Real Variables II - MATH6321

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Contents

Course Information

14/01/2025 Office : Tuesday 11:30 - 12:30 PM, Wednesday 1-2PM
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Chapter 1

Hilbert Spaces (continuation)

Exercise 1.0.1 (Warm up)**.** Show that if $f \in \ell^2(X)$, then utmost countably many values of f are non-zero.

Solution 1.0.1.1*.* Take the sets

$$E_n = \{x \in X \ : \ |f(x)| > 1/n\}$$

and consider their union.

1.1 Orthonormal Sets

Definition 1.1.1. A orthonormal family in a Hilbert space is a family $(u_\alpha)_{\alpha \in A}$ such that $\langle u_\alpha, u_\beta \rangle = 0$ if $\alpha \neq \beta$ and $\|u_\alpha\| = 1$ for each $\alpha, \beta \in A$.

Theorem 1.1.1 (Bessel's Inequality)**.** *Given an orthonormal family* $(u_\alpha)_{\alpha \in A}$ *in a Hilbert space* $\mathcal{H}$*, and* $h \in \mathcal{H}$*, then*

$$\|h\|^2 \geq \sum_{\alpha \in A} |\langle h, u_\alpha \rangle|^2$$

Proof. Let $B \subset A$ be finite, and let

$$g = \sum_{\alpha \in B} \langle h, u_\alpha \rangle u_\alpha$$

We can easily show that, $\langle h - g, g \rangle = 0$. Thus

$$\begin{aligned}
\|h\|^2 &= \|h - g + g\|^2 \\
&= \langle h - g + g, h - g + g \rangle \\
&= \|h - g\|^2 + \|g\|^2 \\
&\geq \|g\|^2
\end{aligned}$$

Now the inequality follows form the definition of summation as the supremum of finite index sums. ☐

Definition 1.1.2. Let $\mathcal{H}$ be a Hilbert space. An orthonormal family $(u_\alpha)_{\alpha\in A}$ is called complete, or an orthonormal basis, if for each $h \in H$,

$$\|h\|^2 = \sum_{\alpha\in A} |\langle h, u_\alpha\rangle|^2$$

Definition 1.1.3. A set $U = \{u_\alpha \ : \ \alpha \in A\}$ is a maximal orthonormal set if for any V with $V \supset U$ and V is orthonormal, then $V = U$.

Theorem 1.1.2. *Let $\mathcal{H}$ be a Hilbert space, $(u_\alpha)_{\alpha\in A}$ an orthonormal family, then the following are equivalent.*

1. *$(u_\alpha)_{\alpha\in A}$ is an orthonormal basis*
2. *span$\{u_\alpha \ : \ \alpha \in A\}$ is dense in $\mathcal{H}$*
3. *$\{u_\alpha\}$ is a maximal orthonormal set*

Proof. ($1 \implies 2$) Let h be given. Consider for any $B \subset A$,

$$g = \sum_{\alpha\in B} \langle h, u_\alpha\rangle u_\alpha$$

then we recall $\langle h - g, g\rangle = 0$. And thus

$$\|h\|^2 = \|h - g\|^2 + \|g\|^2$$

We know from equality in Bessel's inequality that, for given $\varepsilon > 0$ we can choose B such that $\|h\|^2 - \|g\|^2 < \varepsilon^2$. Hence

$$\|h - g\|^2 = \|h\|^2 - \|g\|^2 < \varepsilon^2$$

Thus, we can find a g that is arbitrarily close to h.

($\neg 3 \implies \neg 2$). Assuming 3 is wrong, there exists $u \in \mathcal{H}$, such that $\|u\| = 1$, and $\langle u, u_\alpha\rangle = 0$ for each $\alpha \in A$. Next, for any finite $B \subset A$, and any $c_\alpha \in \mathbb{C}$, we look at

$$\begin{aligned}
\|u - \sum_{\alpha\in B} c_\alpha u_\alpha\| &= \|u\|^2 + \|\sum_{\alpha\in B} c_\alpha u_\alpha\|^2 \\
&= \|u\|^2 + \sum_{\alpha\in B} |c_\alpha|^2 \\
&\geq \|u\|^2 = 1
\end{aligned}$$

Thus $u \notin \overline{\text{span}\{u_\alpha\}}$.

($\neg 1 \implies \neg 3$). Assume there is $h \in \mathcal{H}$ such that

$$\|h\|^2 > \sum_{\alpha \in A} |\langle h, u_\alpha \rangle|^2$$

We know that $A_o = \{\alpha \in A \ : \ \langle h, u_\alpha \rangle \neq 0\}$ is at most countable from Exercise 1.0.1. We can find $A_1 \subset A_2 \subset \dots$ each finite where

$$A_n = \{\alpha \in A_o \ : \ |\langle h, u_\alpha \rangle| \geq \frac{1}{n}\}$$

and

$$A_o = \bigcup_{n=1}^{\infty} A_n$$

Let $g_n = \sum_{\alpha \in A_n} \langle h, u_\alpha \rangle u_\alpha$. By monotone convergence theorem, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sum_{\alpha \in A} |\langle h, u_\alpha \rangle|^2 < \sum_{\alpha \in A_N} |\langle h, u_\alpha \rangle|^2 + \varepsilon$$

Thus, for $m \geq n \geq N$

$$\begin{aligned}
\|g_m - g_n\|^2 &= \| \sum_{\alpha \in A_m \setminus A_n} \langle h, u_\alpha \rangle u_\alpha \|^2 \\
&= \sum_{\alpha \in A_m \setminus A_n} |\langle h, u_\alpha \rangle|^2 \\
&= \sum_{\alpha \in A_m} |\langle h, u_\alpha \rangle|^2 - \sum_{\alpha \in A_n} |\langle h, u_\alpha \rangle|^2 \\
&= \sum_{\alpha \in A} |\langle h, u_\alpha \rangle|^2 - \sum_{\alpha \in A_n} |\langle h, u_\alpha \rangle|^2 \\
&< \varepsilon
\end{aligned}$$

