Necessary Stability Criterion for Unstructured Mesh Upwinding FVTD Schemes for Maxwell's Equations

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Abstract: A new stability criterion applicable to explicit upwind FVTD schemes for solving Maxwell's equations on unstructured meshes is derived. This criterion is based on two-norm estimates of specially constructed matrices representing the surface integration on the sum-total of all the facets of each arbitrary finite-volume associated with the finite-volume spatial discretization. The new stability criterion gives a time-step that is larger than the time-step calculated using previously published stability criteria. On structured meshes the new criterion gives the same time-step limit as the von Neumann analysis. Implementation of the method incurs a small computational expense at the beginning of each run of the algorithm. The resulting time-step is mesh dependent. The method is generalizable but the extent to which it can be generalized to other time-evolving physical phenomenon is not considered in this paper.

Keywords: Finite-Volume Time-Domain, Unstructured Mesh, Maxwell's Equations, Stability Criterion

1. Introduction

One of the classical drawbacks of using explicit time-stepping numerical schemes is that a stringent time-step limit must be adhered to for stability. For structured meshes this time-step limit can usually be obtained using von Neumann analysis, but it is not possible to use von Neumann analysis with unstructured meshes. Thus, several authors have used analysis based on the energy-norm in the mesh to obtain estimates for the time-step bound. Previously published maximum time-step bounds have been sufficient to ensure stability but not tight bounds: they generally restrict the time-step to a value that is smaller than necessary. Obviously, using a smaller time-step than is necessary for stability increases execution time for any particular problem, but in addition, a time-step that is too small compared to the necessary limit may also result in poorer accuracy of solution. A sufficient maximum time-step criterion for an FVTD scheme applicable to Maxwell's equations was presented in [1, 2]. The time-step limit given in [2] is

$$\Delta t \le \min_i \frac{V_i}{cA_i}, \tag{1}$$

for an unstructured mesh and

$$\Delta t \le \frac{h}{2c},$$

for a structured mesh, where $1 \le i \le N$ is a number identifying the elements in the unstructured mesh, V_i and A_i are the volume and total facet area for the i-th element respectively, and h is the edge-size of the elements on a structured cubical mesh. On the other hand, in [1] the time-step limit for an unstructured mesh is given as

$$\Delta t \leq \min_i \frac{2V_i}{cA_i}, \tag{2}$$

twice that of (1) reported in [2]. Unfortunately, as was stated in [1], the bound given by (2) is merely a sufficient condition and not necessary: a larger time-step is possible.

In this paper we derive the necessary stability criterion for the first-order Euler explicit scheme which can then be easily extended for higher-order time integration schemes [3]. As in [1], the derivation is based on the natural physical constraint that in a mesh that is free of sources of energy the total energy in the mesh should not increase with time. The electromagnetic energy in a particular region, $\Omega \subset \mathbb{R}^3$, gives rise to the mathematical concept of an energy-norm which can be calculated as

$$\left\| \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} \right\| = \sqrt{\frac{1}{2} \int_{\Omega} \epsilon \boldsymbol{E} \cdot \boldsymbol{E} + \boldsymbol{\mu} \boldsymbol{H} \cdot \boldsymbol{H} dv} \,. \tag{3}$$

The key difference in the derivation that allows us to obtain the necessary criterion is that we express the summation of fluxes over facets as a single matrix operator for which the norm can be determined numerically. Thus, once an arbitrary mesh is generated, the necessary time-step limit is obtained by computing a simple formula over each element in the mesh at the beginning of each FVTD run.

2. The Time-Step Criterion in Terms of Energy

Suppose we have a domain $\Omega \subset \mathbb{R}^3$ upon which is specified an unstructured mesh $\Omega = \bigcup_{i=1}^{N} \Omega_i$, where Ω_i are the elements of the mesh each having a volume V_i. Here we suppose that the electromagnetic parameters $\boldsymbol{\epsilon}_i$ and $\boldsymbol{\mu}_i$ are constants on each element Ω_i. The finite-volume time-domain method is formulated in terms a generalized solution vector containing the electric and magnetic field vectors: $\boldsymbol{u}(\boldsymbol{x}) = (\boldsymbol{E}^T(\boldsymbol{x})\ \boldsymbol{H}^T(\boldsymbol{x}))^T$ and solved for the averaged values $\boldsymbol{u}_i = \frac{1}{V_i} \int_{\Omega_i} \boldsymbol{u}(\boldsymbol{x}) dv$ on each element. An equivalent discrete energy-norm over the domain Ω can be written as

