A Well-Conditioned Solution to the 1D Inverse Scattering Problem using the Distorted Born Iterative Method

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Abstract: A well-conditioned formulation of the electromagnetic inverse scattering problem using the Distorted Born iterative method is presented for 1D problems. The conditioning of the discrete problem is optimized using a set of adaptive whole-domain complex exponential basis functions for expansion of the object function. The method is derived using a generalized version of the method of moments and can easily be extended to higher dimensional scattering problems.

Keywords: Inverse imaging, Distorted Born, regularization

1. Introduction

The purpose of this paper is to present a well-conditioned algorithm for solving the inverse electromagnetic scattering problem in 1D using the Distorted Born iterative method. Similar to previous implementations of the Distorted Born method we solve the continuous problem using a method of moments (MoM) discretization [1]. As the inverse problem is ill-posed, discretization of the problem generally yields an ill-conditioned system of linear equations. While previous research has considered a pulse-basis expansion of the unknown permittivity distribution in addition to regularization [1], we suggest the use of whole-domain, adaptive basis functions which can be dynamically altered at each iteration in order to ensure a well-conditioned linear system. The advantage of this technique is that it does not require the selection of an appropriate regularization parameter. Instead, we perform regularization through an optimization over the basis function parameter space to yield a well-conditioned system. The formulation of this new method requires a multitude of scattering experiments each at carefully selected frequencies. Effectively, these experiments alter the kernel of the integral operator resulting in what we refer to as a *generalized* version of the moment method. The motivation for our adaptive basis function approach stems from the fact that under the Born Approximation, the system of equations produced is diagonal and hence perfectly conditioned. This paper first demonstrates this perfect conditioning under the Born Approximation in Section 2. The Distorted Born iterative technique is over-viewed in Section 3 while in Section 4, the results of Section 2 are extended to the iterative method. Finally, numerical results are presented to demonstrate the effectiveness of the method.

For simplicity, the work discussed focuses solely on the inverse scattering problem in 1D and in the usual manner of iteratively solving the inverse scattering problem we make use of the data and domain equations [2]. As the forward problem (solving the data equation) is a second kind integral equation, its conditioning is not in question and consequently we concentrate on the first-kind inverse problem (solving the domain equation) whose smooth kernel poses the basis for ill-conditioning.

2. A Generalized MoM Solution to the 1D Scattering Problem under the Born Approximation

Consider the 1D integral equation of potential scattering:

$$E(x) = E^{inc}(x) + k_o^2 \int_{-\infty}^{\infty} (\varepsilon(x') - 1) E(x') G(x, x') dx' \tag{1}$$

where $E(x)$ is the transverse component of the electric field, $E^{inc}(x)$ is the incident electric field, $\varepsilon(x)$ is the unknown relative permittivity as a function of position and where k_o is the wavenumber of free space. Throughout this paper an $e^{i\omega t}$ time dependence is suppressed where $i = \sqrt{-1}$ and ω is the radial frequency of the electric field. For a one-dimensional problem the free-space Green's function $G(x, x')$ is:

$$G(x, x') = \frac{1}{2ik_o} e^{-ik_o|x-x'|} = \frac{1}{2ik_o} e^{-ik_o s(x-x')} \tag{2}$$

where $s = 1$ if $x > x'$ and $s = -1$ if $x < x'$. For the incident field, we consider plane waves propagating in either the positive or negative x direction *i.e.*:

$$E^{inc}(x) = e^{-ik_o s' x} \tag{3}$$

where the direction of propagation is negative for $s = 1$ and positive for $s = 1$. Assuming a permittivity contrast which is spatially bounded to a domain $D = [x_1, x_2]$, the infinite integral in (1) collapses to D. Applying the Born Approximation, namely that scattering is weak and the field within the domain D may be approximated by the incident field, (1) becomes:

$$E(x, k_o, s, s') - E^{inc}(x, k_o, s) = -\frac{ik_o}{2} e^{-ik_o sx} \int_{x_1}^{x_2} \delta\varepsilon(x') e^{-ik_o(s'-s)x'} dx' . \tag{4}$$

