MATH6303 - Modern Algebra II
Homework 5

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1. **Solution:** By the properties of tensor product

$$2 \otimes 1 = 2(1 \otimes 1) = 1 \otimes 2$$

Since $2 = 0$ in $\mathbb{Z}/2\mathbb{Z}$, we get that $2 \otimes 1 = 1 \otimes 2 = 1 \otimes 0 = 0$ in $\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z}$.

Now let $1 \otimes 2k \in \mathbb{Z}/2\mathbb{Z} \otimes 2\mathbb{Z}$ be an arbitrary non-zero tensor. By the properties of the tensor product, $1 \otimes 2k = k(1 \otimes 2)$, and thus $1 \otimes 2$ generate $\mathbb{Z}/2\mathbb{Z} \otimes 2\mathbb{Z}$.

Now to show that $1 \otimes 2 \neq 0$ in $\mathbb{Z}/22\mathbb{Z} \otimes \mathbb{Z}$, by the universal property of the tensor products, we just need to find a $\mathbb{Z}$-module homomorphism, $\phi : \mathbb{Z}/2\mathbb{Z} \times 2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ such that $\phi(1, 2) \neq 0$. Define

$$\phi(x, y) = x\frac{y}{2} \mod 2$$

Then for $a, b \in \mathbb{Z}$, $x, y \in \mathbb{Z}/2\mathbb{Z}$, and $p, q \in 2\mathbb{Z}$,

$$\begin{aligned}
\phi(ax + y, bp + q) &= (ax + y)\frac{(bp + q)}{2} \mod 2 \\
&= \left(b(ax + y)\frac{p}{2} \mod 2 + (ax + y)\frac{q}{2} \mod 2\right) \mod 2 \\
&= \left(abx\frac{p}{2} \mod 2 + by\frac{p}{2} \mod 2 + ax\frac{q}{2} \mod 2 + y\frac{q}{2} \mod 2\right) \mod 2 \\
&= ab\phi(x, p) + b\phi(y, p) + a\phi(x, q) + \phi(y, q)
\end{aligned}$$

where the last sum is in $\mathbb{Z}/2\mathbb{Z}$. Thus, we see that ϕ is bilinear with $\phi(1, 2) = 1 \in \mathbb{Z}/2\mathbb{Z}$. Hence $1 \otimes 2 \neq 0$ in $\mathbb{Z}/2\mathbb{Z} \otimes 2\mathbb{Z}$.

2. **Solution:** For the sake of contradiction, assume that there exist $v, w \in \mathbb{R}^2$ such that $e_1 \otimes e_2 + e_2 \otimes e_1 = v \otimes w$. Since e_1, e_2 is a basis of $\mathbb{R}^2$, let $v = v_1 e_1 + v_2 e_2$ and $w = w_1 e_1 + w_2 e_2$ for $v_i, w_j \in \mathbb{R}$. Then

$$\begin{aligned} v \otimes w = (v_1 e_1 + v_2 e_2) \otimes (w_1 e_1 + w_2 e_2) = v_1 w_1 (e_1 \otimes e_1) + v_1 w_2 (e_1 \otimes e_2) \\ + v_2 w_1 (e_2 \otimes e_1) + v_2 w_2 (e_2 \otimes e_2) \end{aligned}$$

Since $v \otimes w = e_1 \otimes e_2 + e_2 \otimes e_1$, and $\{e_i \otimes e_j \ : \ i, j \in \{1, 2\}\}$ forms a basis for $\mathbb{R}^2 \otimes \mathbb{R}^2$, this forces $v_1 w_1 = v_2 w_2 = 0$. Without loss of generality, assume that $v_1 = 0$ and $w_2 = 0$. But this forces $v \otimes w = v_2 w_1 (e_2 \otimes e_1)$, and we get a contradiction. We can show that the other cases also leads to a contradiction, and hence our assumption that $e_1 \otimes e_2 + e_2 \otimes e_1$ is a simple tensor is false.