$$\|\boldsymbol{u}\| = \sqrt{\frac{1}{2} \sum_{i=1}^{N} V_i (\boldsymbol{\epsilon}_i \boldsymbol{E}_i \cdot \boldsymbol{E}_i + \boldsymbol{\mu}_i \boldsymbol{H}_i \cdot \boldsymbol{H}_i)} \,, \tag{4}$$

where $\boldsymbol{E}_i$ represents the averaged value of the electric field over the element i, and similarly for the magnetic field vector $\boldsymbol{H}_i$.

Consider now the Euler approximation for the time-dependent Maxwell's equations cast as a conservation law (see [4]). It can be written concisely as

$$\boldsymbol{u}^{n+1} = \boldsymbol{u}^n - \Delta t L \boldsymbol{u}^n, \tag{5}$$

where L represents the discretization of the spatial derivatives. More specifically, for the case of the FVTD method, L represents the integration of the fluxes over the facets of each element. We are interested in the maximum value of Δt that keeps the scheme stable. A numerical scheme is L^2-stable if the energy doesn't grow in time; that is, if the following is true:

$$\|\boldsymbol{u}^{n+1}\|^2 \leq \|\boldsymbol{u}^n\|^2. \tag{6}$$

We can define an inner product for Maxwell's equations as $(\boldsymbol{u}, \boldsymbol{w}) = \sum_{i=1}^{N} V_i \boldsymbol{u}_i^T \alpha_i \boldsymbol{w_i}$, where

$$\alpha_i = \begin{pmatrix} \boldsymbol{\epsilon}_i & 0 \\ 0 & \boldsymbol{\mu}_i \end{pmatrix}$$

and $\boldsymbol{\epsilon}_i$, $\boldsymbol{\mu}_i$ are permittivity and permeability matrices for volume i. The energy-norm is obtained as $\|u\| = \sqrt{(\boldsymbol{u}, \boldsymbol{u})}$. It can be easily verified, that $(\boldsymbol{u}, \boldsymbol{w})$ satisfies the mathematical properties of an inner product.

Taking the inner product of (5) with $\boldsymbol{u}^n$ we get $(\boldsymbol{u}^{n+1}, \boldsymbol{u}^n) - (\boldsymbol{u}^n, \boldsymbol{u}^n) = -\Delta t(L\boldsymbol{u}^n, \boldsymbol{u}^n)$, and using the property that $(\boldsymbol{u}^{n+1}, \boldsymbol{u}^n) = \frac{1}{2}(\boldsymbol{u}^{n+1}, \boldsymbol{u}^{n+1}) + \frac{1}{2}(\boldsymbol{u}^n, \boldsymbol{u}^n) - \frac{1}{2}(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n, \boldsymbol{u}^{n+1} - \boldsymbol{u}^n)$ we can rewrite this as

$$(\boldsymbol{u}^{n+1}, \boldsymbol{u}^{n+1}) - (\boldsymbol{u}^n, \boldsymbol{u}^n) - (\boldsymbol{u}^{n+1} - \boldsymbol{u}^n, \boldsymbol{u}^{n+1} - \boldsymbol{u}^n) = -2\Delta t(L\boldsymbol{u}^n, \boldsymbol{u}^n).$$

This last equation together with $(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n, \boldsymbol{u}^{n+1} - \boldsymbol{u}^n) = (\Delta t L\boldsymbol{u}^n, \Delta t L\boldsymbol{u}^n)$ and the energy constraint (6) give

$$(\boldsymbol{u}^{n+1}, \boldsymbol{u}^{n+1}) - (\boldsymbol{u}^n, \boldsymbol{u}^n) = \Delta t^2(L\boldsymbol{u}^n, L\boldsymbol{u}^n) - 2\Delta t(L\boldsymbol{u}^n, \boldsymbol{u}^n) \leq 0.$$

