

A Study in Fourier Analysis

From circle, through the line, to the complex

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Fourier Series

» Structure and Topology of \mathbb{T}

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$$g: \mathbb{T} \rightarrow [0, 1) := [x] \rightarrow \{x\} \quad (1)$$

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- * Define Lebesgue measure μ on \mathbb{T} as

$$\mu(A) = \lambda(g(A)) \quad (3)$$

where λ is the Lebesgue measure on \mathbb{R} .

» Functions in \mathbb{T}

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- * Also by the Lebesgue measure on \mathbb{T} , we say $f \in L^p(\mathbb{T})$ if the corresponding function in $[0, 1)$ is in $L^p[0, 1)$.
- * For any two functions $f, g \in L^1(\mathbb{T})$, their convolution, $(f * g)(x)$ defined as

$$(f * g)(x) = \int_0^1 f(x - y)g(y) \, dy \quad (4)$$

is again in $L^1(\mathbb{T})$

» **Fourier Coefficients**

- * For $f \in L^1(\mathbb{T})$, and $n \in \mathbb{Z}$ we define the n^{th} Fourier coefficient of f as

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- * Since we are interested in the convergence of the Fourier series, we will define the symmetric and Cesàro partial sums of the Fourier series respectively as

$$S_N(x) = \sum_{n=-N}^N \hat{f}(n) e^{2\pi i n x} \quad \text{and} \quad \sigma_N(x) = \frac{1}{N} \sum_{n=0}^{N-1} S_n(x)$$

» **Summability Kernel**

* A collection of functions $K_N \in L^1(\mathbb{T})$ are called a summability kernel if it satisfies the following properties

1. $\int_0^1 K_N(x) dx = 1$
2. $\int_0^1 |K_N(x)| dx \leq C$ for some constant $C > 0$
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* We prove that if K_N is a summability kernel in $L^1(\mathbb{T})$, then $(f * K_N)(x)$ converge to $f(x)$ in $L^1(\mathbb{T})$. That is

$$\int_0^1 |f(x) - (f * K_N)(x)| dx \rightarrow 0 \quad (5)$$

» Fejér Kernel and Cesàro Convergence

* Fejér kernel defined as

$$\Delta_N(x) = \sum_{n=-N}^N \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x} = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=-n}^n e^{2\pi i k x}$$

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- * Therefore if $f, g \in L^1(\mathbb{T})$ such that $\hat{f} = \hat{g}$, then $f \stackrel{a.e}{=} g$

» **Fourier Series in $L^2(\mathbb{T})$**

- * Since \mathbb{T} is identified with the finite measure space $[0, 1)$, we get that $L^2(\mathbb{T}) \subset L^1(\mathbb{T})$. Therefore the definition of Fourier coefficients and series in $L^1(\mathbb{T})$ holds good in $L^2(\mathbb{T})$.

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- * Moreover we see that if $f \in L^2(\mathbb{T})$, since the Fejér kernel, $\Delta_N(x) \leq N$, its Cesàro partial sum, $\sigma_N = (f * \Delta_N) \in L^2(\mathbb{T})$

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- * The same results follow for functions in $L^p(\mathbb{T})$

» Fejér's Theorem and Pointwise Convergence

* (Fejér's Theorem) If $f \in L^1(\mathbb{T})$, then

$$\lim_{N \rightarrow \infty} \sigma_N(x) = \frac{f(x^-) + f(x^+)}{2} \quad (6)$$

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* Therefore if f is continuous then the Cesàro partial sum converge pointwise to f everywhere.

Fourier Transforms in \mathbb{R}

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- * (Riemann Lebesgue Lemma) \hat{f} is uniformly continuous and

$$\lim_{|t| \rightarrow \infty} \hat{f}(t) = 0 \quad (8)$$

» **Fourier Inversion**

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- * Generalizing further we get that if $f, \hat{f} \in L^1(\mathbb{R})$ then

$$\check{\check{f}} \stackrel{a.e}{=} f \quad (11)$$

» **Fourier transforms in $L^2(\mathbb{R})$**

- * We consider the space $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Since it is a subspace of $L^1(\mathbb{R})$, the definition of Fourier transform and inverse transform in $L^1(\mathbb{R})$ holds good in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

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- * (Plancherel's Theorem) If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then

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- * Now since the collection of compactly supported continuous functions in \mathbb{R} , $C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for all $1 \leq p < \infty$, we get that $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$.

