MATH 6321 - Theory of functions of a real variable
Homework 9

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1. **Solution:** Let $f \in L^1(\mathbb{R})$ and $x \in \mathbb{R}$ be a Lebesgue point. Then by the definition, we have

$$\lim_{r \to 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(x) - f(y)| \; dm(y) \; = \; 0$$

Thus for every $\varepsilon > 0$, there is a $r_\varepsilon > 0$ such that for all $r < r_\varepsilon$,

$$\frac{1}{m(B_r(x))} \int_{B_r(x)} |f(x) - f(y)| \; dm(y) < \varepsilon$$

Since $| \int f \; d\mu| < \int |f| \; d\mu$, we get

$$\begin{aligned}
\Big| f(x) - \frac{1}{m(B_r(x))} \int_{B_r(x)} f(y) \; dm(y) \Big| & \\
&= \Big| \frac{1}{m(B_r(x))} \int_{B_r(x)} f(x) \; dm(y) - \frac{1}{m(B_r(x))} \int_{B_r(x)} f(y) \; dm(y) \Big| \\
&\leq \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(x) - f(y)| \; dm(y) \\
&< \varepsilon
\end{aligned}$$

and thus

$$|f(x)| - \Big| \frac{1}{m(B_r(x))} \int_{B_r(x)} f(y) \; dm(y) \Big| \; < \varepsilon$$

which gives

$$|f(x)| < \frac{1}{m(B_r(x))} \int_{B_r(x)} |f| \, dm + \varepsilon$$

Since $\varepsilon > 0$ was chosen arbitrarily, taking $\varepsilon \to 0$ and taking supremum over all $r > 0$ gives

$$|f(x)| \leq \sup_{r>0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f| \, dm = \mathcal{M}f(x)$$

2. **Solution:** For the sake of contradiction, assume that there exists $c_1 > c_2 > 0$ such that for $A_i := \{x \in \mathbb{R} \ : \ |f(x)| \geq c_i\}$, we have $c_i\mu(A_i) = \|f\|_1$. Then

   $$\|f\|_1 \geq \int |f|\chi_{A_i} \, d\mu \geq \int c_i\chi_{A_i} \, d\mu = c_i\mu(A_i) = \|f\|_1$$

   Thus

   $$\int_{A_i} |f| - c_i \, d\mu = 0 \tag{1}$$

   Since $\|f\|_1, c_i > 0$, by assumption, we see $\mu(A_i) > 0$. Moreover $|f(x)| - c_i > 0$ for all $x \in A_i$ by definition. Thus *Equation* 1 forces $|f(x)| = c_i$ almost everywhere in A_i. But since by definition $A_1 \subset A_2$, this gives a contradiction as $|f|$ cannot be a.e equal to c_1 and c_2 simultaneously in A_2 unless $c_1 = c_2$.

3. not finished

   Solution: Let $x \in \mathbb{R}$ and $g_x : [x-1, x+1] \to \mathbb{R} : g_x(y) := |f(x) - f(y)|^2$. Note that since $f \in L^2([x-1, x+1])$, by Holder's inequality, $f \in L^1([x-1, x+1])$. Also

   $$\begin{aligned} \int_{[x-1,x+1]} |g_x| \, dm &= \int_{[x-1,x+1]} |f(x) - f(y)|^2 \, dm(y) \\ &\leq \int_{[x-1,x+1]} |f(x)|^2 + 2|f(x)||f(y)| + |f(y)|^2 \, d\mu \\ &\leq 2|f(x)|^2 + 2|f(x)|\|f\|_1 + \|f\|_2 \\ &< \infty \end{aligned}$$

   Thus $g_x \in L^1([x-1, x+1])$. Thus almost every $y \in [x-1, x+1]$ is a Lebesgue point of g_x by the Lebesgue differentiation theorem.

4. **Solution:** We know that $\mu(A) := \int_A f \; dm$, defines a measure on $\mathbb{R}$. By the properties of the measure μ, for any $x < y \in \mathbb{R}$,

$$\int_{(x,y]} f \; dm = \int_{(-\infty,y]} f \; dm - \int_{(-\infty,x]} f \; dm = 0 - 0 = 0$$

Any open interval $(x, y) = \cup_{n=1}^{\infty}(x, y - \frac{1}{n})$. By the continuity of the measure μ from below, we get

$$\int_{(x,y)} f \; dm = \lim_{n\to\infty} \int_{(x,y-\frac{1}{n}]} f \; dm = 0$$

Since $f \in L^1(m)$, we know that almost all $x \in \mathbb{R}$ are Lebesgue points of f. That is for almost every $x \in \mathbb{R}$,

$$f(x) = \lim_{r\to 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} f \; dm = \lim_{r\to 0} 0 = 0$$

Thus we are done.