Master's Project

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1 Preface

I plan to write and detail everything(almost) I study and learn for my Master's project into these latex files. I assume it will be much easier to track whatever I have learned and to have a good overview of the topic with this note taking. Also since I am doing it on LaTeX I am sure it will save me from the last minute rush to type everything out and make the report of the project. I will start with Fourier series and will introduce new concepts when they are required as we go along. So, let us start

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2 Preliminaries

2.1 Measure theory

Definition 2.1 – σ-algebra A collection Σ of subsets of a set X is called a σ-algebra if it satisfy the following properties

1. $X \in \Sigma$
2. If $E \in \Sigma$, then $E^c \in \Sigma$.
3. If $E_1, E_2, \dots$ are elements of Σ, then $\bigcup_{i=1}^{\infty} E_i \in \Sigma$.

3 Fourier Series in $L^1(\mathbb{T})$

3.1 Definition and basic properties

We'll begin by reviewing the definition of L^p space since we'll be mostly working on functions from these space.

Definition 3.1 – L^p function A real valued function f defined on a lebesgue measure space S is called an L^p function on S or a p-integrable function on S if

$$\left(\int_S |f(x)|^p\right)^{1/p} < \infty$$

For such a function the integral above is called the p-norm of the function f and often denoted by $\|f\|_{L^p(S)}$ or $\|f\|_p$ if the space is known.

Although we've defined general L^p spaces, we'll mostly be concerned about L^1 and L^2 functions in $\mathbb{R}$ and $\mathbb{T} = [0, 1)$.

Now let's define what a periodic function is.

Definition 3.2 – Periodic function A function $f : \mathbb{R} \to \mathbb{R}$ is called periodic with period p(or p-periodic) if $f(x + p) = f(x)$ for all $x \in \mathbb{R}$

Although we're definining real valued periodic functions the definition holds well for any function f from the reals to a space where equality is defined. Also note that if we know the value of a p periodic function f in a closed-open (or open-close or closed) interval of length p say $[x, x + p)$, then using the definition of periodic function we can get the function value at the whole of $\mathbb{R}$(we leave it for the reader to verify). That is, we can identify any p periodic function f on $\mathbb{R}$ with the restriction of f onto an interval of length p. We can also identify f with the values in a unit circle in $\mathbb{C}$. Specifically we'll be working with 1-periodic functions identified by their restrictions in $\mathbb{T}$.

Definition 3.3 – Continuous functions in $\mathbb{T}$ Continuous functions on $\mathbb{T}$, are precisely those functions which are continuous in $\mathbb{R}$ with period 1. We'll denote the collection of continuous functions in $\mathbb{T}$ with $C(\mathbb{T})$.

Now let's prove an important result of periodic functions.

Lemma 3.1 If $f : \mathbb{R} \to \mathbb{R}$ is of period 1 and $\int_0^1 f(x)dx$ exists, then for any real number a,

$$\int_a^{a+1} f(x)dx = \int_0^1 f(x)dx$$

Proof. Let $a = n + b$, where $0 \leq b < 1$ and n is an integer. Then since f has period 1,

$$\int_a^{a+1} f(x)dx = \int_{n+b}^{n+b+1} f(x)dx = \int_b^{b+1} f(x+n)dx = \int_b^{b+1} f(x)dx$$

and,

$$\begin{aligned}
\int_b^{b+1} f(x)dx &= \int_b^1 f(x)dx + \int_1^{b+1} f(x)dx \\
&= \int_b^1 f(x)dx + \int_0^b f(x+1)dx \\
&= \int_b^1 f(x)dx + \int_0^b f(x)dx \\
&= \int_0^1 f(x)dx
\end{aligned}$$

Hence the result.

□

Now we'll define fourier coefficients of a periodic function $f \in L^1(\mathbb{T})$

Definition 3.4 – Fourier coefficient Let $f \in L^1(\mathbb{T})$, i.e $\int_{\mathbb{T}} f < \infty$. Then for each integer n we define the n^{th} fourier coefficient, $\widehat{f}(n)$ as

$$\widehat{f}(n) = \int_0^1 f(x)e^{-2\pi inx}dx$$

$\widehat{f}(n)$ is finite and well defined for each n since $f \in L^1(\mathbb{T})$, since

$$|\widehat{f}(n)| \leq \int_0^1 |f(x)e^{-2\pi inx}|dx \leq \int_0^1 |f(x)||e^{-2\pi inx}|dx = \int_0^1 |f(x)|dx < \infty$$

Once we have the fourier coefficients of a function at hand we can combine them together to make a series called the fourier series. We'll be investigating the conditions at which this series converges to our initial function f.

Also note that the map which takes f to $\widehat{f}$ is linear since if $f, g \in L^1(\mathbb{T})$, then

$$\widehat{f+g}(n) = \int_0^1 (f+g)(x)e^{-2\pi inx}\, dx = \int_0^1 f(x)e^{-2\pi inx}\, dx + \int_0^1 g(x)e^{-2\pi inx}\, dx = \widehat{f}(n) + \widehat{g}(n)$$

and for some $\lambda \in \mathbb{R}$,

$$\widehat{\lambda f}(n)\, dx = \int_0^1 \lambda f(x)e^{-2\pi inx}\, dx = \lambda \int_0^1 f(x)e^{-2\pi inx}\, dx = \lambda \widehat{f}(n)$$

Definition 3.5 – Fourier series Given a function $f \in L^1(\mathbb{T})$, the fourier series of the function f is defined as

$$\sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{2\pi inx}$$

where $\widehat{f}(n)$ is the n^{th} fourier coefficient as defined in 3.4

Proposition 3.1 – Properties of Fourier coefficinets Suppose that $f \in L^1(\mathbb{T})$.

(a) If a is a real number and $g(x) = f(x+a)$ for all x, then $\widehat{g}(n) = \widehat{f}(n)e^{2\pi ina}$ for all $n \in \mathbb{Z}$.

(b) if b is an integer and $h(x) = f(x)e^{2\pi ibx}$ for all x, then $\widehat{h}(n) = \widehat{f}(n-b)$ for all $n \in \mathbb{Z}$.

(c) if $j(x) = f(-x)$ for all x, then $\widehat{j}(n) = \widehat{f}(-n)$

Proof. Given $f(x) \in L^1(\mathbb{T})$ and n^{th} Fourier coefficient of f,

$$\widehat{f}(n) = \int_0^1 f(x)e^{-2\pi inx}.$$

(a) Then, the n^{th} Fourier coefficient of $g(x) = f(x+a)$ is

$$\begin{aligned}
\widehat{g}(n) &= \int_0^1 g(x)e^{-2\pi inx}dx \\
&= \int_0^1 f(x+a)e^{-2\pi inx}dx \\
&= \int_0^1 f(x)e^{-2\pi in(x-a)}dx \\
&= e^{2\pi ina}\int_0^1 f(x)e^{-2\pi inx}dx \\
&= e^{2\pi ina}\widehat{f}(n)
\end{aligned}$$

(b) If $h(x) = f(x)e^{2\pi ibx}$, then

$$\widehat{h}(n) = \int_0^1 f(x)e^{-2\pi i(n-b)x}dx = \widehat{f}(n-b)$$

(c) $j(x) = f(-x)$, then

$$\begin{aligned}
\widehat{j}(n) &= \int_0^1 f(-x)e^{-2\pi inx}dx \\
&= -\int_0^{-1} f(y)e^{2\pi iny}dy && \text{by } y = -x \\
&= \int_{-1}^0 f(y)e^{2\pi iny}dy \\
&= \int_0^1 f(y)e^{2\pi iny}dy && \text{by lemma 3.1} \\
&= \widehat{f}(-n)
\end{aligned}$$

□

3.2 Convolution

Now we'll define another important operation with function called the convolution of two functions.

