In a first attempt a range of assets were simulated using univariate symmetric Student’s t-distribution with different degrees of freedom and allocated to the following investment horizons using value-at-risk as risk measure:

\* daily returns

\* weekly returns

\* monthly returns

\* yearly returns

For the process of asset allocation we used a specific value-at-risk that is dependent on the length of the investment horizon:

To define value-at-risk, we must specify two parameters: the time horizon $\tau$ and the level of confidence $\alpha$. If $\alpha$ were independent of $\tau$, the optimization results for different horizons would not be comparable. E.g., $\alpha = 99\%$ value-at-risk of yearly returns vs. daily returns measures, roughly, the worst annual loss expected in 100 years vs. the worst daily loss expected in 100 days $\approx$ 5 months. E.g., suppose we sell insurance against a major earthquake in Zurich against an insurance premium. Suppose such an earthquake occurs, on average, once in a hundred years and causes a loss of CHF 1 million within one day. In this case, 99% 1-year value-at-risk for this “investment” is CHF 1 million, but 99% 1-day value-at-risk is almost zero. To “see” the earthquake in the daily value-at-risk, we must consider it at a level of at least $1-\frac{1}{26000}$ (assuming that there are 260 trading days per year). To cover the same time period (we use 30000 days) for each investment horizon $\tau$ , we therefore let $\alpha$ depend on $\tau$ as follows:

$$\mathrm{VaR}\_{\alpha}=F^{-1}\left(1-\frac{\tau}{26000}\right)$$

In Figure 12 it shown that the weight of the normally distributed asset shrinks when the investment horizon is increased. Said phenomenon corresponds with the knowledge obtained earlier, as the heavy tails feature diminishes when returns are aggregated. This means short-term investors can take less tail risk Thus, the weights are more evenly distributed throughout the range of assets. Although this first attempt illustrates the relationship between short term and long term risks it also has several drawbacks:

\* Independence copula (no tail correlation)

\* Symmetric distribution model (no skewness)

The next step of the portfolio optimization process was conducted using an arbitrary multivariate skewed Student’s t-distribution. In this stage, we make the following assumptions:

\* The absolute log-returns are not autocorrelated

\* The degrees of freedom of the single assets remain constant across all investment horizons

The following parameters were used for the multivariate model:

$$\nu=3.5$$

$$\boldsymbol{\mu}=\begin{bmatrix}-0.0625 \\ -0.0125 \\ 0.0125 \\ -0.0375\end{bmatrix}$$

$$\boldsymbol{\gamma}=0.025+2\boldsymbol{\mu}$$

$$\boldsymbol{\gamma}=\begin{bmatrix}-0.1 \\ 0 \\ 0.05 \\ -0.05\end{bmatrix}$$

$$\Sigma=\begin{bmatrix}1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{bmatrix}$$

Where $\nu$ denotes the degrees of freedom parameters, $\boldsymbol{\gamma}$ the skewing parameter and $\boldsymbol{\mu}$, $\Sigma$ the location-scale parameters. The parameters were chosen that $\boldsymbol{\gamma}$ is a linear combination from $\boldsymbol{\mu}$, which allows to model an efficient frontier that is independent of the risk measure. This approach has computing advantages as the portfolio optimization calculations can be done using the mean-variance method, which is less complex than optimizing value-at-risk or expected shortfall and does not require an iterative optimization algorithm e.g. Nelder-Mead.

\*\*Mathematical proof:\*\*

$$\boldsymbol{\gamma}=a+b\boldsymbol{\mu}$$

$$\overline{R}=\boldsymbol{w}^{\top}(\boldsymbol{\mu}+\boldsymbol{\gamma})$$

$$\boldsymbol{w}^{\top}\boldsymbol{\gamma}=a+b(\overline{R}-\boldsymbol{w}^{\top}\boldsymbol{\gamma})$$

$$\boldsymbol{w}^{\top}\boldsymbol{\gamma}=\frac{a+b\overline{R}}{1+b}$$

Where $\boldsymbol{\mu}$, $\boldsymbol{\gamma}$ are the location and skewness parameters of a constructed portfolio and $\overline{R}$ is the expected portfolio return. Thus, the portfolio parameters are only dependent upon three constants: the linear transformation constants $a, b$ and the expected return. Therefore, the efficient frontier is the same for any risk measure.

In figure 13 it is shown that the optimal portfolio allocation for an investor that considers expected shortfall, changes significantly as the investment horizon increases. Especially equities are weighted differently as they are the riskiest of the bunch and thus can be tolerated more the longer the investment horizon. It is also demonstrated that volatility is not a suitable risk measure as there is no way to include the investment horizon in the calculations. However, the changes in the weights happen rather fast as the x-axis had to be log-scaled to highlight the impact of the investment duration.