RESEARCH STATEMENT

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1. Introduction

Let $\{x\}$ denote the fractional part of the real number x. The saw-tooth functions $S_+(x) = \{x\}$ and $S_-(x) = \{x\} - 1$ satisfy a simple and interesting property, namely for any real-valued trigonometric polynomial g without a constant term, an integration by parts shows

$$g(x) = \int_0^1 S_+(x-u)g'(u)du = \int_0^1 S_-(x-u)g'(u)du$$

for any real x. Since $S_+ \geq 0$, $S_- \leq 0$ and the average of $S_\pm$ on $[0,1]$ equals $\pm 1/2$, the inequality

$$|g(x)| \leq \frac{1}{2} \max_y g'(y) \tag{1}$$

follows.

Denote by $\widehat{f}(t) := \int_{\mathbb{R}} f(x)e^{-2\pi ixt}dx$ the Fourier transform of an integrable function f. In the 1930's, A. Beurling was interested whether an analogue of (1) is true for functions with a 'spectral gap', i.e., real valued f with a Fourier transform $\widehat{f}$ whose support does not intersect the interval $(-1,1)$. Since the Fourier coefficients of $B_\pm$ are $a_n^\pm = (2\pi in)^{-1}$ for $n \neq 0$, he was led to search for integrable $\varphi_+ \geq 0$ and $\varphi_- \leq 0$ such that

$$\widehat{\varphi_+}(t) = \widehat{\varphi_-}(t) = (2\pi it)^{-1} \text{ for } |t| \geq 1 \tag{2}$$

and $\varphi_\pm(0) = \pm 1/2$.

Two results from distribution theory guide the construction of these functions. First, the generalized inverse Fourier transform of $(\pi it)^{-1}$ is the signum function $\operatorname{sgn}(x)$, and secondly, by the Paley-Wiener theorem, the Fourier transform of a function with support in $[-1,1]$ is an entire function of finite exponential type 2π, i.e., an entire function that grows no faster in the complex plane than $\exp(2\pi|z|)$ times a subexponential term.

Indeed, Beurling found that the entire function

$$B(x) = \frac{\sin^2 \pi x}{\pi^2}\Big(\sum_{n=0}^\infty (x-n)^{-2} - \sum_{n=1}^\infty (x+n)^{-2} - 2x^{-1}\Big)$$

has finite exponential type 2π and satisfies $B(x) \geq \operatorname{sgn}(x)$ and $-B(-x) \leq \operatorname{sgn}(x)$. Moreover, the functions $\varphi_+(x) := 2^{-1}(\operatorname{sgn}(x) + B(-x)) \geq 0$ and $\varphi_-(x) := 2^{-1}(\operatorname{sgn}(x) - B(x)) \leq 0$ are real line analogues of $S_\pm$, namely, they are integrable and satisfy (2); using this, Beurling concluded that (1) holds for real-valued f with a spectral gap containing $(-1,1)$.

Since Beurling's investigation, the functions $\varphi_\pm$ have been quite useful in the proofs of certain inequalities. A striking application is Vaaler's short and beautiful proof of Hilbert's general inequality. Since

$$\int_{\mathbb{R}} (\pm\varphi_{\pm}(x)) \Big| \sum_{n=1}^{N} a_n e^{2\pi i \lambda_n x} \Big|^2 dx \geq 0,$$

expanding the square and interchanging integration and summation leads after an application of (2) to

$$\Big| \sum_{m \neq n} \frac{a_m \overline{a_n}}{\lambda_m - \lambda_n} \Big| \leq \pi \sum_n |a_n|^2,$$

provided $|\lambda_m - \lambda_n| \geq 1$ for $m \neq n$.

In my research, I am interested in problems where these techniques can be applied. I introduce some problems below for which I have found extensions. The corresponding preprints have been posted at www.math.ubc.ca/∼flittman/research/research.htm.

2. Interpolation and Approximation

An immediate generalization of the results described above can be obtained if lower and upper approximations of exponential type 2π to $\mathrm{sgn}(x)x^n$ are known. For technical reasons, it is easier to approximate the truncated powers x_+^n which equal x^n for positive x and 0 for negative x. Since $\mathrm{sgn}(x)x^n - 2x_+^n = -x^n$ (which is a function of exponential type zero), both problems are equivalent.

Best approximations can often be characterized by a condition of the form that the approximated function and its best approximation coincide at an explicitly given discrete set. Hence approximations are often constructed by proving a general interpolation theorem and then specifying the set of interpolation points.

