Question 1.1

Does there exist an infinite σ-algebra which has only countably many members?

Question 1.4

Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences in $[-\infty, \infty]$. Prove the following:

(a) $\limsup_{n\to\infty}(-a_n) = -\liminf_{n\to\infty} a_n$

(b) $\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$

provided none of the sums is of the form $\infty - \infty$.

(c) If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n$.

Show by an example that strict inequality can hold in part (b).

Question 1.5

(a) Suppose that $f : X \to [-\infty, \infty]$ and $g : X \to [\infty, \infty]$ are measurable. Prove that the sets

$$\{x : f(x) < g(x)\} \text{ and } \{x : f(x) = g(x)\}$$

are measurable.

(b) Prove that the set of points at which a sequence of a measurable real-valued function converges to a finite limit is measurable.

Question 1.7

Suppose $f_n : X \to [-\infty, \infty]$ is measurable for $n \in \mathbb{N}$, $f_1 \geq f_2 \geq \cdots \geq 0$, $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in X$, and $f_1 \in L^1(\mu)$. Prove that

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$$

and show that this conclusion does not follow if the condition "$f_1 \in L^1(\mu)$" is omitted.

Question 1.8

Put $f_n = \chi_E$ if n is odd, $f_n = 1 - \chi_E$ if n is even. What is the relevance of this example to Fatou's lemma?

Question 1.12

Suppose $f \in L^1(\mu)$. Prove that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $\int_E |f| \, d\mu < \varepsilon$ whenever $\mu(E) < \delta$.

Question 2.1

Let $(f_n)_{n=1}^{\infty}$ be a sequence of real nonnegative functions on $\mathbb{R}$, and consider the following four statements:

(a) If f_1 and f_2 are upper semicontinuous, then $f_1 + f_2$ is upper semicontinuous.

(b) If f_1 and f_2 are lower semicontinuous, then $f_1 + f_2$ is lower semicontinuous.

(c) If each f_n is upper semicontinuous, then $\sum_{n=1}^{\infty} f_n$ is upper semicontinuous.

(d) If each f_n is lower semicontinuous, then $\sum_{n=1}^{\infty} f_n$ is lower semicontinuous.

Show that three of these are true and that one is false. What happens if the word "nonnegative" is omitted? Is the truth of the statements affected if $\mathbb{R}$ is replaced by a general topological space?

Question 2.2

Let f be an arbitrary complex function on $\mathbb{R}$, and define

$$\varphi(x, \delta) = \sup \{|f(s) - f(t)| : s, t \in (x - \delta, x + \delta)\},$$
$$\varphi(x) = \inf\{\varphi(x, \delta) : \delta > 0\}$$

Prove that φ is upper semicontinuous, that f is continuous at a point x if and only if $\varphi(x) = 0$, and hence the set of points of continuity of an arbitrary complex function is a G_δ set.

Question 2.3

Let X be a metric space, with metric ρ. For any nonempty $E \subset X$, define

$$\rho_E(x) = \inf\{\rho(x, y) : y \in E\}$$

Show that ρ_E is a uniformly continuous function on X. If A and B are disjoint nonempty closed subsets of X, examine the relevance of the function

$$f(x) = \frac{\rho_A(x)}{\rho_A(x) + \rho_B(x)}$$

to Urysohn's lemma.

Question 2.5

Let E be Cantor's familiar "middle thirds" set. Show that $\mu(E) = 0$, even though E and $\mathbb{R}$ have the same cardinality.

Question 2.7

If $0 < \varepsilon < 1$, construct an open set $E \subset [0, 1]$ which is dense in $[0, 1]$, such that $\mu(E) = \varepsilon$. To say that A is dense in B means that the closure of A contains B.

Question 2.11

Let μ be a regular Borel measure on a compact Hausdorff space X and assume $\mu(X) = 1$. Prove that there is a compact set $K \subset X$ (the carrier or support of μ) such that $\mu(K) = 1$ but $\mu(H) < 1$ for every proper compact subset H of K. *Hint:* Let K be the intersection of all compact K_α with $\mu(K_\alpha) = 1$. Show that every open set V which contains K also contains some K_α. Regularity of μ is needed. Show that K^c is the largest open set in X whose measure is 0.

Question 2.21

If X is compact and $f : X \to (-\infty, \infty)$ is upper semicontinuous, prove that f attains its maximum at some point of X.

