First Name: ____________________

Last Name: ____________________

Signature: ____________________

Last 4 digits student I.D.: ____________________

Math 6320 Practice Final Exam

December, 2024

One hour and forty minutes

University of Houston

Instructions:

1. Put your name, signature and last 4 digits of your I.D. No. in the blanks above.
2. There are **three questions** in this exam. Answer the questions in the spaces provided, using the backs of pages or the blank pages at the end for overflow or rough work.
3. Your grade will be influenced by how clearly you present your solutions. **Justify your solutions carefully** by referring to definitions and results from class where appropriate.
4. **This is a closed book exam.**

1. Recall that a Borel measure μ on the Borel algebra B of a topological space X is regular *on a set* $A \in B$ if for all $\varepsilon > 0$ there exists a compact set K and an open set V with $K \subset A \subset V$ such that $K \subset A \subset V$ and $\mu(V \setminus K) < \varepsilon$. Let μ be a finite Borel measure on a **compact metric space** (X, d).

   (a) Let K be a compact subset of X, and $V_n = \{y \in X : \inf_{x \in K} d(x, y) < 1/n\}$. Show that $\mu(K) = \lim_{n \to \infty} \mu(V_n)$.

(b) Show that the class of sets on which μ is regular forms a σ-algebra.

(c) Use the preceding parts to conclude that μ is regular (on all Borel sets).

2. Let f be a real-valued measurable function in $L^1(m)$, where m is the Lebesgue measure on $\mathbb{R}$, and the measurability is with respect to the σ-algebra $\mathcal{M}$ as in Rudin's definition using the Riesz representation theorem. Prove that if $\int_{[a,b)} f\,dm = 0$ for each $0 \le a < b \le 1$, then there exists a set $E \subset [0,1)$ of measure $m(E) = 1$ and $f(x) = 0$ for each $x \in E$. Hint: An indirect proof starts by assuming that the set $E = \{x \in [0,1) : f(x) = 0\}$ does not have $m(E) = 1$, so then $m(E^c \cap [0,1)) > 0$. Derive a contradiction, using the regularity of the Lebesgue measure.

3. Let (X, M, μ) be a measure space, $f \in L^1(\mu)$, $g \in L^\infty(\mu)$. Show that $\|fg\|_1 = \|f\|_1 \|g\|_\infty$ implies that the set

$$E = \{x \in X : f(x) \neq 0, |g(x)| < \|g\|_\infty\}$$

has measure $\mu(E) = 0$.