

# MAT 128B: Project I

Joel Aguayo, Joel Barnett, and Doug Kubota

February 20, 2018

## 1 Introduction

This project explores Julia and Mandelbrot sets. Group members Aguayo, Barnett and Kubota jointly worked on each section.

Given the function  $\phi(z) = z^2 + c$ , where  $c \in \mathbb{C}$  is some constant, we consider the problem of finding the fixed points of  $\phi(z)$  using an iterative method. The usual process starts with some  $z_0 \in \mathbb{C}$  and uses the relation  $z_{k+1} = \phi(z_k)$  to generate a sequence  $\{z_n\}$  which may, or may not, converge to a fixed point of  $\phi(z)$ , depending on the choice of  $z_0$ .

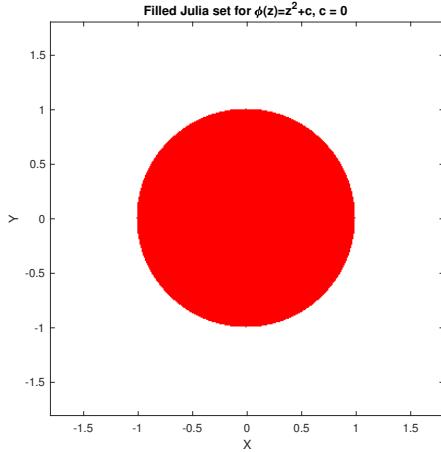
Julia sets study the set of initial points which generate a sequence that remains bounded. That is, we define the filled Julia set defined by  $\{z \in \mathbb{C} : \text{the sequence } z_{k+1} = \phi(z_k), \text{ with initial value } z_0 = z, \text{ remains bounded}\}$ , and define the Julia set to be the boundary of the filled Julia set.

## 2 Julia Sets

### 2.1 Filled Julia Sets on the Unit Disc

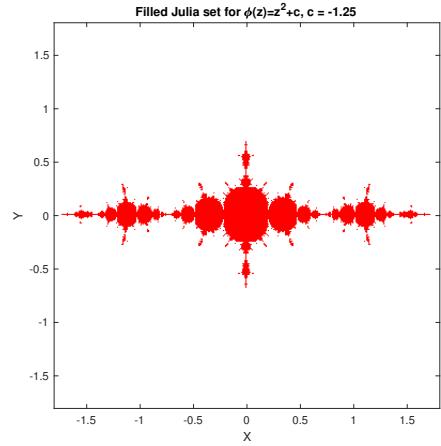
We start with the simplest case:  $\phi(z) = z^2$ , which has two fixed points  $u = 1$  and  $v = 0$ . Clearly, if  $|z_0| \leq 1$ , the sequence will converge because  $|z_1| = |\phi(z_0)| = |(z_0)^2| \leq 1$ , which implies  $|z_2| \leq 1$  and so forth. Also, it is clear that if  $|z_0| > 1$ , the sequence will diverge, because the modulus of the following terms will continue to grow. Thus, the filled Julia set for  $\phi(z)$  is the unit disc  $D^2 = \{z : |z| \leq 1\}$ , and the Julia set is the boundary of this disc.