Question 2.22

Suppose that X is a metric space, with metric d, that $f : X \to [0, \infty]$ is lower semicontinuous, and that $f(p) < \infty$ for at least one $p \in X$. For $n \in \mathbb{N}$, $x \in X$, define

$$g_n(x) = \inf\{f(p) + n \cdot d(x, p) : p \in X\}$$

and prove that

1. $|g_n(x) - g_n(y)| \leq n \cdot d(x, y)$,
2. $0 \leq g_1 \leq g_2 \leq \cdots \leq f$,
3. $\lim\limits_{n\to\infty} g_n(x) = f(x)$ for all $x \in X$.

Thus f is the pointwise limit of an increasing sequence of continuous functions. Note that the converse is almost trivial.

Question 3.1

Prove that the supremum of any collection of convex functions on (a, b) is convex on (a, b) if it is finite and that pointwise limits of sequences of convex functions are convex. What can you say about upper and lower limits of sequences of convex functions?

Question 3.4

Suppose that f is a complex measurable function on X, μ is a positive measure on X, and

$$\varphi(p) = \int_X |f|^p \, d\mu = \|f\|_p^p \quad \text{with } 0 < p < \infty$$

Let $E = \{p : \varphi(p) < \infty\}$. Assume $\|f\|_\infty > 0$.

(a) If $r < p < s$ and $r, s \in E$, then $p \in E$.

(b) Prove that $\log \varphi$ is convex in the interior of E and that φ is continuous on E.

(c) By (a), E is connected. Is E necessarily open or closed? Can E consist of a single point? Can E be any connected subset of $(0, \infty)$?

(d) If $r < p < s$, prove that $\|f\|_p \leq \max\{\|f\|_r, \|f\|_s\}$. Show that this implies the inclusion $L^r(\mu) \cap L^s(\mu) \subset L^p(\mu)$.

(e) Assume that $\|f\|_r < \infty$ for some $r < \infty$. Prove that $\lim\limits_{p\to\infty} \|f\|_p = \|f\|_\infty$.

Question 3.5

Assume, in addition to the hypothesis in the previous exercise, that $\mu(X) = 1$.

(a) Prove that $\|f\|_r \leq \|f\|_s$ if $0 < r < s \leq \infty$.

(b) Under what conditions does it happen that $0 < r < s \leq \infty$ and $\|f\|_r = \|f\|_s < \infty$?

(c) Prove that $L^s(\mu) \subset L^r(\mu)$ if $0 < r < s$. Under what conditions do these two spaces contain the same functions?

(d) Assume that $\|f\|_r < \infty$ for some $r > 0$, and prove that

$$\lim_{p\to\infty} \|f\|_p = \exp \int_X \log |f| \, d\mu$$

if $\exp\{-\infty\}$ is defined to be 0.

Question 3.7

For some measures, the relation $r < s$ implies $L^r(\mu) \subset L^s(\mu)$; for others, the inclusion is reversed; and there are some for which $L^r(\mu)$ does not contain $L^s(\mu)$ if $r \neq s$. Give examples of these situations, and find conditions on μ under which these situations will occur.

Question 3.10

Suppose $(f_n)_{n=1}^{\infty} \in L^p(\mu)$, $\lim_{n\to\infty} \|f_n - f\|_p = 0$, and $\lim_{n\to\infty} f_n = g$. What relation exists between f and g?

Question 3.14a

Suppose $1 < p < \infty$, $f \in L^p((0,\infty))$, relative to the Lebesgue measure, and

$$F(x) = \frac{1}{x}\int_0^x f(t)\,d\mu \quad (0 < x < \infty)$$

Prove Hardy's inequality

$$\|F\|_p \le \frac{p}{p-1}\|f\|_p$$

which shows that the mapping $f \mapsto F$ carries L^p to L^p.

Question 3.14d

Suppose $1 < p < \infty$, $f \in L^p((0,\infty))$, relative to the Lebesgue measure, and

$$F(x) = \frac{1}{x}\int_0^x f(t)\,d\mu \quad (0 < x < \infty)$$

If $f > 0$ and $f \in L^1$, prove that $F \notin L^1$.

Question 4.1

If M is a closed subspace of H, prove that $M = (M^{\perp})^{\perp}$. Is there a similar true statement for subspaces M which are not necessarily closed?