Definition 3.6 Let $f, g \in L^1(\mathbb{T})$, then the convolution of f and g is

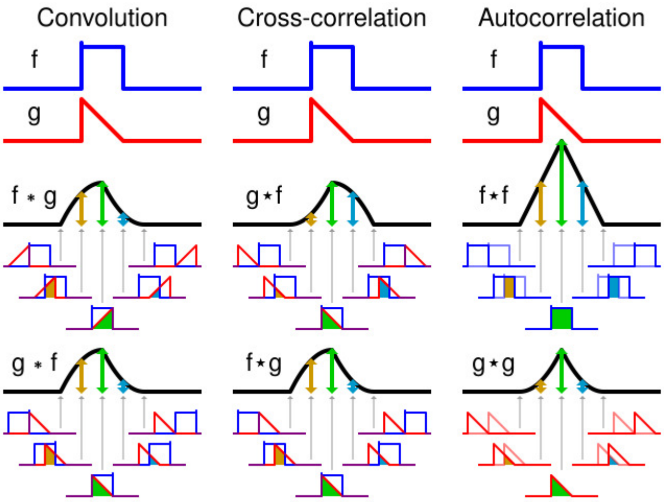

Figure 1: convolution

defined as

$$f * g(x) = \int_0^1 f(y)g(x-y)dy$$

Convolution can be thought of as taking the moving average of a function with another function. Refer figure 1.

Proposition 3.2 – Properties of convolution Let $f, g \in L^1(\mathbb{T})$, then

(a) $f * g = g * f$

(b) $f * (g + h) = f * g + f * h$

(c) $(cf) * g = c(f * g)$

(d) $f * (g * h) = (f * g) * h$

Proof. We'll just prove the commutativity and will leave the rest for the reader to verify. (Hint: Use properties of integration)

Put $v = x - y$, we get $dv = -dy$ and

$$\begin{aligned} f * g(x) &= \int_0^1 f(y)g(x-y)dy \\ &= -\int_x^{x-1} f(x-v)g(v)dy \\ &= \int_{x-1}^{x} g(v)f(x-v)dy \\ &= \int_0^1 g(v)f(x-v)dy && \text{by lemma 3.1} \\ &= g * f(x) \end{aligned}$$

□

We'll prove another important result that the convolution of two $L^1(\mathbb{T})$ functions is again in $L^1(\mathbb{T})$.

Theorem 3.1 Let $f, g \in L^1(\mathbb{T})$. Then $h = f * g \in L^1(\mathbb{T})$, and $\widehat{h}(n) = \widehat{f}(n)\widehat{g}(n)$.

Proof.

$$\begin{aligned} \int_0^1 |h(x)|dx &= \int_0^1 \left| \int_0^1 f(y)g(x-y)\ dy \right| dx \\ &\leq \int_0^1 \int_0^1 |f(y)g(x-y)|\ dy\ dx \\ &= \int_0^1 \int_0^1 |f(y)g(x-y)|\ dx\ dy && \text{by Tonelli's theorem} \\ &= \int_0^1 \left(\int_0^1 |g(x-y)|dx \right) |f(y)|dy \\ &= \|f\|_1 \|g\|_1 \end{aligned}$$

Note that we're using Tonelli's theorem here to interchange the limits of integration since the space is a finite measure space. This proves that $h = f * g \in L^1(\mathbb{T})$.

To prove the next part,

$$
\begin{aligned}
\widehat{h}(n) &= \int_0^1 \left(\int_0^1 f(y)g(x-y)dy \right) e^{-2\pi inx} dx \\
&= \int_0^1 f(y) \left(\int_0^1 g(x-y)e^{-2\pi inx} dx \right) dy && \text{by Tonelli's theorem} \\
&= \int_0^1 f(y)\widehat{g}(n)e^{-2\pi inx} dy && \text{by proposition 3.1(a)} \\
&= \widehat{f}(n)\widehat{g}(n)
\end{aligned}
$$

□

3.3 Partial sums of Fourier series

Given a function f in the $\mathbb{T}$, we are interested in the convergence of fourier series of f. We'll discuss about the convergence of the symmetric partial sum of the Fourier series.

Definition 3.7 – Symmetric partial sum of a Fourier series Given a function $f \in Ł^1(\mathbb{T})$ with its fourier series, $\sum_{-\infty}^{\infty} \widehat{f}(n)e^{2\pi inx}$, we define the n^{th} symmetric parial sum of the fourier series as

$$S_N(x) = \sum_{n=-N}^{N} \widehat{f}(n)e^{2\pi inx}$$

But it may happen that the summetric partial sum of the Fourier seies may not converge. To deal with this we'll define another partial sum called the Cesáro partial sum.

Definition 3.8 – Cesáro partial sum of Fourier series Given a function $f \in Ł^1(\mathbb{T})$ with its Fourier series, $\sum_{-\infty}^{\infty} \widehat{f}(n)e^{2\pi inx}$, we define the n^{th} Cesáro parial sum of its Fourier series as

$$\sigma_N(x) = \frac{1}{N} \sum_{n=0}^{N-1} S_n(x)$$

where $S_n(x)$ is the symmetric partial sum of the Fourier series as defined in 3.7

For an example, $\{-1^n\}$ is a sequence whose symmetric partial sums do not converge but the Cesáro partial sums converge to $\frac{1}{2}$. Also if the symmetric partial sums of a series converge, then the Cesáro partial sums will also converge to the same limit. (Prove it!)

Now we'll show that the Cesáro partial sum can be rewritten to another form which will help our proofs down the road.

Lemma 3.2 If $\sigma_N(x)$ is the N^{th} Cesáro partial sum of the Fourier series of a function $f \in L^1(\mathbb{T})$, then

$$\sigma_N(x) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) \widehat{f}(n) e^{2\pi i n x}$$

Proof. We'll prove the result for a general series so that it'll help us also in Fourier series.

