Thm: Every field is contained in an algebraically closed field.

Pf (general case):

ring of polys.

Let $R = F[\{x_f : f \in F[x] \text{ is nonconstant and monic}\}]$,

and let $\mathfrak{a} = (\{f(x_f) : f \in F[x] \text{ is nonconstant and monic}\})$

ideal in $R$

Claim: $\mathfrak{a}$ is a proper ideal. If not then

$\exists n \in \mathbb{N}$, $a_1, a_2, \dots, a_n \in R$ s.t.

$$a_1 f_1(x_{f_1}) + a_2 f_2(x_{f_2}) + \dots + a_n f_n(x_{f_n}) = 1.$$

By Kron.'s thm there is an extension of $F$ where $f_1, \dots, f_n$ have roots $\alpha_1, \dots, \alpha_n$. Then, since the eqn. above must hold for all choices of the vars. involved, we would have (with $x_{f_1} = \alpha_1$, $x_{f_2} = \alpha_2, \dots, x_{f_n} = \alpha_n$):

$$0 = a_1(\dots) f_1(\alpha_1) + a_2(\dots) f_2(\alpha_2) + \dots + a_n(\dots) f_n(\alpha_n) = 1,$$

which is a contradiction. This establishes the claim.

Since $\mathfrak{a}$ is a proper ideal of $R$, it is contained in a maximal ideal $\mathfrak{m}$ (here we are using Zorn's lemma).

Let $K_1 = R/\mathfrak{m}$. The map $F \to K_1$, $\alpha \mapsto \alpha + \mathfrak{m}$ is not identically $0$, so it's injective, which gives a field extension $K_1/F$, where $f(x_f) = 0$, for every nonconst.

monic poly. $f$ in $F[x]$.

Repeat the construction countably many times to obtain a seq. of field exts.

$$F \subseteq K_1 \subseteq K_2 \subseteq \cdots$$

with the prop. that every poly. w/ coeffs in $K_n[x]$ has a root in $K_{n+1}$.

Then let $K = \bigcup_{n=1}^{\infty} K_n$.

If $f \in K[x]$ then $f \in K_n[x]$ for some $n$, so $f$ has a root in $K_{n+1}$.

Therefore $K$ is alg. closed. $\blacksquare$

Thm: Every field $F$ has an algebraic closure.

Pf: Let $K$ be an alg. closed extension of $F$.

Let $\overline{F} = \{\alpha \in K : \alpha \text{ is algebraic over } F\}$.

If $f$ is a poly w/ coeffs. in $F$ then $\exists$ splitting field $F'$ of $f$ in $K$.

Then $F'/F$ is algebraic, so $F' \subseteq \overline{F}$. $\blacksquare$

Thm: If $F$ is a field and if $L_1, L_2$ are both alg. closures of $F$, then $L_1 \cong L_2$.

Pf: Let $\mathcal{S}$ $\neq \phi$ be the collection of all nonzero field homs.

$\tau : K_1 \to L_2$, where $F \subseteq K_1 \subseteq L_1$.

Partial order on $\mathcal{S}$: $\tau_1 \leq \tau_2$ if $\tau_2$ extends $\tau_1$.

Suppose $\{\tau_i : K_i \to L_2\}$ is a chain in $\mathcal{S}$.

Define $K = \cup K_i$, and define $\tau : K \to L_2$ by

$\tau(\alpha) = \tau_i(\alpha)$, $\forall \alpha \in K_i$. Then $\tau$ is an upp. bd. for the chain.

By Zorn's lemma, $\exists$ max. elem $\tilde{\tau} : \tilde{K} \to L_2$.

Claims: i) $\tilde{K} = L_1$

(next time)...

ii) $\tilde{\tau}$ is onto.