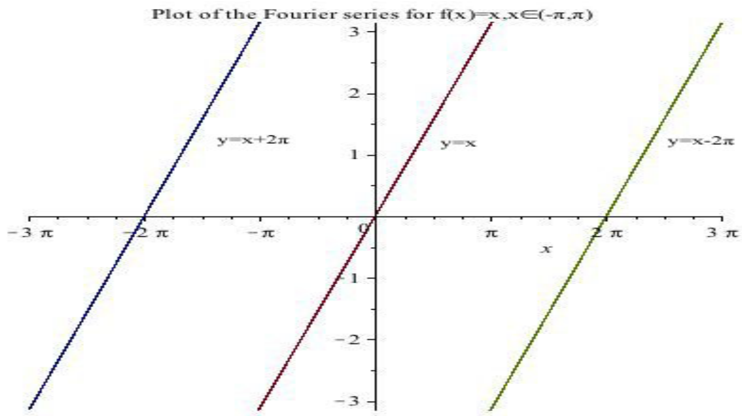


Figure 1: Continuation of the function $f(x)$ defined on $[-\pi, \pi]$

function is continuous on the interval $[-\pi, \pi]$, its extension to $\mathbb{R}$ has discontinuities at the points $(2n + 1)\pi$, $n \in \mathbb{Z}$. We will later see that this is a typical situation.

Other well known examples of periodic functions with period 2π are $\sin nx$ and $\cos nx$, where $n \in \mathbb{Z}$. Any (finite) linear combination

$$\sum_{n\in I}(a_n \cos nx + b_n \sin nx), \tag{2.2}$$

where $I \subset \mathbb{Z}$ is a finite subset, is also a periodic function. We take this example as a motivation for the following definition.

Definition 2.1. A series of the form

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{2.3}$$

is said to be a **trigonometric series**. On the set of points x where a trigonometric series converges, it defines a function f, whose value at x is the sum of the series for that value of x,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \tag{2.4}$$

Such a trigonometric series is said to be the **Fourier series** for f.

Remark 2.2. Since $\sin(-nx) = -\sin(nx)$ and $\cos(-nx) = \cos(nx)$, a trigonometric series does not require terms with negative n. The contribution with $n = 0$ is represented by the first term in the trigonometric series. The factor $\frac{1}{2}$ is conventional and turns out to be useful.

Proposition 2.3. *Assume that the Fourier series* (2.3) *converges at $x \in \mathbb{R}$. Then it converges at $x + 2\pi k$ for every $k \in \mathbb{Z}$ and, moreover, $f(x) = f(x + 2\pi k)$.*

Proof. Since each of the terms appearing in the Fourier series is periodic, with period 2π, the series converges for all x whenever it converges on $[-\pi, \pi]$. Hence, given $f(x)$ with $x \in [-\pi, \pi]$ defined by its Fourier series,

$$\begin{aligned} f(x + 2\pi) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n(x + 2\pi) + b_n \sin n(x + 2\pi)) \\ &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \\ &= f(x), \end{aligned} \tag{2.5}$$

so that the series defines a periodic function of x with period 2π. $\square$

Now assume that the trigonometric series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{2.6}$$

converges on $[-\pi, \pi]$ and defines a function f,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \quad \text{for all } x \in [-\pi, \pi]. \tag{2.7}$$

Given the function f, how can one determine the coefficients a_n and b_n?

We first need the a useful quantity.

Definition 2.4. The **Kronecker delta** is defined by

$$\delta_{mn} = \begin{cases} 1, & \text{if } m = n \\ 0, & \text{if } m \neq n \end{cases} \quad \text{for all } m, n \in \mathbb{Z}. \tag{2.8}$$

With this we we get the following result.

Proposition 2.5. *For all $m, n \in \mathbb{Z}$,*

$$\begin{aligned} \int_{-\pi}^{\pi} \sin mx \sin nx \, \mathrm{d}x &= \pi\delta_{mn}, \\ \int_{-\pi}^{\pi} \cos mx \cos nx \, \mathrm{d}x &= \pi\delta_{mn}, \\ \int_{-\pi}^{\pi} \sin mx \cos nx \, \mathrm{d}x &= 0. \end{aligned} \tag{2.9}$$

Proof.

$$\begin{aligned} \int_{-\pi}^{\pi} \sin mx \sin nx \, \mathrm{d}x &= \frac{1}{2}\int_{-\pi}^{\pi} \big(\cos(n-m)x - \cos(n+m)x\big) \, \mathrm{d}x \\ &= \frac{1}{2}\begin{cases} \left[x - \frac{\sin(n+m)x}{n+m}\right]_{-\pi}^{\pi} & \text{if } m = n \\ \left[\frac{\sin(n-m)x}{n-m} - \frac{\sin(n+m)x}{n+m}\right]_{-\pi}^{\pi} & \text{if } m \neq n \end{cases} \\ &= \begin{cases} \pi & \text{if } m = n \\ 0 & \text{if } m \neq n \ . \end{cases} \\ &= \pi\delta_{mn}. \end{aligned} \tag{2.10}$$

The other cases are similar. □

These integrals can be used to find the coefficients a_n and b_n in the Fourier series of a function.

Proposition 2.6. *Assume that the trigonometric series*

$$f(x) = \frac{1}{2}a_0 + \sum_{m=1}^{\infty}(a_m \cos mx + b_m \sin mx) \tag{2.11}$$

converges for all $x \in [-\pi, \pi]$. Then the **Euler-Fourier formulae**

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, \mathrm{d}x \quad n = 0, 1, 2, \dots, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, \mathrm{d}x \quad n = 1, 2, 3, \dots, \end{aligned} \tag{2.12}$$

hold.

We remark that the coefficients in (2.12) are usually called **Euler-Fourier coefficients**.

Proof. We use (2.11), multiply with $\sin nx$ and integrate over the interval $[-\pi, \pi]$ with the help of Proposition 2.5,

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x &= \frac{1}{2}a_0\int_{-\pi}^{\pi}\sin nx\,\mathrm{d}x + \int_{-\pi}^{\pi}\sum_{m=1}^{\infty}(a_m\cos mx + b_m\sin mx)\sin nx\,\mathrm{d}x\\ &= \frac{1}{2}a_0\Big[-\frac{\cos nx}{n}\Big]_{-\pi}^{\pi}\\ &\quad+\sum_{m=1}^{\infty}a_m\int_{-\pi}^{\pi}\cos mx\sin nx\,\mathrm{d}x + \sum_{m=1}^{\infty}b_m\int_{-\pi}^{\pi}\sin mx\sin nx\,\mathrm{d}x\\ &= \sum_{m=1}^{\infty}b_m\pi\delta_{mn} = \pi b_n.\end{aligned}\tag{2.13}$$

Similarly, for $n \neq 0$,

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x &= \frac{1}{2}a_0\int_{-\pi}^{\pi}\sin nx\,\mathrm{d}x + \int_{-\pi}^{\pi}\sum_{m=1}^{\infty}(a_m\cos mx + b_m\sin mx)\cos nx\,\mathrm{d}x\\ &= \frac{1}{2}a_0\Big[-\frac{\cos nx}{n}\Big]_{-\pi}^{\pi}\\ &\quad+\sum_{m=1}^{\infty}a_m\int_{-\pi}^{\pi}\cos mx\cos nx\,\mathrm{d}x + \sum_{m=1}^{\infty}b_m\int_{-\pi}^{\pi}\sin mx\cos nx\,\mathrm{d}x\\ &= \sum_{m=1}^{\infty}a_m\pi\delta_{mn} = \pi a_n\end{aligned}\tag{2.14}$$

When $n = 0$,

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x &= \frac{1}{2}a_0\int_{-\pi}^{\pi}\mathrm{d}x + \int_{-\pi}^{\pi}\sum_{m=1}^{\infty}(a_m\cos mx + b_m\sin mx)\,\mathrm{d}x\\ &= \frac{1}{2}a_0\,2\pi + \sum_{m=1}^{\infty}a_m\int_{-\pi}^{\pi}\cos mx\,\mathrm{d}x + \sum_{m=1}^{\infty}b_m\int_{-\pi}^{\pi}\sin mx\,\mathrm{d}x\\ &= \pi a_0.\end{aligned}\tag{2.15}$$

□

2.2 Fourier Theorems

So far, we considered trigonometric series with given Euler-Fourier coefficients. If the series converges, it defines a periodic function from which the Euler-Fourier coefficients can be recovered with the help of the Euler-Fourier formulae. Now we we look at the following problem: a periodic function f (or simply a function on the in terval $[-\pi, \pi]$) is given and we apply the Euler-Fourier formulae to obtain coefficients a_n and b_n. Using these coefficients we can set up the trigonometric series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \tag{2.16}$$

We may then ask:

- Does the series converge?
- Is it the Fourier series of the original function f?

A theorem giving conditions under which a trigonometric series converges to the function used in the definition of its coefficients is said to be a **Fourier Theorem**.

Remark 2.7. Examples exist to show that the trigonometric series corresponding to a function f may not converge to $f(x)$, or may not even converge at all.

The Fourier Theorem which we shall use gives sufficient conditions for the convergence of a Fourier series; the conditions are by no means necessary. Nor are they the most general ones guaranteeing convergence. But they do include ideas from the example shown in Figure 2.1, which motivates the following.

Definition 2.8. A function f defined on an interval $[a, b]$ is said to be **piecewise continuous**, if the interval can be subdivided by a finite number of points,

$$a = x_0 < x_1 < \ldots < x_n = b, \tag{2.17}$$

such that

(i) f is continuous on each open subinterval (x_{k-1}, x_k), $k = 1 \ldots, n$,

(ii) f has a finite limit as the end-points of each subinterval are approached from within the subinterval.

Remark 2.9. We use the following notation:

$$\begin{aligned} f(x_k^+) &= \lim_{x \searrow x_k} f(x), \\ f(x_k^-) &= \lim_{x \nearrow x_k} f(x). \end{aligned} \tag{2.18}$$

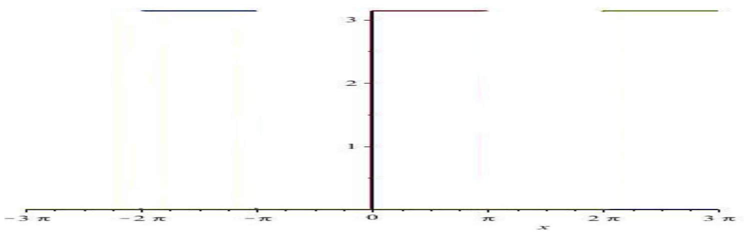
(Here the first limit is the limit $x \to x_k$ taken with the condition $x > x_k$, whereas the second limit is taken with the condition $x < x_k$.)

Note that the definition of a piecewise continuous function does **not** require that $f(x_k^+) = f(x_k^-)$, but only that these limits exist for all $k = 0, \ldots, n$. This means that a piecewise continuous function are allowed to have finite jumps at the points x_k.

Example 2.10. Determine the Fourier series for the function f defined as

$$f(x) = \begin{cases} 0, & \text{if } -\pi < x < 0 \\ \pi, & \text{if } 0 < x < \pi \end{cases}, \tag{2.19}$$

and $f(x+2\pi) = f(x)$ for all x. We do not specify a value for $f(x_k)$ at the points of discontinuity $x_k = k\pi$, $k \in \mathbb{Z}$. The only condition we impose is that such a value shall be finite.



Restricted to the interval $[a,b] = [-\pi,\pi]$, this function is piecewise continuous when we introduce the subdivision

$$a = x_0 = -\pi < x_1 = 0 < x_2 = b = \pi. \tag{2.20}$$

Calculating the Euler-Fourier coefficients,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} \pi\,\mathrm{d}x = \pi. \tag{2.21}$$

Similarly, for $n \in \mathbb{N}$,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} \pi\cos nx\,\mathrm{d}x \\ &= \left[\frac{\sin nx}{n}\right]_0^{\pi} = 0, \end{aligned} \tag{2.22}$$

and for $n \in \mathbb{N}_0$,

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} \pi\sin nx\,\mathrm{d}x \\ &= \left[\frac{-\cos nx}{n}\right]_0^{\pi} = \frac{1-\cos n\pi}{n} = \frac{1-(-1)^n}{n} \\ &= \begin{cases} 0, & \text{if } n = 2k\,,\ k \in \mathbb{N} \\ \frac{2}{2k-1}, & \text{if } n = 2k-1\,,\ k \in \mathbb{N}. \end{cases} \end{aligned} \tag{2.23}$$

Hence, the trigonometric series associated with f is

$$\frac{\pi}{2} + 2\sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1}. \tag{2.24}$$

We may now ask whether this trigonometric series converges and, if so, whether it converges to the function (2.19). In order to answer this question we need a Fourier Theorem.

Theorem 2.11 (A Fourier Theorem)**.** *Let f be a periodic function with period. 2π. Assume that, when restricted to the interval $[-\pi,\pi]$, the function f and its derivative f' are piecewise continuous on the interval $[-\pi,\pi]$. Let a_n and b_n be given by the Euler-Fourier formulae for f. Then, for $x \in (-\pi,\pi)$, the trigonometric series*

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{2.25}$$

converges to

(i) $f(x)$, if f is continuous at x,

(ii) $\frac{f(x_+)+f(x_-)}{2}$ if f is discontinuous at x,

(iii) $\frac{f(-\pi_+)+f(\pi_-)}{2}$ at $x = \pm\pi$.

We note that at a point of discontinuity, x, where the function jumps from $f(x_-)$ to $f(x_+)$, the trigonometric series converges to the mid-point of the jump. At the end-points $\pm\pi$ of the interval the series converges to the mid-point of the end-values $f(-\pi_+)$ and $f(\pi_-)$.

We can apply this theorem to Example 2.10 as we already identified the function f to be piecewise continuous, and its derivative, $f'(x) = 0$, is also piecewise continuous with the same subdivision of the interval. Hence, the series converges to 0 for $x \in (-\pi, 0)$ and to π for $x \in (-\pi, 0)$. At the origin, it converges to

$$\frac{f(0_+) + f(0_-)}{2} = \frac{\pi + 0}{2} = \frac{\pi}{2}, \tag{2.26}$$

and at $x = \pm\pi$ it converges to

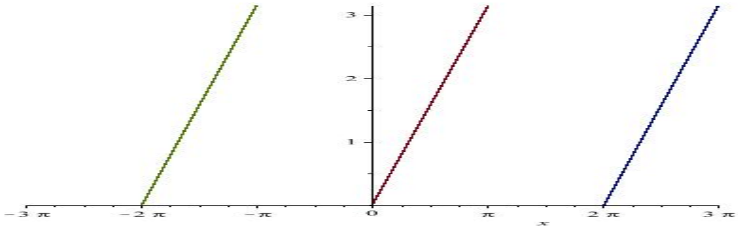
$$\frac{f(-\pi_+) + f(\pi_-)}{2} = \frac{0 + \pi}{2} = \frac{\pi}{2}. \tag{2.27}$$

Above we did not specify a value for f at these points. here we see that the most convenient value to assign to $f(x)$ at the points of discontinuity $x_k = k\pi$, $k \in \mathbb{Z}$, is $\frac{\pi}{2}$; then the series converges to the value of f at these points. If $f(x)$ were assigned any other value at the points of discontinuity, we would have an example of the Fourier series converging to a finite value which is not the value of the function at these points.

Example 2.12. Determine the Fourier series for the function f defined as

$$f(x) = \begin{cases} 0, & \text{if } -\pi < x \leq 0 \\ x, & \text{if } 0 \leq x < \pi \end{cases}, \tag{2.28}$$

with $f(x + 2\pi) = f(x)$ for all x; sum the resulting series.



When restricted to the interval $[a, b] = [-\pi, \pi]$, this function is continuous. Its derivative is the function in Example 2.10, which is piecewise continuous on the same interval with the subdivision (2.20). When restricted to the interval $[-\pi, \pi]$, the function (2.28), therefore, satisfies the assumptions of the Fourier Theorem.

Evaluating the Euler-Fourier coefficients

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{\pi}\int_{0}^{\pi} x\,\mathrm{d}x = \frac{\pi}{2}. \tag{2.29}$$

Similarly, for $n \in \mathbb{N}$,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x = \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,\mathrm{d}x \\ &= \frac{1}{n\pi}\big[x\sin nx\big]_0^{\pi} - \frac{1}{n\pi}\int_0^{\pi}\sin nx\,\mathrm{d}x \\ &= \left[\frac{\cos nx}{n^2\pi}\right]_0^{\pi} = \frac{1}{n^2\pi}[(-1)^n - 1] \\ &= \begin{cases} 0, & \text{if } n = 2k\,,\ k \in \mathbb{N} \\ -\frac{2}{(2k-1)^2\pi}, & \text{if } n = 2k-1\,,\ k \in \mathbb{N}, \end{cases} \end{aligned} \tag{2.30}$$

and

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x = \frac{1}{\pi}\int_{0}^{\pi} x\sin nx\,\mathrm{d}x \\ &= \left[\frac{-x\cos nx}{n\pi}\right]_0^{\pi} + \int_0^{\pi}\cos nx\,\mathrm{d}x \\ &= -\frac{\cos n\pi}{n} = \frac{(-1)^{n+1}}{n}. \end{aligned} \tag{2.31}$$

Hence, the trigonometric series generated from the function f is

$$\frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{\pi n^2}\cos nx + \frac{(-1)^{n+1}}{n}\sin nx\right). \tag{2.32}$$

Since we observed that f defined in (2.28) satisfies the conditions of the Fourier Theorem, the trigonometric series is the Fourier series for f and converges to the value x, $x \in [0, \pi)$ and to the value 0, $x \in (-\pi, 0]$. Note that the function is continuous at $x = 0$. At $x = \pm\pi$, the series converges to

$$\frac{f(-\pi_+) + f(\pi_-)}{2} = \frac{\pi}{2}. \tag{2.33}$$

2.3 Evaluation of some infinite sums

Evaluating the series at particular values will generate formulae that can be used to sum certain specific series. E.g., evaluating (2.32) at $x = 0$, a point of continuity, gives

$$0 = f(0) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{1}{n^2 \pi}[(-1)^n - 1]. \tag{2.34}$$

In other words,

$$-\frac{\pi}{4} = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2}[(-1)^n - 1] \tag{2.35}$$

This can be rearranged to give

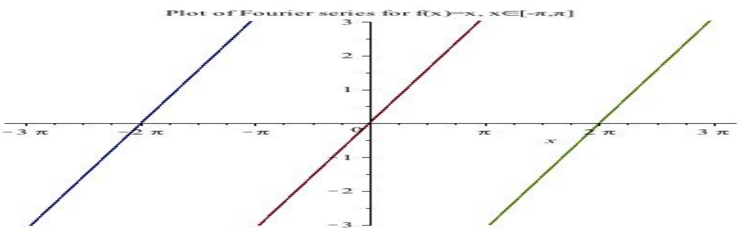
$$\frac{\pi^2}{8} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots \tag{2.36}$$

At $x = \pi$, a point of discontinuity, an application of the Fourier Theorem assigns the value $\frac{\pi}{2}$ (see (2.33)) to the series

$$\frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{1}{n^2 \pi}[(-1)^n - 1] \cos n\pi = \frac{\pi}{4} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}[(-1)^n - 1]. \tag{2.37}$$

This brings us back to (2.35), and finally to (2.36).

Example 2.13. Determine the Fourier series for the function $f(x) = x$, $x \in (-\pi, \pi)$, with $f(x + 2\pi) = f(x)$ for all x.



Calculating the Euler-Fourier formulae we find that

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x \, \mathrm{d}x = 0, \tag{2.38}$$

and for $n \in \mathbb{N}$,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos nx \, \mathrm{d}x = 0; \tag{2.39}$$

in both cases as we are integrating an odd function over a symmetric interval. Furthermore, for $n \in \mathbb{N}$,

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,\mathrm{d}x = \frac{1}{\pi}\int_{0}^{\pi} x\sin nx\,\mathrm{d}x + \frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\,\mathrm{d}x \\ &= \frac{2}{\pi}\int_{0}^{\pi} x\sin nx\,\mathrm{d}x = \frac{2}{\pi}\Big[-\frac{x}{n}\cos nx\Big]_0^{\pi} + \frac{2}{n\pi}\int_0^{\pi}\cos nx\,\mathrm{d}x \\ &= -\frac{2}{n}\cos n\pi + 0 = -\frac{2}{n}(-1)^n. \end{aligned} \tag{2.40}$$

The resulting trigonometric series is

$$2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx. \tag{2.41}$$

Since f and f' are both continuous on $(-\pi,\pi)$, the Fourier Theorem applies and implies that the series converges at all points of the interval to the value x, and at the endpoints $\pm\pi$ to

$$\frac{f(-\pi_+)+f(\pi_-)}{2} = \frac{-\pi+\pi}{2} = 0. \tag{2.42}$$

Remark 2.14. Considering Examples 2.12 and 2.13, we have two distinct series, (2.32) and (2.41), both of which converge to x for $x \in (0,\pi)$ but take distinct values on $(-\pi,0)$.

Evaluating the series (2.41) at $x = \frac{\pi}{2}$, where the function f is continuous and takes the value $\frac{\pi}{2}$, we first note that

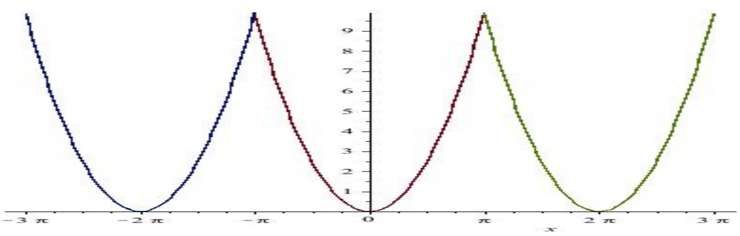
$$\sin n\frac{\pi}{2} = \begin{cases} \sin k\pi = 0, & \text{if } n = 2k, \quad k \in \mathbb{N} \\ \sin(2k-1)\frac{\pi}{2} = (-1)^{k+1}, & \text{if } n = 2k-1, \quad k \in \mathbb{N} \end{cases}. \tag{2.43}$$

Hence,

$$\frac{\pi}{4} = \sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{2k-1}. \tag{2.44}$$

This is another example of a particular infinite series that can be evaluated with the current method. And one final example before we explore some of the relationships between Fourier series, what we may and may not do with them.

Example 2.15. Determine the Fourier series for the function $f(x) = x^2$, $x \in (-\pi,\pi)$, with $f(x+2\pi) = f(x)$ for all x.



The Euler-Fourier formulae give the following coefficients:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \,\mathrm{d}x = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{2\pi^2}{3}, \tag{2.45}$$

and, for $n \in \mathbb{N}$,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx \,\mathrm{d}x = \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos nx \,\mathrm{d}x \\ &= \frac{2}{n\pi}\left[x^2 \sin nx\right]_0^{\pi} - \frac{4}{n\pi}\int_0^{\pi} x \sin nx \,\mathrm{d}x \\ &= -\frac{4}{n\pi}\left[-\frac{\pi}{n}\cos nx\right]_0^{\pi} = \frac{4}{n^2}(-1)^n. \end{aligned} \tag{2.46}$$

Moreover, for $n \in \mathbb{N}$,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx \,\mathrm{d}x = 0, \tag{2.47}$$

since an odd function is integrated over a symmetric interval. The resulting trigonometric series is

$$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx. \tag{2.48}$$

Since both f and f' are continuous over $(-\pi, \pi)$, the Fourier Theorem applies and implies that the series converges to x^2 on this interval and to $\frac{f(-\pi_+)+f(\pi_-)}{2} = \pi^2$ at $x = \pm\pi$.

