4a) $R = C([0,1])$

Let $I = \{f \in R : f(\frac{1}{2}) = 0\}$.

- I is an ideal:

$(I, +)$ is a group: $\forall f, g \in I$, $(f-g)(\frac{1}{2}) = \underbrace{f(\tfrac{1}{2})}_{=0} - \underbrace{g(\tfrac{1}{2})}_{=0} = 0$

$\Rightarrow f - g \in I$.

$\forall f \in I, g \in R$, $(fg)(\frac{1}{2}) = \underbrace{f(\tfrac{1}{2})}_{=0} g(\frac{1}{2}) = 0$.

What can you say about R/I?

Let $\phi : R \to \mathbb{R}$ be def. by $\phi(f) = f(\frac{1}{2})$.

Then ϕ is a surjective ring hom. and

$\ker \phi = I \Rightarrow R/I \cong \mathbb{R}$ (1st isom. thm)

b) Suppose A is a ring, X is a non-empty set,

$R = \{\text{fns } f : X \to A\}$. Then $\forall c \in X$,

$I_c = \{f \in R : f(c) = 0_A\}$ is an ideal of R, and

$R/I_c \cong A$.

Def: A <u>principal ideal domain</u> (PID) is an integral domain in which every ideal is principal.

Def: A <u>maximal ideal</u> M in a ring R is an ideal which is not all of R, and is not contained in any other proper ideal.

Prop: If R has $1 \neq 0$ and $I \subseteq R$ is an ideal then:

$I = R$ if and only if I contains a unit.

Pf: If $I = R$ then $1 \in I$.

If I contains a unit $a \in I$ then $a^{-1} a \in I$ (I is an ideal)

$\Rightarrow R = (1) \subseteq I \Rightarrow R = I$. $\boxtimes$

Cor: If R is a commutative ring with $1 \neq 0$ then

R is a field $\Leftrightarrow$ its only ideals are $\{0\}$ and R.

Cor: If R is a field then any non-trivial ring homom.

$\phi: R \to S$ is injective.

Pf: $\ker \phi$ is an ideal of R and $\ker \phi \neq R$

$\Rightarrow \ker \phi = \{0\}$. $\boxtimes$

Prop: If R has 1 then every proper ideal of R is contained in a maximal ideal.

Pf: Apply Zorn's lemma to the collection of all proper ideals of R, partially ordered by inclusion.

Suppose $I_1 \subseteq I_2 \subseteq \cdots$ are proper ideals of R.

Let $I = \bigcup_{n \in \mathbb{N}} I_n$. Then I is an ideal (check it)

Since $1 \notin I_n$, $\forall n \in \mathbb{N}$, $1 \notin I$, so I is proper. $\boxtimes$

Suppose from here to the end of the lecture that R has $1 \neq 0$:

Prop: Suppose M is an ideal in a commutative ring. Then M is maximal $\Leftrightarrow$ R/M is a field.

Pf: R/M is a field $\Leftrightarrow$ its only ideals are $\{0\}$ and R/M

By the 4th Isom. Thm, there is a bijective corresp. between ideals I with $M \subseteq I \subseteq R$ and ideals of R/M ... $\boxtimes$

Prop: Suppose P is an ideal in a commutative ring R.
Then P is prime if and only if R/P is an ID.

Pf: Suppose P is a prime ideal:

If $(a+P)(b+P) = P$ then

$$ab+P = P \Rightarrow ab \in P \Rightarrow a \in P \text{ or } b \in P$$

(P is a prime ideal)

$$\Rightarrow a+P = P \text{ or } b+P = P,$$

so R/P is an ID.

Suppose R/P is an ID. If $ab \in P$

then $P = ab+P = (a+P)(b+P)$

$$\Rightarrow a+P = 0 \text{ or } b+P = 0$$

$$\Rightarrow a \in P \text{ or } b \in P. \quad \blacksquare$$