Above, $\delta\varepsilon(x)$ is the contrast function used to denote the relative permittivity contrast $\varepsilon(x) - 1$. We now expand the unknown contrast function in terms of $2L$ complex exponential basis functions over the entire imaging domain:

$$\delta\varepsilon(x) \approx \sum_{l=1}^{L} A_l e^{ik_l x} + \sum_{l=1}^{L} B_l e^{-ik_l x} . \tag{5}$$

The spatial frequencies are selected as $k_l = l\Delta k - \Delta k/2$, for $l = 1, \ldots, L$, where Δk is selected as $(2\pi)/(x_2 - x_1)$. The $\Delta k/2$ shift is essential in that it implicitly adds a DC component into each harmonic over the imaging domain thereby eliminating the need for an explicit basis function at DC. Provided that Δk is sufficiently small, this expansion is equivalent to approximating the spectrum of the permittivity contrast with a set of equidistantly spaced, piece-wise functions over the interval $k = [-k_{max}, k_{max}]$ where $k_{max} = k_L$.

Upon substitution of (5) into (4) we obtain a single equation in $2L$ unknowns:

$$\frac{2i}{k_o} e^{ik_o sx} (E(x, k_o, s, s') - E^{inc}(x, k_o, s)) = \sum_{l=1}^{L} A_l \int_{x_1}^{x_2} e^{-i(k_o(s'-s) - k_l)x'} dx' + \sum_{l=1}^{L} B_l \int_{x_1}^{x_2} e^{-i(k_o(s'-s) + k_l)x'} dx' . \tag{6}$$

While a typical approach in obtaining a system of equations from (6) would be to test the equation $2L$ times to obtain a system of linear algebraic equations [3], we take a different approach by consider $2L$ *different* scattering experiments. We use L experiments with an incident field propagating in the positive x direction

where the scattering amplitude is measured at a single location $x_a < x_1$. For each of these experiments the incident field wavenumber is selected as $k_m/2$ for $m = 1, \ldots, L$. Further, we use L experiments where the incident field propagates in the negative x direction (taking corresponding measurements at a single location $x_b > x_2$) again using wavenumbers of $k_m/2$ for $m = 1, \ldots, L$. Corresponding to a frequency change in the incident field, we must enforce the required change in the Green's function within the integral equation. For incidence in the positive x direction this gives the following L algebraic equations:

$$\frac{2i}{k_o} e^{ik_o s x_a}(E(x_a, k_o, s, s') - E^{inc}(x_a, k_o, s)) = \sum_{l=1}^{L} A_l \int_{x_1}^{x_2} e^{-i(k_m - k_l)x'} dx' + \sum_{l=1}^{L} B_l \int_{x_1}^{x_2} e^{-i(k_m + k_l)x'} dx' \tag{7}$$

while for propagation in the negative x direction we obtain:

$$\frac{2i}{k_o} e^{ik_o s x_b}(E(x_b, k_o, s, s') - E^{inc}(x_b, k_o, s)) = \sum_{l=1}^{L} A_l \int_{x_1}^{x_2} e^{i(k_m - k_l)x'} dx' + \sum_{l=1}^{L} B_l \int_{x_1}^{x_2} e^{i(k_m + k_l)x'} dx' \tag{8}$$

Having carefully selected our spatial frequencies such that the functions $e^{ik_m x}$ and $e^{-ik_l x}$ are orthogonal over the imaging domain, the combined system of equations consisting of (7) and (8) is *perfectly conditioned* and may be written as:

$$\begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{B} \end{bmatrix} = \begin{bmatrix} \boldsymbol{F} \\ \boldsymbol{G} \end{bmatrix}, \tag{9}$$

where $\boldsymbol{I}$ is the identity matrix, $\boldsymbol{A} = [A_1, A_2, \ldots, A_L]^T$ and $\boldsymbol{B} = [B_1, B_2, \ldots, B_L]^T$ are vectors of the unknown contrast expansion coefficients and $\boldsymbol{F}$ and $\boldsymbol{G}$ represent vectors of the left-hand side of (7) and (8) respectively for each of the L scattering experiments, appropriately scaled by the width of the imaging domain. Clearly, this demonstrates that by using whole-domain basis functions and a multitude of scattering experiments we are able to produce a perfectly-conditioned system for the Born Approximation solution to the inverse scattering problem. In fact, it can be shown that this is merely a result corresponding to so-called Fourier Imaging techniques [2].

