Internal direct products

Suppose G is a group, $H, K \leq G$, define

$HK = \{hk : h \in H, k \in K\}$. (product of subgroups)

Thm: Suppose $|H|, |K| < \infty$. Then $|HK| = \dfrac{|H| \cdot |K|}{|H \cap K|}$.

Pf: Write $HK = \bigcup_{h \in H} hK$. Note that

$$h_1K = h_2K \iff \overset{\in H}{h_2^{-1}h_1} \in K \iff h_2^{-1}h_1 \in H \cap K$$

$$\iff h_1(H \cap K) = h_2(H \cap K).$$

The # of distinct cosets of the form hK, where $h \in H$, is equal to the number of cosets in $H/H \cap K$.

So $|HK| = \left|H/H\cap K\right| \cdot |K| = \dfrac{|H||K|}{|H \cap K|}$. ∎

Note: This theorem also implies immediately that it is not always the case that $HK \leq G$.

Ex: Let $G = S_3$, $H = \langle (1\,2) \rangle$, $K = \langle (2,3) \rangle$

Then $|G| = 6$, $|H| = 2$, $|K| = 2$, and $|H \cap K| = 1$

$\Rightarrow |HK| = \dfrac{|H||K|}{|H \cap K|} = 4$, so it can't be a subgroup of G, by Lagrange's thm.

Thm: If $H, K \leq G$ then $HK \leq G \iff HK = KH$.

Pf: $\Leftarrow$ Suppose $HK = KH$. Note: $HK \neq \emptyset$.

Want to show that if $a, b \in HK$ then $ab^{-1} \in HK$.

Write $a = h_1 k_1$, $b = h_2 k_2$. Then

$ab^{-1} = h_1(k_1 k_2^{-1} h_2^{-1}) = h_1 h' k'$ for some $h' \in H$, $k' \in K$.

$\in HK$. Therefore $HK \leq G$, by the subgroup criterion.

$\Rightarrow$ Suppose $HK \leq G$. Then:

- $K, H \subseteq HK \Rightarrow KH \subseteq HK$.
- Suppose $a \in HK$. Then $a^{-1} \in HK$

  $\Rightarrow a^{-1} = hk \Rightarrow a = k^{-1}h^{-1} \in KH$.

Conclusion: $HK = KH$. $\blacksquare$

($\forall k \in K$, $kH = Hk$)

Corollary:
- If $K \leq N_G(H)$ then $HK \leq G$.
- If $H \trianglelefteq G$ then $HK \leq G$.

Thm: If $H, K \leq G$, $|H|, |K| < \infty$, $H, K \trianglelefteq G$, and $H \cap K = \{e\}$

then $HK \cong H \times K$. (in this case HK is called the <u>internal direct product</u> of H and K)

Pf: By the corollary from before, $HK \leq G$.

Consider the map $\phi : HK \to H \times K$ defined by $\phi(hk) = (h, k)$.

- ϕ is well defined: $|HK| \overset{\text{Thm. above}}{=} \frac{|H||K|}{|H \cap K|} = |H||K| \Rightarrow$ there is only one way to represent each element of HK as hk.

- ϕ is a hom.: Suppose $h_1 k_1, h_2 k_2 \in HK$.

Want to show: $\phi(h_1 k_1 h_2 k_2) = \phi(h_1 k_1)\phi(h_2 k_2)$

$$= (h_1, k_1)(h_2, k_2) = (h_1 h_2, k_1 k_2)$$

Equivalently: Want to show that, $\forall h_1, h_2 \in H$, $k_1, k_2 \in K$,

$$h_1 k_1 h_2 k_2 = h_1 h_2 k_1 k_2.$$

Also equiv: Show that $\forall h \in H$, $k \in K$,

$$[h, k] = h^{-1} k^{-1} h k = e$$ (h and k commute w/ each other)

<u>commutator</u> of h and k

To see why $[h, k] = e$:

$$h^{-1}(k^{-1} h k) \overset{(H \trianglelefteq G)}{=} h^{-1} h' \text{ for some } h' \in H$$

$$\Rightarrow [h, k] \in H.$$

$$(h^{-1} k^{-1} h) k \overset{K \trianglelefteq G}{=} k' k \text{ for some } k' \in K$$

$$\Rightarrow [h, k] \in K$$

$$\Rightarrow [h, k] = e.$$

Also, ϕ is a surjective map between finite sets with the same cardinality, so it is a bijection.

Conclusion: ϕ is an isom. $\blacksquare$

$\boxed{\text{Exercise: Use this to show that } D_{12} \cong D_6 \times C_2 .}$