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- * Plancherel's theorem asserts that Fourier transform in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is an isometry therefore we can extend Fourier transform to an isometry in $L^2(\mathbb{R})$.

Holomorphic Fourier Transforms

» Extending Domain to \mathbb{C}

- * Fourier transform of certain functions can be extended into a holomorphic functions in certain regions. That is for $z \in \mathbb{C}$,

$$\hat{f}(z) = \int_{\mathbb{R}} f(x) e^{-2\pi izx} dx \quad (13)$$

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- * For example if $f(x) = e^{-|x|}$, then its Fourier transform, $\hat{f}(t) = \frac{1}{1+(2\pi t)^2}$ can be extended into holomorphic function in regions in the complex plane without the points $\pm \frac{i}{2\pi}$.

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- * We will focus on two types of functions in $L^2(\mathbb{R})$
 1. $f(x) = 0, (x < 0)$
 2. $f(x) = 0, (x \notin (-A, A))$

» Paley Wiener Theorem 1

The following statements are equivalent

1. $F \in L^2(\mathbb{R})$ such that F is essentially supported in $(0, \infty)$ and for all $z \in \Pi^+$

$$f(z) = \int_0^\infty F(t) e^{2\pi i t z} dt \quad (14)$$

and

$$\int_0^\infty |F(t)|^2 dt = C < \infty$$

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2. $f \in H(\Pi^+)$ such that f restricted to horizontal lines is uniformly bounded by C in L^2 . That is

$$\sup_{0 < y < \infty} \frac{1}{2\pi} \int_{\mathbb{R}} |f(x + iy)|^2 dx = C < \infty \quad (15)$$

» Paley Wiener Theorem 2

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1. $F \in L^2(\mathbb{R})$ is essentially supported in $(-A, A)$ such that

$$f(z) = \int_{-A}^A F(x) e^{2\pi i z x} dx \quad (16)$$

» Paley Wiener Theorem 2

The following statements are equivalent:

1. $F \in L^2(\mathbb{R})$ is essentially supported in $(-A, A)$ such that

$$f(z) = \int_{-A}^A F(x) e^{2\pi i z x} dx \quad (16)$$

2. $f: \mathbb{C} \rightarrow \mathbb{C}$ is an entire function satisfying $|f(z)| \leq C e^{2\pi A|z|}$ for some constant C , and f restricted to horizontal lines is bounded in L^2 . That is

$$\int_{-\infty}^{\infty} |f(x + iy)|^2 dx < \infty \quad (17)$$

Future Directions

» **Schwartz Class**

- * A smooth function $f: \mathbb{R}^n \rightarrow \mathbb{C}$, f is called a *Schwartz function* if for any given multi index α, β , there exists a positive constant $C_{\alpha, \beta}$ such that

$$\rho_{\alpha, \beta} = \sup_{x \in \mathbb{R}^n} \left| x^\alpha (D^\beta f) x \right| = C_{\alpha, \beta} < \infty \quad (18)$$

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- * Schwartz class is dense in $L^p(\mathbb{R}^n)$ for all $1 \leq p < \infty$.

» **Fourier transforms in \mathbb{R}^n**

* Fourier transform of $f \in \mathcal{S}(\mathbb{R}^n)$, $\hat{f}: \mathbb{R}^n \rightarrow \mathbb{C}^n$ is defined as

$$\hat{f}(\mathbf{t}) = \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-2\pi i \mathbf{t} \cdot \mathbf{x}} d\mathbf{x} \quad (19)$$

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- * Fourier transform is a homeomorphism in $\mathcal{S}(\mathbb{R}^n)$.
- * By Parseval's identity Fourier transform can be extended into whole of \mathbb{R}^n

» **A problem**

Let $n \geq 2$. Does there exist a function $f \in L^{\frac{2n+2}{n+3}}(\mathbb{R}^n)$ such that

$$\hat{f}|_{S^{n-1}} = 0$$

and

$$|1 - |\xi|^2|^{-\frac{1}{2}} f \notin L^2(\mathbb{R}^n)$$