3. **Solution:** Let $v = av'$, then

$$v \otimes v' = av' \otimes v' = a(v' \otimes v') = v' \otimes av' = v' \otimes v$$

Now assume $v \neq av'$ for any $a \in F$. We need to show that $v \otimes v' \neq v' \otimes v$. By the universal property of the tensor products, this is equivalent to finding a bilinear map $\phi : V \times V \to \mathbb{C}$ such that $\phi(v, v') \neq \phi(v', v)$. Define ϕ such that

$$\phi(x, y) = \begin{cases} \alpha\beta, & \text{if } x = \alpha v, y = \beta v' \\ 0, & \text{elsewhere} \end{cases}$$

Then we can verify that ϕ is a bilinear map, such that $\phi(v, v') = 1$. But $(v', v) = (v', 0) + (0, v)$ cannot be represented as a linear combination of $(v, 0), (0, v')$ by assumption that $v \neq av'$ for any $a \in F$. Thus $\phi(v', v) = 0$. This shows that $v \otimes v' \neq v' \otimes v$.

4. not finished

Solution:

(a) Let $p(x) = p_0 + p_1 x, q(x) = q_0 + q_1 x, r(x) = r_0 + r_1 x$ be elements in I. Since ϕ one depends on the first two co-efficients, it is enough to prove that ϕ is a bilinear map on the linear polynomials. By rules of modular addition we get,

$$
\begin{aligned}
\phi(p+r,q) &= \frac{(p_0+r_0)q_1}{2} \mod 2 \\
&= (\frac{p_0q_1}{2} \mod 2 + \frac{r_0q_1}{2} \mod 2) \mod 2 \\
&= (\phi(p,q)+\phi(r,q)) \mod 2
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\phi(p,q+r) &= \frac{p_0(q_1+r_1)}{2} \mod 2 \\
&= (\frac{p_0q_1}{2} \mod 2 + \frac{p_0r_1}{2} \mod 2) \mod 2 \\
&= (\phi(p,q)+\phi(p,r)) \mod 2
\end{aligned}
$$

Now let $\psi : R/I \to \mathbb{Z}/2\mathbb{Z}$ be the natural homomorphism. Then $s \in \mathbb{Z}/2\mathbb{Z}$ can be identified with $\psi^{-1}(s) = s + I \in R/I$, and will have the representative $s + x \in R$. Since $(s+x)p(x) = sp_0 + (sp_1 + p_0)x$, and $(s+x)q(x) = sq_0 + (sq_1 + q_0)x$, by these identification,

$$
\begin{aligned}
\phi(sp,q) = \phi((s+x)p,q) &= \frac{sp_0q_1}{2} \mod 2 \\
&= s\phi(p,q) \mod 2
\end{aligned}
$$

and

$$
\begin{aligned}
\phi(p,sq) = \phi(p,(s+x)q) &= \frac{p_0(sq_1+q_0)}{2} \mod 2 \\
&= \frac{sp_0q_1}{2} \mod 2 + \frac{p_0q_0}{2} \mod 2
\end{aligned}
$$

Since we assumed $p, q \in I$, p_0, q_0 are multiples of 2, therefore $4|p_0q_0$, and hence $\frac{p_0q_0}{2} \mod 2 = 0$. Thus we get

$$
\phi(p,sq) = \frac{sp_0q_1}{2} \mod 2 = s\phi(p,q)
$$

Thus we get that ϕ is a bilinear map $I \times I \to \mathbb{Z}/2\mathbb{Z}$.

(b) Notice that R-module homomorphism $I \otimes I \to \mathbb{Z}/2\mathbb{Z}$ corresponding to the bilinear map ϕ above does precisely this by the universal property of the tensor products.

(c) By the universal property of the tensor products, we just need to find a bilinear map $\phi : I \times I \to \mathbb{Z}/2\mathbb{Z}$ such that $\phi(2,x) \neq \phi(x,2)$. Taking ϕ to be the bilinear map above in the first part of the problem, we see that $\phi(2,x) = 1 \neq 0 = \phi(x,2)$. Hence $2 \otimes x \neq x \otimes 2$.