

# Master's Project

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# Preface

I plan to write and detail everything(almost) I study and learn for my Master's project into these latex files. I assume it will be much easier to track whatever I have learned and to have a good overview of the topic with this note taking. Also since I am doing it on L<sup>A</sup>T<sub>E</sub>X I am sure it will save me from the last minute rush to type everything out and make the report of the project. I will start with Fourier series and will introduce new concepts when they are required as we go along. So, let us start

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# Contents



# Chapter 1

## Fourier Series

Throughout this chapter we will be concerned with periodic functions. We will discuss the Fourier series of functions and their convergence. Before we begin we will formally define periodic functions in  $\mathbb{R}$ .

**Definition 1.0.1** (Periodic function). A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is called periodic with period  $p$  (or  $p$ -periodic) if  $f(x + p) = f(x)$  for all  $x \in \mathbb{R}$ .

From the above definition it is evident that a periodic function in  $\mathbb{R}$  is uniquely determined by its images in a closed open interval of length  $p$ , where  $p$  is the period of the function. Moreover if  $g : \mathbb{R} \rightarrow \mathbb{R}$  is a  $p$ -periodic function,  $f(x) := g(\frac{x}{p})$  is a 1-periodic function. Therefore we'll focus our attention to 1-periodic functions. Now we'll identify 1-periodic functions in  $\mathbb{R}$  with functions in the interval  $[0, 1)$ . By this identification we will be able to use many results from Lebesgue measure theory which are restricted to finite measure spaces.

### 1.1 Structure and Topology of $\mathbb{T}$

Let us define an relation  $\sim$  on  $\mathbb{R}$  where

$$x \sim y \iff x - y \in \mathbb{Z}$$

It is easy to verify that this is an equivalence relation on  $\mathbb{R}$  and the equivalence classes of this relation is denoted by  $\mathbb{T}$  or  $\frac{\mathbb{R}}{\mathbb{Z}}$ . Note that if  $r \in \mathbb{R}$ , then the equivalence class of  $r$ , denoted by  $[r]$  is

$$[r] = r + \mathbb{Z} = \{r + n \mid n \in \mathbb{Z}\} = \{r\} + \mathbb{Z}$$

where  $\{r\}$  is the fractional part of  $r$ .

Moreover  $f : \mathbb{R} \rightarrow \mathbb{T}$  defined as  $f(r) = [r]$  is a surjective map onto  $\mathbb{T}$  and therefore defines the quotient topology on  $\mathbb{T}$ .

**Proposition 1.1.1** (Periodisation of functions in  $\mathbb{T}$ ). *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined as  $f(x) = [x]$  and  $h : \mathbb{T} \rightarrow \mathbb{R}$ . Then  $h \circ f : \mathbb{R} \rightarrow \mathbb{R}$  is a periodic function of period 1.*

By the above theorem we consider functions in  $\mathbb{T}$  as 1-periodic functions in  $\mathbb{R}$ . Also, having endowed  $\mathbb{T}$  with a topology and measure, we are now at a position to discuss continuity of functions in  $\mathbb{T}$ .

**Proposition 1.1.2** (Continuous functions in  $\mathbb{T}$ ). *Let  $g : \mathbb{T} \rightarrow \mathbb{R}$  be a function. Then  $g$  is continuous on  $\mathbb{T}$  if and only if the function  $h : \mathbb{R} \rightarrow \mathbb{R}$  defined as  $h(x) = g([x])$  is a continuous function of period 1.*

*Proof.* ( $\implies$ ) Assume  $g$  is a continuous function in  $\mathbb{T}$ . Then by the definition, if  $U \subset \mathbb{R}$  is an open set then  $g^{-1}(A)$  is an open set in  $\mathbb{T}$ . Since the topology on  $\mathbb{T}$  is defined by the quotient map  $f : \mathbb{R} \rightarrow \mathbb{T}$ , where  $f(x) = [x]$ , we get that  $f^{-1}g^{-1}(A)$  is open in  $\mathbb{R}$ . Since  $h = g \circ f$  by definition, it follows that  $h$  is continuous. Also since  $f(x+1) = f(x)$  for all  $x \in \mathbb{R}$ , it follows that  $h$  is of period 1.

( $\impliedby$ ) Assume that  $h$  is a continuous function on  $\mathbb{R}$ . Then  $h^{-1}(A)$  is an open set in  $\mathbb{R}$  for any open set  $A \in \mathbb{R}$ . Since  $h = g \circ f$  we get  $f^{-1}g^{-1}(A)$  is open in  $\mathbb{R}$ . Again since  $f$  is a quotient map to  $\mathbb{T}$ , we get that  $g^{-1}(A)$  is open in  $\mathbb{T}$ . Therefore  $g$  is a continuous function on  $\mathbb{T}$ .  $\square$

Now by abuse of notation we'll identify  $\mathbb{T}$  with  $[0, 1)$  by the function  $f : [x] \rightarrow \{x\}$ . By this identification we can define Lebesgue measure on  $\mathbb{T}$ . If  $A$  is an open subset of  $\mathbb{T}$  then Lebesgue measure of  $A$  is defined as,  $\mu(A) = \lambda(f(A))$  where the  $\lambda$  is the Lebesgue measure on  $\mathbb{R}$ . Then we see that  $\mathbb{T}$  is a finite Lebesgue measure space.

Recall the definition of  $L^p$  functions for a Lebesgue measure space.

**Definition 1.1.1** ( $L^p$  function). A real valued function  $f$  defined on a Lebesgue measure space  $S$  is called an  $L^p$  function on  $S$  or a  $p$ -integrable function on  $S$  if

$$\left( \int_S |f(x)|^p d\mu \right)^{1/p} < \infty$$

For such a function the integral above is called the  $p$ -norm of the function  $f$  and often denoted by  $\|f\|_{L^p(S)}$  or  $\|f\|_p$  if the space is known.

Let us now prove an important result about periodic functions.

**Lemma 1.1.1.** *If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is of period 1 and  $\int_0^1 f(x)dx$  exists, then for any real number  $a$ ,*

$$\int_a^{a+1} f(x)dx = \int_0^1 f(x)dx$$



*Proof.* Let  $a = n + b$ , where  $0 \leq b < 1$  and  $n$  is an integer. Then since  $f$  has period 1,  $\int_a^{a+1} f(x)dx = \int_{n+b}^{n+b+1} f(x)dx = \int_b^{b+1} f(x+n)dx = \int_b^{b+1} f(x)dx$  and,

$$\begin{aligned} \int_b^{b+1} f(x)dx &= \int_b^1 f(x)dx + \int_1^{b+1} f(x)dx \\ &= \int_b^1 f(x)dx + \int_0^b f(x+1)dx \\ &= \int_b^1 f(x)dx + \int_0^b f(x)dx \\ &= \int_0^1 f(x)dx \end{aligned}$$

Hence the result. □

## 1.2 Fourier Coefficients

Now we'll define fourier coefficients of a periodic function  $f \in L^1(\mathbb{T})$

**Definition 1.2.1** (Fourier coefficient). Let  $f \in L^1(\mathbb{T})$ , i.e  $\int_{\mathbb{T}} f < \infty$ . Then for each integer  $n$  we define the  $n^{th}$  fourier coefficient,  $\widehat{f}(n)$  as

$$\widehat{f}(n) = \int_0^1 f(x)e^{-2\pi inx}dx$$

$\widehat{f}(n)$  is finite and well defined for each  $n$  since  $f \in L^1(\mathbb{T})$ , since

$$\begin{aligned} |\widehat{f}(n)| &\leq \int_0^1 |f(x)e^{-2\pi inx}|dx \\ &\leq \int_0^1 |f(x)||e^{-2\pi inx}|dx \\ &= \int_0^1 |f(x)|dx < \infty \end{aligned}$$

Once we have the fourier coefficients of a function at hand we can combine them together to make a series called the fourier series. We'll be investigating the conditions at which this series converges to our initial function  $f$ .

Also note that the map which takes  $f$  to  $\widehat{f}$  is linear since if  $f, g \in L^1(\mathbb{T})$ , then

$$\begin{aligned}\widehat{f+g}(n) &= \int_0^1 (f+g)(x) e^{-2\pi i n x} dx \\ &= \int_0^1 f(x) e^{-2\pi i n x} dx + \int_0^1 g(x) e^{-2\pi i n x} dx \\ &= \widehat{f}(n) + \widehat{g}(n)\end{aligned}$$

and for  $\lambda \in \mathbb{R}$

$$\begin{aligned}\widehat{\lambda f}(n) &= \int_0^1 \lambda f(x) e^{-2\pi i n x} dx \\ &= \lambda \int_0^1 f(x) e^{-2\pi i n x} dx \\ &= \lambda \widehat{f}(n)\end{aligned}$$

**Definition 1.2.2** (Fourier series). Given a function  $f \in L^1(\mathbb{T})$ , the fourier series of the function  $f$  is defined as

$$\sum_{n \in \mathbb{Z}} \widehat{f}(n) e^{2\pi i n x}$$

where  $\widehat{f}(n)$  is the  $n^{th}$  fourier coefficient as ??

**Proposition 1.2.1** (Properties of Fourier coefficients). Suppose that  $f \in L^1(\mathbb{T})$ .

(a) If  $a$  is a real number and  $g(x) = f(x+a)$  for all  $x$ , then  $\widehat{g}(n) = \widehat{f}(n) e^{2\pi i n a}$  for all  $n \in \mathbb{Z}$ .

(b) If  $b$  is an integer and  $h(x) = f(x) e^{2\pi i b x}$  for all  $x$ , then  $\widehat{h}(n) = \widehat{f}(n-b)$  for all  $n \in \mathbb{Z}$ .

(c) If  $j(x) = f(-x)$  for all  $x$ , then  $\widehat{j}(n) = \widehat{f}(-n)$

*Proof.* Given  $f(x) \in L^1(\mathbb{T})$  and  $n^{th}$  Fourier coefficient of  $f$ ,

$$\widehat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x} dx.$$

(a) Then, the  $n^{th}$  Fourier coefficient of  $g(x) = f(x + a)$  is

$$\begin{aligned}
 \widehat{g}(n) &= \int_0^1 g(x) e^{-2\pi i n x} dx \\
 &= \int_0^1 f(x + a) e^{-2\pi i n x} dx \\
 &= \int_0^1 f(x) e^{-2\pi i n (x-a)} dx \\
 &= e^{2\pi i n a} \int_0^1 f(x) e^{-2\pi i n x} dx \\
 &= e^{2\pi i n a} \widehat{f}(n)
 \end{aligned}$$

(b) If  $h(x) = f(x)e^{2\pi i b x}$ , then

$$\widehat{h}(n) = \int_0^1 f(x) e^{-2\pi i (n-b)x} dx = \widehat{f}(n - b)$$

(c)  $j(x) = f(-x)$ , then

$$\begin{aligned}
 \widehat{j}(n) &= \int_0^1 f(-x) e^{-2\pi i n x} dx \\
 &= - \int_0^{-1} f(y) e^{2\pi i n y} dy && \text{by } y = -x \\
 &= \int_{-1}^0 f(y) e^{2\pi i n y} dy \\
 &= \int_0^1 f(y) e^{2\pi i n y} dy && \text{by ??} \\
 &= \widehat{f}(-n)
 \end{aligned}$$

□

## 1.3 Convolution

Now we'll define another important operation with function called the convolution of two functions.

**Definition 1.3.1.** Let  $f, g \in L^1(\mathbb{T})$ , then the convolution of  $f$  and  $g$  is defined as

$$f * g(x) = \int_0^1 f(x - y) g(y) dy$$

**Proposition 1.3.1** (Properties of convolution). *Let  $f, g \in L^1(\mathbb{T})$ , then*

$$(a) \quad f * g = g * f$$

$$(b) \quad f * (g + h) = f * g + f * h$$

$$(c) \quad (cf) * g = c(f * g)$$

$$(d) \quad f * (g * h) = (f * g) * h$$

*Proof.* We'll just prove the commutativity since the rest are easily verifiable by the properties of integration. Put  $v = x - y$ , we get  $dv = -dy$  and

$$\begin{aligned} f * g(x) &= \int_0^1 f(x - y)g(y) \, dy \\ &= - \int_x^{x-1} f(v)g(x - v) \, dv \\ &= \int_{x-1}^x g(x - v)f(v) \, dv \\ &= \int_0^1 g(x - v)f(v) \, dv && \text{by ??} \\ &= g * f(x) \end{aligned}$$

□

We'll prove another important result that the convolution of two  $L^1(\mathbb{T})$  functions is again in  $L^1(\mathbb{T})$ .

**Proposition 1.3.2.** *Let  $f, g \in L^1(\mathbb{T})$ . Then  $h = f * g \in L^1(\mathbb{T})$ , and  $\widehat{h}(n) = \widehat{f}(n)\widehat{g}(n)$ .*

*Proof.*

$$\begin{aligned} \int_0^1 |h(x)| \, dx &= \int_0^1 \left| \int_0^1 f(y)g(x - y) \, dy \right| \, dx \\ &\leq \int_0^1 \int_0^1 |f(y)g(x - y)| \, dy \, dx \\ &= \int_0^1 \int_0^1 |f(y)g(x - y)| \, dx \, dy && \text{by Tonelli's theorem} \\ &= \int_0^1 \left( \int_0^1 |g(x - y)| \, dx \right) |f(y)| \, dy \\ &= \|f\|_1 \|g\|_1 \end{aligned}$$

Note that we're using Tonelli's theorem here to interchange the limits of integration since the space is a finite measure space. This proves that  $h = f * g \in L^1(\mathbb{T})$ .

To prove the next part,

$$\begin{aligned}
 \widehat{h}(n) &= \int_0^1 \left( \int_0^1 f(y)g(x-y)dy \right) e^{-2\pi inx} dx \\
 &= \int_0^1 f(y) \left( \int_0^1 g(x-y)e^{-2\pi inx} dx \right) dy && \text{by Tonelli's theorem} \\
 &= \int_0^1 f(y)\widehat{g}(n)e^{-2\pi iny} dy && \text{by ??(a)} \\
 &= \widehat{f}(n)\widehat{g}(n)
 \end{aligned}$$

□

## 1.4 Partial sums of Fourier series

Given a function  $f$  in the  $\mathbb{T}$ , we are interested in the convergence of fourier series of  $f$ . We'll discuss about the convergence of the symmetric partial sum of the Fourier series.

**Definition 1.4.1** (Symmetric partial sum of a Fourier series). Given a function  $f \in L^1(\mathbb{T})$  with its fourier series,  $\sum_{-\infty}^{\infty} \widehat{f}(n)e^{2\pi inx}$ , we define the  $n^{th}$  symmetric parial sum of the fourier series as

$$S_N(x) = \sum_{n=-N}^N \widehat{f}(n)e^{2\pi inx}$$

But it may happen that the summetric partial sum of the Fourier seies may not converge. To deal with this we'll define another partial sum called the Cesàro partial sum.

**Definition 1.4.2** (Cesàro partial sum of Fourier series). Given a function  $f \in L^1(\mathbb{T})$  with its Fourier series,  $\sum_{-\infty}^{\infty} \widehat{f}(n)e^{2\pi inx}$ , we define the  $n^{th}$  Cesàro parial sum of its Fourier series as

$$\sigma_N(x) = \frac{1}{N} \sum_{n=0}^{N-1} S_n(x)$$

where  $S_n(x)$  is the symmetric partial sum of the Fourier series in ??

For an example,  $\{-1^n\}$  is a sequence whose symmetric partial sums do not converge but the Cesàro partial sums converge to  $\frac{1}{2}$ . Also if the symmetric partial sums of a series converge, then the Cesàro partial sums will also converge to the same limit. [Apo74, Theorem 8.48 p. 206]

Now we'll show that the Cesàro partial sum can be rewritten to another form which will help our proofs down the road.

**Lemma 1.4.1.** *If  $\sigma_N(x)$  is the  $N^{\text{th}}$  Cesàro partial sum of the Fourier series of a function  $f \in L^1(\mathbb{T})$ , then*

$$\sigma_N(x) = \sum_{n=-N}^N \left(1 - \frac{|n|}{N}\right) \widehat{f}(n) e^{2\pi i n x}$$

*Proof.* We'll prove the result for a general series so that it'll help us also in Fourier series.

Let  $S_N = \sum_{n=-N}^N a_n$  be the  $N^{\text{th}}$  partial sum of the series  $\sum_{-\infty}^{\infty} a_n$ . Then by the definition of Cesàro partial sum,

$$\begin{aligned} \sigma_N &= \frac{1}{N} \sum_{n=0}^{N-1} S_n \\ &= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=-n}^n a_k \\ &= \frac{1}{N} \sum_{k=-N+1}^{N-1} a_k \sum_{n=|k|}^{N-1} 1 \\ &= \frac{1}{N} \sum_{k=-N+1}^{N-1} (N - |k|) a_k \\ &= \sum_{k=-N+1}^{N-1} \left(1 - \frac{|k|}{N}\right) a_k \\ &= \sum_{k=-N}^N \left(1 - \frac{|k|}{N}\right) a_k \end{aligned}$$

Now specifically if we take  $a_k = \widehat{f}(k) e^{2\pi i k x}$ , we get the required result.  $\square$

## 1.5 Summability Kernels

Now we'll define a family of functions called the summability kernels, which we will use heavily in our proofs and simplify it.

