PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 88, Number 4, August 1983

# A PROOF OF A FORMULA IN FOURIER ANALYSIS ON THE SPHERE

JIN-GEN YANG

ABSTRACT. A short and elementary proof of a useful formula in spherical harmonic analysis is provided.

In [3] Sherman proved an integral formula for eigenfunctions of the Laplacian on the sphere $S^n$. He developed a certain theory of Fourier analysis on the basis of this formula. The purpose of this note is to give an elementary short proof of Sherman's formula.

Let $a = (0,\ldots,0,1) \in S^n$, $B = \{(x_1,\ldots,x_{n+1}) \in S^n \mid x_{n+1} = 0\}$. $a$ is the "north pole" and $B$ is the "equator". For any integer $k \geqslant 0$ and $b \in B$, define

$$e_{b,k}(s) = (a + ib, s)^k, \qquad s \in S^n,$$

and

$$f_{b,k}(s) = \operatorname{sgn}(s, a)^{n-1}(a + ib, s)^{-k-n+1}, \qquad s \in S^n - B,$$

where $(\cdot, \cdot)$ is the Euclidean inner product, $i = \sqrt{-1}$. Let $db$ be the normalized Euclidean measure on $B$.

THEOREM (SHERMAN, LEMMA 3.9 OF [3]).

$$\int_B e_{b,k}(s) f_{b,k}(s')\, db = P_k((s, s')) \tag{1}$$

*for all* $s \in S^n$, $s' \in S^n - B$ *and* $k > 0$, *where* $P_k$ *is a polynomial of degree* $k$ *with* $P_k(1) = 1$, *called the (normalized) Gegenbauer polynomial.*

Formula (1) corresponds to formula (1.9) in [3].

PROOF. Denote the left-hand side of (1) by $F(s, s')$. Since $F(-s, -s') = F(s, s')$ we may assume $\operatorname{sgn}(s', a) = 1$. Let $u_\phi$ be the rotation represented by

$$\begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \cos\phi & -\sin\phi \\ & & \sin\phi & \cos\phi \end{bmatrix}.$$

If $F(u_\phi s, u_\phi s') = F(s, s')$ for all $\phi$ such that $(u_\phi s', a) > 0$, then the proof of (1) will be reduced to the case $s' = a$ (in which (1) is the standard integral formula for the

Received by the editors October 25, 1982.
1980 *Mathematics Subject Classification*. Primary 33A45, 33A50; Secondary 26A06.

Gegenbauer polynomial, cf. Theorem 7 of [**2**] or Lemma 4.2 of [**3**]) because we can always find some rotation $u$ of $B$ and some $u_\phi$ such that $u_\phi us' = a$, and it is obvious that $F(us, us') = F(s, s')$. Hence it is enough to prove $\partial F(u_\phi s, u_\phi s')/\partial\phi = 0$.

Any point $b \in B$ can be written in the form

$$b = (c_1\cos\theta, c_2\cos\theta, \dots, c_{n-1}\cos\theta, \sin\theta, 0)$$

where $c_1^2 + \cdots + c_{n-1}^2 = 1$. Let

$$g(c, \theta, \phi, s) = i\cos\theta\sum_{j=1}^{n-1} c_j s_j + i\sin\theta(s_n\cos\phi - s_{n+1}\sin\phi) + (s_n\sin\phi + s_{n+1}\cos\phi).$$

That means $e_{b,k}(u_\phi s) = [g(c, \theta, \phi, s)]^k$ and $f_{b,k}(u_\phi s') = [g(c, \theta, \phi, s')]^{-k-n+1}$. Then for all integers $m$,

$$\frac{\partial(g^m)}{\partial\theta} = mg^{m-1}\left[-i\sin\theta\sum_{j=1}^{n-1} c_j s_j + i\cos\theta(s_n\cos\phi - s_{n+1}\sin\phi)\right].$$

$$\frac{\partial(g^m)}{\partial\phi} = mg^{m-1}\left[i\sin\theta(-s_n\sin\phi - s_{n+1}\cos\phi) + (s_n\cos\phi - s_{n+1}\sin)\right].$$

Hence we obtain a useful relation

$$i\frac{\partial(g^m)}{\partial\phi} - \cos\theta\frac{\partial(g^m)}{\partial\theta} = mg^m\sin\theta. \tag{2}$$

Therefore

$$\begin{aligned}
\frac{\partial F(u_\phi s, u_\phi s')}{\partial\phi} &= \frac{\partial}{\partial\phi}\int_B [g(c, \theta, \phi, s)]^k [g(c, \theta, \phi, s')]^{-k-n+1}\, db \\
&= A\int_{c\in S^{n-2}} dc \int_{-\pi/2}^{\pi/2}\left[\frac{\partial\left(g(c, \theta, \phi, s)^k\right)}{\partial\phi} g(c, \theta, \phi, s')^{-k-n+1}\right. \\
&\qquad \left. + g(c, \theta, \phi, s)^k \frac{\partial\left(g(c, \theta, \phi, s')^{-k-n+1}\right)}{\partial\phi}\right]\cos^{n-2}\theta\, d\theta \\
&= A\int_{c\in S^{n-2}} dc \int_{-\pi/2}^{\pi/2} -i\left[\left(\cos\theta\frac{\partial\left(g(c, \theta, \phi, s)^k\right)}{\partial\theta} + k\sin\theta g(c, \theta, \phi, s)^k\right)\right. \\
&\qquad \times g(c, \theta, \phi, s')^{-k-n+1} + g(c, \theta, \phi, s)^k \\
&\qquad \times\left(\cos\theta\frac{\partial\left(g(c, \theta, \phi, s')^{-k-n+1}\right)}{\partial\theta}\right. \\
&\qquad \left.\left. -(k + n - 1)\sin\theta g(c, \theta, \phi, s')^{-k-n+1}\right)\right] \\
&\qquad \times\cos^{n-2}\theta\, d\theta
\end{aligned}$$

by using (2). Here $dc$ is the ordinary Euclidean measure on $S^{n-2}$, $A$ is constant. We can easily see that the integrand is $\partial Q/\partial\theta$, where

$$Q = -i\left[g(c,\theta,\phi,s)^k g(c,\theta,\phi,s')^{-k-n+1}\cos^{n-1}\theta\right].$$

Therefore the integral is zero.

ACKNOWLEDGEMENT. The author would like to thank Professor Sigurdur Helgason for correcting some mistakes and giving valuable advice.

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DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139