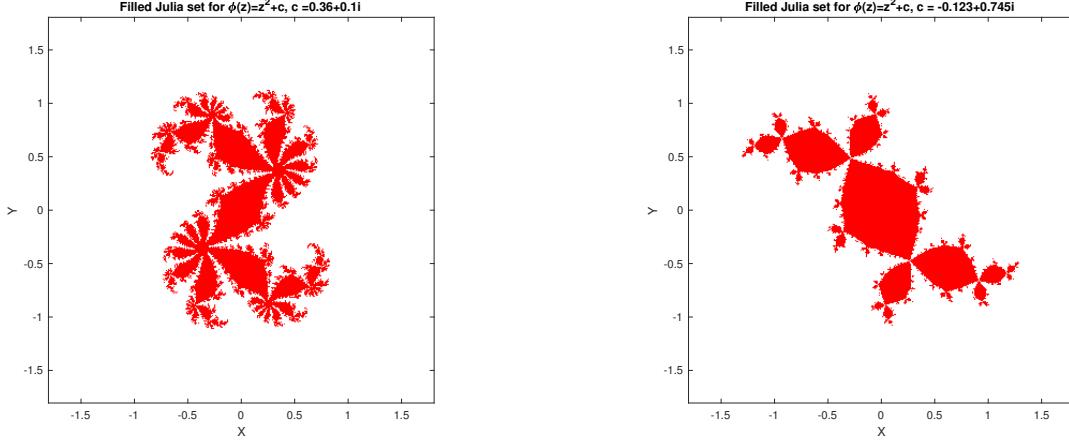
Below is a plot of the set, which was generated with the MATLAB code provided in section 7.



## 2.2 Various Julia Sets

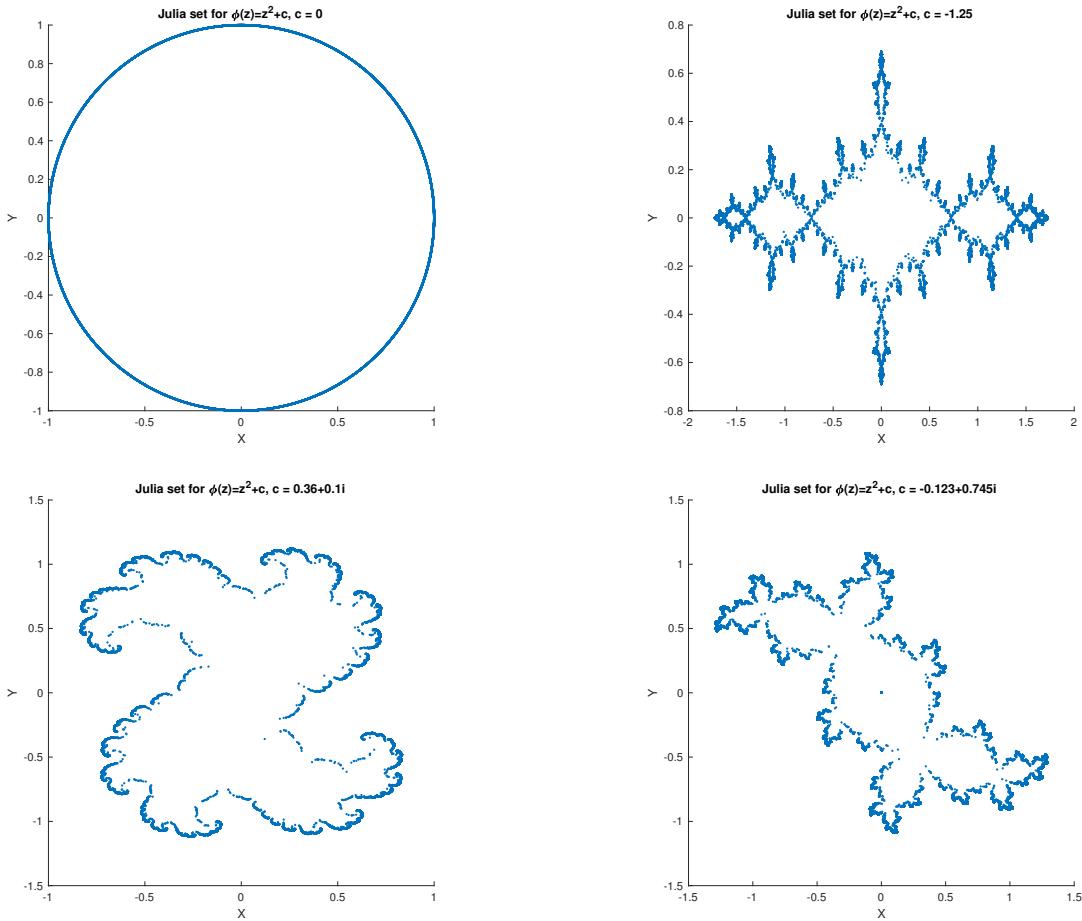
We also graph the Julia set for the function  $\phi(z) = z^2 + c$ , for  $c = -1.25, 0.36 + 0.1i, -0.123 + 0.745i$  and display their graph's below. Note that the code we use to generate these filled Julia sets checks the magnitude of the sequence elements determined by  $z_n = \phi(z_{n-1})$ . If  $|z_n| > 2$  for some  $n \in \mathbb{N}$ , we assume that the sequence is diverging. The algorithm checks all  $z_0 \in [-2, 2] \times [-2, 2]$  region in the complex plane (spaced 0.01 units apart) and plots those which lead to bounded sequences for  $z_n$  as red pixels, and those that diverge as white pixels.





