2a) <u>Rational canonical form</u>

$R = F[x]$, F a field, V a fin.-gen. R-module,

$T: V \to V$ the lin. trans. determined $T(v) = x \cdot v$.

Invariant factor decomp:

$V \cong R^r \oplus R/(a_1) \oplus \cdots \oplus R/(a_\ell)$,

$a_1, \ldots, a_\ell \in F[x]$, $\deg(a_i) \geq 1$, $a_1 | a_2 | \cdots | a_\ell$.

Observations:

- If we assume all a_i's are monic, then they are unique.
- V is a fin-dim. v.s. over F, but $R = F[x]$ is inf. gen. over F.

  $\Rightarrow r = 0 \Rightarrow V$ is a torsion $\overset{=F[x]}{R}$-module

  i.e. $\forall v \in V$, $\exists f \in F[x]$ s.t. $f \cdot v = f(T)(v) = 0$.
- $\mathrm{Ann}(V) = \{f \in F[x] : f \cdot v = 0, \forall v \in V\}$

  is an ideal in R

  $\Rightarrow \mathrm{Ann}(V) = (m_T)$ for some monic $m_T \in F[x]$.

m_T is called the <u>minimal polynomial for T</u>.

From the inv. fact. decomp., $m_T = a_\ell$,

and $a_i | m_T$, $\forall 1 \leq i \leq \ell$.

- Each invariant factor corresponds to a T-invariant subspace (submodule obtained by projection onto that coordinate).

Now write

$$a_1(x) = \sum_{i=0}^{k} b_i x^i \quad (b_k = 1).$$

Then $\{1, x, \dots, x^{k-1}\}$ is an F-basis for

$$F[x]/(a_1).$$

W.r.t. this basis, the matrix for mult. by x is:

$$\mathcal{C}_{a_1} = \begin{pmatrix} 0 & 0 & & & 0 & -b_0 \\ 1 & 0 & & & \vdots & -b_1 \\ 0 & 1 & \ddots & & \vdots & \vdots \\ \vdots & 0 & & \ddots & \vdots & \vdots \\ \vdots & \vdots & & & 0 & \vdots \\ 0 & 0 & \cdots & 0 & 1 & -b_{k-1} \end{pmatrix}$$

companion matrix of a_1

Scratch work

$1 \mapsto x$

$x \mapsto x^2$

$x^2 \mapsto x^3$

$\vdots$

$x^{k-2} \mapsto x^{k-1}$

$x^{k-1} \mapsto x^k = -b_0 - b_1 x - b_2 x^2 - \cdots - b_{k-1} x^{k-1}$

Choosing bases in an analogous way for each invariant factor, and reinterpreting this in terms of V and T, we can choose a basis for V s.t. the matrix for T is

$$\left(\begin{array}{c|c|c|c} \mathcal{C}_{a_1} & 0 & 0 & 0 \\ \hline 0 & \mathcal{C}_{a_2} & 0 & 0 \\ \hline 0 & 0 & \ddots & 0 \\ \hline 0 & 0 & 0 & \mathcal{C}_{a_\ell} \end{array}\right)$$

Rational canonical form for T.

By uniqueness of inv. fact. decomp., it is not difficult to show that rat. can. form associated to a lin. trans. is unique. Important facts:

- The rational canonical form is computable and has entries in F.
- The rational canonical forms of two lin. transformations on V are the same iff they are similar transformations.

2b) Jordan canonical form

Same setup, but use the use the elem. div. decomp. to write

$$V \cong R/(p_1^{a_1}) \oplus \cdots \oplus R/(p_k^{a_k}),$$ (as before, $r=0$)

$p_i \in F[x]$ irreducible.

Assume for simplicity that each invariant factor from before factors in $F[x]$ as a product of linear factor.

Suppose $a_1(x) = \prod_{i=1}^{s} (x-\lambda_i)^{\alpha_i}$, $\lambda_i \in F$, $\alpha_i \in \mathbb{N}$, $\lambda_i \neq \lambda_j$ for $i \neq j$.

Then $R/(a_1) \cong R/(x-\lambda_1)^{\alpha_1} \oplus \cdots \oplus R/(x-\lambda_s)^{\alpha_s}$.

(side note: this is how you get the elem. div. decomp. from the inv. fact. decomp.)

Consider one of these factors, $R/(x-\lambda_1)^{\alpha_1}$, and choose the F-basis

$$(x-\lambda_1)^{\alpha_1-1}, (x-\lambda_1)^{\alpha_1-2}, \ldots, (x-\lambda_1), 1$$

for the F-v.s. $R/(x-\lambda_1)^{\alpha_1}$.

The matrix for mult. by $x = \lambda_1 + (x-\lambda_1)$ w.r.t. this basis is:

$$J_1 = \begin{pmatrix} \lambda_1 & 1 & 0 & \cdots & 0 \\ 0 & \lambda_1 & 1 & & \vdots \\ & 0 & \lambda_1 & & \\ & & 0 & \ddots & 0 \\ & 0 & & \ddots & 1 \\ & & & 0 & \lambda_1 \end{pmatrix}$$

Jordan block of size α_1 w.r.t. λ_1

Scratch work:

$$x = \lambda_1 + (x-\lambda_1)$$

$$\begin{aligned} 1 &\mapsto \lambda_1 \cdot 1 + (x-\lambda_1) \\ (x-\lambda_1) &\mapsto \lambda_1(x-\lambda_1) + (x-\lambda_1)^2 \\ (x-\lambda_1)^2 &\mapsto \lambda_1(x-\lambda_1)^2 + (x-\lambda_1)^3 \\ &\vdots \\ (x-\lambda_1)^{\alpha_1-1} &\mapsto \lambda_1(x-\lambda_1)^{\alpha_1-1} \end{aligned}$$

$(x-\lambda_1)^{\alpha_1} = 0$ in $R/(x-\lambda_1)^{\alpha_1}$

Doing this for each factor allows us to choose a basis for V w.r.t. which the matrix for T is:

$$\begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_k \end{pmatrix},$$

where each J_i is a Jordan block. This mat. is unique up to reordering blocks.

Jordan canonical form for T

One more result:

chan. poly. of T.

Thm (Cayley-Hamilton thm): $m_T \mid \chi_T$

Pf: Choose basis s.t. T is in Jordan can. form.

Note that $\det \begin{pmatrix} x-\lambda_i & & (*) \\ & \ddots & \\ 0 & & x-\lambda_i \end{pmatrix} = (x-\lambda_i)^{a_i}$. ☒