

MATH 6320

Theory of Functions of a Real Variable
Fall 2024

First name: _____ Last name: _____

Points:

Assignment 9, due Thursday, November 14, 11:30am

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let $(X; \mathcal{M}; \mu)$ be a measure space and f be a complex-valued measurable function on X . Let

$$\|f\|_p = \left(\int_X |f|^p d\mu \right)^{1/p}$$

and $E = \{p > 0 : \|f\|_p < \infty\}$. Assume $1 \in E$.

(a) Show that if $r < p < s$, and $r, s \in E$, then

$$\|f\|_p \leq \max\{\|f\|_r, \|f\|_s\};$$

so $p \in E$. Hint: Hölder's inequality.

(b) Assume that $r \in E$ for some $r > 0$ and prove

$$\lim_{p \downarrow 1} \|f\|_p = \|f\|_1 :$$

Problem 2

Suppose $1 < p < \infty$, $(f_n)_{n=1}^\infty$ is a sequence in $L^p(\mu)$, $\|f_n - f\|_p \rightarrow 0$ and $f_n \rightarrow g$ pointwise almost everywhere. What is the relationship between f and g and why?

Problem 3

Let $(X; \mathcal{M}; \mu)$ be a measure space and $f \in L^p(\mu)$, $(f_n)_{n=1}^\infty$ a sequence in $L^p(\mu)$ with $f_n \rightarrow f$ pointwise almost everywhere and $\|f_n\|_p \rightarrow \|f\|_p$. Show that $\|f - f_n\|_p \rightarrow 0$ using Egorov's theorem, splitting $X = A \cup B$ such that $\int_A |f|^p d\mu < \epsilon$ and $f_n \rightarrow f$ uniformly on B with $\mu(B) < \delta$, in combination with Fatou's lemma applied to $\int_B |f_n|^p d\mu$ which shows

$$\limsup_n \int_A |f_n|^p d\mu = 0 :$$