Let $S_N = \sum_{n=-N}^{N} a_n$ be the N^{th} partial sum of the series $\sum_{-\infty}^{\infty} a_n$. Then by the definition of Cesáro partial sum,

$$\begin{aligned}
\sigma_N &= \frac{1}{N} \sum_{n=0}^{N-1} S_n \\
&= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=-n}^{n} a_k \\
&= \frac{1}{N} \sum_{k=-N+1}^{N-1} a_k \sum_{n=|k|}^{N-1} 1 \\
&= \frac{1}{N} \sum_{k=-N+1}^{N-1} (N - |k|)\, a_k \\
&= \sum_{k=-N+1}^{N-1} \left(1 - \frac{|k|}{N}\right) a_k \\
&= \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N}\right) a_k
\end{aligned}$$

Now speficially if we take $a_k = \widehat{f}(k) e^{2\pi i k x}$, we get the required result. $\square$

3.4 Summability Kernels

Now we'll define a family of functions called the sumambility kernels, which we will use heavily in our proofs and simplify it.

Definition 3.9 – Summability kernel A sequence of functions $K_N \in L^1(\mathbb{T})$ is called a summability kernel or an approximation identity if

(a) $\int_0^1 K_N(x)dx = 1$

(b) $\int_0^1 |K_N(x)|dx \leq C$ for some constant $C > 0$

(c) $\lim_{N\to\infty} \int_\delta^{1-\delta} |K_N(x)|dx = 0$

Now we'll define one of the important summability kernels.

Definition 3.10 – Fejér kernel Fejér kernel is defined as a collection of functions Δ_N where for each $N \in \mathbb{N}$, $\Delta_N : \mathbb{R} \to \mathbb{R}$ is defined as

$$\Delta_N(x) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x}$$

Notice that each Δ_N is a 1−periodic function.

We'll prove that Fejér kernel as defined in 3.10 satify the properties in 3.9. But before that we'll explore some properties of Fejér kernel so that it'll aid us in our proof.

Proposition 3.3 – Properties of Fejér kernel If $\Delta_N(x)$ is defined as in 3.10, then the following hold true

(a)

$$\Delta_N(x) = 1 + 2\sum_{n=1}^{N}(1 - \frac{n}{N}) \cos 2\pi n x$$

(b)

$$\int_0^1 \Delta_N(x)dx = 1$$

(c)

$$\Delta_N(x) = \Delta_N(1 - x)$$

(d) For $0 \leq \delta \leq \frac{1}{2}$,

$$\int_{\delta}^{\frac{1}{2}} \Delta_N(x)dx = \int_{\frac{1}{2}}^{1-\delta} \Delta_N(x)dx$$

and therefore,

$$\int_{0}^{\frac{1}{2}} \Delta_N(x)dx = \int_{\frac{1}{2}}^{1} \Delta_N(x)dx = \frac{1}{2}$$

(e)

$$\Delta_N(x) = \begin{cases} \frac{1}{N}\left(\frac{\sin(\pi N x)}{\sin(\pi x)}\right)^2, & \text{if } x \notin \mathbb{Z} \\ N, & \text{if } x \in \mathbb{Z} \end{cases}$$

(f) If $0 \leq x \leq \frac{1}{2}$, then

$$\Delta_N(x) \leq \min\left(N, \frac{1}{4Nx^2}\right)$$

(g) If $0 < \delta \leq \frac{1}{2}$, then

$$\int_{\delta}^{\frac{1}{2}} \Delta_N(x) < \frac{1}{4N\delta}$$

(h)

$$(\Delta_N * f)(x) = \sigma_N(x)$$

where $\sigma_n(x)$ is the n^{th} Cesáro partial sum of the Fourier series of f as defined in 3.8

Proof. (a) This follows straight from De Moivre's formula that $e^{ix} = \cos(x) + i\sin(x)$ and $e^{ix} + e^{-ix} = 2\cos(x)$.

(b) By previous result,

$$\Delta_N(x) = 1 + 2\sum_{n=1}^{N}\left(1 - \frac{|n|}{N}\right)\cos(2\pi n x)$$

Therefore,

$$\int_0^1 \Delta_N(x)\ dx = \int_0^1 1\ dx + 2\sum_{n=1}^{N}\left(1-\frac{|n|}{N}\right)\int_0^1 cos(2\pi nx)\ dx$$
$$= 1 + 0$$

(c) This follows from the fact that $\cos(2\pi n(1-x)) = \cos(2\pi n - 2\pi nx) = \cos(2\pi nx)$ in the last result.

(d) From 3.3, we know that $\Delta_N(x) = \Delta_N(1-x)$. Therefore by change of variables,

$$\int_\delta^{\frac{1}{2}} \Delta_N(x)dx = \int_\delta^{\frac{1}{2}} \Delta_N(1-x)dx = -\int_{1-\delta}^{\frac{1}{2}} \Delta_N(y)dy = \int_{\frac{1}{2}}^{1-\delta} \Delta_N(y)dy$$

Also from previous result, we know

$$\int_0^{\frac{1}{2}} \Delta_N(x)dx + \int_{\frac{1}{2}}^1 \Delta_N(x)dx = \int_0^1 \Delta_N(x)dx = 1$$

Hence

$$\int_0^{\frac{1}{2}} \Delta_N(x)dx = \int_{\frac{1}{2}}^1 \Delta_N(x)dx = \frac{1}{2}$$

(e) If $x \in \mathbb{N}$ then $e^{2\pi inx} = 1$ for all n and then,

$$\sum_{n=0}^{N-1} e^{2\pi inx} = \sum_{n=0}^{N-1} 1 = N$$

Hence the last case is solved. But if $x \notin \mathbb{N}$ then from the finite sum of geometric series,

$$\sum_{n=0}^{N-1} e^{2\pi inx} = \frac{e^{2\pi iNx}-1}{e^{2\pi ix}-1}$$
$$= \frac{e^{\pi iNx}}{e^{\pi ix}} \times \frac{e^{\pi iNx}-e^{-\pi iNx}}{e^{\pi ix}-e^{-\pi ix}}$$
$$= e^{\pi i(N-1)x}\frac{\sin(\pi Nx)}{\sin(\pi x)}$$

Since $|e^{ix}| = 1$ for all $x \in \mathbb{R}$, we'll get

$$\left|\sum_{n=0}^{N-1} e^{2\pi inx}\right|^2 = \frac{\sin^2(\pi Nx)}{\sin^2(\pi x)}$$

But we also know that,

$$\begin{aligned}
\left|\sum_{n=0}^{N-1} e^{2\pi inx}\right|^2 &= \left(\sum_{n=0}^{N-1} e^{2\pi inx}\right)\left(\sum_{n=0}^{N-1} \overline{e^{2\pi inx}}\right) \\
&= \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} e^{2\pi i(m-n)x} \\
&= \sum_{k=-(N-1)}^{N-1} e^{2\pi ikx} \sum_{\substack{0\le m\le N-1 \\ 0\le n\le N-1 \\ m-n=k}} 1 \\
&= \sum_{k=-(N-1)}^{N-1} e^{2\pi ikx}(N-|k|) \\
&= N\Delta_N(x)
\end{aligned}$$

Which implies that

$$\Delta_N(x) = \frac{1}{N}\frac{\sin^2(\pi Nx)}{\sin^2 \pi x}$$

Hence the proposition.

