# Burnside's Lemma

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**Theorem 0.1.** *Let $G$ be a group, and $G \curvearrowright A$. If we denote the collection of all the orbits of $A$ under the action of $G$ by $A/G$, and the elements of $A$ which are fixed by a $g \in G$ by $A^g$, then*

$$|A/G| = \frac{1}{|G|} \sum_{g \in G} |A^g|$$

*Proof.* Notice that

$$\bigcup_{a \in A} G_a \times a = \{(g, a) \in G \times A \ : \ g \cdot a = a\} = \bigcup_{g \in G} g \times A^g$$

Thus $\sum_{a \in A} |G_a| = \sum_{g \in G} |A^g|$. Moreover, notice that if $ga = b$, then $G_b = gG_a g^{-1}$. Thus $|G_a| = |G_b|$ if $a$ and $b$ are in the same orbits. Also, by the orbit-stabilizer theorem, we know that there are $|G : G_a|$ elements in the orbit of $a$. Thus for each representative $r$ from the distinct orbits of $A$, we see that

$$\sum_{g \in G} |A^g| = \sum_{r \in A/G} |G_r||G : G_r| = \sum_{r \in A/G} |G| = |G||A/G|$$

Hence the theorem follows. $\square$