A Study in Fourier Analysis

From circle, through the line, to the complex

Joel Sleeba

IISER Thiruvananthapuram

May 14, 2023

Fourier Series

» Structure and Topology of $\mathbb{T}$

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$$g : \mathbb{T} \to [0,1) := [x] \to \{x\} \tag{1}$$

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* Define Lebesgue measure μ on $\mathbb{T}$ as

$$\mu(A) = \lambda(g(A)) \tag{3}$$

where λ is the Lebesgue measure on $\mathbb{R}$.

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* Also by the Lebesgue measure on $\mathbb{T}$, we say $f \in L^p(\mathbb{T})$ if the corresponding function in $[0, 1)$ is in $L^p[0, 1)$.
* For any two function $f, g \in L^1(\mathbb{T})$, their convolution, $(f * g)(x)$ defined as

$$(f * g)(x) = \int_0^1 f(x - y)g(y)\ dy \tag{4}$$

is again in $L^1(\mathbb{T})$

» Fourier Coefficients

* For $f \in L^1(\mathbb{T})$, and $n \in \mathbb{Z}$ we define the n^{th} Fourier coefficient of f as

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* Since we are interested in the convergence of the Fourier series, we will define the symmetric and Cesàro partial sums of the Fourier series respectively as

$$S_N(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{2\pi i n x} \quad \text{and} \quad \sigma_N(x) = \frac{1}{N} \sum_{n=0}^{N-1} S_n(x)$$

» Summability Kernel

* A collection of functions $K_N \in L^1(\mathbb{T})$ are called a summability kernel if it satisfies the following properites
  1. $\int_0^1 K_N(x)dx = 1$
  2. $\int_0^1 |K_N(x)|dx \leq C$ for some constant $C > 0$
  3. $\lim_{N\to\infty} \int_\delta^{1-\delta} |K_N(x)|dx = 0$

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  3. $\lim_{N\to\infty} \int_\delta^{1-\delta} |K_N(x)|dx = 0$
- We prove that if K_N is a summability kernel in $L^1(\mathbb{T})$, then $(f * K_N)(x)$ converge to $f(x)$ in $L^1(\mathbb{T})$. That is

$$\int_0^1 |f(x) - (f * K_N)(x)|\ dx \tag{5}$$

» Fejér Kernel and Cesàro Convergence

* Fejér kernel defined as

$$\Delta_N(x) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e^{2\pi inx} = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=-n}^{n} e^{2\pi ikx}$$

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- Then if $f \in L^1(\mathbb{T})$ such that $\hat{f} = 0$, we get that $f \overset{a.e}{=} 0$
- Therefore if $f, g \in L^1(\mathbb{T})$ such that $\hat{f} = \hat{g}$, then $f \overset{a.e}{=} g$

» Fourier Series in $L^2(\mathbb{T})$

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- Moreover we see that if $f \in L^2(\mathbb{T})$, since the Fejér kernel, $\Delta_N(x) \leq N$, its Cesàro partial sum, $\sigma_N = (f * \Delta_N) \in L^2(\mathbb{T})$

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* Then we get that the Cesàro partial sums σ_N converge to f in $L^2(\mathbb{T})$. That is

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- The same results follow for functions in $L^p(\mathbb{T})$

» Fejér's Theorem and Pointwise Convergence

* (Fejér's Theorem) If $f \in L^1(\mathbb{T})$, then

$$\lim_{N\to\infty} \sigma_N(x) = \frac{f(x^-) + f(x^+)}{2} \tag{6}$$

given that $f(x^-)$ and $f(x^+)$, the left limit and right limit of f at x exists.

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* Therefore if f is continuous then the Cesàro partial sum converge pointwise to f everywhere.