Therefore, we conclude that (g_n) is a Cauchy sequence. By completeness of $\mathcal{H}$, $g_n \to g$ for some $g \in \mathcal{H}$. Let $\gamma = h - g$. If $\gamma \in A_o$, then $\gamma \in A_n$ for some $n \in \mathbb{N}$. Thus we'll get that

$$\langle h - g, u_\alpha \rangle = 0$$

verify

If $\gamma \notin A_o$, then $\langle h, u_\gamma \rangle = 0$ and $\langle g_n, u_\gamma \rangle = 0$, so again $\langle , \rangle$ verify

□

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Exercise 1.1.1. Find an orthonormal basis for $\ell^2(X)$

Solution 1.1.1.1. We want an orthonormal family $(u_\alpha)_{\alpha \in A}$ such that

$$\|h\|^2 = \sum_{\alpha \in A} |\langle h, u_\alpha \rangle|^2$$

But here $\|h\|^2 = \sum_{x \in X} |h(x)|^2$, and

$$\langle h, g \rangle = \sum_{x \in X} h(x)\overline{g(x)}$$

If we choose $u_x = \chi_x$, then we see that this satisfy our required properties.

We formulate consequences of the characterization of orthonormal bases.

Theorem 1.1.3. *Let $(u_\alpha)_{\alpha \in A}$ be an orthonormal basis for a Hilbert space $\mathcal{H}$. Let*

$$A_o = \{\alpha \ : \ \langle h, \alpha \rangle \neq 0\}$$

Use the connection between inner product and evaluation from above to see that A_o is at most countable.

Then enumerating A_o by $A_o = \{\alpha_1, \alpha_2, \ldots\}$ and considering

$$h_n = \sum_{j=1}^{n} \langle h, u_{\alpha_j} \rangle u_{\alpha_j}$$

gives $h_n \to h$.

Proof. We saw that h_n forms a cauchy sequence bases on the last oproof and choose

$$A_j = \{\alpha_k \ : \ 1 \leq k \leq j\}$$

Let $g = \lim_{n \to \infty} h_n$. Then the continuity of the inner product gives $\langle h - g, u_\alpha \rangle = 0$ for each $\alpha \in A$. Thus

$$\|h - g\|^2 = \sum_{\alpha \in A} |\langle h - g, u_\alpha \rangle|^2 = 0$$

so $h = g$. □

Corollary 1.1.3.1. *If $(u_\alpha)_{\alpha \in \mathbb{N}}$ is an orthonormal basis for a Hilbert space $\mathcal{H}$, then for each $h \in \mathcal{H}$,*

$$h = \sum_{n \in \mathbb{N}} \langle h, u_n \rangle u_\alpha$$

Conversely, if $\sum_{n \in \mathbb{N}} |c_n|^2 < \infty$, then

$$\sum_{n \in \mathbb{N}} c_n u_n \in \mathcal{H}$$

Proof. verify ◻

Remark 1.1.1. We can also take $\mathbb{Z}$ instead of $\mathbb{N}$ in the above cases.

Example 1.1.1. Consider $L^2([-\pi,\pi])$, then

$$e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$$

defines an orthonormal basis $(e_n)_{n\in\mathbb{Z}}$ for $L^2([-\pi,\pi])$

Proof. It is clear that the above is an orthonormal family. We show that finite linear combinations of e_ns are dense in $L^2([-\pi,\pi])$. Find a $g \in C([-\pi,\pi])$, $\varepsilon/3$ away from g. Now find a h periodic which is $\frac{\varepsilon}{3}$ away from g. Now use Stone-Weierstrass theorem. ◻

Corollary 1.1.3.2 (Reisz-Fischer Theorem)**.** *If $f \in L^2([-\pi,\pi])$, then*

$$f = \sum_{n\in\mathbb{Z}} \langle f, e_n\rangle e_n$$

and $\|f\|^2 = \sum_{n\in\mathbb{Z}} |\langle f, e_n\rangle^2|$. *Moreover, if* $c \in \ell^2(\mathbb{Z})$, *then*

$$g = \sum_{n\in\mathbb{Z}} c_n e_n \in L^2([-\pi,\pi])$$

Remark 1.1.2. For $f \in L^2([-\pi,\pi])$, $c_n = \langle f, e_n\rangle$ are called Fourier coefficients.

Theorem 1.1.4. *Let $\mathcal{H}$ be a Hilbert space, then $\mathcal{H}$ has an orthonormal basis.*

Proof. We will show this using Zorn's lemma. Let

$$\mathscr{S} = \{U \subset H \ : \ U \text{ is an orthonormal set}\}$$

It is easy to see that $\mathscr{S}$ is nonempty. Order $\mathscr{S}$ by set inclusion. Let $\mathscr{C}$ be a chain in $\mathscr{S}$, then

$$U_{\mathscr{C}} = \bigcup_{C\in\mathscr{C}} C$$

will be an orthonormal set in $\mathscr{S}$. (Use the standard arguments to see this). Therefore $U_{\mathscr{C}}$ is the upper bound for the chain $\mathscr{C}$. Hence we see that $\mathscr{S}$ has a maximal element by the Zorn's lemma. Hence $\mathcal{H}$ has an orthonormal basis. ◻

Definition 1.1.4. Let $\mathcal{H}, \mathcal{K}$ be Hilbert space. A linear map $V : \mathcal{H} \to \mathcal{K}$ is an isometry if

$$\langle Vh, Vg \rangle_{\mathcal{K}} = \langle h, g \rangle_{\mathcal{H}}$$

for each $h, g \in \mathcal{H}$. If V is onto, we say V is unitary. Then we say $\mathcal{H}$ and $\mathcal{K}$ are isomorphic. (From preservation of preservation of norm, V is one-one).

