MATH 6303 - Modern Algebra II
Final Exam

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1. **Solution:**

   (a) Given that $G(A) = \sum_{n \in \mathbb{N}} a_n A^n$, where the (i, j)-th entry of the partial sums converge to the (i, j)-th entry of $G(A)$. Let $G_N(A) = \sum_{i=1}^{N} a_i A^i$, be the N-th partial sum. We'll show that $PG_N(At)P^{-1} = G_N(PAtP^{-1}) = G_N(PAP^{-1}t)$. The first equality follows easily by the distributivity of the matrix multiplication as,

   $$PG_N(At)P^{-1} = P\Big(\sum_{i=1}^{N} a_i A^i t\Big)P^{-1} = \sum_{i=1}^{N} a_i PA^i tP^{-1} = G_N(PAtP^{-1})$$

   Since $t = tI$, where I is the identity matrix, it commutes with P^{-1}, and we get

   $$G_N(PAtP^{-1}) = \sum_{i=1}^{N} a_i PA^i tP^{-1} = \sum_{i=1}^{N} a_i PA^i P^{-1} tI = G_N(PAP^{-1}t)$$

   Since G is defined as the limit of the partial sums G_N, whenever the limit exists, the equality will be preserved for G as well.

   (b) Let $A = A_1 \oplus A_2 \oplus \ldots \oplus A_m$. Then

   $$A = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_m \end{pmatrix} \quad \text{and} \quad A^n = \begin{pmatrix} A_1^n & & \\ & \ddots & \\ & & A_m^n \end{pmatrix}$$

   Thus by the definition of G,

   $$G(A) = \begin{pmatrix} G(A_1) & & \\ & \ddots & \\ & & G(A_m) \end{pmatrix} \tag{1}$$

which proves what we need since $At = A_1 t \oplus A_2 t \oplus \ldots \oplus A_m t$ and

$$G(At) = \begin{pmatrix} G(A_1 t) & & \\ & \ddots & \\ & & G(A_m t) \end{pmatrix}$$

(c) This is the special case of part (b) by taking $A_i = [z_i]$.

2. **Solution:** By the definition and since $AB = BA$,

$$\exp(A)\exp(B) = \Big(I + A + \frac{A^2}{2!} + \frac{A^3}{3!} \ldots\Big)\Big(I + B + \frac{B^2}{2!} + \frac{B^3}{3!} \ldots\Big)$$

and

$$\exp(A + B) = I + A + B + \frac{(A+B)^2}{2!} + \frac{(A+B)^3}{3!} + \ldots$$

Thus we see that the nth term in the above summation is

$$\begin{aligned} \frac{(A+B)^n}{n!} &= \frac{1}{n!}\sum_{i=0}^{n} \binom{n}{i} A^{n-i}B^i \\ &= \sum_{i=o}^{n} \frac{1}{i!(n-i)!} A^{n-i}B^i \\ &= \sum_{i=0}^{n} \frac{A^{n-i}}{(n-i)!} \times \frac{B^i}{i!} \end{aligned}$$

which is precisely the term in $\exp(A)\exp(B)$ whose powers sum to n. Since this is true for all $n \in \mathbb{N}$, the power series agree and thus $\exp(A)\exp(B) = \exp(A + B)$.

3. **Solution:** Let Nt be as given. Then we see that

$$(Nt)^2 = \begin{bmatrix} 0 & 0 & t^2 & & & \\ & 0 & 0 & t^2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 0 & 0 & t^2 \\ & & & & 0 & 0 \\ & & & & & 0 \end{bmatrix}$$

and similarly for the rest of the powers of Nt. Hence by the definition of $\exp(Nt)$, we get that

$$\exp(Nt) = \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \dots & \dots & \frac{t^{r-1}}{(r-1)!} \\ & 1 & t & \frac{t^2}{2!} & & \vdots \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & t & \frac{t^2}{2!} \\ & & & & 1 & t \\ & & & & & 1 \end{bmatrix}$$

