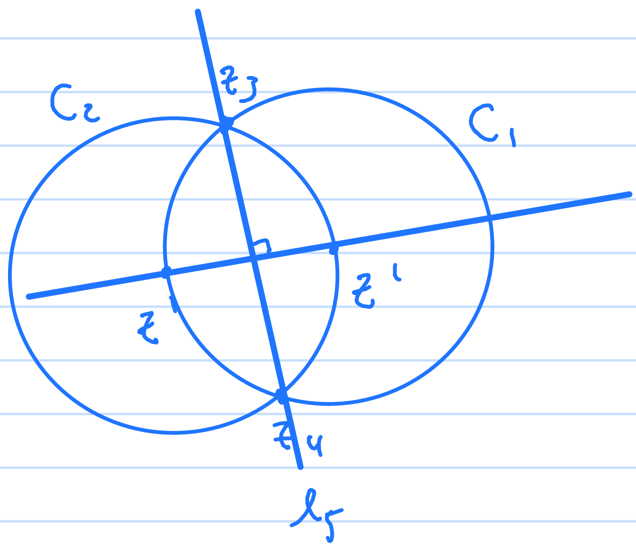
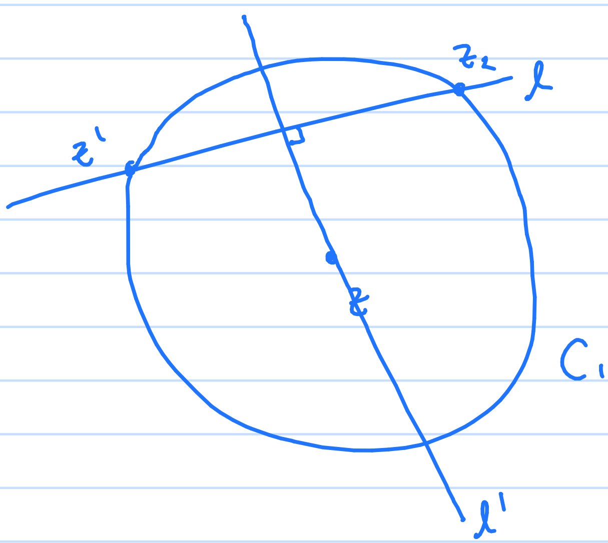
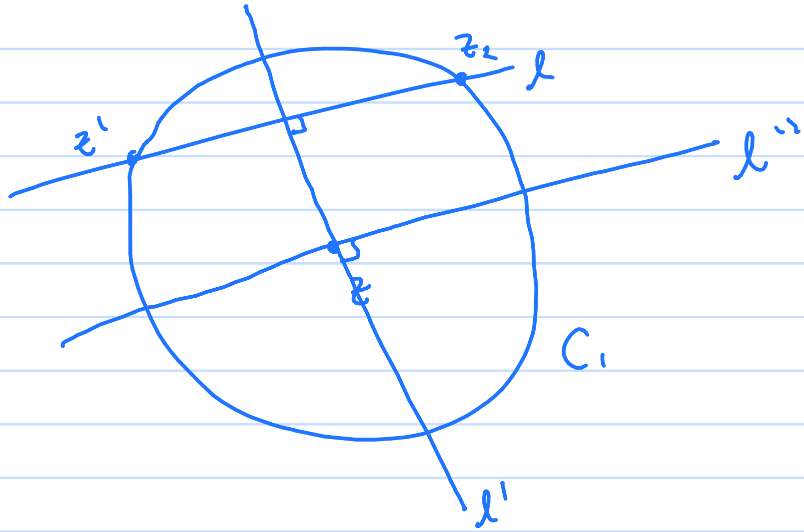
SC1) If $z, z' \in \mathcal{C}$, can construct perp. bis. of line seg. $\overline{zz'}$.



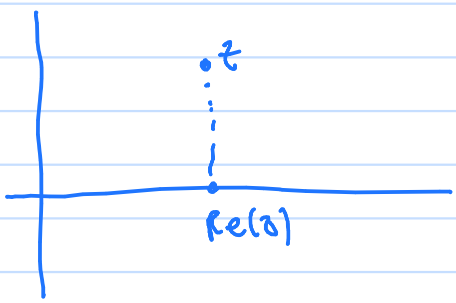
SC2) If l is a line and $z \in \mathcal{C}$, can construct line l' passing through z and perp. to l.

