Mean-variance analysis (MVA), also known as modern portfolio theory, is a mathematical approach to the problem of asset allocation in a portfolio. It was introduced by Harry M. Markowitz in an article published in the “Journal of Finance” in March 1952. Markowitz divided the process of portfolio selection into two separate stages: the first stage is concerned with gathering current knowledge and beliefs about the future performance of the assets, the second stage focuses on portfolio selection basing on relevant data collected in the first stage. Mean-variance analysis is the mathematical framework used to execute the second stage. Markowitz dismissed the idea of return maximization as an adequate criterion to allocate a portfolio and instead pointed to the risk-reducing property of diversification, whereby selecting assets with uncorrelated price behavior reduces the variance of the portfolio. In the context of MVA, variance is the proxy for risk and increased variance is the premium for selecting assets with higher expected returns. Mean-variance analysis focuses on maximizing the expected return on a portfolio for a set level of variance by finding a solution on the efficient frontier in the risk-return space (@Luenberger).

**\*\*Mathematical Definition:\*\***

\newline

Variance of a portfolio:

$$\sigma^2 = \boldsymbol{w}^{\top} \Sigma \boldsymbol{w}$$

Where $\Sigma$ is the covariance matrix of the expected returns on the $N$ selected assets and $\boldsymbol{w}$ is the unknown vector expressing their weights $w\_1, \dots, w\_N$ in the portfolio.

The vector $\boldsymbol{w}$ is determined such that a predetermined expected return on the portfolio $r\_{\text{pf}}$ is reached while minimizing $\sigma^2$. The quadratic matrix equation can be formulated as follows:

$$\underset{\boldsymbol{w}}{\mathrm{arg\min}} \ \frac{\sigma^2}{2} = \underset{\boldsymbol{w}}{\mathrm{arg\min}} \ \frac{1}{2} \boldsymbol{w}^{\top} \Sigma \boldsymbol{w}$$

Where $\underset{\boldsymbol{w}}{\mathrm{arg\min}}$ minimizes the expression through the best $\boldsymbol{w}$.

The above equation is subject to the constraints:

$$\begin{cases} \boldsymbol{w}^{\top} \boldsymbol{r} = r\_{\text{pf}} & (1) \\ \boldsymbol{w}^{\top} \boldsymbol{1} = 1 & (2) \end{cases}$$

The first equation minimizes the portfolio’s half variance $\frac{\sigma^2}{2}$; $\frac{\sigma^2}{2}$ being a linear transformation of $\sigma^2$, the solution set to the first expression also minimizes the portfolio’s full variance $\sigma^2$. This specific formulation further eases the quadratic matrix equation, which turns into a linear expression when using the method of Lagrange multipliers. Condition (1) requires that the portfolio with expected returns on single assets $\boldsymbol{r}^{\top} = r\_1, \dots, r\_n$ generates the investor’s chosen aggregated expected return $r\_{\text{pf}}$ and condition (2) ensures that the weights of the portfolio’s assets sum up to one. This constrained minimization problem can be expressed by the Lagrange function:

$$\mathcal{L}(\boldsymbol{w}, \lambda\_1, \lambda\_2) = \frac{1}{2} \boldsymbol{w}^{\top} \Sigma \boldsymbol{w} - \lambda\_1(\boldsymbol{w}^{\top} \boldsymbol{r} - r\_{\text{pf}}) - \lambda\_2(\boldsymbol{w}^{\top} \boldsymbol{1} - 1)$$

Where $\lambda\_1$ and $\lambda\_2$ are the Lagrange multipliers.

The solution $\boldsymbol{w}$ is found by solving the system of partial differential equations:

$$\boldsymbol{\nabla}\_{\boldsymbol{w}, \lambda\_1, \lambda\_2}\mathcal{L}(\boldsymbol{w}, \lambda\_1, \lambda\_2) = \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{w}}, \frac{\partial \mathcal{L}}{\partial \lambda\_1}, \frac{\partial \mathcal{L}}{\partial \lambda\_2} \right) = \left( \boldsymbol{0}, 0 ,0 \right)$$

Because of the first order partial derivatives, the quadratic expression of the $\frac{\sigma^2}{2}$-minimization part in $\mathcal{L}$ becomes linear, leaving a system of $N + 2$ linear equations with $N + 2$ variables:

$$\begin{cases} \Sigma \boldsymbol{w} - \lambda\_1 \boldsymbol{r} - \lambda\_2 \boldsymbol{1} = \boldsymbol{0} \\ \boldsymbol{w}^{\top} \boldsymbol{r} - r\_{\text{pf}} = 0 \\ \boldsymbol{w}^{\top} \boldsymbol{1} - 1 = 0 \end{cases}$$

This system of linear equations has the explicit, unique solution $\boldsymbol{w}^\* = \Sigma^{-1} (\lambda\_1 \boldsymbol{r} + \lambda\_2 \boldsymbol{1})$, where

$$ \lambda\_1 = \frac{( \boldsymbol{1}^{\top} \Sigma^{-1} \boldsymbol{1}) r\_{\text{pf}} - \boldsymbol{r}^{\top} \Sigma^{-1} \boldsymbol{1}}{( \boldsymbol{r}^{\top} \Sigma^{-1} \boldsymbol{r}) (\boldsymbol{1}^{\top} \Sigma^{-1} \boldsymbol{1}) - ( \boldsymbol{r}^{\top} \Sigma^{-1} \boldsymbol{1})^2}$$

$$ \lambda\_2 = \frac{\boldsymbol{r}^{\top} \Sigma^{-1} \boldsymbol{r} - ( \boldsymbol{r}^{\top} \Sigma^{-1} \boldsymbol{1} ) r\_{\text{pf}}}{( \boldsymbol{r}^{\top} \Sigma^{-1} \boldsymbol{r}) (\boldsymbol{1}^{\top} \Sigma^{-1} \boldsymbol{1}) - ( \boldsymbol{r}^{\top} \Sigma^{-1} \boldsymbol{1})^2}$$

The above process can be reformulated as a $r\_{\text{pf}}$-maximization problem for a given level of $\sigma^2$. If the chosen $\sigma^2$ is the variance of the portfolio found for the $\sigma^2$-minimization problem, all else being equal the $r\_{\text{pf}}$-maximization problem leads to the same solution $\boldsymbol{w}^\*$.

MVA in its original version has the main disadvantage of having variance as its risk metric, and thus it is prone to the same pitfalls that can be incurred when basing financial risk management methods on volatility. However, MVA can be modified to implement another risk metric, provided the latter is coherent with the (expected) returns distribution followed by the assets.