(f) We'll first see that $\sin(\pi x) \ge 2x$ whenever $0 \le x \le \frac{1}{2}$. But this is because the $h(x) := \sin(\pi x) - 2x = 0$ only for $x = 0$ and $x = 2$ and the derivative of h, $h'(x) := \pi\cos(\pi x) - 2 = 0$, only for a unique real number $r \in [0, \frac{1}{2}]$.Now since f is smooth, f cannot have more roots in $[0, \frac{1}{2}]$. Hence $\sin(\pi x) \ge 2x$ for all $x \in [0, \frac{1}{2}]$

Now we'll get to the main proof. Assume $0 < \delta \le \frac{1}{2}$. Then by previous result, when $0 < x \le \frac{1}{2}$,

$$\begin{aligned}
\Delta_N(x) &= \frac{1}{N}\left(\frac{\sin(\pi Nx)}{\sin(\pi x)}\right)^2 \\
&\le \frac{1}{N\sin^2(\pi x)} \\
&\le \frac{1}{4Nx^2} && \text{since } \sin(\pi x) \ge 2x
\end{aligned}$$

We also know that

$$\Delta_N(x) = 1 + 2\sum_{n=1}^{N}\left(1 - \frac{n}{N}\right)\cos(2\pi nx)$$

But this shows that $\Delta_N(x)$ is maximum when $\cos(2\pi nx)$ is maximum, i.e at $x = 0, 1$. At both cases $\cos(2\pi nx) = 1$. Also

$$1 + 2\sum_{n=1}^{N}\left(1 - \frac{n}{N}\right) = 1 + 2N - \frac{2}{N}\sum_{n=1}^{N} n = 1 + 2N - \frac{2}{N}\frac{N(N+1)}{2} = N$$

Therefore, $\Delta_N(x) \leq N$ for all $0 \leq x \leq 1$ Hence combining this with the result above, we get that while $0 \leq x \leq \frac{1}{2}$

$$\Delta_N(x) \leq \min\left(N, \frac{1}{4Nx^2}\right)$$

(g) From last result we know that when $0 < \delta \leq \frac{1}{2}$,

$$\Delta_N(x) \leq \frac{1}{4Nx^2}$$

Then,

$$\int_{\delta}^{\frac{1}{2}} \Delta_N(x)dx \leq \int_{\delta}^{\frac{1}{2}} \frac{1}{4Nx^2}dx \leq \int_{\delta}^{\infty} \frac{1}{4Nx^2}dx = \frac{1}{4N\delta}$$

(h)

$$
\begin{aligned}
(\Delta_N * f)(x) &= \int_0^1 \Delta_N(y) f(x-y)\, dy \\
&= \int_0^1 \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e^{2\pi i n y} f(x-y)\, dy \\
&= \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) \int_0^1 f(x-y) e^{2\pi i n y}\, dy \\
&= \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x} \int_0^1 f(x-y) e^{-2\pi i n (x-y)}\, dy \\
&= \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x} \int_x^{x-1} -f(v) e^{-2\pi i n v}\, dv \\
&= \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x} \int_{x-1}^{x} f(v) e^{-2\pi i n v}\, dv \\
&= \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x} \int_0^1 f(v) e^{-2\pi i n v}\, dv && \text{by lemma 3.1} \\
&= \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x} \widehat{f}(n) \\
&= \sigma_N(x)
\end{aligned}
$$

□

Proposition 3.4 Fejér kernel as defined in 3.10 is a summability kernel as in definition 3.9

Proof. To prove that Fejér kernel is a summability kernel, we'll verify the three properties given in 3.9.

1. We proved that $\int_0^1 \Delta_N(x) = 1$ at proposition 3.3
2. From proposition 3.3, we know that

$$
\Delta_N(x) = \begin{cases} \frac{1}{N}\left(\frac{\sin(\pi N x)}{\sin(\pi x)}\right)^2, & \text{if } x \notin \mathbb{Z} \\ N, & \text{if } x \in \mathbb{Z} \end{cases}
$$

Since $N \in \mathbb{N}$, this implies $\Delta_N(x) \geq 0$. Therefore,

$$\int_0^1 |\Delta_N(x)| \, dx = \int_0^1 \Delta(x) dx = 1$$

This proves the 2^{nd} condition for the summability kernel.

3. To prove that

$$\lim_{N\to\infty} \int_\delta^{1-\delta} |\Delta_N(x)| \, dx = 0$$

we note that from proposition 3.3 if $0 < \delta \leq \frac{1}{2}$,

$$\int_\delta^{\frac{1}{2}} \Delta_N(x) dx \leq \frac{1}{4N\delta}$$

and

$$\int_{\frac{1}{2}}^{1-\delta} \Delta_N(x) = \int_\delta^{\frac{1}{2}} \Delta_N(x)$$

Hence,

$$\int_\delta^{1-\delta} \Delta_N(x) = \int_\delta^{\frac{1}{2}} \Delta_N(x) + \int_{\frac{1}{2}}^{1-\delta} \Delta_N(x) = 2\int_\delta^{\frac{1}{2}} \Delta_N(x) \leq \frac{1}{2N\delta}$$

Therefore, for $0 < \delta \leq \frac{1}{2}$, we have

$$\lim_{N\to\infty} \int_\delta^{1-\delta} |\Delta_N(x)| = \lim_{N\to\infty} \int_\delta^{1-\delta} \Delta_N(x) = 0$$

If $\frac{1}{2} < \delta < 1$ then by change of variable $y = 1 - x$

$$\lim_{N\to\infty} \int_\delta^{1-\delta} \Delta_N(x) = \lim_{N\to\infty} -\int_{1-\delta}^{\delta} \Delta_n(y) = 0$$

And therefore for all $0 < \delta < 1$

$$\lim_{N\to\infty} \int_\delta^{1-\delta} |\Delta_N(x)| = 0$$

Which completes the proof that Fejeŕ kernel is a summability kernel.

□

3.5 Convergence of Fourier Series

Now with the help of summability kernels and convolution we'll prove an important theorem which will serve as a backbone for the discussion of convergence forward.

Theorem 3.2 – Convergence to convolution of summability kernels If $f \in L^1(\mathbb{T})$ and K_N is a summability kernel then $f * K_N$ converges to f in L^1 norm. That is

$$\lim_{N\to\infty} \int_0^1 |f(x) - (f * K_N)(x))| = 0$$

Proof.

$$f * K_N(x) = \int_0^1 f(x-y)K_N(y)dy$$

Since $\int_0^1 K_N(y)dy = 1$, by the 2^{nd} property of summability kernel,

$$f(x) - f * K_N(x) = \int_0^1 (f(x) - f(x-y))\, K_N(y)dx$$

Now then,

$$\begin{aligned}
\int_0^1 |f(x) - K_N(x)|dx &= \int_0^1 \left| \int_0^1 (f(x) - f(x-y))K_N(y)dy \right| dx \\
&\le \int_0^1 \int_0^1 |f(x) - f(x-y)K_N(y)|\, dy\ dx \\
&= \int_0^1 |K_N(y)| \int_0^1 |f(x) - f(x-y)|dx\ dy \qquad \text{by Tonelli's theorem} \\
&= \int_{-\delta}^{1-\delta} \qquad \text{by lemma 3.1} \\
&= \int_{-\delta}^{\delta} + \int_{\delta}^{1-\delta} = I_1 + I_2
\end{aligned}$$

We'll show that for a given ϵ we can find an N such that for all $n > N,\ I_1 + I_2 < \epsilon$

Since $f \in L^1(\mathbb{T})$ we can find $\delta > 0$ such that $\int_0^1 |f(x+\delta) - f(x)|dx = \epsilon/2C$, where C is the constant in the second conditon of summability kernel. The proof can be found in any measure theory textbook.

