

Exs: (Cont. from last time)

1) $K = \mathbb{Q}(\zeta_5)$, $F = \mathbb{Q}$.

$$f(x) = \min_{\mathbb{Q}}(\zeta_5) = \Phi_5(x) = x^4 + x^3 + x^2 + x + 1,$$

so $[K:\mathbb{Q}] = 4$.

Also $f(x) = \prod_{a=1}^4 (x - \zeta_5^a)$, so K is the spl. field of f .

By our thm., K/F is Galois, so $|\text{Aut}(K/F)| = 4$.

Any $\sigma \in \text{Aut}(K/F)$ is uniquely determined by $\sigma(\zeta_5)$, and there are 4 possibilities:

$$\sigma(\zeta_5) = \zeta_5^a, \quad 1 \leq a \leq 4.$$

All of these must occur, since $|\text{Aut}(K/F)| = 4$.

Let $\tau \in \text{Aut}(K/F)$ be determined by $\tau(\zeta_5) = \zeta_5^2$.

$$\text{Then } \tau^2(\zeta_5) = \tau(\zeta_5^2) = (\tau(\zeta_5))^2 = (\zeta_5^2)^2 = \zeta_5^4$$

$$\tau^3(\zeta_5) = \tau(\tau^2(\zeta_5)) = \tau(\zeta_5^4) = (\tau(\zeta_5))^4 = (\zeta_5^2)^4 = \zeta_5^3$$

$$\tau^4(\zeta_5) = \tau(\tau^3(\zeta_5)) = \tau(\zeta_5^3) = \zeta_5.$$

The FTGT gives us the correspondence:

$$\begin{array}{ccc} \{id\} & & \mathbb{Q}(\sqrt[4]{5}) \\ \downarrow & & \downarrow \\ H_1 = \langle \gamma^2 \rangle & & K_{H_1} \\ \downarrow & & \downarrow \\ \text{Gal}(K/F) = \langle \gamma \rangle \cong C_4 & & \mathbb{Q} \end{array}$$

To figure out what K_{H_1} is:

$$\gamma^2(\sqrt[4]{5}) = \gamma(\gamma(\sqrt[4]{5})) = \gamma(\sqrt[4]{5}^2) = \sqrt[4]{5}^4 = \sqrt[4]{5}^{-1}$$

$$\Rightarrow \gamma^2(\sqrt[4]{5} + \sqrt[4]{5}^{-1}) = \sqrt[4]{5} + \sqrt[4]{5}^{-1}$$

$$\text{Also, } \sqrt[4]{5} + \sqrt[4]{5}^{-1} = \sqrt[4]{5} + \sqrt[4]{5}^3 = -1 - \sqrt[4]{5}^2 - \sqrt[4]{5}^3 \notin \mathbb{Q}, \text{ because}$$

$\{1, \sqrt[4]{5}, \sqrt[4]{5}^2, \sqrt[4]{5}^3\}$ is a \mathbb{Q} -basis for K .

$$\text{Therefore } \sqrt[4]{5} + \sqrt[4]{5}^{-1} \in K_{H_1} \setminus \mathbb{Q} \Rightarrow K_{H_1} = \mathbb{Q}(\sqrt[4]{5} + \sqrt[4]{5}^{-1}).$$

\uparrow
 $[K_{H_1} : \mathbb{Q}] = 2$

2) K is splitting field of $x^4 - 4$, over \mathbb{Q} .

$$K/\mathbb{Q} \text{ is Galois, and } K = \mathbb{Q}(\sqrt{2}, i) \Rightarrow [K : \mathbb{Q}] = 4.$$