## 2.3 Boundaries of Filled Julia Sets

We also can form the Julia set by plotting the boundary of the filled Julia sets from section 2.2. Boundary points are found by noticing that if  $\phi(z) = z^2 + c$ , then  $z = \pm\sqrt{\phi(z) - c}$ . So, if  $z_k = \phi(z_{k-1})$  we get that  $z_k = \pm\sqrt{\phi(z_k) - c} = \pm\sqrt{z_{k+1} - c}$ , which provides an iterative method to move backwards in the sequences used to generate filled Julia sets. See 7.1 for the sample code which generates the boundaries of the filled Julia sets from 2.2, as shown below.



### 3 Computing Fractal Dimensions

The fractal dimension is used to describe fractals. Fractals generally have a non-integer dimension, so the fractal dimension value is used as an index to describe how the fractals pattern changes while it is scaled. It can be used to measure the degree of roughness of a shape. If we place our shape on a grid, its length can be viewed as the total amount of boxes that overlap the shape. As we increase or decrease the size of our shape (keeping the grid the same), the number of boxes touching the shape will increase or decrease respectively. The length which has now increased or decreased, has a scaling factor. This scaling factor raised to the value of the fractal dimension is used to compute its new size. Simple math can be used to compute the fractal dimension as well.

Let  $M$  be our scaling factor and  $N$  be our "length" scaling factor. Our fractal dimension is  $D$  such that:

$$M^D = N \Rightarrow D = \log_M(N)$$

### 4 Orbits

While the filled Julia set studies whether or not the initial starting point  $z_0$  will produce a bounded or converging sequence  $\{z_n\}$  under  $\phi$ , it can be worthwhile to investigate the sequence  $\{z_n\}$  itself. We do this by looking at the orbit of a particular  $z_0$ , where the orbit (denoted by  $orb(z_0)$ ) is given by the set of all elements in the sequence  $\{z_n\}$  generated by  $\phi$ .

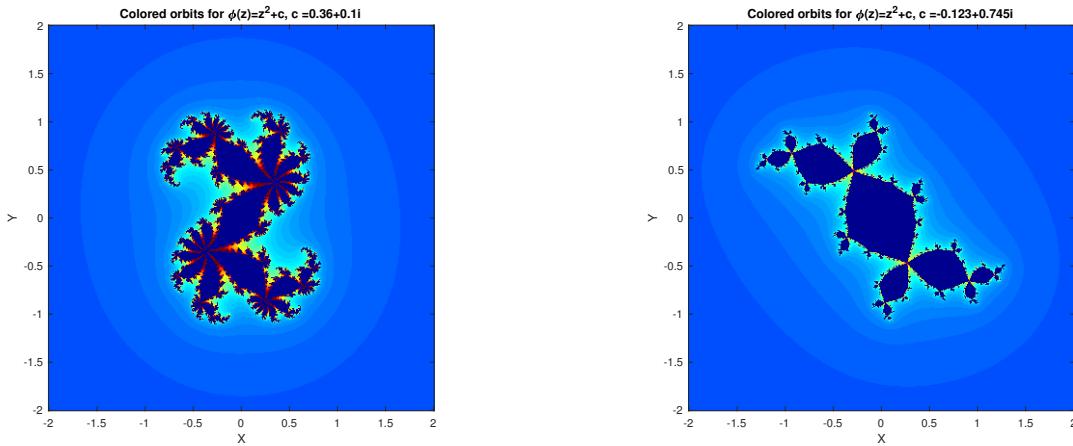
#### 4.1 Connectedness of the Filled Julia Set