Proposition 1.1.1. *A linear map $V : \mathcal{H} \to \mathcal{K}$ is an isometry if and only if for every $h \in \mathcal{H}$,*

$$\|Vh\|_{\mathcal{K}} = \|h\|_{\mathcal{H}}$$

Proof. One way is easy. That is if the inner product is preserved, then the norm is preserved. Conversely, we use the polarization identity.

$$\langle h, g \rangle = \frac{1}{4} \sum_{j=1}^{4} i^j \|h + i^j g\|^2, \quad (i = \sqrt{-1})$$

□

Example 1.1.2. Let $\mathcal{H} = \ell^2(\mathbb{N})$. Define

$$S : \mathcal{H} \to \mathcal{H} := (x_1, x_2, \ldots) \mapsto (0, x_1, x_2, \ldots, x_n)$$

Then S is clearly an isometry, but not unitary since nothing maps to $(1, x_1, x_2, \ldots)$

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Example 1.1.3. Given an infinite dimensional Hilbert space with orthonormal basis $(u_n)_{n \in \mathbb{N}}$, show that $\{u_n\}$ is not compact.

Proof. Since $\|u_\alpha - u_\beta\| = \sqrt{2}$, take $\frac{1}{\sqrt{2}}$ radius balls around each u_α to get a collection of open balls that cover the set with no finite subcover.

Another way to see is to use the sequential compactness criterion and see that the sequence (u_n) does not have any convergent subsequence. Since this is a metric space, sequential compactness is equivalent to compactness. □

Theorem 1.1.5 (Every Hilbert space is $\ell^2(A)$). *Let $\mathcal{H}$ be a Hilbert space, $(u_\alpha)_{\alpha \in A}$ is an orthonormal basis, then there is a unitary map $U : \mathcal{H} \to \ell^2(A)$ such that $U(u_\alpha) = \chi_\alpha$*

Proof. We first note that by linearity, if $p \in \operatorname{span}\{u_\alpha \;:\; \alpha \in A\}$, then $U(p)$ is determined by χ_α. Next, by Bessel's inequality,

$$\|U(p)\|_{\ell^2(A)} \leq \|p\|$$

Hence U is bounded. Hence it can be continuously extended to $\mathcal{H} = \overline{\operatorname{span}\{u_\alpha\}}$ as a limit of sequences. Also, by the equivality in the Bessel's inequality, we get that U is an isometry, hence one-to-one.

Now it remains to show that U is onto. Given $g \in \ell^2(A)$, we know that there exists at most a countable set $\{\alpha_1, \alpha_2, \ldots\} = A_0$ such that $g(\alpha_i) \neq 0$. Consider

$$h_n = \sum_{j=1}^{n} g(\alpha_j) u_{\alpha_j}$$

then,

$$u(h_n)(\alpha) = \begin{cases} g(\alpha_j), & \alpha_j \in A_0 \\ 0, & \text{otherwise} \end{cases}$$

Moreover,

$$\left\| U(h_n) - g \right\|_{\ell^2(A)}^2 = \sum_{j=n+1}^{\infty} |g(\alpha_j)|^2 \to 0$$

Now if

$$h = \sum_{j=1}^{\infty} g(\alpha_j) u_{\alpha_j} \in \mathcal{H} \quad (\text{ since } g \in \ell^2(A))$$

we get

$$\|h - h_n\|^2 = \|U(h_n) - g\|_{\ell^2(A)}^2 \to 0$$

and the injectivity of U shows that $U(h) = g$. □

Chapter 2

Banach Space Techniques

Definition 2.0.1. If X is a real or complex normed vector space with a norm, and the complete in the topology induced by the norm, it is called a Banach space.

Definition 2.0.2. If X, Y are normed vector spaces over $\mathbb{R}$ or $\mathbb{C}$, $\Lambda : X \to Y$ linear, then the norm of the operator

$$\|\Lambda\| = \sup\{\|\Lambda x\| \ : \ \|x\| < 1\}$$

If $\|\Lambda\| < \infty$, then we say that Λ is bounded.

Proposition 2.0.1. *Given $\Lambda : X \to Y$, a linear map between normed linear spaces, the following are equivalent*

(1) Λ is bounded

(2) Λ is continuous

(3) Λ is continuous at some $x_o \in X$

Proof. $(1 \implies 2)$

$$\|\Lambda(x-y)\| \leq \|\Lambda\|\|x-y\|$$

gives $\|\Lambda\|$-Lipschitz continuity.

$(2 \implies 3)$ Follows from the definition.

$(3 \implies 1)$ For each $\varepsilon > 0$, there is $\delta > 0$ such that for each $x \in X$, with $\|x - x_o\| < \delta$, then $\|\Lambda x - \Lambda x_o\| < \varepsilon$. Thus for $\|y\| < \delta$, by linearity of Λ, we get

$$\|\Lambda y\| = \|\Lambda(x_o + y) - \Lambda x_o\| < \varepsilon$$

Again using linearity, we get for $\|y'\| < 1$,

$$\|\Lambda y'\| < \frac{\varepsilon}{\delta} < \infty$$

Now since $\overline{B_1(0)} \subset B_2(0)$, we see that $\|\Lambda\| < \frac{2\varepsilon}{\delta} < \infty$. $\square$

2.1 Consequence of Baire category theorem

Theorem 2.1.1 (Baire Category Theorem). *If (X, d) is a complete metric space, and $V_1, V_2, \ldots$ are dense subsets, then*

$$\bigcap_{n=1}^{\infty} V_n$$

is dense in X.

Proof. We show that for any non-empty open set $W \subset X$,

$$\bigcap_{n=1}^{\infty} V_j \cap W \neq \emptyset$$

We write $B_r(x) = \{y \in X \ : \ d(x, y) < r\}$. Since V_1 is dense and open, $V_1 \cap W$ is open and dense in W. Thus we can find an $r_1 > 0$ such that $\overline{B_{r_1}(x_1)} \subset W \cap V_1$. (First find an $r' > 0$ such that $B_{r'}(x_1) \subset W \cap V_1$. Then take $r_1 = \frac{r'}{2}$).

