Note: If K/F is a field extension then K can be thought of naturally as an F-vector space.

Recall from linear algebra:

- If V is an F-vector space then V has an F-basis.
- Any two F-bases for V as an F-vector space have the same cardinality
- The cardinality of any such basis is called the <u>dimension of V as an F-vector space</u>.

Def: If K/F is a field extension then the <u>degree</u> of the extension, denoted $[K:F]$ is defined to be the dimension of K as an F-vector space.

Thm (Tower Law): Suppose L/K and K/F are field extensions, and that $\mathcal{A}$ is a K-basis for L and $\mathcal{B}$ is an F-basis for K. Then:

i) $\{\alpha\beta : \alpha \in \mathcal{A}, \beta \in \mathcal{B}\}$ is an F-basis for L.

ii) $[L:F] = [L:K] \cdot [K:F]$.

(result from linear algebra)

One more basic result:

Thm (Kronecker's thm.): If F is a field and $f \in F[x]$ is irreducible, then there is an extension of F in which f has a root.

Pf: Let $K = F[x]/(f)$. (see ex. 3 from before)

Let $\alpha = x + (f) \in K$. Write $f(x) = \sum_{i=0}^{n} a_i x^i$.

Then $f(\alpha) = \sum_{i=0}^{n} a_i (x + (f))^i$

$$= \sum_{i=0}^{n} a_i (x^i + (f))$$

$$= \sum_{i=0}^{n} a_i x^i + (f) = f(x) + (f) = 0 + (f). \quad \blacksquare$$

Def: If $f \in F[x]$ is irreducible then the <u>splitting field</u> of f over F is the smallest field extension of F which contains all roots of f.

i.e., the splitting field of f is the smallest field extension of f where $f(x)$ factors as a product of linear factors. (it is well-defined - see textbook)

Algebraic extensions

Suppose K/F is a field extension. An element $\alpha \in K$ is algebraic over F if it is the root of a poly. in $F[x]$.

We say that K/F is an algebraic extension if every element of K is algebraic over F.

Lemma: If $[K:F] < \infty$ then K/F is algebraic.

Pf: Suppose $\alpha \in K$. Let $n = [K:F]$. Then $\{1, \alpha, \ldots, \alpha^n\}$ is F-linearly dependent. So $\exists\, a_0, \ldots, a_n \in F$ s.t. $\sum_{i=0}^{n} a_i \alpha^i = 0$. ($\alpha$ is the root of $f(x) = \sum_{i=0}^{n} a_i x^i$). ∎

Note: The converse of this is not true.

Ex: $K = \mathbb{Q}(2^{1/2}, 2^{1/3}, 2^{1/5}, 2^{1/7}, 2^{1/11}, \ldots)$. Then $K/\mathbb{Q}$ is algebraic, but $[K:\mathbb{Q}] = \infty$.

If $\alpha \in K$ is algebraic over F then the minimal polynomial of α over F is the monic poly. $f_\alpha \in F[x]$ of smallest degree for which $f_\alpha(\alpha) = 0$. The degree of α over F is defined to be $\deg f_\alpha$.

Lemma: Suppose K/F and $\alpha \in K$ is algebraic over F.

Then: i) f_α is irred. over F

ii) If $g \in F[x]$, $g(\alpha)=0$ then $f_\alpha \mid g$.

Pf: i) Suppose $f_\alpha = f_1 f_2$ in $F[x]$. Then

$0 = f_\alpha(\alpha) = f_1(\alpha) f_2(\alpha) \Rightarrow \deg f_1 = 0$ or $\deg f_2 = 0$.

ii) Suppose $g \neq 0$, write $g(x) = q(x) f(x) + r(x)$,

$r=0$ or $\deg r < \deg g$. Then

$0 = g(\alpha) = q(\alpha) f(\alpha) + r(\alpha) = r(\alpha) \Rightarrow r = 0$. $\boxtimes$