Question 4.2

Let $(x_n)_{n=1}^{\infty}$ be a linearly independent set of vectors in H. Show that the following construction yields an orthonormal set $(u_n)_{n=1}^{\infty}$ such that $\{x_1, x_2, \ldots, x_N\}$ and $\{u_1, u_2, \ldots, u_N\}$ have the same span for all N.

Put $u_1 = \frac{x_1}{\|x_1\|}$. Having $u_1, \dots, u_{n-1}$ define

$$v_n = x_n - \sum_{i=1}^{n-1} \langle x_n, u_i \rangle u_i, \quad u_n = \frac{v_n}{\|v_n\|}.$$

Note that this leads to a proof of the existence of a maximal orthonormal set in separable Hilbert spaces which makes no appeal to the Hausdorff maximality principle.

Question 4.3

Show that $L^p(T)$ is separable if $1 \le p < \infty$, but $L^\infty(T)$ is not separable.

Question 4.4

Show that H is separable if and only if H contains a maximal orthonormal system which is at most countable.

Question 4.5

If $M = \{x : Lx = 0\}$, where L is a continuous linear functional on H, prove that $M^\perp$ is a vector space of dimension 1.

Question 4.7

Suppose $(a_n)_{n=1}^\infty$ is a sequence of positive numbers such that $\sum a_n b_n < \infty$ whenever $b_n \ge 0$ and $\sum b_n^2 < \infty$. Prove that $\sum a_n^2 < \infty$.

Question 4.9

If $A \subset [0, 2\pi]$ and A is measurable, prove that

$$\lim_{n\to\infty} \int_A \cos nx \, dx = \lim_{n\to\infty} \int_A \sin nx \, dx = 0.$$

Question 5.2

Prove that the unit ball (open or closed) is convex in every normed linear space.

Question 5.6

Let f be a bounded linear functional on a subspace M of a Hilbert space H. Prove that f has a unique norm-preserving extension to a bounded linear functional on H, and that extension vanishes on $M^{\perp}$.

Question 5.8

Let X be a normed linear space, and let X^* be its dual space with the norm

$$\|f\| = \sup\{|f(x)| : \|x\| \leq 1\}$$

(a) Prove that X^* is a Banach space.

(b) Prove that the mapping $f \mapsto f(x)$ is, for each $x \in X$, a bounded linear functional on X^*, of norm $\|x\|$.

(c) Prove that the sequence $(\|x_n\|)_{n=1}^{\infty}$ is bounded if $(x_n)_{n=1}^{\infty}$ is a sequence in X such that $(f(x_n))_{n=1}^{\infty}$ is bounded for every $f \in X^*$.

Question 5.9

Let c_0, l^1, and l^∞ be the Banach spaces consisting of all complex sequences $x = (\xi_n)_{n=1}^{\infty}$, defined as follows:

$$x \in l^1 \text{ if and only if } \|x\|_1 = \sum |\xi_i| < \infty$$
$$x \in l^\infty \text{ if and only if } \|x\|_\infty = \sup |\xi_i| < \infty$$

and c_0 is the subspace of l^∞ consisting of all $x \in l^\infty$ for which $\lim_{i \to \infty} \xi_i = 0$.

(a) If $y = (\eta_i)_{i=1}^{\infty} \in l^1$ and $\Lambda x = \sum \xi_i \eta_i$ for every $x \in c_0$. Then Λ is a bounded linear functional on c_0, and $\|\Lambda\| = \|y\|_1$. Moreover every $\Lambda \in (c_0)^*$ is obtained in this way. In short, $(c_0)^* = l^1$.

(b) In the same sense $(l^1)^* = l^\infty$.

(c) Every $y \in l^1$ induces a bounded linear functional on l^∞. However, this does not give all of $(l^\infty)^*$, since $(l^\infty)^*$ contains nontrivial functionals that vanish on all of c_0.

(d) c_0 and l^1 are separable but l^∞ is not.

Question 5.10

If $\sum \alpha_i \xi_i$ converges for every sequence $(\xi)_{n=1}^\infty$ such that $\lim\limits_{i\to\infty} \xi_i = 0$, prove that $\sum |\alpha_i| < \infty$.