Now if

$$J = \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 \\ & \lambda & 1 & \dots & \vdots \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{bmatrix}$$

is an elementary Jordan matrix with eigenvalue λ, then since $Jt = \lambda It + Nt$,

$$\exp(Jt) = \exp(\lambda It + Nt) = \exp(\lambda It)\exp(Nt) = e^{\lambda t}\exp(Nt)$$

which gives

$$\exp(Jt) = e^{\lambda t} \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \dots & \dots & \frac{t^{r-1}}{(r-1)!} \\ & 1 & t & \frac{t^2}{2!} & & \vdots \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & t & \frac{t^2}{2!} \\ & & & & 1 & t \\ & & & & & 1 \end{bmatrix} \tag{2}$$

4. **Solution:** From example 3, we see that for the given matrices P, D

$$P^{-1}DP = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}$$

where $J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is the elementary Jordan matrix. From Equation 1, we get that

$$\exp(P^{-1}DP) = \begin{bmatrix} \exp(J) & 0 \\ 0 & \exp(J) \end{bmatrix}$$

Moreover, Equation 2 for $\lambda = t = 1$, shows that

$$\exp(P^{-1}DP) = \begin{bmatrix} \exp(J) & 0 \\ 0 & \exp(J) \end{bmatrix} = e^1 \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} e & e & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & e & e \\ 0 & 0 & 0 & e \end{bmatrix}$$

Then

$$\begin{aligned} \exp(D) = P\exp(P^{-1}DP)P^{-1} &= \begin{bmatrix} 0 & 1 & 2 & 0 \\ 2 & 0 & -2 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} e & e & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & e & e \\ 0 & 0 & 0 & e \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -2 & 2 \end{bmatrix} \\ &= \begin{bmatrix} e & 2e & -4e & 4e \\ 2e & -e & 4e & -8e \\ e & 0 & e & -2e \\ 0 & e & -2e & 3e \end{bmatrix} \end{aligned}$$

5. **Solution:** If $A = \mathbf{0}$, then $A^n = \mathbf{0}$ matrix for all $n \in \mathbb{N}$. Since $B + \mathbf{0} = B$ for any matrix B, form the definition of e^A, we get that $e^{\mathbf{0}} = I$. Moreover since we know that $\exp(A + B) = \exp(A)\exp(B)$, we get that

$$\exp(A)\exp(-A) = \exp(A - A) = \exp(0) = I$$
$$I = \exp(0) = \exp(-A + A) = \exp(-A)\exp(A)$$

Thus $\exp(A)$ is a nonsingular matrix with inverse $\exp(-A)$ for all $A \in M_n(K)$.

6. **Solution:** Let $A = UTU^{-1}$, where U is a invertible matrix and T is the corresponding Jordan representation of A. Additionally assume that $T = T_1 \oplus T_2 \oplus \ldots \oplus T_n$, where each T_i are elementary Jordan blocks with eigenvalue λ_i. Then

$$\begin{aligned} e^A = e^{UTU^{-1}} &= I + UTU^{-1} + \frac{(UTU^{-1})^2}{2!} + \frac{(UTU^{-1})^3}{3!} + \ldots \\ &= U\Big(I + T + \frac{T^2}{2!} + \frac{T^3}{3!} + \ldots\Big) \\ &= Ue^T U^{-1} \end{aligned}$$

Also notice that $\det(e^{T_i}) = e^{\lambda_i}$ by setting $t = 1$ in Equation 2. Thus

$$\det(e^A) = \det(e^T) = \prod_{i=1}^{n} \det(T_i) = \prod_{i=1}^{n} e^{\lambda_i} = e^{\sum_{i=1}^{n} \lambda_i} = e^{\mathrm{tr}(T)} = e^{\mathrm{tr}(A)}$$

7. **Solution:** We know that $\exp(A + B) = \exp(A)\exp(B)$. For a fixed $A \in M_n(K)$, $\Phi : K \to GL_n(K) := t \to \exp(At)$ is a well defined map since $\exp(A) \in GL_n(K)$ for all $A \in M_n(K)$. Moreover

$$\Phi(t + r) = \exp(A(t + r)) = \exp(At + Ar) = \exp(At)\exp(Ar)$$

shows that Φ is a group homomorphism.