Then,

$$|I_1| \le \frac{\epsilon}{2C} \int_{-\delta}^{\delta} |K_N(y)|dy \le \frac{\epsilon}{2C} \int_0^1 |K_N(y)|dy \le \frac{\epsilon}{2}$$

For I_2, we see that

$$\int_0^1 |f(x) - f(x-y)|dx \leq \int_0^1 |f(x)|dx + \int_0^1 |f(x-y)|dx = 2\|f\|_1$$

Then,

$$|I_2| \leq 2\|f\|_1 \int_\delta^{1-\delta} |K_N(y)|dy$$

But by the 3^{rd} property of the summability kernel, we know that the integral in the above converges to zero. Therefore there exists an N such that for an $n > N, |I_2| < \epsilon/2$, which completes our proof. □

Corollary 3.1 Convergence of Cesáro sum of functions in $L^1(\mathbb{T})$ If $f \in L^(\mathbb{T})$ then $\sigma_N(x)$, the Cesáro sum of the Fourier series of f converge to $f(x)$ in L^1 norm. That is,

$$\lim_{N\to\infty} \int_0^1 |f(x) - \sigma_N(x)| = 0$$

Proof. Since we know that Fejér kernel, $\Delta_N(x)$ is a summability kernel by proposition 3.4 and that $(\Delta_N * f)(x) = \sigma_N(x)$ by prop 3.3, the result follows from theorem 3.2. □

Theorem 3.3 – Fejér's Theorem Let $f \in L^1(\mathbb{T})$ and $f(a^-) = \lim_{x\to a^-} f(x)$ and $f(a^+) = \lim_{x\to a^+} f(x)$ exist and are finite, then

$$\lim_{N\to\infty} \sigma_N(x) = \frac{f(x^-) + f(x^+)}{2}$$

Proof. Let ϵ be given. Since we assumed $f(x^+)$ and $f(x^-)$ exist and are finite, we can take $\delta < \frac{1}{2}$ small enough such that $|f(x-u) - f(x^-)| < \epsilon$ for $0 \leq u \leq \delta$ and $|f(x+u) - f(x^+)| < \epsilon$ for $1-\delta \leq u \leq 1$.

$$\sigma_N(x) = \int_0^1 f(x-u)\Delta_N(u)du = \int_0^\delta + \int_\delta^{1-\delta} + \int_{1-\delta}^1 = I_1 + I_2 + I_3$$

Then,

$$I_1 = \int_0^\delta (f(x-u) - f(x^-))\Delta_N(u)du + \int_0^\delta f(x^-)\Delta_N(u)du = T_1 + T_1^{'}$$

where by the property of summability kernels as in 3.9, and by our choice of δ, we have

$$|T_1| \leq \int_0^\delta |f(x-u) - f(x^-)|\Delta_N(u)du < \epsilon \int_0^\delta \Delta_N(x) \leq \epsilon$$

Therefore T_1 converge to 0 as $N \to \infty$.

Again, from the proposition 3.3, we get

$$\frac{1}{2} \geq \int_0^\delta \Delta_N(x)dx = \int_0^{\frac{1}{2}} \Delta_N(x)dx - \int_\delta^{\frac{1}{2}} \Delta_N(x)dx \geq \frac{1}{2} - \frac{1}{4N\delta}$$

which then implies that,

$$\frac{f(x^-)}{2} \ \geq \ T_1' = f(x^-)\int_0^\delta \Delta_N(x)dx \ \geq \ \frac{f(x^-)}{2} - \frac{f(x^-)}{4N\delta}$$

Therefore,

$$0 \geq T_1' - \frac{f(x^-)}{2} \geq -\frac{f(x^-)}{4N\delta}$$

and hence,

$$\left|T_1' - \frac{f(x^-)}{2}\right| \leq \left|\frac{f(x^-)}{4N\delta}\right|$$

which implies that $I_1 = T_1 + T_1'$ converge to $\frac{f(x^-)}{2}$ as $N \to \infty$.

Now by proposition 3.3 we know that if $0 < x \leq \frac{1}{2}$, then

$$\Delta_N(x) \leq \frac{1}{4Nx^2}$$

which implies that

$$|I_2| \leq \frac{1}{4N\delta^2}\int_\delta^{1-\delta} f(x-u)du \leq \frac{1}{4N\delta^2}\int_0^1 |f(u)|du = \frac{\|f\|_1}{4N\delta^2}$$

Therefore I_2 converge to 0 as $N \to \infty$

Now we'll prove that I_3 converge to $\frac{f(x^+)}{2}$. For this we'll split I_3 into T_3 and T_3' like we did with I_1.

$$I_3 = \int_{1-\delta}^1 (f(x-u) - f(x^+))\Delta_N(u)du + f(x^+)\int_{1-\delta}^1 \Delta_N(u)du = T_3 + T_3'$$

Then,

$$|T_3| \leq \int_{1-\delta}^1 |((f(x-u) - f(x^+))|\Delta_N(u)du < \epsilon \int_{1-\delta}^1 \Delta_N(x)dx \leq \epsilon$$

Therefore T_3 converge to 0 as $N \to \infty$. Also

$$\int_{1-\delta}^{1} \Delta_N(x)dx = -\int_{\delta}^{0} \Delta_N(x)dx = \int_{0}^{\delta} \Delta_N(x)dx$$

Therefore by the same inequality we used for T_1' in 3.3, we'll get

$$\left| T_3' - \frac{f(x^+)}{2} \right| \leq \left| \frac{f(x^+)}{4N\delta} \right|$$

which implies that $I_3 = T_3 + T_3'$ converge to $\frac{f(x^+)}{2}$ as $N \to \infty$

Therefore since $\sigma_N(x) = I_1 + I_2 + I_3$, by the algebra of limits,

$$\lim_{N \to \infty} \sigma_N(x) = \frac{f(x^-) + f(x^+)}{2}$$

$\square$

Theorem 3.4 – Hardy Tauberian Theorem Let $\sum_{n=1}^{\infty} a_n$ is Cesáro summable to a, then if there exist a constant C such that

$$|a_n| \leq \frac{C}{n}$$

for all n, then the series $\sum_{n=1}^{\infty} a_n$ converge to a

4 Fourier Series in $L^p(\mathbb{T})$

Before we go into the general L^p, we'll first recall some importatnt results from functional analysis which are essential for the discussion of further topics

Recall that in definiton 3.1, we've discussed what is an L^p function for a general space S. Here the space is $\mathbb{T}$ and $L^p(\mathbb{T})$ are precisely the set of all L^p functions in $\mathbb{T}$.

