MATH 6321 - Functions of a real variable
Homework I

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1. **Solution:** For the sake of contradiction, assume that $M^\perp$ has dimension more than one. Then the Gram-Schmidt orthonormalization procedure guarantees the existence of orthonormal vectors $a, b \in M^\perp$. Now consider the vector $L(b)a - L(a)b$. Since $M^\perp$ is a subspace, we see that $L(b)a - L(a)b \in M^\perp$. Moreover

$$L(L(b)a - L(a)b) = L(b)L(a) - L(a)L(b) = \mathbf{0}$$

Hence $L(b)a - L(a)b \in M$. Thus $L(b)a - L(a)b = 0$ and since $L(a) \neq 0 \neq L(b)$, as $a, b \in M^\perp$, we see that

$$b = \frac{L(b)}{L(a)}a$$

But this contradicts our assumption that a, b are orthonormal. Hence we see that $M^\perp$ is a one dimensional subspace.

2. **Solution:** Let $f_k(t) = e^{ikt}$ for $k \in \mathbb{Z}$. Then we get that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f_k(t)\ dt = \begin{cases} 1, & \text{k} = 0 \\ 0, & \text{otherwise} \end{cases}$$

Also when $k \neq 0$,

$$\begin{aligned}
\frac{1}{N}\sum_{n=1}^{N} f_k(2\pi n\alpha) &= \frac{1}{N}\sum_{n=1}^{N} e^{i2\pi\alpha nk} \\
&= \frac{1}{N}\frac{e^{i2\pi\alpha Nk}-1}{e^{i2\pi\alpha k}-1} \\
&\leq \frac{1}{N}\frac{2}{e^{i2\pi\alpha k}-1}
\end{aligned}$$

Since α is irrational, the denominator above cannot be zero, and we get that the

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f_k(2\pi n\alpha) = 0$$

If $k = 0$, then $f_k(t) = e^0 = 1$ and we get

$$\frac{1}{N}\sum_{n=1}^{N} f_k(2\pi n\alpha) = 1$$

making the limit also equal to 1. Hence we have showed that the given equality

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f_k(2\pi n\alpha) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_k(t)\ dt \tag{1}$$

holds for all $f_k(t) = e^{ikt}$, where $k \in \mathbb{Z}$.

Now, we know that the family of sets $f_k(t) = \frac{1}{\sqrt{2\pi}}e^{ikt}$ forms an orthonormal basis for $L^2([-\pi,\pi])$. Moreover, we know that every 2π periodic continuous function can be embedded into $L^2([-\pi,\pi])$. Hence if f is any 2π periodic continuous function, then there exists $a_j \in \mathbb{C}$ such that

$$f(t) = \sum_{j=1}^{J} a_j f_{k_j}(t)$$

Then by the properties of integration,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\ dt \ = \ \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{j=1}^{J} a_j f_{k_j}(t)\ dt \ = \ \sum_{j=1}^{J} a_j \frac{1}{2\pi}\int_{-\pi}^{\pi} f_{k_j}(t)\ dt$$

Moreover by Equation 1, for each f_{k_j}, and $\varepsilon > 0$, there is a $N_{k_j} \in \mathbb{N}$ such that for all $N > N_{k_j}$

$$\left| \frac{1}{N} \sum_{n=1}^{N} f_{k_j}(2\pi n\alpha) - \frac{1}{2\pi} \int_{-\pi}^{\pi} f_{k_j}(t)\ dt \right| < \frac{\varepsilon}{2^j |a_j|}$$

Let $N_f = \max\{N_{k_j}\ :\ 1 \leq j \leq J\}$, then for all $N > N_f$

$$\begin{aligned}
\left| \frac{1}{N} \sum_{n=1}^{N} f(2\pi n\alpha) - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\ dt \right| &= \left| \frac{1}{N} \sum_{n=1}^{N} \sum_{j=1}^{J} a_j f_{k_j}(2\pi n\alpha) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{j=1}^{J} a_j f_{k_j}(t)\ dt \right| \\
&= \left| \sum_{j=1}^{J} a_j \frac{1}{N} \sum_{n=1}^{N} f_{k_j}(2\pi n\alpha) - \sum_{j=1}^{J} a_j \frac{1}{2\pi} \int_{-\pi}^{\pi} f_{k_j}(t)\ dt \right| \\
&\leq \sum_{j=1}^{J} |a_j| \left| \frac{1}{N} \sum_{n=1}^{N} f_{k_j}(2\pi n\alpha) - \frac{1}{2\pi} \int_{-\pi}^{\pi} f_{k_j}(t)\ dt \right| \\
&< \sum_{j=1}^{J} \frac{\varepsilon}{2^j} \\
&< \varepsilon
\end{aligned}$$

Since $\varepsilon > 0$ was chosen arbitrary, this proves that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(2\pi n\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\ dt$$

3. **Solution:** Consider $P_2(\mathbb{R})$ the set of polynomials with real coefficients as a subspace of the inner product space $L^2([-1,1])$. Clearly $1, x, x^2$ is a basis for $P_2(\mathbb{R})$. Now by Gram-Schmidt orthonormalization, we see that $\left\{\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x, \sqrt{\frac{5}{2}}x^2\right\}$ is an orthonormal basis for $P_2(\mathbb{R})$. Now we find the projection of the cubic polynomial x^3 to $P_2(\mathbb{R})$. We know that this projection will be

$$\left\langle x^3, \frac{1}{\sqrt{2}} \right\rangle \frac{1}{\sqrt{2}} + \left\langle x^3, \sqrt{\frac{3}{2}}x \right\rangle \sqrt{\frac{3}{2}}x + \left\langle x^3, \sqrt{\frac{5}{2}}x^2 \right\rangle \sqrt{\frac{5}{2}}x^2 = 0 + \frac{3}{5}x + 0x^2$$

Also we know that $\|x^3 - f\|$ will be minimum for $f \in P_2(\mathbb{R})$, when f is the projection of x^3 to $P_2(\mathbb{R})$. Hence

$$\min_{a,b,c\ \in\ \mathbb{R}} \int_{-1}^{1} |x^3 - a - bx - cx^2|^2 \ dx = \int_{-1}^{1} \left| x^3 - \frac{3}{5}x \right|^2 \ dx = \frac{8}{175}$$