The complex nature of Julia Sets brings up the question of whether or not the filled Julia set is connected (in the topological sense). Surprisingly, there is a simple necessary and sufficient condition for connectedness. Julia and Fatou showed that the filled Julia set for  $\phi(z) = z^s + c$  is connected if, and only if, 0 is in the filled Julia set. Using this condition, we develop an algorithm that checks whether or not the  $orb(0)$  for a particular  $\phi$  is bounded. Boundedness implies that 0 is in the filled Julia set for  $\phi$ , so the set is connected. From an algorithmic standpoint, it is difficult to determine with complete certainty that an orbit is bounded since the orbit may remain bounded for many terms and then diverge, or it might grow quickly and then cycle and remain bounded. So, our algorithm checks the first 1000 terms of  $orb(0)$  and if the terms remain bounded by 100, we conclude that  $orb(0)$  is bounded. Below is the MATLAB output from our algorithm (see section 7.2) which checks the boundedness of  $orb(0)$

```
>> Orb_0(0)
The sequence orb(0) converges to a root; thus, the filled Julia set is connected
Orb(0) remained bounded for 1000 iterations. We assume the filled set is connected
>> Orb_0(-1.25)
Orb(0) remained bounded for 1000 iterations. We assume the filled Julia set is connected
>> Orb_0(0.36+0.1*i)
Orb(0) remained bounded for 1000 iterations. We assume the filled Julia set is connected
>> Orb_0(-0.123+745*i)
The orb(0) is unbounded, so the filled Julia set is not connected
```

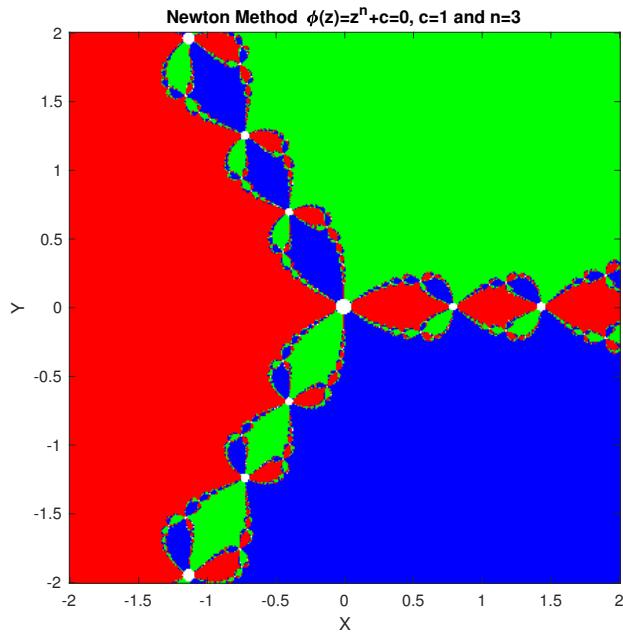
## 4.2 Coloring Divergent Orbits

We can also study how fast an orbit diverges. We do this by computing the sequence  $z_n$  generated by *phi*. When  $|z_k| > 1000$  for some  $k$ , we assign a color to the matrix element corresponding to the location of  $z_k$  in the complex plane. For colors that diverge at similar points in the sequence (I.e for similar  $k$  values), the color is similar. Printing this gives a good idea at how fast points near the boundary of the filled Julia set are diverging. Below is the algorithm, and a sample output for  $\phi(z) = z^2 + 0.36 + 0.1 \cdot i$ .