The method described now builds on an idea of Holt and Vaaler. Starting with an entire function F, they define an entire interpolation G of the signum function which satisfies

$$G(x) - \mathrm{sgn}(x) = F(x)H(x) \tag{3}$$

with a positive function H; the only restriction is that F be the uniform limit on compact sets in $\mathbb{C}$ of polynomials having only real roots (e.g., any product of terms $\sin(ax+b)$ is such a function). Equation (3) shows that the points where $G(x) = \mathrm{sgn}(x)$ are exactly the zeros of F.

A large part of my research deals with interpolation theorems for x_+^n. Denote by $\mathcal{L}[g]$ the two-sided Laplace transform of a real-valued function g. Starting with F and g connected by $F(z)\mathcal{L}[g](z) = 1$ in some vertical open strip S of the complex plane, one defines

$$G_n(z) := \frac{F(z)}{z} \int_{-\infty}^{0} e^{-zt} g^{(n+1)}(t)dt \qquad \text{for } z \in S. \tag{4}$$

The following is an example of the kind of theorem that can be obtained:

Theorem 2.1. *Let F and g as above. If $0 \in S$ and $g^{(n)}(0) = 0$, then G_n is an entire function, and*

$$G_n(x) - x_+^n = F(x)H_n(x) \tag{5}$$

holds with a function H_n satisfying $g^{(n+1)}(0)H_n(x) > 0$ for all $x \in \mathbb{R}$.

FIGURE 1. $\mathscr{G}_{n,\beta_n}(x)$ for $n = 0$, $n = 1$ and $n = 7$

Repeated integrations by parts in (4) show that H_n is essentially the product of $\operatorname{sgn}(x)x^n$ with the one-sided Laplace transform of

$$T_n g(t) := g(t) - \sum_{j=0}^{n-1} \frac{g^{(j)}(0)}{j!} t^j .$$

A proof of Theorem 2.1 requires an investigation of the number of real zeros of $T_n g$. The following theorem bounds the number of these zeros.

Theorem 2.2. *Let g be as above, and let $P \not\equiv 0$ be a real polynomial. The function $g + P$ has at most deg$(P) + 2$ zeros on the real line (counted with multiplicity).*

Theorem 2.1 makes the construction of best lower and upper approximation of exponential type 2π to x_+^n reasonably straightforward. Such approximations require interpolation points which are not sign changes of $G_n - x_+^n$. The interpolation points are in fact double zeros at a translate of the integers, hence the function F of Theorem 2.1 is

$$\mathfrak{F}_\alpha(x) = \pi^{-2} \sin^2 \pi(x - \alpha).$$

Lower and upper approximation have different sets of interpolation points $\alpha_n + \mathbb{Z}$ and $\beta_n + \mathbb{Z}$, respectively. It turns out that α_n and β_n are distinct zeros of the nth Bernoulli polynomial B_n (with a modification for $n = 0$ and $n = 1$). In this case, the function G_n in (4) can be represented as a combination of special functions. We write ψ for the logarithmic derivative of the Euler Gamma function.

Theorem 2.3. *The function*

$$\mathscr{G}_{n,\alpha}(z) := \frac{\sin^2 \pi(z-\alpha)}{\pi^2} z^n \left[\psi'(\alpha - z) + \sum_{j=0}^{n} B_j(\alpha) z^{-j-1}\right]$$

is the unique best upper (lower) $L^1(\mathbb{R})$–approximation from the class of functions of exponential type 2π to x_+^n for $\alpha = \beta_n$ ($\alpha = \alpha_n$).

It should be mentioned that Theorem 2.1 gives an explicit representation for the (non-onesided) best $L^1(\mathbb{R})$-approximation to x_+^n. By a theorem of Nagy, the interpolating set in this case is also a translate of the integers. These points turn out to be simple zeros of $G_n(x) - x_+^n$, hence one takes

$$F_\theta(x) = \pi^{-1} \sin \pi(x - \theta)$$

in Theorem 2.1. The correct value of θ is a zero of the nth Euler polynomial E_n. The interpolation in (4) becomes

Theorem 2.4. *Let $\theta_n = 0$ for even n and $\theta_n = 1/2$ for odd n. The function*

$$\mathfrak{G}_n(z) := \frac{\sin \pi(z-\theta_n)}{\pi} z^n \Big[\psi(2^{-1}(\theta_n - z)) - \psi(\theta_n - z) + \log 2 - \frac{1}{2}\sum_{j=0}^{n} E_j(\theta_n) z^{-j-1}\Big]$$

is the unique best $L^1(\mathbb{R})$–approximation of exponential type π to x_+^n.

