Department of Mathematics University of Houston

MATH 6321

Theory of Functions of a Real Variable
Spring 2025

First name: ______________________ **Last name:** ________________________ **Points:**

Assignment 7, due Thursday, March 28, 10am

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let μ be the Lebesgue measure on $[0,1]$ and ν the counting measure on the σ-algebra of Lebesgue-measurable sets. Show that although $\mu \ll \nu$ and μ is bounded, there is no $h \in L^1(\nu)$ such that $d\mu = h d\nu$.

Problem 2

Let $(X, \mathcal{M})$ be a measurable space and ν, λ be finite (positive) measures, and μ be a σ-finite measure on $\mathcal{M}$. Show that if $\nu \ll \lambda$ and $\lambda \ll \mu$ then the corresponding Radon-Nikodym derivatives satisfy the Leibnitz rule

$$\frac{d\nu}{d\mu} = \frac{d\nu}{d\lambda}\frac{d\lambda}{d\mu}$$

which holds μ-almost everywhere, and if $\nu \ll \lambda$ as well as $\lambda \ll \nu$, then

$$\frac{d\nu}{d\lambda} = \left(\frac{d\lambda}{d\nu}\right)^{-1} .$$

holds λ-almost everywhere.

Problem 3

Let μ be a complex Borel measure on $[0, 2\pi]$ and let for $n \in \mathbb{Z}$, $e_n(t) = e^{-int}$ and

$$\hat{\mu}(n) = \int e_n d\mu .$$

Show that if $\lim_{n\to\infty} \hat{\mu}(n) = 0$, then $\lim_{n\to-\infty} \hat{\mu}(n) = 0$ as well. Hint: The assumption implies that $\lim_{n\to\infty} \hat{\nu}(n) = 0$ where $d\nu = f d\mu$ for any trigonometric polynomial f, and hence for any continuous f, and hence for any bounded f, and hence for $\nu = |\mu|$.