# A Study in Fourier Analysis

## From circle, through the line, to the complex

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# Fourier Series

# » Structure and Topology of $\mathbb{T}$

- Defining $\mathbb{T}$ as the set of equivalence class of the relation $x \sim y \iff x - y \in \mathbb{Z}$ and identifying classes in $\mathbb{T}$ with their representative element in $[0, 1)$ as $[x] \to \{x\}$, where $\{x\}$ is the fractional part of $x$.
- Endow $\mathbb{T}$ with quotient topology by the map $f : \mathbb{R} \to \mathbb{T} := x \to [x]$
- Lebesgue measure on $\mathbb{T}$ is defined by the Lebesgue measure of its identification in $[0, 1)$.

# » Functions in $\mathbb{T}$

- Functions in $\mathbb{T}$ are identified with periodic functions in $\mathbb{R}$ with period $1$ this again can be completely characterized by their values in $[0, 1)$.
- By the quotient topology in $\mathbb{T}$, we see that continuous functions in $\mathbb{T}$ can identified with continuous functions in $\mathbb{R}$ with period $1$.
- Also by the Lebesgue measure defined on $\mathbb{T}$, we say $f \in L^p(\mathbb{T})$ if the corresponding function in $[0, 1)$ is in $L^p[0, 1)$.
- For any two function $f, g \in L^1(\mathbb{T})$, their convolution, $(f * g)(x)$ as

$$(f * g)(x) = \int_0^1 f(x-y)g(y)\ dy$$

is again in $L^1(\mathbb{T})$

# » Fourier Coefficients

* For $f \in L^1(\mathbb{T})$, and $n \in \mathbb{Z}$ we define the $n^{th}$ Fourier coefficient of $f$ as

$$\hat{f}(n) = \int_0^1 f(x)e^{-2\pi inx}\,dx$$

* Also the Fourier series of $f \in L^1(\mathbb{T})$ is defined as

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi inx}$$

* Since we are interested in the convergence of the Fourier series, we will define the symmetric and Cesàro partial sums of the Fourier seres respectively as

$$S_N(x) = \sum_{n=-N}^{N} \hat{f}(n)e^{2\pi inx} \quad \text{and} \quad \sigma_N(x) = \frac{1}{N}\sum_{n=0}^{N-1} S_n(x)$$

# » Summability Kernel

- A collection of functions $K_N \in L^1(\mathbb{T})$ are called a summability kernel if it satisfies the following properites
  1. $\int_0^1 K_N(x)dx = 1$
  2. $\int_0^1 |K_N(x)|dx \leq C$ for some constant $C > 0$
  3. $\lim_{N\to\infty} \int_\delta^{1-\delta} |K_N(x)|dx = 0$
- We prove that if $K_N$ is a summability kernel in $L^1(T)$, then $(f * K_N)(x)$ converge to $f(x)$ in $L^1(\mathbb{T})$. That is

$$\int_0^1 |f(x) - (f * K_N)(x)| \; dx$$

# » Fejér Kernel and Cesàro Convergence

- Fejér kernel defined as

$$\Delta_N(x) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x}$$

  is a summability kernel we get that $(f * \Delta_N)$ converge to $f$ in $L^1(\mathbb{T})$
- Moreover we see that $(f * \Delta_N)(x) = \sigma_N(x)$ and therefore the Cesàro partial sums of the Fourier series of $f$ converge to $f$ in $L^1(\mathbb{T})$.

# » Fourier Series in $L^2(\mathbb{T})$

- Since $\mathbb{T}$ is identified with the finite measure space $[0,1)$, we get that $L^2(\mathbb{T}) \subset L^1(\mathbb{T})$. Theorefore the Fourier coefficients and series can be defined the same way as in $L^1(\mathbb{T})$.
- Moreover we see that if $f \in L^2(\mathbb{R})$, since the Fejér kernel, $\Delta_N \in L^\infty(\mathbb{T})$, its Cesàro partial sum, $\sigma_N = (f * \Delta_N) \in L^2(\mathbb{T})$
- As in $L^1(\mathbb{T})$, we get that the Cesàro partial sums $\sigma_N$ converge to $f$ in $L^2(\mathbb{T})$. That is

$$\lim_{N\to\infty} \int_0^1 |f(x) - \sigma_N(x)| \ dx = 0$$

- The same results follow for functions in $L^p(\mathbb{T})$

# » Fejér Theorem and Pointwise Convergence

Write if needed

# Fourier Transforms in $\mathbb{R}$

# » Fourier transforms in $L^1(\mathbb{R})$

* For any $f \in L^1(\mathbb{R})$, the Fourier transform of $f$ is defined as

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i t x} \, dx$$

* $\hat{f}$ is uniformly continuous and

$$\lim_{|t| \to \infty} \hat{f}(t) = 0$$

# Riemann Lebesgue Lemma

# » Fourier Inversion

* Let $f \in L^1(\mathbb{R})$, then the inverse Fourier transform is defined as

$$\check{f}(t) = \int_{-\infty}^{\infty} f(x) e^{2\pi i t x} \, dx$$

* We see that if $f \in L^1(\mathbb{R})$, continuous at $x \in \mathbb{R}$ and its Fourier transform $\hat{f} \in L^1(\mathbb{R})$, then

