

Isomorphism Theorems for rings:

1st) If $\phi: R \rightarrow S$ is a ring homom. then $\ker \phi$ is an ideal of R and $R/\ker \phi \cong \phi(R)$.

(isomorphic as a ring)

Also: If $I \subseteq R$ is an ideal, then it is the kernel of the ring homom. $\gamma: R \rightarrow R/I$.

$$a \mapsto a + I$$

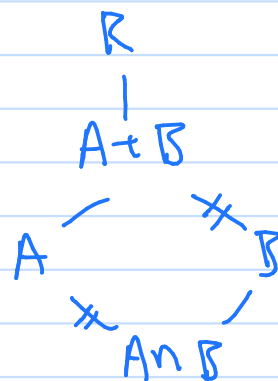
2nd) Suppose $A \subseteq R$ and $B \subseteq R$ is an ideal. Then

$A+B = \{a+b : a \in A, b \in B\}$ is a subring of R (since B is an ideal)

$A \cap B$ is an ideal of A , and

$$A+B/B \cong A/A \cap B$$

(ring isom.)



3rd) Suppose $I, J \subseteq R$ are ideals, $I \subseteq J$.

Then J/I is an ideal of R/I and

$$(R/I)/(J/I) \cong R/J$$

(ring isom.)



4th) If $I \subseteq R$ is an ideal then the correspondence $A \leftrightarrow A/I$ is an inclusion preserving bijection between the set of all subrings of A that contain I and the set of subrings of R/I . Also, A is an ideal of R if and only if A/I is an ideal of R/I .

Notation related to ideals:

Suppose $I, J \subseteq R$ are ideals:

$$1) I+J = \{a+b : a \in I, b \in J\}$$

This is an subring:

- It is a group under \checkmark

- $\forall a_1, a_2 \in I, b_1, b_2 \in J,$

$$(a_1+b_1)(a_2+b_2) = \underbrace{(a_1 a_2)}_{\in I \cap J} + \underbrace{(a_1 b_2 + b_1 a_2)}_{\in I+J} + b_1 b_2$$

$\in I \cap J$
since they are ideals

$$\in I+J.$$

It's an ideal: $\forall r \in R, a \in I, b \in J$

$$r(a+b) = \underbrace{ra}_{\in I} + \underbrace{rb}_{\in J}$$

$\in I, \in J$, since they are ideals

$$(a+b)r \in I+J. \checkmark$$

2) IJ is the set of all finite sums of elements of the form ab , $a \in I$, $b \in J$.

3) $I^n =$ collection of all finite sums of elems. of the form $a_1 a_2 \dots a_n$, $a_i \in A$.

4) If $A \subseteq R$ the ideal generated by A , denoted (A) , is the smallest ideal containing A .

If I is an ideal and $I = (\{a\})$ for some $a \in R$, then I is a principal ideal.

5) Suppose R has $1 \neq 0$. Then the left ideal generated by $A \subseteq R$ is

$$RA = \{r_1 a_1 + \dots + r_n a_n : n \in \mathbb{N}, r_1, \dots, r_n \in R, \text{ and } a_1, \dots, a_n \in A\}$$

The right ideal generated by $A \subseteq R$ is

$$AR = \{a_1 r_1 + \dots + a_n r_n : n \in \mathbb{N}, r_1, \dots, r_n \in R, \text{ and } a_1, \dots, a_n \in A\}$$

The ideal generated by $A \subseteq R$ is

$$(A) = RAR = \{r_1 a_1 s_1 + r_2 a_2 s_2 + \dots + r_n a_n s_n : n \in \mathbb{N}, r_1, \dots, r_n \in R, s_1, \dots, s_n \in R, a_1, \dots, a_n \in A\}$$

Special case: If R is commutative and $A = \{a\}$,
 $(A) = (a) = \{ra : r \in R\}$.

Also, if R is commutative then

$$b \in (a) \iff (b) \subseteq (a)$$

$$\iff b = ra \text{ for some } r \in R.$$

In this case, a divides b , or

that b is a multiple of a .

("to contain is to divide")

6) An ideal $P \subseteq R$ is prime if $P \neq R$ and if whenever $ab \in P$, it must be the case that $a \in P$ or $b \in P$.