3. The Distorted Born Iterative Method

The pertinent theory of the Distorted Born iterative technique may be found in [1] and is summarized herein. As is common to most iterative techniques for solving the inverse scattering problem we consider equation (1) as two distinct equations used alternatively in a two-step updating procedure [2]. First, the *data equation* is used to update the permittivity contrast within the imaging domain at the nth iteration from field computed at the $(n-1)^{th}$ iteration.

$$E(x) - E_b^{(n-1)}(x) = k_o^2 \int_{x_1}^{x_2} \delta\varepsilon^{(n)}(x') E_b^{(n-1)}(x') G_b^{(n-1)}(x, x') dx' \qquad x \notin D \tag{10}$$

where $G_b^{(n-1)}$ is the numerical Green's function for the background permittivity from the previous iteration $\delta\varepsilon^{(n-1)}$ which satisfies the integral equation:

$$G_b^{(n)}(x, x') = G(x, x') + k_o^2 \int_{x_1}^{x_2} \delta\varepsilon^{(n)}(x'') G_b^{(n)}(x'', x') G(x, x'') dx'' , \tag{11}$$

and where $E_b^{(n-1)}(x)$ is the total field at the observation point x in the presence of the contrast $\delta\varepsilon^{(n-1)}$, and where $E(x)$ is simply the true total field which is a measurable quantity at x. To compute the numerical Green's function we must first solve (11) for all source points x' when x is located inside the imaging domain, *i.e.,* we solve the integral equation:

$$G_b^{(n)}(x, x') = G(x, x') + k_o^2 \int_{x_1}^{x_2} \delta\varepsilon^{(n)}(x'') G_b^{(n)}(x'', x') G(x, x'') dx'' \qquad x \in D , \tag{12}$$

which is a second kind integral equation and may be solved without difficulty. Having computed the numerical Green's function within the imaging domain we may directly solve (11) for it's value at any location in space.

The *domain equation* is then used to update the field within the imaging region from collected data outside of the domain D:

$$E_b^{(n)}(x) = E^{inc}(x) + k_o^2 \int_{x_1}^{x_2} \delta\varepsilon^{(n-1)}(x') E_b^{(n)}(x') G(x, x') dx' \qquad x \in D \tag{13}$$

Hence we may solve the inverse problem by repeating the following procedure:

- Solve for $\delta\varepsilon^{(n)}$ from (10) using the field computed for iteration $n-1$. (In the case of $n = 1$ we approximate the field using the Born Approximation, hence $\delta\varepsilon^{(0)} = 0$, $E_b^{(0)} = E^{inc}$ and $G_b^{(0)} = G$.)

- Solve for the numerical Green's function at the observation points outside the imaging domain using first equation (12) and then equation (11) and solve for the updated field within the imaging domain using equation (13) from the current contrast function. Additionally, directly compute $E_b^{(n)}$ at the observation points x considering (13) when $x \notin D$.

4. A Well-Conditioned MoM formulation for the Distorted Born Iterative method

The iterative procedure summarized in the previous section makes use of two integral equations for iteratively solving the non-linear inverse imaging problem under the linearizing assumption of the Born Approximation. The domain equation (13) (which has the same form as (12)) is a second-kind integral equation and typically does not pose any numerical problems in solving. For instance it can readily be solved by expanding the unknown field quantity into pulse basis functions and using point-matching. Consequently, the solution to equations (11) and (13) will not be discussed further. Conversely, the data equation is a first-kind integral equation with a smooth kernel making it an ill-posed problem in the sense of Hadamard [2]. The solutions to such problems undoubtedly lend themselves to numerical difficulties such as ill-conditioning. We consider therefore, the solution to the domain equation under the Born Approximation presented in Section 2, which, despite the ill-posedness of the problem gave an ideally conditioned matrix. It is clear however, that the orthogonal property used to produce this ideally conditioned system will vanish at subsequent iterations if the basis function expansion (5) is used. Instead, for iterations $n > 1$ we expand the contrast in the modified set of basis functions:

$$\delta\varepsilon^{(n)}(x) \approx \sum_{l=1}^{L} A_l^{(n)} e^{i\alpha^{(n)} k_l x} + \sum_{l=1}^{L} B_l^{(n)} e^{-i\alpha^{(n)} k_l x} \tag{14}$$

