After the qualitative observations of previous subsections, we now quantify the parameters that describe the fat tails and skewness of the return distributions. As evaluated before the normal distribution is not a suitable choice to model daily log-returns. A more adequate option could be the Student’s t-distribution as it exhibits fat tails that differ for the degrees of freedom parameter as shown in figure 8. The Student’s t-distribution converges towards the normal distribution as the degrees of freedom parameter is increased. In figure 9 it is demonstrated that the Student’s t-distribution approximates the log-returns well as almost all observations lie within the confidence interval. It was assumed that the S&P500 follows a Student’s t-distribution with three degrees of freedom respectively four for the JPY/USD exchange rate.

Although, the Student t-distribution incorporates kurtosis, it neglects asymmetry as it is a symmetric distribution. Thus, the skewed Student’s t-distribution was a logical continuation for the modeling process. At first the data was fitted to univariate skewed Student’s t-distributions to evaluate whether the additional skewing parameter was of significance. The fitting process included the usage of the ghyp-package, which was implemented by Wolfgang Breymann and David Luethi. The package focuses on the generalized hyperbolic distributions and especially their contributions to finance and risk management. The optimizer uses a maximum likelihood algorithm to fit a univariate, generalized hyperbolic distribution to provided sample data. In figures 10-11 it is shown that the tails were better approximated by the skewed Student’s t-distribution compared to both the normal distribution and the symmetric Student’s t-distribution. As the normal distribution underestimates the frequency of extreme events in both the lower and upper tail, the symmetric Student’s t-distribution overestimates the upper tail for the S&P500 and overestimates the lower tail for the JPY/USD exchange rate. Further the significance of the skewing parameter was examined by applying a log-likelihood ratio test that tests whether the simpler model is the true underlying model (@GaryKing). The simpler model in this case is the symmetric Student’s t-distribution as it incorporates one less parameter as the skewed Student’s t-distribution. In table 4 it is shown that the p-values of the log-likelihood ratio test are both significant and thus the null hypothesis, which states that the parameter $\gamma = 0$ explains the data, was rejected. As a second measure of goodness of fit a value-at-risk forecast backtest was concluded to evaluate whether violations against the value-at-risk forecast occur more often than suggested. First, the number of violations against the value-at-risk forecast were counted for the normal distribution, the symmetric Student’s t-distribution and the skewed Student’s t-distribution. Second, a binomial test was conducted to test whether the ratio of violations is higher than suggested by an arbitrary significance level. As shown in tables 5-8 the skewed Student’s t-distribution does not violate the value-at-risk forecast for any of the significance levels, thus making it the best available choice to model log-returns.

With the information obtained by assessing the characteristics of log-returns and the accurate approximation of the univariate skewed Student’s t-distribution, the multivariate skewed Student’s t-distribution was chosen to model the behavior of log-returns. The multivariate skewed Student’s t-distribution is a suitable choice as it accounts both for extreme values in the tails and also their skewed appearance. Additionally, it incorporates the tail correlation structure that was examined in the chapter before, which is crucial when modeling non-linear dependence. It also has computing advantages as linear combinations of the marginal distributions can be described as a univariate skewed Student’s -t distribution as explained in the following equations:

$$\overline{\mu}=\boldsymbol{w}^\top \boldsymbol {\mu}$$

$$\overline{\sigma}=\sqrt{\boldsymbol{w}^\top \Sigma \boldsymbol{w}}$$

$$\overline{\gamma}=\boldsymbol{w}^\top \boldsymbol{\gamma}$$

The estimation method used to estimate the multivariate skewed Student’s t-distribution uses a multi-cycle, expectation, conditional estimation (MCECM) algorithm to identify the unknown parameters. It adopts an iterative approach by updating the starting values for the parameters after each iteration. In table 9 the fitted parameters are summarized, the first parameter refers to $nu$ whereas the rest of the parameters are the individual skewing parameters.