MATH 6321 - Theory of functions of a real variable
Homework 9

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1. **Solution:** Let $f \in L^1(\mathbb{R})$ and $x \in \mathbb{R}$ be a Lebesgue point. Then by the definition, we have

$$\lim_{r\to 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(x) - f(y)| \; dm(y) \; = \; 0$$

Thus for every $\varepsilon > 0$, there is a $r_\varepsilon > 0$ such that for all $r < r_\varepsilon$,

$$\frac{1}{m(B_r(x))} \int_{B_r(x)} |f(x) - f(y)| \; dm(y) < \varepsilon$$

Since $|\int f \; d\mu| < \int |f| \; d\mu$, we get

$$\begin{aligned} &\left| f(x) - \frac{1}{m(B_r(x))} \int_{B_r(x)} f(y) \; dm(y) \right| \\ &\qquad = \left| \frac{1}{m(B_r(x))} \int_{B_r(x)} f(x) \; dm(y) - \frac{1}{m(B_r(x))} \int_{B_r(x)} f(y) \; dm(y) \right| \\ &\qquad \leq \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(x) - f(y)| \; dm(y) \\ &\qquad < \varepsilon \end{aligned}$$

and thus

$$|f(x)| - \left| \frac{1}{m(B_r(x))} \int_{B_r(x)} f(y) \; dm(y) \right| \; < \varepsilon$$

which gives

$$|f(x)| < \frac{1}{m(B_r(x))} \int_{B_r(x)} |f| \; dm + \varepsilon$$

Since $\varepsilon > 0$ was chosen arbitrarily, taking $\varepsilon \to 0$ and taking supremum over all $r > 0$ gives

$$|f(x)| \leq \sup_{r>0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f| \; dm = \mathcal{M}f(x)$$

2. **Solution:** For the sake of contradiction, assume that there exists $c_1 > c_2 > 0$ such that for $A_i := \{x \in \mathbb{R} \; : \; |f(x)| \geq c_i\}$, we have $c_i \mu(A_i) = \|f\|_1$. Then

$$\|f\|_1 \geq \int |f| \chi_{A_i} \; d\mu \geq \int c_i \chi_{A_i} \; d\mu = c_i \mu(A_i) = \|f\|_1$$

Thus

$$\int_{A_i} |f| - c_i \; d\mu = 0 \tag{1}$$

Since $\|f\|_1, c_i > 0$, by assumption, we see $\mu(A_i) > 0$. Moreover $|f(x)| - c_i > 0$ for all $x \in A_i$ by definition. Thus *Equation* 1 forces $|f(x)| = c_i$ almost everywhere in A_i. But since by definition $A_1 \subset A_2$, this gives a contradiction as $|f|$ cannot be a.e equal to c_1 and c_2 simultaneously in A_2 unless $c_1 = c_2$.

3. **Solution:** Let $x \in \mathbb{R}$ and $f \in L^2(\mathbb{R})$. Then $f \in L^2(B_2(x)) \subset L^1(B_2(x))$ (2-radius ball). We know that

$$\lim_{1>r\to 0} \frac{1}{m(B_r(y))} \int_{B_r(y)} |f(y) - f| \; d\mu = 0 \tag{2}$$

for almost all $y \in B_1(x)$. As a consequence for almost every $y \in B_1(x)$,

$$\lim_{1>r\to 0} \frac{1}{m(B_r(y))} \int_{B_r(y)} f \; d\mu = f(y) \tag{3}$$

Moreover, as a consequence of Holders inequality, we have $f, \overline{f}, |f|^2 \in L^1(B_2(x))$. Let $A_f, A_{\overline{f}}, A_{|f|^2}$ be the measure zero subsets of $B_1(x)$, where *Equation* 2 does not hold for $f, \overline{f}, |f|^2$. Let $A = A_f \cup A_{\overline{f}} \cup A_{|f|^2}$. Then $m(A) = 0$, and for all $y \in B_1(x) \setminus A$ and $r < 1$.

$$\begin{aligned}
\frac{1}{m(B_r(y))} \int_{B_r(y)} |f(y) - f|^2 \, dm &= \frac{1}{m(B_r(y))} \int_{B_r(y)} (f(y) - f)(\overline{f(y)} - \overline{f}) \, dm \\
&= \frac{1}{m(B_r(y))} \int_{B_r(y)} |f(y)|^2 - \overline{f(y)} f - \overline{f} f(y) + |f|^2 \, dm \\
&= |f(y)|^2 - \frac{\overline{f(y)}}{m(B_r(y))} \int_{B_r(y)} f \, dm \\
&\quad - \frac{f(y)}{m(B_r(y))} \int_{B_r(y)} \overline{f} \, dm + \frac{1}{m(B_r(y))} \int_{B_r(y)} |f|^2 \, dm
\end{aligned}$$

Then, taking limit as $r \to 0$, *Equation* 3 gives

$$\begin{aligned}
\lim_{1 > r \to 0} \frac{1}{m(B_r(y))} \int_{B_r(y)} |f(y) - f|^2 \, dm &= |f(y)|^2 - \overline{f(y)} f(y) - f(y) \overline{f(y)} + |f(y)|^2 \\
&= |f(y)|^2 - |f(y)|^2 - |f(y)|^2 + |f(y)|^2 \\
&= 0
\end{aligned}$$

Thus for almost every $y \in B_1(x)$,

$$\lim_{1 > r \to 0} \frac{1}{m(B_r(y))} \int_{B_r(y)} |f(y) - f|^2 \, dm = 0$$

Since $x \in \mathbb{R}$ was chosen arbitrarily, this holds true for all $x \in \mathbb{R}$, and thus

$$\lim_{r \to 0} \frac{1}{m(B_r(y))} \int_{B_r(y)} |f(y) - f|^2 \, dm = 0$$

for almost every $y \in \mathbb{R}$.

4. **Solution:** We know that $\mu(A) := \int_A f \, dm$, defines a measure on $\mathbb{R}$. By the properties of the measure μ, for any $x < y \in \mathbb{R}$,

$$\int_{(x,y]} f \, dm = \int_{(-\infty,y]} f \, dm - \int_{(-\infty,x]} f \, dm = 0 - 0 = 0$$

Any open interval $(x, y) = \cup_{n=1}^{\infty} (x, y - \frac{1}{n})$. By the continuity of the measure

μ from below, we get

$$\int_{(x,y)} f \ dm = \lim_{n\to\infty} \int_{(x,y-\frac{1}{n}]} f \ dm = 0$$

Since $f \in L^1(m)$, we know that almost all $x \in \mathbb{R}$ are Lebesgue points of f. That is for almost every $x \in \mathbb{R}$,

$$f(x) = \lim_{r\to 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} f \ dm = \lim_{r\to 0} 0 = 0$$

Thus we are done.