Since $x = 0$ is a point of continuity the series converges to $f(0) = 0$, i.e.,

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos 0, \tag{2.49}$$

which can be rewritten as

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}. \tag{2.50}$$

Evaluating at $x = \pi$, however, we have

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi, \tag{2.51}$$

or

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}. \tag{2.52}$$

2.4 Differentiation and integration of Fourier series

Compare the Fourier series of Examples 2.13 and 2.15, defined on $(-\pi, \pi)$, i.e.,

$$x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx \qquad \text{and} \qquad x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx. \tag{2.53}$$

The Fourier series for x may be derived from that for x^2 by differentiating. Does such a relationship hold in general? Whereas continuity of the periodic function is understandably an important condition for the differentiability of a Fourier series, integration is possible under more general conditions - this is to be expected as the process of integration introduces a factor of n in the denominator, thereby improving the convergence of the series. However, when integrating a series, the integrated series will not in general be a Fourier series for the l.h.s. because the r.h.s. will contain a term $\frac{1}{2}a_0 x$, unless $a_0 = 0$. Hence

Theorem 2.16. *Let f be piecewise continuous on $(-\pi, \pi)$. Then, whether the trigonometric series corresponding to f,*

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \tag{2.54}$$

converges or not,

$$\int_{-\pi}^{x} f(t)\, dt = \frac{1}{2}a_0(x + \pi) + \sum_{n=1}^{\infty} \frac{1}{n}[a_n \sin nx - b_n(\cos nx - \cos n\pi)], \tag{2.55}$$

holds for $x \in (-\pi, \pi)$.

The proof of this theorem rests on some results from integration theory that not everyone will have covered; its other details are similar to those for the following result.

Theorem 2.17. *Let f be a continuous function on $[-\pi, \pi]$ such that $f(-\pi) = f(\pi)$. Let f' and f'' be piecewise continuous on $(-\pi, \pi)$. Then the Fourier series*

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \quad x \in [-\pi, \pi] \tag{2.56}$$

is differentiable at each point $x \in [-\pi, \pi]$ where f'' exists, with

$$f'(x) = \sum_{n=1}^{\infty} n(-a_n \sin nx + b_n \cos nx) \tag{2.57}$$

at these points.

Proof. f' satisfies the conditions of the Fourier Theorem and hence is represented by a Fourier series at each point x where $f''(x)$ exists. Given such a point, f' is continuous at that point, allowing

$$f'(x) = \frac{1}{2}\alpha_0 + \sum_{n=1}^{\infty}(\alpha_n \cos nx + \beta_n \sin nx), \tag{2.58}$$

with Euler-Fourier coefficients

$$\alpha_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos nx \,\mathrm{d}x \quad \text{and} \quad \beta_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\sin nx \,\mathrm{d}x. \tag{2.59}$$

More explicitly,

$$\alpha_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\,\mathrm{d}x = \frac{1}{\pi}\big[f(x)\big]_{-\pi}^{\pi} = \frac{1}{\pi}[f(\pi) - f(-\pi)] = 0, \tag{2.60}$$

and, for $n \in \mathbb{N}$,

$$\begin{aligned} \alpha_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos nx \,\mathrm{d}x \\ &= \frac{1}{\pi}\Big([f(x)\cos nx]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} f(x) n \sin nx \,\mathrm{d}x\Big) \\ &= nb_n, \end{aligned} \tag{2.61}$$

as well as

$$\begin{aligned} \beta_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\sin nx \,\mathrm{d}x \\ &= \frac{1}{\pi}\Big([f(x)\sin nx]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f(x) n \cos nx \,\mathrm{d}x\Big) \\ &= -na_n. \end{aligned} \tag{2.62}$$

Hence the result. □

Remark 2.18. The series (2.57) converges to $f'(x)$ where this is continuous. At points x where f' jumps, it converges to $\frac{f'(x_-)+f'(x_+)}{2}$.

2.5 Fourier sine and cosine series

Another observation that can be made with (2.53) is that x is an odd function and its Fourier series is a series of odd functions, $\sin nx$, wereas x^2 is an even function whose Fourier series is a series of even functions, $\cos nx$. Is that a coincidence or predictable? If predictable, can we generalise this to our advantage?

Theorem 2.19. *Let f satisfy the conditions of the Fourier Theorem 2.11. Then*

(i) Letting f be an even function, i.e., $f(-x) = f(x)$, $x \in (-\pi, \pi)$,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx, \quad \textit{where} \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(t)\cos nt \, \mathrm{d}t. \tag{2.63}$$

The Fourier series reduces to the so-called **Fourier cosine series**.

(ii) Letting f be an odd function, i.e., $f(-x) = -f(x)$, $x \in (-\pi, \pi)$,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad \textit{where} \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(t)\sin nt \, \mathrm{d}t. \tag{2.64}$$

The Fourier series reduces to the so-called **Fourier sine series**.

Proof. From the Fourier Theorem 2.11,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \tag{2.65}$$

where the Euler-Fourier coefficients are:

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt \, \mathrm{d}t = \frac{1}{\pi}\Big(\int_0^{\pi} f(t)\cos nt \, \mathrm{d}t + \int_{-\pi}^{0} f(t)\cos nt \, \mathrm{d}t\Big) \\ &= \frac{1}{\pi}\Big(\int_0^{\pi} f(t)\cos nt \, \mathrm{d}t + \int_{\pi}^{0} f(-u)\cos(-nu) \, \mathrm{d}(-u)\Big) \\ &= \frac{1}{\pi}\int_0^{\pi}\big(f(t) + f(-t)\big)\cos nt \, \mathrm{d}t \end{aligned} \tag{2.66}$$

and

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt \, \mathrm{d}t = \frac{1}{\pi}\Big(\int_0^{\pi} f(t)\sin nt \, dt + \int_{-\pi}^{0} f(t)\sin nt \, \mathrm{d}t\Big) \\ &= \frac{1}{\pi}\Big(\int_0^{\pi} f(t)\sin nt \, \mathrm{d}t + \int_{\pi}^{0} f(-u)\sin(-nu) \, \mathrm{d}(-u)\Big) \\ &= \frac{1}{\pi}\int_0^{\pi}\big(f(t) - f(-t)\big) \, \mathrm{d}t. \end{aligned} \tag{2.67}$$

Now,

(i) If f is even, $f(x) = f(-x)$, then $b_n = 0$ and

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(t)\cos nt \, \mathrm{d}t. \tag{2.68}$$

The Fourier series reduces to the **Fourier cosine series**.

(ii) If f is odd, $f(x) = -f(-x)$, then $a_n = 0$ and

$$b_n = \frac{2}{\pi}\int_0^\pi f(t)\sin nt\,\mathrm{d}t. \tag{2.69}$$

The Fourier series reduces to the **Fourier sine series**.

□

Example 2.20. Determine the Fourier series for the functions f and g, defined by

(a) $f(x) = x\sin x$,

(b) $g(x) = x\cos x$,

for $x \in [-\pi, \pi]$. Since f is even,

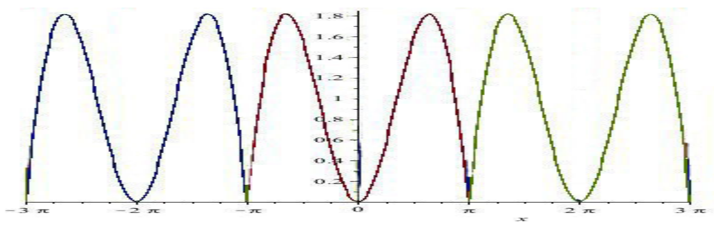
$$f(-x) = (-x)\sin(-x) = x\sin x = f(x), \tag{2.70}$$

we will obtain a Fourier cosine series, wheres g is odd,

$$g(-x) = (-x)\cos(-x) = -x\cos x = -g(x), \tag{2.71}$$

resulting in a Fourier sine series.

(a) Sketching the function f is helped by knowing its derivative, $f'(x) = x\cos x + \sin x$, so that, e.g., $f'(0) = 0$ and $f'(\pi) = \pi\cos\pi = -\pi$.



The Euler-Fourier coefficients for the Fourier cosine series are

$$\begin{aligned} a_0 &= \frac{2}{\pi}\int_0^\pi t\sin t\,\mathrm{d}t \\ &= \frac{2}{\pi}\Big(\big[-t\cos t\big]_0^\pi + \int_0^\pi \cos t\,\mathrm{d}t\Big) \\ &= \frac{2}{\pi}(-\pi\cos\pi) = 2, \end{aligned} \tag{2.72}$$

and for $n \in \mathbb{N}$,

$$a_n = \frac{2}{\pi}\int_0^\pi t\sin t\cos nt\,\mathrm{d}t = \frac{1}{\pi}\int_0^\pi t[\sin(n+1)t - \sin(n-1)t]\,\mathrm{d}t. \tag{2.73}$$

The integrand assumes a different form for $n = 1$ so that a_1 has to be evaluated separately,

$$\begin{aligned} a_1 &= \frac{1}{\pi}\int_0^\pi t\sin 2t\,\mathrm{d}t \\ &= \frac{1}{\pi}\Big(\Big[-\frac{t}{2}\cos 2t\Big]_0^\pi + \int_0^\pi \frac{\cos 2t}{2}\,\mathrm{d}t\Big) \\ &= \frac{1}{\pi}\Big(-\frac{\pi}{2}\cos 2\pi\Big) = -\frac{1}{2}. \end{aligned} \tag{2.74}$$

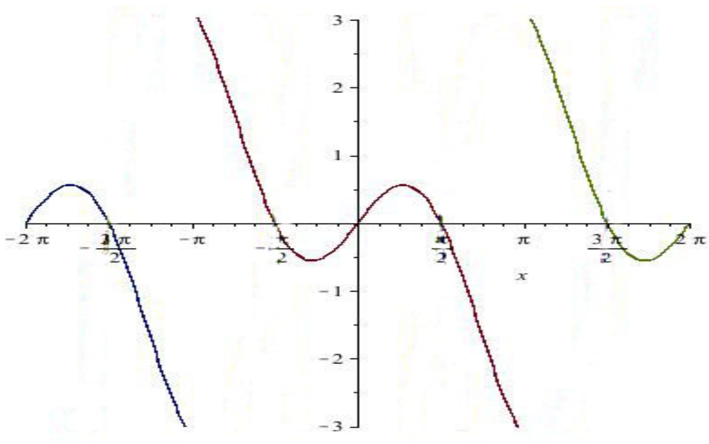
For $n \geq 2$,

$$\begin{aligned} a_n &= \frac{1}{\pi}\Big(\Big[-\frac{t}{n+1}\cos(n+1)t\Big]_0^\pi + \int_0^\pi \frac{\cos(n+1)t}{n+1}\,\mathrm{d}t \\ &\qquad + \Big[\frac{t}{n-1}\cos(n-1)t\Big]_0^\pi - \int_0^\pi \frac{\cos(n-1)t}{n-1}\,\mathrm{d}t\Big) \\ &= \frac{1}{\pi}\Big(-\frac{\pi}{n+1}\cos(n+1)\pi + \frac{\pi}{n-1}\cos(n-1)\pi\Big) \\ &= (-1)^{n+1}\Big(\frac{1}{n-1} - \frac{1}{n+1}\Big) = \frac{2(-1)^{n+1}}{n^2-1}. \end{aligned} \tag{2.75}$$

Hence the Fourier cosine series is

$$x\sin x = 1 - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos nx. \tag{2.76}$$

(b) In order to sketch the function g we note that its derivative is $g'(x) = \cos x - x\sin x$.



The Euler-Fourier coefficients for the Fourier sine series of his function are

$$b_n = \frac{2}{\pi}\int_0^\pi t\cos t\sin nt\,\mathrm{d}t = \frac{1}{\pi}\int_0^\pi t[\sin(n+1)t + \sin(n-1)t]\,\mathrm{d}t. \tag{2.77}$$

Again, the integrand assumes a different form for $n = 1$ so that b_1 has to be evaluated separately,

$$b_1 = \frac{1}{\pi}\int_0^{\pi} t \sin 2t \,\mathrm{d}t = -\frac{1}{2} \tag{2.78}$$

which, incidentally, is the same as a_1 in (2.74). The case $n \geq 2$ is similar to (2.75),

$$b_n = (-1)^n\Big(\frac{1}{n-1} + \frac{1}{n+1}\Big) = \frac{2n(-1)^n}{n^2-1}. \tag{2.79}$$

Hence, the Fourier sine series is

$$x \cos x = -\frac{1}{2}\sin x + 2\sum_{n=2}^{\infty}\frac{(-1)^n n}{n^2-1}\sin nx. \tag{2.80}$$

The Fourier Theorem guarantees convergence of both series (2.76) and (2.80) for $x \in (-\pi, \pi)$, as the functions are continuous there.

More significantly, the Fourier sine and cosine series that we have developed depend only on the values the function assumes on $[0, \pi)$. Since the cosine series represents an even function, knowing its values on $[0, \pi)$ implies, via the relation $f(-x) = f(x)$, the values on $(-\pi, 0)$, Likewise, a sine series represents an odd function, and the relation $f(-x) = -f(x)$ determines the values of f on $(-\pi, 0)$ once they are given on $[0, \pi)$.

On the other hand, a function f given on $[0, \pi)$ can be extended to $(-\pi, \pi)$ in two ways: either as an even function, f_{ev}, or as an odd fucntion, f_{odd}. These are defined by

$$f_{\mathrm{ev}}(x) = \begin{cases} f(x), & x \in [0, \pi) \\ f(-x), & x \in (-\pi, 0) \end{cases} \tag{2.81}$$

and

$$f_{\mathrm{odd}}(x) = \begin{cases} f(x), & x \in [0, \pi) \\ -f(-x), & x \in (-\pi, 0) \end{cases} \tag{2.82}$$

The Fourier series for f_{ev} is a cosine series, whereas the one for f_{odd} is a sine series. Although $f_{\mathrm{ev}}(x) = f_{\mathrm{odd}}(x)$ for every $x \in [0, \pi)$, the functions are different on $(-\pi, \pi)$ and therefore have different 2π-periodic extensions.

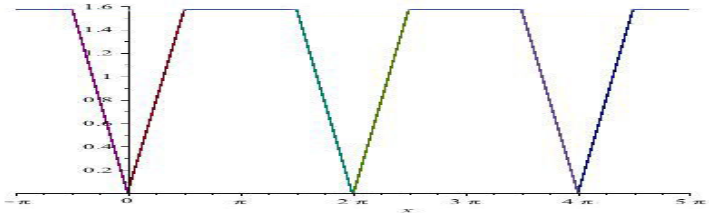
Example 2.21. The function f is defined on $(0, \pi)$ by

$$f(x) = \begin{cases} x, & x \in (0, \frac{\pi}{2}) \\ \frac{\pi}{2}, & x \in (\frac{\pi}{2}, \pi) \end{cases}. \tag{2.83}$$

Its even and odd extensions are

$$f_{\mathrm{ev}}(x) = \begin{cases} |x|, & x \in (-\frac{\pi}{2}, \frac{\pi}{2}) \\ \frac{\pi}{2}, & x \in (-\pi, -\frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi) \end{cases}, \tag{2.84}$$

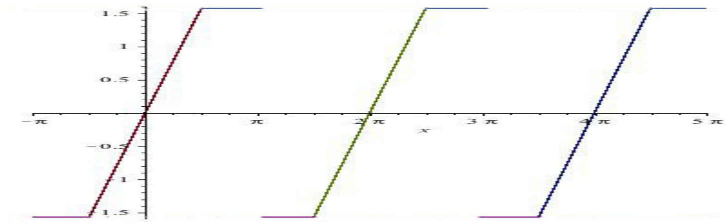
with (periodically externded) graph



and

$$f_{\text{odd}}(x) = \begin{cases} x, & x \in (-\frac{\pi}{2}, \frac{\pi}{2}) \\ -\frac{\pi}{2}, & x \in (-\pi, -\frac{\pi}{2}) \\ \frac{\pi}{2}, & x \in (\frac{\pi}{2}, \pi) \end{cases} \tag{2.85}$$

with (periodically extended) graph



The Fourier cosine series for f_{ev} has coefficients

$$a_0 = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} t\,\mathrm{d}t + \frac{2}{\pi}\frac{\pi}{2}\int_{\frac{\pi}{2}}^{\pi} \mathrm{d}t = \frac{2}{\pi}\Big(\Big[\frac{t^2}{2}\Big]_0^{\frac{\pi}{2}} + \frac{\pi^2}{4}\Big) = \frac{3\pi}{4}, \tag{2.86}$$

and, for $n \in \mathbb{N}$,

$$\begin{aligned} a_n &= \frac{2}{\pi}\Big(\int_0^{\frac{\pi}{2}} t\cos nt\,\mathrm{d}t + \frac{\pi}{2}\int_{\frac{\pi}{2}}^{\pi} \cos nt\,\mathrm{d}t\Big) \\ &= \frac{2}{\pi}\Big(\Big[\frac{t\sin nt}{n}\Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{\sin nt}{n}\,\mathrm{d}t + \frac{\pi}{2}\Big[\frac{\sin nt}{n}\Big]_{\frac{\pi}{2}}^{\pi}\Big) \\ &= \frac{1}{n}\sin\frac{n\pi}{2} + \frac{2}{n\pi}\Big[\frac{\cos nt}{n}\Big]_0^{\frac{\pi}{2}} - \frac{1}{n}\sin\frac{n\pi}{2} \\ &= \frac{2}{n^2\pi}\Big(\cos\frac{n\pi}{2} - 1\Big) \end{aligned} \tag{2.87}$$

Hence the Fourier cosine series is

$$f_{\text{ev}}(x) = \frac{3\pi}{8} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\Big(1 - \cos\frac{n\pi}{2}\Big)\cos nx. \tag{2.88}$$

In contrast, the Fourier sine series for f_{odd} has coefficients

$$\begin{aligned}
b_n &= \frac{2}{\pi}\Big(\int_0^{\frac{\pi}{2}} t\sin nt\,\mathrm{d}t + \frac{\pi}{2}\int_{\frac{\pi}{2}}^{\pi} \sin nt\,\mathrm{d}t\Big)\\
&= \frac{2}{\pi}\Big(\Big[-\frac{t\cos nt}{n}\Big]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \frac{\cos nt}{n}\,\mathrm{d}t + \frac{\pi}{2}\Big[-\frac{t\cos nt}{n}\Big]_{\frac{\pi}{2}}^{\pi}\Big)\\
&= \frac{2}{\pi}\Big(-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \Big[\frac{\sin nt}{n^2}\Big]_0^{\frac{\pi}{2}} - \frac{\pi}{2n}\cos n\pi + \frac{\pi}{2n}\cos\frac{n\pi}{2}\Big)\\
&= \frac{2}{\pi n^2}\sin\frac{n\pi}{2} + \frac{(-1)^{n+1}}{n}.
\end{aligned} \tag{2.89}$$

Hence the Fourier sine series is

$$f_{\text{odd}}(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\frac{\pi}{2}}{n^2}\sin x + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx. \tag{2.90}$$

Note that the two series (2.85) and (2.90) converge to the same value for $x \in [0, \pi)$!

2.6 Postscript

There are some further useful properties of Fourier series of which we intend to mention a few.

2.6.1 Linearity

Proposition 2.22. *Let f and g be two functions, both defined on the interval $[-\pi, \pi]$, with corresponding trigonometric series*

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) \quad \textit{and} \quad \frac{1}{2}\alpha_0 + \sum_{n=1}^{\infty}(\alpha_n\cos nx + \beta_n\sin nx), \tag{2.91}$$

respectively. Let μ and ν be two (real) constants. Then the function $\mu f(x) + \nu g(x)$ has a corresponding trigonometric series

$$\frac{1}{2}(\mu a_0 + \nu\alpha_0) + \sum_{n=1}^{\infty}((\mu a_n + \nu\alpha_n)\cos nx + (\mu b_n + \nu\beta_n)\sin nx). \tag{2.92}$$

The proof of this Proposition is obvious from the definition of trigonometric series.

Remark 2.23. (i) One can conclude from this Proposition that the set of trigonometric series is a **vector space**.

(ii) If one restricts the Proposition to piecewise continuous functions with piecewise continuous derivative, the Fourier Theorem applies and the Fourier series converge. The set of such functions is a vector space, as well as the set of the corresponding Fourier series. One concludes that both these vector spaces have a **basis** consisting of the functions $\{1, \sin nx, \cos nx;\ n \in \mathbb{N}\}$. This basis is infinite, implying that the vector space is **infinite-dimensional**.

2.6.2 Complex series

As one can easily see, the linearity shown in Proposition 2.22 can be extended to complex coefficients. Hence, for $n \in \mathbb{N}$,

$$c_n := \frac{a_n - \mathrm{i}b_n}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\big(\cos nx - \mathrm{i}\sin nx\big)\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{e}^{-\mathrm{i}nx}\,\mathrm{d}x, \tag{2.93}$$

implying

$$c_{-n} = \frac{a_n + \mathrm{i}b_n}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{e}^{\mathrm{i}nx}\,\mathrm{d}x. \tag{2.94}$$

For $n = 0$ we define $c_0 := \frac{a_0}{2}$. Conversely,

$$a_n = c_n + c_{-n} \quad \text{and} \quad b_n = \mathrm{i}(c_n - c_{-n}). \tag{2.95}$$

With these definitions, the Fourier series for a function f takes the form

$$\begin{aligned} f(x) &= c_0 + \sum_{n=1}^{\infty}\Big((c_n + c_{-n})\cos nx + \mathrm{i}(c_n - c_{-n})\sin nx\Big) \\ &= c_0 + \sum_{n=1}^{\infty}\Big(c_n\,\mathrm{e}^{\mathrm{i}nx} + c_{-n}\,\mathrm{e}^{-\mathrm{i}nx}\Big). \\ &= \sum_{n=-\infty}^{\infty} c_n\,\mathrm{e}^{\mathrm{i}nx}. \end{aligned} \tag{2.96}$$

This is the **complex form of Fourier series**.

