

# A Study in Fourier Analysis

From circle, through the line, to the complex

**Joel Sleeba**

IISER Thiruvananthapuram

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# Fourier Series

» Structure and Topology of  $\mathbb{T}$ 

- \* Defining  $\mathbb{T}$  as the set of equivalence class of the relation  $x \sim y \iff x - y \in \mathbb{Z}$  and identifying classes in  $\mathbb{T}$  with their representative element in  $[0, 1)$  as  $[x] \rightarrow \{x\}$ , where  $\{x\}$  is the fractional part of  $x$ .
- \* Endow  $\mathbb{T}$  with quotient topology by the map  $f: \mathbb{R} \rightarrow \mathbb{T} := x \rightarrow [x]$
- \* Lebesgue measure on  $\mathbb{T}$  is defined by the Lebesgue measure of its identification in  $[0, 1)$ .

» Functions in  $\mathbb{T}$ 

- \* Functions in  $\mathbb{T}$  are identified with periodic functions in  $\mathbb{R}$  with period 1 this again can be completely characterized by their values in  $[0, 1)$ .
- \* By the quotient topology in  $\mathbb{T}$ , we see that continuous functions in  $\mathbb{T}$  can be identified with continuous functions in  $\mathbb{R}$  with period 1.
- \* Also by the Lebesgue measure defined on  $\mathbb{T}$ , we say  $f \in L^p(\mathbb{T})$  if the corresponding function in  $[0, 1)$  is in  $L^p[0, 1)$ .
- \* For any two functions  $f, g \in L^1(\mathbb{T})$ , their convolution,  $(f * g)(x)$  as

$$(f * g)(x) = \int_0^1 f(x - y)g(y) \, dy$$

is again in  $L^1(\mathbb{T})$

» **Fourier Coefficients**

- \* For  $f \in L^1(\mathbb{T})$ , and  $n \in \mathbb{Z}$  we define the  $n^{\text{th}}$  Fourier coefficient of  $f$  as

$$\hat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x} dx$$

- \* Also the Fourier series of  $f \in L^1(\mathbb{T})$  is defined as

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}$$

- \* Since we are interested in the convergence of the Fourier series, we will define the symmetric and Cesàro partial sums of the Fourier series respectively as

$$S_N(x) = \sum_{n=-N}^N \hat{f}(n) e^{2\pi i n x} \quad \text{and} \quad \sigma_N(x) = \frac{1}{N} \sum_{n=0}^{N-1} S_n(x)$$

» **Summability Kernel**

- \* A collection of functions  $K_N \in L^1(\mathbb{T})$  are called a summability kernel if it satisfies the following properties

1.  $\int_0^1 K_N(x) dx = 1$
2.  $\int_0^1 |K_N(x)| dx \leq C$  for some constant  $C > 0$
3.  $\lim_{N \rightarrow \infty} \int_\delta^{1-\delta} |K_N(x)| dx = 0$

- \* We prove that if  $K_N$  is a summability kernel in  $L^1(\mathbb{T})$ , then  $(f * K_N)(x)$  converge to  $f(x)$  in  $L^1(\mathbb{T})$ . That is

$$\int_0^1 |f(x) - (f * K_N)(x)| dx$$

## » Fejér Kernel and Cesàro Convergence

- \* Fejér kernel defined as

$$\Delta_N(x) = \sum_{n=-N}^N \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x}$$

is a summability kernel we get that  $(f * \Delta_N)$  converge to  $f$  in  $L^1(\mathbb{T})$

- \* Moreover we see that  $(f * \Delta_N)(x) = \sigma_N(x)$  and therefore the Cesàro partial sums of the Fourier series of  $f$  converge to  $f$  in  $L^1(\mathbb{T})$ .
- \* Therefore if  $f, g \in L^1(\mathbb{T})$  such that  $\hat{f} = \hat{g}$ , then  $f \stackrel{a.e}{=} g$

» **Fourier Series in  $L^2(\mathbb{T})$** 

- \* Since  $\mathbb{T}$  is identified with the finite measure space  $[0, 1)$ , we get that  $L^2(\mathbb{T}) \subset L^1(\mathbb{T})$ . Therefore the Fourier coefficients and series can be defined the same way as in  $L^1(\mathbb{T})$ .
- \* Moreover we see that if  $f \in L^2(\mathbb{R})$ , since the Fejér kernel,  $\Delta_N \in L^\infty(\mathbb{T})$ , its Cesàro partial sum,  $\sigma_N = (f * \Delta_N) \in L^2(\mathbb{T})$
- \* As in  $L^1(\mathbb{T})$ , we get that the Cesàro partial sums  $\sigma_N$  converge to  $f$  in  $L^2(\mathbb{T})$ . That is

$$\lim_{N \rightarrow \infty} \int_0^1 |f(x) - \sigma_N(x)| \, dx = 0$$

- \* The same results follow for functions in  $L^p(\mathbb{T})$



## » Fejér Theorem and Pointwise Convergence

If  $f \in L^1(\mathbb{T})$ , then

$$\lim_{N \rightarrow \infty} \sigma_N(x) = \frac{f(x^-) + f(x^+)}{2}$$

given that  $f(x^-)$  and  $f(x^+)$ , the left limit and right limit of  $f$  at  $x$  exists.