$$\check{\hat{f}}(x) = f(x)$$

* Generalizing further we get that that if $f, \hat{f} \in L^1(\mathbb{R})$ then

$$\check{\hat{f}} \overset{a.e}{=} f$$

# » Fourier transforms in $L^2(\mathbb{R})$

* We consider the space $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Since it is a subspace of $L^1(\mathbb{R})$, the definition of Fourier transform and inverse transform holds good in the smaller space.
* (Plancherel's Theorem) If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then

$$\int_{\mathbb{R}} |f(x)|^2 \; dx = \int_{\mathbb{R}} |\hat{f}(t)|^2 \; dt$$

* Now since the collection of compactly supported functions in $\mathbb{R}$, $C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for all $1 \leq p < \infty$, we get that $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$.
* Moreover we see that the Fourier transform of functions in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ form a dense subset in $L^2(\mathbb{R})$. Plancherel's theorem asserts that Fourier transform in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is an isometry therefore we can extend Fourier transform to an isometry in $L^2(\mathbb{R})$.

# Homolorphic Fourier Transforms

# » Extending Domain to $\mathbb{C}$

* Fourier transform of certain functions can be extended into a holomorphic functions in certain regions. That is for $z \in \mathbb{C}$,

$$\hat{f}(z) = \int_{\mathbb{R}} f(x) e^{-2\pi i z x}\, dx$$

will be holomorphic in certain regions in $\mathbb{C}$.

* For example if $f(x) = e^{-|x|}$, then its Fourier transform, $\hat{f}(t) = \frac{1}{1+(2\pi t)^2}$ can be extended into holomorphic function in regions in the complex plane without the points $\pm\frac{i}{2\pi}$.
* We will focus on two types of functions in $L^2(\mathbb{R})$
    1. $f(x) = 0, (x < 0)$
    2. $f(x) = 0, (x \notin (-A, A))$

# » Paley Wiener Theorem 1

* Let $F \in L^2(\mathbb{R})$ such that $F(x) = 0$ in $(-\infty, 0)$. Then $f : \Pi^+ \to \mathbb{C}$ defined as,

$$f(z) = \int_0^\infty F(x) e^{2\pi i t z}\, dz$$

is in $H(\Pi^+)$ if and only if

$$\sup_{0<y<\infty} \frac{1}{2\pi} \int_{\mathbb{R}} f(x+iy)\, dx = \int_0^\infty |F(x)|^2\, dx < \infty$$

# » Paley Wiener Theorem 2

The following statements are equivalent:

1. $f : \mathbb{C} \to \mathbb{C}$ is an entire function satisfying $|f(z)| \leq Ce^{2\pi A|z|}$ and
$$\int_{-\infty}^{\infty} |f(x+iy)|^2 \, dx < \infty$$
2. There exist an $F \in L^2(\mathbb{R})$ is compactly supported in $[-A, A]$ such that
$$f(z) = \int_{-A}^{A} F(x) e^{2\pi izx} \, dx$$

# » Consequence of Paley Wiener Theorems

# Future Directions

# » Schwartz Class

- A smooth function $f\colon \mathbb{R}^n \to \mathbb{C}$, $f$ is called a *Schwartz function* if for any given multi index $\alpha, \beta$, there exists a positive constant $C_{\alpha,\beta}$ such that

$$\rho_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} \left| x^\alpha (D^\beta f) x \right| = C_{\alpha,\beta} < \infty$$

- Here $\rho_{\alpha,\beta}(f)$ is called *Schwartz seminorm of f*. The collection of all such functions is called the *Schwartz space* of $\mathbb{R}^n$ and is denoted by $\mathscr{S}(\mathbb{R}^n)$.
- Schwartz class is dense in $L^p(\mathbb{R}^n)$ for all $1 \leq p < \infty$.

# » Fourier transforms in $R^n$

* Fourier transform of $f \in \mathscr{S}(\mathbb{R}^n)$, $\hat{f} : \mathbb{R}^n \to \mathbb{C}^n$ is defined as

$$\hat{f}(\mathbf{t}) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \mathbf{t} \cdot \mathbf{x}} \, dx$$

* Parseval's identity holds in Schwartz class

$$\|\hat{f}\|_2 = \|f\|_2$$

* Fourier transform is a homeomorphism in $\mathscr{S}(\mathbb{R}^n)$.
* By Parseval's identity Fourier transform can be extended into whole of $\mathbb{R}^n$

# » Restriction Conjecture