We inductively proceed by taking $x_n \in V_n \cap B_{r_{n-1}}(x_{n-1})$ such that $\overline{B_{r_n}(x_n)} \subset V_n \cap B_{r_{n-1}}(x_{n-1})$. Without loss of generality, choose $0 < r_n < \frac{1}{n}$. This gives a sequence which satisfies for $i, j > n$ that $x_i, x_j \in B_{r_n}(x_n) \implies d(x_i, x_j) < 2r_n < \frac{2}{n}$. Hence x_n is Cauchy. By completeness $x_n \to x \in \overline{B_{r_n}(x_n)} \subset V_n \cap W$ for all n. Thus $x \in W$ and

$$x \in \bigcap_{n=1}^{\infty} V_n$$

□

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Corollary 2.1.1.1. *Let $G_1, G_2, \ldots$ be a sequence of dense G_δ subsets, then $\cap_{n=1}^{\infty} G_n$ is dense and G_δ.*

Theorem 2.1.2 (Banach-Steinhaus theorem). *If X is a Banach space and Y a normed vector space. Let $(T_\alpha)_{\alpha \in A}$ be a family of bounded linear maps from $X \to Y$. Then either of the two holds,*

(1) $\exists M \geq 0$ such that $\|T_\alpha\| \leq M$ for all $\alpha \in A$.

(2) The set $\{x \in X \ : \ \sup_\alpha \|T_\alpha x\| = \infty\}$ is a dense G_δ set.

Corollary 2.1.2.1. *With X, Y, T_α as above, if for each $x \in X$, $\sup_{\alpha \in A} \|T_\alpha x\| < \infty$, then there is a $M \geq 0$ such that*

$$\sup_{\alpha \in A} \|T_\alpha\| \leq M$$

We study consequences before looking at the proof of Theorem 2.1.2.

Exercise 2.1.1. Suppose (a_n) is a sequence such that for each $(b_n) \in \ell^2(\mathbb{N})$, $\sum_{n\in\mathbb{N}} a_n b_n < \infty$, then $a \in \ell^2(\mathbb{N})$.

Proof. To see this, take

$$T_n : \ell^2(\mathbb{N}) \to \mathbb{C} := T_n(b) \mapsto \sum_{i=1}^{n} \overline{a_i} b_i$$

and observe

$$\begin{aligned} |T_n(b)| &\leq \Big(\sum_{i=1}^{n} |a_i|^2\Big)^{\frac{1}{2}} \Big(\sum_{i=1}^{n} |b_j|^2\Big)^{\frac{1}{2}} \\ &\leq \Big(\sum_{i=1}^{n} |a_i|^2\Big)^{\frac{1}{2}} \|b\| \end{aligned}$$

Hence each T_n is linear (by definition) and bounded by the inequality above. Let $(b_n) \in \ell^2(\mathbb{N})$, then for $n \in \mathbb{N}$, we get

$$\begin{aligned} |T_n(b)| &\leq \sum_{i=1}^{n} |a_i b_i| \\ &\leq \sum_{i=1}^{\infty} |a_i b_i| < \infty \end{aligned}$$

Now a direct application of Theorem 2.1.2 gives that three is a $M \geq 0$ such that $|T_n b| \leq M\|b\|$ for each $n \in \mathbb{N}$. Hence if we consider

$$T : \ell^2(\mathbb{N}) \to \mathbb{C} := b \mapsto \sum_{i=1}^{\infty} a_i b_i$$

we find that $\|T\| \leq M$. By Reisz representation theorem for Hilbert space s, there is a $c \in \ell^2(\mathbb{N})$ such that $T(b) = \langle b, c\rangle$. Choosing $b = e_m$, we get $T(e_m) = \bar{c_m} = a_m$. We conclude that $a \in \ell^2(\mathbb{N})$. □

Proof of Banach-Steinhaus Theorem 2.1.2. Let $\phi_\alpha : X \to [0,\infty) := x \to \|T_\alpha x\|$. Look at

$$\begin{aligned} |\phi_\alpha(x) - \phi_\alpha(y)| &= |\|T_\alpha x\| - \|T_\alpha y\|| \\ &\leq \|T_\alpha(x-y)\| \\ &\leq \|T_\alpha\|\|x-y\| \end{aligned}$$

Hence ϕ_α is (lower-semi) continuous.

Therefore, we can define a lower semi-continuous function

$$\phi(x) = \sup_{\alpha \in A} \phi_\alpha(x)$$

Thus, for $n \in \mathbb{N}$, $V_n = \{x \in X \ : \ \phi(x) > n\}$ is open. If each V_n is dense in X, then $G = \cap_{n=1}^{\infty} V_n$ is a dense G_δ set, and $\phi(G) = \{\infty\}$. Otherwise, if for some $n \in \mathbb{N}$, one of V_n is not dense. Then that particular V_n^c contains a non-empty open set. Choosing a $B_\delta(y) \subset W$ centered at y, we get that for $x \in X$, $\|x - y\| < \delta$, we have $\phi_\alpha(x) = \|T_\alpha x\| \leq n$ for each $\alpha \in A$. This implies that there is an $M \geq 0$ for which

$$\sup_{\alpha \in A} \|T_\alpha\| \leq M$$

□

We investigate more consequences of Banach-Steinhaus' theorem.

Theorem 2.1.3 (Open mapping theorem)**.** *Let X, Y be Banach spaces, $T : X \to Y$ is bounded, linear. If T is onto and $U \subset X$ is open, then $T(U)$ is open.* review this

Proof. We claim it is equivalent to show $T(B_1(0)) \subset B_\delta(0)$ for some $\delta > 0$. The statement then follows by choosing for U open, a vector $u \in U$ with $\varepsilon > 0$ such that $B_\varepsilon(u) \subset U$. In that case, if $y \in Y$ satisfies

$$\|y - Tu\| < \varepsilon\delta$$

or

$$\|\frac{y - Tu}{\varepsilon}\| < \delta$$

By the inclusion, we get $z \in X$ such that $\|z\| < 1$ and $Tz = \frac{y-Tu}{z}$. Solving for y, gives $y = T(\varepsilon z + y)$. Letting $w = \varepsilon z + u$, then $\|w - u\| = \varepsilon\|z\| < \varepsilon$ and $Tw = y$. We have found for each y near Tu a vector $w \in U$ which maps to y. Let $U \subset X$ be open. Fix $u \in U$. Then there is $\delta > 0$ with $\{x \ : \ \|x - u\| < \delta\} \subset U$. We also know that if $y \in Y$

