MATH 6320

Theory of Functions of a Real Variable
Fall 2024

First name: ______________________ **Last name:** ________________________ **Points:**

Assignment 8, due Thursday, November 7, 11:30am

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let f be a measurable complex-valued function on a locally compact Hausdorff space X with a Borel measure μ. Let for $w \in \mathbb{C}$, $\epsilon > 0$,

$$A_{w,\epsilon} = \{x \in X : |f(x) - w| < \epsilon\}$$

and

$$R_f = \{w \in \mathbb{C} : \text{ for each } \epsilon > 0, \mu(A_{w,\epsilon}) > 0\}.$$

Show that R_f is a closed subset of $\mathbb{C}$.

Problem 2

A step function on $\mathbb{R}$ is a finite linear combination of characteristic functions of bounded intervals in $\mathbb{R}$. Let m be the Lebesgue measure and let $f \in L^1(m)$. Show that for any $\epsilon > 0$, there is a step function g such that $\int |f - g| dm < \epsilon$. Hint: First prove this for a function f which is bounded and for which $A : \{x \in \mathbb{R} : f(x) \neq 0\}$ has finite measure $m(A) < \infty$. Then use that for a general f, $f\chi_B$ has these properties where $B = \{x \in \mathbb{R} : \frac{1}{n} \leq |f(x) \leq n\}$ and consider $n \to \infty$.

Problem 3

Let $f \in L^1(m)$, where m is the Lebesgue measure on $\mathbb{R}$ For $t \in \mathbb{R}$, let $f_t : x \mapsto f(x - t)$. Show that

$$\lim_{t \to 0} \int |f_t - f| dm = 0\,.$$

Hint: Use the preceding problem.