Department of Mathematics University of Houston

MATH 6321

Theory of Functions of a Real Variable
Spring 2025

First name: ______________________ **Last name:** ________________________ **Points:**

Assignment 2, due Thursday, February 6, 10am

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let $1 < p < \infty$. Prove that for $f, g \in L^p(\mu)$, if $\|f\|_p = \|g\|_p = 1$ and $f \neq g$ (they differ on a set of positive measure), then $\|(f+g)/2\|_p < 1$.

Problem 2

Consider $C([0,1])$, equipped with the sup-norm. Show that the affine subspace

$$M = \left\{ f \in C([0,1]) : \int_0^{1/2} f dt - \int_{1/2}^1 f dt = 1 \right\}$$

is a closed and convex subset of $C([0,1])$ which does not contain an element of minimal norm.

Problem 3

Consider $L^1([0,1])$ (as space of equivalence classes of functions that are identical up to sets with vanishing Lebesgue measure m) and

$$M = \left\{ f \in L^1([0,1]) : \int_{[0,1]} f dm = 1 \right\}$$

and show that M has infinitely many elements of minimal norm (viewed as functions, they differ on sets of non-zero measure).

Problem 4

Consider $C([0,1])$, equipped with the sup-norm. Let for $n \in \mathbb{N}$,

$$X_n = \{ f \in C([0,1]) : \text{ there is } x \in [0,1] \text{ s.th. for each } y \in [0,1] : |f(x) - f(y)| \leq n|x-y| \}.$$

Prove that each X_n has empty interior. Use this to prove that there is a G_δ set of nowhere differentiable functions in $C([0,1])$. Hint: Why is each X_n closed? To show X_n has empty interior, take $f \in X_n$ and use uniform continuity to approximate it with a piecewise linear, continuous function g. Next, consider $h(x) = g(x) + \epsilon \sin(Nx)$ or functions that have similar oscillatory behavior.