This gives us a condition for the maximum time-step, based on the nonincreasing energy stability criterion for the Euler scheme, that depends on spatial discretization:

$$\Delta t(L\boldsymbol{u}^n, L\boldsymbol{u}^n) \leq 2(L\boldsymbol{u}^n, \boldsymbol{u}^n). \tag{7}$$

3. Application to FVTD Solution of Maxwell Equations

We can write the FVTD scheme for Maxwell's equations with Euler explicit time integration and first order spatial upwinding, as [1, 2]

$$\boldsymbol{u}_i^{n+1} = \boldsymbol{u}_i^n - \Delta t \frac{1}{V_i} \sum_{j=1}^{m_i} A_i(j) \left(T_i^+(j) B_i^+(j) \boldsymbol{u}_i^n + T_i^-(j) B_i^-(j) \boldsymbol{u}_{i_j}^n \right), \tag{8}$$

where m_i is the number of facets defining finite-volume Ω_i, and $A_i(j)$ is the area of the j^{th} facet of the i^{th} volume. The subscript i_j denotes the element neighboring facet j. The transmission operators are given as

$$T_i^{\pm}(j) = \alpha_i^{-1} \begin{pmatrix} \frac{2Y_i^{\mp}(j)}{Y_i^+(j)+Y_i^-(j)} I & 0 \\ 0 & \frac{2Z_i^{\mp}(j)}{Z_i^+(j)+Z_i^-(j)} I \end{pmatrix}$$

for facets between dielectrics, where the permittivity and permeability are scalars

$$Y_i^+(j) = \frac{1}{Z_i^+(j)} = \sqrt{\frac{\epsilon_i}{\mu_i}}, Y_i^-(j) = \frac{1}{Z_i^-(j)} = \sqrt{\frac{\epsilon_{i_j}}{\mu_{i_j}}}.$$

For facets located on a perfect electric conductor (PEC) these become

$$T_i^+(j) = \alpha_i^{-1} \begin{pmatrix} 2I & 0 \\ 0 & 0 \end{pmatrix}, T_i^-(j) = 0,$$

and for a facet at the external boundary of the mesh, we have

$$T_i^+(j) = \alpha_i^{-1} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, T_i^-(j) = 0.$$

The flux splitting operators are given as

$$B_i(j)^+ = \frac{1}{2} \begin{pmatrix} -S_i^2(j) & -S_i(j) \\ S_i(j) & -S_i^2(j) \end{pmatrix}, \quad B_i(j)^- = \frac{1}{2} \begin{pmatrix} S_i^2(j) & -S_i(j) \\ S_i(j) & S_i^2(j) \end{pmatrix}, \tag{9}$$

where the matrix operator $S_i(j)$ applied to an arbitrary vector $\boldsymbol{a}$ produces the cross-product of the outward normal $\hat{\boldsymbol{n}}_i(j)$, the normal to the j-th facet of element i, with $\boldsymbol{a}$, that is, $S_i(j)\boldsymbol{a} = \hat{\boldsymbol{n}}_i(j) \times \boldsymbol{a}$.

For simplicity, we now consider only the case when we have no PEC boundaries and the same $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ for all elements. We first write

$$(L\boldsymbol{u}, \boldsymbol{u}) = \frac{1}{2} \sum_{i=1}^{N} \sum_{k=1}^{m_i} A_i(k) \left(B_i^+(k)\boldsymbol{u}_i + B_i^-(k)\boldsymbol{u}_{i_k} \right) \cdot \boldsymbol{u}_i \tag{10}$$

$$(L\boldsymbol{u}, L\boldsymbol{u}) = \frac{1}{2} \sum_{i=1}^{N} \frac{c}{V_i} \sum_{k=1}^{m_i} A_i(k) \left(B_i^+(k)\boldsymbol{u}_i + B_i^-(k)\boldsymbol{u}_{i_k} \right) \cdot \sum_{j=1}^{m_i} A_i(j) \left(B_i^+(j)\boldsymbol{u}_i + B_i^-(j)\boldsymbol{u}_{i_j} \right) \tag{11}$$