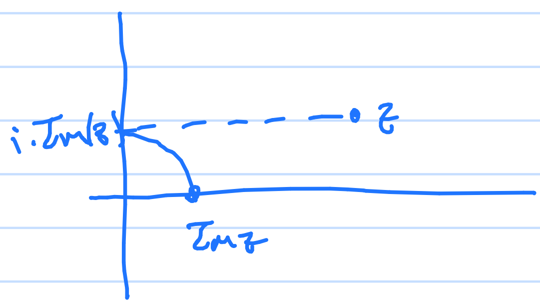


SC3) If ℓ is a line and $z \in \mathcal{C}$, can construct line ℓ'' passing through z and par. to ℓ.



SC4) $z \in \mathcal{C} \Leftrightarrow Re(z), Im(z) \in \mathcal{C}$





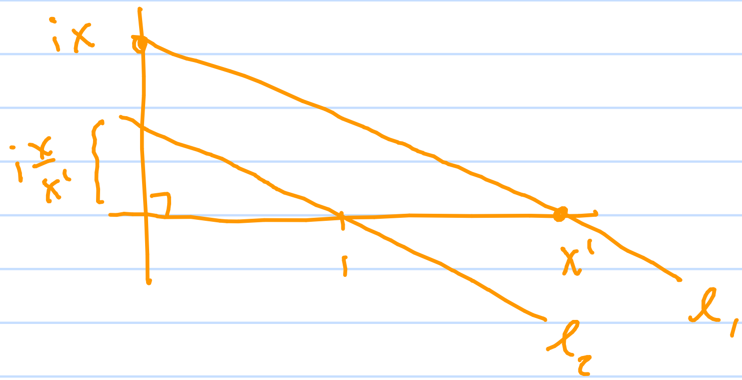
SC5) $z, z' \in \mathcal{C} \Rightarrow -z \in \mathcal{C}$ and $z+z' \in \mathcal{C}$

$-z \in \mathcal{C}$ is easy.

To see that $z+z' \in \mathcal{C}$: Note that $2 \in \mathcal{C}$. Apply the recipe used to construct z from $\{0,1\}$, but instead starting from $\{1,2\}$. This constructs $z+1$. Then apply the recipe used to construct z', but starting from $\{z, z+1\}$. This constructs $z+z'$.

S(6) If $x, x' \in \mathcal{C} \cap \mathbb{R}$, $x' \neq 0$, then $x/x' \in \mathcal{C}$

Case when $x' > 1$:



S(7) $z, z' \in \mathcal{C}$, $z' \neq 0 \Rightarrow z/z' \in \mathcal{C}$

Write $z = x + iy$, $z' = x' + iy'$ then

$$\frac{z}{z'} = \frac{(x+iy)(x'-iy')}{x'^2 + y'^2}$$

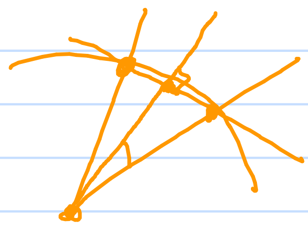
$$= \frac{(xx' + yy')}{x'^2 + y'^2} + i \frac{(x'y - xy')}{x'^2 + y'^2} \in \mathcal{C}$$

by (S(4))-(S(6)).

Thm: $\mathcal{C}$ is a field.

A few more constructions: Let Θ be the collection of constructible angles.

SC8) If $\theta \in \Theta$ then $\frac{\theta}{2} \in \Theta$

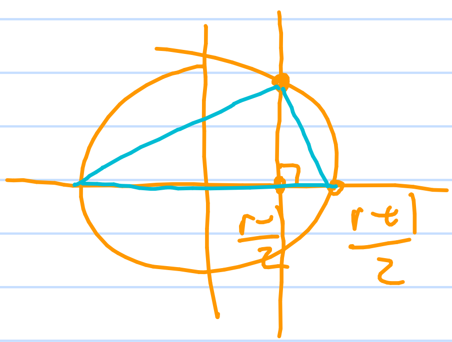


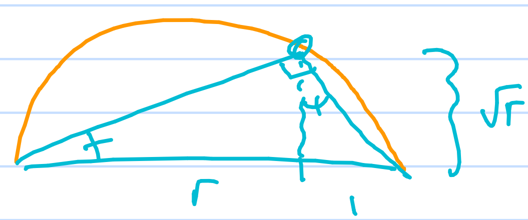
SC9) $\theta \in \Theta \iff \cos\theta, \sin\theta \in \mathcal{C}$

SC10) $z \in \mathcal{C} \iff |z| \in \mathcal{C} \cap (0,\infty)$ and $\arg z \in \Theta$.