where $\alpha^{(n)}$ is an iteration dependent, real-valued parameter which is used to dynamically modify the basis function expansion of the unknown contrast function with the sole purpose of minimizing the condition number of the matrix created by the generalized MoM discretization of the data equation. Specifically, if we substitute the expansion (14) into (10) and making the dependence on the measurement location and incident field direction explicit via the parameters s and s' we obtain:

$$\begin{aligned}\frac{2i}{k_o} e^{ik_o sx}(E(x, k_o, s, s') - E^{inc}(x, k_o, s)) &= \sum_{l=1}^{L} A_l^{(n)} k_o^2 \int_{-\infty}^{\infty} e^{i\alpha^{(n)} k_l x'} E^{(n-1)}(x', k_o, s, s') G_b^{(n-1)}(x, x', k_o, s) dx' \quad . \\ &+ \sum_{l=1}^{L} B_l^{(n)} k_o^2 \int_{-\infty}^{\infty} e^{i\alpha^{(n)} k_l x'} E_{s'}^{(n-1)}(x') G_b^{(n-1)}(x, x', k_o, s) dx'\end{aligned} \tag{15}$$

The testing procedure is identical to the generalized moment method presented in Section 2 yielding a linear system which we write in compact form as:

$$\begin{bmatrix} \boldsymbol{E}_{11}^{(n-1)} & \boldsymbol{E}_{12}^{(n-1)} \\ \boldsymbol{E}_{21}^{(n-1)} & \boldsymbol{E}_{22}^{(n-1)} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{A}^{(n)} \\ \boldsymbol{B}^{(n)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}^{(n-1)} \\ \boldsymbol{G}^{(n-1)} \end{bmatrix} \tag{16}$$

where $\boldsymbol{E}_{11}^{(n-1)}$ and $\boldsymbol{E}_{21}^{(n-1)}$ are respective matrix representations of those terms in (15) which the coefficients $\boldsymbol{A}^{(n)}$ for the two sets of experiments while $\boldsymbol{E}_{12}^{(n-1)}$ and $\boldsymbol{E}_{22}^{(n-1)}$ correspond to the terms multiplying $\boldsymbol{B}^{(n)}$. Equation (16) reduces to equation (10) for $n = 1$ under the Born Approximation with $\alpha^{(1)} = 1$

It is clear that unlike matrix equation (9) resulting from the Born Approximation, the matrix in (16) will, in general, be full. Also in general, it will be poorly conditioned if the parameter $\alpha^{(n)} = 1$ is selected. Therefore, we minimize the condition number $C^{(n)}$ of the matrix by performing an optimization over the parameter $\alpha^{(n)}$. Experience has shown that the function $C^{(n)}(\alpha^{(n)})$ is not unimodal as shown in Fig. 2 and hence a global optimization routine is required.

5. Numerical Results

The iterative procedure using the basis function expansion previously described was implemented tested. Herein we show the results for a relative permittivity contrast selected to be the positive cycle of sinusoid with period 0.4 metres centred over the range $x = [-0.1, 0.1]$ having an amplitude of 2. Data was acquired at the locations $x = -0.4$ and $x = 0.4$ metres while the imaging domain was restricted to $D = [-0.3, 0.3]$ metres. The contrast was expanded using 10 basis functions and as optimization of the matrix condition number is over a single parameter, a direct search of the parameter space was performed over the range $[1, 1.6]$. The algorithm was terminated once the norm $\|\delta\varepsilon^{(n)} - \delta\varepsilon^{(n-1)}\|_2$ dropped below a specified tolerance. For convergence we have heuristically found that the optimal $\alpha^{(n)}$ must be selected at each iteration, although there may be many parameter values that give a sufficiently well conditioned matrix. The results of the iterative procedure, along with the profile error from iteration to iteration are shown in Fig. 1 while the system condition number versus parameter value α are shown for the first and second iterations in Fig. 2.