Theorem 4.1 – Holder's Inequality Let $1 \leq p \leq \infty$ and q such that $1/p + 1/q = 1$ (for convention we will assume that the tuples $(1, \infty)$, and $(\infty, 1)$, satify the above relation) then for lebesgue measure space S and functions $f \in L^p(S)$, and $g \in L^q(S)$

$$\left| \int_S f(x)g(x)dx \right| \leq \|f\|_p \|g\|_q$$

where

$$\|f\|_p = \left(\int_s |f(x)|^p \right)^{\frac{1}{p}}$$

as in the definition of L^p norm in 3.1.

Theorem 4.2 – Minkowski's Inequality Let $1 \leq p \leq \infty$ then for a lebesgue measure space S and a fucntion $f \in L^p(S)$, then

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

The proofs of the above two theorems can be found in any measure theory textbook and since it is a common proof, we'll ommit it from detailing it here.

4.1 Fourier Series in $L^2(\mathbb{T})$

Before we proceed with the fourier coefficients and fourier series for $L^2(\mathbb{T})$ functions we first prove some important results.

Proposition 4.1 If $f \in L^2(\mathbb{T})$, then

$$\lim_{\delta \to 0} \int_0^1 |f(x + \delta) - f(x)|^2 dx = 0$$

Proof. From measure theory we know that continuous functions are dense in $L^2(\mathbb{T})$. Then given any ϵ there exists a function $g \in C(\mathbb{T})$ such that $\|f-g\|_2 < \epsilon$. Then,

$$f(x+\delta) - f(x) = (f(x+\delta) - g(x+\delta)) - (f(x) - g(x)) + (g(x+\delta) - g(x))$$

Now by triangle inequality, (i.e minkowski's inequality for $p=2$),

$$\|f(x+\delta) - f(x)\|_2 = \|f(x+\delta) - g(x+\delta)\|_2 - \|f(x) - g(x)\|_2 + \|g(x+\delta) - g(x)\|_2$$

Now since g is continuous on $\mathbb{T}$ it is uniformly continuous (since it is continuous on $\mathbb{R}$, it'll be continuous on $[0,1]$, a compact set) and therefore δ can be taken such that $\|g(x+\delta) - g(x)\|_2 < \epsilon$. Hence the theorem. $\square$

> **Proposition 4.2** If $f, g \in L^2(\mathbb{T})$, then their convolution, $f * g$ is continuous and moreover $\|f * g\|_\infty \leq \|f\|_2 \|g\|_2$

Proof.

$$(f*g)(x+\delta) - (f*g)(x) = \int_0^1 f(u)(g(x+\delta-u) - g(x-u))du$$

Therefore by Cauchy Shwarz inequality,

$$\begin{aligned}
|f*g(x+\delta) - f*g(x)| &= \left| \int_0^1 f(u)(g(x+\delta-u) - g(x-u))du \right| \\
&\leq \|f\|_2 \left(\int_0^1 |g(x+\delta-u) - g(x-u)|^2 du \right)^{1/2}
\end{aligned}$$

which converge to zero as $\delta \to 0$ by proposition 4.1. Therefore $f * g$ is continuous in $\mathbb{T}$. Also,

$$\begin{aligned}
|f*g(x)| &= \left| \int_0^1 f(u)g(x-u)du \right| \\
&\leq \left(\int_0^1 |f(u)|^2 du \right)^{1/2} \left(\int_0^1 |g(u)|^2 du \right)^{1/2} \\
&= \|f\|_2 \|g\|_2
\end{aligned}$$

Hence the theorem. $\square$

Now that we've proved some important results, we'll define the fourier coeffients $\widehat{f}(n)$ the same way we defined them for $Ł^1(\mathbb{T})$ in definition 3.4 as

$$\hat{f}(n) = \int_0^1 f(x)e^{-2\pi inx}dx$$

Note that $\widehat{f}(n)$ is a finite quantity since $|e^{ix}| = 1$ and,

$$|\widehat{f}(n)| = \left|\int_0^1 f(x)e^{-2\pi inx}dx\right| \leq \|f\|_2 \left(\int_0^1 |e^{-2\pi inx}|^2 dx\right)^{1/2} = \|f\|_2$$

Hence we define the fourier series the same way as in definition 3.5. Then we'll investigate if the Fourier series of $L^2(\mathbb{T})$ functions are Cesáro summable to f. In fact this is true.

Theorem 4.3 – Fourier series of $L^2(\mathbb{T})$ functions are Cesáro summable
If $f \in L^2(\mathbb{T})$, and $\sigma_N(x)$ is the N^{th} Cesáro partial sum of the Fourier series of f, then

$$\lim_{N\to\infty} \int_0^1 |f(x) - \sigma_N(x)|^2 \, dx = 0$$

Proof. From proposition 3.3 we know that

$$\sigma_N(x) = \int_0^1 \Delta_N(u)f(x-u) \, du$$

Hence,

$$f(x) - \sigma_N(x) = \int_0^1 (f(x) - f(x-u))\Delta_N(u) \, du$$

By Holder's inequality as in 4.1 applied to $(f(x)-f(x-u))\sqrt{\Delta_N(u)}$ and $\sqrt{\Delta_N(u)}$

$$\begin{aligned}|f(x) - \sigma_N(x)| &= \left|\int_0^1 (f(x) - f(x-u))\Delta_N(u)\right| \, du \\ &\leq \left(\int_0^1 |f(x) - f(x-u)|^2 \Delta_N(u) \, du\right)^{1/2} \left(\int_0^1 \Delta_N(u) \, du\right)^{1/2}\end{aligned}$$

Since by 3.3, $\int_0^1 \Delta_N(x)dx = 1$ and by Tonelli's theorem,

$$\begin{aligned}
\int_0^1 |f(x) - \sigma_N(x)|^2 dx &= \int_0^1 \int_0^1 |f(x) - f(x-u)|^2 \Delta_N(u)\ du\ dx \\
&= \int_0^1 \Delta_N(u) \int_0^1 |f(x) - f(x-u)|^2\ dx\ du \\
&= \int_{-\delta}^{\delta} + \int_{\delta}^{1-\delta} \\
&= I_1 + I_2
\end{aligned}$$

Also from proposition 4.1, given ϵ, we can find δ such that

$$\int_0^1 |f(x) - f(x-\delta)|^2\ dx < \epsilon$$

Then for that choice of δ,

$$|I_1| \le \epsilon \int_{-\delta}^{\delta} \Delta_N(u)du \le \epsilon \int_0^1 \Delta_N(u)du = \epsilon$$

