MATH6321 - Theory of functions of a real variable
Homework 8

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1. **Solution:** Let X_n be a countable collection of finite measurable sets such that $X = \cup_{n=1}^{\infty} X_n$. Additionally assume that $X_n \subset X_{n+1}$ for each $n \in \mathbb{N}$. If this is not the case, then we can modify the collection by taking $Y_i = \cup_{n=1}^{i} X_n$. Let $E_n = \{x \in X_n \ : \ |g(x)| < n\}$. Clearly each E_n is measurable and $E_n \subset E_{n+1}$. Let $g_n = g\chi_{E_n}$ and define linear functionals

$$T_n : L^p(X) \to \mathbb{C} := f \mapsto \int f g_n \ d\mu$$

Since

$$\int |g_n|^q \ d\mu = \int |g\chi_{E_n}|^q \ d\mu = \int |g|^q \chi_{E_n} \ d\mu \leq n^q \mu(E_n)$$

$\|g_n\|_q \leq n\mu(E_n)^{\frac{1}{q}} < \infty$. Thus by Holder's inequality, $\|T_n\| \leq n\mu(E_n)^{\frac{1}{q}}$ and thus T_n are bounded linear functionals on $L^p(X)$.

Now we claim that $\sup_n |T_n(f)| < \infty$, for all $f \in L^p(X)$. Let $f \in L^p(X)$, then

$$|T_n(f)| = \left| \int f g_n \ d\mu \right| \leq \int |f g \chi_{E_n}| \ d\mu \leq \int |fg| \ d\mu < \infty$$

where the last inequality is by our assumption. Thus by uniform boundedness principle, we get that

$$\sup_n \|T_n\| < \infty$$

Let $\sup_n \|T_n\| = M$. Again since $fg_n \to fg$ pointwise in n, and since $fg \in L^p(X)$, by dominated convergence theorem,

$$T_n(f) = \int f g_n \ d\mu \to \int fg \ d\mu = T(f)$$

Now let $f \in L^p(X)$. Since $T(f) = \lim_{n\to\infty} T_n(f)$

$$|T(f)| = |\lim_{n\to\infty} T_n(f)| = \lim_{n\to\infty} |T_n(f)| \leq \lim_{n\to\infty} M\|f\|_p = M\|f\|_p$$

Thus $\|T\| \leq M$, and hence we see that T is a bounded linear functional on $L^p(X)$.

By the duality of $L^p(X)$ and $L^q(X)$, there exist a unique $h \in L^q(X)$, such that

$$T(f) = \int fg\ d\mu = \int fh\ d\mu = \Lambda_h$$

We claim that the set $E = \{x \in X\ :\ h \neq g\}$ has measure zero. If not, then for some $n \in \mathbb{N}$, $\mu(E \cap X_n) \neq 0$. Let $f = \chi_{E\cap X_n} \frac{\overline{g-h}}{|g-h|}$. Then clearly $f \in L^p(X)$, and

$$T(f) - \Lambda(f) = \int \chi_{E\cap X_n} \frac{\overline{g-h}}{|g-h|}(g-h)\ d\mu = \int_{E\cap X_n} |g-h|\ d\mu$$

Since $|g-h|$ is a positive function on $\chi_{E\cap X_n}$ and $\mu(E \cap X_n) \neq 0$, by a result we proved in the first semester mid-term exam

$$T(f) - \Lambda(f) = \int_{E\cap X_n} |g-h|\ d\mu > 0$$

But this contradicts our assumption that $T = \Lambda_h$. Thus $\mu(E) = 0$, and $h = g$ almost everywhere forcing $g \in L^q(X)$

2. **Solution:** Since the following result seems plays an integral role, we state is separately.

Proposition 0.1. *If $\{f_n\}$ is uniformly integrable, then $\{|f_n|\}$ is uniformly integrable.*

Proof. We'll show that $\{|f_n|\}$ is uniformly integrable whenever $\{f_n\}$ is uniformly integrable. Since

$$\left|\int f\ d\mu\right| = \left|\int \mathrm{Re}(f)\ d\mu + i\int \mathrm{Im}(f)\ d\mu\right| \geq \left|\int \mathrm{Re}(f)\ d\mu\right|, \left|\int \mathrm{Im}(f)\ d\mu\right|$$

we see that if

$$\left|\int_E f\ d\mu\right| < \varepsilon$$

then

$$\left|\int_E \operatorname{Re}(f)\ d\mu\right|, \left|\int_E \operatorname{Im}(f)\ d\mu\right| < \varepsilon$$

Thus we see that if $\{f_n\}$ is uniformly integrable, then $\{\operatorname{Re}(f_n)\}, \{\operatorname{Im}(f_n)\}$ are uniformly integrable with the same (ε, δ) pair as in $\{f_n\}$.