To determine $\text{Gal}(K/\mathbb{Q})$:

$$\text{min}_{\mathbb{Q}}(\sqrt{2}) = x^2 - 2,$$

$$\text{min}_{\mathbb{Q}}(i) = x^2 + 1,$$

$$\begin{array}{c|c} \mathbb{Q}(\sqrt{2}, i) & \\ \hline \mathbb{Q}(\sqrt{2}, i) & \leftarrow i \notin \mathbb{Q}(\sqrt{2}) \\ \mathbb{Q}(\sqrt{2}) & \\ \hline \mathbb{Q} & \end{array}$$

so any element of $\text{Gal}(K/\mathbb{Q})$ must satisfy

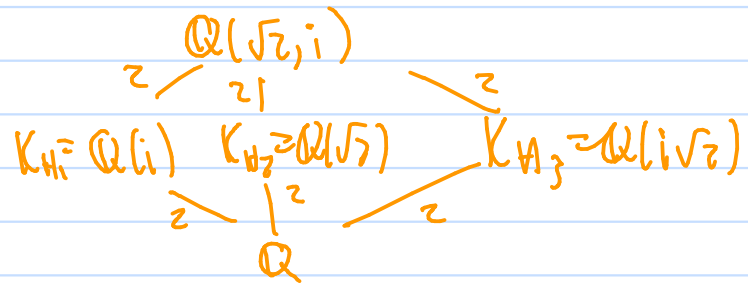
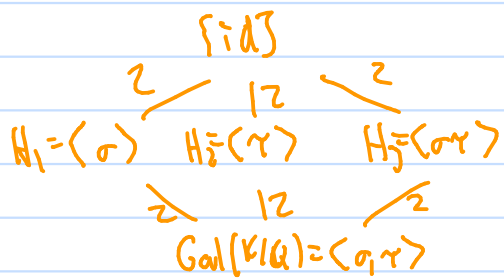
$$\sqrt{2} \mapsto \pm \sqrt{2} \quad \text{and} \quad i \mapsto \pm i.$$

There are only 4 possibilities, so they all have to extend to auts. of K which fix \mathbb{Q} .

Let $\sigma, \tau \in \text{Gal}(K/\mathbb{Q})$ be def by:

$$\sigma: \begin{cases} \sqrt{2} \mapsto -\sqrt{2} \\ i \mapsto i \end{cases}, \quad \tau: \begin{cases} \sqrt{2} \mapsto \sqrt{2} \\ i \mapsto -i \end{cases}$$

Then $\text{Gal}(K/\mathbb{Q}) = \{\text{id}, \sigma, \tau, \sigma\tau\}$.



3) K be spl. field of $x^3 - 2$ over \mathbb{Q} .

$$x^3 - 2 = \prod_{i=0}^2 (x - \sqrt[4]{3} \varphi_j^i).$$

$$\sqrt[4]{3}, \varphi_j = \frac{\sqrt[4]{3} \varphi_j}{\sqrt[4]{3}} \in K \Rightarrow K = \mathbb{Q}(\sqrt[4]{3}, \varphi_j) = \mathbb{Q}(\sqrt[4]{3}) \mathbb{Q}(\varphi_j)$$

Also, $[K:\mathbb{Q}] \leq 3 \cdot 2 = 6$, but $2, 3 \mid [K:\mathbb{Q}]$

$$\Rightarrow [K:\mathbb{Q}] = 6$$

$$\Rightarrow |\text{Gal}(K/\mathbb{Q})| = 6.$$

To determine $\text{Gal}(K/\mathbb{Q})$:

$$\text{min}_{\mathbb{Q}}(\sqrt[4]{3}) = x^3 - 2$$

$$\text{min}_{\mathbb{Q}}(\varphi_j) = \Phi_3(x) = x^2 + x + 1,$$

So any element of $\text{Gal}(K/\mathbb{Q})$ must satisfy

$$\sqrt[4]{3} \mapsto \begin{cases} \sqrt[4]{3} \\ \sqrt[4]{3} \varphi_j \\ \sqrt[4]{3} \varphi_j^2 \end{cases} \quad / \quad \varphi_j \mapsto \begin{cases} \varphi_j \\ \varphi_j^2 \end{cases}.$$

Since there are 6 possibilities, all choices must extend to auto. (cont next time...)