Fourier Transforms in $\mathbb{R}$

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$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i t x} \, dx \tag{7}$$

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* (Riemann Lebesgue Lemma) $\hat{f}$ is uniformly continuous and

$$\lim_{|t| \to \infty} \hat{f}(t) = 0 \tag{8}$$

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* Generalizing further we get that if $f, \hat{f} \in L^1(\mathbb{R})$ then

$$\check{\hat{f}} \overset{a.e}{=} f \tag{11}$$

» Fourier transforms in $L^2(\mathbb{R})$

* We consider the space $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Since it is a subspace of $L^1(\mathbb{R})$, the definition of Fourier transform and inverse transform in $L^1(\mathbb{R})$ holds good in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

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* (Plancherel's Theorem) If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then

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* Now since the collection of compactly supported continuous functions in $\mathbb{R}$, $C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for all $1 \leq p < \infty$, we get that $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$.

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* Plancherel's theorem asserts that Fourier transform in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is an isometry therefore we can extend Fourier transform to an isometry in $L^2(\mathbb{R})$.

Holomorphic Fourier Transforms

» Extending Domain to $\mathbb{C}$

* Fourier transform of certain functions can be extended into a holomorphic functions in certain regions. That is for $z \in \mathbb{C}$,

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* For example if $f(x) = e^{-|x|}$, then its Fourier transform, $\hat{f}(t) = \frac{1}{1+(2\pi t)^2}$ can be extended into holomorphic function in regions in the complex plane without the points $\pm\frac{i}{2\pi}$.

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* We will focus on two types of functions in $L^2(\mathbb{R})$
  1. $f(x) = 0, (x < 0)$
  2. $f(x) = 0, (x \notin (-A, A))$

» Paley Wiener Theorem 1

The following statements are equivalent

1. $F \in L^2(\mathbb{R})$ such that F is essentially supported in $(0, \infty)$ and for all $z \in \Pi^+$

$$f(z) = \int_0^\infty F(t) e^{2\pi i t z}\, dt \tag{14}$$

and

$$\int_0^\infty |F(t)|^2\, dt = C < \infty$$

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and
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2. $f \in H(\Pi^+)$ such that f restricted to horizontal lines is uniformly bounded by C in L^2. That is
$$\sup_{0<y<\infty} \frac{1}{2\pi} \int_{\mathbb{R}} |f(x+iy)|^2\, dx = C < \infty \tag{15}$$

» Paley Wiener Theorem 2

The following statements are equivalent:

1. $F \in L^2(\mathbb{R})$ is essentially supported in $(-A, A)$ such that

$$f(z) = \int_{-A}^{A} F(x) e^{2\pi izx}\, dx \tag{16}$$

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The following statements are equivalent:

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$$f(z) = \int_{-A}^{A} F(x) e^{2\pi izx}\, dx \tag{16}$$

2. $f : \mathbb{C} \to \mathbb{C}$ is an entire function satisfying $|f(z)| \leq Ce^{2\pi A|z|}$ for some constant C, and f restricted to horizontal lines is bounded in L^2. That is
$$\int_{-\infty}^{\infty} |f(x + iy)|^2\, dx < \infty \tag{17}$$

Future Directions

» Schwartz Class

* A smooth function $f\colon \mathbb{R}^n \to \mathbb{C}$, f is called a *Schwartz function* if for any given multi index α, β, there exists a positive constant $C_{\alpha,\beta}$ such that

$$\rho_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} \left| x^{\alpha} (D^{\beta} f) x \right| = C_{\alpha,\beta} < \infty \tag{18}$$

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* Schwartz class is dense in $L^p(\mathbb{R}^n)$ for all $1 \leq p < \infty$.

» Fourier transforms in R^n

* Fourier transform of $f \in \mathscr{S}(\mathbb{R}^n)$, $\hat{f} : \mathbb{R}^n \to \mathbb{C}^n$ is defined as

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* Fourier transform is a homeomorphism in $\mathscr{S}(\mathbb{R}^n)$.
* By Parseval's identity Fourier transform can be extended into whole of $\mathbb{R}^n$

» A problem

Let $n \geq 2$. Does there exist a function $f \in L^{\frac{2n+2}{n+3}}(\mathbb{R}^n)$ such that

$$\hat{f}|_{S^{n-1}} = 0$$

and

$$\left|1 - |\xi|^2\right|^{-\frac{1}{2}} f \notin L^2(\mathbb{R}^n)$$