2.6.3 Arbitrary periods

The functions we have been discussing have all been periodic with period 2π. Can the theory be adapted to cater for functions that are periodic of period $2L$, where L is any finite positive number? If a function f is defined on an interval $[-L, L]$ one can change variables from $x \in [-L, L]$ to

$$t = \frac{\pi}{L}x \in [-\pi, \pi]. \tag{2.97}$$

Hence, the function

$$\tilde{f}(t) = f\Big(\frac{L}{\pi}t\Big) \tag{2.98}$$

is defined on the interval $[-\pi, \pi]$. If f is piecewise continuous with piecewise continuous derivative, the same holds true for the function $\tilde{f}$, so that the Fourier Theorem can be applied. Hence, $\tilde{f}$ has a Fourier series representation,

$$\tilde{f}(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nt + b_n\sin nt). \tag{2.99}$$

Changing variables as in (2.97) leads to a (modified) Fourier series for the $2L$-periodic function f,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\Big(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\Big). \tag{2.100}$$

The Euler-Fourier formulae for $\tilde{f}$ convert to corresponding formulae for f,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f\Big(\frac{L}{\pi}t\Big)\cos nt\,\mathrm{d}t = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,\mathrm{d}x \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f\Big(\frac{L}{\pi}t\Big)\sin nt\,\mathrm{d}t = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,\mathrm{d}x. \end{aligned} \tag{2.101}$$

In the complex form introduced above the (modified) Fourier series is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n\,\mathrm{e}^{\mathrm{i}\frac{n\pi}{L}x}, \tag{2.102}$$

with coefficients

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\,\mathrm{e}^{-\mathrm{i}\frac{n\pi}{L}x}\,\mathrm{d}x. \tag{2.103}$$

One may now wonder what happens in the limit $L \to \infty$? One the one hand, one would thus treat 'any' real function as the interval $[-L, L]$ would approach $\mathbb{R}$. On the other hand, it is not quite clear what would happen to the Fourier series (2.102) and the Euler-Fourier formulae (2.103). These question will lead us to the next section.

3 The Fourier Transform

3.1 Introduction

We have been discussing the idea of a complex Fourier series, first on $(-\pi, \pi)$ and then on $(-L, L)$, see (2.102) and (2.103). Hence, using (2.103) in (2.102) the Fourier series of the $2L$-periodic function f takes the form

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{2L} \int_{-L}^{L} f(y)\, \mathrm{e}^{-\mathrm{i}\frac{n\pi}{L}y}\, \mathrm{d}y\; \mathrm{e}^{\mathrm{i}\frac{n\pi}{L}x}. \tag{3.1}$$

With the limit $L \to \infty$ in mind, we introduce the notation

$$u_n = \frac{n\pi}{L} \quad \text{so that} \quad u_{n+1} - u_n = \frac{\pi}{L}, \tag{3.2}$$

and find

$$f(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (u_{n+1} - u_n)\Big(\int_{-L}^{L} f(y)\, \mathrm{e}^{-\mathrm{i}u_n y}\, \mathrm{d}y\Big)\mathrm{e}^{\mathrm{i}u_n x}. \tag{3.3}$$

The sum over n with the variable u_n looks like an approximation of an integral in the sense of a Riemann sum. Hence, in the limit $L \to \infty$ one would expect the expression (3.3) to approach

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Big(\int_{-\infty}^{\infty} f(y)\, \mathrm{e}^{-\mathrm{i}uy}\, \mathrm{d}y\Big)\mathrm{e}^{\mathrm{i}ux}\, \mathrm{d}u. \tag{3.4}$$

Our discussion of the limiting process was, of course, entirely heuristic. However, the expression (3.4) is a motivation for an important definition.

Definition 3.1. The **Fourier Transform** of function f is defined, for all real s, by

$$\mathcal{F}[f](s) = \hat{f}(s) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\mathrm{e}^{\mathrm{i}sx}\, \mathrm{d}x. \tag{3.5}$$

Remark 3.2. (i) We will use both notations, $\mathcal{F}[f](s)$ and $\hat{f}(s)$, for the Fourier transform of a function f interchangeably. The former puts an emphasis on the fact that a function f is mapped into another function $\mathcal{F}[f](s)$, whereas the latter notation is simpler.

(ii) The definition of the Fourier transform varies slightly from source to source. You may see $\mathrm{e}^{-\mathrm{i}sx}$ replacing $\mathrm{e}^{\mathrm{i}sx}$, and/or factors of 1 or $\frac{1}{2\pi}$ instead of $\frac{1}{\sqrt{2\pi}}$.

(iii) The x-integral extends over all $\mathbb{R}$; its existence is not always guaranteed. The function f must be *integrable*, meaning that

$$\lim_{\substack{a\to-\infty \\ b\to\infty}} \int_a^b f(x)\, \mathrm{d}x \quad \text{and} \quad \lim_{\substack{a\to-\infty \\ b\to\infty}} \int_a^b |f(x)|\, \mathrm{d}x \tag{3.6}$$

both exist.

Example 3.3. A rectangular pulse is described by

$$f(x) = \begin{cases} 1 & \text{if } |x| < a \\ 0, & \text{if } |x| > a \end{cases}, \tag{3.7}$$

where $a > 0$ is a constant. Determine $\hat{f}(s)$.

We first note that

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} |f(x)|\,\mathrm{d}x = \int_{-a}^{a} \mathrm{d}x = 2a \tag{3.8}$$

is finite, and hence f is integrable. With the definition of the Fourier transform we calculate:

$$\begin{aligned} \mathcal{F}[f](s) &= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \mathrm{e}^{\mathrm{i}sx}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \left[\frac{\mathrm{e}^{\mathrm{i}sx}}{\mathrm{i}s} \right]_{-a}^{a} \\ &= \sqrt{\frac{2}{\pi}} \frac{1}{2\mathrm{i}s} \left(\mathrm{e}^{\mathrm{i}sa} - \mathrm{e}^{-\mathrm{i}sa} \right) = \sqrt{\frac{2}{\pi}} \frac{\sin as}{s}. \end{aligned} \tag{3.9}$$

There is a group of theorems that helps in evaluating the transforms of functions.

Theorem 3.4. *The Fourier transform is linear, i.e.,*

$$\mathcal{F}[af + bg](s) = a\mathcal{F}[f](s) + b\mathcal{F}[g](s), \tag{3.10}$$

for any functions f and g such that $\hat{f}$ and $\hat{g}$ exist, and arbitrary complex constants a and b.

Proof. By definition,

$$\begin{aligned} \mathcal{F}[af + bg](s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [af(x) + bg(x)]\mathrm{e}^{\mathrm{i}sx}\,\mathrm{d}x \\ &= a\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\mathrm{e}^{\mathrm{i}sx}\,\mathrm{d}x \right) + b\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)\mathrm{e}^{\mathrm{i}sx}\,\mathrm{d}x \right) \\ &= a\mathcal{F}[f](s) + b\mathcal{F}[g](s). \end{aligned} \tag{3.11}$$

□

Theorem 3.5. *Let f be a function with Fourier transform $\hat{f}$. Then, for all $a \in \mathbb{R}$,*

$$\mathcal{F}[\mathrm{e}^{\mathrm{i}xa} f](s) = \mathcal{F}[f](s + a) = \hat{f}(s + a). \tag{3.12}$$

Proof. By definition,

$$\begin{aligned} \mathcal{F}[e^{ixa} f](s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}xa} f(x)\mathrm{e}^{\mathrm{i}sx}\,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\mathrm{e}^{\mathrm{i}(s+a)x}\,\mathrm{d}x \\ &= \hat{f}(s + a). \end{aligned} \tag{3.13}$$

□

Theorem 3.6. *Let f be a function with Fourier transform $\hat{f}$. Let $a > 0$ be a constant and define a function g_a by $g_a(x) = f(\frac{x}{a})$. Then,*

$$\mathcal{F}[g_a](s) = a\hat{f}(sa). \tag{3.14}$$

Proof. By definition,

$$\mathcal{F}[g_a](s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f\Big(\frac{x}{a}\Big) \mathrm{e}^{isx} \, \mathrm{d}x = \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \mathrm{e}^{\mathrm{i}(as)t} \, \mathrm{d}t = a\hat{f}(as). \tag{3.15}$$

□

Example 3.7. Calculate the Fourier transform of

$$g(x) = \begin{cases} \sin \omega x, & \text{if } |x| < 2\pi/\omega \\ 0, & \text{if } |x| > 2\pi/\omega \end{cases}, \tag{3.16}$$

using Theorems 3.4 – 3.6.

As a first step, rewrite g as

$$g(x) = \frac{1}{2\mathrm{i}} (\mathrm{e}^{\mathrm{i}\omega x} - \mathrm{e}^{-\mathrm{i}\omega x}) f(x), \tag{3.17}$$

where f is the function of Example 3.3, taking $a = \frac{2\pi}{\omega}$. Hence,

$$\hat{f}(s) = \sqrt{\frac{2}{\pi}} \frac{\sin \frac{2\pi}{\omega} s}{s}. \tag{3.18}$$

Then, using Theorem 3.4 and Theorem 3.5,

$$\begin{aligned} \hat{g}(s) &= \sqrt{\frac{2}{\pi}} \frac{1}{2\mathrm{i}} \Big(\frac{\sin \frac{2\pi}{\omega}(s+\omega)}{s+\omega} - \frac{\sin \frac{2\pi}{\omega}(s-\omega)}{s-\omega} \Big) \\ &= \sqrt{\frac{2}{\pi}} \frac{1}{2\mathrm{i}} \sin \frac{2\pi s}{\omega} \Big(\frac{1}{s+\omega} - \frac{1}{s-\omega} \Big) \\ &= \sqrt{\frac{2}{\pi}} \frac{\mathrm{i}\omega}{s^2 - \omega^2} \sin \frac{2\pi s}{\omega}. \end{aligned} \tag{3.19}$$

The Fourier transform is used in evaluating the characteristic function of an important probability distribution.

Example 3.8. Determine $\hat{f}$, where f defines the Normal distribution (or: Gaussian function),

$$f(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}x^2}. \tag{3.20}$$

Then, by completing the square in the exponent,

$$\hat{f}(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2}x^2} \mathrm{e}^{\mathrm{i}sx} \, \mathrm{d}x = \frac{1}{2\pi} \mathrm{e}^{-\frac{1}{2}s^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-\mathrm{i}s)^2} \, \mathrm{d}x. \tag{3.21}$$

Changing variables to $u = x - \mathrm{i}s$, one would expect

$$\hat{f}(s) = \frac{1}{2\pi}\mathrm{e}^{-\frac{1}{2}s^2}\int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2}\,\mathrm{d}u. \tag{3.22}$$

A subtlety here might go unnoticed: the x-integration was along the whole real line, but u is complex. In terms of u, therefore, the integration is along a line in the complex plane parallel to the real line and going through $-\mathrm{i}s$. In order to get to the r.h.s. of (3.21) one has to shift this line back to the real line (in the variable u). That this is possible is a consequence of Cauchy's Theorem, which will be discussed in MT2900.

To evaluate the integral that now remains, either use the normalisation of the Normal distribution or use the double integral technique,

$$\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{1}{2}u^2}\,\mathrm{d}u\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{1}{2}v^2}\,\mathrm{d}v = \int_0^{2\pi}\int_0^{\infty}\mathrm{e}^{-\frac{1}{2}r^2}r\,\mathrm{d}r\,\mathrm{d}\theta = 2\pi\left[-e^{\frac{1}{2}r^2}\right]_0^{\infty} = 2\pi, \tag{3.23}$$

using polar coordinates for $(u, v) \in \mathbb{R}^2$. Hence,

$$\hat{f}(s) = \frac{1}{2\pi}\mathrm{e}^{-\frac{1}{2}s^2}\sqrt{2\pi} = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}s^2}. \tag{3.24}$$

The normal distribution (Gaussian function) replicates itself under Fourier transform, i.e., it is *self-replicating.*

In the same way as Fourier series corresponding to even and odd functions take a special form, Fourier transforms of even and odd functions are Fourier cosine and sine transforms, respectively.

Definition 3.9. The **Fourier cosine transform** and the **Fourier sine transform** of the function f are defined, for all real s, by

$$\mathcal{F}_c[f](s) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos(sx)\,\mathrm{d}x, \tag{3.25}$$

and

$$\mathcal{F}_s[f](s) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin(sx)\,\mathrm{d}x, \tag{3.26}$$

respectively.

We now have the following statement.

Theorem 3.10. *Let f be a function on $\mathbb{R}$ with Fourier transform $\mathcal{F}[f]$. Then, letting f be*

(i) An even function, $f(x) = f(-x)$ for all $x \in \mathbb{R}$, $\mathcal{F}[f] = \mathcal{F}_c[f]$,

(ii) An odd function, $f(x) = -f(-x)$ for all $x \in \mathbb{R}$, $\mathcal{F}[f] = \mathrm{i}\mathcal{F}_s[f]$.

Proof.

$$
\begin{aligned}
\mathcal{F}[f(x)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \mathrm{e}^{\mathrm{i}sx} \, \mathrm{d}x. \\
&= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(x) \mathrm{e}^{\mathrm{i}sx} \, \mathrm{d}x - \frac{1}{\sqrt{2\pi}} \int_{\infty}^{0} f(-y) \mathrm{e}^{-\mathrm{i}sy} \, \mathrm{d}y \\
&= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} [f(x) \mathrm{e}^{\mathrm{i}sx} + f(-x) \mathrm{e}^{-\mathrm{i}sx}] \, \mathrm{d}x.
\end{aligned} \tag{3.27}
$$

(i) If f is an even function,

$$
\begin{aligned}
\mathcal{F}[f](s) &= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(x) \big(\mathrm{e}^{\mathrm{i}sx} + \mathrm{e}^{-\mathrm{i}sx}\big) \, \mathrm{d}x \\
&= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos(sx) \, \mathrm{d}x = \mathcal{F}_c[f](s).
\end{aligned} \tag{3.28}
$$

(ii) Similarly, if f is an odd function,

$$
\begin{aligned}
\mathcal{F}[f](s) &= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(x) \big(\mathrm{e}^{\mathrm{i}sx} - \mathrm{e}^{-\mathrm{i}sx}\big) \, \mathrm{d}x \\
&= \mathrm{i}\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin(sx) \, \mathrm{d}x = \mathrm{i}\mathcal{F}_s[f](s).
\end{aligned} \tag{3.29}
$$

□

In order to find further properties of the Fourier transform it is convenient to focus on a class of 'nice' functions.

Definition 3.11. A function f defined on $\mathbb{R}$ with the following properties,

(i) f is ∞-often continuously differentiable, i.e., $\frac{\mathrm{d}^n f}{\mathrm{d}x^n}$ exists and is continuous for all $n \in \mathbb{N}$;

(ii) f and all of its derivatives drop off faster than any polynomial as $x \to \pm\infty$, i.e.,

$$
\lim_{x \to \pm\infty} \left| x^m \frac{\mathrm{d}^n f}{\mathrm{d}x^n}(x) \right| = 0, \tag{3.30}
$$

for all $n, m \in \mathbb{N}$.

is said to be of **Schwartz class**. The set of Schwartz class function is denotes as $\mathcal{S}(\mathbb{R})$.

Example 3.12. An example of a Schwartz class function is the Gaussian function

$$
f(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}x^2}. \tag{3.31}
$$

In order to see that both properties demanded in Definition 3.11 are fulfilled it is enough to notice that, following the product rule of differentiation,

$$\frac{\mathrm{d}^n f}{\mathrm{d}x^n}(x) = p_n(x)\,\mathrm{e}^{-\frac{1}{2}x^2}, \tag{3.32}$$

where $p_n(x)$ is a polynomial of degree n, e.g.,

$$p_1(x) = -\frac{1}{\sqrt{2\pi}}\,x, \quad p_2(x) = \frac{1}{\sqrt{2\pi}}(x^2-1), \ldots \tag{3.33}$$

Hence, derivatives of arbitrary high order exist and, moreover, if multiplied with a monomial x^m are of the form of a polynomial multiplied by the Gaussian function. Hence, the limit of this product as $|x| \to \infty$ vanishes.

Theorem 3.13. *Let $f \in \mathcal{S}(\mathbb{R})$ and $n \in \mathbb{N}$, then*

(i)

$$\mathcal{F}\left[\frac{\mathrm{d}^n f}{\mathrm{d}x^n}\right](s) = (-\mathrm{i}s)^n \mathcal{F}[f](s); \tag{3.34}$$

(ii)

$$\mathcal{F}\left[x^n f\right](s) = (-\mathrm{i})^n \frac{\mathrm{d}^n}{\mathrm{d}s^n}\mathcal{F}[f](s); \tag{3.35}$$

Proof. (i) This we prove by induction. We first observe that (3.34) is trivially true when $n = 0$. Next,

$$\begin{aligned}\mathcal{F}\left[\frac{\mathrm{d}^{k+1} f}{\mathrm{d}x^{k+1}}\right](s) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\mathrm{d}^{k+1} f}{\mathrm{d}x^{k+1}}(x)\,\mathrm{e}^{\mathrm{i}sx}\,\mathrm{d}x. \\ &= \frac{1}{\sqrt{2\pi}}\left(\left[\frac{\mathrm{d}^k f}{\mathrm{d}x^k}(x)\mathrm{e}^{\mathrm{i}sx}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{\mathrm{d}^k f}{\mathrm{d}x^k}(x)\,\mathrm{i}s\,\mathrm{e}^{\mathrm{i}sx}\,\mathrm{d}x\right) \\ &= (-\mathrm{i}s)\mathcal{F}\left[\frac{\mathrm{d}^k f}{\mathrm{d}x^k}\right](s).\end{aligned} \tag{3.36}$$

The boundary term after having integrated by parts vanishes due to the property (ii) in Definition 3.11. Hence, assuming (3.34) for $n = k$ implies the validity of (3.34) for $n = k+1$.

(ii) A direct calculation shows that

$$\begin{aligned}\frac{\mathrm{d}^n}{\mathrm{d}s^n}\mathcal{F}[f](s) &= \frac{\mathrm{d}^n}{\mathrm{d}s^n}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\mathrm{e}^{\mathrm{i}sx}\,\mathrm{d}x\right) \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\left(\frac{\mathrm{d}^n}{\mathrm{d}s^n}\mathrm{e}^{\mathrm{i}sx}\right)\mathrm{d}x. \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)(\mathrm{i}x)^n\mathrm{e}^{\mathrm{i}sx}\,\mathrm{d}x \\ &= \mathrm{i}^n\mathcal{F}[x^n f(x)](s).\end{aligned} \tag{3.37}$$

□

Remark 3.14. Fourier transforms convert derivatives into a multiplication with monomials, and vice versa. This property can be utilised to convert (certain) differential equations with constant coefficients into algebraic equations. The latter can often be solved more easily. In order to find the solution of the differential equation one only needs to know how to obtain the function whose Fourier transform is the solution of the algebraic equation. Hence the need for an **inverse Fourier transform**.

Theorem 3.15 (Fourier Integral Theorem for Schwartz class functions)**.** *Let $f \in \mathcal{S}(\mathbb{R})$. Then $\mathcal{F}[f] = \hat{f} \in \mathcal{S}(\mathbb{R})$ and*

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(s)\, \mathrm{e}^{-\mathrm{i}xs}\, \mathrm{d}s. \tag{3.38}$$

We shall not give a proof here, which requires techniques that go well beyond the scope of this course.

Remark 3.16. The expression given in (3.38) can be seen as an inverse Fourier transform,

$$\mathcal{F}^{-1}[g](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(s)\, \mathrm{e}^{-\mathrm{i}xs}\, \mathrm{d}s, \tag{3.39}$$

since the statement of Theorem 3.15 then reads $\mathcal{F}^{-1}[\mathcal{F}[f]] = f$. Another way to view this Theorem is to rewrite (3.38) as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(y)\, \mathrm{e}^{\mathrm{i}ys}\, \mathrm{d}y \right) \mathrm{e}^{-\mathrm{i}xs}\, \mathrm{d}s, \tag{3.40}$$

and to note that this is, up to the signs in the exponents, the heuristic expression in (3.4).

3.2 Convolutions and Parseval's Theorem

The convolution is an operation that combines two functions, f and g, into a single function $h = f * g$.

Definition 3.17. Let f, g be functions on $\mathbb{R}$. Their **convolution** is the function

$$f * g(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) g(x-y)\, \mathrm{d}y \tag{3.41}$$

on $\mathbb{R}$, whenever the integral exists

Proposition 3.18. *If $f, g \in \mathcal{S}(\mathbb{R})$, then $f * g \in \mathcal{S}(\mathbb{R})$.*

It is not too difficult to prove this proposition. One only needs to know how to differentiate under an integral, and to consider limits as $|x| \to \infty$ (not done here). Some properties of convolutions are easy to prove (also not done here).

Remark 3.14. Fourier transforms convert derivatives into a multiplication with monomials, and vice versa. This property can be utilised to convert (certain) differential equations with constant coefficients into algebraic equations. The latter can often be solved more easily. In order to find the solution of the differential equation one only needs to know how to obtain the function whose Fourier transform is the solution of the algebraic equation. Hence the need for an **inverse Fourier transform**.

Theorem 3.15 (Fourier Integral Theorem for Schwartz class functions)**.** *Let* $f \in \mathcal{S}(\mathbb{R})$*. Then* $\mathcal{F}[f] = \hat{f} \in \mathcal{S}(\mathbb{R})$ *and*

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(s)\, \mathrm{e}^{-\mathrm{i}xs}\, \mathrm{d}s. \tag{3.38}$$

We shall not give a proof here, which requires techniques that go well beyond the scope of this course.

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It is not too difficult to prove this proposition. One only needs to know how to differentiate under an integral, and to consider limits as $|x| \to \infty$ (not done here). Some properties of convolutions are easy to prove (also not done here).

Proposition 3.19. *The convolution is a product satisfying the following rules:*

1. $f * g = g * f$ *(commutative law);*
2. $f * (g_1 + g_2) = f * g_1 + f * g_2$ *(distributive law);*
3. $(f * g) * h = f * (g * h)$ *(associative law);*
4. $f * 0 = 0 * f = 0$ *(zero element).*

One might miss here a statement about a unit element, $e(x)$, for the convolution that would satisfy $f * e(x) = f(x)$ for every function $f(x)$. One guess would be that $e(x) = 1$. However, choosing the rectangular pulse,

$$f(x) = \begin{cases} x, & |x| < a \\ 0, & \text{otherwise} \end{cases}, \tag{3.42}$$

shows that

$$f * e(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e(x - y)\, \mathrm{d}y = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} y\, \mathrm{d}y = 0. \tag{3.43}$$

Hence $f * e \neq f$. Moreover, $f * e = 0$ with neither $f = 0$ nor $e = 0$ – not what we normally expect from a product. We will return to the question of a unit element later.

Convolutions are particularly interesting in the light of Fourier transforms.

Theorem 3.20 (Convolution Theorem)**.** *Let $f, g \in \mathcal{S}(\mathbb{R})$, then $\mathcal{F}[f * g] = \mathcal{F}[f]\, \mathcal{F}[g]$.*

Proof.

$$\begin{aligned} \mathcal{F}[f * g](s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f * g(x) \mathrm{e}^{\mathrm{i}sx}\, \mathrm{d}x \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(y) g(x - y)\, \mathrm{d}y \right) \mathrm{e}^{\mathrm{i}sx}\, \mathrm{d}x \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) \left(\int_{-\infty}^{\infty} g(x - y) \mathrm{e}^{\mathrm{i}sx}\, \mathrm{d}x \right) \mathrm{d}y. \end{aligned} \tag{3.44}$$

The interchange of the order of integration is justified for functions $f, g \in \mathcal{S}(\mathbb{R})$. Hence, a change of variables, $v = x - y$ in the x-integrations gives

$$\mathcal{F}[f * g](s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) \mathrm{e}^{\mathrm{i}sy} \left(\int_{-\infty}^{\infty} g(v) \mathrm{e}^{\mathrm{i}sv}\, \mathrm{d}v \right) \mathrm{d}y = \mathcal{F}[f](s)\, \mathcal{F}[g](s). \tag{3.45}$$

□

The important observation here is that the Fourier transform turns a convolution of functions into a multiplication.