3. Applications

The generalizations of the results mentioned in the introduction follow now. Let $h(t)$ be a hermitian function on $\mathbb{R}$, i.e. $h(-t) = \overline{h(t)}$. What are the optimal bounds $L(h)$ and $U(h)$ such that

$$-L(h)\sum_{\nu=1}^{N} |a_\nu|^2 \le \sum_{\substack{\mu,\nu=1 \\ \mu\ne\nu}}^{N} a_\nu \overline{a}_\mu h(\lambda_\nu - \lambda_\mu) \le U(h)\sum_{\nu=1}^{N}|a_\nu|^2 \tag{6}$$

holds for all $N \in \mathbb{N}$, all sequences $\{a_\nu\}_{\nu=1}^N$ of complex numbers, and all sequences of real numbers$\{\lambda_\nu\}_{\nu=1}^N$ with $|\lambda_\nu - \lambda_\mu| \ge 1$ for $\nu \ne \mu$?

Theorem 2.3 yields sharp bounds $L(h_m)$ and $U(h_m)$ for the functions $h_m(t) := (it)^{-m}$ for $m \in \mathbb{N}$, namely

$$L(h_m) = (2\pi)^m (m!)^{-1} B_m(\alpha_{m-1}),$$
$$U(h_m) = -(2\pi)^m (m!)^{-1} B_m(\beta_{m-1}).$$

A classical question in approximation theory asks for the rate of appoximation of functions in a given class by functions from another class. Theorem 2.3 can be used to give bounds for the best one-sided $L^1(\mathbb{R})$–approximation by functions of type δ to functions f with an nth derivative having finite total variation $V_{f^{(n)}}$. The case $n = 0$ was treated independently by J. D. Vaaler and by D. Dryanov.

Define the error function $E^+(\delta, f)$, which is the infimum of $||A - f||_1$ taken over all A of type δ satisfying $A \ge f$ on the real line. $E^-(\delta, f)$ is defined analogously with the inequality reversed.

Theorem 3.1. *Let $n \in \mathbb{N}_0$ and assume that $f^{(n-1)}$ is locally absolutely continuous. The estimate*

$$E^{\pm}(\delta, f) \le \pi V_{f^{(n)}} \delta^{-n-1}$$

holds for all $\delta > 0$.

4. Future Work

I would like to find out if the interpolatory ideas described above can be used to find best approximations in other situations. As was shown in Section 2, best one-sided $L^1(\mathbb{R})$–approximations to x_+^n can be constructed in essentially the same way as best $L^1(\mathbb{R})$–approximations. This is certainly not true for arbitrary functions, but it could be true for functions that are 'regular enough'. In particular, it is worth investigating functions f with the property that the best $L^1(\mathbb{R})$–approximation to f by functions of type π has a nodal set which is a translate of the integers. This includes the class of functions which satisfy the assumptions of certain Markov-type theorems by Krein and by Szökefalvi-Nagy.

In 1981, S. W. Graham and J. D. Vaaler used best one-sided approximations to $x_+^0 e^{-\lambda x}$ to establish quantitative Tauberian theorems for positive measures α with support in $[0, \infty)$.

If the Laplace transform $T(s)$ of α is analytic in $\Re s > r > 0$, has a pole at $s = r$, and a continuous extension to $\{s : \Re s \geq r \text{ and } |\Im s| < T\}$, they proved sharp bounds for the limsup and liminf of $e^{-xr}\alpha([0, x])$ in terms of r and T. I work on extending Graham and Vaalers theorem to measures α_k that are obtained by integrating k times a positive measure α (integrating the bounds of Graham and Vaaler k times leads to bounds that are not sharp). Finding sharp bounds involves best one-sided $L^1(\mathbb{R})$–approximations for the class of functions $x_+^n e^{-\lambda x}$. It is likely that these approximations can be found with the interpolatory approach of Section 2.

The approximations introduced in Section 2 have the property that they interpolate the approximated function at a translate of the integers. In general, best approximations will not have this interpolation property, e.g., the best upper approximation in the sense of Section 2 to $f(x) = x_+^0 - 20x_+^3$ does not interpolate $f(x)$ at a translate of the integers. I work on extensions of Theorem 2 which allow the construction of interpolants to such functions.