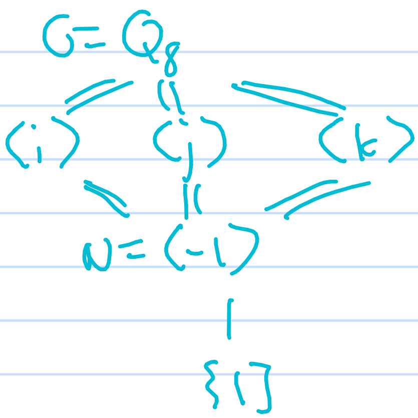
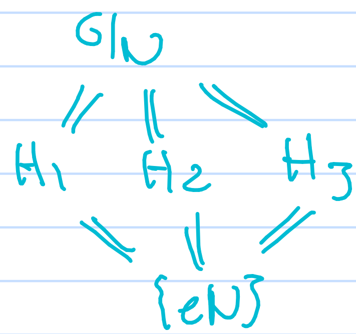
4th isom. thm (lattice thm): If $N \trianglelefteq G$ then there is an inclusion preserving bijection between subgroups $A \le G$ which contain N, and subgroups of G/N. This bijection also preserves indices of respective subgroups.

Exs:

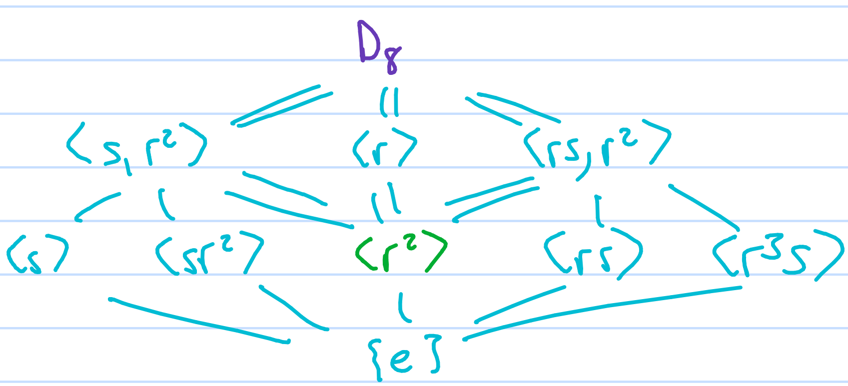
1) $G = Q_8$, $N = \{\pm 1\}$

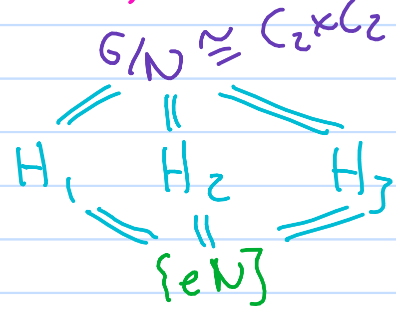




2) $G = D_8$, $N = \langle r^2 \rangle$

$$\left(\begin{array}{l} \text{check that } N \trianglelefteq G: \\ r r^2 r^{-1} = r^2 \in N \\ s r^2 s^{-1} = r^{-2} = r^2 \in N \end{array} \right)$$





Idea of pf: Consider the map $\pi : G \to G/N$ defined by $\pi(g) = gN$. (this is called the **natural projection**)

This is the map that does everything in the statement of the thm. ☒

Fundamental theorem for finitely generated Abelian groups (FTFGAG)

Def. For $r \geq 0$, the group

$$\mathbb{Z}^r = \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{r\text{-times}} \quad \text{is called the } \underline{\text{free}}$$

<u>Abelian group of rank r</u>.

Thm (FTFGAG, <u>Invariant factor decomposition</u>):

If G is a F.G.A.G. then $\exists\, r \geq 0$, $s \geq 0$, and $n_1, \ldots, n_s \geq 2$ s.t.: $\quad (\mathbb{Z}_{n_i} = C_{n_i})$

i) $G \cong \mathbb{Z}^r \times \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_s}$, and

ii) $n_{i+1} \mid n_i$, $\forall\ 1 \leq i < s$.