Applying the inverse Fourier transform, $\mathcal{F}^{-1}$, to both sides of the convolution theorem yields

$$f * g(x) = \mathcal{F}^{-1}[\hat{f}\,\hat{g}](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(s)\,\hat{g}(s)\,\mathrm{e}^{-\mathrm{i}sx}\,\mathrm{d}s. \tag{3.46}$$

Now choosing $x = 0$ leads to the identity

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)g(-y)\,\mathrm{d}y = f * g(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(s)\,\hat{g}(s)\,\mathrm{d}s. \tag{3.47}$$

Remembering that the Fourier transform involves a factor of $\mathrm{e}^{\mathrm{i}sx}$, whose complex conjugate is $\overline{\mathrm{e}^{\mathrm{i}sx}} = \mathrm{e}^{-\mathrm{i}sx}$, gives the following statement.

Theorem 3.21 (Parseval's Relation)**.** *Let $f, g \in \mathcal{S}(\mathbb{R})$ with Fourier transforms $\hat{f}$ and $\hat{g}$. Then*

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,\mathrm{d}x = \int_{-\infty}^{\infty} \hat{f}(s)\overline{\hat{g}(s)}\,\mathrm{d}s. \tag{3.48}$$

Proof. From (3.47) we know that

$$\int_{-\infty}^{\infty} f(x)h(-x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \hat{f}(s)\,\hat{h}(s)\,\mathrm{d}s. \tag{3.49}$$

Define $g(x) := \overline{h(-x)}$, hence $h(-x) = \overline{g(x)}$. Therefore, the l.h.s. of (3.49) is the same as the l.h.s. of (3.48). Furthermore,

$$\begin{aligned} \hat{h}(s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(x)\,\mathrm{e}^{\mathrm{i}sx}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{g(-x)}\,\mathrm{e}^{\mathrm{i}sx}\,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{g(y)}\,\mathrm{e}^{-\mathrm{i}sy}\,\mathrm{d}y = \overline{\hat{g}(s)}. \end{aligned} \tag{3.50}$$

Using this on the r.h.s. of (3.49) gives the r.h.s. of (3.48). □

Remark 3.22. One could have attempted to prove Parseval's relation as follows,

$$\begin{aligned} \int_{-\infty}^{\infty} \hat{f}(s)\overline{\hat{g}(s)}\,\mathrm{d}s &= \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\mathrm{e}^{\mathrm{i}sx}\,\mathrm{d}x\right)\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{g(y)}\mathrm{e}^{-\mathrm{i}ys}\,\mathrm{d}y\right)\mathrm{d}s \\ &= \int_{-\infty}^{\infty} f(x) \int_{-\infty}^{\infty} \overline{g(y)} \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}s(y-x)}\,\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}x. \end{aligned} \tag{3.51}$$

This would give (3.48), if

$$g(x) = \int_{-\infty}^{\infty} g(y)\Big(\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}s(y-x)}\,\mathrm{d}s\Big)\mathrm{d}y \tag{3.52}$$

were true. In order to give this relation a mathematical meaning one has to introduce a novel concept that generalises the notation of a function. This will be done in the next paragraph.

First, however, we want to see how Parselval's Theorem can be utilised to calculate certain integrals.

Example 3.23. In Example 3.3 we saw that the Fourier transform of the rectangular pulse,

$$f(x) = \begin{cases} 1 & \text{if } |x| < a \\ 0, & \text{if } |x| > a \end{cases}, \tag{3.53}$$

is

$$\hat{f}(s) = \sqrt{\frac{2}{\pi}} \frac{\sin as}{s}. \tag{3.54}$$

Hence, with $g = f$ the l.h.s. of (3.48) is

$$\int_{-\infty}^{\infty} f(x)\overline{f(x)}\,\mathrm{d}x = \int_{-a}^{a} \mathrm{d}x = 2a, \tag{3.55}$$

whereas its r.h.s. is

$$\int_{-\infty}^{\infty} \hat{f}(s)\overline{\hat{f}(s)}\,\mathrm{d}s = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2 as}{s^2}\,\mathrm{d}s. \tag{3.56}$$

Therefore, from Parseval's Theorem we get the following identity,

$$\int_{-\infty}^{\infty} \frac{\sin^2 as}{s^2}\,\mathrm{d}s = a\pi. \tag{3.57}$$

3.3 The Dirac delta

When we discussed the question of a unit element for the convolution we encountered the condition

$$f * e(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)e(x-y)\,\mathrm{d}y = f(x) \tag{3.58}$$

that should hold for all reasonable functions f. Comparing this to (3.52), one would expect that

$$e(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}xs}\,\mathrm{d}s. \tag{3.59}$$

The exponential function on the r.h.s. can readily be integrated; however, due to the infinite integration range, that way we cannot obtain a meaningful expression.

In a more conventional notation we are seeking a function $\delta(x)$ such that

$$\int_{-\infty}^{\infty} f(y)\delta(x-y)\,\mathrm{d}y = f(x) \tag{3.60}$$

for all reasonable functions f, and

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}xs}\,\mathrm{d}s. \tag{3.61}$$

However, there is no ordinary functions with the desired properties of $\delta(x)$. To see this, we use

$$f(x) = \begin{cases} 1, & a \le x \le a+\epsilon \\ 0, & \text{otherwise} \end{cases} \tag{3.62}$$

in (3.60). Here we think of $\epsilon > 0$ as being very small. Hence, (3.60) reads

$$f(x) = \int_{-\infty}^{\infty} f(y)\delta(x-y)\,\mathrm{d}y = \int_{a}^{a+\epsilon} \delta(x-y)\,\mathrm{d}y = \int_{a-x}^{a+\epsilon-x} \delta(u)\,\mathrm{d}u. \tag{3.63}$$

If $a \le x \le a+\epsilon$, the value of this expression should be 1, and 0 otherwise. As $a \le x \le a+\epsilon$ is equivalent to $a - x \le 0 \le a + \epsilon - x$, $\delta(u)$ must be concentrated near $u = 0$. In the limit $\epsilon \to 0$, $\delta(u)$ would have to be a function that is zero when $u \neq 0$ and only has a non-zero value at $u = 0$. The integral of such a function would, however, not be 1, but 0. The conclusion is that a unit element for the convolution must be something that is not an ordinary function. Such generalisations of functions do exist and have been extensively studied; they are known as **distributions**. Here we only want to introduce the distribution that has the properties (3.60) and (3.61), if interpreted suitably.

Definition 3.24. Let f be a continuous function and $a \in \mathbb{R}$. The map that assigns the value $f(a)$ to the function f,

$$\delta_a : f \mapsto \delta_a[f] = f(a), \tag{3.64}$$

is said to be the **Dirac delta distribution**.

Remark 3.25. (i) Instead of the *Dirac delta distribution* one often speaks of the *Dirac delta function*, despite the fact that it is not an ordinary function, as we have seen.

(ii) One also often retains the notation used in (3.60),

$$\delta_x[f] = \int_{-\infty}^{\infty} f(y)\delta(x-y)\,\mathrm{d}y = f(x). \tag{3.65}$$

(iii) When $a = 0$ one simply writes $\delta_0 = \delta$. Hence,

$$\delta[f] = \int_{-\infty}^{\infty} f(y)\delta(y)\,\mathrm{d}y = f(0). \tag{3.66}$$

(iv) In the theory of distributions the function f on which the distribution is evaluated is said to be a *test function.*

Proposition 3.26. *The Dirac delta distribution (function) is a linear map, i.e., if f, g are continuous functions and $\mu, \nu \in \mathbb{R}$,*

$$\delta_a[\mu f + \nu g] = \mu\delta_a[f] + \nu\delta_a[g]. \tag{3.67}$$

Proof. The proof is straight forward,

$$\delta_a[\mu f + \nu g] = \mu f(a) + \nu g(a) = \mu \delta_a[f] + \nu \delta_a[g]. \tag{3.68}$$

□

Although δ cannot be a function, it can be approximated by functions and then a limit has to be taken.

Example 3.27. Here are some popular examples of functions $\delta^{(\epsilon)}$ that approximate δ. They involve a parameter $\epsilon > 0$ and a limit,

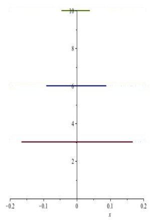
$$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} f(y)\delta^{(\epsilon)}(y - x)\,\mathrm{d}y = f(x). \tag{3.69}$$

It is important to note here that the limit is outside of the integral, i.e., the integral has to be evaluated first, and only then the limit can be performed.

1. Let

$$\delta^{(\epsilon)}(x) = \begin{cases} 0, & |x| > \epsilon \\ \frac{2}{\epsilon}, & |x| < \epsilon \end{cases}. \tag{3.70}$$

Its graph is sketched here for some values of ϵ:



This function has the property that

$$\int_{-\infty}^{\infty} f(y)\delta^{(\epsilon)}(y - x)\,\mathrm{d}y = \int_{-\infty}^{\infty} f(u + x)\delta^{(\epsilon)}(u)\,\mathrm{d}u = \frac{2}{\epsilon}\int_{-\epsilon}^{\epsilon} f(x + u)\,\mathrm{d}u. \tag{3.71}$$

With a Taylor expansion,

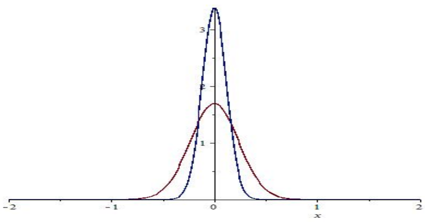
$$f(x + u) = f(x) + uf'(x) + O(u^2), \tag{3.72}$$

(under which conditions on f is this possible?), one finds that (3.69) holds.

2. Let

$$\delta^{(\epsilon)}(x) = \frac{1}{\sqrt{2\pi}\epsilon} \mathrm{e}^{-\frac{x^2}{2\epsilon^2}}, \tag{3.73}$$

whose graph is sketched here for some values of ϵ:



This function has the property that

$$\begin{aligned}\int_{-\infty}^{\infty} f(y)\delta^{(\epsilon)}(y-x)\,\mathrm{d}y &= \frac{1}{\sqrt{2\pi}\epsilon}\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{(x-y)^2}{2\epsilon^2}} f(y)\,\mathrm{d}y \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{u^2}{2}} f(\epsilon u + x)\,\mathrm{d}u.\end{aligned} \tag{3.74}$$

Here a Taylor expansion gives

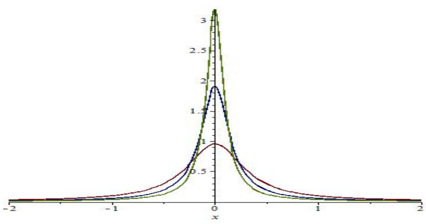
$$f(x+\epsilon u) = f(x) + O(\epsilon), \tag{3.75}$$

This, together with the result (3.23), shows that (3.69) holds.

3. Let

$$\delta^{(\epsilon)}(x) = \frac{1}{\pi\epsilon}\frac{1}{1+\frac{x^2}{\epsilon^2}}, \tag{3.76}$$

whose graph is sketched here for some values of ϵ:



This function has the property that

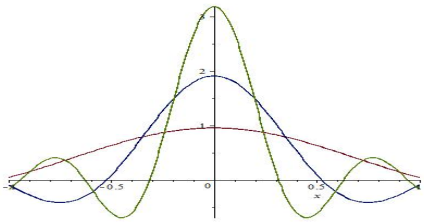
$$\begin{aligned}\int_{-\infty}^{\infty} f(y)\delta^{(\epsilon)}(y-x)\,\mathrm{d}y &= \frac{1}{\pi\epsilon}\int_{-\infty}^{\infty} \frac{f(y)}{1+\frac{(y-x)^2}{\epsilon^2}}\,\mathrm{d}y \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(x+\epsilon u)}{1+u^2}\,\mathrm{d}u.\end{aligned} \tag{3.77}$$

Again, the Taylor expansion (3.75) has to be used, together with the integral

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{1+u^2}\,\mathrm{d}u = 1, \tag{3.78}$$

to show that (3.69) holds.

4. Let
$$\delta^{(\epsilon)}(x) = \frac{\sin\frac{x}{\epsilon}}{\pi x}, \tag{3.79}$$
whose graph is sketched here for some values of ϵ:



This function has the property that
$$\begin{aligned}\int_{-\infty}^{\infty} f(y)\delta^{(\epsilon)}(y-x)\,\mathrm{d}y &= \int_{-\infty}^{\infty} \frac{\sin\frac{(y-x)}{\epsilon}}{\pi(y-x)} f(y)\,\mathrm{d}y \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin u}{u} f(x+\epsilon u)\,\mathrm{d}u.\end{aligned} \tag{3.80}$$
Again, the Taylor expansion (3.75) has to be used, together with the integral
$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin u}{u}\,\mathrm{d}u = 1, \tag{3.81}$$
to show that (3.69) holds.

Remark 3.28. All of the above examples have in common that they use a function $h(x)$ which is known to satisfy
$$\int_{-\infty}^{\infty} h(x)\,\mathrm{d}x = 1, \tag{3.82}$$
and then define
$$\delta^{(\epsilon)}(x) = \frac{1}{\epsilon} h\Big(\frac{x}{\epsilon}\Big). \tag{3.83}$$
Such a choice will always lead to (3.69).

Remark 3.29. Going back to the last example, we note that
$$\delta^{(\epsilon)}(x) = \frac{\sin\frac{x}{\epsilon}}{\pi x} = \frac{1}{2\pi \mathrm{i} x}\left(\mathrm{e}^{\mathrm{i}\frac{x}{\epsilon}} - \mathrm{e}^{-\mathrm{i}\frac{x}{\epsilon}}\right) = \frac{1}{2\pi}\int_{-\frac{1}{\epsilon}}^{\frac{1}{\epsilon}} \mathrm{e}^{\mathrm{i}xt}\,\mathrm{d}t \tag{3.84}$$
As $\epsilon \to 0$, we formally obtain
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}xt}\,\mathrm{d}t = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}xt}\,\mathrm{d}t = \mathcal{F}[\frac{1}{\sqrt{2\pi}}](x). \tag{3.85}$$

Hence, in a way that needs to be defined suitably,

$$\delta(x) = \mathcal{F}[\frac{1}{\sqrt{2\pi}}](x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}xt}\,\mathrm{d}t, \tag{3.86}$$

i.e., the Dirac delta distribution/function is the Fourier transform of a constant.

Revisiting (3.52), we see that the r.h.s. contains the Fourier representation (3.86) of the Dirac delta. Knowing the latter allows us to accept the alternative attempt to prove Parseval's relation in Remark 3.22 as a valid approach.

3.4 An application of the Fourier transform

One of the areas where Fourier transforms are often applied are inhomogeneous differential equations. As an example we consider the second order ordinary differential equation

$$-\frac{\mathrm{d}^2y}{\mathrm{d}x^2}(x) + \kappa^2 y(x) = \rho(x), \tag{3.87}$$

where κ is a constant and $\rho(x)$ is the inhomogeneity.

The first step in finding a solution $y(x)$ of this equation is to consider the related equation

$$-\frac{\mathrm{d}^2G}{\mathrm{d}x^2}(x) + \kappa^2 G(x) = \delta(x), \tag{3.88}$$

where the general inhomogeneity is replaced by a Dirac delta. The solution $G(x)$ is said to be the **Green's function** of equation (3.87).

We claim that, once the Greens function is kown, the solution of (3.87) is given by a convolution,

$$y(x) = \sqrt{2\pi}\rho * G(x) = \int_{-\infty}^{\infty} \rho(y)G(x-y)\,\mathrm{d}y. \tag{3.89}$$

To prove this statement, we insert it into (3.87),

$$\begin{aligned}-\frac{\mathrm{d}^2y}{\mathrm{d}x^2}(x) + \kappa^2 y(x) &= \int_{-\infty}^{\infty} \rho(y)\left(-\frac{\mathrm{d}^2G}{\mathrm{d}x^2}(x-y) + \kappa^2 G(x-y)\right)\mathrm{d}y \\ &= \int_{-\infty}^{\infty} \rho(y)\delta(x-y)\,\mathrm{d}y \\ &= \rho(x).\end{aligned} \tag{3.90}$$

In a next step we need to find the Green's function. Due to the singular nature of the r.h.s. of (3.88), solving this equations appears to be even more difficult than solving (3.87). Here taking Fourier transforms on both sides of (3.88) helps. According to Theorem 3.13 we know that $\mathcal{F}[-\frac{\mathrm{d}^2G}{\mathrm{d}x^2}](s) = s^2\hat{G}(s)$, whereas Theorem 3.4 implies that $\mathcal{F}[\kappa^2 G](s) = \kappa^2\hat{G}(s)$. Moreover, $\mathcal{F}[\delta](s) = \frac{1}{\sqrt{2\pi}}$. Hence from (3.88) we conclude that

$$\big(s^2 + \kappa^2\big)\hat{G}(s) = \frac{1}{\sqrt{2\pi}}, \tag{3.91}$$

or

$$\hat{G}(s) = \frac{1}{\sqrt{2\pi}} \frac{1}{s^2 + \kappa^2}. \tag{3.92}$$

Knowing the Fourier transform of the Green's function, we obtain the Green's function itself as an inverse Fourier transform of $\hat{G}(s)$,

$$G(x) = \mathcal{F}^{-1}[\hat{G}](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{G}(s) \mathrm{e}^{-\mathrm{i}sx} \,\mathrm{d}s = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}sx}}{s^2 + \kappa^2} \,\mathrm{d}s. \tag{3.93}$$

From one homework question we know that

$$\mathcal{F}[\mathrm{e}^{-a|x|}](s) = \sqrt{\frac{2}{\pi}} \frac{a}{s^2 + a^2}, \quad a > 0. \tag{3.94}$$

Hence, we conclude that

$$\hat{G}(s) = \frac{1}{2\kappa} \mathcal{F}[\mathrm{e}^{-\kappa|x|}](s). \tag{3.95}$$

This means that the Green's function is

$$G(x) = \frac{1}{2\kappa} \mathrm{e}^{-\kappa|x|}. \tag{3.96}$$

Using this result in (3.88) finlly leads to the solution

$$y(x) = \frac{1}{2\kappa} \int_{-\infty}^{\infty} \rho(y) \mathrm{e}^{-\kappa|x-y|} \,\mathrm{d}y \tag{3.97}$$

of the inhomogeneous differential equation (3.87).

This method can be successfully applied to many inhomogeneous, linear ordinary or partial differential equations.

4 Ordinary Differential Equations with Variable Coefficients

Our aim in this section is to develop methods to solve some linear ordinary differential equations (ODE). Whereas (1.1) is a general ODE, a linear equation is of the form

$$a_0(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n}(x) + a_1(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}}(x) + \cdots + a_{n-1}(x)\frac{\mathrm{d}y}{\mathrm{d}x}(x) + a_n(x)y(x) = f(x). \tag{4.1}$$

It is homogeneous when $f(x) = 0$ and inhomogeneous otherwise. The functions $a_k(x)$, $k = 0, \ldots, n$, are arbitrary coefficient functions. If they are constants one says that the ODE has constant coefficients; it has variable coefficients otherwise.

We will first study practical methods to solve a certain type of such equations that have polynomial coefficients, and later look at some of the theoretical foundations of these methods.

4.1 Cauchy-Euler equations

We consider differential equations that contain the function $y(x)$, its derivatives and the variable x only in the combination

$$x^x\frac{\mathrm{d}^k y}{\mathrm{d}x^k}(x), \quad k = 0, \ldots, n, \tag{4.2}$$

i.e., the coefficient functions in (4.1) are of the form $a_k(x) = a_k x^{n-k}$ with constants a_k.

Definition 4.1. An equation of the form

$$a_0 x^n\frac{\mathrm{d}^n y}{\mathrm{d}x^n}(x) + a_1 x^{n-1}\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}}(x) + \cdots + a_{n-1}x\frac{\mathrm{d}y}{\mathrm{d}x}(x) + a_n y(x) = f(x), \tag{4.3}$$

where the a_k, $k = 0, 1, \ldots, n$ are constants, and f is a given function, is said to be a **Cauchy-Euler equation**.

There are two approaches to solving such an equation. The first one relies on the familiar strategy of reducing an unfamiliar problem to one that is familiar. In this case, an ODE with variable coefficients is turned into an ODE with constant coefficients through a change of variables. The second method, however, avoids the change of variables but probes closely the particular form of the terms (4.2) in the Cauchy-Euler equations.

4.1.1 Method A

Assuming $x > 0$, the substitution $x = \mathrm{e}^t$ will reduce a Cauchy-Euler equation to a linear ODE with constant coefficients. We use the notation

$$y(x) = y(\mathrm{e}^t) =: \tilde{y}(t). \tag{4.4}$$

The chain rule then gives

$$\frac{\mathrm{d}\tilde{y}}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}y}{\mathrm{d}x}. \tag{4.5}$$

or, symbolically,

$$\frac{\mathrm{d}}{\mathrm{d}t} = x\frac{\mathrm{d}}{\mathrm{d}x}. \tag{4.6}$$

Lemma 4.2 (Version I)**.** *Let $n \in \mathbb{N}$, then*

$$x^n\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\mathrm{d}}{\mathrm{d}t} - 1\Big)\ldots\Big(\frac{\mathrm{d}}{\mathrm{d}t} - (n-1)\Big)\tilde{y}. \tag{4.7}$$

Proof. We use mathematical induction. The case $n = 1$ is covered by (4.5). Now assume the statement to be true for $n = k$. Then

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t} - 1\right)\ldots\left(\frac{\mathrm{d}}{\mathrm{d}t} - (k-1)\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} - k\right)\tilde{y} \\
&\quad = \left(\frac{\mathrm{d}}{\mathrm{d}t} - k\right)\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t} - 1\right)\ldots\left(\frac{\mathrm{d}}{\mathrm{d}t} - (k-1)\right)\tilde{y}\right] \\
&\quad = \left(x\frac{\mathrm{d}}{\mathrm{d}x} - k\right)\left(x^k\frac{\mathrm{d}^k y}{\mathrm{d}x^k}\right) \\
&\quad = x\left(kx^{k-1}\frac{\mathrm{d}^k y}{\mathrm{d}x^k} + x^k\frac{\mathrm{d}^{k+1} y}{\mathrm{d}x^{k+1}}\right) - kx^k\frac{\mathrm{d}^k y}{\mathrm{d}x^k} \\
&\quad = x^{k+1}\frac{\mathrm{d}^{k+1} y}{\mathrm{d}x^{k+1}}.
\end{aligned} \tag{4.8}$$

Hence the statement is also true for $n = k + 1$. □

To see this lemma at work we consider the following example.

