Department of Mathematics University of Houston

MATH 6321

Theory of Functions of a Real Variable
Spring 2025

First name: ______________________ **Last name:** ________________________ **Points:**

Assignment 4, due Thursday, February 20, 10am

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Construct a bounded linear functional on a subspace M of some $L^1(\mu)$ which has two distinct norm-reserving linear extensions to $L^1(\mu)$.

Problem 2

Let X be a normed linear space and X^* its dual space, consisting of all bounded linear functionals on X, and each $f \in X^*$ has norm $\|f\| = \sup_{x \in X: \|x\| \leq 1} |f(x)|$. You may assume X^* is a Banach space and that for each $x \in X$, $i_x : X^* \to \mathbb{C}$, $i_x(f) = f(x)$, maps X isometrically to $(X^*)^*$ via $x \mapsto i_x$. Prove that for any sequence $(x_n)_{n \in \mathbb{N}}$, the sequence of norms $(\|x_n\|)_{n \in \mathbb{N}}$ is bounded if and only if for each $f \in X^*$, the sequence $(f(x_n))_{n \in \mathbb{N}}$ is bounded.

Problem 3

Let c_0, ℓ^1 and ℓ^∞ be given by complex sequences such that

$$x \in \ell^1 \text{ if and only if } \|x\|_1 = \sum_{n=1}^{\infty} |x_n| < \infty$$

and

$$x \in \ell^\infty \text{ if and only if } \|x\|_\infty = \sup_{n \in \mathbb{N}} |x_n| < \infty$$

and

$$x \in c_0 \text{ if and only if } \|x\|_\infty < \infty \text{ and } \lim_{n \to \infty} x_n = 0\,.$$

You may assume that these spaces are Banach when equipped with the respective norm. Show that if Λ is a bounded linear functional on c_0, then there is $y \in \ell^1$ such that for each $x \in c_0$, $\Lambda(x) = \sum_{n=1}^{\infty} x_n y_n$.