Implementation of Surface Impedance Boundary Conditions in Upwind Finite Difference Techniques

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Abstract - General dispersive surface impedance boundary conditions (SIBCs) are formulated in the time domain in conjunction with an upwind finite difference method for the solution of the Maxwell equations. The associated convolution integral is implemented using an efficient recursive technique. The technique is validated on a test problem consisting of a short Gaussian pulse impinging on a lossy dispersive half-space. The reflection coefficient obtained via this new technique is compared with the exact analytic solution and and the results agree very well.

1. Introduction

The use of the FDTD technique for the numerical analysis of time domain electromagnetic problems has become quite common and can produce useful results for the engineering community [1]. Although the centered space and time leap-frog scheme, implemented on an interlaced grid, seems to be the most popular differencing scheme, it is not the only effective scheme which can be applied to the Maxwell equations. Many other techniques, originating in the fluid dynamics community, can be applied [2, 3, 4, 5]. The class of split-flux upwind schemes, considered herein, are effective methods which can be implemented on body-fitted curvilinear coordinate systems [6]. Some of the advantages of these upwind methods are the spatial and temporal collocation of the electric and magnetic field vectors as well as superior numerical dispersion characteristics. In this paper we develop a method of implementing dispersive surface impedance boundary conditions with first and second-order upwind methods. The method we formulate is consistent with the characteristic-based nature of the upwind schemes and takes full advantage of the collocation of the electric and magnetic field vectors at the impedance boundary.

2. Characteristic-Based Upwind Schemes

It is well known that the Maxwell curl equations can be written in conservation law form as $\partial_t \boldsymbol{u} + \boldsymbol{A}\partial_x \boldsymbol{u} + \boldsymbol{B}\partial_y \boldsymbol{u} + \boldsymbol{C}\partial_z \boldsymbol{u} = \boldsymbol{F}$ where $\boldsymbol{u} = \left[\boldsymbol{E}(\boldsymbol{x}, t)\ \boldsymbol{H}(\boldsymbol{x}, t)\right]^T$ is the solution vector of electric and magnetic fields. Each of the matrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$, can be diagonalized; *i.e.*we can find a matrix $\boldsymbol{S}$, consisting of the right eigenvectors of $\boldsymbol{A}$, such that $\boldsymbol{RAS} = \Lambda$,where $\boldsymbol{R} = \boldsymbol{S}^{-1}$ and Λ is a diagonal matrix of the ordered eigenvalues of the matrix $\boldsymbol{A}$. In one spatial dimension we can define new characteristic variables:

$$\boldsymbol{w} = \boldsymbol{Ru}\,,\ \boldsymbol{u} = \boldsymbol{Sw} = \boldsymbol{S}\left[w^- \ w^+\right]^T \qquad (1)$$

which remain constant along the characteristic directions shown in figure 1.

In one spatial dimension, the upwind differencing method can be applied to hyperbolic systems of the form $\partial_t \boldsymbol{u} + \boldsymbol{A}\partial_x \boldsymbol{u} = \boldsymbol{0}$ via the flux vector splitting technique of Steger and Warming [2]. Thus if we define the diagonal matrices of positive and negative eigenvalues of $\boldsymbol{A}$ as Λ^+, and Λ^- then $\boldsymbol{A}$ can be split as $\boldsymbol{S}(\Lambda^+ + \Lambda^-)\boldsymbol{Ru} = (\boldsymbol{A}^+ + \boldsymbol{A}^-)\boldsymbol{u}$ and we can write the predictor-corrector upwind difference scheme of Warming and Beam [3] as

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$$u_j^{\overline{n+1}} = u_j^n - \rho[A^+(u_j^n - u_{j-1}^n) + A^-(u_{j+1}^n - u_j^n)], \tag{2}$$

$$\begin{aligned} u_j^{n+1} = {} & \frac{1}{2}(u_j^n + u_j^{\overline{n+1}}) - \frac{\rho}{2}(A^+(u_j^{\overline{n+1}} - u_{j-1}^{\overline{n+1}}) + A^+(u_j^n - 2u_{j-1}^n + u_{j-2}^n)) \\ & - \frac{\rho}{2}(A^-(u_{j+1}^{\overline{n+1}} - u_j^{\overline{n+1}}) - A^-(u_{j+2}^n - 2u_{j+1}^n + u_j^n)) \end{aligned} \tag{3}$$

where $\rho = \Delta t/\Delta x$. This scheme is second order accurate in both space and time and it has a Courant number of 2. With slight modification the predictor part of this scheme can be used on its own as a first order accurate upwind differencing technique. Both methods extend easily to the case of two and three dimensional space.