**Definition 1.5.1** (Summability kernel). A collection of functions  $K_N \in L^1(\mathbb{T})$  is called a summability kernel or an approximation identity if

- (a)  $\int_0^1 K_N(x) dx = 1$
- (b)  $\int_0^1 |K_N(x)| dx \leq C$  for some constant  $C > 0$
- (c)  $\lim_{N \rightarrow \infty} \int_\delta^{1-\delta} |K_N(x)| dx = 0$

Now we'll define one of the important summability kernels.

**Definition 1.5.2** (Fejér kernel). Fejér kernel is defined as a collection of functions  $\Delta_N$  where for each  $N \in \mathbb{N}$ ,  $\Delta_N : \mathbb{R} \rightarrow \mathbb{R}$  is defined as

$$\Delta_N(x) = \sum_{n=-N}^N \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x}$$

Notice that each  $\Delta_N$  is a 1-periodic function.

We'll prove that Fejér kernel as in ?? satisfy the properties in ?. But before that we'll explore some properties of Fejér kernel so that it'll aid us in our proof.

**Proposition 1.5.1** (Properties of Fejér kernel). *If  $\Delta_N(x)$  is as in ??, then the following hold true*

(a)

$$\Delta_N(x) = 1 + 2 \sum_{n=1}^N \left(1 - \frac{n}{N}\right) \cos 2\pi n x$$

(b)

$$\int_0^1 \Delta_N(x) dx = 1$$

(c)

$$\Delta_N(x) = \Delta_N(1 - x)$$

(d) For  $0 \leq \delta \leq \frac{1}{2}$ ,

$$\int_\delta^{\frac{1}{2}} \Delta_N(x) dx = \int_{\frac{1}{2}}^{1-\delta} \Delta_N(x) dx$$

and therefore,

$$\int_0^{\frac{1}{2}} \Delta_N(x) dx = \int_{\frac{1}{2}}^1 \Delta_N(x) dx = \frac{1}{2}$$

(e)

$$\Delta_N(x) = \begin{cases} \frac{1}{N} \left( \frac{\sin(\pi Nx)}{\sin(\pi x)} \right)^2, & \text{if } x \notin \mathbb{Z} \\ N, & \text{if } x \in \mathbb{Z} \end{cases}$$

(f) If  $0 \leq x \leq \frac{1}{2}$ , then

$$\Delta_N(x) \leq \min \left( N, \frac{1}{4Nx^2} \right)$$

(g) If  $0 < \delta \leq \frac{1}{2}$ , then

$$\int_{\delta}^{\frac{1}{2}} \Delta_N(x) dx < \frac{1}{4N\delta}$$

(h)

$$(\Delta_N * f)(x) = \sigma_N(x)$$

where  $\sigma_n(x)$  is the  $n^{\text{th}}$  Cesàro partial sum of the Fourier series of  $f$  as in ??.

*Proof.* (a) This follows straight from De Moivre's formula that  $e^{ix} = \cos(x) + i \sin(x)$  and  $e^{ix} + e^{-ix} = 2 \cos(x)$ .

(b) By previous result,

$$\Delta_N(x) = 1 + 2 \sum_{n=1}^N \left( 1 - \frac{|n|}{N} \right) \cos(2\pi nx)$$

Therefore,

$$\begin{aligned} \int_0^1 \Delta_N(x) dx &= \int_0^1 1 dx + 2 \sum_{n=1}^N \left( 1 - \frac{|n|}{N} \right) \int_0^1 \cos(2\pi nx) dx \\ &= 1 + 0 \end{aligned}$$

(c) This follows from the fact that  $\cos(2\pi n(1-x)) = \cos(2\pi n - 2\pi nx) = \cos(2\pi nx)$  in the last result.



- (d) From ??, we know that  $\Delta_N(x) = \Delta_N(1 - x)$ . Therefore by change of variables,

$$\int_{\delta}^{\frac{1}{2}} \Delta_N(x) dx = \int_{\delta}^{\frac{1}{2}} \Delta_N(1 - x) dx = - \int_{1-\delta}^{\frac{1}{2}} \Delta_N(y) dy = \int_{\frac{1}{2}}^{1-\delta} \Delta_N(y) dy$$

Also from previous result, we know

$$\int_0^{\frac{1}{2}} \Delta_N(x) dx + \int_{\frac{1}{2}}^1 \Delta_N(x) dx = \int_0^1 \Delta_N(x) dx = 1$$

Hence

$$\int_0^{\frac{1}{2}} \Delta_N(x) dx = \int_{\frac{1}{2}}^1 \Delta_N(x) dx = \frac{1}{2}$$

- (e) If  $x \in \mathbb{N}$  then  $e^{2\pi i n x} = 1$  for all  $n$  and then,

$$\sum_{n=0}^{N-1} e^{2\pi i n x} = \sum_{n=0}^{N-1} 1 = N$$

Hence the last case is solved. But if  $x \notin \mathbb{N}$  then from the finite sum of geometric series,

$$\begin{aligned} \sum_{n=0}^{N-1} e^{2\pi i n x} &= \frac{e^{2\pi i N x} - 1}{e^{2\pi i x} - 1} \\ &= \frac{e^{\pi i N x}}{e^{\pi i x}} \times \frac{e^{\pi i N x} - e^{-\pi i N x}}{e^{\pi i x} - e^{-\pi i x}} \\ &= e^{\pi i (N-1)x} \frac{\sin(\pi N x)}{\sin(\pi x)} \end{aligned}$$

Since  $|e^{ix}| = 1$  for all  $x \in \mathbb{R}$ , we'll get

$$\left| \sum_{n=0}^{N-1} e^{2\pi i n x} \right|^2 = \frac{\sin^2(\pi N x)}{\sin^2(\pi x)}$$

But we also know that,

$$\begin{aligned}
 \left| \sum_{n=0}^{N-1} e^{2\pi i n x} \right|^2 &= \left( \sum_{n=0}^{N-1} e^{2\pi i n x} \right) \left( \sum_{n=0}^{N-1} \overline{e^{2\pi i n x}} \right) \\
 &= \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} e^{2\pi i (m-n)x} \\
 &= \sum_{k=-(N-1)}^{N-1} e^{2\pi i k x} \sum_{\substack{0 \leq m \leq N-1 \\ 0 \leq n \leq N-1 \\ m-n=k}} 1 \\
 &= \sum_{k=-(N-1)}^{N-1} e^{2\pi i k x} (N - |k|) \\
 &= N \Delta_N(x)
 \end{aligned}$$

Which implies that

$$\Delta_N(x) = \frac{1}{N} \frac{\sin^2(\pi N x)}{\sin^2 \pi x}$$

Hence the proposition.

- (f) We'll first see that  $\sin(\pi x) \geq 2x$  whenever  $0 \leq x \leq \frac{1}{2}$ . But this is because the  $h(x) := \sin(\pi x) - 2x = 0$  only for  $x = 0$  and  $x = \frac{1}{2}$  and the derivative of  $h$ ,  $h'(x) := \pi \cos(\pi x) - 2 = 0$ , only for a unique real number  $r \in [0, \frac{1}{2}]$ . Now since  $f$  is smooth,  $f$  cannot have more roots in  $[0, \frac{1}{2}]$ . Hence  $\sin(\pi x) \geq 2x$  for all  $x \in [0, \frac{1}{2}]$

Now we'll get to the main proof. Assume  $0 < \delta \leq \frac{1}{2}$ . Then by previous result, when  $0 < x \leq \frac{1}{2}$ ,

$$\begin{aligned}
 \Delta_N(x) &= \frac{1}{N} \left( \frac{\sin(\pi N x)}{\sin(\pi x)} \right)^2 \\
 &\leq \frac{1}{N \sin^2(\pi x)} \\
 &\leq \frac{1}{4N x^2} \quad \text{since } \sin(\pi x) \geq 2x
 \end{aligned}$$

We also know that

$$\Delta_N(x) = 1 + 2 \sum_{n=1}^N \left( 1 - \frac{n}{N} \right) \cos(2\pi n x)$$

But this shows that  $\Delta_N(x)$  is maximum when  $\cos(2\pi nx)$  is maximum, i.e. at  $x = 0, 1$ . At both cases  $\cos(2\pi nx) = 1$ . Also

$$1 + 2 \sum_{n=1}^N \left(1 - \frac{n}{N}\right) = 1 + 2N - \frac{2}{N} \sum_{n=1}^N n = 1 + 2N - \frac{2}{N} \frac{N(N+1)}{2} = N$$

Therefore,  $\Delta_N(x) \leq N$  for all  $0 \leq x \leq 1$ . Hence combining this with the result above, we get that while  $0 \leq x \leq \frac{1}{2}$

$$\Delta_N(x) \leq \min \left( N, \frac{1}{4Nx^2} \right)$$

(g) From last result we know that when  $0 < \delta \leq \frac{1}{2}$ ,

$$\Delta_N(x) \leq \frac{1}{4Nx^2}$$

Then,

$$\int_{\delta}^{\frac{1}{2}} \Delta_N(x) dx \leq \int_{\delta}^{\frac{1}{2}} \frac{1}{4Nx^2} dx \leq \int_{\delta}^{\infty} \frac{1}{4Nx^2} dx = \frac{1}{4N\delta}$$

(h)

$$\begin{aligned} (\Delta_N * f)(x) &= \int_0^1 \Delta_N(y) f(x-y) dy \\ &= \int_0^1 \sum_{n=-N}^N \left(1 - \frac{|n|}{N}\right) e^{2\pi i n y} f(x-y) dy \\ &= \sum_{n=-N}^N \left(1 - \frac{|n|}{N}\right) \int_0^1 f(x-y) e^{2\pi i n y} dy \end{aligned}$$

Now since the summation is finite we get,

$$= \sum_{n=-N}^N \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x} \int_0^1 f(x-y) e^{-2\pi i n (x-y)} dy$$

By change of variable as  $v = x - y$ ,

$$= \sum_{n=-N}^N \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x} \int_x^{x-1} -f(v) e^{-2\pi i n v} dv$$

And therefore we get,

$$\begin{aligned}
 (\Delta_N * f)(x) &= \sum_{n=-N}^N \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x} \int_{x-1}^x f(v) e^{-2\pi i n v} dv \\
 &= \sum_{n=-N}^N \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x} \int_0^1 f(v) e^{-2\pi i n v} dv \quad \text{by ??} \\
 &= \sum_{n=-N}^N \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x} \hat{f}(n) \\
 &= \sigma_N(x)
 \end{aligned}$$

□

**Proposition 1.5.2.** *Fejér kernel as in ?? is a summability kernel as in ??*

*Proof.* To prove that Fejér kernel is a summability kernel, we'll verify the three properties given in ??.

1. We proved that  $\int_0^1 \Delta_N(x) dx = 1$  at ??
2. From ??, we know that

$$\Delta_N(x) = \begin{cases} \frac{1}{N} \left( \frac{\sin(\pi N x)}{\sin(\pi x)} \right)^2, & \text{if } x \notin \mathbb{Z} \\ N, & \text{if } x \in \mathbb{Z} \end{cases}$$

Since  $N \in \mathbb{N}$ , this implies  $\Delta_N(x) \geq 0$ . Therefore,

$$\int_0^1 |\Delta_N(x)| dx = \int_0^1 \Delta(x) dx = 1$$

This proves the  $2^{nd}$  condition for the summability kernel.

3. To prove that

$$\lim_{N \rightarrow \infty} \int_{\delta}^{1-\delta} |\Delta_N(x)| dx = 0$$

we note that from ?? if  $0 < \delta \leq \frac{1}{2}$ ,

$$\int_{\delta}^{\frac{1}{2}} \Delta_N(x) dx \leq \frac{1}{4N\delta}$$

and

$$\int_{\frac{1}{2}}^{1-\delta} \Delta_N(x) = \int_{\delta}^{\frac{1}{2}} \Delta_N(x)$$

Hence,

$$\int_{\delta}^{1-\delta} \Delta_N(x) = \int_{\delta}^{\frac{1}{2}} \Delta_N(x) + \int_{\frac{1}{2}}^{1-\delta} \Delta_N(x) = 2 \int_{\delta}^{\frac{1}{2}} \Delta_N(x) \leq \frac{1}{2N\delta}$$

Therefore, for  $0 < \delta \leq \frac{1}{2}$ , we have

$$\lim_{N \rightarrow \infty} \int_{\delta}^{1-\delta} |\Delta_N(x)| = \lim_{N \rightarrow \infty} \int_{\delta}^{1-\delta} \Delta_N(x) = 0$$

If  $\frac{1}{2} < \delta < 1$  then by change of variable  $y = 1 - x$

$$\lim_{N \rightarrow \infty} \int_{\delta}^{1-\delta} \Delta_N(x) = \lim_{N \rightarrow \infty} - \int_{1-\delta}^{\delta} \Delta_N(y) = 0$$

And therefore for all  $0 < \delta < 1$

$$\lim_{N \rightarrow \infty} \int_{\delta}^{1-\delta} |\Delta_N(x)| = 0$$

Which completes the proof that Fejer kernel is a summability kernel.

□

## 1.6 Convergence of Fourier Series

Now with the help of summability kernels and convolution we'll prove an important theorem which will serve as a backbone for the discussion of convergence forward.

**Theorem 1.6.1** (Convergence to convolution of summability kernels). *If  $f \in L^1(\mathbb{T})$  and  $K_N$  is a summability kernel then  $f * K_N$  converges to  $f$  in  $L^1$  norm. That is*

$$\lim_{N \rightarrow \infty} \int_0^1 |f(x) - (f * K_N)(x)| = 0$$

*Proof.*

$$f * K_N(x) = \int_0^1 f(x-y)K_N(y)dy$$

Since  $\int_0^1 K_N(y)dy = 1$ , by the  $2^{nd}$  property of summability kernel,

$$f(x) - f * K_N(x) = \int_0^1 (f(x) - f(x-y)) K_N(y)dx$$

Now then,

$$\begin{aligned} \int_0^1 |f(x) - K_N(x)|dx &= \int_0^1 \left| \int_0^1 (f(x) - f(x-y))K_N(y)dy \right| dx \\ &\leq \int_0^1 \int_0^1 |f(x) - f(x-y)K_N(y)| dy dx \\ &= \int_0^1 |K_N(y)| \int_0^1 |f(x) - f(x-y)|dx dy \\ &= \int_{-\delta}^{1-\delta} \quad \quad \quad \text{by lemma ??} \\ &= \int_{-\delta}^{\delta} + \int_{\delta}^{1-\delta} = I_1 + I_2 \end{aligned}$$

Note that the change of order of integration above is justified by Tonelli's theorem, since the space is of finite measure. We'll show that for a given  $\epsilon$  we can find an  $N$  such that for all  $n > N$ ,  $I_1 + I_2 < \epsilon$

Since  $f \in L^1(\mathbb{T})$  we can find  $\delta > 0$  such that  $\int_0^1 |f(x+\delta) - f(x)|dx = \epsilon/2C$ , where  $C$  is the constant in the second condition of summability kernel. The proof can be found in any measure theory textbook.