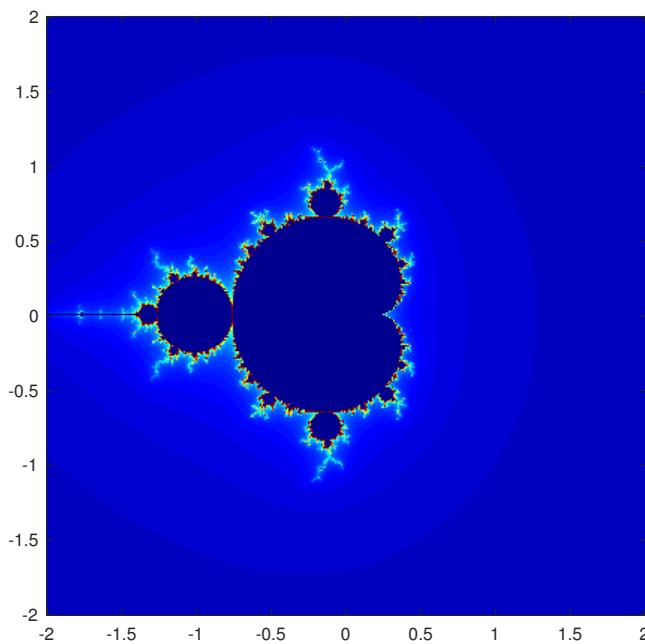
## 5 A Newton's Method Approach

Rather than only look at  $(z) = z^2 + c$  as is done in Julia sets, we can also use Newton's method to find the roots of a function of the form  $f(z) = z^n + c$ . We can then observe which  $z_0$  will produce a sequence induced by Newton's method which converges to one of the  $n$ -roots of  $f$ . Further, by coloring each of these  $z_0$  a different color, we get fractal akin to the Julia set. Below is a sample for the function  $f(z) = z^3 + 1$ , where the red, green, and blue sections represent starting points that converge to  $1$ ,  $e^{\pi/3i}$ , and  $e^{-\pi/3i}$ , respectively.



## 6 Mandelbrot Sets

As opposed to Julia sets which study the boundedness of the sequence  $\{z_n\}$  for fixed  $\phi$ , we can choose to check whether the filled Julia set is connected for varying  $\phi$  by changing the constant  $c$  values. That is, given  $c \in \mathbb{C}$ , define  $\phi(z) = z^2 + c$ , and check  $orb(0)$  for  $\phi$ . Studying which  $c$  produces bounded zero-orbits gives the Mandelbrot set. Below is an image of the Mandelbrot set and an algorithm which computes the set is provided in 7.3.



## 7 MATLAB Code

### 7.1 Julia Sets

Below is the code which prints the filled Julia set:

```
% plots Julia set for z^2 + c
c= 0.36 +0.1*i;    % Define the constant c
phi = @(z) z^2 + c; % Define the function whose fixed points we seek.
fixpt1 = (1 + sqrt(1-4*c))/2;
fixpt2 = (1 - sqrt(1-4*c))/2;

colormap([1 0 0; 1 1 1]);      % Points numbered 1 (inside) will be colored red;
% those numbered 2 (outside) will be colored white.
M = 2*ones(361,361);         % Initialize array of point colors to 2 (white).

for j=1:361                  % Try initial values with imaginary parts between
y = -1.8+ (j-1)*.01;          % -1.8 and 1.8
for i=1:361                  % and with real parts between
x = -1.8 + (i-1)*.01;        % -1.8 and 1.8.
z = x + 1i*y;
zk = z;
iflag1 = 0;                   % iflag1 and iflag2 count the number of iterations
iflag2 = 0;                   % when a root is within 1.e-6 of a fixed point;
kount = 0;                    % kount is the total number of iterations.

while kount < 100 & abs(zk) < 2 & iflag1 < 5 & iflag2 < 5
kount = kount+1;
zk = phi(zk);                % This is the fixed point iteration.
err1 = abs(zk-fixpt1);       % Test for convergence to fixpt1.
if err1 < 1.e-6
iflag1 = iflag1 + 1;
else
iflag1 = 0;
end
err2 = abs(zk-fixpt2);       % Test for convergence to fixpt2.
if err2 < 1.e-6,
iflag2 = iflag2 + 1;
else
iflag2 = 0;
end;
end;
if iflag1 >= 5 | iflag2 >= 5 | kount >= 100    % If orbit is bounded, set this
M(j,i) = 1;                           % point color to 1 (red).
end
end
```

```

end

image([-1.8 1.8],[-1.8 1.8],M), % This plots the results.
title('Filled Julia set for \phi(z)=z^2+c, c =0.36+0.1i')
xlabel('X')
ylabel('Y')
pbaspect([1 1 1]); %keeps the x/y ratio even
axis xy           % If you don't do this, vertical axis is inverted.