Now for the rest, follow the same logic as in functional analysis last semester to see that $T(B_1^X(0))$ is not nowhere dense, and thus $\exists y \in Y, r > 0$ such that

$$B_{4r}(y_0) \subset \overline{T(B_1^X(0))}$$

Choose $y' \in B_{2r}(y_0) \cap T(B_1(0))$. (The fact that this is non-empty follows from the fact that every open ball in the closure must intersect the original set pre-closure). Then there is $x' \in B_1^X(0)$ such that $y' = T(x')$. Now using triangle inequality

$$B_{2r}(y') \subset B_{4r}(y_0) \subset \overline{T(B_1^X(0))}$$

Thus for $y \in B_{2r}(0)$,

$$\begin{aligned} y &= -y' + (y + y') \\ &\in -y' + B_{2r}(y') \\ &\subset -y' + \overline{T(B_1(0))} \\ &= \overline{T(-x' + B_1(0))} \\ &\subset \overline{T(B_2^X(0))} \end{aligned}$$

Now by resclaing with 2, we see that

$$B_r(0) \subset \overline{T(B_1^X(0))}$$

Again by further scaling we see that for all $n \in \mathbb{N}$,

$$B_{r2^{-n}}(0) \subset \overline{T(B_{2^{-n}}(0))}$$

For $y \in B_{\frac{r}{2}}(0)$, there is a $x_1 \in B_{2^{-1}}(0)$ such that

$$\|y - T(x_1)\| < r2^{-1}$$

Now let $y_1 = y - T(x_1)$ and repeat the same procedure to get $x_2 \in B_{2^{-2}(0)}$ such that

$$\begin{aligned} y_2 &:= y_1 - T(x_2) \\ &= y - T(x_1 + x_2) \\ &\in B_{r2^{-3}}(0) \end{aligned}$$

Proceeding inductively, we get $x_n \in B_{2^{-n}}(0)$ such that $y_n = y - T(\sum_{i=1}^{n} x_i) \in B_{r2^{-n-1}(0)}$. That is

$$\|y_n\| = \left\| y - T\left(\sum_{i=1}^{n} x_i\right) \right\| < \frac{r}{2^{n+1}}$$

verify ☐

Corollary 2.1.3.1. *If T is one-one and onto, then T^{-1} is bounded.*

Proof. ☐

30/01/2025

Definition 2.1.1. A set $E \subset X$ is called **nowhere dense** if $\overline{E}$ does not contain a non-empty open set in X. A set is called **first category** if it is a union of nowhere dense sets, otherwise the set is called **second category**.

Theorem 2.1.4 (Baire category theorem, version II)**.** *Let (X,d) be a complete metric space. Then X is not of first category.*

Proof. Let $E_1, E_2, \ldots$ be a sequence of nowhere dense sets. Then $\overline{E_n}$ has an empty interior for all $n \in \mathbb{N}$. Thus $\overline{E_n}^c$ is open and dense. Then by Theorem 2.1.1 $\cap_{n=1}^{\infty}\overline{E_n}^c$ is a dense G_δ set. Thus $\cap_{n=1}^{\infty}\overline{E_n}^c \neq \emptyset$. Thus taking complements, we get

$$\bigcup_{n=1}^{\infty} E_n \subset \bigcup_{n=1}^{\infty} \overline{E_n} \neq X$$

□

missed something here

04/02/2025

Corollary 2.1.4.1. *Let X, Y be Banach spaces. If $T : X \to Y$ is linear, bounded map and T is one-to-one and onto, then T^{-1} is bounded.*

Proof. Use the fact the open mapping theorem gives that T^{-1} is continuous, and that continuity is boundedness in linear spaces. □

Theorem 2.1.5 (Closed graph theorem)**.** *Let X, Y be Banach spaces, then the graph of T, defined as $G(T) = \{(x, Tx) : x \in X\} \subset X \times Y$, under the norm $\|(x,y)\| = \|x\|_X + \|y\|_Y$ is closed if and only if T is bounded.*

Proof. Refer back to functional analysis notes. □

2.2 Applications of Banach-Steinhaus

Let $C_{per}([-\pi,\pi])$ denote continuous functions $f : [-\pi,\pi] \to \mathbb{C}$ such that $f(\pi) = f(-\pi)$. Since each $C_{per}([-\pi,\pi]) \subset L^2([-\pi,\pi])$, each such f has a Fourier series. Let

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{int}\,dt$$

and

$$s_n(t) = \sum_{j=-n}^{n} c_j e^{ijt}$$

We know that $s_n \to f$ in L^2.

But what about pointwise convergence. Let

$$D_n = \sum_{j=-n}^{n} e^{ijt} = \frac{\sin((n+\frac{1}{2})t)}{\sin(\frac{t}{2})}$$

Observe that

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} f_n(t)D_n(x-t)\ dx &= \sum_{j=-n}^{n}\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ijt}\ dt\Big)e^{ijx} \\ &= \sum_{j=-n}^{n} c_j e^{ijx} = s_n(x)\end{aligned}$$

Choose linear functionals $\Lambda_n : C_{per}([-\pi,\pi]) \to \mathbb{C}$ defined as

$$\Lambda_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_n(-t)\ dt = s_n(0)$$

Putting sup norm on $C_{per}([-\pi,\pi])$, we get that Λ_n is linear, bounded, with

$$\begin{aligned}|\Lambda_n(f)| &= \Big|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_n(-t)\ dt\Big| \\ &\le \frac{1}{2\pi}\|f\|_\infty\|D_n\|_1\end{aligned}$$

Read the rest from Rudin

06/02/2025

Exercise 2.2.1. Show that for $L^p(\mu)$, and a bounded linear functional Λ and letting

$$\mathcal{M} = \{f \in L^p(\mu)\ :\ \Lambda(f) = 1\}$$

If $1 < p < \infty$, then M has at most one norm minimizer.

