# Group actions

Suppose $G$ is a group and $A$ is a set. A group action of $G$ on $A$ is map from $G \times A$ to $A$ (denoted $(g,a) \mapsto g \cdot a$ or $ga$) satisfying:

i) $\forall g_1, g_2 \in G$, $a \in A$, $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$ (associativity)

ii) $\forall a \in A$, $1_G \cdot a = a$.

Notation: $G \curvearrowright A$ means $G$ acts on $A$.

Why group actions are important:

1) They allow you to impose a group structure on a set, i.e. to "turn the set" into a group.

   Ex: Rubik's cube group

2) The set $A$ can allow you to "visualize" the group $G$.

   Ex: $G = SL_2(\mathbb{R})$ acts on $A = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$

   by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \dfrac{az+b}{cz+b}$ (fractional linear transformations)

3) Other reasons (see intro to Chapter about group actions)

Thm: Suppose $G \curvearrowright A$. Then:

i) $\forall g \in G$, the map $\pi_g : A \to A$ defined by $\pi_g(a) = g.a$ is an element of $S_A$.

ii) The map $\varphi : G \to S_A$ defined by $\varphi(g) = \pi_g$ is a homomorphism. (called the <u>permutation representation</u> associated to the action).

Pf: i) $\forall g \in G$, $a \in A$

$$\begin{aligned}(\pi_{g^{-1}} \circ \pi_g)(a) &= g^{-1}.(g.a) && (\text{def. of } \pi_{g^{-1}}, \pi_g)\\ &= (g^{-1}g).a && (\text{assoc. of } G \curvearrowright A)\\ &= e_G.a \\ &= a && (\text{identity prop. of } G \curvearrowright A)\end{aligned}$$

Since $\pi_g$ has a well-def. inv. fn, it is a bijection.

ii) follows from associativity of $G \curvearrowright A$. ■

Defs: Let $\varphi$ be the perm. rep. assoc. to $G \curvearrowright A$.

i) If $\ker \varphi = G$ the action is the <u>trivial action</u>.

ii) If $\ker \varphi = \{e\}$ then the group acts <u>faithfully</u>.

iii) $\forall a \in A$, the <u>stabilizer</u> of $a$ is

$$G_a = \mathrm{stab}_G(a) = \{g \in G : g.a = a\}.$$

iv) $\forall a \in A$, the <u>orbit</u> of $a$ is

$$\mathrm{orb}_G(a) = \{g \cdot a : g \in G\}.$$

v) The action is transitive if $\forall a, b \in A$, $\exists g \in G$ s.t. $g \cdot a = b$. (Equivalently: there is only one orbit. i.e. $\forall a \in A$, $\mathrm{orb}_G(a) = A$).

Facts:

1) $\forall a \in A$, $G_a \leq G$.

Pf: $e \in G_a$, so $G_a \neq \emptyset$.

Suppose $g, h \in G_a$. Then:

- $$\begin{aligned} g^{-1} \cdot a &= g^{-1}(g \cdot a) \quad (g \in G_a) \\ &= (g^{-1}g) \cdot a \\ &= e \cdot a = a \implies g^{-1} \in G_a. \end{aligned}$$
- $(gh) \cdot a = g(ha) = ga = a \implies gh \in G_a$. $\blacksquare$

2) The orbits of $G \curvearrowright A$ form a partition of $A$.

Equivalently: The relation

$$a \sim b \iff \exists g \in G \text{ s.t. } ga = b$$

is an equiv. rel. on $A$.

3) $\forall a \in A$, $|\mathrm{orb}_G(a)| = |G : G_a|$.

Pf: Fix $a \in A$. Define

$$\gamma : \mathrm{orb}_G(a) \to G/G_a$$

by $\quad g.a \mapsto gG_a$.

- well def: Suppose $g.a = h.a$

  Then $(h^{-1}g).a = (h^{-1}h).a = e.a = a$

  $\Rightarrow h^{-1}g \in G_a \Rightarrow gG_a = hG_a$.

- bijectivity: follows from a similar argument. $\boxed{\text{Pf}}$

Important examples:

1) Any group $G$ acts on itself by left multiplication: $\forall g \in G$, $a \in G$,

$$g.a = ga.$$

Note: Right mult. is not a group action, in general

$$(gh).a = agh$$

$\neq$ in general.

$$g.(h.a) = g.(ah) = ahg.$$

Note: If $g \in G$ and $ga = a$ $\forall a \in A$ then $g = e$.

So this action is faithful.

This is called the <u>left regular action</u> on $G$.