3. Surface Impedance Boundary Conditions

A general class of surface impedance boundary conditions are written as

$$\hat{n} \times (\hat{n} \times \boldsymbol{E}) = -\psi \otimes \hat{n} \times \boldsymbol{H} \tag{4}$$

where $\hat{n}$ is the unit normal pointing away from the impedance boundary surface, the symbol $\otimes$ denotes convolution, and $\psi(t)$ is the inverse Fourier transform of the frequency dependent surface impedance, $\eta_s(\omega)$. When applying upwind schemes we generally encounter the tangential electric and magnetic field components at a boundary. The discussion of the method is simplified if we consider an impedance boundary on a one-dimensional grid as shown in figure 1. The diagonal arrows represent the characteristic directions upon which the characteristic variables propagate.

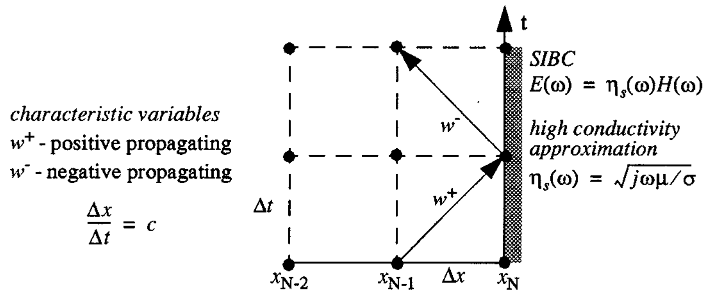

Figure 1 1-D grid showing characteristics and impedance relation at the boundary.

In the time domain the SIBC is a convolution between the tangential electric and magnetic fields at the boundary. For simplicity we now assume the high conductivity approximation for the surface impedance, $\eta_s(\omega) = \sqrt{j\omega\mu/\sigma}$, and following a procedure similar to that found in ref. [1] we arrive at the following relation between the tangential electric and magnetic fields:

$$E^n = \sqrt{\frac{\mu}{\sigma\pi\Delta t}} \sum_{m=0}^{n-1} \left[\int_m^{m+1} \frac{d\alpha}{\alpha} \right] [H^{n-m} - H^{n-m-1}] = \gamma \sum_{m=0}^{n-1} \beta(m)[H^{n-m} - H^{n-m-1}] \tag{5}$$

where $\gamma = \sqrt{\mu/\sigma\pi\Delta t}$ and $\beta(m)$ can be approximated using Prony's method as in ref. [1], that is

$$\beta(m) \cong \sum_{i=1}^{10} a_i e^{m\alpha_i} \tag{6}$$

where the coefficients a_i and the exponents α_i can be found in [1]. After some manipulation the final relation between the electric and magnetic fields can be rewritten as

$$E^n - \gamma\beta(0)H^n = -\gamma\beta(0)H^{n-1} + S(n) \tag{7}$$

where $S(n)$ is recursively defined as

$$S(n) = \gamma\sum_{i=1}^{10} a_i S^i(n), \qquad S^i(n) = e^{\alpha_i}\{H^{n-1} - H^{n-2} + S^i(n-1)\}. \tag{8}$$

4. Implementation into Upwind Scheme

We now describe how to implement the above SIBC into the upwind finite difference scheme previously described. At the boundary identified with index N in figure 1, eq. (7) can be written as

$$E_N^{n+1} - \gamma\beta(0)H_N^{n+1} = \begin{bmatrix} 1 & -\gamma\beta(0) \end{bmatrix} \boldsymbol{u}_N^{n+1} = F^{n+1} \tag{9}$$

where E_N^{n+1} and H_N^{n+1} represent the tangential electric and magnetic fields at time n+1 collocated at the boundary and $F^{n+1} = -\gamma\beta(0)H^n + S(n+1)$ represents a source term which is dependent on previous values of the tangential magnetic field. Introducing the characteristic variables as $\boldsymbol{u} = \boldsymbol{S}\boldsymbol{w}$ we arrive at

$$\begin{bmatrix} 1 & -\gamma\beta(0) \end{bmatrix} \boldsymbol{S}\boldsymbol{w}_N^{n+1} = \begin{bmatrix} \theta_1 & \theta_2 \end{bmatrix} \boldsymbol{w}_N^{n+1} = \theta_1 \overset{-}{w}{}_N^{n+1} + \theta_2 \overset{+}{w}{}_N^{n+1} = F^{n+1}. \tag{10}$$

