2nd motivation for talking about tensor products:

Extension of scalars: Given an R-module M and a ring S ← (commutative w/ id) containing R (with $1_R = 1_S$), is it possible to "embed" M in an S-module? i.e. is there an S-module N and an injective R-mod. hom. $\varphi: M \to N$?

Exs: 1a) Suppose $\Lambda \subseteq \mathbb{R}^n$ is a lattice (Λ is discrete and $\mathbb{R}^n/\Lambda$ has a compact meas. fund. dom.). Then Λ is a free $\mathbb{Z}$-module of rank n. Suppose $\vec{\lambda}_1, \ldots, \vec{\lambda}_n$ is a generating set, so

$$\Lambda = \{m_1\vec{\lambda} + \cdots + m_n\vec{\lambda}_n : m_i \in \mathbb{Z}\}$$

Then Λ is a sub-$\mathbb{Z}$-module of the $\mathbb{Q}$-module

$$\Lambda = \{r_1\vec{\lambda} + \cdots + r_n\vec{\lambda}_n : r_i \in \mathbb{Q}\}.$$

(1b) More abstractly if $R \subseteq S$ then the free R-module R^n can be embedded into the S-module S^n by the canonical inclusion $R^n \hookrightarrow S^n$ (which is an R-mod. hom.).

2) Suppose $R=\mathbb{Z}$ and suppose G is a finite Abel. gp. (i.e. a finite R-module). Let $|G|=n$.

If N is any $\mathbb{Q}$-module and $\varphi: G \to N$ is a homom. Then $\forall g \in G$,

$$n\varphi(g) = \varphi(ng) = \varphi(0) = 0 \Rightarrow \varphi(g) = 0 \Rightarrow \varphi \equiv 0.$$

So there is no way to embed a group w/ more than 1 elem. into a $\mathbb{Q}$-module.

Back to general situ: $R \subseteq S$ comm. rings, $1_R = 1_S$, M an R-module. Then $S \otimes_R M$ is an S-module, w/ scal. mult. def. by $s(s' \otimes m) = (ss') \otimes m$, and extended linearly.

Also, the map $i: M \to S \otimes_R M$ defined by $i(m) = 1 \otimes m$ is an R-module homom.

Thm. (univ. prop. for ext. of scalars):

If L is an S-module and if $\varphi: M \to L$ is an R-mod. homom. then $\exists$ a unique S-mod. homom. $\Phi: S \otimes_R M \to L$ s.t. this diagram commutes:

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[dr, "\varphi"'] & S \otimes_R M \arrow[d, "\Phi"] \\ & L \end{tikzcd}$$

Cor: $M/\ker i$ is the unique largest quotient of M which can be embedded into an S-module.

Pf: • $M/\ker(\varphi) \xrightarrow{i} S \otimes_R M$ is an inj. R-mod. hom.

• If L is an S-module and if $\varphi: R \to L$ is an R-mod. hom (so $\varphi: R/\ker\varphi \to L$ is injective) then by the univ. prop. $\ker i \leq \ker\varphi \Rightarrow R/\ker\varphi \leq R/\ker i$. ▣

Ex. 2 above, revisited: if G is a finite Abel. gp. then

$$\mathbb{Q} \otimes_{\mathbb{Z}} G = 0$$

$$\left(r \otimes g = \left(n \tfrac{r}{n}\right) \otimes g = \left(\tfrac{r}{n}\right) \otimes ng = 0 \right).$$

In this the univ. prop. implies that any homom. from G to a $\mathbb{Q}$-v.s. is the zero map.

2a) $S \otimes_R R$

• As an R-module: $S \otimes_R R$ (as an R-module) is generated by $\{ s \otimes 1 : s \in S \}$.

In fact: $S \otimes_R R = \{ s \otimes 1 : s \in S \}$.

- The map $\varphi : S \times R \to S$, $(s,r) \mapsto sr$ is R-bilinear. By the

univ. prop. for tensor products, φ determines an R-mod

hom. $\Phi : S \otimes_R R \to S$, which is surjective, $S \otimes_R R \neq 0$.

- As an S-module, $S \otimes_R R$ is gen. by $1 \otimes 1 \neq 0$,

so $S \otimes_R R \cong S$, as S-modules.

b) $S \otimes_R R^n \cong S^n$ (as S-modules)

follows from the fact that

$$S \otimes_R (M_1 \oplus M_2) \cong (S \otimes_R M_1) \oplus (S \otimes_R M_2)$$