Example 4.3. Solve the differential equation

$$x^3\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + 3x^2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} = 8x^2, \quad x > 0. \tag{4.9}$$

Letting $x = e^t > 0$ and using Lemma 4.2, the l.h.s. of (4.9) is

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t} - 1\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} - 2\right)\tilde{y} + 3\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t} - 1\right)\tilde{y} + \frac{\mathrm{d}\tilde{y}}{\mathrm{d}t} \\
&\quad = \left(\frac{\mathrm{d}^3\tilde{y}}{\mathrm{d}t^3} - 3\frac{\mathrm{d}^2\tilde{y}}{\mathrm{d}t^2} + 2\frac{\mathrm{d}\tilde{y}}{\mathrm{d}t}\right) + 3\left(\frac{\mathrm{d}^2\tilde{y}}{\mathrm{d}t^2} - \frac{\mathrm{d}\tilde{y}}{\mathrm{d}t}\right) + \frac{\mathrm{d}\tilde{y}}{\mathrm{d}t} \\
&\quad = \frac{\mathrm{d}^3\tilde{y}}{\mathrm{d}t^3}.
\end{aligned} \tag{4.10}$$

Hence, (4.9) is turned into the linear differential equation,

$$\frac{\mathrm{d}^3\tilde{y}}{\mathrm{d}t^3} = 8e^{2t}, \tag{4.11}$$

in t with constant coefficients. This can be integrated directly.

$$\begin{aligned}\frac{\mathrm{d}^2\tilde{y}}{\mathrm{d}t^2} &= 4\mathrm{e}^{2t} + 2c_1,\\ \frac{\mathrm{d}\tilde{y}}{\mathrm{d}t} &= 2\mathrm{e}^{2t} + 2c_1 t + c_2,\\ \tilde{y}(t) &= \mathrm{e}^{2t} + c_1 t^2 + c_2 t + c_3,\end{aligned} \tag{4.12}$$

where c_1, c_2 and c_3 are arbitrary constants. However, we were given an equation in x, where $t = \log x$. Hence,

$$y(x) = x^2 + c_1(\log x)^2 + c_2(\log x) + c_3. \tag{4.13}$$

Example 4.4. Solve the differential equation

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4x\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = x, \quad x > 0. \tag{4.14}$$

Letting $x = \mathrm{e}^t$ and using Lemma 4.2, the l.h.s. of (4.14) is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t} - 1\right)\tilde{y} - 4\frac{\mathrm{d}\tilde{y}}{\mathrm{d}t} + 4\tilde{y} = \frac{\mathrm{d}^2\tilde{y}}{\mathrm{d}t^2} - 5\frac{\mathrm{d}\tilde{y}}{\mathrm{d}t} + 4\tilde{y}, \tag{4.15}$$

such that (4.14) is turned into the linear differential equation,

$$\frac{\mathrm{d}^2\tilde{y}}{\mathrm{d}t^2} - 5\frac{\mathrm{d}\tilde{y}}{\mathrm{d}t} + 4\tilde{y} = \mathrm{e}^t, \tag{4.16}$$

in t with constant coefficients.

Solving the complementary equation,

$$\frac{\mathrm{d}^2\tilde{y}}{\mathrm{d}t^2} - 5\frac{\mathrm{d}\tilde{y}}{\mathrm{d}t} + 4\tilde{y} = 0, \tag{4.17}$$

with $\tilde{y} = \mathrm{e}^{mt}$ for some index m) leads to the auxiliary equation

$$m^2 - 5m + 4 = (m-1)(m-4) = 0, \tag{4.18}$$

such that $m = 1$ and $m = 4$, giving the complementary solution

$$\tilde{y}_c(t) = c_1\mathrm{e}^t + c_2\mathrm{e}^{4t}, \tag{4.19}$$

with arbitrary constants c_1 and c_2.

For the particular integral, since e^t is a solution of the complementary equation, let $\tilde{y}_p(t) = At\mathrm{e}^t$ with some constant A. Then

$$\frac{\mathrm{d}\tilde{y}_p}{\mathrm{d}t} = A\mathrm{e}^t + At\mathrm{e}^t \quad \text{and} \quad \frac{\mathrm{d}^2\tilde{y}_p}{\mathrm{d}t^2} = 2A\mathrm{e}^t + At\mathrm{e}^t. \tag{4.20}$$

Hence $y_p(t)$ is a solution provided that

$$(2+t)A\mathrm{e}^t - 5(1+t)A\mathrm{e}^t + 4tA\mathrm{e}^t = -3A\mathrm{e}^t = \mathrm{e}^t. \tag{4.21}$$

Hence, $A = -\frac{1}{3}$, and thus a particular solution is

$$\tilde{y}_p(t) = -\frac{t}{3}\mathrm{e}^t, \tag{4.22}$$

so that the general solution is

$$\tilde{y}(t) = c_1\mathrm{e}^t + c_2\mathrm{e}^{4t} - \frac{t}{3}\mathrm{e}^t. \tag{4.23}$$

In terms of the variable x, the general solution of the differential equation (4.14) is

$$y(x) = c_1 x + c_2 x^4 - \frac{x}{3}\log x. \tag{4.24}$$

The method can be extended to a slightly more general type of equations as, e.g., in the following case.

Example 4.5. Solve the differential equation

$$(1+2x)^2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 6(1+2x)\frac{\mathrm{d}y}{\mathrm{d}x} + 16y = 8(1+2x)^2, \quad 1+2x > 0. \tag{4.25}$$

Consider the substitution $1+2x = e^t > 0$ and let $y(x) = y(\frac{1}{2}(\mathrm{e}^t - 1)) = \tilde{y}(t)$. Then

$$2\frac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{e}^t = (1+2x), \tag{4.26}$$

and

$$2\frac{\mathrm{d}\tilde{y}}{\mathrm{d}t} = 2\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = (1+2x)\frac{\mathrm{d}y}{\mathrm{d}x}. \tag{4.27}$$

Iterating this gives

$$(1+2x)\frac{\mathrm{d}}{\mathrm{d}x}\left((1+2x)\frac{\mathrm{d}y}{\mathrm{d}x}\right) = 2\frac{\mathrm{d}}{\mathrm{d}t}\left(2\frac{\mathrm{d}\tilde{y}}{\mathrm{d}t}\right), \tag{4.28}$$

so that, using (4.27),

$$(1+2x)^2\frac{\mathrm{d}^2 y}{dx^2} = 4\frac{\mathrm{d}^2\tilde{y}}{\mathrm{d}t^2} - 4\frac{\mathrm{d}\tilde{y}}{\mathrm{d}t} = 2^2\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t} - 1\right)\tilde{y}. \tag{4.29}$$

Hence, rewriting (4.14) in terms of the t-variable, we have,

$$\frac{\mathrm{d}^2\tilde{y}}{\mathrm{d}t^2} - 4\frac{\mathrm{d}\tilde{y}}{\mathrm{d}t} + 4\tilde{y} = 2\mathrm{e}^{2t}. \tag{4.30}$$

Solving the complementary equation,

$$\frac{\mathrm{d}^2\tilde{y}}{\mathrm{d}t^2} - 4\frac{\mathrm{d}\tilde{y}}{\mathrm{d}t} + 4\tilde{y} = 0, \tag{4.31}$$

by substituting $\tilde{y} = \mathrm{e}^{mt}$, for some index m, gives the auxiliary equation

$$m^2 - 4m + 4 = (m-2)^2 = 0, \tag{4.32}$$

with a double root at $m = 2$. Hence, the complementary solution is

$$\tilde{y}_c(t) = c_1\mathrm{e}^{2t} + c_2 t\mathrm{e}^{2t}, \tag{4.33}$$

with arbitrary constants c_1 and c_2.

For the particular integral, let $\tilde{y}_p(t) = At^2\mathrm{e}^{2t}$ with some constant A, since both e^{2t} and $t\mathrm{e}^{2t}$ appear in the complementary solution. Then

$$\frac{\mathrm{d}\tilde{y}_p}{\mathrm{d}t} = 2At\mathrm{e}^{2t} + 2At^2\mathrm{e}^{2t} \quad \text{and} \quad \frac{\mathrm{d}^2\tilde{y}_p}{\mathrm{d}t^2} = 2A\mathrm{e}^{2t} + 8At\mathrm{e}^{2t} + 4At^2\mathrm{e}^{2t}. \tag{4.34}$$

Hence $\tilde{y}_p(t)$ is a solution provided that

$$(2A + 8At + 4At^2)\mathrm{e}^{2t} - 4(2At + 2At^2)\mathrm{e}^{2t} + 4At^2\mathrm{e}^{2t} = 2A\mathrm{e}^{2t} = 2\mathrm{e}^{2t}, \tag{4.35}$$

Hence $a = 1$ and a particular solution is

$$\tilde{y}_p(t) = t^2\mathrm{e}^{2t}. \tag{4.36}$$

The general solution hence is

$$\tilde{y}(t) = (c_1 + c_2 t + t^2)\mathrm{e}^{2t}. \tag{4.37}$$

Finally, in terms of the variable x the general solution of (4.25) is

$$y(x) = (1+2x)^2(c_1 + c_2\log(1+2x) + (\log(1+2x))^2). \tag{4.38}$$

Method A has the advantage that it always provides the solution: put in the necessary ingredients and the recipe always produces the results. However, there is an alternative method that exploits the specific form of Cauchy-Euler equations as expressed in Lemma 4.2 in a slightly different way, avoiding the change of variables $x = \mathrm{e}^t$.

4.1.2 Method B

Consider a simple power x^p, $p \in \mathbb{N}$. Then, for $r \in \mathbb{N}$ with $r < p$,

$$x^r\frac{\mathrm{d}^r}{\mathrm{d}x^r}x^p = x^r[p(p-1)\ldots(p-(r-1))x^{p-r}] = \frac{p!}{(p-r)!}x^p. \tag{4.39}$$

Hence, the l.h.s. of a Cauchy-Euler equation can be rewritten with $y = x^p$, $p > n$, as

$$\begin{aligned} a_0 x^n \frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_1 x^{n-1} \frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \ldots + a_n y &= a_0 \frac{p!}{(p-n)!} x^p + a_1 \frac{p!}{(p-n+1)!} x^p + \ldots + a_n x^p \\ &= p! \left(\frac{a_0}{(p-n)!} + \frac{a_1}{(p-n+1)!} + \cdots + \frac{a_n}{p!} \right) x^p. \end{aligned} \tag{4.40}$$

A solution of a homogeneous Cauchy-Euler equation can therefore be found if the last line of (4.40) is zero. Further, similar solutions may exist with $p < n$, or with $p \notin \mathbb{N}$.

Example 4.6. Solve the differential equation

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 4x \frac{\mathrm{d}y}{\mathrm{d}x} + 4y = x, \quad x > 0, \tag{4.41}$$

of Example 4.4. Solving the complementary equation,

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 4x \frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 0, \tag{4.42}$$

we let $y(x) = x^\alpha$, with some index $\alpha \in \mathbb{R}$. Then

$$x^2(\alpha(\alpha-1)x^{\alpha-2}) - 4x(\alpha x^{\alpha-1}) + 4x^\alpha = (\alpha^2 - 5\alpha + 4)x^\alpha = 0, \tag{4.43}$$

if

$$\alpha^2 - 5\alpha + 4 = (\alpha - 4)(\alpha - 1) = 0, \tag{4.44}$$

i.e., if $\alpha = 1$ and $\alpha = 4$. Hence the general solution of the complementary equationis

$$y(x) = c_1 x + c_2 x^4, \tag{4.45}$$

with arbitrary constants c_1 and c_2. This is, of course, equivalent to (4.19).

A particular solution $y_p(x)$ is again found in analogy to Example 4.4, only working with the variable x. Hence, following (4.22),

$$y_p(x) = -\frac{x}{3} \log x, \tag{4.46}$$

giving the general solution (4.24)

$$y(x) = c_1 x + c_2 x^4 - \frac{x}{3} \log x. \tag{4.47}$$

In a similar way we can rework Example 4.3.

Example 4.7. Solve the differential equation (4.9) of Example 4.3,

$$x^3\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 3x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} = 8x^2, \quad x > 0. \tag{4.48}$$

Considering the complementary equation

$$x^3\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 3x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} = 0, \tag{4.49}$$

and letting $y_c(x) = x^\alpha$ for some index $\alpha \in \mathbb{R}$, we obtain the condition,

$$x^3(\alpha(\alpha-1)(\alpha-2)x^{\alpha-3}) + 3x^2(\alpha(\alpha-1)x^{\alpha-2}) + x(\alpha x^{\alpha-1}) = 3\alpha^3x^\alpha = 0, \tag{4.50}$$

for which $\alpha = 0$ is a triple root. Hence, the only solution of the form x^α is $y(x) = x^0 = 1$. In order to generate further solutions, we multiply by $\log x$ and $(\log x)^2$. Hence the general solution of the complementary equation is

$$y_c(x) = c_1 + c_2\log x + c_3(\log x)^2, \tag{4.51}$$

with arbitrary constants c_1, c_2 and c_3.

For the particular solution $y_p(x)$, since x^2 does not appear in the complementary solution, we can use the form $y_p(x) = Ax^2$ with some constant A. Then

$$\frac{\mathrm{d}y_p}{\mathrm{d}x} = 2Ax, \quad \frac{\mathrm{d}^2y_p}{\mathrm{d}x^2} = 2A, \quad \frac{\mathrm{d}^3y_p}{\mathrm{d}x^3} = 0, \tag{4.52}$$

giving that $y_p(x)$ is a solution provided that

$$6Ax^2 + 2Ax^2 = 8x^2, \tag{4.53}$$

i.e., $A = 1$. Therefore, the general solution is

$$y(x) = c_1 + c_2\log x + c_3(\log x)^2 + x^2. \tag{4.54}$$

Not surprisingly, this is the same as (4.13).

4.2 Solution in series: δ-method

We saw that a particular type of ordinary differential equations with variable coefficients, the Cauchy-Euler equations, can be solved using methods that are, more or less, based on familiar approaches. When it comes to other types of equations, such approaches are unlikely to succeed, and more general methods are required. The approach that we want to look at now assumes that the differential equations possess solutions that have a representation in terms of a power series.

More precisely, it is assumed that solutions can be found that are of the form,

$$y(x) = x^c\sum_{n=0}^{\infty} a_nx^n, \tag{4.55}$$

Considering the complementary equation

$$x^3 \frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + 3x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} = 0, \tag{4.49}$$

and letting $y_c(x) = x^\alpha$ for some index $\alpha \in \mathbb{R}$, we obtain the condition,

$$x^3(\alpha(\alpha-1)(\alpha-2)x^{\alpha-3}) + 3x^2(\alpha(\alpha-1)x^{\alpha-2}) + x(\alpha x^{\alpha-1}) = 3\alpha^3 x^\alpha = 0, \tag{4.50}$$

for which $\alpha = 0$ is a triple root. Hence, the only solution of the form x^α is $y(x) = x^0 = 1$. In order to generate further solutions, we multiply by $\log x$ and $(\log x)^2$. Hence the general solution of the complementary equation is

$$y_c(x) = c_1 + c_2 \log x + c_3(\log x)^2, \tag{4.51}$$

with arbitrary constants c_1, c_2 and c_3.

For the particular solution $y_p(x)$, since x^2 does not appear in the complementary solution, we can use the form $y_p(x) = Ax^2$ with some constant A. Then

$$\frac{\mathrm{d}y_p}{\mathrm{d}x} = 2Ax, \quad \frac{\mathrm{d}^2 y_p}{\mathrm{d}x^2} = 2A, \quad \frac{\mathrm{d}^3 y_p}{\mathrm{d}x^3} = 0, \tag{4.52}$$

giving that $y_p(x)$ is a solution provided that

$$6Ax^2 + 2Ax^2 = 8x^2, \tag{4.53}$$

i.e., $A = 1$. Therefore, the general solution is

$$y(x) = c_1 + c_2 \log x + c_3(\log x)^2 + x^2. \tag{4.54}$$

Not surprisingly, this is the same as (4.13).

4.2 Solution in series: δ-method

We saw that a particular type of ordinary differential equations with variable coefficients, the Cauchy-Euler equations, can be solved using methods that are, more or less, based on familiar approaches. When it comes to other types of equations, such approaches are unlikely to succeed, and more general methods are required. The approach that we want to look at now assumes that the differential equations possess solutions that have a representation in terms of a power series.

More precisely, it is assumed that solutions can be found that are of the form,

$$y(x) = x^c \sum_{n=0}^{\infty} a_n x^n, \tag{4.55}$$

where $a_0 \neq 0$ so that the **index** $c \in \mathbb{R}$ is the smallest exponent that occurs in the solution. Hence, the solution is assumed to be a product of a power x^c, where c need not be an integer, and a function that is represented by a power series. The task then is to determine the index $c \in \mathbb{R}$ as well as the coefficients $a_n \in \mathbb{R}$, $n \in \mathbb{N}_0$. A brute force way of doing this would be to insert (4.55) into the differential equation and rearrange the result in such a way that one can potentially extract the a_n and c.

Doing this in a slightly more organised way can be achieved with the help of the expression,

$$\delta = x\frac{\mathrm{d}}{\mathrm{d}x}, \tag{4.56}$$

which is a **differential operator**, i.e., an object that acts ("operates") on functions. Here, we retain x as the fundamental variable (not switching to t as we did for the Cauchy-Euler equation). We then need four results. The first is simply a rewrite of Lemma 4.2 in terms of δ, using (4.6).

Lemma 4.8 (Version II of Lemma 4.2)**.** *Let $n \in \mathbb{N}$, then*

$$x^n\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = \delta(\delta-1)\dots(\delta-(n-1))y. \tag{4.57}$$

The second result describes the action of δ on powers.

Lemma 4.9. *Let $p \in \mathbb{N}_0$ and $\lambda \in \mathbb{R}$, then*

$$\delta^p x^\lambda = \lambda^p x^\lambda. \tag{4.58}$$

Proof. Use mathematical induction on p: (i) If $p = 0$, then $x^\lambda = x^\lambda$ is obvious; (ii) if the statement is true for $p = k$, then

$$\delta^{k+1}x^\lambda = \delta(\delta^k x^\lambda) = x\frac{\mathrm{d}}{\mathrm{d}x}\left(\lambda^k x^\lambda\right) = x\lambda^k(\lambda x^{\lambda-1}) = \lambda^{k+1}x^\lambda. \tag{4.59}$$

Hence, the statement holds for $p = k+1$. ☐

The third result is a generalisation to polynomials.

Lemma 4.10. *Let P be a polynomial of degree n,*

$$P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n, \tag{4.60}$$

and let $\lambda \in \mathbb{R}$. Then

$$P(\delta)x^\lambda = P(\lambda)x^\lambda. \tag{4.61}$$

Proof. This is a straight forward calculation using Lemma 4.9,

$$\begin{aligned} P(\delta)x^\lambda &= \sum_{k=0}^{n} a_k\delta^{n-k}x^\lambda = \sum_{k=0}^{n} a_k(\lambda^{n-k}x^\lambda) \\ &= \left(\sum_{k=0}^{n} a_k\lambda^{n-k}\right)x^\lambda = P(\lambda)x^\lambda. \end{aligned} \tag{4.62}$$

☐

The final result is a certain product rule.

Lemma 4.11. *Let $p \in \mathbb{N}$ and $y(x)$ be a differentiable function, then*

$$\delta(x^p y) = p x^p y + x^p \delta y. \tag{4.63}$$

Proof. This is a simple calculation,

$$\delta(x^p y) = x \frac{\mathrm{d}}{\mathrm{d}x}(x^p y) = x p x^{p-1} y + x x^p \frac{\mathrm{d}y}{\mathrm{d}x} = p x^p y + x^p \delta y. \tag{4.64}$$

□

In order to see how the above results can be used to find the solutions of an ODE we consider an example.

Example 4.12. Let the differential equation,

$$4x \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0, \tag{4.65}$$

be given. This is not a Cauchy-Euler equation. If we want to use δ by applying Lemma 4.8 we need to multiply this equation with x,

$$4x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}y}{\mathrm{d}x} + xy = 0, \tag{4.66}$$

giving

$$4\delta(\delta - 1)y + 2\delta y + xy = 0, \tag{4.67}$$

or

$$(4\delta^2 - 2\delta)y + xy = 0. \tag{4.68}$$

We now *assume* that a solution of the form (4.55) exists. We then get

$$\begin{aligned}(4\delta^2 - 2\delta)y &= \sum_{n=0}^{\infty} a_n (4\delta^2 - 2\delta) x^{n+c} = \sum_{n=0}^{\infty} a_n (4(n+c)^2 - 2(n+c)) x^{n+c} \\ &= \sum_{n=0}^{\infty} a_n 2(n+c)(2n+2c-1) x^{n+c},\end{aligned} \tag{4.69}$$

and

$$xy = \sum_{m=0}^{\infty} a_m x^{m+1+c} = \sum_{n=1}^{\infty} a_{n-1} x^{n+c}. \tag{4.70}$$

Adding the r.h.s. of (4.69) and (4.70) gives,

$$a_0 2c(2c-1)x^c + \sum_{n=1}^{\infty} \left(a_n 2(n+c)(2n+2c-1) + a_{n-1}\right) x^{n+c}, \tag{4.71}$$

which is required to vanish if (4.55) is to represent a solution. In order to achieve this all coefficients of powers x^{n+c}, $n = 0, 1, 2, \ldots$, in (4.71) need to vanish:

(i) $2c(2c-1)a_0 = 0$;

(ii) $a_n 2(n+c)(2n+2c-1)) + a_{n-1} = 0$, for all $n \in \mathbb{N}$.

These conditions can be solved as follows:

(i) Since it was assumed that $a_0 \neq 0$ one obtains the **indicial equation**,

$$c(2c-1) = 0, \tag{4.72}$$

with solutions $c_1 = 0$ and $c_2 = \frac{1}{2}$.

(ii) The second condition has to be solved for the two indices c_1 and c_2 separately.

(a) With $c_1 = 0$, condition (ii) reads

$$a_n = -\frac{1}{2n(2n-1)} a_{n-1}, \quad n \in \mathbb{N}. \tag{4.73}$$

Here a_{n-1} can be expressed in the same way, with $n-1$ instead of n. This process can be iterated,

$$\begin{aligned} a_n &= -\frac{1}{2n(2n-1)} \left(-\frac{1}{2(n-1)(2(n-1)-1)} a_{(n-1)-1} \right) \\ &= \frac{(-1)^2}{2n(2n-1)(2n-2)(2n-3)} a_{n-2} = \dots \\ &= \frac{(-1)^n}{2n(2n-1)\dots 2\cdot 1} a_0 = \frac{(-1)^n}{(2n)!} a_0. \end{aligned} \tag{4.74}$$

This gives the first solution

$$\begin{aligned} y_1(x) &= x^0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} a_0 x^n = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (\sqrt{x})^{2n} \\ &= a_0 \cos\sqrt{x}. \end{aligned} \tag{4.75}$$

(b) With $c_2 = \frac{1}{2}$, condition (ii) reads

$$a_n = -\frac{1}{(2n+1)2n} a_{n-1}, \quad n \in \mathbb{N}. \tag{4.76}$$

Iterating as above,

$$\begin{aligned} a_n &= -\frac{1}{(2n+1)2n} \left(-\frac{1}{(2(n-1)+1)2(n-1)} a_{(n-1)-1} \right) \\ &= \frac{(-1)^2}{(2n+1)2n(2n-1)(2n-2)} a_{n-2} = \dots \\ &= \frac{(-1)^n}{(2n+1)2n\dots 3\cdot 2} a_0 = \frac{(-1)^n}{(2n+1)!} a_0. \end{aligned} \tag{4.77}$$

The second solution hence is

$$\begin{aligned} y_2(x) &= x^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} a_0 x^n = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (\sqrt{x})^{2n+1} \\ &= a_0 \sin \sqrt{x}. \end{aligned} \tag{4.78}$$

Both $y_1(x)$ and $y_2(x)$ satisfy the equation (4.65). Since this is a linear ODE, so does any linear combination of them. The general solution therefore, is

$$y(x) = d_1 \cos \sqrt{x} + d_2 \sin \sqrt{x}, \tag{4.79}$$

where d_1 and d_2 are arbitrary constants.