To prove I_2 is also bounded, we'll use a small trick aided by Minkowski's inequality. We know that $\|f-g\|_p \le \|f\|_p + \|g\|_p$. Therefore $\|f-g\|_p \le 2\max\{\|f\|_p, \|g\|_p\}$ and $\|f-g\|_p^p \le 2\max\{\|f\|_p^p, \|g\|_p^p\}$, and finally, $\|f-g\|_p^p \le 2(\|f\|_p^p + \|g\|_p^p)$.Then for $p = 2$,

$$\int_0^1 |f(x) - f(x-u)|^2 dx \le 2(\|f\|_2^2 + \|f\|_2^2) = 4\|f\|_2^2$$

Therefore by proposition 3.3, we get

$$|I_2| = \int_{\delta}^{1-\delta} \Delta_N(u) \int_0^1 |f(x) - f(x-u)|^2\ dx\ du \le 4\|f\|_2^2 \int_{\delta}^{1-\delta} \Delta_N(u)du$$

But by proposition 3.3

$$\int_{\delta}^{1-\delta} \Delta_N(u)du = 2\int_{\delta}^{1/2} \Delta_N(x) \le \frac{1}{2N\delta}$$

Which implies,

$$|I_2| \le \frac{2\|f\|_2^2}{N\delta}$$

Therefore I_2 converge to 0 as $N \to \infty$. Hence the theorem. □

4.2 Fourier Series in $L^p(\mathbb{T})$

Proposition 4.3 If $f \in L^p(\mathbb{T})$, then

$$\lim_{\delta \to 0} \int_0^1 |f(x+\delta) - f(x)|^p dx = 0$$

Proof. From measure theory we know that continuous functions are dense in $L^p(\mathbb{T})$. Then given any ϵ there exists a function $g \in C(\mathbb{T})$ such that $\|f - g\|_p < \epsilon$. Then,

$$f(x+\delta) - f(x) = (f(x+\delta) - g(x+\delta)) - (f(x) - g(x)) + (g(x+\delta) - g(x))$$

Now by triangle inequality, (i.e minkowski's inequality for $p = 2$),

$$\|f(x+\delta) - f(x)\|_p = \|f(x+\delta) - g(x+\delta)\|_p - \|f(x) - g(x)\|_p + \|g(x+\delta) - g(x)\|_p$$

Now since g is continuous on $\mathbb{T}$ it is uniformly continuous (since it is continuous on $\mathbb{R}$, it'll be continuous on $[0,1]$, a compact set) and therefore δ can be taken such that $\|g(x+\delta) - g(x)\|_p < \epsilon$. Hence the theorem. □

Proposition 4.4 If $f \in L^p(\mathbb{T})$ and $g \in L^q(\mathbb{T})$, then their convolution, $f * g$ is continuous and moreover $\|f * g\|_\infty \leq \|f\|_p \|g\|_q$

Proof.

$$(f * g)(x+\delta) - (f * g)(x) = \int_0^1 f(u)(g(x+\delta-u) - g(x-u))du$$

Therefore by Minkowski's inequality from theorem 4.2,

$$\begin{aligned} |f * g(x+\delta) - f * g(x)| &= \left| \int_0^1 f(u)(g(x+\delta-u) - g(x-u))du \right| \\ &\leq \|f\|_p \left(\int_0^1 |g(x+\delta-u) - g(x-u)|^q du \right)^{1/q} \end{aligned}$$

which converge to zero as $\delta \to 0$ by proposition 4.3. Therefore $f * g$ is continuous in $\mathbb{T}$. Also,

$$\begin{aligned} |f * g(x)| &= \left| \int_0^1 f(u) g(x-u) du \right| \\ &\leq \left(\int_0^1 |f(u)|^p du \right)^{1/p} \left(\int_0^1 |g(u)|^q du \right)^{1/q} \\ &= \|f\|_p \|g\|_q \end{aligned}$$

Hence the theorem. □

Now that we've proved some important results, we'll define the fourier coeffients $\widehat{f}(n)$ the same way we defined them for $Ł^1(\mathbb{T})$ in definition 3.4 as

$$\hat{f}(n) = \int_0^1 f(x)e^{-2\pi inx}dx$$

Note that $\widehat{f}(n)$ is a finite quantity since $|e^{ix}| = 1$ and,

$$|\widehat{f}(n)| = \left|\int_0^1 f(x)e^{-2\pi inx}dx\right| \leq \|f\|_p \left(\int_0^1 |e^{-2\pi inx}|^q dx\right)^{1/q} = \|f\|_p$$

where $1/p + 1/q = 1$.

Hence we define the fourier series the same way as in definition 3.5. Then we'll investigate if the Fourier series of $L^p(\mathbb{T})$ functions are Cesáro summable to f. In fact this is true.

Theorem 4.4 – Fourier series of $L^p(\mathbb{T})$ functions are Cesáro summable
If $f \in L^p(\mathbb{T})$, and $\sigma_N(x)$ is the N^{th} Cesáro partial sum of the Fourier series of f, then

$$\lim_{N\to\infty} \int_0^1 |f(x) - \sigma_N(x)|^p \; dx = 0$$

Proof. From proposition 3.3 we know that

$$\sigma_N(x) = \int_0^1 \Delta_N(u) f(x-u) \; du$$

Hence,

$$f(x) - \sigma_N(x) = \int_0^1 (f(x) - f(x-u))\Delta_N(u) \; du$$

By Holder's inequality as in 4.1 applied to $(f(x) - f(x-u))(\Delta_N(u))^{1/p}$ and $(\Delta_N(u))^{1/q}$

$$\begin{aligned} |f(x) - \sigma_N(x)| &= \left|\int_0^1 (f(x) - f(x-u))\Delta_N(u)\right| \; du \\ &\leq \left(\int_0^1 |f(x) - f(x-u)|^p \Delta_N(u) \; du\right)^{1/p} \left(\int_0^1 \Delta_N(u) \; du\right)^{1/q} \end{aligned}$$

Since by 3.3, $\int_0^1 \Delta_N(x)dx = 1$ and by Tonelli's theorem,

$$\begin{aligned}
\int_0^1 |f(x) - \sigma_N(x)|^2 dx &= \int_0^1 \int_0^1 |f(x) - f(x-u)|^p \Delta_N(u)\ du\ dx \\
&= \int_0^1 \Delta_N(u) \int_0^1 |f(x) - f(x-u)|^p\ dx\ du \\
&= \int_{-\delta}^{\delta} + \int_{\delta}^{1-\delta} \\
&= I_1 + I_2
\end{aligned}$$

Also from proposition 4.3, given ϵ, we can find δ such that

$$\int_0^1 |f(x) - f(x-\delta)|^p\ dx < \epsilon$$

Then for that choice of δ,

$$|I_1| \leq \epsilon \int_{-\delta}^{\delta} \Delta_N(u)du \leq \epsilon \int_0^1 \Delta_N(u)du = \epsilon$$