The flux-splitting operators, $B_i^+(k)$ and $B_i^-(k)$, when applied to the field value at the center of an element give the flux at facet k which when summed over all facets of the element give zero [2]. That is, we have $\sum_{k=1}^{m_i} A_i(k) \left(B_i^+(k)\boldsymbol{u}_i + B_i^-(k)\boldsymbol{u}_i \right) = 0$, which can be written as

$$\sum_{k=1}^{m_i} A_i(k) B_i^+(k)\boldsymbol{u}_i = - \sum_{k=1}^{m_i} A_i(k) B_i^-(k)\boldsymbol{u}_i \tag{12}$$

Combining (11) with (12) allows us to write

$$(L\boldsymbol{u}, L\boldsymbol{u}) = \frac{1}{2} \sum_{i=1}^{N} \frac{c}{V_i} \left[\sum_{k=1}^{m_i} A_i(k) B_i^-(k)(\boldsymbol{u}_{i_k} - \boldsymbol{u}_i) \right] \cdot \left[\sum_{j=1}^{m_i} A_i(j) B_i^-(j)(\boldsymbol{u}_{i_j} - \boldsymbol{u}_i) \right],$$

whereas combining (10) with (12) gives

$$(L\boldsymbol{u}, \boldsymbol{u}) = \frac{1}{2} \sum_{i=1}^{N} \boldsymbol{u}_i \cdot \sum_{k=1}^{m_i} A_i(k) \left(B_i^-(k)(\boldsymbol{u}_{i_k} - \boldsymbol{u}_i) \right).$$

For our case,because $S^T = -S$, we have $B = B^T$ we can write

$$B\boldsymbol{a} \cdot \boldsymbol{b} = \frac{1}{2} B\boldsymbol{b} \cdot \boldsymbol{b} + \frac{1}{2} B\boldsymbol{a} \cdot \boldsymbol{a} - \frac{1}{2} B(\boldsymbol{b} - \boldsymbol{a}) \cdot (\boldsymbol{b} - \boldsymbol{a}),$$

and therefore

$$\begin{aligned} &\boldsymbol{u}_i \cdot \sum_{k=1}^{m_i} A_i(k) \left(B_i^-(k)(\boldsymbol{u}_{i_k} - \boldsymbol{u}_i) \right) = \\ &\tfrac{1}{2} \sum_{k=1}^{m_i} A_i(k) \left[\boldsymbol{u}_{i_k} \cdot B_i^-(k)\boldsymbol{u}_{i_k} - \boldsymbol{u}_i \cdot B_i^-(k)\boldsymbol{u}_i - (\boldsymbol{u}_{i_k} - \boldsymbol{u}_i) \cdot B_i^-(k)(\boldsymbol{u}_{i_k} - \boldsymbol{u}_i) \right]. \end{aligned}$$

Introducing the expressions for $(L\boldsymbol{u}, \boldsymbol{u})$ and $(L\boldsymbol{u}, L\boldsymbol{u})$ in to formula (7) we get

$$\begin{aligned} &\Delta t \sum_{i=1}^{N} \tfrac{c}{V_i} \left[\sum_{k=1}^{m_i} A_i(k) B_i^-(k)(\boldsymbol{u}_{i_k} - \boldsymbol{u}_i) \right] \cdot \left[\sum_{k=1}^{m_i} A_i(k) B_i^-(k)(\boldsymbol{u}_{i_k} - \boldsymbol{u}_i) \right] \leq \\ &\sum_{i=1}^{N} \sum_{k=1}^{m_i} A_i(k) \left[\boldsymbol{u}_{i_k} \cdot B_i^-(k)\boldsymbol{u}_{i_k} - \boldsymbol{u}_i \cdot B_i^-(k)\boldsymbol{u}_i - (\boldsymbol{u}_{i_k} - \boldsymbol{u}_i) \cdot B_i^-(k)(\boldsymbol{u}_{i_k} - \boldsymbol{u}_i) \right], \end{aligned}$$