Question 5.11

For $0 < \alpha \leq 1$, let Lip α denote the space of all complex functions f on $[a,b]$ for which

$$M_f = \sup_{s\neq t} \frac{|f(s) - f(t)|}{|s-t|^\alpha} < \infty.$$

Prove that Lip α is a Banach space, if $\|f\| = |f(a)| + M_f$ or if $\|f\| = M_f + \sup\limits_{x\in[a,b]} |f(x)|$.

Question 5.16

Suppose that X and Y are Banach, and suppose Λ is a linear mapping of X into Y, with the following property: For every sequence $(x_n)_{n=1}^\infty$ in X for which $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} \Lambda x_n$ exist, it is true that $y = \Lambda x$. Prove that Λ is continuous. Observe that there exist nonlinear mappings (of $\mathbb{R}$ onto $\mathbb{R}$, for instance) whose graph is closed although they are not continuous: $f(x) = \frac{1}{x}$ if $x \neq 0$ and $f(0) = 0$.

Question 5.17

If μ is a positive measure, each $f \in L^\infty(\mu)$ defines a multiplication operator M_f on $L^2(\mu)$ into $L^2(\mu)$, such that $M_f(g) = fg$. Prove that $\|M_f\| \leq \|f\|_\infty$. For which measures μ is it true that $\|M_f\| = \|f\|_\infty$ for all $f \in L^\infty(\mu)$? For which $f \in L^\infty(\mu)$ does M_f map $L^2(\mu)$ onto $L^2(\mu)$?

Question 5.18

Suppose $(\Lambda_n)_{n=1}^{\infty}$ is a sequence of bounded linear transformations from a normed linear space X to a Banach space Y, suppose that $\|\Lambda_n\| \leq M < \infty$ for all $n \in \mathbb{N}$, and suppose there is a dense set $E \subset X$ such that $(\Lambda_n x)_{n=1}^{\infty}$ converges for each $x \in E$. Prove that $(\Lambda_n x)_{n=1}^{\infty}$ converges for each $x \in X$.

Question 6.2

Let μ and λ be positive and σ-finite measures on a σ-algebra $\mathcal{M}$ in a set X. Prove that there is a unique pair of complex measures λ_a and λ_s on $\mathcal{M}$ such that

1. $\lambda = \lambda_a + \lambda_s$
2. $\lambda << \mu$
3. $\lambda_s \perp \mu$
4. If λ is finite, then so are λ_a and λ_s

Question 6.3

Prove that the vector space $M(X)$ of all complex regular Borel measures on a locally compact Hausdorff space X is a Banach space if $\|\mu\| = |\mu|(X)$. *Hint:* Compare to Question 5.8

Question 6.4

Suppose $1 \leq p \leq \infty$, and q is the exponent conjugate to p. Suppose μ is a positive σ-finite measure and g is a measurable function such that $fg \in L^1(\mu)$ for every $f \in L^p(\mu)$. Prove that $g \in L^q(\mu)$.

Question 6.5

Suppose X consists of two points a and b; define $\mu(\{a\}) = 1$, $\mu(\{b\}) = \mu(X) = \infty$, and $\mu(\phi) = 0$. Is it true, for this μ, that $L^\infty(\mu)$ is the dual space of $L^1(\mu)$.

Question 6.10

Let $(X, \mathcal{M}, \mu)$ be a positive measure space. Call a set $\Phi \subset L^1(\mu)$ uniformly integrable if to each $\varepsilon > 0$ corresponds a $\delta > 0$ such that

$$\left| \int_E f \, d\mu \right| < \varepsilon$$

whenever $f \in \Phi$ and $\mu(E) < \delta$.

(a) Prove that every finite subset of $L^1(\mu)$ is uniformly integrable.

(b) Prove the following convergence theorem of Vitali: If $\mu(X) < \infty$, $(f_n)_{n=1}^{\infty}$ in uniformly integrable, $\lim_{n\to\infty} f_n(x) = f(x)$ a.e., and $|f(x)| < \infty$ a.e., then $f \in L^1(\mu)$ and

$$\lim_{n\to\infty} \int_X |f_n - f| \, d\mu = 0.$$

Question 6.13

Let $L^\infty = L^\infty(m)$, where m is Lebesgue measure on $I = [0, 1]$. Show that there is a bounded linear functional $\Lambda \neq 0$ on L^∞ that is 0 on $C(I)$, and that therefore there is no $q \in L^1(m)$ that satisfies $\Lambda f = \int_I fg \, dm$ for every $f \in L^\infty$. Thus $(L^\infty)^* \neq L^1$.

