Department of Mathematics University of Houston

MATH 6321

Theory of Functions of a Real Variable
Spring 2025

First name: ______________________ **Last name:** ________________________ **Points:**

Assignment 6, due Thursday, March 20, 10am

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let $X = \{a, b\}$ and μ a measure on X with $\mu(\{a\}) = 1$, $\mu(\{b\}) = \infty$. Describe $L^1(\mu)$, and $L^\infty(\mu)$. Is it true that there is an isometric isomorphism between the space of bounded linear functionals on $L^1(\mu)$ and $L^\infty(\mu)$? Explain a reason for your answer.

Problem 2

Let $\mathcal{M}$ be the collection of all subsets of $[0, 1]$ such that either E or $[0, 1] \setminus E$ is at most countable. Let μ be the counting measure on the σ-algebra $\mathcal{M}$ (no need to prove its properties). Let $g(x) = x$, then show that g is not $\mathcal{M}$-measurable, but for each $f \in L^1(\mu)$,

$$\Lambda : f \mapsto \int fg d\mu$$

defines a bounded linear functional.

Problem 3

Consider $L^\infty(m)$, with m the Lebesgue measure on $I = [0, 1]$. Show that there is a bounded linear functional Λ on $L^\infty(m)$ that is non-zero but vanishes on all of $C(I)$. Why can you conclude that such a Λ cannot be of the form $\Lambda_g : f \mapsto \int_I fg dm$ with $g \in L^1(m)$?