SC11) $z \in \mathcal{C} \Rightarrow z^{1/2} \in \mathcal{C}$

Let $r = |z|$, $\theta \in \arg z$. We have to show that $\theta/2 \in \Theta$ (it is) and that $r^{1/2} \in \mathcal{C}$.





Thm: $\alpha \in \mathcal{C} \iff \exists\, n \in \mathbb{N},\ \alpha_1, \ldots, \alpha_n \in \mathbb{C}$ s.t. $[\mathbb{Q}(\alpha_1):\mathbb{Q}] = 2$,

i) $\forall\, 2 \le i \le n$, $[\mathbb{Q}(\alpha_1, \ldots, \alpha_i) : \mathbb{Q}(\alpha_1, \ldots, \alpha_{i-1})] = 2$, and

ii) $\mathbb{Q}(\alpha) = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$.

Answers to some of the questions posed by the Greeks:

1) Can't "double the cube".

$\sqrt[3]{2}$ is a root of the irreducible poly $x^3 - 2 \in \mathbb{Q}[x]$,

so $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3 \neq 2^n$, $\forall\, n \in \{0, 1, \ldots\}$.

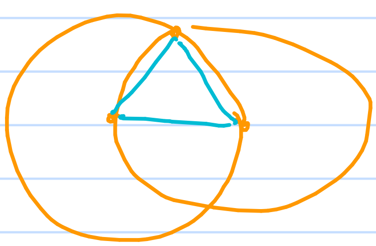
2) Can't "square the circle".

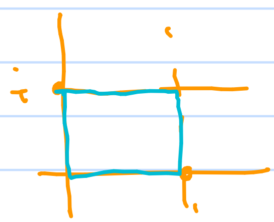
(hard)

Can't construct $\sqrt{\pi}$, because π is transcendental,

so $[\mathbb{Q}(\pi) : \mathbb{Q}] = \infty$

3) Regular n-gons:

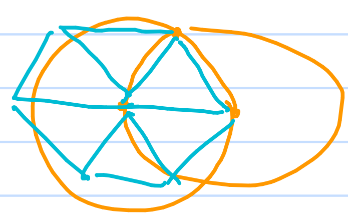




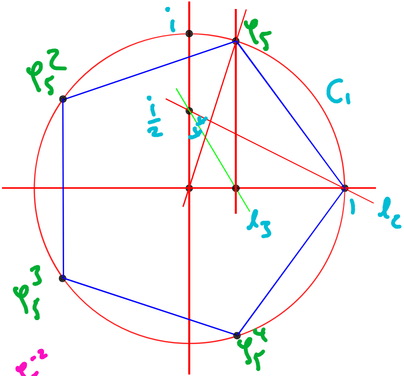
5-gon: next page

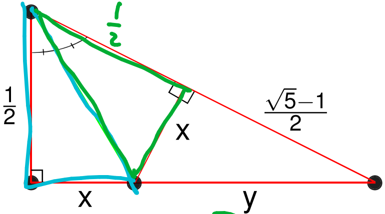
5-gon: next page

6-gon:



Ex. 5: Regular pentagon





$$\left.\begin{aligned} x + y &= 1, \\ x^2 + \left(\tfrac{\sqrt{5}-1}{2}\right)^2 &= y^2 \end{aligned}\right\} \Rightarrow \left(\frac{\sqrt{5}-1}{2}\right)^2 = 1-2x$$

$$\Rightarrow x = \frac{-1+\sqrt{5}}{4}.$$

$$\Rightarrow x = \mathrm{Re}(e^{2\pi i/5})$$

$$0 = \underbrace{\varphi_5^4}_{=\varphi_5^{-1}} + \underbrace{\varphi_5^3}_{=\varphi_5^{-2}} + \varphi_5^2 + \varphi_5 + 1$$

$$\Rightarrow (\varphi_5 + \varphi_5^{-1})^2 + \overbrace{(\varphi_5 + \varphi_5^{-1})}^{=2\mathrm{Re}(\varphi_5)} - 1 = 0 \Rightarrow \mathrm{Re}(\varphi_5) = \frac{-1+\sqrt{5}}{4}$$