Then,

$$|I_1| \leq \frac{\epsilon}{2C} \int_{-\delta}^{\delta} |K_N(y)|dy \leq \frac{\epsilon}{2C} \int_0^1 |K_N(y)|dy \leq \frac{\epsilon}{2}$$

For  $I_2$ , we see that

$$\int_0^1 |f(x) - f(x-y)|dx \leq \int_0^1 |f(x)|dx + \int_0^1 |f(x-y)|dx = 2\|f\|_1$$

Then,

$$|I_2| \leq 2\|f\|_1 \int_{\delta}^{1-\delta} |K_N(y)|dy$$

But by the  $3^{rd}$  property of the summability kernel, we know that the integral in the above converges to zero. Therefore there exists an  $N$  such that for an  $n > N$ ,  $|I_2| < \epsilon/2$ , which completes our proof.  $\square$

This leads to one important theorem,

**Theorem 1.6.2.** *Convergence of Cesàro sum of functions in  $L^1(\mathbb{T})$  If  $f \in L^1(\mathbb{T})$  then  $\sigma_N(x)$ , the Cesàro sum of the Fourier series of  $f$  converge to  $f(x)$  in  $L^1$  norm. That is,*

$$\lim_{N \rightarrow \infty} \int_0^1 |f(x) - \sigma_N(x)| = 0$$

*Proof.* Since we know that Fejér kernel,  $\Delta_N(x)$  is a summability kernel by proposition ?? and that  $(\Delta_N * f)(x) = \sigma_N(x)$  by prop ??, the result follows from theorem ??.  $\square$

**Corollary 1.6.2.1.** *If  $f \in L^1(\mathbb{T})$  and  $\hat{f}(n) = 0$  for all  $n \in \mathbb{N}$ . Then  $f(x) = 0$  almost everywhere in  $\mathbb{T}$ .*

*Proof.* Since  $\hat{f}(n) = 0$  for all  $n \in \mathbb{N}$ , the Cesàro partial sum (??) will be identically zero. Then by ?? it follows that

$$\int_{\mathbb{T}} |f(x)| dx = 0$$

Hence the result.  $\square$

**Corollary 1.6.2.2.** *If  $f, g \in L^1(\mathbb{T})$  and  $\hat{f}(n) = \hat{g}(n)$  for all  $n \in \mathbb{N}$ , then  $f = g$  almost everywhere.*

*Proof.* This follows from taking  $f - g$  in above corollary.  $\square$

**Theorem 1.6.3** (Fejér's Theorem). *Let  $f \in L^1(\mathbb{T})$  and  $f(a^-) = \lim_{x \rightarrow a^-} f(x)$  and  $f(a^+) = \lim_{x \rightarrow a^+} f(x)$  exist and are finite, then*

$$\lim_{N \rightarrow \infty} \sigma_N(x) = \frac{f(x^-) + f(x^+)}{2}$$

*Proof.* Let  $\epsilon$  be given. Since we assumed  $f(x^+)$  and  $f(x^-)$  exist and are finite, we can take  $\delta < \frac{1}{2}$  small enough such that  $|f(x - u) - f(x^-)| < \epsilon$  for  $0 \leq u \leq \delta$  and  $|f(x - u) - f(x^+)| < \epsilon$  for  $1 - \delta \leq u \leq 1$ . Notice that while the bounds for  $u$  seems obvious for  $f(x^-)$ , for  $f(x^+)$  we are using the fact that  $f$  is periodic with period 1, and hence  $f$  has the same value in  $[1 - \delta, 1]$ , and  $[-\delta, 0]$ .

$$\sigma_N(x) = \int_0^1 f(x - u) \Delta_N(u) du = \int_0^\delta + \int_\delta^{1-\delta} + \int_{1-\delta}^1 = I_1 + I_2 + I_3$$

Then,

$$I_1 = \int_0^\delta (f(x-u) - f(x^-))\Delta_N(u)du + \int_0^\delta f(x^-)\Delta_N(u)du = T_1 + T_1'$$

where by the property of summability kernels as in ??, and by our choice of  $\delta$ , we have

$$|T_1| \leq \int_0^\delta |f(x-u) - f(x^-)|\Delta_N(u)du < \epsilon \int_0^\delta \Delta_N(x) \leq \epsilon$$

Therefore  $T_1$  converge to 0 as  $N \rightarrow \infty$ .

Again, from the proposition ??, we get

$$\frac{1}{2} \geq \int_0^\delta \Delta_N(x)dx = \int_0^{\frac{1}{2}} \Delta_N(x)dx - \int_\delta^{\frac{1}{2}} \Delta_N(x)dx \geq \frac{1}{2} - \frac{1}{4N\delta}$$

which then implies that,

$$\frac{f(x^-)}{2} \geq T_1' = f(x^-) \int_0^\delta \Delta_N(x)dx \geq \frac{f(x^-)}{2} - \frac{f(x^-)}{4N\delta}$$

Therefore,

$$0 \geq T_1' - \frac{f(x^-)}{2} \geq -\frac{f(x^-)}{4N\delta}$$

and hence,

$$\left| T_1' - \frac{f(x^-)}{2} \right| \leq \left| \frac{f(x^-)}{4N\delta} \right|$$

which implies that  $I_1 = T_1 + T_1'$  converge to  $\frac{f(x^-)}{2}$  as  $N \rightarrow \infty$ .

Now by proposition ?? we know that if  $0 < x \leq \frac{1}{2}$ , then

$$\Delta_N(x) \leq \frac{1}{4Nx^2}$$

which implies that

$$|I_2| \leq \frac{1}{4N\delta^2} \int_\delta^{1-\delta} f(x-u)du \leq \frac{1}{4N\delta^2} \int_0^1 |f(u)|du = \frac{\|f\|_1}{4N\delta^2}$$

Therefore  $I_2$  converge to 0 as  $N \rightarrow \infty$

Now we'll prove that  $I_3$  converge to  $\frac{f(x^+)}{2}$ . For this we'll split  $I_3$  into  $T_3$  and  $T_3'$  like we did with  $I_1$ .

$$I_3 = \int_{1-\delta}^1 (f(x-u) - f(x^+))\Delta_N(u)du + f(x^+) \int_{1-\delta}^1 \Delta_N(u)du = T_3 + T_3'$$



Then by our choice of  $\delta$ ,

$$|T_3| \leq \int_{1-\delta}^1 |((f(x-u) - f(x^+))\Delta_N(u)du| < \epsilon \int_{1-\delta}^1 \Delta_N(x)dx \leq \epsilon$$

Therefore  $T_3$  converge to 0 as  $N \rightarrow \infty$ . Also

$$\int_{1-\delta}^1 \Delta_N(x)dx = - \int_{\delta}^0 \Delta_N(x)dx = \int_0^{\delta} \Delta_N(x)dx$$

Therefore by the same inequality we used for  $T_1'$  from proposition ??, we'll get

$$\left| T_3' - \frac{f(x^+)}{2} \right| \leq \left| \frac{f(x^+)}{4N\delta} \right|$$

which implies that  $I_3 = T_3 + T_3'$  converge to  $\frac{f(x^+)}{2}$  as  $N \rightarrow \infty$

Therefore since  $\sigma_N(x) = I_1 + I_2 + I_3$ , by the algebra of limits,

$$\lim_{N \rightarrow \infty} \sigma_N(x) = \frac{f(x^-) + f(x^+)}{2}$$

□

Fejér's theorem show that the Cesàro partial sum of the a continuous function converge pointwise to itself.

**Corollary 1.6.3.1.** *If  $f \in C(\mathbb{T})$  and  $\sigma_N(x)$  is the Cesàro partial sum of the Fourier series of  $f$ , then*

$$\lim_{N \rightarrow \infty} \sigma_N(x) = f(x)$$

**Theorem 1.6.4** (Hardy Tauberian Theorem). *Let  $\sum_{n=1}^{\infty} a_n$  is Cesàro summable to  $a$ , then if there exist a constant  $C$  such that*

$$|a_n| \leq \frac{C}{n}$$

*for all  $n$ , then the series  $\sum_{n=1}^{\infty} a_n$  converge to  $a$*

*Proof.* See [Mon14, Theorem 4.32 p. 123].

□



## Chapter 2

# Fourier Series in $L^p(\mathbb{T})$

Before we go into the general  $L^p$ , we'll first recall some important results from functional analysis which are essential for the discussion of further topics.

Recall that in ??, we've discussed what is an  $L^p$  function for a general space  $S$ . Here the space is  $\mathbb{T}$  and  $L^p(\mathbb{T})$  are precisely the set of all  $L^p$  functions in  $\mathbb{T}$ .

**Theorem 2.0.1** (Holder's Inequality). *Let  $1 \leq p \leq \infty$  and  $q$  such that  $1/p + 1/q = 1$  (for convention we will assume that the tuples  $(1, \infty)$ , and  $(\infty, 1)$ , satisfy the above relation) then for lebesgue measure space  $S$  and functions  $f \in L^p(S)$ , and  $g \in L^q(S)$*

$$\left| \int_S f(x)g(x)dx \right| \leq \|f\|_p \|g\|_q$$

where

$$\|f\|_p = \left( \int_S |f(x)|^p \right)^{\frac{1}{p}}$$

as in ??.

**Theorem 2.0.2** (Minkowski's Inequality). *Let  $1 \leq p \leq \infty$  then for a lebesgue measure space  $S$  and a function  $f \in L^p(S)$ , then*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

The proofs of the above two theorems can be found in any measure theory textbook and since it is a common proof, we'll omit it from detailing it here.

## 2.1 Fourier Series in $L^2(\mathbb{T})$

Before we proceed with the fourier coefficients and fourier series for  $L^2(\mathbb{T})$  functions we first prove some important results.

**Proposition 2.1.1.** *If  $f \in L^2(\mathbb{T})$ , then*

$$\lim_{\delta \rightarrow 0} \int_0^1 |f(x + \delta) - f(x)|^2 dx = 0$$

*Proof.* From measure theory we know that continuous functions are dense in  $L^2(\mathbb{T})$ . Then given any  $\epsilon$  there exists a function  $g \in C(\mathbb{T})$  such that  $\|f - g\|_2 < \epsilon$ . Then,

$$f(x + \delta) - f(x) = (f(x + \delta) - g(x + \delta)) - (f(x) - g(x)) + (g(x + \delta) - g(x))$$

Now by triangle inequality, (i.e minkowski's inequality for  $p = 2$ ),

$$\|f(x + \delta) - f(x)\|_2 = \|f(x + \delta) - g(x + \delta)\|_2 + \|f(x) - g(x)\|_2 + \|g(x + \delta) - g(x)\|_2$$

Now since  $g$  is continuous on  $\mathbb{T}$  it is uniformly continuous (since it is continuous on  $\mathbb{R}$ , it'll be continuous on  $[0, 1]$ , a compact set) and therefore  $\delta$  can be taken such that  $\|g(x + \delta) - g(x)\|_2 < \epsilon$ . Hence the theorem.  $\square$

**Proposition 2.1.2.** *If  $f, g \in L^2(\mathbb{T})$ , then their convolution,  $f * g$  is continuous and moreover  $\|f * g\|_\infty \leq \|f\|_2 \|g\|_2$*

*Proof.*

$$(f * g)(x + \delta) - (f * g)(x) = \int_0^1 f(u)(g(x + \delta - u) - g(x - u))du$$

Therefore by Cauchy Shwarz inequality,

$$\begin{aligned} |f * g(x + \delta) - f * g(x)| &= \left| \int_0^1 f(u)(g(x + \delta - u) - g(x - u))du \right| \\ &\leq \|f\|_2 \left( \int_0^1 |g(x + \delta - u) - g(x - u)|^2 du \right)^{1/2} \end{aligned}$$

which converge to zero as  $\delta \rightarrow 0$  by ???. Therefore  $f * g$  is continuous in  $\mathbb{T}$ . Also,

$$\begin{aligned} |f * g(x)| &= \left| \int_0^1 f(u)g(x - u)du \right| \\ &\leq \left( \int_0^1 |f(u)|^2 du \right)^{1/2} \left( \int_0^1 |g(u)|^2 du \right)^{1/2} \\ &= \|f\|_2 \|g\|_2 \end{aligned}$$

Hence the theorem.  $\square$

Now that we've proved some important results, we'll define the fourier coefficients  $\hat{f}(n)$  the same way we defined them for  $L^1(\mathbb{T})$  in ?? as

$$\hat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x} dx$$

Note that  $\hat{f}(n)$  is a finite quantity since  $|e^{ix}| = 1$  and,

$$|\hat{f}(n)| = \left| \int_0^1 f(x) e^{-2\pi i n x} dx \right| \leq \|f\|_2 \left( \int_0^1 |e^{-2\pi i n x}|^2 dx \right)^{1/2} = \|f\|_2$$

Hence we define the fourier series the same way as in ?. Then we'll investigate if the Fourier series of  $L^2(\mathbb{T})$  functions are Cesàro summable to  $f$ . In fact this is true.

**Theorem 2.1.1** (Fourier series of  $L^2(\mathbb{T})$  functions are Cesàro summable). *If  $f \in L^2(\mathbb{T})$ , and  $\sigma_N(x)$  is the  $N^{th}$  Cesàro partial sum of the Fourier series of  $f$ , then*

$$\lim_{N \rightarrow \infty} \int_0^1 |f(x) - \sigma_N(x)|^2 dx = 0$$

*Proof.* From ?? we know that

$$\sigma_N(x) = \int_0^1 \Delta_N(u) f(x-u) du$$

Hence,

$$f(x) - \sigma_N(x) = \int_0^1 (f(x) - f(x-u)) \Delta_N(u) du$$

By Holder's inequality in ?? applied to  $(f(x) - f(x-u))\sqrt{\Delta_N(u)}$  and  $\sqrt{\Delta_N(u)}$

$$\begin{aligned} |f(x) - \sigma_N(x)| &= \left| \int_0^1 (f(x) - f(x-u)) \Delta_N(u) du \right| \\ &\leq \left( \int_0^1 |f(x) - f(x-u)|^2 \Delta_N(u) du \right)^{1/2} \left( \int_0^1 \Delta_N(u) du \right)^{1/2} \end{aligned}$$

Since by ??,  $\int_0^1 \Delta_N(x) dx = 1$  and by Tonelli's theorem,

$$\begin{aligned} \int_0^1 |f(x) - \sigma_N(x)|^2 dx &= \int_0^1 \int_0^1 |f(x) - f(x-u)|^2 \Delta_N(u) du dx \\ &= \int_0^1 \Delta_N(u) \int_0^1 |f(x) - f(x-u)|^2 dx du \\ &= \int_{-\delta}^{\delta} + \int_{\delta}^{1-\delta} \\ &= I_1 + I_2 \end{aligned}$$

Also from ??, given  $\epsilon$ , we can find  $\delta$  such that

$$\int_0^1 |f(x) - f(x-\delta)|^2 dx < \epsilon$$

Then for that choice of  $\delta$ ,

$$|I_1| \leq \epsilon \int_{-\delta}^{\delta} \Delta_N(u) du \leq \epsilon \int_0^1 \Delta_N(u) du = \epsilon$$

To prove  $I_2$  is also bounded, we'll use a small trick aided by Minkowski's inequality. We know that  $\|f - g\|_p \leq \|f\|_p + \|g\|_p$ . Therefore  $\|f - g\|_p \leq 2 \max\{\|f\|_p, \|g\|_p\}$  and  $\|f - g\|_p^p \leq 2 \max\{\|f\|_p^p, \|g\|_p^p\}$ , and finally,  $\|f - g\|_p^p \leq 2(\|f\|_p^p + \|g\|_p^p)$ . Then for  $p = 2$ ,

$$\int_0^1 |f(x) - f(x-u)|^2 dx \leq 2(\|f\|_2^2 + \|f\|_2^2) = 4\|f\|_2^2$$

Therefore by ??, we get

$$|I_2| = \int_{\delta}^{1-\delta} \Delta_N(u) \int_0^1 |f(x) - f(x-u)|^2 dx du \leq 4\|f\|_2^2 \int_{\delta}^{1-\delta} \Delta_N(u) du$$

But by ??

$$\int_{\delta}^{1-\delta} \Delta_N(u) du = 2 \int_{\delta}^{1/2} \Delta_N(x) \leq \frac{1}{2N\delta}$$

Which implies,

$$|I_2| \leq \frac{2\|f\|_2^2}{N\delta}$$

Therefore  $I_2$  converge to 0 as  $N \rightarrow \infty$ . Hence the theorem.  $\square$

## 2.2 Fourier Series in $L^p(\mathbb{T})$

**Proposition 2.2.1.** *If  $f \in L^p(\mathbb{T})$ , then*

$$\lim_{\delta \rightarrow 0} \int_0^1 |f(x + \delta) - f(x)|^p dx = 0$$

*Proof.* From measure theory we know that continuous functions are dense in  $L^p(\mathbb{T})$ . Then given any  $\epsilon$  there exists a function  $g \in C(\mathbb{T})$  such that  $\|f - g\|_p < \epsilon$ . Then,

$$f(x + \delta) - f(x) = (f(x + \delta) - g(x + \delta)) - (f(x) - g(x)) + (g(x + \delta) - g(x))$$

Now by triangle inequality, (i.e minkowski's inequality for  $p = 2$ ),

$$\|f(x + \delta) - f(x)\|_p = \|f(x + \delta) - g(x + \delta)\|_p + \|f(x) - g(x)\|_p + \|g(x + \delta) - g(x)\|_p$$

Now since  $g$  is continuous on  $\mathbb{T}$  it is uniformly continuous (since it is continuous on  $\mathbb{R}$ , it'll be continuous on  $[0, 1]$ , a compact set) and therefore  $\delta$  can be taken such that  $\|g(x + \delta) - g(x)\|_p < \epsilon$ . Hence the theorem.  $\square$

**Proposition 2.2.2.** *If  $f \in L^p(\mathbb{T})$  and  $g \in L^q(\mathbb{T})$ , then their convolution,  $f * g$  is continuous and moreover  $\|f * g\|_\infty \leq \|f\|_p \|g\|_q$*

*Proof.*

$$(f * g)(x + \delta) - (f * g)(x) = \int_0^1 f(u)(g(x + \delta - u) - g(x - u))du$$

Therefore by Minkowski's inequality from ??,

$$\begin{aligned} |f * g(x + \delta) - f * g(x)| &= \left| \int_0^1 f(u)(g(x + \delta - u) - g(x - u))du \right| \\ &\leq \|f\|_p \left( \int_0^1 |g(x + \delta - u) - g(x - u)|^q du \right)^{1/q} \end{aligned}$$

which converge to zero as  $\delta \rightarrow 0$  by ??. Therefore  $f * g$  is continuous in  $\mathbb{T}$ . Also,

$$\begin{aligned} |f * g(x)| &= \left| \int_0^1 f(u)g(x - u)du \right| \\ &\leq \left( \int_0^1 |f(u)|^p du \right)^{1/p} \left( \int_0^1 |g(u)|^q du \right)^{1/q} \\ &= \|f\|_p \|g\|_q \end{aligned}$$

Hence the theorem.  $\square$

Now that we've proved some important results, we'll define the fourier coefficients  $\hat{f}(n)$  the same way we defined them for  $L^1(\mathbb{T})$  in ?? as

$$\hat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x} dx$$

Note that  $\hat{f}(n)$  is a finite quantity since  $|e^{ix}| = 1$  and,

$$|\hat{f}(n)| = \left| \int_0^1 f(x) e^{-2\pi i n x} dx \right| \leq \|f\|_p \left( \int_0^1 |e^{-2\pi i n x}|^q dx \right)^{1/q} = \|f\|_p$$

where  $1/p + 1/q = 1$ .

