AAA - 1st Assignment

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Exercise 1 (A)

$$y = \phi(\sum_{i=0}^{m} x_i * w_i)$$

$$\phi(x) = \begin{cases} 1 \;\; if \;\; x > 0 \\ 0 \;\; if \;\; x < 0 \end{cases}$$

Looking at the pipeline from output to input we can write the function as:

$$y = \phi(x_1 * w_{21} + x_2 * w_{22} + w_{23} * \phi(x_1 * w_{11} + x_2 * w_{12} + b_1) + b_2) = \phi(x_1 + x_2 - 2 * \phi(x_1 + x_2 - 1.5) - 0.5)$$

To calculate the separation planes:
$\phi(x_1 + x_2 - 1.5) \leq 0 \Leftrightarrow x_1 + x_2 + 1.5 \leq 0 \Leftrightarrow x_1 + x_2 \leq 1.5$

From the previous calculation we can see that:

$$\phi(x_1 + x_2 - 2 * \phi(x_1 + x_2 - 1.5) - 0.5) = \begin{cases} \phi(x_1 + x_2 - 2 * 0 - 0.5) & \text{if } x_1 + x_2 \leq 1.5 \\ \phi(x_1 + x_2 - 2 * 1 - 0.5) & \text{if } x_1 + x_2 > 1.5 \end{cases}$$

To calculate the intervals where $y > 0$ we just have to develop the equation:

$\phi(x_1 + x_2 - 2 * \phi(x_1 + x_2 - 1.5) - 0.5) > 0$

$$\begin{cases} \phi(x_1 + x_2 - 0.5) > 0 & \text{if } x_1 + x_2 \leq 1.5 \\ \phi(x_1 + x_2 - 2.5) > 0 & \text{if } x_1 + x_2 > 1.5 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x_1 + x_2 - 0.5 > 0 & \text{if } x_1 + x_2 \leq 1.5 \\ x_1 + x_2 - 2.5 > 0 & \text{if } x_1 + x_2 > 1.5 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x_1 + x_2 > 0.5 & \text{if } x_1 + x_2 \leq 1.5 \\ x_1 + x_2 > 2.5 & \text{if } x_1 + x_2 > 1.5 \end{cases}$$

We can see that the areas where $y = 1$ is where the following condition is met: $0.5 < x_1 + x_2 \leq 1.5$.
We do not consider the second case because $x_1 + x_2 > 2.5$ is out of the domain of this problem.

The separating planes are, therefore:

- $x_2 = 0.5 - x_1$
- $x_2 = 1.5 - x_1$

Exercise 1 (B)

if $x_1 = 0$ and $x_2 = 0$:
$y = \phi(0 + 0 - 2 * \phi(0 + 0 - 1.5) - 0.5) = \phi(2 * \phi(-1.5) - 0.5) = \phi(-0.5) = 0$

if $x_1 = 1$ and $x_2 = 0$:
$y = \phi(1 + 0 - 2 * \phi(1 + 0 - 1.5) - 0.5) = \phi(1 + 2 * \phi(-0.5) - 0.5) = \phi(0.5) = 1$

if $x_1 = 0$ and $x_2 = 1$:
$y = \phi(0 + 1 - 2 * \phi(0 + 1 - 1.5) - 0.5) = \phi(1 + 2 * \phi(-0.5) - 0.5) = \phi(0.5) = 1$

if $x_1 = 1$ and $x_2 = 1$:
$y = \phi(1+1-2*\phi(1+1-1.5)-0.5) = \phi(2-2*\phi(0.5)-0.5) = \phi(2-2-0.5) = 0$

x_1	x_2	*output*
0	0	0
1	0	1
0	1	1
1	1	0

Table 1: Truth Table

Exercise 1 (C)

Using sigmoid function as the activation function, with $a = 1$ we have:

$$\phi(x) = \frac{1}{1 + e^{-x}}$$

The derivative of this function is

$$\phi^{'}(x) = \phi(x) * (1 - \phi(x))$$

The learning correction is given by the function:

$$\Delta w_{ij}(n) = \eta \cdot \delta_j(n) \cdot y_i(n),$$

such that, for this scenario $\eta = 0.1$ and the gradient

$$\delta_j(n) = \begin{cases} e_j(n) \cdot \phi_j^{'}(v_j(n)) & \text{if j is output neuron} \\ \phi_j^{'}(v_j(n)) \cdot \sum_k \delta_k(n) \cdot w_{kj}(n) & \text{if j is hidden neuron} \end{cases}$$

The starting conditions of the network in consideration are:

$$w_{11} = w_{12} = w_{21} = w_{22} = 1; \;\; w_{23} = -2$$

Using the scenario where $x_1 = 0; x_2 = 0$; in the forward steps we have that

$$y_1 = \phi(x_1 \cdot w_{11} + x_2 \cdot w_{12} - 1.5) = \phi(0 \cdot 1 + 0 \cdot 1 - 1.5) = \phi(-1.5) \approx 0.1824$$

$$y_2 = \phi(x_1 w_{21} + x_2 \cdot w_{22} + w_{23} \cdot y_1 - 0.5) = \phi(-2 \cdot 0.1824 - 0.5) \approx 0.2963$$

Since the desired output $d_j = 0$, we get an error, $e_2(0) = 0 - 0.2963 = -0.2963$.

Starting the backward step we calculate the learning corrections of each weight. For that we can calculate each local gradient, to simplify further maths:

$$\delta_2(0) = e_2(0) \cdot \phi^{'}(v_j(n)) = -y_2 \cdot y_2 \cdot (1 - y_2) = -0.2963^2 \cdot (1 - 0.2963) = -0.06178$$

Calculating the learning corrections:

$$\Delta w_{21}(0) = 0.1 \cdot \delta_2(0) \cdot x_1 \cdot w_{21} = 0.1 \cdot \delta_2(0) \cdot 0 \cdot 1 = 0$$

$$\Delta w_{22}(0) = 0.1 \cdot \delta_2(0) \cdot x_2 \cdot w_{22} = 0.1 \cdot \delta_2(0) \cdot 0 \cdot 1 = 0$$

$$\Delta w_{23}(0) = 0.1 \cdot \delta_2(0) \cdot y_1 = 0.1 \cdot -0.06178 \cdot 0.1824 = -0.001127$$

$$\Delta w_{11}(0) = 0.1 \cdot \delta_1(0) \cdot x_1 \cdot w_{11} = 0.1 \cdot \delta_1(0) \cdot 0 \cdot 1 = 0$$

$$\Delta w_{12}(0) = 0.1 \cdot \delta_2(0) \cdot x_2 \cdot w_{12} = 0.1 \cdot \delta_1(0) \cdot 0 \cdot 1 = 0$$

Finally, to update the weights:

$$w_{11}(1) = w_{11}(0) + \Delta w_{11}(0) = 1 + 0 = 1$$

$$w_{12}(1) = w_{12}(0) + \Delta w_{12}(0) = 1 + 0 = 1$$

$$w_{21}(1) = w_{21}(0) + \Delta w_{21}(0) = 1 + 0 = 1$$

$$w_{22}(1) = w_{22}(0) + \Delta w_{22}(0) = 1 + 0 = 1$$

$$w_{23}(1) = w_{23}(0) + \Delta w_{23}(0) = -2 - 0.001127 = -2.001127$$