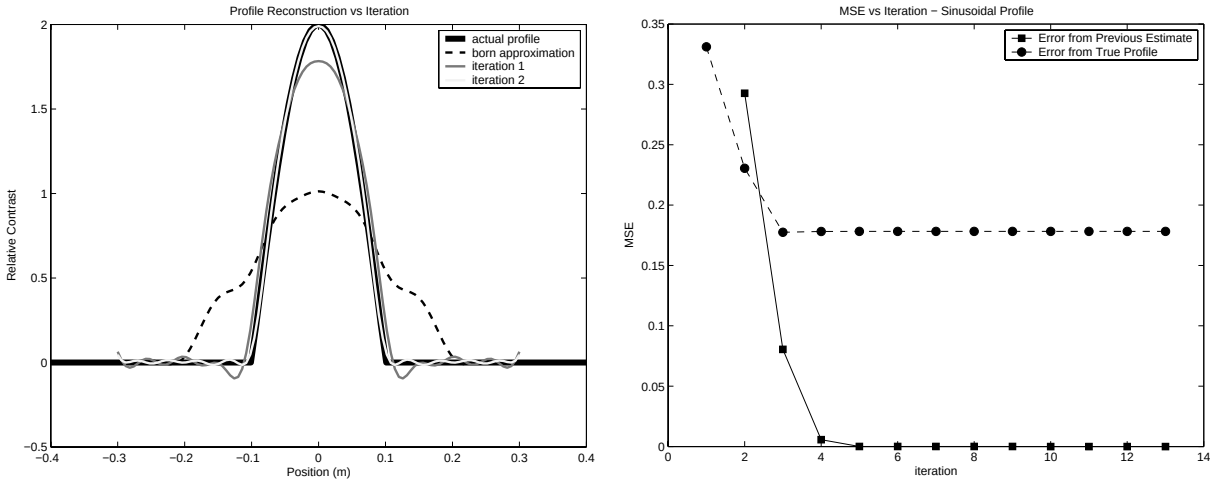

Figure 1. Iterative reconstruction results (left) and profile error (right)

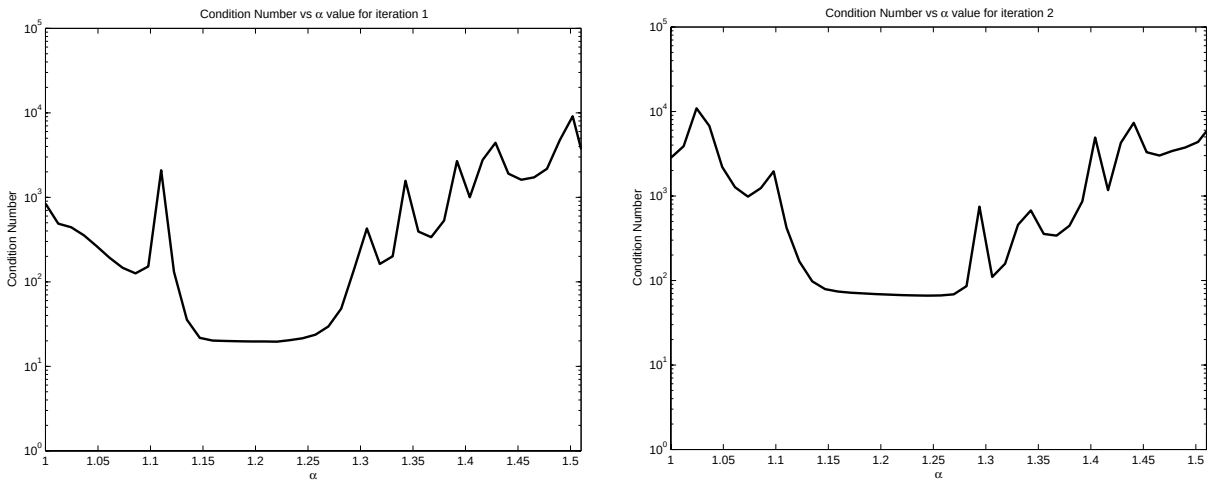

Figure 2. Condition number versus parameter value for iteration 1 (left) and iteration 2 (right)

6. Conclusions

Herein we have shown that it is possible to avoid the use of explicit regularization in the iterative MoM solution to the inverse scattering problem using the Distorted Born iterative method. The selection of adaptive basis functions and a global optimization over the basis parameter α ensures a well-conditioned matrix at each iteration. We have also implemented this algorithm for the (Non-Distorted) iterative Born method and are currently investigating the benefits of associating a unique parameter α_i with each of the i basis functions. Our current concerns are convergence to high-contrast profiles but this seems to be a problem which may be inherent in the Iterative Born techniques [2]. The proposed method can easily be expanded to higher dimensions and work in this direction is currently being undertaken.

7. References

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