```

Below is the code which prints the boundary of a filled Julia set:

```

% plots Julia set for z^2 +c
c= 0.36+.1*i;    % Define the function whose fixed points we seek.

phiInv= @(z) sqrt(z-c); %this is the inverse iteration of a point

%these holds the real values of the points on the boundary of the filled Julia sets
Rphi=[200^2];
Iphi=[200^2];
arraycount=0;

for j=1:201          % Try initial values with imaginary parts between
y = -1+ (j-1)*.01;   % -1 and 1
for i=1:201          % and with real parts between
x = -1 + (i-1)*.01; % -1 and 1.
arraycount=arraycount+1; %increment the Iphi/Rphi arraycount
zk=x+y*i;           %set zk to the point in the [-1,1]x[-1,1] region
kcount=0;             %counter for reverse iteration

%run reverse iteration untill it reaches boundary, kick out if it
%diverges
while kcount < 100 && abs(zk) < 2
n=randi(2,1);
y=imag(zk-c);
kcount = kcount+1;
zk=(-1)^n*phiInv(zk);

end
%zk is now on the boundary, so set the real and imaginary arrays to its
%value there
if abs(zk)<2
Rphi(arraycount)=real(zk);
Iphi(arraycount)=imag(zk);
end
end
end

```

```
%plot the Rphi vs Iphi
scatter(Rphi,Iphi,'.')
title('Julia set for \phi(z)=z^2+c, c =0.36+0.1i')
xlabel('X')
ylabel('Y')
pbaspect([1 1 1]); %keep aspect ration square
```

## 7.2 Orbits

This is the function for computing the  $\text{orb}(0)$  implemented in MATLAB:

```
function Orb_0(const)
% Checks if 0 is in the filled Julia Set for a given \phi(z)=z^2+c
% by computing the orbit of 0.

%Define the c and initial z_0 (for orb(0), use z0=0;
c=const;
z0=0;

%Define the functions and fixed points.
phi = @(z) z^2 + c;
fixpt1 = (1 + sqrt(1-4*c))/2;
fixpt2 = (1 - sqrt(1-4*c))/2;

iflag1 = 0;      % iflag1 and iflag2 count the number of iterations
iflag2 = 0;      % when a root is within 1.e-6 of a fixed point;
kount = 0;       % kount is the total number of iterations.
zk=z0;          %calculates the orb(z_0).

while kount < 1000 && abs(zk) < 100 && iflag1 < 5 && iflag2 < 5
kount = kount+1;
zk = phi(zk);           % This is the fixed point iteration.
err1 = abs(zk-fixpt1); % Test for convergence to fixpt1.
if err1 < 1.e-6
iflag1 = iflag1 + 1;
else
iflag1 = 0;
end
err2 = abs(zk-fixpt2); % Test for convergence to fixpt2.
if err2 < 1.e-6
iflag2 = iflag2 + 1;
else
iflag2 = 0;
end
end
```

```

if iflag1 >= 5 || iflag2 >= 5    % If orb(0) converges to a root,
fprintf('The sequence orb(0) converges to a root; thus, the filled Julia set is connected\n'
elseif kount >= 1000 %otherwise, if the orbit remained bounded for large iterations
fprintf('Orb(0) remained bounded for 1000 iterations. We assume the filled Julia set is connected.\n'
else
fprintf('The orb(0) is unbounded, so the filled Julia set is not connected \n');
end
end

