Department of Mathematics University of Houston

MATH 6321

Theory of Functions of a Real Variable
Spring 2025

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 9, due Thursday, April 17, 10am

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let $f \in L^1(\mathbb{R})$ and Mf be the maximal function associated with f as defined in the lecture. Show that for each Lebesgue point $x \in \mathbb{R}$, we have $|f(x)| \leq Mf(x)$.

Problem 2

Let μ be a Borel measure on $\mathbb{R}$, $f \in L^1(\mu)$ and $\|f\|_1 > 0$. Show that there is at most one number $c > 0$ such that

$$\mu(\{x \in \mathbb{R} : |f(x)| \geq c\}) = \frac{1}{c}\|f\|_1 .$$

Problem 3

Let $f \in L^2(\mathbb{R})$, and m is the Lebesgue measure. Prove that for almost every $x \in \mathbb{R}$,

$$\lim_{r \to 0} \frac{1}{2r} \int_{[x-r,x+r]} |f - f(x)|^2 dm = 0 .$$

Problem 4

Prove that if $f \in L^1(\mathbb{R})$ and for each $x \in \mathbb{R}$,

$$\int_{(-\infty,x]} f dm = 0,$$

then $f(x) = 0$ for almost every $x \in \mathbb{R}$.