Solution 2.2.1.1. Assume we have two minimizers for the norm, $f, g \in M$, then we know from Problem 1 in Assignment 2, if $f \neq g$, then

$$\|(f+g)/2\| < \frac{1}{2}\|f\| + \frac{1}{2}\|g\| = \|f\|$$

This contradicts our assumption that f is a norm minimizer.

2.3 Application of open mapping theorem

Let $\mathbb{T} = [-\pi, \pi]$, and $f \in L^1(\mathbb{T})$. Then

$$c_n := \hat{f}(n) = \frac{1}{2\pi} \int_{\mathbb{T}} f \bar{e_n} \, dm$$

where $e_n(t) = e^{int}$. c_n are called the n-th Fourier coefficient of f.

Theorem 2.3.1 (Riemann - Lebesgue lemma)**.** *If $f \in L^1(\mathbb{T})$, then*

$$\lim_{n \to \pm\infty} \hat{f}(n) = 0$$

Proof. Let $\epsilon > 0$ be given. Take trigonometric polynomials

$$P = \left\{ p(t) = \sum_{k=-m}^{m} p_k e^{ikt} \ : \ m \in \mathbb{N} \right\}$$

which are dense in $L^1(\mathbb{T})$, so we can choose $p \in P$ such that $\|f - p\|_1 < \varepsilon$. For sufficiently large $|n|$, that is $|n| > m$, we also have

$$\frac{1}{2\pi} \int_{\mathbb{T}} (f - p) \bar{e_n} \, dm - \frac{1}{2pi} \int_{\mathbb{T}} f \bar{e_n} \, dm = \hat{f}(n)$$

We also observe that for any $n \in \mathbb{Z}$,

$$|\hat{f}(n) - \hat{p}(n)| = \left| \frac{1}{2\pi} \int_{\mathbb{T}} (f - p) \bar{e_n} \, dm \right| \leq \frac{1}{2\pi} \|f - p\|_1 < \frac{\varepsilon}{2\pi} < \varepsilon$$

By comparing $\hat{p}(n) \to 0$ as $n \to 0$, with $\hat{f}(n)$, we get that

$$\lim_{n \to \pm\infty} \sup |\hat{f}(n)| < \varepsilon$$

Since $\varepsilon > 0$, we get our proof. □

Let $c_{\mathbf{0}} = \{\phi : \mathbb{Z} \to \mathbb{C} \ : \ \lim_{n \to \pm\infty} \phi(n) = 0\}$. We conclude that computing the Fourier coefficients gives us a linear map $\Lambda : L^1(\mathbb{T}) \to c_{\mathbf{0}}$, such that

$$\Lambda(f)(n) = \hat{f}(n)$$

We know Λ is continuous by

$$\begin{aligned} |\Lambda(f)(n)| &= \left| \frac{1}{2\pi} \int_{\mathbb{T}} f \bar{e_n} \, dm \right| \\ &\leq \frac{1}{2\pi} \int_{\mathbb{T}} |f| \, dm \\ &= \frac{1}{2\pi} \|f\|_1 \end{aligned}$$

Now taking supremums over $n \in \mathbb{Z}$, we get $\|\Lambda(f)\|_\infty \leq \frac{1}{2\pi} \|f\|_1$. Again by the usual trick, we see that $\|\Lambda\| \leq \frac{1}{2\pi}$.

Theorem 2.3.2. Λ *as defined above is not onto.*

Proof. We establish that Λ is one to one. This follows easily from Weierstrass and Lusin by using trigonometric polynomials, and then continuous functions to approximate $L^1(\mathbb{T})$.

Now asssume that Λ^{-1} is bounded, then for each $\hat{f} \in c_{\mathbf{0}}$,

$$\|\Lambda^{-1}(\hat{f})\| \leq \|f\|$$

If we choose

$$\hat{f}(m) = \begin{cases} \frac{1}{2\pi}, & \|m\| \leq n \\ 0, & \text{else} \end{cases}$$

the coefficients corresponding to the n-th Dirichlet's kernel. Then since Λ is one-to-one, we get $\Lambda^{-1}(\hat{D_n}) = D_n$. But we know that $\|D_n\|_1 \to \infty$, as $n \to \infty$, while $\|\hat{D_n}\|_\infty = 1$. Thus we see that Λ^{-1} is unbounded. We conclude by the open mapping theorem that Λ is not onto. □

Corollary 2.3.2.1. *We see that the range of* Λ *is not a closed subspace of* c_0*, by the same proof.*

2.4 Hahn-Banach Theorem

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Example 2.4.1. Kadison Singer problem.

2.5 Applications of Hahn-Banach Theorem

For the god given Hahn-Banach theorem, refer Chapter 1 of last semester's functional notes.

Proposition 2.5.1. *Let* $D = \{z \in \mathbb{C} \ : \ |z| < 1\}$*, and let* $\partial D = S^1$*. Now, if* $p(z) = \sum_{j=1}^{n} p_j z^j$*, then*

$$\max\{|p(z)| \ : \ z \in \bar{D}\} = \max\{|p(z)| \ : \ z \in S^1\}$$

Proof. Since p is continuous and $\bar{D}$ is compact, the maximum is attained at $\bar{D}$. Assume $z_0 \in D$ is where the maximum is achieved. Then if we rewrite $p(z) = \sum_{j=1}^{n} q_j (z - z_0)^j$, for $0 < r < 1$ such that $z_0 + re^{i\theta} \in D$ for any $\theta \in [0, 2\pi)$,

$$\int_0^{2\pi} p(z_0 + re^{i\theta}) \frac{d\theta}{2\pi} = \sum_{j=0}^{n} \int_0^{2\pi} q_j (re^{i\theta}) \frac{d\theta}{2\pi} = q_0 = p(z_0)$$

This using the max property (abs value of integral is $\leq$ integral of abs value) of the integral forces $p(z_0) = p(z_0 + re^{i\theta})$ for all $\theta \in [0, 2\pi]$. Use again the fact that $p(z)$ is a polynomial and see. Then p will be take a constant value q_0 in D. $\square$

Example 2.5.1. Let $A \subset C(\bar{D})$ be a subspace containing all polynomials, and all such functions for which maximum modulus holds.