Hence we define the fourier series the same way as in ??. Then we'll investigate if the Fourier series of  $L^p(\mathbb{T})$  functions are Cesàro summable to  $f$ . In fact this is true.

**Theorem 2.2.1** (Fourier series of  $L^p(\mathbb{T})$  functions are Cesàro summable). *If  $f \in L^p(\mathbb{T})$ , and  $\sigma_N(x)$  is the  $N^{th}$  Cesàro partial sum of the Fourier series of  $f$ , then*

$$\lim_{N \rightarrow \infty} \int_0^1 |f(x) - \sigma_N(x)|^p dx = 0$$

*Proof.* From ?? we know that

$$\sigma_N(x) = \int_0^1 \Delta_N(u) f(x-u) du$$

Hence,

$$f(x) - \sigma_N(x) = \int_0^1 (f(x) - f(x-u)) \Delta_N(u) du$$

By Holder's inequality as in ?? applied to  $(f(x) - f(x-u))(\Delta_N(u))^{1/p}$  and  $(\Delta_N(u))^{1/q}$

$$\begin{aligned} |f(x) - \sigma_N(x)| &= \left| \int_0^1 (f(x) - f(x-u)) \Delta_N(u) du \right| \\ &\leq \left( \int_0^1 |f(x) - f(x-u)|^p \Delta_N(u) du \right)^{1/p} \left( \int_0^1 \Delta_N(u) du \right)^{1/q} \end{aligned}$$



Since by ??,  $\int_0^1 \Delta_N(x) dx = 1$  and by Tonelli's theorem,

$$\begin{aligned} \int_0^1 |f(x) - \sigma_N(x)|^p dx &= \int_0^1 \int_0^1 |f(x) - f(x-u)|^p \Delta_N(u) du dx \\ &= \int_0^1 \Delta_N(u) \int_0^1 |f(x) - f(x-u)|^p dx du \\ &= \int_{-\delta}^{\delta} + \int_{\delta}^{1-\delta} \\ &= I_1 + I_2 \end{aligned}$$

Also from ??, given  $\epsilon$ , we can find  $\delta$  such that

$$\int_0^1 |f(x) - f(x-\delta)|^p dx < \epsilon$$

Then for that choice of  $\delta$ ,

$$|I_1| \leq \epsilon \int_{-\delta}^{\delta} \Delta_N(u) du \leq \epsilon \int_0^1 \Delta_N(u) du = \epsilon$$

To prove  $I_2$  is also bounded, we'll use a small trick aided by Minkowski's inequality. We know that  $\|f - g\|_p \leq \|f\|_p + \|g\|_p$ . Therefore  $\|f - g\|_p \leq 2 \max\{\|f\|_p, \|g\|_p\}$  and  $\|f - g\|_p^p \leq 2 \max\{\|f\|_p^p, \|g\|_p^p\}$ , and finally,  $\|f - g\|_p^p \leq 2(\|f\|_p^p + \|g\|_p^p)$ . Then,

$$\int_0^1 |f(x) - f(x-u)|^p dx \leq 2(\|f\|_p^p + \|f\|_p^p) = 4\|f\|_p^p$$

Therefore by ??, we get

$$|I_2| = \int_{\delta}^{1-\delta} \Delta_N(u) \int_0^1 |f(x) - f(x-u)|^p dx du \leq 4\|f\|_p^p \int_{\delta}^{1-\delta} \Delta_N(u) du$$

But by ??

$$\int_{\delta}^{1-\delta} \Delta_N(u) du = 2 \int_{\delta}^{1/2} \Delta_N(x) \leq \frac{1}{2N\delta}$$

Which implies,

$$|I_2| \leq \frac{2\|f\|_p^p}{N\delta}$$

Therefore  $I_2$  converge to 0 as  $N \rightarrow \infty$ . Hence the theorem.  $\square$

If looked close enough one can see that the proof of convergence of fourier series in  $L^p(\mathbb{T})$  is almost the same as in  $L^2(\mathbb{T})$ . This is in fact true and  $L^2$  convergence is just a special case of  $L^p$  convergence.



# Chapter 3

## Fourier Transform

### 3.1 Definiton and basic properties

While defining Fourier series we were mainly focused on periodic functions. Now we'll try to expand that into another set of functions. We'll be interested on functions in  $L^1(\mathbb{R})$ , that is those real or complex valued functions  $f(x)$  in  $\mathbb{R}$  for which

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty$$

. For those functions in  $L^1(\mathbb{R})$  the integral above will be called the  $L^1$  norm of the function  $f$  and will be denoted by  $\|f\|_{L^1(\mathbb{R})}$  or in short  $\|f\|_1$ . Also note that the notations  $\int_{\mathbb{R}}$  and  $\int_{-\infty}^{\infty}$  means the same and we might use them interchangeably as we see fit.

Analogous to what we did in finding the  $n^{th}$  fourier coefficient in ??, we'll define the fourier transform of  $f$

**Definition 3.1.1** (Fourier transform of a function  $f$ ). Let  $f \in L^1(\mathbb{R})$ , then we define the Fourier transform of  $f$ ,  $\hat{f}: \mathbb{R} \rightarrow \mathbb{C}$  as

$$\hat{f}(t) = \int_{\mathbb{R}} f(x) e^{-2\pi i t x} dx$$

Note that while we say  $\hat{f}$  is the Fourier transform of the function  $f$ , the term "Fourier transform" is also used for the map which takes  $f$  to  $\hat{f}$ .

Also note that  $\hat{f}(t)$  is a finite quantity (real or complex) for all  $t \in \mathbb{T}$  since  $f \in L^1(\mathbb{T})$  and  $|e^{-2\pi i t x}| = 1$  implies

$$|\hat{f}(t)| \leq \int_{-\infty}^{\infty} |f(x)| dx < \infty$$

This shows that the fourier transform of a function in  $L^1(\mathbb{R})$  is in  $L^\infty(\mathbb{R})$ . We state this as a seperate result.

**Proposition 3.1.1.** *If  $f \in L^1(\mathbb{R})$ , then  $\widehat{f} \in L^\infty(\mathbb{R})$ , i.e*

$$\sup_{t \in \mathbb{R}} |\widehat{f}(t)| \leq \infty$$

By the linearity of the integral we can also show that for functions  $f, g \in L^1(\mathbb{R})$  and scalars  $\mu, \nu$ ,  $\widehat{\mu f + \nu g}(t) = \mu \widehat{f}(t) + \nu \widehat{g}(t)$ .

Now we'll prove some important properties of Fourier transforms. Note that this will almost remind you of the properties of Fourier coefficients in ??.

**Proposition 3.1.2** (Properties of Fourier transform). *If  $f \in L^1(\mathbb{R})$  and  $\widehat{f}$  is the Fourier transform of  $f$  as in ??, then*

(a) *If  $a \in \mathbb{R}$  and  $g(x) = f(x + a)$  for all  $x \in \mathbb{R}$ , then  $g \in L^1(\mathbb{R})$  and  $\widehat{g}(t) = e^{2\pi i t a} \widehat{f}(t)$  for all  $t$ .*

(b) *If  $b \in \mathbb{R}$  and  $h(x) = e^{2\pi i b x} f(x)$ , then  $h \in L^1(\mathbb{R})$  and  $\widehat{h}(t) = \widehat{f}(t - b)$  for all  $t$ .*

(c) *If  $c \in \mathbb{R}$  is not 0, and  $j(x) = f(cx)$ , then  $j \in L^1(\mathbb{R})$  and  $\widehat{j}(t) = \frac{\widehat{f}(t/c)}{|c|}$  for all  $t$ .*

(d) *if  $l(x) = \overline{f(x)}$ , then  $l \in L^1(\mathbb{R})$  and  $\widehat{l}(t) = \overline{\widehat{f}(-t)}$*

*Proof.* Note that by appropriate change of variable we can see that all the above functions  $g, h, j, l$  are in  $L^1(\mathbb{R})$ . We'll prove the other properties.

(a) By the change of variable  $y = x + a$ , we get that

$$\widehat{g}(t) = \int_{\mathbb{R}} g(x) e^{-2\pi i t x} dx = \int_{\mathbb{R}} f(x + a) e^{-2\pi i t x} dx = e^{2\pi i t a} \int_{\mathbb{R}} f(y) e^{-2\pi i t y} dy$$

which is equal to  $e^{2\pi i t a} \widehat{f}(t)$

(b)

$$\widehat{h}(t) = \int_{\mathbb{R}} h(x) e^{-2\pi i t x} dx = \int_{\mathbb{R}} f(x) e^{-2\pi i (t-b)x} dx = \widehat{f}(t - b)$$

(c) Here we'll need to be careful because  $c$  maybe negative. Assume  $c > 0$ , then by a change of variable  $y = cx$ , we get

$$\widehat{j}(t) = \int_{-\infty}^{\infty} j(x) e^{-2\pi i t x} dx = \int_{-\infty}^{\infty} f(cx) e^{-2\pi i t x} dx = \frac{1}{c} \int_{-\infty}^{\infty} f(y) e^{-2\pi i \frac{t}{c} y} dy$$

Then if  $c > 0$ ,  $\widehat{j}(t) = \frac{\widehat{f}(t/c)}{c}$ . Now if  $c < 0$  the limits of integration will reverse, i.e

$$\int_{-\infty}^{\infty} f(cx)e^{-2\pi itx} dx = \frac{1}{c} \int_{\infty}^{-\infty} f(y)e^{-2\pi i \frac{t}{c}y} dy = \frac{1}{-c} \int_{-\infty}^{\infty} f(y)e^{-2\pi i \frac{t}{c}y} dy$$

Which shows that if  $c \neq 0$ ,  $\widehat{j}(t) = \frac{\widehat{f}(t/c)}{|c|}$ .

(d) Since we know that integral of the conjugate is the conjugate of the integral,

$$\widehat{l}(t) = \int_{\mathbb{R}} \overline{f(x)}e^{-2\pi itx} dx = \int_{\mathbb{R}} \overline{f(x)e^{-2\pi i(-t)x}} dx = \overline{\int_{\mathbb{R}} f(x)e^{-2\pi i(-t)x} dx} = \overline{\widehat{f}(-t)}$$

□

Now we'll prove an important theorem

**Theorem 3.1.1** (Riemann Lebesgue Lemma). *Let  $f \in L^1(\mathbb{R})$ . Then*

$$\lim_{t \rightarrow \pm\infty} \widehat{f}(t) = 0$$

*Proof.* We know that

$$\widehat{f}(t) = \int_{\mathbb{R}} f(x)e^{-2\pi itx} dx$$

By change of variables  $x = y + \frac{1}{2t}$ , we see that the above equation becomes

$$\widehat{f}(t) = \int_{\mathbb{R}} f(y + \frac{1}{2t})e^{-2\pi it(y + \frac{1}{2t})} dy = - \int_{\mathbb{R}} f(y + \frac{1}{2t})e^{-2\pi ity} dy$$

Therefore

$$2\widehat{f}(t) = \int_{\mathbb{R}} (f(x + \frac{1}{2t}) - f(x))e^{-2\pi itx} dx$$

and therefore

$$|\widehat{f}(t)| \leq \frac{1}{2} \int_{\mathbb{R}} |f(x + \frac{1}{2t}) - f(x)| dx$$

Now by ?? as  $t \rightarrow \infty$ ,  $\frac{1}{2t} \rightarrow 0$  and therefore  $\widehat{f}(t) \rightarrow 0$ . Hence proved. □

Convolution was one important operation which helped us in the theory of Fourier series. We will define the convolution of two functions in  $L^1(\mathbb{R})$  the same way we defined it for functions in  $L^1(\mathbb{T})$  in ?. That is if  $f, g \in L^1(\mathbb{R})$ , we define the convolution of  $f$  and  $g$  as

$$(f * g)(x) = \int_{\mathbb{R}} f(x - y)g(y)dy$$

But unlike  $L^1(\mathbb{T})$ ,  $L^1(\mathbb{R})$  is not finite measure space. So we'll need to make sure this definition of convolution is well defined for functions in  $L^1(\mathbb{R})$ .

**Proposition 3.1.3.** *For functions  $f, g \in L^1(\mathbb{R})$ ,  $f(x-y)g(y)$  is Lebesgue measurable in the product space  $\mathbb{R}^2$ . Moreover  $f(x-y)g(y) \in L^1(\mathbb{R}^2)$ . And therefore by Fubini's theorem  $\int_{\mathbb{R}} f(x-y)g(y)dx$  exists and is well defined for all  $x \in \mathbb{R}$ .*

*Proof.* Consider  $h, k : \mathbb{R}^2 \rightarrow \mathbb{R}$  defined as  $h(x, y) = f(x)$  and  $k(x, y) = g(y)$ . Then  $h^{-1}(a, \infty) = f^{-1}(a, \infty) \times \mathbb{R}$  and  $k^{-1}(b, \infty) = \mathbb{R} \times g^{-1}(b, \infty)$  and therefore  $h, k$  are measurable functions. Moreover we know from the theory of measurable functions that the product of measurable functions is measurable. Therefore  $F(x, y) = h(x, y)k(x, y) = f(x)g(y)$  is measurable. Also the map  $T(x, y) = (x - y, y)$  is linear. Hence  $H(x, y) = F \circ T(x, y) = f(x - y)g(y)$  is measurable.

Also  $H \in L^1(\mathbb{R}^2)$  implies

$$\int_{\mathbb{R}^2} |H(x, y)| dx dy = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(x - y)| dx \right) |g(y)| dy = \int_{\mathbb{R}} \|f\|_1 |g(y)| dy = \|f\|_1 \|g\|_1$$

Hence  $f(x - y)g(y) \in L^1(\mathbb{R}^2)$  and the result follows.  $\square$

Almost all the properties of the convolution in  $L^1(\mathbb{T})$  has its analogues in  $L^1(\mathbb{R})$ , but we will only quote the most important one, that is the convolution is commutative,  $f * g = g * f$ . One can easily prove this using a change of variable like we did in ??

We'll also prove that for functions  $f, g \in L^1(\mathbb{R})$ ,  $f * g \in L^1(\mathbb{R})$ .

**Proposition 3.1.4.** *Let  $f, g \in L^1(\mathbb{R})$ , then  $f * g \in L^1(\mathbb{R})$  and  $\widehat{f * g}(t) = \widehat{f}(t)\widehat{g}(t)$ .*

*Proof.* By the definition of the convolution of  $f$  and  $g$ ,

$$\begin{aligned} \int_{\mathbb{R}} |(f * g)(x)| dx &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(x - y)g(y) dy \right| dx \\ &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x - y)g(y)| dy dx \end{aligned}$$

Now by Fubini's theorem we know that last integral is equal to

$$\int_{\mathbb{R}^2} f(x - y)g(y)$$

and by previous theorem it is less than or equal to  $\|f\|_1 \|g\|_1$ .

For the fourier transform

$$\begin{aligned}
 \widehat{f * g}(t) &= \int_{\mathbb{R}} \widehat{f * g}(x) e^{-2\pi i t x} dx \\
 &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-y) g(y) e^{-2\pi i t x} dy dx \\
 &= \int_{\mathbb{R}} g(y) e^{-2\pi i t y} \int_{\mathbb{R}} f(x-y) e^{-2\pi i t (x-y)} dx dy \\
 &= \int_{\mathbb{R}} g(y) e^{-2\pi i t y} \widehat{f}(t) dy \\
 &= \widehat{f}(t) \widehat{g}(t)
 \end{aligned}$$

Note that the change of variable is justified since we know that for all  $x \in \mathbb{R}$ ,  $|e^{-2\pi i t x}| = 1$  and hence the integral is absolutely convergent by the proof similar to last theorem.  $\square$

Now we'll prove a result which has close relations with the Plancherel's theorem we'll prove in ??.