Now assume $\{f_n\}$ is a set of real valued functions with $f_n = f_n^+ - f_n^-$, where $f_n^+(x) = \max\{f_n(x), 0\}, f_n^-(x) = \max\{-f_n(x), 0\}$. Notice that for $P_n = \{x \in X\ :\ f_n(x) \geq 0\}$ and $N_n = X \setminus P_n$, $f_n^+ = f_n \chi_{P_n}$ and $f_n^- = f_n \chi_{N_n}$. Now let $\varepsilon > 0$ be given. Then there is a $\delta > 0$ such that for all f_n

$$\mu(E) < \delta \implies \left|\int_E f_n\ d\mu\right| < \varepsilon$$

But since

$$\left|\int_E f_n^+\ d\mu\right| = \left|\int_E f_n \chi_{P_n}\ d\mu\right| = \left|\int_{E \cap P_n} f_n\ d\mu\right|$$

and $\mu(E \cap P_n) \leq \mu(E)$, we get that

$$\mu(E) < \delta \implies \left|\int_E f_n^+\ d\mu\right| < \varepsilon$$

Thus we see that $\{f_n^+\}$ is uniformly integrable when $\{f_n\}$ is uniformly integrable with again the same (ε, δ) pair as in $\{f_n\}$. Using a similar reasoning, we can show that $\{f_n^-\}$ is also uniformly integrable with the same (ε, δ) pair as $\{f_n\}$.

Now let $\delta > 0$ be such that for all f_n

$$\mu(E) < \delta \implies \left|\int_E f_n\ d\mu\right| < \frac{\varepsilon}{4}$$

Then what we have done before, we see that

$$\left|\int_E \operatorname{Re}(f_n)\ d\mu\right|, \left|\int_E \operatorname{Im}(f_n)\ d\mu\right| < \frac{\varepsilon}{4}$$

and since $\operatorname{Re}(f_n), \operatorname{Im}(f_n)$ are real valued functions, we get

$$\left|\int_E \operatorname{Re}(f_n)^+\ d\mu\right|, \left|\int_E \operatorname{Re}(f_n)^-\ d\mu\right|, \left|\int_E \operatorname{Im}(f_n)^+\ d\mu\right|, \left|\int_E \operatorname{Im}(f_n)^-\ d\mu\right| < \frac{\varepsilon}{4}$$

Since

$$\int_E |f_n|\, d\mu = \int_E \mathrm{Re}(f_n)^+\, d\mu + \int_E \mathrm{Re}(f_n)^-\, d\mu + \int_E \mathrm{Im}(f_n)^+\, d\mu + \int_E \mathrm{Im}(f_n)^-\, d\mu$$

by triangle inequality, we get that

$$\left| \int_E |f_n|\, d\mu \right| = \int_E |f_n|\, d\mu < \varepsilon$$

Thus we get that $\{|f_n|\}$ is uniformly integrable whenever $\{f_n\}$ is uniformly integrable. □

Let $\varepsilon > 0$ be given. Since $\{|f_n|\}$ is uniformly integrable by the above proposition, let $\delta > 0$ be such that whenever $\mu(E) < \delta$,

$$\int_E |f_n|\, d\mu < \frac{\varepsilon}{3}$$

for all f_n. Since $f_n \to f$ pointwise almost everywhere, by Egoroff's theorem, there exists $E_\varepsilon \in \mathcal{M}$ such that f_n converges uniformly to f on E_ε and $\mu(E_\varepsilon^c) < \delta$. Now let $N \in \mathbb{N}$ such that for all $n > N$, $\|f_n - f\|_\infty < \frac{\varepsilon}{3\mu(E_\varepsilon)}$ in E_ε. Then for all $n > N$,

$$\begin{aligned}
\int |f_n - f|\, d\mu &= \int_{E_\varepsilon} |f_n - f|\, d\mu + \int_{E_\varepsilon^c} |f_n - f|\, d\mu \\
&\leq \frac{\varepsilon}{3} + \int_{E_\varepsilon^c} |f_n|\, d\mu + \int_{E_\varepsilon^c} |f|\, d\mu \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \int_{E_\varepsilon^c} |f|\, d\mu
\end{aligned}$$

where the last inequality is because $\mu(E_\varepsilon^c) < \delta$. Note that we'll be done, if we show that $\int_{E_\varepsilon^c} |f|\, d\mu < \frac{\varepsilon}{3}$. For this notice that since $f_n \to f$ almost everywhere,

$$|f| = \liminf |f_n|$$

almost everywhere. Thus

$$\int_{E_\varepsilon^c} |f|\, d\mu = \int_{E_\varepsilon^c} \liminf |f_n|\, d\mu \leq \liminf \int_{E_\varepsilon^c} |f_n|\, d\mu \leq \liminf \frac{\varepsilon}{3} = \frac{\varepsilon}{3}$$

due to Fatou's lemma and our choice of δ. Thus f_n converge to f in L_1 norm and by the completeness of the space, we get $f \in L^1(\mu)$.

3. **Solution:** Notice that because of problem 2, we'll be done if we show that $\{f_n\}$ is uniformly integrable. Again, see that by Holder inequality for f_n and the constant function 1

$$\left|\int_E f_n \, d\mu\right| \leq \int_E |f_n| \, d\mu \leq \Big(\int_E |f_n|^p \, d\mu\Big)^{\frac{1}{p}} \Big(\int_E 1^q \, d\mu\Big)^{\frac{1}{q}} \leq \|f_n\|_p \mu(E)^{\frac{1}{q}}$$

But since we know that $\|f_n\|_p^p < C$ for all $n \in \mathbb{N}$, we get

$$\left|\int_E f_n \, d\mu\right| \leq C^p \mu(E)^{\frac{1}{q}}$$

Now let $\varepsilon > 0$ be given. Then for $E \in \mathcal{M}$ with $\mu(E) < \delta = (\frac{\varepsilon}{C^p})^q$ the above inequality gives

$$\left|\int_E f_n \, d\mu\right| \leq C^p \mu(E)^{\frac{1}{q}} < \varepsilon$$

Thus we see that $\{f_n\}$ is uniformly integrable, and thus the result follows.