Department of Mathematics University of Houston

MATH 6321

Theory of Functions of a Real Variable
Spring 2025

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 10, due Friday, April 25, noon

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let $f : \mathbb{R} \to [0, \infty)$ and

$$A(f) = \{(x, y) \in \mathbb{R}^2 : 0 < y < f(x)\}.$$

Show that if f is Lebesgue-measurable, then $A(f)$ is a Lebesgue-measurable set in $\mathbb{R}^2$, and if m_2 is the Lebesgue measure on $\mathbb{R}^2$, then

$$m_2(A(f)) = \int_{\mathbb{R}} f\,dm\,.$$

Hint: Approximate f by a sequence of simple functions.

Problem 2

Let $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$, for $1 \le p \le \infty$. Show that

$$f * g(x) = \int_{\mathbb{R}} f(x - y) g(y)\,dm(y)$$

is defined for almost every x, that $f * g \in L^p(\mathbb{R})$, and that

$$\|f * g\|_p \le \|f\|_1 \|g\|_p\,.$$

Problem 3

Use Fubini's theorem and the identity

$$\frac{1}{x} = \int_0^\infty e^{-xt}\,dt$$

to prove

$$\lim_{A \to \infty} \int_0^A \frac{\sin(x)}{x}\,dx = \frac{\pi}{2}\,.$$