**Proposition 3.1.5.** *Let  $f, g \in L^1(\mathbb{R})$ , then*

$$\int_{\mathbb{R}} \widehat{f}(t) g(t) dt = \int_{\mathbb{R}} f(t) \widehat{g}(t) dt$$

*Proof.* Since  $f, g \in L^1(\mathbb{R})$  from ?? we get that  $\widehat{f}, \widehat{g} \in L^\infty(\mathbb{R})$ . Then we get that

$$\begin{aligned}
 \int_{\mathbb{R}} \widehat{f}(t) g(t) dt &= \int_{\mathbb{R}} g(t) \int_{\mathbb{R}} f(x) e^{-2\pi i t x} dx dt \\
 &= \int_{\mathbb{R}} f(x) \int_{\mathbb{R}} g(t) e^{-2\pi i t x} dt dx \\
 &= \int_{\mathbb{R}} f(x) \widehat{g}(x) dx
 \end{aligned}$$

where the change of order of integration is justified by Fubini's theorem since  $\widehat{f}$  is uniformly bounded by some constant  $M$  and we know that  $g \in L^1(\mathbb{R})$ .  $\square$

## 3.2 Fourier Inverse Transform

For  $L^1$  periodic functions, we were able to represent them using their Fourier series. We wish to do the same for functions in  $L^1(\mathbb{R})$ , by trying to represent

them using their Fourier transform. We hope that

$$f(x) = \int_{\mathbb{R}} \widehat{f}(t) e^{2\pi i t x} dt$$

But note that for the above integral to make sense we require  $\widehat{f} \in L^1(\mathbb{R})$ . Therefore we'll assume that  $\widehat{f} \in L^1(\mathbb{R})$  for the later part of the chapter. Still we might encounter difficulties in convergence, like we had in Fourier series. We might need to introduce analogous techniques to Fejér kernel to define the integral. As a first step towards that we'll consider functions like

$$B(x) = \int_{-T}^T b(t) e^{2\pi i t x} dt$$

For the Fourier transform, these functions will serve as analogue for trigonometric polynomials in Fourier series.

We'll now prove an important property of functions  $B(x)$  as defined above.

**Proposition 3.2.1.** *Let  $b \in L^1[-T, T]$ , and  $f \in L^1(\mathbb{R})$ , then*

$$B(x) = \int_{-T}^T b(t) e^{2\pi i t x} dt \implies (f * B)(x) = \int_{-T}^T b(t) \widehat{f}(t) e^{2\pi i t x} dx$$

*Proof.* Since  $|e^{2\pi i t x}| = 1$ , and  $b \in L^1[-T, T]$ , by triangle inequality

$$|B(x)| \leq \int_{-T}^T |b(t)| dt < \infty$$

Thus  $B$  is a bounded function in the finite interval  $[-T, T]$ . Hence  $|(f * B)(x)| \leq \|f\|_1 \|b\|_1$  for all  $x$ . Moreover,

$$\begin{aligned} (f * B)(x) &= \int_{\mathbb{R}} f(x - y) B(y) dy \\ &= \int_{\mathbb{R}} f(x - y) \int_{-T}^T b(t) e^{2\pi i t y} dt dy \\ &= \int_{-T}^T b(t) e^{2\pi i t x} \int_{\mathbb{R}} f(x - y) e^{-2\pi i t(x - y)} dy dt \\ &= \int_{-T}^T b(t) \widehat{f}(t) e^{2\pi i t x} dt \end{aligned}$$

Note that the change of integral is justified because we already know that the integral is finite since  $|(f * B)(x)| \leq \infty$ .  $\square$



Now we'll prove some propositions which will help us in  $L^2(\mathbb{R})$ .

**Proposition 3.2.2.** *We have*

$$\int_{-\infty}^{\infty} \left( \frac{\sin(\pi x)}{\pi x} \right)^2 dx = 1$$

*Proof.* from (b) and (e) of ??, and ?? we get that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{N} \left( \frac{\sin(\pi N x)}{\sin(\pi x)} \right)^2 dx = 1$$

We can write this integral as

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{N} \left( \frac{\sin(\pi N x)}{\pi x} \right)^2 dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\sin^2 \pi N x}{N} \left( \frac{1}{\sin^2 \pi x} - \frac{1}{(\pi x)^2} \right) dx = I_1 + I_2 = 1$$

For  $I_1$ , we see that by the change of variable  $y = Nx$ , it becomes

$$I_1 = \int_{-\frac{N}{2}}^{\frac{N}{2}} \left( \frac{\sin(\pi x)}{\pi x} \right)^2 dx$$

. And for  $I_2$ , we see that the difference of fractions inside paranthesis is

$$\frac{(\pi x)^2 - \sin^2(\pi x)}{(\pi x \sin(\pi x))^2} = \frac{(\pi x - \sin(\pi x))(\pi x + \sin(\pi x))}{(\pi x \sin(\pi x))^2}$$

Here  $\pi x - \sin(\pi x)$  has a zero of order 3, and  $\pi x + \sin(\pi x)$  has a zero of order 1 at  $x = 0$ . Hence the numerator has a zero of order 4 at  $x = 0$ . Again the denominator also has a zero of order 0 at  $x = 0$ . Then the expression has a bounded value at  $x = 0$ . Also everywhere else in the interval  $-\frac{1}{2} \leq x \leq \frac{1}{2}$ , the expression is continuous being a product and quotient of continuous function. Hence the expression is bounded in the same interval. Hence  $|I_2| \leq \frac{C}{N}$ , for some constant  $C$ . Therefore as  $N \rightarrow \infty$ ,  $I_1$  tends to our desired integral and  $I_2 \rightarrow 0$ . Since  $I_1 + I_2 = 1$  our result follows.  $\square$

Let's define the  $L^1(\mathbb{R})$  analogue of Fejér kernel.

**Definition 3.2.1.** For real numbes  $T > 0$ , we define

$$\Delta_T(x) = \begin{cases} \frac{1}{T} \left( \frac{\sin \pi T x}{\pi x} \right)^2, & \text{if } x \neq 0 \\ T, & \text{if } x = 0 \end{cases}$$

**Proposition 3.2.3** (Properties of  $\Delta$  function). *Let  $T > 0$ , and  $\Delta(x)$  be as defined above, then the following holds*

(a)

$$\int_{\mathbb{R}} \Delta_T(x) \, dx = 1$$

(b)

$$\Delta(-x) = \Delta(x)$$

(c)

$$\int_{-T}^T \left(1 - \frac{|t|}{T}\right) e^{2\pi i t x} \, dt = \Delta_T(x)$$

*Proof.* (a) From ??,

$$\int_{\mathbb{R}} \left(\frac{\sin(\pi x)}{\pi x}\right)^2 \, dx = 1$$

Put  $x = Ty$ , then  $dx = T \, dy$  and we get the desired result

(b) This is clear from the definition, since  $\sin^2(-x) = \sin^2(x)$ .

(c) To prove this first we'll find the Fourier transform of the function  $g$ , where  $g(x) = 1 - |x|$  for  $|x| \leq 1$  and  $g(x)$  is 0 everywhere else.

$$\begin{aligned} \widehat{g}(t) &= \int_{-1}^1 (1 - |x|) e^{-2\pi i t x} \, dx = 2 \int_0^1 (1 - x) \cos(2\pi t x) \, dx \\ &= 2 \left[ (1 - x) \frac{\sin(2\pi t x)}{2\pi t} \right]_0^1 + \int_0^1 \frac{\sin(2\pi t x)}{\pi t} \, dx \\ &= \left[ -\frac{\cos(2\pi t x)}{2(\pi t)^2} \right]_0^1 = \frac{1 - \cos(2\pi t)}{2(\pi t)^2} = \left( \frac{\sin(\pi t)}{\pi t} \right)^2 \end{aligned}$$

Now from the (c) of ??, we get that the fourier transform of the function  $h(x) = g(Tx)$ ,

$$\widehat{h}(t) = \frac{1}{T} \left( \frac{\sin(\pi T t)}{\pi t} \right)^2 = \Delta(x)$$

Note that the above holds true for values  $x \neq 0$  and when  $x = 0$ , the integral reduces to

$$\frac{1}{T} \int_{-T}^T (T - |t|) \, dt = \frac{2}{T} \int_0^T (T - t) \, dt = T = \Delta(0)$$

Hence the proof. □

From the above properties we conclude the following result.

**Corollary 3.2.0.1.** *If  $f \in L^1(\mathbb{R})$ , and  $\Delta_T$  is as in ??, then*

$$(f * \Delta_T)(x) = \int_{-T}^T \left(1 - \frac{|t|}{T}\right) \widehat{f}(t) e^{2\pi i t x} dt$$

*Proof.* From last proposition we get that

$$\Delta_T(x) = \int_{-T}^T \left(1 - \frac{|t|}{T}\right) e^{-2\pi i t x} dx$$

Using the result of ??, on  $\Delta_T$ , we get the result.  $\square$

Now we prove the  $L^1(\mathbb{R})$  analogue of the Fejer's theorem (??)

**Theorem 3.2.1.** *Let  $f \in L^1(\mathbb{R})$ . If  $f$  is continuous at  $x$ , then*

$$\lim_{T \rightarrow \infty} \int_{-T}^T \left(1 - \frac{|t|}{T}\right) \widehat{f}(t) e^{2\pi i t x} dt = f(x)$$

*Proof.* Let  $\epsilon$  be given, since we know that  $f$  is continuous at  $x$ , there exist a  $\delta$ , such that  $|f(x - y) - f(x)| < \epsilon$ , when  $|y| < \delta$ . Also since we know that  $\int_{\mathbb{R}} \Delta_T(x) dx = 1$  by ?? and by ??,

$$\begin{aligned} \int_{-T}^T \left(1 - \frac{|t|}{T}\right) \widehat{f}(t) e^{2\pi i t x} dt - f(x) &= (f * \Delta_T)(x) - f(x) \\ &= \int_{\mathbb{R}} f(x - y) \Delta_T(y) dy - \int_{\mathbb{R}} f(x) \Delta_T(y) dy \\ &= \int_{\mathbb{R}} (f(x - y) - f(x)) \Delta_T(y) dy \end{aligned}$$

Now we'll evaluate the last integral into  $I_1, I_2, I_3$ , where  $I_1$  is when  $-\infty < y \leq -\delta$ ,  $I_2$  is when  $-\delta < y < \delta$  and  $I_3$  is  $\delta \leq y < \infty$ . When  $y$  is in  $I_1$  and  $I_3$ , we see from the ?? that since  $|y| > |\delta|$  in  $I_1$  and  $I_3$ ,  $|\Delta_T(y)| \leq \frac{1}{T(\pi\delta)^2}$ . Thus

$$\begin{aligned} |I_1| &\leq \frac{1}{T\pi^2\delta^2} \int_{-\infty}^{-\delta} |f(x - y)| dy + |f(x)| \int_{-\infty}^{-\delta} \Delta_T(y) dy \\ |I_3| &\leq \frac{1}{T\pi^2\delta^2} \int_{\delta}^{\infty} |f(x - y)| dy + |f(x)| \int_{\delta}^{\infty} \Delta_T(y) dy \end{aligned}$$

where the first integral in both the inequalities is less than  $\|f\|_1$  which is finite, and second integral

$$\int_{-\infty}^{-\delta} \Delta_T(y) dy = \int_{\delta}^{\infty} \Delta_T(y) dy \leq \int_{\delta}^{\infty} \frac{1}{T\pi^2\delta^2} dx = \frac{1}{T\pi^2\delta}$$

Therefore  $I_1, I_3 \rightarrow 0$ , as  $T \rightarrow \infty$ . Now in  $I_2$ , by the continuity of  $f$  at  $x$ ,

$$|I_2| \leq \epsilon \int_{-\delta}^{\delta} \Delta_T(y) dy \leq \epsilon \int_{\mathbb{R}} \Delta_T(y) dy = \epsilon$$

Therefore  $I_2$  can also be made arbitrarily small. Hence the theorem.  $\square$

Now we'll prove some results which will help us to prove the general Fourier inversion theorem. The below result is analogous to the fact that Cesàro sum of a convergent sequence converges to the same limit as the original sequence, given it exists.

**Proposition 3.2.4.** *Let  $f \in L^1[-T, T]$  and let*

$$I(T) = \int_{-T}^T f(t) dt, \quad J(T) = \int_{-T}^T \left(1 - \frac{|t|}{T}\right) f(t) dt$$

*Then*

$$\lim_{T \rightarrow \infty} I(T) = \lim_{T \rightarrow \infty} J(T)$$

*Proof.* We'll first show that  $\lim_{T \rightarrow \infty} I(T)$  exists. But since  $f \in L^1(\mathbb{R})$ ,

$$|I(T)| \leq \int_{-T}^T |f(t)| dt \leq \|f\|_1$$

Therefore  $\lim_{T \rightarrow \infty} I(T) \leq \|f\|_1$ , and the limit exists. Moreover we see that by Fubini's theorem,

$$\frac{1}{T} \int_0^T I(u) du = \frac{1}{T} \int_0^T \int_{-u}^u f(t) dt du = \frac{1}{T} \int_{-T}^T f(t) \left( \int_{|t|}^T 1 du \right) dt = J(T)$$

Since we assumed that  $\lim_{T \rightarrow \infty} I(T) = a$ , given  $\epsilon > 0$ , there exists an  $N$  such that  $|I(u) - a| < \epsilon$  for all  $u \geq N$ . Now we can write

$$J(T) - a = \frac{1}{T} \int_0^T I(u) - a du = \frac{1}{T} \int_0^N I(u) - a du + \frac{1}{T} \int_N^T I(u) - a du = E_1 + E_2$$

Now since  $I(u)$  is bounded for all  $u$ , the integral in  $E_1$  is finite and therefore  $E_1 \rightarrow 0$  as  $T \rightarrow \infty$ . As for  $E_2$ ,

$$|E_2| \leq \frac{1}{T} \int_N^T |I(u) - a| du < \frac{1}{T} \int_N^T \epsilon du = \frac{T - N}{T} \epsilon < \epsilon$$

Therefore  $E_2$  converge to 0 as  $\epsilon$  can be made arbitrarily small. Hence the result.  $\square$

As a result of the above result combined with ??, we get

**Corollary 3.2.1.1.** *Let  $f \in L^1(\mathbb{R})$  and  $f$  is continuous at  $x$ . If*

$$\lim_{T \rightarrow \infty} \int_{-T}^T \widehat{f}(t) e^{2\pi i t x} dt$$

*exists, then it is equal to  $f(x)$ .*

Since we'll be working a lot with functions  $f$  which are continuous everywhere in the real line, we explicitly state the next theorem.

**Theorem 3.2.2.** *If  $f \in L^1(\mathbb{R})$ ,  $f$  is continuous in  $\mathbb{R}$ , and if  $\widehat{f} \in L^1(\mathbb{R})$ , then*

$$\int_{\mathbb{R}} \widehat{f}(t) e^{2\pi i t x} dt = f(x)$$

Note that the condition that  $\widehat{f} \in L^1(\mathbb{R})$  in the above theorem guarantee the existence of the limit from the corollary above.

Now we'll prove one important theorem

**Theorem 3.2.3.** *If  $f \in L^1(\mathbb{R})$  and  $\sigma_T(x) = (f * \Delta_T)(x)$ , where  $\Delta_T$  is as in ??, then*

$$\lim_{T \rightarrow \infty} \int_{-\infty}^{\infty} |f(x) - \sigma_T(x)| dx = 0$$

*Proof.* From ??, we know that  $\int_{\mathbb{R}} \Delta_T(x) dx = 1$ , therefore

$$f(x) - \sigma_T(x) = \int_{\mathbb{R}} (f(x) - f(x-u)) \Delta_T(u) du$$

which implies

$$|f(x) - \sigma_T(x)| \leq \int_{\mathbb{R}} |f(x) - f(x-u)| \Delta_T(u) du$$

and therefore

$$\int_{\mathbb{R}} |f(x) - \sigma_T(x)| dx \leq \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x) - f(x-u)| \Delta_T(u) du dx$$

Again we see that

$$\int_{\mathbb{R}} \Delta_T(u) \int_{\mathbb{R}} |f(x) - f(x-u)| dx du \leq \int_{\mathbb{R}} \Delta_T(u) \cdot 2\|f\|_1 du = 2\|f\|_1$$

and since  $\mathbb{R}^2$  is a sigma finite measure space by Fubini-Tonelli theorem, we can change the order of integration. Then

$$\int_{\mathbb{R}} |f(x) - \sigma_T(x)| \, dx \leq \int_{\mathbb{R}} \Delta_T(u) \int_{\mathbb{R}} |f(x) - f(x-u)| \, dx \, du$$

Now we'll analyse the last integral. Let us denote the inner integral by  $I(u)$ . Since  $f \in L^1(\mathbb{R})$ , by ?? given any  $\epsilon$ , there exists a  $\delta$  such that for all  $|u| < \delta$ ,

$$I(u) = \int_{\mathbb{R}} |f(x) - f(x-u)| \, dx < \epsilon$$

Moreover,

$$I(u) \leq \int_{\mathbb{R}} |f(x)| + |f(x-u)| \, du = 2\|f\|_1$$

We'll split the integral into three intervals and analyse there. For that let

$$\int_{\mathbb{R}} \Delta_T(u) I(u) \, du = \int_{-\infty}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{\infty} = I_1 + I_2 + I_3$$

Then by the definition of  $\Delta_T$  in ??,

$$I_1 \leq 2\|f\|_1 \int_{-\infty}^{-\delta} \Delta_T(u) \, du \leq 2\|f\|_1 \int_{-\infty}^{-\delta} \frac{1}{T\pi^2 u^2} \leq \frac{2\|f\|_1}{T\pi^2 \delta}$$

$$I_3 \leq 2\|f\|_1 \int_{\delta}^{\infty} \Delta_T(u) \, du \leq 2\|f\|_1 \int_{\delta}^{\infty} \frac{1}{T\pi^2 u^2} \leq \frac{2\|f\|_1}{T\pi^2 \delta}$$

And by the choice of  $\delta$  and ??,  $I_2$  can be estimated as

$$I_2 \leq \epsilon \int_{-\delta}^{\delta} \Delta_T(u) \, du \leq \epsilon \int_{\mathbb{R}} \Delta_T(u) \, du = \epsilon$$

Since  $\epsilon$  can be chosen arbitrarily small and for the corresponding  $\delta$ , we see that  $I_1, I_3 \rightarrow 0$  as  $T \rightarrow \infty$ . Hence the result.  $\square$

Now armed with the above theorem we can conclude some important results.

