In the previous sub-section, we have modelled the returns of various asset classes (equities, bonds, FX rates and commodities) by symmetric and skewed Student’s t-distributions. In this sub-section, we discuss optimal asset allocations for these asset classes. As risk measures, we use value-at-risk and expected shortfall.

To define value-at-risk and expected shortfall, we must specify two parameters: the time horizon $T$ ($T=1, 5, 22, 250$ for daily, weekly, monthy, yearly returns) and the level of confidence $\alpha$. If $\alpha$ were chosen independently of $T$, the optimization results for different horizons would not be comparable. E.g., $\alpha=99\%$ value-at-risk of yearly returns vs. daily returns measures, roughly, the worst annual loss expected in 100 years vs. the worst daily loss expected in 100 days, or approximately 5 months. However, to cover the same time period of 100 years, i.e. 25,000 days, we should instead compare the 99% value-at-risk of yearly returns with the $1-\frac{1}{25000} = 99.996\%$ value-at-risk of daily returns, and 99% expected shortfall of yearly returns with 99.996% expected shortfall of daily returns.

To illustrate this point, suppose we sell insurance against a major earthquake in Zürich against an insurance premium. Suppose such an earthquake occurs, on average, once in a hundred years and causes a loss of CHF 1 billion within one day. In this case, 99% 1-year expected shortfall for this “investment” is CHF 1 billion, which is the same as the 99.996% 1-day expected shortfall. By comparison, the 99% 1-day expected shortfall would only be CHF $\frac{\mathrm{CHF 1 billion }}{250} = \mathrm{CHF 4 million}$.

In the following, we let $\alpha$ depend on $\tau$ as follows: $\alpha = 1 - \frac{\tau}{30000}$. Thus, our risk measure is sensitive to losses that occur once in up to 30,000 days, or 117 years (we choose 30,000 days, because $1-\frac{1}{30000} = 99.997\%$ VaR corresponds to a 4-standard-deviation event for normal distributions).

In a first step, we model the asset classes by symmetric t-distributions in sub-section 3.3.2, choosing a variety of degrees of freedom within the same portfolio. In this case, we solve the optimization problem numerically, using Monte Carlo simulations. As a risk measure, we use value-at-risk (the results for expected shortfall would be very similar in this case). We find that long-term investors can tolerate more tail-risk than short term investors. I.e., the optimal weights of assets with fat left tails are small for short-term investors, and larger for longer-term investors.

While symmetric t-distributions capture the fat tails of real market return distributions, the analysis of sub-section 3.3.2 has at least three shortcomings:

a. It does not account for skewness

b. It does not account for tail correlations between the assets

c. It assumes that the returns at different days are independent of each other, thereby ignoring the autocorrelation of absolute returns described above

We therefore supplement the analysis of sub-section 3.3.2 by a second, more realistic analysis in sub-section 3.3.3. There, we use the multivariate student-t distribution with 3.5 degrees of freedom to model market returns, thereby resolving issues (a) and (b). We then use analytical methods to optimize expected shortfall.

As for issue (c), the autocorrelation of absolute returns indicates that aggregating daily returns to $T$-day returns does not lead to normal return distributions as quickly as expected by the central limit theorem. I.e., the number of deegrees of freedom of the t-distribution does not go to infinity as quickly as if the returns on different days were completely independent of each other. We do not have enough historical data to measure reliably how the number of degrees of freedom grows with the investment horizon. We therefore make the conservative assumption that the aggregation of daily returns to $T$-day returns does not change the number of degrees of freedom at all, but that it merely changes the dispersion.

Based on this, in subsection 3.3.3, we optimize expected shortfall for the same multivariate skewed t-distribution as for daily returns, but at different levels of confidence, with higher levels again corresponding to shorter investment horizons. Despite our conservative assumption, the results are again consistent with those of subsection 3.3.2: the optimal asset allocation for short-term investors gives smaller weights to assets with heavy left tails, and higher weights to assets with thin left tails.