It is also easy to check that the following is true:

$$\sum_{i=1}^{N}\sum_{k=1}^{m_i} A_i(k)\left[\boldsymbol{u}_{i_k}\cdot B_i^-(k)\boldsymbol{u}_{i_k} - \boldsymbol{u}_i\cdot B_i^-(k)\boldsymbol{u}_i\right] = -\sum_{i=1}^{N_b}\sum_{k=1}^{m_i^s} A_i(k)\boldsymbol{u}_i\cdot B_i^-(k)\boldsymbol{u}_i \geq 0, \qquad (13)$$

where N_b is the number of elements with facets on the domain boundary, and m_i^s is the number of facets of i-th element on that boundary. Hence with (13) we can write the inequality as

$$\begin{aligned}\sum_{i=1}^{N}\frac{c\Delta t}{V_i}\left[\sum_{k=1}^{m_i} A_i(k)B_i^-(k)(\boldsymbol{u}_{i_k}-\boldsymbol{u}_i)\right]\cdot\left[\sum_{j=1}^{m_i} A_i(j)B_i^-(j)(\boldsymbol{u}_{i_j}-\boldsymbol{u}_i)\right] \leq \\ -\sum_{i=1}^{N}\sum_{k=1}^{m_i} A_i(k)(\boldsymbol{u}_{i_k}-\boldsymbol{u}_i)\cdot B_i^-(k)(\boldsymbol{u}_{i_k}-\boldsymbol{u}_i) \leq 2(L\boldsymbol{u},\boldsymbol{u})\end{aligned} \qquad (14)$$

This is the fundamental global inequality that imposes the stability constraint on Δt. It is not a simple task to derive a global constraint on Δt based on this formula. Therefore, we have to make due with imposing this inequality locally on a finite-volume by finite-volume basis. This removes the summation over all finite-volumes but leaves the inner summations. In order to get a manageable constraint for Δt, even limiting ourselves to a local constraint, requires that we somehow remove the inner summations over facets while keeping the formula exact. We proceed by first constructing a block-diagonal matrix $Z_i = diag\{-B_i^-(k)\}_{k=1}^{m_i}$ as well as a block-row vector of m_i identity matrices $W = \{I, ..., I\}$ where the dimension of W is $6 \times 6m_i$ and I is the 6×6 identity matrix. We also construct a column vector made up of the solution vector differences across each facet:

$$\boldsymbol{x} = vector\{\boldsymbol{u}_{i_k} - \boldsymbol{u}_i\}_{k=1}^{m_i},$$

which is a vector of length $6m_i$. Hence a new local inequality, based on (14), can be written concisely using these constructions as

$$\frac{c\Delta t}{V_i}(WZ_i\boldsymbol{x}, WZ_i\boldsymbol{x}) \leq (Z_i\boldsymbol{x}, \boldsymbol{x}), \;\; \forall i.$$

Also, because $B_i^-(k)^T = B_i^-(k)$, we have also $Z_i = Z_i^T$. Hence the square-root of Z_i can be expressed a $Z_i = Q^T\Lambda_i Q = Q^T\sqrt{\Lambda_i}Q\; Q^T\sqrt{\Lambda_i}Q = \sqrt{Z_i}\sqrt{Z_i}$, where $\Lambda_i = diag\{\lambda_j\}_{j=1}^{6m_i}$ is the diagonal matrix of eigenvalues of Z_i, and $\sqrt{\Lambda_i} = diag\{\sqrt{\lambda_j}\}_{j=1}^{6m_i}$. Now defining the new variable $\boldsymbol{y} = \sqrt{Z_i}\boldsymbol{x}$, which means that $\sqrt{Z_i}\boldsymbol{y} = \boldsymbol{x}$ and therefore $\boldsymbol{y} \notin Ker(\sqrt{Z_i})$, we have

$$\frac{c\Delta t}{V_i}(W\sqrt{Z_i}\boldsymbol{y}, W\sqrt{Z_i}\boldsymbol{y}) \leq (\boldsymbol{y}, \boldsymbol{y}),$$

or

$$\frac{V_i}{c\Delta t} \geq \frac{(W\sqrt{Z_i}\boldsymbol{y}, W\sqrt{Z_i}\boldsymbol{y})}{(\boldsymbol{y}, \boldsymbol{y})}, \;\forall i.$$