**Corollary 3.2.3.1.** *If  $f \in L^1(\mathbb{R})$  and  $\widehat{f}(t) = 0$  for all  $t$ , then  $f$  is 0 almost everywhere. That is*

$$\int_{\mathbb{R}} |f(x)| \, dx = 0$$

*Proof.* Since we know that  $\widehat{f}(t) = 0$  for all  $t \in \mathbb{R}$ , by ??,  $\sigma_T = (f * \Delta_T) = 0$  (zero function) for all  $T$ . Then by previous theorem the result follows.  $\square$

Applying the corollary above to  $f - g$  where  $f, g \in L^1(\mathbb{R})$ , we get

**Corollary 3.2.3.2.** *Let  $f, g \in L^1(\mathbb{R})$  and  $\widehat{f} = \widehat{g}$ . Then  $f = g$  almost everywhere. That is*

$$\int_{\mathbb{R}} |f(x) - g(x)| \, dx = 0$$

.

Now we can explicitly state the most important theorem in this section. The Fourier inversion theorem. The theorem states that given any function  $\widehat{f} \in L^1(\mathbb{R})$ , it is almost everywhere equal to the Fourier transform of some function  $f \in L^1(\mathbb{R})$ .

**Theorem 3.2.4.** *Let  $f, \widehat{f} \in L^1(\mathbb{R})$ . If*

$$g(x) = \int_{\mathbb{R}} \widehat{f}(t) e^{2\pi i t x} \, dx$$

*then  $f = g$  almost everywhere.*

*Proof.* Since it is given that  $\widehat{f} \in L^1(\mathbb{R})$ ,  $g$  is well defined and  $g(x)$  exists for all points  $x \in \mathbb{R}$ . Also at points of continuity of  $f$ ,  $g(x) = f(x)$  by ???. But since  $f \in L^1(\mathbb{R})$ , it is continuous almost everywhere. Therefore  $f = g$  almost everywhere. □

### 3.3 Fourier Transform in $L^2(\mathbb{R})$

We are interested in defining the Fourier transform for functions in  $L^2(\mathbb{R})$  because a lot of important relations appear when the space is  $L^2(\mathbb{R})$ . But here arise a new problem. The way we defined Fourier transforms for  $L^1$  functions were by directly using the fact that since their  $L^1$  norm is finite, the integral in ??? is well defined. But since there is no containment between the spaces  $L^1(\mathbb{R})$  and  $L^2(\mathbb{R})$ , we will not be able to define the Fourier transform for all  $L^2(\mathbb{R})$  functions. Therefore we restrict ourselves to the space  $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ . Since our new space  $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$  is a subspace of  $L^1(\mathbb{R})$ , the Fourier transform of all functions  $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$  is well defined.

Now we'll discuss about the convolution of functions in  $L^2(\mathbb{R})$ . We'll follow the same definition as before in  $L^1(\mathbb{R})$ . (For formula refer to ???). It is natural to think if such a definition is well defined. In fact by the Hölder's inequality (??)

one can show that it is well defined. Since if  $f, g \in L^2(\mathbb{R})$ , then

$$\begin{aligned} |f * g(x)| &= \left| \int_{\mathbb{R}} f(x-y)g(y) dy \right| \\ &\leq \int_{\mathbb{R}} |f(x-y)g(y)| dy \\ &\leq \left( \int_{\mathbb{R}} |f(x-y)|^2 dy \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} |g(y)|^2 dy \right)^{\frac{1}{2}} \\ &= \|f\|_2 \|g\|_2 \end{aligned}$$

We can now discuss about the convolution of  $L^2(\mathbb{R})$  functions. We'll now show that the convolution of two functions in  $L^2(\mathbb{R})$  is uniformly continuous. This proposition will be a stepping stone to the celebrated Plancherel's Theorem.

**Proposition 3.3.1.** *If  $f, g \in L^2(\mathbb{R})$ , then  $f * g$  is uniformly continuous.*

*Proof.* Let  $h(x) = (f * g)(x)$ . Then

$$h(x + \delta) - h(x) = \int_{\mathbb{R}} (f(x + \delta - y) - f(x - y))g(y) dy$$

By Holder's inequality,

$$\begin{aligned} |h(x + \delta) - h(x)| &\leq \int_{\mathbb{R}} |(f(x + \delta - y) - f(x - y))g(y)| dy \\ &\leq \left( \int_{\mathbb{R}} |f(x + \delta - y) - f(x - y)|^2 dy \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} |g(y)|^2 dy \right)^{\frac{1}{2}} \end{aligned}$$

where the first integral converge to zero by ?? and the last integral is a constant. Since the first integral is over all of  $\mathbb{R}$ , it is independent of  $x$  and hence the result follows.  $\square$

Now we'll prove the famous Plancherel's theorem.

**Theorem 3.3.1** (Plancherel's Theorem). *If  $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ , then*

$$\int_{\mathbb{R}} |f(x)|^2 dx = \int_{\mathbb{R}} |\widehat{f}(t)|^2 dt$$

*Proof.* Let  $g(x) = \overline{f(-x)}$ . Then  $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ . Put  $h(x) = (f * g)(x)$ . Since  $f, g \in L^1(\mathbb{R})$ , by ??,  $h \in L^1(\mathbb{R})$ . Since  $f, g \in L^2(\mathbb{R})$ , by ??,  $h$  is uniformly continuous. Hence by ??

$$h(x) = \lim_{T \rightarrow \infty} \int_{-T}^T \left(1 - \frac{|t|}{T}\right) \widehat{h}(t) e^{2\pi i t x} dt$$



for all  $x$ . In particular, for  $x = 0$  we get

$$h(0) = \lim_{T \rightarrow \infty} \int_{-T}^T \left(1 - \frac{|t|}{T}\right) \widehat{h}(t) dt$$

By the ??  $\widehat{h}(t) = \widehat{f}(t)\widehat{g}(t)$ . By the definition of  $g$ , and combining (c), (d) from ??, we get that  $\widehat{g}(t) = \overline{\widehat{f}(t)}$ . Since

$$h(0) = (f * g)(0) = \int_{\mathbb{R}} f(y)g(-y) dy = \int_{\mathbb{R}} |f(y)|^2 dy$$

we get that

$$\int_{\mathbb{R}} |f(x)|^2 dx = \lim_{T \rightarrow \infty} \int_{-T}^T \left(1 - \frac{|t|}{T}\right) |\widehat{f}(t)|^2 dt$$

Since the integral in the right is always positive, the limit is an increasing limit. Then for any  $0 < \epsilon < 1$ , we have

$$(1 - \epsilon) \int_{-\epsilon T}^{\epsilon T} |\widehat{f}(t)|^2 dt \leq \int_{-T}^T \left(1 - \frac{|t|}{T}\right) |\widehat{f}(t)|^2 dt \leq \int_{\mathbb{R}} |\widehat{f}(t)|^2 dt$$

Then as  $T \rightarrow \infty$ ,

$$(1 - \epsilon) \int_{\mathbb{R}} |\widehat{f}(t)|^2 dt \leq \int_{\mathbb{R}} |f(t)|^2 dt \leq \int_{\mathbb{R}} |\widehat{f}(t)|^2 dt$$

Since  $\epsilon$  can be made arbitrary small, the result follows.  $\square$

Also by ??,  $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$  is dense in  $L^2(\mathbb{R})$ . Then by Plancherel's theorem Fourier transform in  $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$  can be extended into an equinorm transform  $\mathcal{F} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ . When working with  $L^2(\mathbb{R})$ , by Fourier transform we'll mean this  $\mathcal{F}$ . Also note that  $\mathcal{F}(L^1(\mathbb{R}) \cap L^2(\mathbb{R}))$  is dense in  $L^2(\mathbb{R})$ . The proof can be seen at [Rud87, Theorem 9.13 p. 186] or [Pin09, Theorem 2.4.1 p. 128]



## Chapter 4

# Holomorphic Fourier Transforms

### 4.1 Motivation and Basic Results

In last chapter we explored the Fourier transforms of functions in  $\mathbb{R}$ . Frequently the fourier transform of functions in  $\mathbb{R}$  can be extended to a holomorphic function in certain regions. For example say  $f(x) = e^{-|x|}$ , then  $\widehat{f}(t) = \frac{1}{1+(2\pi t)^2}$ . Since,

$$\begin{aligned}\widehat{f}(t) &= \int_{\mathbb{R}} e^{-|x|} e^{-2\pi i t x} dx = \int_{-\infty}^0 e^{x(1-2\pi i t)} dx + \int_0^{\infty} e^{-x(1+2\pi i t)} dx \\ &= \left[ \frac{e^{x(1-2\pi i t)}}{1-2\pi i t} \right]_{-\infty}^0 - \left[ \frac{e^{-x(1+2\pi i t)}}{1+2\pi i t} \right]_0^{\infty} \\ &= \frac{1}{1-2\pi i t} + \frac{1}{1+2\pi i t} \\ &= \frac{1}{1+(2\pi t)^2}\end{aligned}$$

Clearly  $w(z) = \frac{1}{1+(2\pi z)^2}$  is a holomorphic extension of  $\widehat{f}$  into the regions of the complex plane without the points  $z = \pm \frac{i}{2\pi}$ .

We will study two classes of functions which can be extended in this manner. The first class of such functions is

$$f(z) = \int_0^{\infty} F(t) e^{2\pi i t z} dt$$

where  $z \in \Pi^+ = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$  and  $F \in L^2(\mathbb{R})$  is a function which vanishes on  $(-\infty, 0)$ . The second class of functions will be

$$f(z) = \int_{-A}^A F(t) e^{2\pi i t z} dt$$

where  $0 < A < \infty$  and  $F \in L^2(-A, A)$ . We'll now prove some important results regarding these two classes of functions

**Proposition 4.1.1.** *Let  $F \in L^2(\mathbb{R})$  such that  $F$  vanishes at  $(-\infty, 0)$ . Then  $f : \Pi^+ \rightarrow \mathbb{C}$  defined as,*

$$f(z) = \int_0^\infty F(t) e^{2\pi i t z} dt$$

*is holomorphic in the upper half plane, i.e  $f \in H(\Pi^+)$ . Moreover  $f$  restricted to horizontal lines in  $\Pi^+$  is bounded in the  $L^2$  norm and the bound is independent of the horizontal line. That is if  $z = x + iy$ , then*

$$\sup_{0 < y < \infty} \int_{-\infty}^\infty |f(x + iy)|^2 dx < \infty$$

*Proof.* Let  $z \in \Pi^+$ . Then there exists a  $\delta$  such that  $0 < \delta < \text{Im}(z)$ . Since  $\Pi^+$  is open, there exists a sequence  $z_n$  in  $\Pi^+$  such that  $\delta < \text{Im}(z_n)$ , and  $z_n \rightarrow z$ . Also

$$\begin{aligned} |e^{2\pi i t z_n} - e^{2\pi i t z}|^2 &= |(e^{2\pi i t z_n} - e^{2\pi i t z})^2| \\ &= |e^{4\pi i t z_n} + e^{4\pi i t z} - 2e^{2\pi i t(z+z_n)}| \\ &\leq |e^{4\pi i t z_n}| + |e^{4\pi i t z}| + 2|e^{2\pi i t(z+z)}| \\ &= e^{-4\pi t \text{Im}(z_n)} + e^{-4\pi t \text{Im}(z)} + 2e^{-2\pi t \text{Im}(z_n+z)} \\ &\leq e^{-4\pi t \delta} + e^{-4\pi t \delta} + 2e^{-4\pi t \delta} \\ &= 4e^{-4\pi t \delta} \end{aligned}$$

Now since

$$\int_0^\infty 4e^{-4\pi t \delta} dt = \frac{4}{2\pi \delta}$$

the integrand is  $L^1$  in  $(0, \infty)$  and therefore by dominated convergence theorem,

$$\lim_{n \rightarrow \infty} \int_0^\infty |e^{2\pi i t z_n} - e^{2\pi i t z}|^2 dt = 0$$

Now therefore if  $w \in \Pi^+$  and  $w_n \rightarrow w$ , then

$$\begin{aligned} f(w_n) - f(w) &= \int_0^\infty F(t)(e^{2\pi i t w_n} - e^{2\pi i t w}) dt \\ &\leq \|F\|_2 \|e^{2\pi i t w_n} - e^{2\pi i t w}\|_2 \end{aligned}$$

By assumption  $\|F\|_2$  is a finite quantity and from above result  $\|e^{2\pi i t w_n} - e^{2\pi i t w}\|_2 \rightarrow 0$  as  $w_n \rightarrow w$ . Therefore  $f(w_n) \rightarrow f(w)$ . Since our choice of  $w$  was arbitrary, this implies that  $f$  is continuous everywhere in  $\Pi^+$ . Also if  $\gamma$  is any triangle in  $\Pi^+$ , then  $\gamma$  can be parametrized by  $s \in [a, b]$  as  $\gamma(s) = x(s) + iy(s)$ , where  $y(s) > 0$  since  $\gamma \in \Pi^+$ . Now let  $M = \inf_{s \in [a, b]} |\gamma'(s)|$  (here we take only differentiable points in  $\gamma$  for infimum, since the non-differentiable points will be measure zero and adds nothing to the integral). Also since  $\gamma \in \Pi^+$ , there exist an  $m > 0$  such that  $y(s) > m$  for all  $s$ . Then we get that

$$\begin{aligned} \int_\gamma \int_0^\infty |F(t)e^{itz}| dt d|z| &= \int_a^b \int_0^\infty |F(t)|e^{-ty(s)} dt |\gamma'(s)| ds \\ &\leq M \int_a^b \int_0^\infty |F(t)|e^{-mt} dt ds \\ &= M(b-a) \int_0^\infty |F(t)|e^{-mt} dt \\ &\leq M(b-a) \|F\|_{L^2((0, \infty))} \cdot \left( \int_0^\infty (e^{-mt})^2 dt \right)^{1/2} \end{aligned}$$

Therefore by Fubini's Theorem we can change the order of integration and we get,

$$\begin{aligned} \int_\gamma f(z) dz &= \int_\gamma \int_0^\infty F(t)e^{2\pi i tz} dt dz \\ &= \int_0^\infty F(t) \int_\gamma e^{2\pi i tz} dz dt \\ &= \int_0^\infty F(t) \cdot 0 dt \\ &= 0 \end{aligned}$$

Then by Morera's theorem,  $f$  is holomorphic everywhere in the upper half plane.