In the example above the difference between the two indices $c_1 - c_2 = -\frac{1}{2}$ is non-integer. Later we will see that this difference being integer or not plays a role when it comes to power series solutions. Therefore, we want to examine an example where this difference is an integer.

Example 4.13. Solve the differential equation

$$x(1+x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - (1-x)\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0. \tag{4.80}$$

This is not a Cauchy-Euler equation. If we want to use δ by applying Lemma 4.8 we need to multiply this equation with x,

$$(1+x)x^2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - (1-x)x\frac{\mathrm{d}y}{\mathrm{d}x} - xy = 0, \tag{4.81}$$

or

$$(1+x)\delta(\delta-1)y - (1-x)\delta y - xy = 0. \tag{4.82}$$

Separating those terms involving x explicitly from those with no specific x-dependence,

$$(\delta^2 - 2\delta)y + x(\delta^2 - 1)y = 0. \tag{4.83}$$

The following notation is useful,

$$F(\delta)y := \delta(\delta-2)y + x(\delta-1)(\delta+1)y, \tag{4.84}$$

so that the differential equation (4.80) is equivalent to $F(\delta)y = 0$.

We now *assume* that a solution of the form (4.55) exists. We then get

$$\begin{aligned} F(\delta)y &= \sum_{n=0}^{\infty} a_n \delta(\delta-2)x^{n+c} + x\sum_{n=0}^{\infty} a_n(\delta-1)(\delta+1)x^{n+c} \\ &= \sum_{n=0}^{\infty} a_n(n+c)(n+c-2)x^{n+c} + \sum_{n=0}^{\infty} a_n(n+c-1)(n+c+1)x^{n+c+1} \\ &= \sum_{n=0}^{\infty} a_n(n+c)(n+c-2)x^{n+c} + \sum_{m=1}^{\infty} a_{m-1}(m+c-2)(m+c)x^{m+c} \\ &= a_0 c(c-2)x^c + \sum_{n=1}^{\infty} \big((n+c)(n+c-2)a_n + (n+c)(n+c-2)a_{n-1}\big)x^{n+c}. \end{aligned} \tag{4.85}$$

In order for this expression to be zero, all coefficients of powers x^{n+c}, $n = 0, 1, 2, \ldots$ need to vanish:

(i) $c(c-2)a_0 = 0$;

(ii) $(n+c)(n+c-2)(a_n + a_{n-1}) = 0$, for all $n \in \mathbb{N}$.

These conditions can be solved as follows:

(i) Since it was assumed that $a_0 \neq 0$ one obtains the indicial equation,

$$c(c-2) = 0, \tag{4.86}$$

with solutions $c_1 = 0$ and $c_2 = 2$. (We note that the difference between the indices is integer.)

(ii) The second condition has to be solved for the two indices c_1 and c_2 separately.

(a) With $c_1 = 0$, condition (ii) reads

$$n(n-2)(a_n + a_{n-1}) = 0, \quad n \in \mathbb{N}. \tag{4.87}$$

When $n = 1$ the condition is that $a_1 = -a_0$, and this is non-zero. However, when $n = 2$, no condition is imposed on the coefficient a_2. In the case $n \geq 3$ one finds that,

$$a_n = -a_{n-1} = a_{n-2} = \cdots = (-1)^{n-2} a_2, \tag{4.88}$$

by iteration. Since there is no condition on a_2, we are free to choose it as $a_2 = 0$; hence $a_n = 0$ for all $n \geq 2$. Hence, in this case the solution (4.55) is

$$y_1(x) = a_0 x^{c_1} + a_1 x^{1+c_1} = a_0(1-x). \tag{4.89}$$

(b) With $c_1 = 2$, condition (ii) reads

$$n(n+2)(a_n + a_{n-1}) = 0, \quad n \in \mathbb{N}. \tag{4.90}$$

Here the facto $n(n+2) \neq 0$, so that the condition $a_n = -a_{n-1}$ leads to

$$a_n = -a_{n-1} = a_{n-2} = \cdots = (-1)^n a_0, \tag{4.91}$$

by iteration. Hence, in this case the solution (4.55) is

$$y_2(x) = x^2 \sum_{n=0}^{\infty} (-1)^n a_0 x^n = a_0 x^2 \sum_{n=0}^{\infty} (-x)^n = a_0 \frac{x^2}{1+x}, \qquad |x| < 1. \tag{4.92}$$

Here we made use of the *geometric series*,

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \tag{4.93}$$

that converges when $|x| < 1$.

Therefore, the general solution of (4.80) is

$$y(x) = d_1(1-x) + d_2\frac{x^2}{1+x}, \tag{4.94}$$

where d_1 and d_2 are arbitrary constants.

4.3 Solution in series: theory

Above we saw in some examples that seeking a solution of a linear homogeneous ODE in the form (4.55) of a series can be a viable approach. It is, however, by no means guaranteed that this will always be the case. We now want to explore conditions under which it is known that solutions in series exist. For that purpose we are restricting our attention to the case of second order equations, for which we will use the notation,

$$P(x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}(x) + Q(x)\frac{\mathrm{d}y}{\mathrm{d}x}(x) + R(x)y(x) = 0. \tag{4.95}$$

At points $x \in \mathbb{R}$ where $P(x) \neq 0$ we introduce the functions

$$p(x) := \frac{Q(x)}{P(x)} \qquad \text{and} \qquad q(x) := \frac{R(x)}{P(x)}. \tag{4.96}$$

For the following we need to single out some $x \in \mathbb{R}$.

Definition 4.14. A point $x_0 \in \mathbb{R}$ is said to be an **ordinary point** of the differential equation (4.95) if there exists an interval (a, b) such that $x_0 \in (a, b)$ and the functions $p(x)$ and $q(x)$ defined in (4.96) have Taylor series expansions about x_0,

$$p(x) = \sum_{n=0}^{\infty} \alpha_n (x-x_0)^n \quad \text{and} \quad q(x) = \sum_{n=0}^{\infty} \beta_n (x-x_0)^n, \tag{4.97}$$

that converge in (a, b). A point $x_0 \in \mathbb{R}$ that is not an ordinary point is said to be a **singular point**.

Certain singular points play a particular role.

Definition 4.15. A singular point $x_0 \in \mathbb{R}$ is said to be an **regular singular point** of the differential equation (4.95) if there exists an interval (c, d) such that $x_0 \in (c, d)$ and the functions $(x-x_0)p(x)$ and $(x-x_0)^2q(x)$ have Taylor series expansions about x_0

$$(x-x_0)p(x) = \sum_{n=0}^{\infty} \gamma_n (x-x_0)^n \quad \text{and} \quad (x-x_0)^2q(x) = \sum_{n=0}^{\infty} \delta_n (x-x_0)^n, \tag{4.98}$$

that converge in (c, d).

Remark 4.16. (i) Note that the condition (4.98) for a regular singular point is weaker than the one (4.97) for an ordinary point. In the case of a regular singular point at x_0 condition (4.98) can be phrased for p, q as

$$\begin{aligned} p(x) &= \sum_{n=0}^{\infty} \gamma_n (x-x_0)^{n-1} = \frac{\gamma_0}{x-x_0} + \sum_{n=0}^{\infty} \gamma_{n+1}(x-x_0)^n, \\ q(x) &= \sum_{n=0}^{\infty} \delta_n (x-x_0)^{n-2} = \frac{\delta_0}{(x-x_0)^2} + \frac{\delta_1}{x-x_0} + \sum_{n=0}^{\infty} \delta_{n+2}(x-x_0)^n, \end{aligned} \tag{4.99}$$

so that p and/or q may be singular at a regular singular point x_0.

(ii) From (4.98) one concludes that at a regular singular point x_0 the limits

$$\lim_{x\to x_0} (x-x_0)p(x) = \gamma_0 \quad \text{and} \quad \lim_{x\to x_0} (x-x_0)^2 q(x) = \delta_0 \tag{4.100}$$

are both finite. This often is a practical criterion to decide whether x_0 is a regular singular point.

(iii) The motivation for singling out regular singular points is the following: Suppose for simplicity that $x_0 = 0$ is a regular singular point and divide (4.95) by $P(x)$. After a multiplication with x^2 the equation becomes

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}(x) + (xp(x))x\frac{\mathrm{d}y}{\mathrm{d}x}(x) + (x^2 q(x))y(x) = 0. \tag{4.101}$$

This looks like a Cauchy-Euler equation whose coefficients are multiplied by the functions $xp(x)$ and $x^2q(x)$ that have convergent power series representations. The hope is that such an equation has a solution that is a product of a solution of a Cauchy-Euler equation, $y_{C-E}(x) = x^c$, and a convergent power series.

Example 4.17. (a) In Example 4.6 the coefficient functions are $P(x) = x^2$, $Q(x) = -4x$ and $R(x) = 4$, hence $p(x) = -\frac{4}{x}$ and $q(x) = \frac{4}{x^2}$. The only singular point is $x_0 = 0$. Since

$$\lim_{x\to 0} x\left(-\frac{4}{x}\right) = -4 \quad \text{and} \quad \lim_{x\to 0} x^2 \frac{4}{x^2} = 4, \tag{4.102}$$

this is a regular singular point.

(b) In Example 4.13 the coefficient functions are $P(x) = x(1+x)$, $Q(x) = x-1$ and $R(x) = -1$, hence $p(x) = \frac{x-1}{x(x+1)}$ and $q(x) = -\frac{1}{x(x+1)}$. The singular points are $x_1 = 0$ and $x_2 = -1$. Since

$$\lim_{x\to 0} x\frac{x-1}{x(1+x)} = -1 \quad \text{and} \quad \lim_{x\to 0} x^2 \frac{-1}{x(1+x)} = 0, \tag{4.103}$$

as well as

$$\lim_{x\to -1} (x+1)\frac{x-1}{x(1+x)} = 2 \quad \text{and} \quad \lim_{x\to -1} (x+1)^2 \frac{x-1}{x(1+x)} = 0, \tag{4.104}$$

both are regular singular points.

The definitions above are valid for finite points but how does one test the behaviour of the point at infinity? The answer is to use a suitable transformation.

Definition 4.18. The **point at infinity** is said to be an ordinary (a regular singular) point of the differential equation (4.95) when zero is an ordinary (a regular singular) point for the equation derived under the transformation $t = x^{-1}$.

For the following statement we need the concept of a *formal solution* of a differential equation. This is a power series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^{n+c} \tag{4.105}$$

that, when inserted in the equation (4.95) and terms are formally rearranged, produces a solution but where convergence is not known or not given. We will also apply a shift of variables, $x - x_0 \mapsto x$, and therefore only focus on the case $x_0 = 0$.

Theorem 4.19. *Let the differential equation* (4.95) *have an ordinary or a regular singular point at* $x_0 = 0$. *Then there exists at least one formal power series solution of the form* (4.105) *with* $a_0 \neq 0$. *Unless the roots of the indicial equation differ by an integer, there are two linearly independent formal power series solutions of the differential equation.*

A stronger result, where the existence of a formal solution is replaced by a statement concerning the convergence of that formal power series solution, is given when a condition on the solutions of the indicial equation is satisfied. We recall that for a second order ODE (4.95) the indicial equation is a quadratic equation, $I(c) = 0$, with a quadratic polynomial $I(c)$.

Theorem 4.20. *Let the second-order, linear ODE* (4.95) *have an ordinary or a regular singular point at the origin, and let* c *be a root of the indicial equation,* $I(c) = 0$, *such that* $I(c + n) = 0$ *for no* $n \in \mathbb{N}_0$. *Then the formal power series* (4.105) *converges to a solution of* (4.95) *for* $0 < |x| < \sigma$, *with some* $\sigma > 0$.

Remark 4.21. (i) Although power series converge in intervals, $x \in (x_0 - R, x_0 + R)$, in the case of a regular singular point at x_0 it may be that the formal solution (4.105) converges to an actual solution for $x_0 < x < x_0 + \sigma$, or $x_0 - \sigma < x < x_0$.

(ii) This theorem covers the solutions corresponding to both indices, c_1 and c_2, when the roots of the indicial equation do not differ by an integer, i.e., $c_1 - c_2 \notin \mathbb{Z}$. When the indices do differ by a natural number the theorem covers the solution related to the larger index, say $c_1 > c_2$, since in that case $c_2 = c_1 + n$ with $n < 0$.

When the roots of the indicial equation do differ by an integer, the following result holds.

Theorem 4.22. *Given that the roots c_1 and c_2 of the indicial equation of a second-order, linear ODE* (4.95) *having an ordinary or a regular singular point at $x_0 = 0$ differ by a non-negative integer, $c_2 = c_1 - n$ with $n \in \mathbb{N}_0$, there exists a basis of solutions of the form*

$$y_1(x) = \sum_{k=1}^{\infty} a_k x^{k+c_1} \quad \textit{and} \quad y_2(x) = \sum_{k=1}^{\infty} b_k x^{k+c_2} + C\, y_1(x) \log x, \tag{4.106}$$

where C is a constant and the power series are convergent in a neighbourhood of the origin.

We have met, and will only meet now, the case with $C = 0$. The logarithmic term will arise when the roots of the indicial equation are equal or when they differ by an integer such that one of the coefficients becomes infinite - the two cases that are not required now. The final result is a technical one about radii of convergence.

Theorem 4.23. *The radius of convergence of a solution in series is at least as large as the smaller of the radii of convergence of the series* (4.98)*.*

4.3.1 Examples

To turn from the theory to the practice of the method, we no longer need to assume that the series solutions exist: we know the theorems that guarantee their existence.

The first example demonstrates a case where the roots of the indicial equation differ by a non-zero integer.

Example 4.24. Solve the ODE

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 4x \frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 + 6)y = 0. \tag{4.107}$$

The term in $x^2 y$ prevents the equation from being of Cauchy-Euler type. However, the derivatives are partnered with the correct powers of x so that we may use Lemma 4.8 directly.

The coefficient functions are $P(x) = x^2$, $Q(x) = -4x$ and $R(x) = x^2 + 6$. Hence,

$$p(x) = -\frac{4}{x} \quad \text{and} \quad q(x) = 1 + \frac{6}{x^2}. \tag{4.108}$$

Therefore, $x_0 = 0$ is a regular singular point and all other $x \in \mathbb{R}$ are ordinary points. Since $xp(x) = -4$ and $x^2 q(x) = x^2 + 6$ are finite sums the radii of convergence are infinite. From Theorem 4.23 we conclude that there exists a solution in series with an infinite radius of convergence.

With the help of Lemma 4.8 the ODE (4.107) is turned into the equation

$$\delta(\delta - 1)y - 4\delta y + (x^2 + 6)y = 0. \tag{4.109}$$

We rearrange the terms, separating those with explicit x-dependence from those without, and introducing

$$F(\delta)y(x) := (\delta^2 - 5\delta + 6)y + x^2 y = (\delta - 3)(\delta - 2)y + x^2 y. \tag{4.110}$$

Theorem 4.22. *Given that the roots c_1 and c_2 of the indicial equation of a second-order, linear ODE* (4.95) *having an ordinary or a regular singular point at $x_0 = 0$ differ by a non-negative integer, $c_2 = c_1 - n$ with $n \in \mathbb{N}_0$, there exists a basis of solutions of the form*

$$y_1(x) = \sum_{k=0}^{\infty} a_k x^{k+c_1} \quad \textit{and} \quad y_2(x) = \sum_{k=0}^{\infty} b_k x^{k+c_2} + C\, y_1(x) \log x, \tag{4.106}$$

where C is a constant and the power series are convergent in a neighbourhood of the origin.

We have met, and will only meet now, the case with $C = 0$. The logarithmic term will arise when the roots of the indicial equation are equal or when they differ by an integer such that one of the coefficients becomes infinite - the two cases that are not required now. The final result is a technical one about radii of convergence.

Theorem 4.23. *The radius of convergence of a solution in series is at least as large as the smaller of the radii of convergence of the series* (4.98).

4.3.1 Examples

To turn from the theory to the practice of the method, we no longer need to assume that the series solutions exist: we know the theorems that guarantee their existence.

The first example demonstrates a case where the roots of the indicial equation differ by a non-zero integer.

Example 4.24. Solve the ODE

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 4x \frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 + 6)y = 0. \tag{4.107}$$

The term in $x^2 y$ prevents the equation from being of Cauchy-Euler type. However, the derivatives are partnered with the correct powers of x so that we may use Lemma 4.8 directly.

The coefficient functions are $P(x) = x^2$, $Q(x) = -4x$ and $R(x) = x^2 + 6$. Hence,

$$p(x) = -\frac{4}{x} \quad \text{and} \quad q(x) = 1 + \frac{6}{x^2}. \tag{4.108}$$

Therefore, $x_0 = 0$ is a regular singular point and all other $x \in \mathbb{R}$ are ordinary points. Since $xp(x) = -4$ and $x^2 q(x) = x^2 + 6$ are finite sums the radii of convergence are infinite. From Theorem 4.23 we conclude that there exists a solution in series with an infinite radius of convergence.

With the help of Lemma 4.8 the ODE (4.107) is turned into the equation

$$\delta(\delta - 1)y - 4\delta y + (x^2 + 6)y = 0. \tag{4.109}$$

We rearrange the terms, separating those with explicit x-dependence from those without, and introducing

$$F(\delta)y(x) := (\delta^2 - 5\delta + 6)y + x^2 y = (\delta - 3)(\delta - 2)y + x^2 y. \tag{4.110}$$

The solution in series associated with the regular singular point $x_0 = 0$ has the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+c}, \quad a_0 \neq 0, \tag{4.111}$$

which, when inserted in (4.110), gives

$$\begin{aligned} F(\delta)y(x) &= \sum_{n=0}^{\infty} a_n(\delta-3)(\delta-2)x^{n+c} + \sum_{n=0}^{\infty} a_n x^{n+c+2} \\ &= \sum_{n=0}^{\infty} a_n(n+c-3)(n+c-2)x^{n+c} + \sum_{k=2}^{\infty} a_{k-2}x^{k+c} \\ &= a_0(c-3)(c-2)x^c + a_1(c-2)(c-1)x^{c+1} \\ &\qquad + \sum_{n=2}^{\infty}((n+c-3)(n+c-2)a_n + a_{n-2})x^{n+c} \\ &= 0. \end{aligned} \tag{4.112}$$

This condition is satisfied if all the coefficients of the powers of x are zero. Hence we require:

(i) $(c-3)(c-2)a_0 = 0$,

(ii) $(c-2)(c-1)a_1 = 0$,

(iii) $(n+c-3)(n+c-2)a_n + a_{n-2} = 0$, for all $n \geq 2$.

These conditions can be solved as follows:

(i) Since it was required that $a_0 \neq 0$, the indicial equation

$$(c-3)(c-2) = 0 \tag{4.113}$$

yields the two indices $c_1 = 3$ and $c_2 = 2$.

(ii) For the index $c_1 = 3$ we obtain $a_1 = 0$, but for the second index $c_2 = 2$ the condition $0 \cdot a_1 = 0$ leaves a_1 indeterminate. We are free to choose $a_1 = 0$ here too.

(iii) The two values for the index lead to conditions

$$n(n \pm 1)a_n = -a_{n-2}, \quad \text{for } n \geq 2. \tag{4.114}$$

These are so-called two-step recurrence relations. If we begin with an even coefficient, a_{2k} say, we will step down through the even coefficients to reach $a_0 \neq 0$, whereas if we begin with an odd coefficient, a_{2k+1} say, we step down via the odd coefficients to reach $a_1 = 0$. Hence, we need to treat the odd and even coefficients separately.

There are four cases in which we need to solve the two-step recurrence relation (4.114): even and odd n for each of the two indices.

1. Let $c_1 = 3$, where (4.114) reads $n(n+1)a_n = -a_{n-2}$. Hence:

   (a) If $n = 2k+1$, $k \geq 1$, this gives

$$\begin{aligned} a_{2k+1} &= \frac{-1}{(2k+2)(2k+1)} a_{2k-1} \\ &= \dots \\ &= \frac{2(-1)^k}{(2k+2)!} a_1 = 0. \end{aligned} \tag{4.115}$$

   (b) If $n = 2k$, $k \geq 1$, we have

$$\begin{aligned} a_{2k} &= \frac{-1}{(2k+1)(2k)} a_{2(k-1)} \\ &= \dots \\ &= \frac{(-1)^k}{(2k+1)!} a_0 \neq 0. \end{aligned} \tag{4.116}$$

   The series solution associated with the index $c_1 = 3$ therefore is

$$\begin{aligned} y_1(x) &= a_0 x^3 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)!} a_0 x^{2k+3} \\ &= a_0 x^2 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \\ &= a_0 x^2 \sin x \, . \end{aligned} \tag{4.117}$$

2. Let $c_2 = 2$, where (4.114) reads $n(n-1)a_n = -a_{n-2}$. Hence:

   (a) If $n = 2k+1$, $k \geq 1$, this gives

$$\begin{aligned} a_{2k+1} &= \frac{-1}{(2k+1)(2k)} a_{2k-1} \\ &= \dots \\ &= \frac{(-1)^k}{(2k+1)!} a_1 = 0. \end{aligned} \tag{4.118}$$

   (b) If $n = 2k$, $k \geq 1$, we have

$$\begin{aligned} a_{2k} &= \frac{-1}{(2k)(2k-1)} a_{2(k-1)} \\ &= \dots \\ &= \frac{(-1)^k}{(2k)!} a_0 \neq 0. \end{aligned} \tag{4.119}$$

The series solution associated with the index $c_2 = 2$ therefore is

$$\begin{aligned} y_2(x) &= a_0 x^2 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} a_0 x^{2k+2} \\ &= a_0 x^2 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \\ &= a_0 x^2 \cos x. \end{aligned} \tag{4.120}$$

By Theorem 4.22 the two solutions, $y_1(x)$ and $y_2(x)$, are linearly independent. Hence the general solution is

$$y(x) = x^2(d_1 \sin x + d_2 \cos x), \tag{4.121}$$

with arbitrary constants d_1 and d_2.

Example 4.25. The final example plays an important role when expressing the three-dimensional Laplacian.