Therefore if  $f$  is continuous then the Cesàro partial sum converge pointwise to  $f$

# Fourier Transforms in $\mathbb{R}$

» **Fourier transforms in  $L^1(\mathbb{R})$** 

- \* For any  $f \in L^1(\mathbb{R})$ , the Fourier transform of  $f$  is defined as

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i t x} dx$$

- \*  $\hat{f}$  is uniformly continuous and

$$\lim_{|t| \rightarrow \infty} \hat{f}(t) = 0$$

» **Fourier Inversion**

- \* Let  $f \in L^1(\mathbb{R})$ , then the inverse Fourier transform is defined as

$$\check{f}(t) = \int_{-\infty}^{\infty} f(x) e^{2\pi i t x} dx$$

- \* We see that if  $f \in L^1(\mathbb{R})$ , continuous at  $x \in \mathbb{R}$  and its Fourier transform  $\hat{f} \in L^1(\mathbb{R})$ , then

$$\check{\check{f}}(x) = f(x)$$

- \* Generalizing further we get that that if  $f, \hat{f} \in L^1(\mathbb{R})$  then

$$\check{\check{f}} \stackrel{a.e}{=} f$$

» **Fourier transforms in  $L^2(\mathbb{R})$** 

- \* We consider the space  $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ . Since it is a subspace of  $L^1(\mathbb{R})$ , the definition of Fourier transform and inverse transform holds good in the smaller space.
- \* (Plancherel's Theorem) If  $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ , then

$$\int_{\mathbb{R}} |f(x)|^2 dx = \int_{\mathbb{R}} |\hat{f}(t)|^2 dt$$

- \* Now since the collection of compactly supported functions in  $\mathbb{R}$ ,  $C_c(\mathbb{R})$  is dense in  $L^p(\mathbb{R})$  for all  $1 \leq p < \infty$ , we get that  $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$  is dense in  $L^2(\mathbb{R})$ .
- \* Plancherel's theorem asserts that Fourier transform in  $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$  is an isometry therefore we can extend Fourier transform to an isometry in  $L^2(\mathbb{R})$ .

# Homomorphic Fourier Transforms

» Extending Domain to  $\mathbb{C}$ 

- \* Fourier transform of certain functions can be extended into a holomorphic functions in certain regions. That is for  $z \in \mathbb{C}$ ,

$$\hat{f}(z) = \int_{\mathbb{R}} f(x) e^{-2\pi izx} dx$$

will be holomorphic in certain regions in  $\mathbb{C}$ .

- \* For example if  $f(x) = e^{-|x|}$ , then its Fourier transform,  $\hat{f}(t) = \frac{1}{1+(2\pi t)^2}$  can be extended into holomorphic function in regions in the complex plane without the points  $\pm \frac{i}{2\pi}$ .
- \* We will focus on two types of functions in  $L^2(\mathbb{R})$ 
  1.  $f(x) = 0, (x < 0)$
  2.  $f(x) = 0, (x \notin (-A, A))$

## » Paley Wiener Theorem 1

The following statements are equivalent

1.  $f \in H(\Pi_+)$  such that

$$\sup_{0 < y < \infty} \frac{1}{2\pi} \int_{\mathbb{R}} f(x + iy) \, dx = C < \infty$$

2. There exist an  $F \in L^2(\mathbb{R})$  such that  $F$  is supported in  $(0, \infty)$  and for all  $z \in \Pi^+$

$$f(z) = \int_0^\infty F(t) e^{2\pi itz} \, dt$$

and

$$\int_0^\infty |F(t)|^2 \, dt = C$$



## » Paley Wiener Theorem 2

The following statements are equivalent:

1.  $f: \mathbb{C} \rightarrow \mathbb{C}$  is an entire function satisfying  $|f(z)| \leq Ce^{2\pi A|z|}$  and

$$\int_{-\infty}^{\infty} |f(x + iy)|^2 dx < \infty$$

2. There exist an  $F \in L^2(\mathbb{R})$  is supported in  $(-A, A)$  such that

$$f(z) = \int_{-A}^A F(x) e^{2\pi izx} dx$$

# » Consequence of Paley Wiener Theorems

## Future Directions

» **Schwartz Class**

- \* A smooth function  $f: \mathbb{R}^n \rightarrow \mathbb{C}$ ,  $f$  is called a *Schwartz function* if for any given multi index  $\alpha, \beta$ , there exists a positive constant  $C_{\alpha, \beta}$  such that

$$\rho_{\alpha, \beta} = \sup_{x \in \mathbb{R}^n} |x^\alpha (D^\beta f)x| = C_{\alpha, \beta} < \infty$$

- \* Here  $\rho_{\alpha, \beta}(f)$  is called *Schwartz seminorm of  $f$* . The collection of all such functions is called the *Schwartz space* of  $\mathbb{R}^n$  and is denoted by  $\mathcal{S}(\mathbb{R}^n)$ .
- \* Schwartz class is dense in  $L^p(\mathbb{R}^n)$  for all  $1 \leq p < \infty$ .

» **Fourier transforms in  $\mathbb{R}^n$** 

- \* Fourier transform of  $f \in \mathcal{S}(\mathbb{R}^n)$ ,  $\hat{f}: \mathbb{R}^n \rightarrow \mathbb{C}^n$  is defined as

$$\hat{f}(\mathbf{t}) = \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-2\pi i \mathbf{t} \cdot \mathbf{x}} d\mathbf{x}$$

- \* Parseval's identity holds in Schwartz class

$$\|\hat{f}\|_2 = \|f\|_2$$

- \* Fourier transform is a homeomorphism in  $\mathcal{S}(\mathbb{R}^n)$ .
- \* By Parseval's identity Fourier transform can be extended into whole of  $\mathbb{R}^n$

## » Restriction Conjecture