Now consider  $z = x + iy$ , then  $f$  can be written as

$$f(x + iy) = \int_0^\infty F(t)e^{2\pi i t(x+iy)} dt = \int_0^\infty F(t)e^{-2\pi ty}e^{2\pi i tx} dt$$

Now for a fixed  $y \in \mathbb{R}$ , consider  $g_y(t) = F(t)e^{-2\pi ty}$ . Then from above equation  $f(x + iy) = \widehat{g_y}(x)$ . By Holder's inequality we get that

$$\begin{aligned} \int_0^\infty |g_y(t)| dt &= \int_0^\infty |F(t)e^{-2\pi ty}| dt \\ &\leq \left( \int_0^\infty |F(t)|^2 dt \right)^{\frac{1}{2}} \left( \int_0^\infty e^{-4\pi ity} dt \right)^{\frac{1}{2}} \end{aligned}$$

where both of the integrals are finite. Therefore  $g_y \in L^1(0, \infty)$ . Again since  $0 < e^{-x} \leq 1$  when  $0 \leq x < \infty$ , we know that

$$\int_0^\infty |g_y(t)|^2 dt = \int_0^\infty |F(t)|^2 |e^{-2\pi ty}|^2 dt \leq \int_0^\infty |F(t)|^2 dt$$

which is again finite. Therefore  $g \in L^2(0, \infty)$ . Therefore by Plancherel's theorem (refer theorem ??) on  $g_y$  we get that,

$$\int_{-\infty}^\infty |f(x + iy)|^2 dx = \int_{-\infty}^\infty |\widehat{g_y}(x)|^2 dx = \int_{-\infty}^\infty |g_y(t)|^2 dt = \int_0^\infty |g_y(t)|^2 dt$$

Note that the change in limit in the last integral is because  $g_y(t) = 0$  when  $t < 0$ . But from last inequality

$$\int_0^\infty |g_y(t)|^2 dt \leq \int_0^\infty |F(t)|^2 dt$$

which is finite since  $F \in L^2(0, \infty)$ , and independent of  $y$ . Therefore

$$\sup_{0 < y < \infty} \int_{-\infty}^\infty |f(x + iy)|^2 dx \leq \int_0^\infty |F(t)|^2 dt$$

□

**Proposition 4.1.2.** *Let  $0 < A < \infty$  and  $F \in L^2(-A, A)$ . Then  $f : \mathbb{C} \rightarrow \mathbb{C}$  defined as*

$$f(z) = \int_{-A}^A F(t)e^{2\pi itz} dt$$

*is an entire function which satisfy*

$$|f(z)| \leq Ce^{2\pi A|z|}$$

*for some constant  $C$  and  $f$  when restricted to horizontal lines is bounded in the  $L^2$  norm. That is if  $f(z)$  is written as  $f(x + iy)$  then*

$$\int_{-\infty}^\infty |f(x + iy)|^2 dx < \infty$$

*Proof.* Like in proof of the above theorem for  $z \in \mathbb{C}$  we write  $z = x + iy$ . Then

$$f(x + iy) = \int_{-A}^A F(t) e^{2\pi i t(x+iy)} dt = \int_{-A}^A F(t) e^{-2\pi t y} e^{2\pi i t x} dt$$

Therefore since  $|e^{2\pi i t x}| = 1$ ,

$$|f(x + iy)| \leq \int_{-A}^A |F(t)| e^{-2\pi t y} dt \leq e^{2\pi A|y|} \int_{-A}^A |F(t)| dt$$

We know that  $F \in L^2(-A, A)$ . But since  $(-A, A)$  under Lebesgue measure is a finite measure space,  $f \in L^1(-A, A)$ . And therefore the last integral above is finite. Let's call it  $C$ . Also since  $e^{\text{Im}(z)} \leq e^z$ , it follows that

$$|f(z)| \leq C e^{2\pi A|z|}$$

Now consider points  $z \in \mathbb{C}$ . Then for  $\delta \in \mathbb{C}$ ,

$$\begin{aligned} f(z + \delta) - f(z) &= \int_{-A}^A F(t) (e^{it(z+\delta)} - e^{itz}) dt \\ &\leq \left( \int_{-A}^A |F(t)|^2 dt \right)^{\frac{1}{2}} \left( \int_{-A}^A |e^{it(z+\delta)} - e^{itz}|^2 dt \right)^{\frac{1}{2}} \end{aligned}$$

Now the first integral is finite since  $F \in L^2(-A, A)$ . Again since  $e^{itz}$  is a continuous function in a finite measure space it is in  $L^2(-A, A)$  and by a change of variable  $r = At$ , we can use proposition ?? and conclude that the second integral converge to zero as  $\delta \rightarrow 0$ . Therefore  $f$  is continuous in the whole of  $\mathbb{C}$ .

Moreover if  $\gamma$  is any closed path in  $\mathbb{C}$ , and  $\Omega$  is the interior of the space enclosed by  $\gamma$ , then  $(-A, A) \times \Omega$  is a finite measure space. Therefore by Fubini's Theorem we change of order of integration and

$$\int_{\gamma} f(z) dz = \int_{\gamma} \int_{-A}^A F(t) e^{2\pi i t z} dt dz = \int_{-A}^A F(t) \int_{\gamma} e^{2\pi i t z} dz dt = \int_{-A}^A F(t) \cdot 0 dt = 0$$

since  $e^{2\pi i t z}$  is entire. Then by Morera's theorem,  $f$  is entire.

We'll now prove that restrictions of  $f$  in to horizontal lines are bounded in the  $L^2$  norm. For this let

$$g_y(t) = F(t) e^{-2\pi t y}$$

Then from the first equation in the proof, we get that

$$f(x + iy) = \int_{-A}^A g_y(t) e^{2\pi i t x} dt$$

which implies that  $f$  restricted to horizontal lines is the inverse Fourier transform of  $g_y$  in the domain  $(-A, A)$ . Also by the definition of  $g_y$ ,

$$\int_{-A}^A |g_y(t)|^2 dt = \int_{-A}^A |F(t)|^2 e^{-4\pi ty} dt \leq e^{|4\pi Ay|} \int_{-A}^A |F(t)|^2 dt$$

Since by assumption  $F \in L^2(-A, A)$ ,  $g_y \in L^2(-A, A)$ . Since  $(-A, A)$  is a finite measure space  $g_y \in L^1(-A, A)$ . Now using Plancherel's theorem (??), we get that

$$\int_{-\infty}^{\infty} |f(x + iy)|^2 dx = \int_{-\infty}^{\infty} |\widehat{g_y}(x)|^2 dx = \int_{-A}^A |g_y(t)|^2 dt$$

Therefore,  $f$  restricted to horizontal lines is bounded in  $L^2$ .  $\square$

## 4.2 Paley Wiener Theorems

What is interesting is that the converse of the above two results is also true. These are known as the classical Paley Wiener theorems. With the help of these we can extend the definition of Fourier transforms to holomorphic functions in the upper half plane and entire functions of exponential type (functions where  $|f(z)| \leq Ce^{2\pi A|z|}$ ).

**Theorem 4.2.1** (Paley Wiener Theorem). *Let  $f \in H(\Pi^+)$  and*

$$\sup_{0 < y < \infty} \frac{1}{2\pi} \int_{\mathbb{R}} |f(x + iy)|^2 dx = C < \infty$$

*Then there exists an  $F \in L^2(0, \infty)$  such that for  $z \in \Pi^+$ ,*

$$f(z) = \int_0^{\infty} F(t) e^{2\pi itz} dt$$

*and*

$$\int_0^{\infty} |F(t)|^2 dt = C$$

*Note.* Let's analyse this intuitively. Rewriting the second equation in the theorem, we see that

$$f(x + iy) = e^{-2\pi ty} \int_0^{\infty} F(t) e^{2\pi itx} dt$$

which implies

$$\int_0^{\infty} F(t) e^{2\pi itx} dt = e^{2\pi ty} f(x + iy)$$



From this we see that the required function  $F$  is the inverse transform of  $f(x + iy)e^{2\pi ty}$  along the horizontal line  $\text{Im}(z) = y$ . Therefore a good candidate is

$$F(t) = \int_{\mathbb{R}} f(x + iy)e^{2\pi ty}e^{-2\pi itx} dx = \int_{\text{Im}(z)=y} f(z)e^{-2\pi itz} dz$$

Here the left hand side of the equation seems to be depending on only  $t$ , while the right hand side seems to depend on  $y$ . So we will have that the value of  $F(t)$  is independent of  $y$ , i.e.  $F(t)$  will depend only on the choice of the function  $f$  and not the horizontal lines in which  $f$  is evaluated.

*Proof.* We'll first try to prove that our choice of  $F$  from above is independent of  $y$ , i.e. the horizontal lines. Fix  $y$ , where  $0 < y < \infty$  and for each  $\alpha > 0$ , consider the rectangular path  $\Gamma_\alpha$  with vertices at  $\pm\alpha + i$  and  $\pm\alpha + iy$ . Then since  $\Gamma_\alpha \subset \Pi^+$  and for a fixed  $t \in \mathbb{R}$   $f, e^{-2\pi itz} \in H(\Pi^+)$ ,

$$\int_{\Gamma_\alpha} f(z)e^{-2\pi itz} dz = 0 \quad (4.1)$$

For  $\beta \in \mathbb{R}$  and  $I = [1, y]$  if  $1 < y$  (otherwise let  $I = [y, 1]$ ) let

$$\Phi(\beta) = \int_I f(\beta + iu)e^{-2\pi it(\beta + iu)} du$$

Now by Cauchy Schwarz inequality, we get that

$$|\Phi(\beta)|^2 \leq \left( \int_I |f(\beta + iu)|^2 du \right) \left( \int_I e^{4\pi\beta u} dt \right) \quad (4.2)$$

where the last integral is finite being the integral of a continuous function in a compact interval. And as for the first integral, since the space is  $\sigma$ -finite, by Fubini-Tonelli theorem

$$\begin{aligned} \int_{\mathbb{R}} \int_I |f(\beta + iu)|^2 du d\beta &= \int_I \int_{\mathbb{R}} |f(\beta + iu)|^2 d\beta du \\ &\leq C|y - 1| \end{aligned}$$

where  $C$  is the constant in the assumption that restriction of  $f$  into horizontal lines are uniformly bounded in  $L^2(\mathbb{R})$ .

Hence  $\int_I |f(\beta + iu)|^2 du$  is integrable and therefore there exist a sequence  $\{\alpha_j\}$  such that  $\alpha_j \rightarrow \infty$  as  $j \rightarrow \infty$  and

$$\lim_{j \rightarrow \infty} \int_I f(\alpha_j + iu) du = \lim_{j \rightarrow \infty} \int_I f(-\alpha_j + iu) du = 0 \quad (4.3)$$

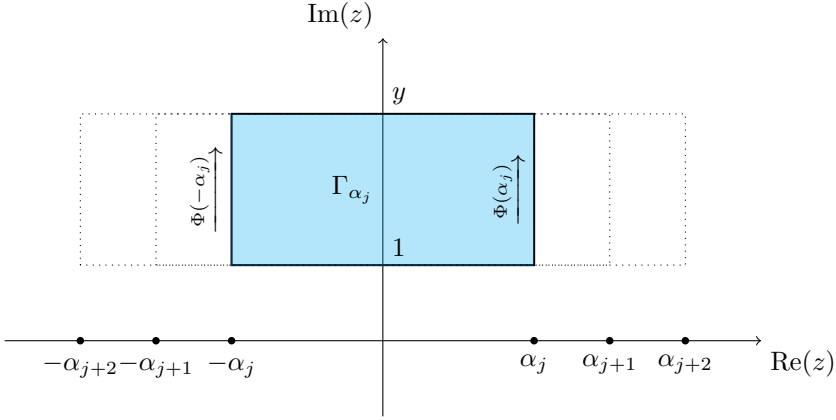


Figure 4.1: Indicative diagram shows the domains of integration

Therefore  $\Phi(\alpha_j) \rightarrow 0, \Phi(-\alpha_j) \rightarrow 0$  as  $j \rightarrow \infty$ .

Also by ??,

$$\begin{aligned} \int_{-\alpha_j+i}^{-\alpha_j+yi} f(z)e^{-2\pi itz} dz + \int_{-\alpha_j+yi}^{\alpha_j+yi} f(z)e^{-2\pi itz} dz &= \int_{-\alpha_j+i}^{\alpha_j+i} f(z)e^{-2\pi itz} dz \\ &+ \int_{\alpha_j+i}^{\alpha_j+yi} f(z)e^{-2\pi itz} dz \end{aligned}$$

Now taking limit as  $j \rightarrow \infty$  on both sides and by ??, we get that

$$\lim_{j \rightarrow \infty} \int_{-\alpha_j+yi}^{\alpha_j+yi} f(z)e^{-2\pi itz} dz = \lim_{j \rightarrow \infty} \int_{-\alpha_j+i}^{\alpha_j+i} f(z)e^{-2\pi itz} dz$$

Therefore,

$$\int_{\mathbb{R}} f(x+iy)e^{-2\pi it(x+iy)} dx = \int_{\mathbb{R}} f(x+i)e^{-2\pi it(x+i)} dx$$

Now for a fixed  $y > 0$  and  $f_y(x) = f(x+iy)$ , take  $F(t) = e^{2\pi ty} \widehat{f_y}(t)$ , where  $\widehat{f_y}$ , is

the Fourier transform of  $f_y$ . Then by previous equality we get that,

$$\begin{aligned}
 F(t) &= e^{2\pi ty} \widehat{f_y}(t) \\
 &= e^{2\pi ty} \int_{\mathbb{R}} f(x + iy) e^{-2\pi itx} dx \\
 &= \int_{\mathbb{R}} f(x + iy) e^{-2\pi t(x + iy)} dx \\
 &= \int_{\mathbb{R}} f(x + i) e^{-2\pi t(x + i)} dx \\
 &= e^{2\pi t} \int_{\mathbb{R}} f(x + i) e^{-2\pi itx} dx \\
 &= e^{2\pi t} \widehat{f_1}(t)
 \end{aligned}$$

Thus we see that  $F(t)$  is independent of our choice of horizontal line.

Now by Plancherel's theorem (??),  $\|f_y\|_2 = \|\widehat{f_y}\|_2$ , and for every  $y > 0$

$$\int_{\mathbb{R}} e^{-4\pi ty} |F(t)|^2 dt = \int_{\mathbb{R}} |\widehat{f_y}(t)|^2 dt = \int_{\mathbb{R}} |f_y(t)|^2 dt \leq C$$

where  $C$  is the constant in the assumption of the theorem. Now by monotone convergence theorem, as  $y \rightarrow \infty$ , this shows that  $F(t) = 0$  almost everywhere in  $(-\infty, 0)$ . Again as  $y \rightarrow 0$ , we get that

$$\int_0^\infty |F(t)|^2 dt \leq C$$

Again since  $\widehat{f_y}(t) = F(t)e^{-2\pi ty}$  and  $F(t) = 0$  almost everywhere in  $(-\infty, 0)$ , by Cauchy Schwarz inequality, we get

$$\int_{\mathbb{R}} |\widehat{f_y}(t)| dt = \int_0^\infty F(t) e^{-2\pi ty} dy < \infty$$

Hence  $\widehat{f_y} \in L^1(\mathbb{R})$  and by Fourier inversion

$$f(z) = f_y(x) = \int_{\mathbb{R}} \widehat{f_y}(t) e^{2\pi itx} dt = \int_0^\infty F(t) e^{2\pi it(x + iy)} dt = \int_0^\infty F(t) e^{2\pi itz} dz$$

And from ?? it follows that

$$\int_0^\infty |F(t)|^2 dt = C$$

□

**Theorem 4.2.2** (Paley Wiener Theorem). *Suppose  $A$  and  $C$  are positive constants and  $f$  is an entire function such that for all  $z \in \mathbb{C}$ ,*

$$|f(z)| \leq Ce^{2\pi A|z|}$$

*and restriction of  $f$  into horizontal lines is in  $L^2$ , i.e*

$$\int_{-\infty}^{\infty} |f(x + iy)|^2 dx < \infty$$

*Then there exists an  $F \in L^2(-A, A)$  such that for all  $z \in \mathbb{C}$ ,*

$$f(z) = \int_{-A}^A F(t) e^{2\pi itz} dt$$

*Proof.* Before we start analysing on  $f$ , we'll define  $f_\epsilon(x) = f(x) e^{-2\pi\epsilon|x|}$

**Step 1:** For each  $\alpha \in \mathbb{R}$  define  $\Gamma_\alpha$  to be the ray,

$$\Gamma_\alpha(s) = se^{i\alpha}$$

where  $0 \leq s < \infty$  and

$$\Pi_\alpha = \{w \mid \operatorname{Re}(we^{i\alpha}) > A\}$$

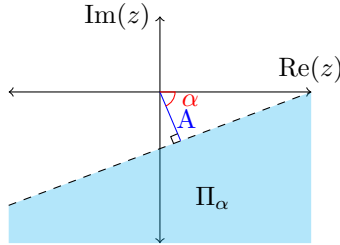


Figure 4.2: Region  $\Pi_\alpha$

By assumption, we know that  $|f(se^{i\alpha})| \leq Ce^{2\pi As}$  and therefore

$$\left| f(se^{i\alpha}) e^{-2\pi w se^{i\alpha}} \right| = \left| f(se^{i\alpha}) e^{-\operatorname{Re}(3\pi w se^{i\alpha})} \right| \leq Ce^{-2\pi[\operatorname{Re}(we^{i\alpha}) - A]s} = Ce^{-2\pi\tau s}$$

where  $\tau > 0$ , since  $w \in \Pi_\alpha$ . Let  $T$  is any triangular path in  $\Pi_\alpha$ . Since  $T$  is compact and  $\operatorname{Re}(we^{i\alpha}) - A$  is continuous and positive in  $\Pi_\alpha$ , there is a  $w = \tau_k$

where  $\operatorname{Re}(we^{i\alpha} - A)$  is minimum and positive. Then

$$\begin{aligned} \int_T |\Phi_\alpha(w)| \, dw &= \int_T e^{i\alpha} \left| \int_0^\infty f(se^{i\alpha}) e^{-2\pi w e^{i\alpha}} \, ds \right| dw \\ &\leq \int_T e^{i\alpha} \int_0^\infty |f(se^{i\alpha}) e^{-2\pi w e^{i\alpha}}| \, ds \, dw \\ &\leq \int_T e^{i\alpha} \int_0^\infty C e^{-2\pi \tau_k s} \, ds \, dw \\ &\leq C e^{i\alpha} \mu(T^\circ) \int_0^\infty e^{-2\pi \tau_k s} \, ds \end{aligned}$$

is finite where  $\mu(T^\circ)$  is the lebesgue measure of the interior of the triangle  $T$ . Now we can use Fubini-Tonelli theorem and get

$$\begin{aligned} \int_T \Phi_\alpha(w) \, dw &= \int_T e^{i\alpha} \int_0^\infty f(se^{i\alpha}) e^{-2\pi w se^{i\alpha}} \, ds \, dw \\ &= e^{i\alpha} \int_0^\infty f(se^{i\alpha}) \int_T e^{-2\pi w se^{i\alpha}} \, dw \, ds \\ &= e^{i\alpha} \int_0^\infty f(se^{i\alpha}) \cdot 0 \, ds \\ &= 0 \end{aligned}$$

