The goal of this paper is to model the “fat tails” of the return distributions of financial assets and, based on this, to understand how the length of the investment horizon impacts portfolio allocation among various assets for different levels of risk. Addressing this problem requires essentially to (1) establish a measure of financial risk and (2) to build a model for representing the evolution of each asset’s value and the effects of financial risk.

The notion of financial risk plays a crucial role in defining an investment strategy. In the case of this paper, a constant rebalancing strategy is followed, where the portfolio return is defined as the weighted average of the single asset returns. The shorter the time horizon of the investment strategy, the more the investor needs to consider the likelihood and the effects of large economic and financial downturns on the portfolio’s value. Growing globalization and interconnectedness of financial systems increase the chances of acute, but local issues to spread to other geographic areas and to start chain reactions affecting different markets and actors. Another characteristic of crashes is their potential to undo long periods of profitability in a very short time. The 2007-2008 financial crisis is a prime example of these concepts. The crisis started with the failures of subprime loans in the US. mortgage market and spread globally to affect virtually every financial market and asset class; as the crisis developed, even long-standing and often successful financial firms were forced into bankruptcy, restructuring, acquisition, or bailout programs after suffering large losses. To reflect these events quantitatively requires establishing a financial risk metric which is intuitive, computable, and gives a concrete, realistic indication of the loss a portfolio would incur in relation to an occurrence, the latter being specified by its assumed likelihood.

In the case of this paper, modeling the value of an asset means to analyze its historical returns and to determine a statistical distribution which approximates its behavior as closely as possible. This step entails not oversimplifying the mathematical approximation for the sake of the model’s elegance, since a slightly different statistical distribution could cause market events with significant bearing on an asset to be under- or overrepresented. It is important that the historical data include as many instances of important market movements as possible to better reflect their likelihood and impact. The reference data must also be correctly interpreted and, where appropriate, transformed, to ensure the same property is being quantified and used to model all instruments, e.g. in the case where an asset’s price uses an otherwise unusual notation.

This paper continues and builds upon the investigation of a preliminary paper, some key points of which are summarized here:

1. We go beyond volatility as a risk measure. While at the core of important theories in the field of mathematical finance, risk measurement in terms of volatility has several drawbacks. One disadvantage of volatility from a conceptual standpoint is that it equally weighs positive and negative deviations from the mean. From a quantitative point of view, a common critique to volatility-based risk management is that it does not give a concrete indication of the size of possible losses. Value-at-risk (VaR) is used to measure the minimal potential loss over a defined time frame $\tau$ with a stated confidence level $1 - \alpha$, $\alpha$ being the significance level.

2. We go beyond the use of the normal (or Gaussian) distribution. While its flexibility and tame mathematical nature make it enticing, it carries substantial disadvantages. The heavy-tailed distributions of returns on various financial instruments are in stark contrast with their normal approximations. The higher the frequency at which the returns on an asset are being measured, the more evident extreme movements are. For instance, approximating daily returns on a stock market index with a normal distribution underrepresents the proportion of tail-events that actually occurred in the index’ lifetime. The Student’s t-distribution allows, by changing the degrees of freedom $\nu$, to adjust how heavy its tails are, and is used to model returns generated by eight instruments from four different asset classes.

3. Data analysis indicates that riskier investments, e.g. stock indices, display heavier tails than assets considered safer, such as government bonds. Also, extreme, negative downturns are more frequent in risky instruments, whereas their safer counterparts are more often subject to market surges. Further, by analyzing correlations between the assets, it is noticed that the safer assets negatively correlate to risky investments. These facts seem to confirm the notion of safe haven assets being bought in times of distress or uncertainty in the financial markets.

4. Logarithmic returns are used for ease of aggregation to compute returns over longer time horizons and to test the assumption of returns being log-normally distributed, i.e. that log returns are normally distributed. By performing the Anderson-Darling normality test, it is established that log returns are generally not normally distributed. However, the result changes according to the timespan and the instrument being considered: the longer the investment period and the less volatile the asset, the more the normality assumption is plausible.

5. Empirical validation of modern portfolio theory for optimizing a portfolio demonstrates that the framework works (1) with Student’s t-distributed instruments and variance as a risk metric and (2) with normally distributed instruments and VaR as a risk metric. However, the optimal weights differ from the mean-variance optimized results when one asset is normally distributed, the other is Student’s t-distributed, and simultaneously VaR is used as the risk metric. For this reason, further weight optimizations are done numerically.

6. To make the optimal portfolio weights for different investment horizons $\tau$ comparable, the significance level of the VaR is calculated as a function of $\tau$ over the duration of the total period of data being considered: $\alpha= \frac{\tau}{Duration}$. This modified version of VaR accounts for the risk-reducing effect of longer investment horizons. The optimization indicates a stronger appetite for instruments with lower degrees of freedom, i.e. riskier investments, with growing investment horizon.

In this context, the secondary goal is the improvement of both non-normal and non-volatility numerical results obtained in the preliminary paper. Compared to the preliminary paper, this paper introduces new approaches for quantifying financial risk, modeling assets behavior, and determining optimal model parameters.