Since the characteristic variables are constant along the characteristic directions as shown in figure 1, we have $\overset{+}{w}{}_N^{n+1} = \overset{+}{w}{}_{N-1}^{n}$ and therefore we can solve for the outgoing characteristic variable at the boundary, $\overset{-}{w}{}_N^{n+1}$, in eq. (10) as $\overset{-}{w}{}_N^{n+1} = \theta_1^{-1}(F^{n+1} - \theta_2 \overset{+}{w}{}_{N-1}^{n})$. The characteristic variable $\overset{-}{w}{}_{N-1}^{n}$ is obtained via the transformation $\boldsymbol{w}_{N-1}^n = \boldsymbol{R}\boldsymbol{u}_{N-1}^n$ and finally the solution vector at the boundary at time n+1 is found as

$$\boldsymbol{u}_N^{n+1} = \boldsymbol{S}\begin{bmatrix} \theta_1^{-1}\left(F^{n+1} - \theta_2\begin{bmatrix} R_{21} & R_{22} \end{bmatrix}\boldsymbol{u}_{N-1}^n\right) \\ \begin{bmatrix} R_{21} & R_{22} \end{bmatrix}\boldsymbol{u}_{N-1}^n \end{bmatrix}. \tag{11}$$

This method can be coupled to either the first order or second order upwind difference equations described previously.

5. Test Problem

We choose the same test problem as that described in ref. [1]. That is, a sharp Gaussian pulse impinging on a half space of conductivity σ = 2 S/m and σ = 20 S/m. The frequency domain reflection coefficient obtained via the method described herein, and the exact analytic reflection coefficient are shown in figures 2 and 3. As can be seen the results agree very well.

6. Conclusions

A method of implementing dispersive SIBCs in upwind finite difference methods has been developed and validated on a simple test problem where the analytic solution is known. Using upwind schemes has the advantage that the field vectors are collocated in space and in time on the numerical grid (including the surface of the impedance boundary).

Acknowledgments

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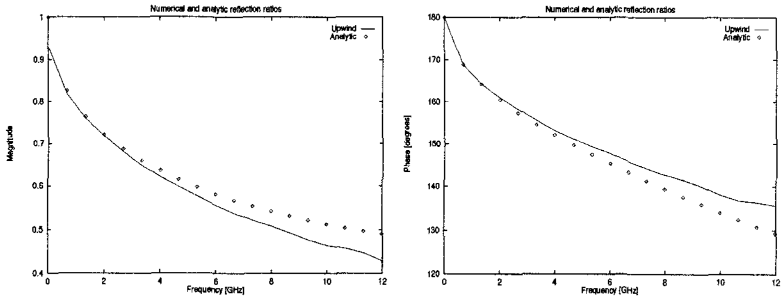

Figure 2 Reflection coefficient for $\sigma = 2$ S/m.

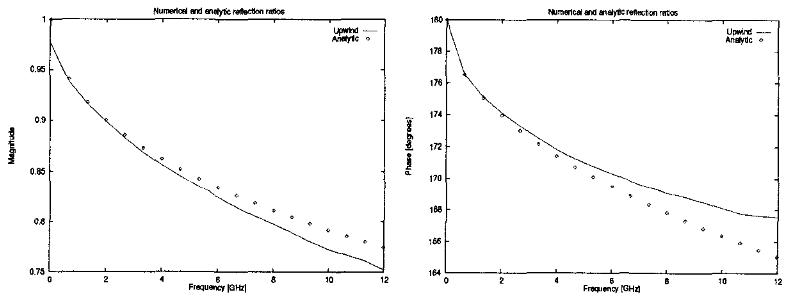

Figure 3 Reflection coefficient for $\sigma = 20$ S/m.