MATH 6303 - Modern Algebra II
Homework I

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1. **Solution:** Let F be a finite field of characteristic p. Then F_p can be embedded naturally to F. Hence we see that F is a field extension over F. Then F is a vector space over the field F_p. Moreover since F is finite, there is an F_p-basis for F. Let $\beta = \{f_1, f_2, \dots, f_n\}$ be an F_p-basis for F. Then $F \cong F_p^n$ by the fact that every vector space is isomorphic to its co-ordinate space. Thus we see that $|F| = |F_p|^n = p^n$.

2. **Solution:** We know that $[\mathbb{Q}(i) : Q] = 2$ and $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3 = [\mathbb{Q}(\sqrt[3]{3}) : \mathbb{Q}]$. Now if $x^3 - 2$ is irreducible over $\mathbb{Q}(i)$, then $\mathbb{Q}(i, \sqrt[3]{2}) = \mathbb{Q}(i)$ (This equality may be read as isomorphism and here we are considering $\sqrt[3]{2}$ to be a root of the polynomial $x^3 - 2$). But this would mean that $\mathbb{Q}(i)$ is a field extension of $\mathbb{Q}(\sqrt[3]{2})$. That is $\mathbb{Q}(\sqrt[3]{2})$ is a linear subspace of $\mathbb{Q}(i)$. But this is impossible as a vector space of dimension 3 cannot be a subspace of dimension 2.

3. **Solution:** Clearly $\mathbb{Q}(\sqrt{2}, \sqrt{3}) \supset \mathbb{Q}(\sqrt{2} + \sqrt{3})$ since $\sqrt{2} + \sqrt{3} \in \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Now since $(\sqrt{2} + \sqrt{3})^{-1} = \sqrt{3} - \sqrt{2}$, we see that
$$\sqrt{2} = (\sqrt{2} + \sqrt{3}) - (\sqrt{3} - \sqrt{2}) \ \in \ \mathbb{Q}(\sqrt{2} + \sqrt{3})$$
$$\sqrt{3} = (\sqrt{2} + \sqrt{3}) + (\sqrt{3} - \sqrt{2}) \ \in \ \mathbb{Q}(\sqrt{2} + \sqrt{3})$$
Thus $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2}+\sqrt{3})$. Moreover, $\sqrt{2} \notin \mathbb{Q}(\sqrt{3})$. Otherwise $\sqrt{2} = a + b\sqrt{3}$ for some $a, b \in \mathbb{Q}$, which is a contradiction. Thus we see that $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{3})] = 2$ since $x^2 - 2$ is a polynomial in $\mathbb{Q}(\sqrt{3})[x]$ with root $\sqrt{2}$. Hence we see that
$$[\mathbb{Q}(\sqrt{2} + \sqrt{3}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{3})][\mathbb{Q}(\sqrt{3}) : \mathbb{Q}] = 2 \times 2 = 4$$
Moreover by taking higher powers of $\sqrt{2} + \sqrt{3}$, we see that $x^4 - 10x^2 + 1$ is an irreducible polynomial satisfied by $\sqrt{2} + \sqrt{3}$.

4. **Solution:** For the sake of contradiction, assume that $\sqrt[3]{2} \in \mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\alpha_i \notin \mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_{i-1})$ for all $1 \leq i \leq n$. Then there is a smallest $m \leq n$ such that $\sqrt[3]{2} \in \mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_m)$. Then $\sqrt[3]{2} = a + \alpha_m b$, where $a, b \in \mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_{m-1})$. Hence

$$2 = (a + \alpha_m b)^3 = a^3 + 3a^2\alpha_m b + 3a\alpha_m^2 b^2 + b^3 \in \mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_{m-1})$$

But this forces $\alpha_m \in \mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_{m-1})$. This is a contradiction since $\alpha_m \notin \mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_{m-1})$ by our assumption.