Finally, this last inequality can be written in terms of the original summations over the facets as

$$\frac{V_i}{c\Delta t} \geq \frac{\left[\sum_{k=1}^{m_i}\sqrt{A_i(k)}\sqrt{-B_i^-(k)}\boldsymbol{y}_k\right]\cdot\left[\sum_{j=1}^{m_i}\sqrt{A_i(j)}\sqrt{-B_i^-(j)}\boldsymbol{y}_j\right]}{\sum_{k=1}^{m_i}(\boldsymbol{y}_k, \boldsymbol{y}_k)}, \;\forall i \qquad (15)$$

¿From the above equation we can evaluate the maximum Δt for stability by numerically evaluating the right hand side over all the elements. This calculation can be simplified considerably if we use the property that $S_i^3(k) = -S_i(k)$ and notice that

$$[-B_i^-(k)]^2 = \frac{1}{4}\begin{pmatrix} S_i^2(k) & -S_i(k) \\ S_i(k) & S_i^2(k)\end{pmatrix} \times \begin{pmatrix} S_i^2(k) & -S_i(k) \\ S_i(k) & S_i^2(k)\end{pmatrix} = \frac{1}{2}\begin{pmatrix} -S_i^2(k) & S_i(k) \\ -S_i(k) & -S_i^2(k)\end{pmatrix} = -B_i^-(k),$$

hence $\sqrt{-B_i^-(k)} = -B_i^-(k)$. This means that the eigenvalues of the operator $-B_i^-(k)$ are only 1, with multiplicity 2, and 0, with multiplicity 4. Now an eigenvector of $-B_i^-(k)$ which corresponds to the eigenvalue 1 can be written as

$$\begin{pmatrix} \hat{\boldsymbol{a}} \\ \hat{\boldsymbol{b}} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -S^2 & S \\ -S & -S^2 \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{a}} \\ \hat{\boldsymbol{b}} \end{pmatrix}, \quad \begin{matrix} 2\hat{\boldsymbol{a}} = -S^2\hat{\boldsymbol{a}} + S\hat{\boldsymbol{b}} \\ S\hat{\boldsymbol{a}} = -2\hat{\boldsymbol{b}} - S^2\hat{\boldsymbol{b}} \end{matrix},$$

giving that any vector $\hat{e} = (\hat{\boldsymbol{a}}^T\ \hat{\boldsymbol{b}}^T)^T$ with $\hat{\boldsymbol{a}} = S\hat{\boldsymbol{b}}$ is an eigenvector.

Hence we can choose an eigenvector as $\hat{\boldsymbol{e}}_k = \frac{1}{\sqrt{2}}\left(\hat{\boldsymbol{n}}_k \times \hat{\boldsymbol{b}}_k, \hat{\boldsymbol{b}}_k\right)^T$, where we choose $\hat{\boldsymbol{b}}_k$ as an arbitrary vector in the plane of the k^{th} facet ($\hat{\boldsymbol{n}}_k \cdot \hat{\boldsymbol{b}}_k = 0$). We can construct two orthogonal vectors $\hat{\boldsymbol{b}}_k^1$ and $\hat{\boldsymbol{b}}_k^2$ by the formula $\hat{\boldsymbol{b}}_k^2 = \hat{\boldsymbol{n}}_k \times \hat{\boldsymbol{b}}_k^1$. Finally both eigenvectors can be written as:

$$\hat{\boldsymbol{e}}_k^1 = \frac{1}{\sqrt{2}}\left(\hat{\boldsymbol{n}}_k \times \hat{\boldsymbol{b}}_k, \hat{\boldsymbol{b}}_k\right)^T, \ \ \hat{\boldsymbol{e}}_k^2 = \frac{1}{\sqrt{2}}\left(-\hat{\boldsymbol{b}}_k, \hat{\boldsymbol{n}}_k \times \hat{\boldsymbol{b}}_k\right)^T.$$

The four eigenvectors corresponding to the zero eigenvalue of $-B_i^-(k)$ can be written as

$$\hat{\boldsymbol{e}}_k^{3,4} = \frac{1}{\sqrt{2}}\left(\hat{\boldsymbol{n}}_k, \pm\hat{\boldsymbol{n}}_k\right)^T, \ \ \hat{\boldsymbol{e}}_k^{5,6} = \frac{1}{\sqrt{2}}\left(-\hat{\boldsymbol{n}}_k \times \hat{\boldsymbol{b}}_k^{1,2}, \hat{\boldsymbol{b}}_k^{1,2}\right)^T.$$