For example, let $A(\mathbb{D})$ be the closure of the space of polynomials with $\|\cdot\|_\infty$ on S^1. Then for any $f \in A(\mathbb{D})$, there exist a sequence of polynomials p_n such that $p_n \to f$ uniformly, and then $\|p_n\|_\infty \to \|f\|_\infty$ by uniform convergence.

By max-modulus property of the polynomials and uniform convergence,

$$\|f\|_{\infty,\partial\mathbb{D}} = \|f\|_{\infty,\mathbb{D}}$$

Moreover if $\|f\|_{\infty,\partial\mathbb{D}} = 0$ for some $f \in A(\mathbb{D})$, then the maximum modulus principle forces $\|f\|_{\infty,\mathbb{D}} = 0$. Hence we see that the restriction of f to the circle is a injective linear map. Thus we can identify $A(\mathbb{D})$ with a closed subspace of $C(\partial\mathbb{D})$. 18/02/2025

Example 2.5.2. Consider $M : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}) := (x_n) \mapsto (\frac{x_n}{n})$. Show that M is not onto.

Proof. If $Mx = 0$, this forces $x = 0$. Thus we see that M is injective. If M were onto, then M would have a bounded inverse by open mapping theorem. Then working with the orthonormal basis $\{e_n\}$, we see that $M^{-1}(e_n) = ne_n$, which shows that $\|M^{-1}\| = \infty$, a contradiction. $\square$

18/03/2025

Chapter 3

Complex Measures

20/03/2025

3.1 Consequence of Radon-Nikodym Theorem

Theorem 3.1.1. *If μ, ν are positive σ-finite measures such that $\nu \ll \mu$, then there is a positive measurable function h such that $d\nu = h\mu$*

Theorem 3.1.2 (Hahn-Decomposition Theorem)**.** *Let μ be a real-valued complex measure (signed measure) on a measurable space $(X, \mathcal{M})$. Then there are two sets A, B such that $A \cup B = X, A \cap B = \emptyset$ and*

$$\mu_+(E) := \mu(E \cap A), \quad \mu_-(E) = \mu(E \cap B)$$

with $\mu_+ \perp \mu_-$ and $\mu_+ + \mu_- = \mu$, and $\mu_+ + \mu_- = |\mu|$.

Moreover, if $\mu = \mu_1 - \mu_2$ with μ_1, μ_2 being positive measures, then for any $E \in \mathcal{M}$ we have $\mu_1(E) \geq \mu_+(E), \mu_2(E) \geq \mu_-(E)$

Proof. Since μ is a complex measure, $\mu \ll |\mu|$ and by Radon-Nikodym, there is a $h \in L^1(\mu)$ with $h(x) \in \{1, 2\}$ (polar decomposition) such that $d\mu = hd|\mu|$.

Let $A = h^{-1}(1), B = X \setminus A$. We find that $d\mu_+ = \frac{1}{2}(d|\mu| + d\mu) = \frac{1}{2}(|h|d|\mu| + hd|\mu|) = h_+ d|\mu|$, and similarly $\mu_- = h_- d|\mu|$. The rest follows easily. $\square$

3.2 Bounded linear functionals on L^p

Note. Let μ be a positive measure, $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Fixing $g \in L^1(\mu)$. Holder's inequality gives that for any $f \in L^p(\mu)$,

$$|\int fg \; d\mu| \leq \|f\|_p \|g\|_q$$

So that $\Lambda_g : L^p(\mu) \to \mathbb{C} := f \to \int fg \; d\mu$ is a bounded linear functional. Thus, we have a map $\Lambda : L^q(\mu) \to L^p(\mu)^* := g \mapsto \Lambda_g$.

For $1 \leq p < \infty$, the converse is true, too

Lemma 3.2.1. *If μ is σ-finite on $(X, \mathcal{M})$, then there is a $\omega \in L^1(\mu)$ such that $\forall x \in X : 0 < \omega(x) < 1$.*

Proof. Choose a partition E_j of X such that $\mu(E_j) < \infty$ for each $j \in \mathbb{N}$. Let

$$\omega = \sum_{n\in\mathbb{N}} \frac{1}{2^n} \frac{1}{1+\mu(E_n)} \chi_{E_n}$$

Since E_j is a partition, we get that

$$\begin{aligned}
\int |\omega| \; d\mu &= \int \omega \; d\mu \\
&= \int \sum_{n\in\mathbb{N}} \frac{1}{2^n} \frac{1}{1+\mu(E_n)} \chi_{E_n} \; d\mu \\
&= \sum_{n\in\mathbb{N}} \frac{1}{2^n} \frac{1}{1+\mu(E_n)} \mu(E_n) \\
&\leq \sum_{n\in\mathbb{N}} \frac{1}{2^n} = 1
\end{aligned}$$

Hence $\omega \in L^1(\mu)$ is the required function. □

Corollary 3.2.0.1. *If μ is σ-finite, then $\tilde{\mu}$ given by $d\tilde{\mu} = \omega d\mu$ is finite.*

Theorem 3.2.1. *Let μ be a σ-finite measure, $1 \leq p < \infty$, q as usual. If $\Lambda \in L^p(\mu)^*$, then there is $g \in L^q(\mu)$ such that*

$$\Lambda = \Lambda_g$$

and $\|\Lambda\| = \|g\|_q$

Proof. Begin by assuming μ is finite. Let $\Lambda : L^p(\mu) \to \mathbb{C}$ be a bounded linear functional. Notice that $\chi_E \in L^p(\mu)$ for each $E \in \mathcal{M}$. Consider $\lambda(E) = \Lambda(\chi_E)$. Let $\{E_j\}_{n=1}^{\infty}$ be a partition of E. We find

$$\begin{aligned}
\lambda\Big(\bigcup_{j=1}^{n} E_j\Big) &= \Lambda\Big(\sum_{j=1}^{n} \xi_{E_j}\Big) \\
&= \sum_{j=1}^{n} \Lambda(\chi_{E_j}) \\
&= \sum_{j=1}^{n} \lambda(E_j)
\end{aligned}$$

We conclude λ is finitely additive. Note

$$\left\|\chi_E - \chi_{\cup_{j=1}^n E_j}\right\|_p = \Big(\mu(\bigcup_{j=n+1}^{\infty} E_j)\Big)^{\frac{1}{p}}$$

Using monotone convergence and boundedness of Λ, we get

$$\Lambda(\chi_{\cup_{j=1}^n E_j}) \to \Lambda(\chi_E)$$

Thus λ is a measure and $\lambda \ll \mu$ by the definition of λ. By Radon-Nikodym, we have $g \in L^1(\mu)$ with $d\lambda = gd\mu$.

