2a) Suppose G is free Abelian group of rank n and suppose $H \leq G$ also has rank n. By the Stacked bases thm., we can choose gens. $x_1, \dots, x_n$ for G, and $y_1, \dots, y_n$ for H s.t.

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad A = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}, \quad a_i \in \mathbb{Z},$$

$$a_1 \mid a_2 \mid \cdots \mid a_n.$$

Define $\varphi : G \to \mathbb{Z}/a_1\mathbb{Z} \times \cdots \times \mathbb{Z}/a_n\mathbb{Z}$

by $\varphi(m_1 x_1 + \cdots + m_n x_n) = (m_1, \dots, m_n)$.

This is a homom. with

$$\ker \varphi = H.$$

1st isom. thm $\Rightarrow G/H \cong \mathbb{Z}/a_1\mathbb{Z} \times \cdots \times \mathbb{Z}/a_n\mathbb{Z}$

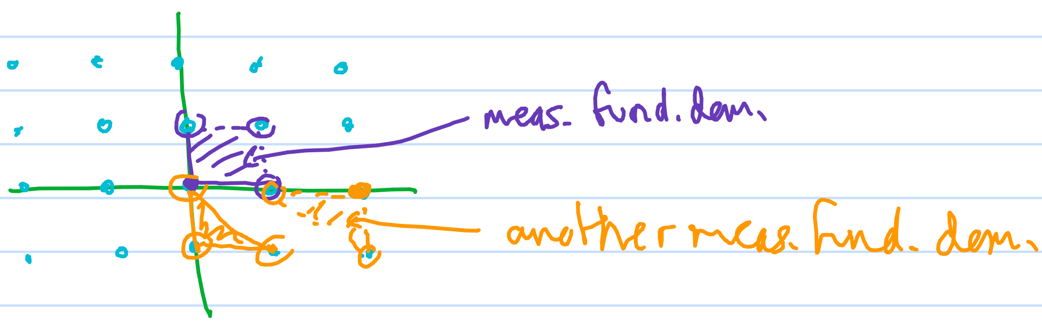
$$\Rightarrow |\det A| = |G : H|.$$

2b) Suppose $\Lambda \subseteq \mathbb{R}^n$ is a discrete subgroup of rank n (a lattice in $\mathbb{R}^n$).

A measurable subset $F_\Lambda \subseteq \mathbb{R}^n$ which is a complete set of distinct coset reps. for $\mathbb{R}^n/\Lambda$ is called a <u>measurable fundamental domain</u> for Λ.

Ex: $n=2$, $\Lambda = \mathbb{Z}$-submodule of $\mathbb{R}^2$ gen. by $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$

Λ:



All meas. fund. doms. for Λ have the same volume, by translation invariance of Lebesgue measure λ.

Now suppose $\Gamma \le \Lambda$ also has rank n. By Ex. 2a, can choose bases $x_1, \dots, x_n$ for Λ and $y_1, \dots, y_n$ for Γ s.t.

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad A = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}, \quad a_i \in \mathbb{Z},$$

$a_1 \mid a_2 \mid \cdots \mid a_n$.

Let $F_\Lambda = \left\{ \sum_{i=1}^{n} t_i x_i : 0 \le t_i < 1, 1 \le i \le n \right\}$, and

$F_\Gamma = \left\{ \sum_{i=1}^{n} s_i y_i : 0 \le s_i < 1, 1 \le i \le n \right\}$.

Then $\lambda(F_\Gamma) = \int_{F_\Gamma} 1 \, d\vec{s}$ $\quad (\vec{s} = A\vec{t})$

$$= \int_{F_\Lambda} |\det A| \, d\vec{t} = |\det A| \cdot \lambda(F_\Lambda)$$

$$\Longrightarrow \frac{\lambda(F_\Gamma)}{\lambda(F_\Lambda)} = |\det A| = |\Lambda : \Gamma| .$$

Another way to see this: By the 3rd isom. thm,

$$\left(\mathbb{R}^n/\Gamma\right) \Big/ \left(\Lambda/\Gamma\right) \cong \mathbb{R}^n/\Lambda .$$

From this, you can deduce that F_Γ is a disjoint union of $|\Lambda : \Gamma|$ translates of $\mathbb{R}^n/\Lambda$, so the result also follows from translation invariance of Leb meas.

Fund. Thm. for Fin. Gen. Modules over a PID:

(invariant factor decomposition):

If R is a PID and M is a finitely generated R-module then

$$M \cong R^r \oplus R/(a_1) \oplus \cdots \oplus R/(a_m),$$

where $r \in \{0,1,\dots\}$, $a_1,\dots,a_m \in R \setminus R^\times$,

and $a_1 \mid a_2 \mid \cdots \mid a_m$.

This decomp. is unique.

(Elem. div. decomp.):

$$M \cong R^r \oplus R/(p_1^{a_1}) \oplus \cdots \oplus R/(p_k^{a_k}),$$

where $p_1^{a_1},\dots,p_k^{a_k}$ are powers of (not nec. dist.) prime elements $p_1,\dots,p_k \in R$.

Pf: (invariant factor decomp, existence)

Suppose $\operatorname{rank}(M)=n$, and that M is generated by $x_1,\dots,x_n$.

The map $\varphi: R^n \to M$ defined by

$$(r_1,\dots,r_n) \mapsto \sum_{i=1}^{n} r_i x_i$$

is a surjective R-module homom.

By the stacked bases thm, $\ker(\varphi)$ is a free R-module of rank $m \leq n$ and $\exists$ basis $y_1, \dots, y_n$ for R^n and $a_1, \dots, a_m \in R$, $a_1 | \cdots | a_m$ ($a_i \neq 0$, $a_i \notin R^\times$), s.t.

$$\ker \varphi = \left(\begin{array}{ccc|c} a_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & a_m & \end{array} \right) R^n .$$

i.e., $a_1 y_1, \dots, a_m y_m$ is a basis for $\ker \varphi$.

By the 1st Isom thm,

$$M = \varphi(R^n) \cong R^n / \ker \varphi \cong R^{n-m} \oplus R/(a_1) \oplus \cdots \oplus R/(a_m).$$

∎

Applications:

1) If $R = \mathbb{Z}$ then this gives the FTFGAG.