Hence by Morera's theorem, we get that  $\Phi_\alpha$  is analytic everywhere in  $\Pi_\alpha$ . For our further analysis we'll be focusing on  $\Phi_0$  and  $\Phi_\pi$ , therefore we'll explicitly write them.

$$\boxed{\Phi_0(w) = \int_0^\infty f(s) e^{-2\pi w s} \, ds, \quad \Phi_\pi(w) = - \int_{-\infty}^0 f(s) e^{-2\pi w s} \, ds}$$

Now for  $\alpha = 0$  and a triangle  $T$  in  $\{w \in \mathbb{C} \mid \operatorname{Re}(w) > 0\}$ , by Cauchy Schwarz inequality we get,

$$\begin{aligned} \int_T |\Phi_0(w)| \, dw &\leq \int_T \int_0^\infty |f(s) e^{-2\pi w s}| \, ds \, dw \\ &\leq \int_T \|f\|_2 \left( \int_0^\infty |e^{-2\pi w s}|^2 \, ds \right)^{\frac{1}{2}} dw \end{aligned}$$

where the inner integral is finite since there exists a  $\tau \in T$  such that

$$\operatorname{Re}(\tau) = \inf\{\operatorname{Re}(t) \mid t \in T\}$$

and therefore

$$\int_0^\infty |e^{-2\pi ws}|^2 ds \leq \int_0^\infty e^{-4\pi \tau s} ds < \infty$$

Hence the integral is finite, and we can use Fubini-Tonelli theorem to interchange the order of the integration to get,

$$\begin{aligned} \int_T \Phi_0(w) dw &\leq \int_T \int_0^\infty f(s) e^{-2\pi ws} ds dw \\ &= \int_0^\infty f(s) \int_T e^{-2\pi ws} dw ds \\ &= 0 \end{aligned}$$

Then Morera's theorem asserts that  $\Phi_0$  is analytic in  $\{w \in \mathbb{C} \mid \operatorname{Re}(w) > 0\}$ . By similar argument we can show that  $\Phi_\pi$  is analytic in  $\{w \in \mathbb{C} \mid \operatorname{Re}(w) < 0\}$ . Now we can write our candidate for  $F(t)$  as,

$$\begin{aligned} F(t) &= \Phi_0(it) - \Phi_\pi(it) \\ &= \int_0^\infty f(x) e^{-2\pi itx} dx + \int_{-\infty}^0 f(x) e^{-2\pi itx} dx \\ &= \int_{-\infty}^\infty f(x) e^{-2\pi itx} dx \end{aligned}$$

**Step 2:** But we have to show that  $F(t)$  vanishes almost everywhere outside  $(-A, A)$ . Let  $f_\epsilon(x) = f(x) e^{-2\pi \epsilon |x|}$  where  $\epsilon > 0$ , then

$$\int_{-\infty}^\infty |f(x) - f_\epsilon(x)|^2 dx = \int_{-\infty}^\infty (1 - e^{-2\pi \epsilon |x|})^2 |f(x)|^2 dx$$

Since  $(1 - e^{-2\pi \epsilon |x|})^2 |f(x)|^2 < |f(x)|^2$  and  $f|_{\mathbb{R}} \in L^2(\mathbb{R})$ , by Lebesgue dominated convergence theorem, we get  $f_\epsilon \rightarrow f$  in  $L^2(\mathbb{R})$ . Hence it is enough to show that for all  $t$  such that  $A < |t|$ ,

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}} f_\epsilon(x) e^{-2\pi itx} dx = 0$$

Since

$$\begin{aligned} \Phi_0(\epsilon + it) &= \int_0^\infty f(x) e^{-2\pi(\epsilon + it)x} dx \\ &= \int_0^\infty \underbrace{f(x) e^{-2\pi \epsilon |x|}}_{f_\epsilon(x)} e^{-2\pi itx} dx \\ &= \int_0^\infty f_\epsilon(x) e^{-2\pi itx} dx \end{aligned}$$

and

$$\begin{aligned}\Phi_\pi(-\epsilon + it) &= - \int_{-\infty}^0 f(x) e^{-2\pi(-\epsilon + it)x} dx \\ &= - \int_{-\infty}^0 \underbrace{f(x) e^{-2\pi\epsilon|x|}}_{f_\epsilon(x)} e^{-2\pi itx} dx \\ &= \int_{-\infty}^0 f_\epsilon(x) e^{-2\pi itx} dx\end{aligned}$$

we get that,

$$\int_{\mathbb{R}} f_\epsilon(x) e^{-2\pi itx} dx = \Phi_0(\epsilon + it) - \Phi_\pi(-\epsilon + it)$$

Therefore it is enough to show that for  $|t| > A$ , the right side of above equation converge to 0 as  $\epsilon \rightarrow 0$ . For that we'll show that  $\Phi_0$  and  $\Phi_\pi$  are analytic continuations of  $\Phi_{\frac{\pi}{2}}$  if  $t > A$  or  $\Phi_{-\frac{\pi}{2}}$  if  $t < -A$  on the intersections of their respective domains. Then  $\Phi_0(\epsilon + it) + \Phi_\pi(-\epsilon + it) \rightarrow 0$  as  $\epsilon \rightarrow 0$  will follow from the fact that every  $\Phi_\alpha$  is continuous.

For that we'll show that for any  $\alpha, \beta$  such that  $0 < \beta - \alpha < \pi$ , and  $\Phi_\alpha, \Phi_\beta$  defined on  $\Pi_\alpha$  and  $\Pi_\beta$  respectively

$$\Phi_\alpha(x) = \Phi_\beta(x), \quad x \in \Pi_\alpha \cap \Pi_\beta$$

Let  $\gamma = \frac{\alpha + \beta}{2}, \eta = \cos \frac{\beta - \alpha}{2}$ , and let  $w = |w|e^{-i\gamma}$  then,

$$\operatorname{Re}(we^{i\alpha}) = \operatorname{Re}(|w|e^{i(\alpha - \gamma)}) = |w| \cos\left(\frac{\alpha - \beta}{2}\right) = \operatorname{Re}(|w|e^{i(\beta - \gamma)}) = \operatorname{Re}(we^{i\beta})$$

Then  $w \in \Pi_\alpha \cap \Pi_\beta$  if  $|w| > A/\eta$ . Now for a fixed  $r \in \mathbb{R}$  consider the circular path  $\Gamma_r = re^{it}$  were  $\alpha \leq t \leq \beta$  and the integral

$$\int_{\Gamma_r} f(z) e^{-2\pi wz} dz$$

Also we get that

$$\operatorname{Re}(-wz) = \operatorname{Re}(-|w|r e^{i(t - \gamma)}) = -|w|r \cos(t - \gamma) \leq -|w|r\eta$$

Since  $|z| = r$  in  $\Gamma_r$  and  $|f(z)| \leq Ce^{2\pi A|z|}$  by assumption,

$$|f(z) e^{-2\pi wz}| \leq Ce^{-2\pi(|w|\eta - A)r}$$

Therefore if  $|w| > A/\eta$ ,

$$\left| \int_{\Gamma_r} f(z) e^{-2\pi wz} dz \right| = \int_{\Gamma_r} |f(z) e^{-2\pi wz}| dz \leq Ce^{-2\pi(|w|\eta - A)r} \ell(\Gamma_r)$$

where  $\ell(\Gamma_r)$  is the arc length of  $\Gamma_r$ . Now when  $r \rightarrow \infty$ , since the arc length increases linearly and the exponent decays exponentially, we get that

$$\lim_{r \rightarrow \infty} \int_{\Gamma_r} f(z) e^{-2\pi w z} dz = 0$$

□



# Chapter 5

## Further Topics

From our understanding our Fourier analysis in the circle, line and for special classes of holomorphic functions, we can step into a world of further analysis where answers to seemingly simple problems are still hidden. In this chapter we will discuss some of them and try to understand what tools are required for them. This will be a brief outlook into the future of the topic presenting some open problems and probing directions for further research.

### 5.1 Fourier Transforms in $\mathbb{R}^n$

Before we begin our analysis on  $\mathbb{R}^n$ , unlike  $\mathbb{R}$ , we define a new class of functions called the Schwartz functions.

**Definition 5.1.1** (Schwartz Class). A smooth function  $f : \mathbb{R}^n \rightarrow \mathbb{C}$ ,  $f$  is called a *Schwartz function* if for any given multi index  $\alpha, \beta$ , there exists a positive constant  $C_{\alpha, \beta}$  such that

$$\rho_{\alpha, \beta} = \sup_{x \in \mathbb{R}^n} |x^\alpha (D^\beta f)x| = C_{\alpha, \beta} < \infty$$

Here  $\rho_{\alpha, \beta}(f)$  is called *Schwartz seminorm of  $f$* . The collection of all such functions is called the *Schwartz space* of  $\mathbb{R}^n$  and is denoted by  $\mathcal{S}(\mathbb{R}^n)$ . See [Gra14, Section 2.2] for the definition of multi index and derivatives.

We also get that  $\mathcal{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$  for any  $1 \leq p < \infty$  [Gra14, Proposition 2.2.6 p. 106]. Moreover  $\mathcal{S}(\mathbb{R}^n)$  is dense in  $L^p(\mathbb{R}^n)$ . Therefore if we define Fourier transform in  $\mathcal{S}(\mathbb{R}^n)$ , using Plancherel's theorem we'll be able to define Fourier transform for all  $L^p(\mathbb{R}^n)$ .

**Definition 5.1.2.** Let  $f \in \mathcal{S}(\mathbb{R}^n)$ , then we define the Fourier transform of  $f$ , as

$$\widehat{f}(t) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i t \cdot x} dx$$

where  $x \cdot y$  denote the standard inner product in  $\mathbb{R}^n$ .

Notice that  $\widehat{f}$  is well defined since  $\mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R})$ . Therefore

$$|\widehat{f}(t)| \leq \int_{\mathbb{R}^n} |f(x) e^{-2\pi i t \cdot x}| dx \leq \int_{\mathbb{R}^n} |f(x)| dx = \|f\|_1$$

The reason why Schwartz class is an attractive space for Fourier analysis is because Fourier transform is a homeomorphism on the Schwartz class [Gra14, Corollary 2.2.15]

## 5.2 Restriction Conjecture

One celebrated conjecture in Fourier analysis is the 'Restriction conjecture' proposed by Elias M Stein.

**Problem 5.2.1** (Restriction Conjecture). *The restriction problem asks when the inequality*

$$\|\widehat{f}|_{S^{n-1}}\|_{L^q(S^{n-1})} \leq C_{n,p,q} \|f\|_{L^p}$$

*holds. Where  $f \in L^p(\mathbb{R})$ ,  $S^{n-1}$  is the unit sphere in  $\mathbb{R}^n$  and  $C_{n,p,q}$  is a constant which depend only on  $n, p$  and,  $q$ .*

When  $q = 2$ , the best possible result is given by the Stein-Tomas theorem.

**Theorem 5.2.1** (Stein Tomas Theorem). *For every  $n \geq 2$ , there is a constant  $C_{n,p}$  such that for all  $f \in L^p(\mathbb{R}^n)$ ,*

$$\|\widehat{f}|_{S^{n-1}}\|_{L^2(S^{n-1})} \leq C_{n,p} \|f\|_{L^p}$$

*where  $p \leq \frac{2n+2}{n+3}$*

*Proof.* See [MS13, Theorem 11.1 p. 288]

□

## 5.3 Another problem

Stein Tomas theorem can be used utilized to analyse the following problem from [Gra+17, Problem.no 3]. But since analysing that is beyond the scope of our understanding, we state the problem and wrap up.

**Problem 5.3.1** (Functions whose Fourier transforms vanishes on the sphere).

Let  $n \geq 2$ . Does there exist a function  $f \in L^{\frac{2n+2}{n+3}}(\mathbb{R}^n)$  such that

$$\widehat{f}|_{S^{n-1}} = 0$$

and

$$|1 - |\xi|^2|^{-\frac{1}{2}} f \notin L^2(\mathbb{R}^n)$$



# Appendix A

## Results from Measure Theory

Here we'll discuss some important results from measure theory which are essential for our subject. We already defined what is an  $L^p$  function in a given space at ??.

**Proposition A.0.1.** *Continuous functions in  $\mathbb{T}$ , (refer ??) are dense in  $L^p(\mathbb{T})$  for  $1 \leq p < \infty$ .*

*Proof.* This is a direct consequence of [Rud87, Theorem 3.14 on p. 69]. Since  $\mathbb{T}$  is identified with  $[0, 1)$ , all continuous functions in  $\mathbb{T}$  are compactly supported.  $\square$

**Proposition A.0.2.** *Let  $C_c(\mathbb{R})$  be the set of all compactly supported continuous functions in  $\mathbb{R}$ , then  $C_c(\mathbb{R})$  is dense in  $L^p(\mathbb{R})$ . This is [Rud87, Theorem 3.14 p. 69].*

**Proposition A.0.3.**  *$L^1(\mathbb{R}) \cap L^2(\mathbb{R})$  is dense in  $L^2(\mathbb{R})$ .*

*Proof.* Let  $C_c(\mathbb{R})$  denote the set of compactly supported continuous functions in  $\mathbb{R}$ . Since every function is continuous and compactly supported,  $C_c(\mathbb{R}) \subset L^p(\mathbb{R})$  for all  $1 \leq p < \infty$ . Therefore  $C_c(\mathbb{R}) \subset L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ . Then by [Rud87, Theorem 3.14 on p. 69]  $C_c(\mathbb{R})$  is dense in  $L^2(\mathbb{R})$  and therefore  $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$  is dense in  $L^2(\mathbb{R})$ .  $\square$

If you follow the proof of the above theorem close enough, you'll see that we can make a stronger claim. Since,  $C_c(\mathbb{R}) \subset L^p(\mathbb{R})$  for all  $1 \leq p < \infty$ ,

$$C_c(\mathbb{R}) \subset \bigcap_{1 \leq p < \infty} L^p(\mathbb{R})$$

and therefore again by [Rud87, Theorem 3.14 on p. 69],  $\bigcap_{1 \leq p < \infty} L^p(\mathbb{R})$  is dense in  $L^q(\mathbb{R})$  for all  $1 \leq q < \infty$ . We will state the generalization of this as a separate result.

**Proposition A.0.4.** *If  $f \in L^p(\mathbb{R})$ , then*

$$\lim_{\delta \rightarrow 0} \int_{\mathbb{R}} |f(x + \delta) - f(x)|^p dx = 0$$

*Proof.* Since  $f \in L^p(\mathbb{R})$ , for every  $\epsilon > 0$  there exist an  $X$  such that

$$\left( \int_{|x| > X-1} |f(x)|^p dx \right)^{\frac{1}{p}} < \epsilon$$

Therefore by Minkowski's inequality, for  $\delta \leq 1$ ,

$$\left( \int_{\mathbb{R}} |f(x + \delta) - f(x)|^p dx \right)^{\frac{1}{p}} \leq \left( \int_{-X}^X |f(x + \delta) - f(x)|^p dx \right)^{\frac{1}{p}} + 2\epsilon$$

Now since  $C_c(\mathbb{R}) \cap L^p(\mathbb{R})$  are dense in  $L^p(\mathbb{R})$  by ??, there exists a  $g \in C([-X - 1, X + 1]) \cap L^p([-X - 1, X + 1])$  such that

$$\|f - g\|_{L^p([-X-1, X+1])} = \left( \int_{-X-1}^{X+1} |f(x) - g(x)|^p dx \right)^{\frac{1}{p}} < \epsilon$$

Then by Minkowski's inequality

$$\left( \int_{-X}^X |f(x + \delta) - f(x)|^p dx \right)^{\frac{1}{p}} \leq \left( \int_{-X}^X |g(x + \delta) - g(x)|^p dx \right)^{\frac{1}{p}} + 2\epsilon$$

Now since  $g$  is continuous in a compact space  $[-X - 1, X + 1]$ , it is uniformly continuous. and since  $\epsilon$  does not depend on the therefore as  $\delta \rightarrow 0$  the above integral tends to 0. Hence the proof.  $\square$

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