Furthermore, $r, s, n_1, \ldots, n_s$ are unique.

Notation:
- r is the <u>free rank</u> of G
- $n_1, \ldots, n_s$ are the <u>invariant factors</u>

$\swarrow$ (finite groups version, ignoring free rank)

Thm (FTFGAG, Elementary divisor decomposition):

If G is an Abelian gp., $|G| = p_1^{\alpha_1} \cdots p_k^{\alpha_k} < \infty$, $p_1 < p_2 < \cdots < p_k$ distinct primes, $\alpha_1, \ldots, \alpha_k \in \mathbb{N}$. Then:

i) $G \cong G_1 \times \cdots \times G_k$, with $|G_i| = p_i^{\alpha_i}$.

ii) $\forall 1 \leq i \leq k$, $\exists\, t_i \geq 1$, $\beta_{i1} \geq \beta_{i2} \geq \cdots \geq \beta_{it_i} \geq 1$, s.t.

$$G_i \cong \mathbb{Z}_{p_i^{\beta_{i1}}} \times \mathbb{Z}_{p_i^{\beta_{i2}}} \times \cdots \times \mathbb{Z}_{p_i^{\beta_{it_i}}}.$$

iii) This decomp. is unique.

Notation: The numbers $p_i^{\beta_{ij}}$ are called the <u>elementary divisors</u> of G.

We will see that both versions of this thm are equivalent.

Exs: Invariant factor decomposition

1a) $|G| = p$, $\mathbb{Z}_p$

b) $|G| = p_1 p_2$, $\mathbb{Z}_{p_1 p_2}$

c) $|G| = p_1 p_2 \cdots p_k$, $\mathbb{Z}_{p_1 \cdots p_k}$

2a) $|G| = p^2$, $\mathbb{Z}_{p^2}$, $\mathbb{Z}_p \times \mathbb{Z}_p$

b) $|G| = p^3$, $\mathbb{Z}_{p^3}$, $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$, $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$

c) $|G| = p^4$, $\mathbb{Z}_{p^4}$, $\mathbb{Z}_{p^3} \times \mathbb{Z}_p$, $\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$,

$\mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p$, $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$.

d) $|G| = p^k$

# of different Abelian gps. of order p^k

$= p(k) =$ # of ways of writing k as a sum of positive integers.

partition function

Note: $p(k) \sim \frac{1}{4k\sqrt{3}} \exp\left(\pi \sqrt{\frac{2k}{3}}\right)$ as $k \to \infty$.

3) $|G| = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, $p_1 < \cdots < p_k$ primes, $\alpha_i \in \mathbb{N}$,

Fact: By the elem. div. thm,

$G \cong G_1 \times \cdots \times G_k$, $|G_i| = p_i^{\alpha_i}$.

The # of Abelian gps. of order $n = \prod_{i=1}^{k} p(\alpha_i)$.

Exs: Invariant factor decomp $\longrightarrow$ Elem. div. decomp.

1) $G = \mathbb{Z}_{36} \times \mathbb{Z}_{12} \times \mathbb{Z}_3$

$$\cong (\mathbb{Z}_4 \times \mathbb{Z}_9) \times (\mathbb{Z}_4 \times \mathbb{Z}_3) \times \mathbb{Z}_3$$

$$\cong \underbrace{(\mathbb{Z}_4 \times \mathbb{Z}_4)}_{G_1} \times \underbrace{(\mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_3)}_{G_2}$$

Scratch: $36^2 = 2^4 \cdot 3^4$,

Want to write

$G = G_1 \times G_2$, $|G_1| = 2^4$, $|G_2| = 3^4$,

then further decompose

G_1, G_2 into products

of cyclic groups.