Thus, with the single vector $\hat{\boldsymbol{b}}_k^1$ in the plane of the k^{th} facet we can define all eigenvectors of the operator $\sqrt{-B_i^-(k)}$. To efficiently evaluate formula (15) we can decompose each $\boldsymbol{y}_k$ as a sum of eigenvectors, with the only ones taking part being the ones corresponding to non-zero eigenvalues of $\sqrt{-B_i^-(k)}$:

$$\boldsymbol{y}_k = \sum_{s=1}^{2} \alpha_k^s \hat{\boldsymbol{e}}_k^s$$

The value of $\hat{\boldsymbol{b}}_k^1$ can be chosen arbitrarily, for example the edge of the facet. Substituting this decomposition into the inequality (15) we get the formula:

$$\frac{V_i}{c\Delta t} \geq max \frac{\left[\sum\limits_{k=1}^{m_i} \sqrt{A_i(k)} \sum\limits_{s=1}^{2} \alpha_k^s \hat{\boldsymbol{e}}_k^s\right] \cdot \left[\sum\limits_{j=1}^{m_i} \sqrt{A_i(j)} \sum\limits_{s=1}^{2} \alpha_j^s \hat{\boldsymbol{e}}_j^s\right]}{\sum\limits_{k=1}^{m_i} \sum\limits_{s=1}^{2} (\alpha_k^s)^2},$$

simplifying we get the formula which is used to obtain the time-step limit:

$$\frac{V_i}{c\Delta t} \geq max \frac{\sum\limits_{k=1}^{m_i} \sum\limits_{j=1}^{m_i} \sqrt{A_i(k)A_i(j)} \sum\limits_{s_1=1}^{2} \sum\limits_{s_2=1}^{2} \hat{\boldsymbol{e}}_j^{s_1} \cdot \hat{\boldsymbol{e}}_j^{s_2} \alpha_j^{s_1} \alpha_j^{s_2}}{\sum\limits_{k=1}^{m_i} \sum\limits_{s=1}^{2} (\alpha_k^s)^2}.$$

This can be written concisely as

$$\Delta t \leq \min_i \frac{V_i}{c\|G_i\|}, \tag{16}$$

where the elements of the matrix $G_i \in \mathbb{R}^{2m_i \times 2m_i}$ are written as

$$G_i = \left[g_j^i g_k^i \hat{\boldsymbol{e}}^j \cdot \hat{\boldsymbol{e}}^k\right]_{j,k=1}^{2m_i}, \ \ g_{2k-1}^i = g_{2k}^i = \sqrt{A_i(k)}, \ \ \hat{\boldsymbol{e}}^{2k} = \hat{\boldsymbol{e}}_j^2, \ \ \hat{\boldsymbol{e}}^{2k-1} = \hat{\boldsymbol{e}}_j^1; \ \ k = 1..m_i$$

The norm of matrix G_i can be computed relatively quickly because $4 \leq m_i \leq 6$ for a cell-centered FVTD mesh which contains tetrahedrons, prisms, pyramids and hexahedrons.

4. Numerical experiments

To test the increase in the allowed time-step due to our new limit we conducted a wide set of numerical experiments on our cell-centered FVTD code [4]. These were conducted for both unstructured as well as structured meshes. For unstructured tetrahedral mesh we had a 5-15% increase in the allowable timestep over the time-step limit given by formula (2) from [1]. For a structured cubical mesh (16) gives the same result as the von Neumann method applied to the FVTD approximation of Maxwell's equations [2]:

$$\Delta t \leq \frac{h}{2c}.$$

This is a 1.5 times larger time-step than the that allowed by (2) when it is applied to a structured cubical mesh (h taken as the cubical element edge size).

5. Conclusion

The derivation we've provided gives a new necessary time-step limit for the explicit upwinding finite-volume time-domain approximation scheme of Maxwell's equations. This new criterion allows one to mix structured and unstructured meshes, while keeping the maximum time-step allowable for retaining stability. In fact, the time-step limit provided by the formula given herein gives the same time-step limit on structured meshes as does the Von Neumann analysis. This derivation can be easily extended to other FVTD approximations of partial differential equations on unstructured meshes; a subject of future work.

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