# MATH 6321 - Theory of functions of one real variable
# Homework IV

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1. **Solution:** Let $e_0, e_1$ be the usual linearly independent unit norm vectors in $\ell^1(\mathbb{N})$. Let $\mathcal{M} = \operatorname{span}\{e_0\}$, and $\mathcal{N} = \operatorname{span}\{e_0, e_1\}$. Let $\lambda \in \mathbb{C} \setminus \{0\}$ and define a linear functional

$$\phi : \mathcal{M} \to \mathbb{C} := te_0 \mapsto t\lambda, \quad t \in \mathbb{C}$$

Then by the definition of the operator norm, we see that $\|\phi\| = |\lambda|$ Now, let $\phi_1, \phi_2 : \mathcal{N} \to \mathbb{C}$ defined as

$$\phi_1(e_0) = \lambda = \phi_2(e_0)$$
$$\phi_1(e_1) = -\frac{\lambda}{2},\ \phi_2(e_1) = \frac{\lambda}{2}$$

and then linearly extending to $\mathcal{N}$. We see that $\phi_1, \phi_2$ extend $\phi$. And since

$$\begin{aligned}
|\phi_i(ae_0 + be_1)| &= |a\phi_i(e_0) + b\phi_i(e_1)| \\
&= |a\lambda + (-1)^i \frac{\lambda}{2} b| \\
&= |\lambda||a + (-1)^i \frac{b}{2}| \\
&\leq |\lambda|\Big(|a| + \frac{|b|}{2}\Big) \\
&\leq |\lambda|(|a| + |b|) \\
&= |\lambda|\|(a, b)\|_1
\end{aligned}$$

we see that $\|\phi_i\| = |\lambda|$. Now by Hahn-Banach extension theorem, we see that both $\phi_1, \phi_2$ extends to linear functionals on $\ell^1(\mathbb{N})$. By an abuse of notation, call them $\phi_1, \phi_2$. Then we see that $\phi_1, \phi_2$ are extensions of $\phi$, which preserve norm, but that the extension is not unique since $\phi_1(e_1) \neq \phi_2(e_1)$.

2. **Solution:** Let $(x_n)$ be as sequence in $X$. Assume that $(\|x_n\|) < M$. Since $\|x_n\| = \|i_{x_n}\|$, for any $f \in X^*$,

$$\|f(x_n)\| = \|i_{x_n}(f)\| \leq \|i_{x_n}\|\|f\| = \|x_n\|\|f\| < M\|f\|$$

shows that $\|f(x_n)\|$ is a bounded sequence. Since $f \in X^*$ was arbitrary, this holds true for all $f \in X^*$.

Conversely let $\sup_{n\in\mathbb{N}} \|i_{x_n}(f)\| = \sup_{n\in\mathbb{N}} \|f(x_n)\| < \infty$ for all $f \in X^*$. Then by a corollary to Banach-Steinhaus theorem, we see that

$$\sup_{n\in\mathbb{N}} \|x_n\| = \sup_{n\in\mathbb{N}} \|i_{x_n}\| < N$$

for some $N \geq 0$.

3. **Solution:** Let $\Lambda \in \mathbf{c}_0^*$. We claim that the sequence $(y_n) = (\Lambda(e_n)) \in \ell^1$. Let $\theta_j \in [0, 2\pi)$ such that $e^{i\theta_j} y_j = |y_j|$. Then for any $N \in \mathbb{N}$, we have

$$\begin{aligned}
\sum_{j=1}^{N} |y_n| = \sum_{j=1}^{N} |\Lambda(e_j)| &= \sum_{j=1}^{N} e^{i\theta_j} \Lambda(e_j) \\
&= \Lambda\Big(\sum_{j=1}^{N} e^{i\theta_j} e_j\Big) \\
&\leq \|\Lambda\| \Big\| \sum_{j=1}^{N} e^{i\theta_j} e_j \Big\|_{\infty} \\
&= \|\Lambda\|
\end{aligned}$$

Since this is true for all $N \in \mathbb{N}$, taking the limits as $N \to \infty$, the inequality is preserved and we get that $(y_n) \in \ell^1$.

Since any $x \in \mathbf{c}_0$ can be written as $x = \sum_{n\in\mathbb{N}} x_i e_i$, where $x_i \to 0$, by linearity of $\Lambda$, we see that

$$\Lambda(x) = \sum_{n\in\mathbb{N}} x_i \Lambda(e_i) = \sum_{n\in\mathbb{N}} x_i y_i$$