Important examples:

1) Any group G acts on itself by left multiplication: $\forall g \in G$, $a \in G$,

$$g \cdot a = ga.$$

This is called the <u>left regular action</u> on G.

2) Any group G acts on itself by conjugation: $\forall g \in G, a \in G$, define $g \cdot a = gag^{-1}$.

Orbits of this action are called <u>conjugacy classes</u>

Lem: For a group G acting on itself by conjugation, $\forall a \in G$, $|\mathrm{orb}_G(a)| = |G : C_G(a)|$.

Pf: From before, $|\mathrm{orb}_G(a)| = |G : G_a|$.

$$G_a = \{g \in G : \underbrace{gag^{-1} = a}_{\Leftrightarrow\, ga = ag}\} (= N_G(a)) = C_G(a). \ \blacksquare$$

Exs 1: Cayley's Thm: Any group G is isomorphic to a subgroup of S_G.

Pf: Consider the left regular action of G on itself.

The action is faithful, so the permutation representation $\varphi : G \to S_G$ is injective. By the 1st Isom. Thm.,

$$G = G/\ker\varphi \cong \varphi(G) \leq S_G. \ \blacksquare$$

(actually, don't really need this)

Assume $|G|<\infty$.

2) Thm: If p is the smallest prime # dividing $|G|$, then any subgroup of G of index p is normal.

Pf: Suppose $H \leq G$, $|G:H| = p$. Let G act on the quotient space G/H by left multiplication:

$$g\cdot(kH) = (gk)H.$$

Let $\varphi: G \to S_p$ be the associated permutation rep., ($|G/H| = |G:H| = p$)

and let $K = \ker\varphi$.

Suppose $g \in K$. Then $g\cdot H = gH = H \Rightarrow g \in H$.

Therefore $K \leq H$. Note:

$$\left(\begin{array}{c} G \\ | \\ H \\ | \\ K \end{array}\right)$$

- By the 1st isom thm:

  $$G/K \cong \varphi(G) \Rightarrow |G/K| \,\Big|\, |S_p| = p!$$

- $|G:K| = |G:H|\cdot|H:K| = pm.$

Since $pm \mid p! \Rightarrow m \mid (p-1)!$, and since p was the smallest prime dividing $|G|$, we have that $m=1$, so $K=H$. ($|G:K| \,\big|\, |G|$)

Since $K=H$ is the kernel of a homom., $K \trianglelefteq G$. $\blacksquare$

3) Class equation: If $|G|<\infty$ and $g_1, \dots, g_r$ be representatives for the distinct conjugacy classes of cardinality greater than 1.

Then: $$|G| = |Z(G)| + \sum_{i=1}^{r} |G : C_G(g_i)|.$$

Pf: Note that $\forall g \in G$, (G acting on $A=G$ by conjugation)

$$|\mathrm{orb}_G(g)| = 1 \Leftrightarrow g \in Z(G).$$

Also, $\forall g \in G$,

$$|\mathrm{orb}_G(g)| = |G : C_G(g)|.$$ (lemma from before)

Then
$$|G| = \sum_{\mathcal{O} \in A/G} |\mathcal{O}| = |Z(G)| + \sum_{i=1}^{r} |G : C_G(g_i)|. \quad \blacksquare$$

set of all orbits