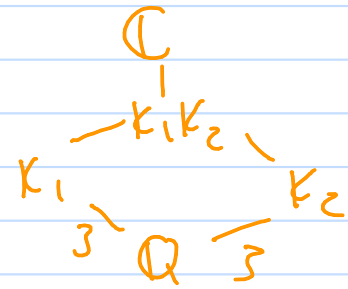
2) $K = \mathbb{C}$, $F = \mathbb{Q}$, $K_1 = \mathbb{Q}(\sqrt[3]{2})$, $K_2 = \mathbb{Q}(\zeta_3 \sqrt[3]{2})$

$(\zeta_3 = e^{2\pi i/3})$

$f(x) = x^3 - 2 = \min_{\mathbb{Q}}(\sqrt[3]{2}) = \min_{\mathbb{Q}}(\zeta_3 \sqrt[3]{2})$

$= (x - 2^{1/3})(x^2 + 2^{1/3}x + 2^{2/3})$



$\Rightarrow \min_{K_1}(\zeta_3 \sqrt[3]{2}) \mid x^2 + 2^{1/3}x + 2^{2/3}$

$\Rightarrow [K_1K_2 : K_1] = 1$ or 2

But $K_1K_2 \not\subseteq \mathbb{R}$ and $K_1 \subseteq \mathbb{R}$,

$K_1 = \{a + b2^{1/3} + c2^{2/3} : a, b, c \in \mathbb{Q}\}$

so $[K_1K_2 : K_1] > 1$

$\Rightarrow [K_1K_2 : K_1] = 2 \Rightarrow [K_1K_2 : \mathbb{Q}] = 6$

One more note: K_1K_2 is the splitting field over $\mathbb{Q}$ of $f(x) = x^3 - 2$.

One more fact: If F is a field and $f \in F[x]$, and if K is a splitting field for f over F then $[K : F] \leq (\deg f)!$.

Splitting fields and algebraic closures

Thm: Suppose $\tau: F_1 \to F_2$ is a field isom., $f_1 \in F_1[x]$, $f_2 = \tau(f_1)$, and that K_1 is a spl. fld. for f_1 over F_1, and K_2 is a spl. fld. for f_2 over F_2. Then τ extends to an isom. $\tau: K_1 \to K_2$.

Pf: Induction on $n = \deg f_1$. If $n=1$ then $K_1 = F_1$ and $K_2 = F_2$.

Suppose $n \geq 2$ and that f_1 has an irred. factor g_1 of $\deg \geq 2$. Let $\alpha_1 \in K_1$ be a root of g_1 and α_2 a root of $\tau(g_1)$ in K_2. Then by Kronecker thm, τ extends to an isom. $\tau: F_1(\alpha_1) \to F_2(\alpha_2)$.

$$\begin{tikzcd} K_1 \arrow[r, "\text{Induction}"] \arrow[d, dash] & K_2 \arrow[d, dash] \\ F_1(\alpha_1) \arrow[r, "\text{Kron. thm}", "\tau"'] \arrow[d, dash] & F_2(\alpha_2) \arrow[d, dash] \\ F_1 \arrow[r, "\tau"'] & F_2 \end{tikzcd}$$

Over $F_1(\alpha_1)$, f_1 factors as $f_1(x) = (x - \alpha_1) h_1(x)$. Then K_1 is a spl. fld. for f_1 over F_1

$\implies$ K_1 is a spl. fld. for h_1 over $F_1(\alpha_1)$.

Sim., K_2 is a spl. fld. for $\tau(h_1)$ over $F_2(\alpha_2)$.

By the inductive hyp. we're done. ∎

Defs: $\overline{F}$ is an <u>algebraic closure</u> of F if $\overline{F}/F$ is algebraic and if every $f \in F[x]$ splits completely in $\overline{F}[x]$.

A field K is <u>algebraically closed</u> if every poly $f \in K[x]$ has a root in K.

Thm: Every field is contained in an algebraically closed field.

"Naive" proof: Let F be a field. How do we construct an algebraic closure? If F is countable, write the collection of non-constant polys in $F[x]$ as $\{f_n\}_{n=1}^{\infty}$

Let K_1 be a spl. field of f_1 over F,

Let K_2 be a spl. field of f_2 over K_1, and

so on $F \subseteq K_1 \subseteq K_2 \subseteq \cdots$. Let $K = \bigcup_{n=1}^{\infty} K_n$

Then K is an algebraic closure of F.

To see K that is algebraically closed:

Suppose $f \in K[x]$, and let α be a root of f in some ext. of K. Then $K(\alpha)/K$ is algebraic, and K/F is algebraic, so $K(\alpha)/F$ is algebraic, so α is algebraic over F.

Issue: This proof doesn't work if F is not countable.