Question 7.1

Show that $|f(x)| \leq (Mf)(x)$ at every Lebesgue point of f if $f \in L^1(\mathbb{R}^k)$.

Question 7.10

If $f \in \text{Lip } 1$ on $[a, b]$, prove that f is absolutely continuous and that $f' \in L^\infty$.

Question 7.11

Assume that $1 < p < \infty$, f is absolutely continuous on $[a, b]$, $f' \in L^p$, and $\alpha = \frac{1}{q}$, where q is the exponent conjugate of p. Prove that $f \in \text{Lip } \alpha$.

Question 7.12

Suppose that $\varphi : [a, b] \to \mathbb{R}$ is nondecreasing.

(a) Show that there is a left-continuous nondecreasing f on $[a, b]$ so that $\{x \in [a, b] : f(x) \neq \varphi(x)\}$ is at most countable. (Left-continuous means: if $a < x \leq b$ and $\varepsilon > 0$, then there is a $\delta > 0$ so that $|f(x) - f(x - t)| < \varepsilon$ whenever $0 < t < \delta$.)

(b) Imitate the proof of Theorem 7.18 to show that there is a positive Borel measure μ on $[a, b]$ for which $f(x) - f(a) = \mu([a, x])$ for $a \leq x \leq b$.

Question 7.14

Show that the product of two absolutely continuous functions on $[a, b]$ is absolutely continuous. Use this to derive a theorem about integration by parts.

Question 7.23

The definition of Lebesgue points applies to individual integrable functions and not to their equivalence classes (section 3.10). However if $F \in L^1(\mathbb{R}^k)$ is one of these equivalence classes, one may call a point $x \in \mathbb{R}^k$ a Lebesgue point of F if there is a complex number, let us call it $(SF)(x)$, such that

$$\lim_{r \to 0} \frac{1}{m(B_r)} \int_{B(x,r)} |f - (SF)(x)| \, dm = 0$$

for one (hence every) $f \in F$. Define $(SF)(x)$ to be 0 at those points $x \in \mathbb{R}^k$ that are not Lebesgue points of F. Prove the following statement: If $f \in F$, and x is a Lebesgue point of f, then x is also a Lebesgue point of F, and $f(x) = (SF)(x)$. Hence $SF \in F$. Thus S "selects" a member of F that has a maximal set of Lebesgue points.

Question 8.2

Suppose f is a Lebesgue measurable nonnegative real function on $\mathbb{R}$ and $A(f)$ is the ordinate set of f. This is the set of all points $(x, y) \in \mathbb{R}^2$ for which $0 < y < f(x)$.

(a) Is it true that $A(f)$ is Lebesgue measurable, in the two-dimensional sense?

(b) If the answer to (a) is affirmative, is the integral of f over $\mathbb{R}$ equal to the measure of $A(f)$?

(c) Is the graph of f a measurable subset of $\mathbb{R}^2$?

(d) If the answer to (c) is affirmative, is the measure of the graph equal to zero?

Question 8.3

Find an example of a positive continuous function f in the open unit square in $\mathbb{R}^2$, whose integral (relative to the Lebesgue measure) is finite but such that $\varphi(x)$ (in the notation of Theorem 8.8) is infinite for some $x \in (0, 1)$.

Question 8.4

Suppose $1 \leq p \leq \infty$, $f \in L^1(\mathbb{R})$, and $g \in L^p(\mathbb{R})$.

(a) Imitate the proof of Theorem 8.14 to show that the integral defining $(f * g)(x)$ exists for almost all x, that $f * g \in L^p(\mathbb{R})$ and that $\|f * g\|_p \leq \|f\|_1 \|g\|_p$.

(b) Show that equality can hold in (a) if $p = 1$ and if $p = \infty$, and find the conditions under which this happens.

(c) Assume $1 < p < \infty$, and equality holds in (a). Show that then either $f = 0$ a.e. or $g = 0$ a.e.

