A copula $C$ is a multivariate cumulative distribution function on the $d$-dimensional unit cube $[0, 1]^{d}$ with standard uniform marginal distributions $U\_n \sim \mathrm{U}(0, 1)$, i.e. a mapping $C: [0, 1]^d \rightarrow [0, 1]$. Copulas are omnipresent and of foremost importance in any field employing multivariate statistics. A copula describes the interdependence among random variables of a statistical model and can be used to produce multivariate distributions with specific dependence structures. The wide use of copulas is justified by Sklar’s theorem: it proves that every multivariate cumulative distribution function (CDF) has a copula, and that copulas can be used to join univariate CDFs to create a specific multivariate distribution. Sklar’s theorem can be summarized as follows (@QRM):

1. There exists a copula $C$ such that, $\forall (x\_1, \dots, x\_d) \in \mathbb{R}$:

$F(x\_1, \dots, x\_d) = C \left(F\_1(x\_1), \dots, F\_d(x\_d)\right)$

Where $F(\cdot)$ is a joint CDF with marginal CDFs $F\_1(\cdot), \dots, F\_d(\cdot)$.

This means that all multivariate distributions have a copula which couples together the marginal CDFs.

2. $C(u\_1, \dots, u\_d) = F(F^{-1}\_1(u\_1), \dots, F^{-1}\_d(u\_d))$

Where $u\_i$ denotes the $i$-th component of the $d$-dimensional random vector $\boldsymbol{U} \sim \mathrm{U}(0, 1)$ and $F^{-1}\_i(\cdot)$ is the inverse CDF $F^{-1}\_i(u\_i) = \mathrm{inf} \{x\_i: F\_d(x\_i) \geq u\_i \}$.

This shows how a copula can be extracted from a joint CDF, and that the dependence is in terms of $u\_i$-quantiles.

\*\*Mathematical Definition:\*\*

For $C$ to be a copula, it has to fulfil three properties:

1. $C(u\_1, \dots, u\_d) = 0$ if $u\_i = 0$ for any $i \in [1, d]$

2. $C(1, \dots, 1, u\_i, 1, \dots, 1) = u\_i \: \forall i \in \{1, \dots, d\}, u\_i \in [0, 1]$

This property ensures that marginal distributions are uniform.

3. $\forall (a\_1, \dots, a\_d), (b\_1, \dots, b\_d) \in [0, 1]^d, a\_i \leq b\_i: \displaystyle \sum\_{i\_1=1}^{2} \dots \sum\_{i\_d=1}^{2} (-1)^{i\_1 + \dots + i\_d} C(u\_{1i\_1}, \dots, u\_{di\_d}) \geq 0$

Where $u\_{j1} = a\_j$ and $u\_{j2} = b\_j \: \forall j \in \{1, \dots, d\}$.

This inequality ensures that for a random vector $(U\_1, \dots, U\_d)^\top$ with CDF $C$, the probability $P(a\_1 \leq U\_1 \leq b\_1, \dots, a\_d \leq U\_d \leq b\_d)$ is non-negative.

In the context of copulas, it is important to delve into the topic of dependence. In statistics, dependence is a concept used to describe how variables interact and to try to establish possible cause-and-effect relationships. In science, this notion of dependence often becomes a concrete description of the mechanism in a phenomenon, commonly referred to as law, e.g. Newton’s laws of motion. However, the dynamics of financial assets follow a very limited and porous set of laws, if any, and most approaches are mere attempts at modeling the behavior of instruments. Nonetheless, even in mathematical finance some facts can be established by analyzing historical data; very often, one of these stylized facts is the interdependence of extreme movements in the returns on a set of instrument. From a statistical perspective, this means that the (absolute) correlations between the variables increase with the distance from the center of the distribution. However, Pearson’s linear correlation can be a misleading dependence measure. One can generate multivariate sets of data with identical correlation matrices, but with observable differences in their behaviors, even when by starting from identical marginal distributions; this fallacy is due to linear correlation not being exclusively a function of the copula of the joint distribution. Still, as explained by McNeil et al. (@QRM), linear correlation is a reliable dependence measure for elliptical distributions, which can be explained by the fact that these distributions are a fully described by their mean vector $\mu$, covariance matrix $\Sigma$ and characteristic generator. Mean and variance being the sole parameters influencing the marginal CDFs, it follows that the copula of the related elliptical distribution solely depends on the correlation matrix and characteristic generator. Furthermore, even copulas of elliptical distribution can exhibit interesting traits. The $t$ copula, based on the elliptical Student’s t-distribution, depends on both the correlation matrix and the degrees of freedom $\nu$; from a practical standpoint, $\nu$ allows to introduce tail dependence. This entails that, as demonstrated by Demarta and McNeil (@Demarta), as long as $\nu < \infty$ there is tail dependence in the multivariate distribution, an indication that extreme outcomes in each direction are dependent on one another. In contrast, joint distributions based on the Gaussian copula, which depends solely on the correlation matrix, are independent in the tails if the correlations are zero. In this paper, modeling financial instruments is approached with a skew Student’s t-distribution based on the GH distribution. The copula of the SST has a third parameter $\boldsymbol{\gamma}$, which introduces the sought skewness in the dependence structure. Figure 2 below shows how the t copula still grows towards the quadrants’ extremities even when $\rho = 0$, as opposed to its Gaussian counterpart.