```

Below is the colored orbit algorithm:

```

% Checks the orbits of z_0 from the region [-2,2]x[-2,2]
% under phi(z)=z^2+c

%Define the c
c=0.36+0.1*i;

%Define the functions and fixed points.
phi = @(z) z^2 + c;
fxpt1 = (1 + sqrt(1-4*c))/2;
fxpt2 = (1 - sqrt(1-4*c))/2;

colormap jet;
M= zeros(401,401);
for j=1:401           % Try initial values with imaginary parts between
y = -2+ (j-1)*.01;    % -2 and 2
for i=1:401           % and with real parts between
x = -2 + (i-1)*.01;   % -2 and 2.
z = x + 1i*y;
zk = z;

iflag1 = 0;      % iflag1 and iflag2 count the number of iterations
iflag2 = 0;      % when a root is within 1.e-6 of a fixed point;
kcount = 0;      % kount is the total number of iterations.

while kcount < 1000 && abs(zk) < 100 && iflag1 < 5 && iflag2 < 5
kcount = kcount+1;
zk = phi(zk);      % This is the fixed point iteration.

err1 = abs(zk-fixpt1); % Test for convergence to fixpt1.
if err1 < 1.e-6
iflag1 = iflag1 + 1;
else
iflag1 = 0;

```

```

end

err2 = abs(zk-fixpt2); % Test for convergence to fixpt2.
if err2 < 1.e-6
iflag2 = iflag2 + 1;
else
iflag2 = 0;
end

end

if abs(zk)>100 %if the orbit diverged, fill in the color map
M(j,i)=kcount; %the color for terms that diverged with similar
%kcounts will have similar color
end
end
end

image([-2 2],[-2 2],M), % This plots the results.
pbaspect([1 1 1]); %keeps the x/y ratio even
axis xy % prevents inverted xy axis

```

### 7.3 Mandelbrot Code

```

% This program generates the mandelbrot set

%Define the functions and fixed points.
phi = @(z,c) z^2 + c;
fixpt1 = @(c) (1 + sqrt(1-4*c))/2;
fixpt2 = @(c) (1 - sqrt(1-4*c))/2;
z0=0;           %madelbrot uses z0=0;
colormap jet;

M= ones(401,401);
for j=1:401          % Try initial values with imaginary parts between
y = -2+ (j-1)*.01;    % -2 and 2
for i=1:401          % and with real parts between
x = -2 + (i-1)*.01;    % -2 and 2.
c = x + 1i*y;

iflag1 = 0;      % iflag1 and iflag2 count the number of iterations
iflag2 = 0;      % when a root is within 1.e-6 of a fixed point;
kcount = 0;      % kount is the total number of iterations.
zk=z0;

```

```

%compute the orb(0) for this particular c value
while kcount < 1000 && abs(zk) < 100 && iflag1 < 10 && iflag2 < 10
kcount = kcount+1;
zk = phi(zk,c); % This is the fixed point iteration.

err1 = abs(zk-fixpt1(c)); % Test for convergence to fixpt1.
if err1 < 1.e-6
iflag1 = iflag1 + 1;
else
iflag1 = 0;
end

err2 = abs(zk-fixpt2(c)); % Test for convergence to fixpt2.
if err2 < 1.e-6
iflag2 = iflag2 + 1;
else
iflag2 = 0;
end

end

if abs(zk)>100 %if the orbit diverged, fill in the color map
M(j,i)=kcount; %the color for terms that diverged with similar
%kcounts will have similar color
end
end
end

image([-2 2],[-2 2],M), % This plots the results.
pbaspect([1 1 1]); %keeps the x/y ratio even
axis xy % prevents inverted xy axis

```