(d) Assume $1 \leq p \leq \infty$, $\varepsilon > 0$, and show that there exist $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$ such that

$$\|f * g\|_p > (1 - \varepsilon)\|f\|_1 \|g\|_p.$$

Question 8.5

Let M be the Banach space of all complex Borel measures on $\mathbb{R}$. The norm in M is $\|\mu\| = |\mu|(\mathbb{R})$. Associate to each Borel set $E \subset \mathbb{R}$ the set

$$E_2 = \{(x, y) \in \mathbb{R}^2 : x + y \in E\}.$$

If $\mu, \lambda \in M$, define their convolution $\mu * \lambda$ to be the set function given by $(\mu * \lambda)(E) = (\mu \times \lambda)(E_2)$ for every Borel set $E \subset \mathbb{R}$; where $\mu \times \lambda$ is as in Definition 8.7.

(a) Prove that $\mu * \lambda \in M$ and that $\|\mu * \lambda\| \leq \|\mu\| \cdot \|\lambda\|$.

(b) Prove that $\mu * \lambda$ is the unique $\nu \in M$ such that

$$\int f \, d\nu = \int \int f(x + y) \, d\mu(x) \, d\lambda(y)$$

for every $f \in C_0(\mathbb{R})$.

(c) Prove that the convolution in M is commutative, associative, and distributive with respect to addition.

(d) Prove the formula

$$(\mu * \lambda)(E) = \int \mu(E - t) \, d\lambda(t)$$

for every $\mu, \lambda \in M$ and every Borel set E. Here $E - t = \{x - t : x \in E\}$.

Question 8.12

Use Fubini's theorem and the relation

$$\frac{1}{x} = \int_0^\infty e^{-xt}\, dt \quad (x > 0)$$

to prove that

$$\lim_{b\to\infty} \int_0^b \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

Question 8.14

Complete the following proof of Hardy's inequality. Suppose $f \geq 0$ on $(0, \infty)$, $f \in L^p$, $1 < p < \infty$, and

$$F(x) = \frac{1}{x} \int_0^x f(t)\, dt.$$

Write $xF(x) = \int_0^x f(t)t^\alpha t^{-\alpha}\, dt$, where $0 < \alpha < \frac{1}{q}$, use Hölder's inequality to get an upper bound for $F(x)^p$, and integrate to obtain

$$\int_0^\infty F^p(x)\, dx \leq (1 - \alpha q)^{1-p}(\alpha p)^{-1} \int_0^\infty f^p(t)\, dt.$$

Show that the best choice of α yields

$$\int_0^\infty F^p(x)\, dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(t)\, dt.$$

Question 8.15

Put $\varphi(t) = 1 - \cos t$ if $0 \leq t \leq 2\pi$, $\varphi(t) = 0$ for all other real t. For $-\infty < x < \infty$, define

$$f(x) = 1, \quad g(x) = \varphi'(x), \quad h(x) = \int_{-\infty}^x \varphi(t)\, dt.$$

Verify the following statements about convolutions of these functions:

1. $(f * g)(x) = 0$ for all x.

2. $(g * h)(x) = (\varphi * \varphi)(x) > 0$ on $(0, 4\pi)$.

3. Therefore $(f * g) * h = 0$, whereas $f * (g * h)$ is a positive constant.

But convolution is supposedly associative, by Fubini's theorem (Question 8.5c). What went wrong?

Question 9.2

Compute the Fourier transform of the characteristic function of an interval. For $n \in \mathbb{N}$, let g_n be the characteristic function of $[-n, n]$, let h be the characteristic function of $[-1, 1]$, and compute $g_n * h$ explicitly. Show that $g_n * h$ is the Fourier transform of a function $f_n \in L^1$; except for a multiplicative constant,

$$f_n(x) = \frac{\sin x \sin nx}{x^2}.$$

Show that $\lim_{n\to\infty} \|f_n\|_1 = \infty$ and conclude that the mapping $f \mapsto \hat{f}$ maps L^1 into a proper subset of C_0. Show, however, that the range of this mapping is dense in C_0.

Question 9.6

Suppose $f \in L^1$, f is differentiable almost everywhere, and $f' \in L^1$. Does it follow that the Fourier transform of f' is $ti\hat{f}(t)$?

Question 9.8

If p and q are conjugate exponents, $f \in L^p$, $g \in L^q$, and $h = f * g$, prove that h is uniformly continuous. If also $1 < p < \infty$, then $h \in C_0$; show that this fails for some $f \in L^1$ and $g \in L^\infty$.