To prove I_2 is also bounded, we'll use a small trick aided by Minkowski's inequality. We know that $\|f-g\|_p \leq \|f\|_p + \|g\|_p$. Therefore $\|f-g\|_p \leq 2\max\{\|f\|_p, \|g\|_p\}$ and $\|f-g\|_p^p \leq 2\max\{\|f\|_p^p, \|g\|_p^p\}$, and finally, $\|f-g\|_p^p \leq 2(\|f\|_p^p + \|g\|_p^p)$.Then,

$$\int_0^1 |f(x) - f(x-u)|^p dx \leq 2(\|f\|_p^p + \|f\|_p^p) = 4\|f\|_p^p$$

Therefore by proposition 3.3, we get

$$|I_2| = \int_{\delta}^{1-\delta} \Delta_N(u) \int_0^1 |f(x) - f(x-u)|^p\ dx\ du \leq 4\|f\|_p^p \int_{\delta}^{1-\delta} \Delta_N(u)du$$

But by proposition 3.3

$$\int_{\delta}^{1-\delta} \Delta_N(u)du = 2\int_{\delta}^{1/2} \Delta_N(x) \leq \frac{1}{2N\delta}$$

Which implies,

$$|I_2| \leq \frac{2\|f\|_p^p}{N\delta}$$

Therefore I_2 converge to 0 as $N \to \infty$. Hence the theorem. □

If looked close enough one can see that the proof of convergence of fourier series in $L^p(\mathbb{T})$ is almost the same as in $L^2(\mathbb{T})$. This is in fact true and L^2 convergence is just a special case of L^p convergence.

5 Fourier Transform

5.1 Definiton and basic properties

While defining Fourier series we were mainly focused on periodic functions. Now we'll try to expand that into another set of functions. We'll be interested on functions in $L^1(\mathbb{R})$, that is those real or complex valued functions $f(x)$ in $\mathbb{R}$ for which

$$\int_{-\infty}^{\infty} |f(x)|\ dx < \infty$$

. For those functions in $L^1(\mathbb{R})$ the integral above will be called the L^1 norm of the function f and will be denoted by $\|f\|_{L^1(\mathbb{R})}$ or in short $\|f\|_1$. Also note that the notations $\int_{\mathbb{R}}$ and $\int_{-\infty}^{\infty}$ means the same and we might use them interchangably as we see fit.

Analogus to what we did in finding the n^{th} fourier coefficient in definition 3.4, we'll define the fourier transform of f

Definition 5.1 – Fourier transform of a function f Let $f \in L^1(\mathbb{R})$, then we define the Fourier tranform of f as

$$\widehat{f}(t) = \int_{\mathbb{R}} f(x)e^{-2\pi itx}\ dx$$

Note that while we say $\widehat{f}$ is the Fourier transform of the function f, the term "Fourier tranform" is also used for the map which takes f to $\widehat{f}$.

Also note that $\widehat{f}(t)$ is a finite quantity (real or complex) for all $t \in \mathbb{T}$ since $f \in L^1(\mathbb{T})$ and $|e^{-2\pi itx}| = 1$ implies

$$|\widehat{f}(t)| \leq \int_{-\infty}^{\infty} |f(x)|\, dx < \infty$$

By the linearity of the integral we can also show that for functions $f, g \in L^2(\mathbb{R})$ and scalars μ, ν, $\widehat{\mu f + \nu g}(t) = \mu\widehat{f}(t) + \nu\widehat{g}(t)$.

Now we'll prove some important properties of Fourier transforms. Note that this will almost remind you of the properties of Fourier coefficinets in proposition 3.1.

Proposition 5.1 – Properties of Fourier transform If $f \in L^1(\mathbb{R})$ and $\widehat{f}$ is the Fourier tranform of f as in definition 5.1, then

(a) If $a \in \mathbb{R}$ and $g(x) = f(x+a)$ for all $x \in \mathbb{R}$, then $g \in L^1(\mathbb{R})$ and $\widehat{g}(t) = e^{2\pi ita}\widehat{f}(t)$ for all t.

(b) If $b \in \mathbb{R}$ and $h(x) = e^{2\pi bx} f(x)$, then $h \in L^1(\mathbb{R})$ and $\widehat{h}(t) = \widehat{f}(t-b)$ for all t.

(c) If $c \in \mathbb{R}$ is not 0, and $j(x) = f(cx)$, then $j \in L^1(\mathbb{R})$ and $\widehat{j}(t) = \frac{\widehat{f}(t/c)}{|c|}$ for all t.

(d) if $l(x) = \overline{f(x)}$, then $l \in L^1(\mathbb{R})$ and $\widehat{l}(t) = \overline{\widehat{f}(-t)}$

Proof. Note that by appropriate change of variable we can see that all the above functions g, h, j, l are in $L^1(\mathbb{R})$. We'll prove the other properties.

(a) By the change of variable $y = x + a$, we get that

$$\widehat{g}(t) = \int_{\mathbb{R}} g(x)e^{-2\pi itx}\, dx = \int_{\mathbb{R}} f(x+a)e^{-2\pi itx}\, dx = e^{2\pi ita} \int_{\mathbb{R}} f(y)e^{-2\pi ity}\, dx$$

which is equal to $e^{2\pi ita}\widehat{f}(t)$

(b)

$$\widehat{h}(t) = \int_{\mathbb{R}} h(x)e^{-2\pi itx}\, dx = \int_{\mathbb{R}} f(x)e^{-2\pi i(t-b)x}\, dx = \widehat{f}(t-b)$$

(c) Here we'll need to be careful because c maybe negative.Assume $c > 0$, then by a change of variable $y = cx$, we get

$$\widehat{j}(t) = \int_{-\infty}^{\infty} j(x)e^{-2\pi itx}\, dx = \int_{-\infty}^{\infty} f(cx)e^{-2\pi itx}\, dx = \frac{1}{c}\int_{-\infty}^{\infty} f(y)e^{-2\pi i\frac{t}{c}y}$$

Then if $c > 0$, $\widehat{j}(t) = \frac{\widehat{f}(t/c)}{c}$. Now if $c < 0$ the limits of integration will reverse, i.e

$$\int_{-\infty}^{\infty} f(cx)e^{-2\pi itx}\, dx = \frac{1}{c}\int_{\infty}^{-\infty} f(y)e^{-2\pi i\frac{t}{c}y}\, dx = \frac{1}{-c}\int_{-\infty}^{\infty} f(y)e^{-2\pi i\frac{t}{c}y}\, dx$$

Which shows that if $c \neq 0$, $\widehat{j}(t) = \frac{\widehat{f}(t/c)}{|c|}$.

(d) Since we know that integral of the conjugate is the conjugate of the integral,

$$\widehat{l}(t) = \int_{\mathbb{R}} \overline{f(x)}e^{-2\pi itx} = \int_{\mathbb{R}} \overline{f(x)e^{-2\pi i(-t)x}} = \overline{\int_{\mathbb{R}} f(x)e^{-2\pi i(-t)x}} = \overline{\widehat{f}(-t)}$$

$\square$

5.2 Fourier tranforms in $L^2(\mathbb{R})$

Similar to how we defined Fourier transforms for functions in $L^1(\mathbb{R})$ in definition 5.1, we can define the same for functions in $L^2(\mathbb{R})$. For any function $f \in L^2(\mathbb{R})$, we define the Fourier transform of f as

$$\widehat{f}(t) = \int_{\mathbb{R}} f(x)e^{-2\pi itx}\, dx$$

.