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}, \tag{4.122}$$

that acts on functions $f(x_1, x_2, x_3)$ of three variables that are given as the three cartesian coordinates of points $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$. Introducing spherical polar coordinates (r, θ, φ) via

$$x_1 = r \sin\varphi \sin\theta, \quad x_2 = r\cos\varphi \sin\theta, \quad x_3 = r\cos\theta, \tag{4.123}$$

the Laplacian can be expressed in these coordinates. If the Laplacian occurs in a differential equation such as, e.g., an eigenvalue equation $\Delta f(x_1, x_2, x_3) = \lambda f(x_1, x_2, x_3)$ one can first transform to spherical polar coordinates and then separate variables, $f(x_1, x_2, x_3) \mapsto R(r)F(\varphi)T(\theta)$. The ODE that emerges for $T(\theta)$, after substituting $x = \cos\theta \in [-1, 1]$, will be the **Legendre differential equation**,

$$(1 - x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + \mu y = 0, \tag{4.124}$$

for the function $y(x) = T(\theta)$, where $x \in [-1, 1]$. Here $\mu \in \mathbb{R}$ is a parameter.

We identify the coefficient functions $P(x) = 1 - x^2$, $Q(x) = -2x$ and $R(x) = \mu$, leading to

$$p(x) = \frac{2x}{x^2 - 1} \qquad \text{and} \qquad q(x) = \frac{\mu}{1 - x^2}. \tag{4.125}$$

The singular points of the Legendre equation are, therefore, $x_0 = \pm 1$. With $x_0 = 1$ we find that

$$(x - 1)p(x) = \frac{2x}{x + 1} \qquad \text{and} \qquad (x - 1)^2 q(x) = \mu\frac{1 - x}{1 + x}, \tag{4.126}$$

both of which are regular near $x_0 = 1$. On the other hand, with $x_0 = -1$ we find that

$$(x + 1)p(x) = \frac{2x}{x - 1} \qquad \text{and} \qquad (x + 1)^2 q(x) = \mu\frac{1 + x}{1 - x}. \tag{4.127}$$

Again, both functions are regular near $x_0 = -1$. Hence, $x_0 = \pm 1$ are both regular singular points. We note that theses are the end-points of the interval $[-1, 1]$ over which solutions are sought.

In the following we will work with the ordinary point $x_0 = 0$ and seek solutions in series of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+c}\,, \quad a_0 \neq 0. \tag{4.128}$$

In order to be able to apply the δ-method we multiply the equation by x^2, giving

$$(1 - x^2)\, x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2x^2\, x\frac{\mathrm{d}y}{\mathrm{d}x} + \mu x^2 y = 0, \tag{4.129}$$

or

$$\begin{aligned} F(\delta)y &:= (1 - x^2)\delta(\delta - 1)y - 2x^2\delta y + \mu x^2 y \\ &= \delta(\delta - 1)y - x^2(\delta^2 + \delta - \mu)y = 0. \end{aligned} \tag{4.130}$$

A more convenient form of this equation can be achieved with redefining the constant μ as

$$\mu = \lambda(\lambda + 1). \tag{4.131}$$

Then (4.130) becomes

$$\delta(\delta - 1)y - x^2(\delta + \lambda + 1)(\delta - \lambda)y = 0. \tag{4.132}$$

Inserting the series representation (4.128) gives

$$\begin{aligned} F(\delta)y(x) &= \sum_{n=0}^{\infty} a_n \delta(\delta - 1)x^{n+c} - x^2 \sum_{n=0}^{\infty} a_n(\delta + \lambda + 1)(\delta - \lambda)x^{n+c} \\ &= \sum_{n=0}^{\infty} a_n(n + c)(n + c - 1)x^{n+c} - \sum_{n=0}^{\infty} a_n(n + c + \lambda + 1)(n + c - \lambda)x^{n+c+2} \\ &= \sum_{n=0}^{\infty} a_n(n + c)(n + c - 1)x^{n+c} - \sum_{k=2}^{\infty} a_{k-2}(k + c + \lambda - 1)(k + c - \lambda - 2)x^{k+c} \\ &= a_0 c(c - 1)x^c + a_1(1 + c)cx^{1+c} \\ &\quad + \sum_{n=2}^{\infty}[(n + c)(n + c - 1)a_n - (n + c + \lambda - 1)(n + c - \lambda - 2)a_{n-2}]x^{n+c} \\ &= 0. \end{aligned} \tag{4.133}$$

This condition is satisfied if all the coefficients of the powers of x are zero. Hence we require:

(i) $c(c - 1)a_0 = 0$,

(ii) $c(c+1)a_1 = 0$,

(iii) $(n+c)(n+c-1)a_n - (n+c+\lambda-1)(n+c-\lambda-2)a_{n-2} = 0$, for all $n \geq 2$.

These conditions can be solved as follows:

(i) Since it was required that $a_0 \neq 0$, the indicial equation

$$c(c-1) = 0 \tag{4.134}$$

yields the two indices $c_1 = 1$ and $c_2 = 0$.

(ii) For the index $c_1 = 1$ we obtain $a_1 = 0$, but for the second index $c_2 = 0$ the condition $0 \cdot a_1 = 0$ leaves a_1 indeterminate. We are free to choose $a_1 = 0$ here too.

(iii) For the index $c_1 = 1$ this condition becomes

$$n(n+1)a_n = (n+\lambda)(n-\lambda-1)a_{n-2}, \quad \text{for } n \geq 2, \tag{4.135}$$

whereas for the index $c_2 = 0$ it is

$$n(n-1)a_n = (n+\lambda-1)(n-\lambda-2)a_{n-2}, \quad \text{for } n \geq 2. \tag{4.136}$$

In both cases these are two-step recurrence relations. If we begin with an even coefficient, a_{2k} say, we will step down through the even coefficients to reach $a_0 \neq 0$, whereas if we begin with an odd coefficient, a_{2k+1} say, we step down via the odd coefficients to reach $a_1 = 0$. Hence, we only need to consider the even coefficients a_{2k}, $k = 0, 1, 2, \ldots$.

The condition (4.135) for the index $c_1 = 1$ and even $n = 2k$ reads

$$a_{2k} = \frac{(2k+\lambda)(2k-1-\lambda)}{(2k+1)2k} a_{2(k-1)}, \quad k = 1, 2, 3, \ldots. \tag{4.137}$$

As we go down from a_{2k} to a_0, at each step we introduce a factor of $(2m+1)(2m)$, $m = k-1, \ldots, 1$ in the denominator, so that the sequence ends in $3 \cdot 2$. In the numerator, the first factor of $2k+\lambda$ spawns a sequence of factors $2m+\lambda$, $m = k-1, \ldots, 1$, ending in $2+\lambda$ as we step from a_2 to a_0, whilst the $2k-\lambda-1$ factor generates the sequence $2m-\lambda-1$, $m = k-1, \ldots, 1$, ending in $1-\lambda$ as we step from a_2 to a_0. This finally gives

$$a_{2k} = \frac{(2k+\lambda)(2k-2+\lambda)\ldots(2+\lambda)(2k-\lambda-1)(2k-3-\lambda)\ldots(1-\lambda)}{(2k+1)!} a_0. \tag{4.138}$$

In the series (4.128) this yields

$$y_1(x) = a_0 x + a_0 \sum_{k=1}^{\infty} \frac{(2k+\lambda)(2k-2+\lambda)\ldots(2+\lambda)(2k-\lambda-1)(2k-3-\lambda)\ldots(1-\lambda)}{(2k+1)!} x^{2k+1}. \tag{4.139}$$

This is an infinite series, in general, that due to the index $c_1 = 1$ involves only odd powers of x.

The second solution arises from the index $c_2 = 0$, for which (4.136) and even $n = 2k$ reads

$$a_{2k} = \frac{(2k-1+\lambda)(2k-2-\lambda)}{2k(2k-1)} a_{2(k-1)}, \quad k = 1, 2, 3, \ldots. \tag{4.140}$$

In a similar way to the previous case, going down from a_{2k} to a_0 leads to

$$a_{2k} = \frac{(2k+\lambda-1)(2k-3+\lambda)\ldots(1+\lambda)(2k-\lambda-2)(2k-4-\lambda)\ldots(-\lambda)}{(2k)!} a_0. \tag{4.141}$$

The emerging series (4.128) hence is

$$y_2(x) = a_0 + a_0 \sum_{k=1}^{\infty} \frac{(2k+\lambda-1)(2k-3+\lambda)\ldots(1+\lambda)(2k-2-\lambda)(2k-4-\lambda)\ldots(-\lambda)}{(2k)!} x^{2k}. \tag{4.142}$$

This is an infinite series, in general, that due to the index $c_2 = 0$ involves only even powers of x.

To summarise, we obtained two solutions in series; one is an infinite series of odd powers, whereas the second is an infinite series of even powers. We still need to invoke the radii of convergence of the solutions in series from Theorem 4.23. From (4.125) we see that both $xp(x)$ and $x^2q(x)$ can be expanded in power series about $x_0 = 0$ by making use of the geometric series in the variable x^2, e.g.,

$$xp(x) = -2x^2 \frac{1}{1-x^2} = -2x^2 \sum_{n=0}^{\infty} x^{2n} = \sum_{n=0}^{\infty} (-2) x^{2n+2}, \tag{4.143}$$

and similarly for $x^2q(x)$. Both series converge for $|x| < 1$ which, by Theorem 4.23, means that the solutions in series converge for, at least, $|x| < 1$ too. It is not known though whether the series converge at $x = \pm 1$. Going back to the motivation for the Legendre equation, the variable x emerged as $x = \cos\theta$, and $x = \pm 1$ correspond to the polar angles, $\theta = 0$ and $\theta = \pi$. As spherical polar coordinates these two values determine the north and the south pole of a unit sphere, respectively. In that context, one often requires solutions that can be extended to the whole unit sphere, including to its poles. This is not guaranteed for the two solutions in series that we found.

Both solutions, however, involve the parameter λ. One may now ask whether under certain conditions on λ this picture might change. As it turns out, it is indeed possible to choose λ so that one of the infinite power series is reduced to a polynomial, involving only a finite number of terms. For such a finite sum no question of convergence arises, and the solution can be extended to the whole unit sphere.

Consider now the case $\lambda \in \mathbb{N}$, an even integer $\lambda = 2m - 2$, in the series with even powers. According to (4.140) this would mean that $a_{2m} = 0$, and hence $a_{2k} = 0$ for every $k \geq m$. Therefore, the series (4.142) reduces to a polynomial of degree $2m - 2 = \lambda$.

Likewise, when $\lambda \in \mathbb{N}$ is an odd integer $\lambda = 2m - 1$, the condition (4.137) on the coefficients in the series with odd powers this implies that $a_{2m} = 0$, and hence $a_{2k} = 0$ for every $k \geq m$.Therefore, the series (4.139) reduces to a polynomial of degree $2m - 1 = \lambda$.

Summarising: when $\lambda = l \in \mathbb{N}$ takes either an odd or an even integer value, either the power series in odd powers of x terminates to give a polynomial, with the one in even powers remaining an infinite power series, or conversely. It is these polynomial solutions of degree $l \in \mathbb{N}$ which occur frequently in the literature.

Definition 4.26. The polynomial solution of the differential equation

$$(1 - x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + l(l+1)y = 0, \quad x \in [-1, 1], \quad l \in \mathbb{N}, \tag{4.144}$$

is said to be the **Legendre polynomial of degree** l, denoted by $P_l(x)$.

The second solution of this Legendre equation is denoted by $Q_l(x)$, but takes infinite value at the points $x = \pm 1$.

5 Sturm–Liouville Systems

This Section aims to bring together the topics already discussed in Sections 1 and 4 and to see how the ideas that we have met may be generalised within the framework of Sturm-Liuoville systems. But we need to establish some concepts and definitions before we can even define a Sturm-Liouville equation, let alone a Sturm-Liouville system.

5.1 Basic definitions

We need a few concepts, some of which are familiar from first-order ODE.

Definition 5.1. The second-order homogeneous linear ODE,

$$P(x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}(x) + Q(x)\frac{\mathrm{d}y}{\mathrm{d}x}(x) + R(x)y(x) = 0, \tag{5.1}$$

with continuously differentiable coefficient functions is said to be **exact** if there exist continuously differentiable functions $A(x)$ and $B(x)$ such that for all functions $y(x)$ that are twice continuously differentiable,

$$P(x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}(x) + Q(x)\frac{\mathrm{d}y}{\mathrm{d}x}(x) + R(x)y(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(A(x)\frac{\mathrm{d}y}{\mathrm{d}x}(x) + B(x)y(x)\right). \tag{5.2}$$

For convenience we introduce the notation:

$$L[y](x) := P(x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}(x) + Q(x)\frac{\mathrm{d}y}{\mathrm{d}x}(x) + R(x)y(x), \tag{5.3}$$

so that the differential equation (5.1) reads $L[y](x) = 0$.

Lemma 5.2. *The equation $L[y] = 0$ is exact if and only if its coefficient functions satisfy*

$$\frac{\mathrm{d}^2 P}{\mathrm{d}x^2}(x) - \frac{\mathrm{d}Q}{\mathrm{d}x}(x) + R(x) = 0. \tag{5.4}$$

Proof. From Definition 5.1, the given equation is exact, if and only if

$$P\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + Q\frac{\mathrm{d}y}{\mathrm{d}x} + Ry = A\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}A}{\mathrm{d}x} + B\right)\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}B}{\mathrm{d}x}y. \tag{5.5}$$

Comparing coefficients of the derivatives yields the conditions

$$P(x) = A(x), \quad Q(x) = \frac{\mathrm{d}A}{\mathrm{d}x}(x) + B(x) \quad \text{and} \quad R(x) = \frac{\mathrm{d}B}{\mathrm{d}x}(x). \tag{5.6}$$

Rearranging these gives

$$R(x) = \frac{\mathrm{d}B}{\mathrm{d}x}(x) = \frac{\mathrm{d}Q}{\mathrm{d}x}(x) - \frac{\mathrm{d}^2 A}{\mathrm{d}x^2}(x) = \frac{\mathrm{d}Q}{\mathrm{d}x}(x) - \frac{\mathrm{d}^2 P}{\mathrm{d}x^2}(x), \tag{5.7}$$

which immediately implies the claim (5.4). $\square$

Definition 5.3. An **integrating factor** for the differential equation (5.1) is a function $\mu(x)$ such that the differential equation $\mu(x)L[y](x) = 0$ is exact.

Lemma 5.4. *A twice-continuously differentiable function $\mu(x)$ is an integrating factor for the differential equation $L[y] = 0$ if and only if it is a solution of the second-order homogeneous linear differential equation*

$$M[\mu](x) := \frac{\mathrm{d}^2}{\mathrm{d}x^2}\big(P(x)\mu(x)\big) - \frac{\mathrm{d}}{\mathrm{d}x}\big(Q(x)\mu(x)\big) + R(x)\mu(x) = 0. \tag{5.8}$$

Proof. Multiply through equation (5.1) by $\mu(x)$,

$$(\mu P)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (\mu Q)\frac{\mathrm{d}y}{\mathrm{d}x} + (\mu R)y = 0, \tag{5.9}$$

and substitute the modified coefficients μP, μQ and μR of the function into the result of Lemma 5.2. □

Using the product rule in (5.8) the equation $M[\mu](x) = 0$ can be brought into the form

$$P(x)\frac{\mathrm{d}^2\mu}{\mathrm{d}x^2}(x) + \Big(2\frac{\mathrm{d}P}{\mathrm{d}x}(x) - Q(x)\Big)\frac{\mathrm{d}\mu}{\mathrm{d}x}(x) + \Big(R(x) - \frac{\mathrm{d}Q}{\mathrm{d}x}(x) + \frac{\mathrm{d}^2 P}{\mathrm{d}x}(x)\Big)\mu(x) = 0. \tag{5.10}$$

Definition 5.5. The differential equation $M[\mu] = 0$ is said to be the **adjoint equation** of the differential equation $L[y] = 0$.

Linking the operators appearing in the original equation and its adjoint one, we have:

Lemma 5.6 (Lagrange's Identity)**.**

$$\mu L[y] - yM[\mu] = \frac{\mathrm{d}}{\mathrm{d}x}\Big[P(x)\mu(x)\frac{\mathrm{d}y}{\mathrm{d}x}(x) - \Big(\frac{\mathrm{d}}{\mathrm{d}x}\big(P(x)\mu(x)\big)\Big)y(x) + Q(x)\mu(x)y(x)\Big]. \tag{5.11}$$

Proof. By the definition (5.3) of L and the version (5.10) of M we obtain

$$\begin{aligned}\mu L[y] - yM[\mu] &= \mu P\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \mu Q\frac{\mathrm{d}y}{\mathrm{d}x} + \mu R y \\ &\quad - yP\frac{\mathrm{d}^2\mu}{\mathrm{d}x^2} - y\left(2\frac{\mathrm{d}P}{\mathrm{d}x} - Q\right)\frac{\mathrm{d}\mu}{\mathrm{d}x} - y\left(R - \frac{\mathrm{d}Q}{\mathrm{d}x} + \frac{\mathrm{d}^2 P}{\mathrm{d}x^2}\right)\mu.\end{aligned} \tag{5.12}$$

Simplifying this gives

$$\begin{aligned}\mu L[y] - yM[\mu] &= \mu P\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - yP\frac{\mathrm{d}^2\mu}{\mathrm{d}x^2} - 2y\frac{\mathrm{d}P}{\mathrm{d}x}\frac{\mathrm{d}\mu}{\mathrm{d}x} - y\frac{\mathrm{d}^2 P}{\mathrm{d}x^2}\mu \\ &\quad + \mu Q\frac{\mathrm{d}y}{\mathrm{d}x} + yQ\frac{\mathrm{d}\mu}{\mathrm{d}x} + y\frac{\mathrm{d}Q}{\mathrm{d}x}\mu \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\mu P\frac{\mathrm{d}y}{\mathrm{d}x} - y\mu\frac{\mathrm{d}P}{\mathrm{d}x} - y\frac{\mathrm{d}\mu}{\mathrm{d}x}P + \mu Q y\right] \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\mu P\frac{\mathrm{d}y}{\mathrm{d}x} - y\frac{\mathrm{d}}{\mathrm{d}x}(\mu P) + \mu Q y\right],\end{aligned} \tag{5.13}$$

which proves Lagrange's identity. □

With

$$A(x) := \mu(x)P(x) \quad \text{and} \quad B(x) := -\frac{\mathrm{d}}{\mathrm{d}x}\big(\mu(x)P(x)\big) + \mu(x)Q(x) \tag{5.14}$$

this immediately implies the following.

Corollary 5.7. $\mu L[y] - yM[\mu] = 0$ *is an exact differential equation for* $y(x)$.

This prompts:

Definition 5.8. A homogeneous linear differential equation which coincides with its adjoint is said to be **self-adjoint**.

Example 5.9. Legendre's differential equation (4.144),

$$L[y] = (1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + l(l+1)y = 0, \tag{5.15}$$

has coefficients $P(x) = 1 - x^2$, $Q(x) = -2x$ and $R(x) = l(l+1)$. According to (5.10) its adjoint equation is

$$\begin{aligned} M[y] &= (1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (-4x+2x)\frac{\mathrm{d}y}{\mathrm{d}x} + \big(l(l+1)+2-2\big)y(x) \\ &= (1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + l(l+1)y(x) \\ &= L[y] = 0, \end{aligned} \tag{5.16}$$

i.e., the Legendre equation is self-adjoint.

Theorem 5.10. *1. The second-order linear differential equation* $L[y] = 0$ *is self-adjoint if and only if it has the form*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(P(x)\frac{\mathrm{d}y}{\mathrm{d}x}(x)\right) + R(x)y(x) = 0, \tag{5.17}$$

i.e., iff $Q = \frac{\mathrm{d}P}{\mathrm{d}x}$.

2. If $L[y] = 0$ *is any second-order linear differential equation,* $hL[y] = 0$ *is a self-adjoint equation where*

$$h(x) = \frac{1}{P(x)}\exp\left\{\int_{x_0}^{x}\frac{Q(t)}{P(t)}\,\mathrm{d}t\right\}, \tag{5.18}$$

if that expression exists.

Proof. 1. Let $L[y] = 0$ be a second-order linear homogeneous ODE as in (5.1), then its adjoint equation has the form (5.10). Assuming $L[y] = M[y]$ and comparing the coefficients of (5.1) and (5.10) leads to the conditions

$$2\frac{\mathrm{d}P}{\mathrm{d}x}(x) - Q(x) = Q(x) \quad \text{and} \quad R(x) - \frac{\mathrm{d}Q}{\mathrm{d}x}(x) + \frac{\mathrm{d}^2P}{\mathrm{d}x}(x) = R(x) \tag{5.19}$$

that are solved by

$$\frac{\mathrm{d}P}{\mathrm{d}x}(x) = Q(x). \tag{5.20}$$

2. Given the equation $L[y] = 0$ as in (5.1), we can multiply through by $\mu(x)$ to give

$$\mu(x)P(x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}(x) + \mu(x)Q(x)\frac{\mathrm{d}y}{\mathrm{d}x}(x) + \mu(x)R(x)y(x) = 0. \tag{5.21}$$

According to 1., this equation is self-adjoint if

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(\mu(x)P(x)\big) = \mu(x)Q(x), \tag{5.22}$$

which, assuming that $P(x) \neq 0$, is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}x}\log\big(\mu(x)P(x)\big) = \frac{1}{\mu(x)P(x)}\frac{\mathrm{d}}{\mathrm{d}x}\big(\mu(x)P(x)\big) = \frac{Q(x)}{P(x)}. \tag{5.23}$$

Hence,

$$\log\big(\mu(x)P(x)\big) = \int_{x_0}^{x}\frac{Q(t)}{P(t)}\,\mathrm{d}t + C, \tag{5.24}$$

where C is a suitable constant. The function $h(x)$ then is given by $\mathrm{e}^{-C}\mu(x)$.

□

5.2 Sturm-Liouville equations and systems

Self-adjoint second order, linear homogeneous ODE (5.17) are of particular interest if the coefficient function $P(x)$ is nowhere negative. Such equations will be the focus of the rest of this section.

Definition 5.11. Let $p(x)$ be a non-negative, continuously differentiable function, and let $\rho(x)$, $q(x)$ be continuous functions where $\rho(x)$ is required to be non-negative. Then a second-order, linear homogeneous differential equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(p(x)\frac{\mathrm{d}y}{\mathrm{d}x}(x)\Big) + \big(\lambda\rho(x) - q(x)\big)y(x) = 0, \tag{5.25}$$

where $\lambda \in \mathbb{R}$ is a parameter, is said to be a **Sturm-Liouville equation**

Remark 5.12. The ODE (5.25) is of the general self-adjoint form (5.17), the only additional condition being that $p(x) = P(x) \geq 0$ for all x. The coefficient function $R(x)$ is only expressed in a seemigly unintuitive way as $R(x) = \lambda\rho(x) - q(x)$, the reason for which will become clearer in the rest of this section.

Example 5.13. 1. The trigonometric differential equation,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}(x) + \lambda y(x) = 0, \tag{5.26}$$

has $p(x) = 1$, $\rho(x) = 1$ and $q(x) = 0$ and is hence a Sturm-Liouville equation. We used this terminology already in Section 1.

