MATH6303 - Modern Algebra II
Homework 4

Joel Sleeba

March 30, 2025

1. **Solution:**

   (a) Since $\mathrm{Tor}(M) \subset M$ by the definition, the distributivity properties of the R-addition and R-multiplication hold. We only need to prove that $\mathrm{Tor}(M)$ is a subgroup of M to show that it is a submodule. Let $g, h \in \mathrm{Tor}(M)$, then there exits $r_g, r_h \in R \setminus \{0\}$ such that $r_g g = r_h h = 0$. Since R is an integral domain, $r_g r_h \neq 0$, and by the commutativity of the ring R,

   $$r_g r_h (g + h) = r_g r_h g + r_g r_h h = r_h r_g g + 0 = 0$$

   Thus $g + h \in \mathrm{Tor}(M)$. To see $-g \in \mathrm{Tor}(M)$, notice that

   $$0 = r_g g = (-r_g)(-g)$$

   Thus $\mathrm{Tor}(M)$ is a subgroup of M, and hence a submodule of M

   (b) Consider the ring $Z_6 := \mathbb{Z}/6\mathbb{Z}$ and the module Z_6 over Z_6. Then $\mathrm{Tor}(Z_6) = \{0, 2, 3, 4\}$. Clearly this is not a submodule since $2 + 3 \notin \mathrm{Tor}(Z_6)$.

   (c) Let $a \in \mathbb{R}$ be a zero divisor such that $ab = 0$ for some $b \neq 0 \in \mathbb{R}$. If $M = \mathrm{Tor}(M)$, we are done. So Let $x \in M \setminus \mathrm{Tor}(M)$. Then $a(bx) = (ab)x = 0$ shows that $bx \in \mathrm{Tor}(M)$. Since $x \notin \mathrm{Tor}(M)$, $bx \neq 0$, and we are done.

2. **Solution:** Let $A_N = \{r \in R \mid rn = 0, \ \forall n \in N\}$ be the annihilator of the submodule N of M. If $a, b \in A_N$, then clearly $a + b, ab \in A_N$ since for any $n \in N$

   $$(a + b)n = an + bn = 0 + 0 = 0$$
   $$(ab)n = a(bn) = a(0) = 0$$

Thus A_N is a subring of R. Now let $c \in R$, then for any $n \in N$, $cn \in N$ since N is a submodule. Also

$$(ac)n = a(cn) = 0$$

thus $ac \in A_N$. Since $(ca)n = c(an) = c0 = 0$, we also get $ca \in A_N$ proving that A_N is a two sided ideal of R.

3. **Solution:** Let I be a right ideal of R and $N_I = \{m \in M \mid am = 0, \ \forall a \in I\}$. By the submodule criterion, we'll be done if we show that $r(x-y) \in N_I$ for all $x, y \in N_I$ and $r \in R$. Let $x, y \in N_I$. Then for any $a \in I$,

$$ar(x-y) = (ar)x - (ar)y = 0 - 0 = 0$$

where $arx, ary = 0$ since $ar \in I$ as I is a right-ideal. Thus N_I is a submodule of M.

4. **Solution:**

   (a) Let I be the annihilator of M in $\mathbb{Z}$, then by the definition of annihilator $n \in I$ if and only if $nm = 0$ for all $m \in M$. A typical element of M is of the form (x, y, z), where $x \in \mathbb{Z}/24\mathbb{Z}$, $y \in \mathbb{Z}/15\mathbb{Z}$, and $z \in \mathbb{Z}/50\mathbb{Z}$. Therefore $n(x, y, z) = (nx, ny, nz) = 0$ if and only if $24|nx, 15|ny$ and $50|nz$. Since this must hold true for all x, y, z, the least positive integer n which satisfy all the three conditions is $n = \mathrm{lcm}(24, 15, 50) = 600$. Hence $I = \langle 600 \rangle = 600\mathbb{Z}$.

   (b) Given that $I = 2\mathbb{Z}$. Let $N_I \subset M$ be the annihliator of I in M. Then by the definition of N_I, $(x, y, z) \in N_I$ (where x, y, z are as before) if and only if $(ax, ay, az) = 0$ for all $a \in I$. Since $I = 2\mathbb{Z}$, for $a = 2n$, this reduces to having $24|2nx, 15|2ny, 50|2nz$ for all $n \in \mathbb{Z}$. Thus, we see that $12|x, 15|y, 25|z$.

   Thus $N_I = \langle (12, 0, 0), \ (0, 0, 25) \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

5. **Solution:** From what we have done in the lecture, we know that submodules of $F[x]$ correspond to the invariant subspaces of T. Thus we'll be done if we show that the invariant subspaces of $T : \mathbb{R}^2 \to \mathbb{R}^2 := (x, y) \mapsto (0, y)$ are precisely $\{0\}, \mathbb{R}^2$, x-axis and the y-axis. Clearly all of these are invariant subspaces by basic verification.

Now let V be any proper non-trivial invariant subspace of V with $(x, y) \in V$. Since V is proper, $V = \text{span}\{(x, y)\}$. If either $x = 0$ or $y = 0$, then V would be x-axis or y-axis respectively. Hence for the sake of contradiction, assume $x, y \neq 0$. But $T(x, y) = (0, y)$ and since $(x, y), (0, y)$ are linearly independent, invariance of V under T forces $V = \mathbb{R}^2$ which contradicts the proper subspace assumption. Hence we are done.