For f simple,

$$\Lambda(f) = \int f \ d\lambda = \int fg \ d\mu := \Lambda_g(f)$$

Now, consider $p = 1$. Then

$$\Big|\int f \ d\lambda\Big| = \Big|\int fg \ d\mu\Big| \leq \|\Lambda\|\|f\|_1$$

□

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Lemma 3.2.2. *Let X be locally compact Hausdorff and $\lambda : C_c(X) \to \mathbb{R}$ be bounded linear functional. Then there are positive bounded linear functionals λ_+, λ_- such that $\lambda = \lambda_+ - \lambda_-$.*

Proof. For this, we find bounded linear functional ρ,

$$|\lambda(f)| \leq \rho(|f|) \leq C\|f\|_\infty$$

and then let $\lambda_+ = \frac{1}{2}(\lambda + \rho)$ and $\lambda_- = \frac{1}{2}(\rho - \lambda)$

We define the map $\rho : C_c(X)^+ \to \mathbb{C} := f \mapsto \sup\{|\lambda(h) \mid h \in C_c(X), |h| \leq f\}$, where $C_c(X)^+$ is the set of non-negative real valued functions in $C_c(X)$. Let $f, g \in C_c(X)^+$, then there is a $h_1, h_2 \in C_c(X)$ such that $|h_1| \leq f, |h_2| \leq g$ with $\rho(f) \leq |\lambda(h_1)| + \varepsilon$ and $\rho(g) \leq |\lambda(h_2)| + \varepsilon$. So, $\rho(f) + \rho(g) \leq |\lambda(h_1)| + |\lambda(h_2)| + 2\varepsilon$. Let $\alpha_1, \alpha_2 \in \{\pm 1\}$ such that $\lambda(\alpha_i h_i) = \alpha_i \lambda(h_i) \geq 0$. Then,

$$|\lambda(\alpha_1 h_1)| + |\lambda(\alpha_2 h_2)| = \lambda(\alpha_1 h_1) + \lambda(\alpha_2 h_2) = \lambda(\alpha_1 h_1 + \alpha_2 h_2)$$

So,

$$\begin{aligned}
\rho(f) + \rho(h) &\leq \lambda(\alpha_1 h_1 + \alpha_2 h_2) + \varepsilon \\
&\leq \rho(|\alpha_1 h_1 + \alpha_2 h_2|) + 2\varepsilon \quad \text{since } \alpha_1 h_2 + \alpha_2 h_2 \leq |\alpha_1 h_1 + \alpha_2 h_2| \\
&\leq \rho(|h_1| + |h_2|) + 2\varepsilon \quad \text{since } \rho \text{ is order preserving} \\
&\leq \rho(f + g) + 2\varepsilon \quad \text{since } \rho \text{ is order preserving}
\end{aligned}$$

Since this holds for any $\varepsilon > 0$, we get $\rho(f+g) \geq \rho(f) + \rho(g)$.

To show the reverse inequality, let $f, g \in C_c(X)^+$, and $h \in C_c(X)$ be such that $|h| \leq f + g$. We define

$$h_1(x) = \begin{cases} \frac{f(x)}{f(x)g(x)} h(x), & f(x) + g(x) > 0 \\ 0, & \text{else} \end{cases}$$

and $h_2(x) = h(x) - h_1(x)$. Then $|h_1| \leq f, |h_2| \leq g$. Moreover, h_1, h_2 are continuous where $f(x) + g(x) \geq 0$. Next,

$$\begin{aligned} |\lambda(x)| &= |\lambda(h_1 + h_2)| \\ &\leq |\lambda(h_1)| + |\lambda(h_2)| \\ &\leq \rho(f) + \rho(g) \end{aligned}$$

Taking supremum over h, we get $\rho(f+g) \leq \rho(f) + \rho(g)$. We have established additivity of ρ for $f, g \geq 0$. For general $f, g \in C_c(X)$, split f, g, h into differences of positive and negative parts and rearrange to apply ρ with linearity. Thus we'll get $\rho(f+g) = \rho(f) + \rho(g)$.

Now to show homogeneity, let $c \in \mathbb{R}$ and $f \in C_c(X)$. If $c < 0$,

$$\begin{aligned} \rho(cf^+) &= -\rho((cf^+)^-) \\ &= -\rho(|c|f^+) \\ &= -|c|\rho(f^+) \\ &= c\rho(f^+) \end{aligned}$$

Again by splitting $f = f^+ - f^-$, we get the homogeneity. Thus we get ρ is linear. $\square$

Lemma 3.2.3. *If ν is a σ-finite regular positive measure on a locally compact Hausdorff space, and μ is a complex measure with $|\mu| \ll \nu$, then μ is regular.*

Proof. Using Radon-Nikodym theorem, for a measurable set E, we have

$$\mu(E) = \int_E h \, d\nu$$

with $h \in L^1(\mu)$. Considering that μ is regular, there are sequences of open sets $V_j \supset E$, $\nu(V_j \setminus E) \overset{j\to\infty}{\longrightarrow} 0$ and compact sets $K_j \subset E$, such that $\nu(E \setminus K_j) \overset{j\to\infty}{\longrightarrow} 0$.

Next, by dominated convergence theorem, $\square$