2. Legendre's differential equation,

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big((1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x}(x)\Big) + l(l+1)y(x) = 0, \tag{5.27}$$

defined on the interval $[-1,1]$, is a Sturm-Liouville equation, with $p(x) = 1 - x^2$, $\lambda = l(l+1)$, $\rho(x) = 1$ and $q(x) = 0$. Note the restriction to the interval $[-1,1]$ as outside of it the function $p(x) = 1 - x^2$ is negative.

3. Bessel's differential equation of order ν,

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(x\frac{\mathrm{d}y}{\mathrm{d}x}(x)\Big) + \Big(\lambda x - \frac{\nu^2}{x}\Big)y(x) = 0 \tag{5.28}$$

is a Sturm–Liouville equation on $\mathbb{R}_+$, with $p(x) = x = \rho(x)$ and $q(x) = \frac{\nu^2}{x}$. Note that $x = 0$ has to be excluded since $q(x)$ is not continuous at $x = 0$.

We now turn to certain Sturm-Liouville equations that are defined in some finite closed intervals $[a,b]$.

Definition 5.14. A Sturm-Liouville equation,

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(p(x)\frac{\mathrm{d}y}{\mathrm{d}x}(x)\Big) + \big(\lambda\rho(x) - q(x)\big)y(x) = 0, \tag{5.29}$$

defined in an interval $[a,b]$ is said to be **regular**, if the functions p and ρ are strictly positive: $p(x) > 0$ and $\rho(x) > 0$ for all $x \in [a,b]$.

We now return to the Sturm-Liouville equations in Example 5.13 to see whether they are regular.

Example 5.15.

1. The trigonometric differential equation (5.26) has both $p(x) = 1 > 0$ and $\rho(x) = 1 > 0$ in every interval $[a,b]$., hence it always is a regular Sturm–Liouville equation.

2. The Legendre differential equation (5.27) is a Sturm-Liouville equation in the finite, closed interval $[-1,1]$. However, $p(\pm 1) = 0$ and hence $p(x)$ is not strictly positive for all $x \in [-1,1]$. Therefore, Legendre's differential equation is a Sturm–Liouville equation but not a regular one.

3. In Bessel's differential equation (5.28), $p(x) = x = \rho(x) > 0$ for $x > 0$. Hence, Bessel's equation is a regular Sturm-Liouville equaton in every interval $[a,b]$ with $0 < a < b < \infty$.

Recalling the examples in Section 1, the general solutions of the Sturm-Liouville equation (1.4) were given by (1.6), (1.8) and (1.10). The concepts of eigenavlues and eigenfunctions only emerged after having introduced boundary conditions. This will now be generalised to general (regular) Stumi-Louville equations.

Definition 5.16. A Sturm–Liouville equation,

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(p(x)\frac{\mathrm{d}y}{\mathrm{d}x}(x)\Big) + \big(\lambda\rho(x) - q(x)\big)y(x) = 0, \tag{5.30}$$

for $x \in [a, b]$, together with boundary conditions imposed on the function $y(x)$ and/or its derivative $y'(x)$ at $x = a$ and $x = b$ is said to be a **Sturm-Liouville system**.

More specifically:

Definition 5.17. A regular Sturm–Liouville equation in a finite, closed interval $[a, b]$ together with boundary conditions of the form

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0 \quad \text{and} \quad \beta_1 y(b) + \beta_2 y'(b) = 0 \tag{5.31}$$

is said to be a **regular Sturm-Liouville system**, if α_i and β_i, $i = 1, 2$, are constants such that not both α_i and not both β_i are zero.

Definition 5.18. A non-trivial solution $y(x)$ of a Sturm–Liouville system is said to be an **eigenfunction** and the corresponding value λ is said to be its associated **eigenvalue**. The set of all eigenvalues of a *regular* Sturm–Liouville system is said to be the **spectrum** of the system.

We now return to the examples of Section 1, all of which involve the regular Sturm-Liouville equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}(x) + \lambda y(x) = 0, \tag{5.32}$$

however, in different intervals $[a, b]$ and with different boundary conditions.

Example 5.19. 1. In Example 1.3 the interval was $[a, b] = [0, \pi]$ and the boundary conditions were $y(0) = 0$ and $y(\pi) = 0$, which are of the form (5.31). Hence this is an example of a regular Sturm-Liouville system. This system has eigenfunctions $y_n(x) = \sin nx$, corresponding to the eigenvalues $\lambda_n = n^2$, $n \in \mathbb{N}$.

2. In Example 1.5 the interval was also $[a, b] = [0, \pi]$ but the boundary conditions were $y(0) + y'(0) = 0$ and $y(\pi) = 0$, which again are of the form (5.31). Hence this is another example of a regular Sturm-Liouville system. This system has eigenfunctions $y_{\beta_n}(x) = \sin\beta_n(\pi - x)$, corresponding to the eigenvalues β_n^2 that are given by the positive solutions of the transcendental equation $\beta = \tan\beta\pi$. In addition, there is one eigenfunction corresponding to a negative eigenvalue.

3. In Example 1.6 the interval was $[a, b] = [-\pi, \pi]$ and the boundary conditions were $y(-\pi) = 0$ and $y(\pi) = 0$, which are of the form (5.31). Hence this is an example of a regular Sturm-Liouville system. This system has eigenfunctions $y_k(x) = \sin kx$, corresponding to the eigenvalues $\lambda_k = k^2$, $k \in \mathbb{N}$, and $y_{k-\frac{1}{2}}(x) = \cos(k - \frac{1}{2})x$, corresponding to the eigenvalues $\lambda_{k-\frac{1}{2}}(x) = (k - \frac{1}{2})^2$, $k \in \mathbb{N}$.

4. In Example 1.7 the interval again was $[a, b] = [-\pi, \pi]$ but the boundary conditions were $y(-\pi) = y(\pi)$ and $y'(-\pi) = y'(\pi)$. These are not of the type (5.31): each boundary condition involves two points and not the single point of each separated boundary condition (5.31). Hence this is *not* a regular Sturm-Liouville system. Nevertheless, this system has eigenfunctions $y_n(x) = E_n \sin nx + F_n \cos nx$ corresponding to the eigenvalues $\lambda_n = n^2$, $n \in \mathbb{N}$, and an eigenfunction $y_0(x) = 1$ corresponding to the eigenvalue $\lambda_0 = 0$.

The last example gives rise to a definition.

Definition 5.20. A **periodic Sturm-Liouville system** is a regular Sturm-Liouville equation, whose coefficients are periodic functions of the independent variable, with period $(b - a)$, on a finite, closed interval $[a, b]$, together with the periodic boundary conditions

$$y(a) = y(b) \quad \text{and} \quad y'(a) = y'(b). \tag{5.33}$$

One may ask whether a Sturm-Liouville system always has eigenvalues and eigenfunctions. The answer, unfortunately, requires the tools of functional analysis.

Theorem 5.21. *A regular or periodic Sturm–Liouville system has an infinite sequence of real eigenvalues and their absolute values are not bounded.*

The specific examples we have considered certainly show this property of the infinite sequence of real eigenvalues. This can be used to your advantage: should you be solving a trigonometric differential equation with boundary conditions are either of the type (5.31) or periodic (5.33), this infinite sequence of eigenvalues should always be present - if it isn't, you've made a mistake!

5.3 Eigenfunctions of Sturm-Liouville systems

Having established what we mean by the term Sturm-Liouville system and given a technical definition of the terms *eigenfunction* and *eigenvalue*, we will focus now on the eigenfunctions, rather than the eigenvalues.

Recall that in Example 1.7 the eigenfunctions were

$$u_0(x) = 1, \qquad u_n(x) = \cos nx, \qquad v_n(x) = \sin nx, \quad n \in \mathbb{N}, \tag{5.34}$$

where $x \in (-\pi, \pi)$. We will see that these eigenfunctions have some important properties that are shared by eigenfunctions of other regular (or periodic) Sturm-Liouville systems. Among them are the integrals in Proposition 2.5 that, for $n \neq m$, can be brought into the form

$$\begin{aligned}
\int_{-\pi}^{\pi} v_m(x) v_n(x)\,\mathrm{d}x &= \int_{-\pi}^{\pi} \sin mx \sin nx\,\mathrm{d}x = 0,\\
\int_{-\pi}^{\pi} u_m(x) u_n(x)\,\mathrm{d}x &= \int_{-\pi}^{\pi} \sin mx \sin nx\,\mathrm{d}x = 0,\\
\int_{-\pi}^{\pi} v_m(x) u_n(x)\mathrm{d}x &= \int_{-\pi}^{\pi} \sin mx \cos nx\,\mathrm{d}x = 0.
\end{aligned} \tag{5.35}$$

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$$\begin{aligned} \int_{-\pi}^{\pi} v_m(x) v_n(x)\, \mathrm{d}x &= \int_{-\pi}^{\pi} \sin mx \sin nx\, \mathrm{d}x = 0, \\ \int_{-\pi}^{\pi} u_m(x) u_n(x)\, \mathrm{d}x &= \int_{-\pi}^{\pi} \cos mx \cos nx\, \mathrm{d}x = 0, \\ \int_{-\pi}^{\pi} v_m(x) u_n(x) \mathrm{d}x &= \int_{-\pi}^{\pi} \sin mx \cos nx\, \mathrm{d}x = 0. \end{aligned} \tag{5.35}$$

The last equation even holds when $n = m$. Below we will identify these equations as *orthogonality relations* of the eigenfunctions. Before that, however, we remark that a Fourier series of a function $f(cx)$ that satisfies the assumptions of a Fourier Theorem can be re-interpreted as

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\bigl(a_n \cos nx + b_n \sin nx\bigr) \\ &= \frac{a_0}{2}u_0(x) + \sum_{n=1}^{\infty}\bigl(a_n u_n(x) + b_n v_n(x)\bigr), \end{aligned} \tag{5.36}$$

i.e., as an *expansion in eigenfunctions* of the Sturm-Liouville system.

To be somewhat more precise we need to introduce a few quantities and their properties. Let $C^1(a,b)$ be the set of continuously differentiable functions in an interval (a,b), and introduce the multiplication with a real number and the addition of functions in the usual sense: if $\lambda, \mu \in \mathbb{R}$ and $f, g \in C^1(a,b)$, then $\lambda f + \mu g \in C^1(a,b)$ is the function that assigns the value $\lambda f(x) + \mu g(x)$ to $x \in (a,b)$. In this way the set $C^1(a,b)$ becomes a real vector space. For vector spaces one can often introduce inner products, as it is well known for $\mathbb{R}^n$ with the dot product as an inner product. The integrals (5.35) can indeed be seen as inner products for $u_n, v_n \in C^1(-\pi,\pi)$, but for more general cases we need a slightly extended version. The examples of Section 1 had in common that the *weight factor* $\rho(x) > 0$ that is present in a general Sturm-Liouville equation (5.25) was $\rho(x) = 1$. In a general regular or periodic Sturm-Liouville system this weight factor has to be made explicit.

Definition 5.22. Let $f, g \in C^1(a,b)$ and let $\rho(x)$ be a positive and continuous function in $[a,b]$, then

$$\langle f, g\rangle_\rho := \int_a^b f(x)g(x)\,\rho(x)\,\mathrm{d}x. \tag{5.37}$$

We say that f and g are **orthogonal** if $\langle f, g\rangle_\rho = 0$.

Without proof we note the following.

Proposition 5.23. *The expression* (5.37) *is an inner product for the real vector space* $C^1(a,b)$.

We need one more quantity.

Definition 5.24. Let $f, g \in C^1(a,b)$, then the determinant

$$W(f,g;x) := \det\begin{pmatrix} f(x) & g(x) \\ f'(x) & g'(x)\end{pmatrix} = f(x)g'(x) - g(x)f'(x) \tag{5.38}$$

is said to be the **Wronskian** of the functions f, g.

Recall that two vectors $u, v \in V$ in a real vector space V are linearly independent, if $\lambda u + \mu v = 0$ for some $\lambda, \mu \in \mathbb{R}$ implies that $\lambda = 0 = \mu$.

Lemma 5.25. *Let $f, g \in C^1(a,b)$. If there exists $x_0 \in [a,b]$ such that $W(f,g;x_0) \neq 0$, the functions f and g are linearly independent.*

Proof. The condition $W(f,g;x_0) \neq 0$ means that the determinant (5.38) is not zero at $x = x_0$. Thus the matrix equation

$$\begin{pmatrix} f(x_0) & g(x_0) \\ f'(x_0) & g'(x_0) \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5.39}$$

only has the trivial solution $\lambda = 0 = \mu$. Hence, the upper component of (5.39),

$$\lambda f(x_0) + \mu g(x_0) = 0, \tag{5.40}$$

also only has the trivial solution $\lambda = 0 = \mu$. If the functions f and g were linearly dependent, the equation

$$\lambda f(x) + \mu g(x) = 0, \quad \text{for all} \quad x \in [a,b], \tag{5.41}$$

would have a non-trivial solution $(\lambda, \mu) \neq (0,0)$. Due to the above reasoning this, however, is not the case when $x = x_0$. Hence, f and g must be linearly independent. □

Remark 5.26. (i) Lemma 5.25 is equivalent to the following statement: If $f, g \in C^1(a,b)$ are linearly dependent, then $W(f,g;x) = 0$ for all $x \in [a,b]$.

(ii) From Lemma 5.25 and its proof one could be tempted to draw the conclusion that the statement *If $f, g \in C^1(a,b)$ are linearly independent, then there exists $x_0 \in [a,b]$ such that* $W(f,g;x_0) \neq 0$ would also hold. This, however, is not true as the mathematician G. Peano first observed: Take, e.g., the interval $[-1,1]$ and choose $f(x) = x^2$, $g(x) = x|x|$. (Despite the occurrence of the absolute values, g is indeed continuously differentiable.) These two functions are linearly independent as, e.g., f is even and g is odd. However, $W(f,g;x) = 0$ for all $x \in [-1,1]$.

The following statement shows that it wasn't mere coincidence that the same symbol was used for the weight function in (5.37) and the function we first met as the coefficient of λ in the definition of a Sturm–Liouville equation (5.25).

Lemma 5.27. *Let $u(x)$ and $v(x)$ satisfy a Sturm–Liouville equation on a closed interval $[a,b]$, with eigenvalues λ and μ respectively. Then*

$$(\lambda - \mu)\langle u, v\rangle_\rho = p(x) W(u,v;x)\big|_a^b, \tag{5.42}$$

where W is the Wronskian of the functions u and v.

Proof. Define the differential operator

$$D[u](x) := \frac{\mathrm{d}}{\mathrm{d}x}\Big(p(x)\frac{\mathrm{d}u}{\mathrm{d}x}(x)\Big) - q(x)u(x), \tag{5.43}$$

acting on a function $u(x)$. Then u and v are eigenfunctions of the Sturm–Liouville equation (5.25) with corresponding eigenvalues λ and μ if and only if

$$D[u](x) + \lambda\rho(x)u(x) = 0 \quad \text{and} \quad D[v](x) + \mu\rho(x)v(x) = 0. \tag{5.44}$$

Multiplying the first equation by v and the second by u gives

$$v(x)D[u](x) + \lambda\rho(x)v(x)u(x) = u(x)D[v](x) + \mu\rho(x)u(x)v(x), \tag{5.45}$$

which can be rearranged as

$$\begin{aligned}(\lambda - \mu)\rho(x)u(x)v(x) &= u(x)D[v](x) - v(x)D[u](x)\\ &= u(x)\frac{\mathrm{d}}{\mathrm{d}x}\Big(p(x)\frac{\mathrm{d}v}{\mathrm{d}x}(x)\Big) - v(x)\frac{\mathrm{d}}{\mathrm{d}x}\Big(p(x)\frac{\mathrm{d}u}{\mathrm{d}x}(x)\Big)\\ &= \frac{\mathrm{d}}{\mathrm{d}x}\Big(p(x)\frac{\mathrm{d}v}{\mathrm{d}x}(x)u(x) - v(x)p(x)\frac{\mathrm{d}u}{\mathrm{d}x}(x)\Big)\\ &= \frac{\mathrm{d}}{\mathrm{d}x}\big(p(x)W(u,v;x)\big).\end{aligned} \tag{5.46}$$

Integrating between the endpoints a and b gives the result. □

This result justifies the identification of the weight function (5.37) with the function appearing in the Sturm–Liouville equation (5.25).

As a further consequence, it is now possible to establish the orthogonality property of the eigenfunctions of a regular or periodic Sturm–Liouville system.

Theorem 5.28. *Let u and v be eigenfunctions of a regular or periodic Sturm-Liouville system with corresponding eigenvalues λ and μ, respectively. If $\lambda \neq \mu$, the eigenfunctions are orthogonal, i.e., $\langle u, v\rangle_\rho = 0$.*

Proof. 1. Regular Sturm-Liouville system: Both eigenfunctions satisfy the boundary conditions (5.31) at $x = a$,

$$\alpha_1 u(a) + \alpha_2 u'(a) = 0 \quad \text{and} \quad \alpha_1 v(a) + \alpha_2 v'(a) = 0, \tag{5.47}$$

where not both α_1 and α_2 are zero. This is a system of two linear equations for the unknowns α_1 and α_2. This has a nontrivial solution if

$$\det\begin{pmatrix} u(a) & u'(a)\\ v(a) & v'(a)\end{pmatrix} = W(u,v;a) = 0. \tag{5.48}$$

Similarly, at the other end-point $x = b$ the same condition holds so that

$$W(u,v;b) = 0. \tag{5.49}$$

With Lemma 5.27 we hence conclude that

$$(\lambda - \mu)\langle u, v\rangle_\rho = p(b)W(u,v;b) - p(a)W(u,v;a) = 0. \tag{5.50}$$

Since it was assumed that $\lambda - \mu \neq 0$, the orthogonality stated in the Theorem follows.

2. Periodic Sturm-Liouville system: Here all relevant functions are periodic, i.e.,

$$p(a) = p(b), \quad u(a) = u(b), \quad u'(a) = u'(b), \quad v(a) = v(b), \quad v'(a) = v'(b). \tag{5.51}$$

This implies that

$$\begin{aligned} p(b)W(u, v; b) &= p(b)\big((u(b)v'(b) - u'(b)v(b)\big) \\ &= p(a)\big((u(a)v'(a) - u'(a)v(a)\big) = p(a)W(u, v; a). \end{aligned} \tag{5.52}$$

Hence, also in the periodic case (5.50) is satisfied and the same conclusion holds.

□

Corollary 5.29. *The orthogonal eigenfunctions of Theorem 5.28 are linearly independent.*

Proof. Assume that there exist $\lambda, \mu \in \mathbb{R}$ such that

$$\lambda u(x) + \mu v(x) = 0 \quad \text{for all} \quad x \in [a, b], \tag{5.53}$$

then:

1. Taking the inner product with u gives

$$0 = \langle u, \lambda u + \mu v\rangle_\rho = \lambda\langle u, u\rangle_\rho + \mu\langle u, v\rangle_\rho = \lambda\langle u, u\rangle_\rho, \tag{5.54}$$

as $\langle u, v\rangle_\rho = 0$. Since $\langle u, u\rangle_\rho = 0$ only for the zero-vector, which u is not, it follows that $\lambda = 0$.

2. Similarly, taking the inner product with v gives

$$0 = \langle v, \lambda u + \mu v\rangle_\rho = \lambda\langle v, u\rangle_\rho + \mu\langle v, v\rangle_\rho = \mu\langle v, v\rangle_\rho, \tag{5.55}$$

and hence $\mu = 0$.

Since (5.53) implies that $\lambda = 0 = \mu$, the vectors u and v are linearly independent. □

From what we now know about the eigenfunctions of a Sturm-Liouville system, we can draw a conclusion about the eigenvalues.

Theorem 5.30. *The eigenvalues of a periodic or regular Sturm-Liouville system are real.*

Proof. Assume that there exists a complex (non-real) eigenvalue λ, with corresponding eigenfunction u_λ,

$$D[u_\lambda] + \lambda\rho u_\lambda = 0. \tag{5.56}$$

Since the functions ρ, p and q are real, the complex conjugate of this equation is

$$D[\overline{u_\lambda}] + \bar{\lambda}\rho\overline{u_\lambda} = 0. \tag{5.57}$$

Hence, the complex conjugate $\overline{u_\lambda}$ of the eigenfunction u_λ is also an eigenfunction, but with eigenvalue $\bar{\lambda}$. As we assumes that $\lambda \neq \bar{\lambda}$, Theorem 5.28 implies that u_λ and $\overline{u_\lambda}$ are orthogonal. Thus

$$0 = \langle u_\lambda, \overline{u_\lambda} \rangle_\rho = \int_a^b u_\lambda(x)\, \overline{u_\lambda(x)}\, \rho(x)\, \mathrm{d}x = \int_a^b |u_\lambda(x)|^2\, \rho(x)\, \mathrm{d}x \tag{5.58}$$

Since $\rho(x) > 0$ for all $x \in [a, b]$ and $|u_\lambda(x)|^2 \geq 0$ for all $x \in [a, b]$, it follows that $|u_\lambda(x)|^2 = 0$, and hence $u_\lambda(x) = 0$, for all $x \in [a, b]$. Since the trivial function cannot be an eigenfunction, our initial assumption that λ was not real is shown to be invalid. □

This result was assumed implicitly in Chapter 1, when only the possibility of real eigenvalues was explored. With hindsight, there was no need to look for complex, non-real ones.

For our final topic in this course we return to the interpretation of Fourier series as the expansion (5.36) of a function in the interval $[-\pi, \pi]$ in eigenfunction of a certain periodic Sturm-Liouville system.

Theorem 5.31. *The eigenfunctions of a regular or periodic Sturm-Liouville system form a basis in the vector space $C^1(a, b)$.*

The proof of this theorem requires somewhat elaborate techniques from Functional Analysis that we do not have at our disposal. We therefore abstain from giving a proof here.

A consequence of this theorem is that if $f \in C^1(a, b)$ is a continuously differentiable function in the interval (a, b) and $\{u_n(x);\ n \in \mathbb{N}\}$ is the set of eigenfunctions of a regular or periodic Sturm-Liouville system in the interval $[a, b]$ with associated eigenvalues $\{\lambda_n;\ n \in \mathbb{N}_0\}$, then f can be expanded in the basis of eigenfunctions, i.e.,

$$f(x) = \sum_{n=0}^{\infty} a_n\, u_n(x), \tag{5.59}$$

where the a_n, $n \in \mathbb{N}_0$, are expansion coefficients. Taking an inner product on both sides with an eigenfunction u_m yields

$$\langle u_m, f \rangle_\rho = \langle u_m, \sum_n a_n u_n \rangle_\rho = \sum_n a_n \langle u_m, u_n \rangle_\rho. \tag{5.60}$$

According to Theorem 5.28 eigenfunctions corresponding to different eigenvalues are orthogonal, i.e.,

$$\langle u_m, u_n \rangle_\rho = 0 \quad \text{if} \quad \lambda_n \neq \lambda_m. \tag{5.61}$$

This implies that

$$\langle u_m, f \rangle_\rho = a_m \langle u_m, u_m \rangle_\rho, \tag{5.62}$$

or

$$a_n = \frac{\langle u_n, f \rangle_\rho}{\langle u_n, u_n \rangle_\rho}. \tag{5.63}$$

We conclude that (5.59) is the generalisation of Fourier series, and that (5.63) is the corresponding generalisation of the Euler-Fourier formulae.