$n=3$) • simple tensors $\leftrightarrow$ "space segments" (type $(3,0)$)

• 3-multilinear forms (type $(0,3)$)

• $(1,2)$-tensors

$$\phi \in V \otimes V \otimes V$$

$$\updownarrow$$

$$\phi : V \times V \times V \to F \quad (3\text{-linear map})$$

$$\updownarrow$$

$$\tilde{\phi} : V \times V \to V^* \cong V$$

$\tilde{\phi}$ is a bilinear map from $V \times V \to V$

This allows us to think about 3 tensors as bilinear maps from $V \times V$ to V.

Ex: $V = \mathbb{R}^3$, $F = \mathbb{R}$,

Levi-Civita symbol

Let $\phi = \sum_{1 \le i,j,k \le 3} \varepsilon_{ijk} (\vec{e}_i \otimes \vec{e}_j \otimes \vec{e}_k)$,

where $\varepsilon_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) = (1,2,3), (2,3,1), \text{ or } (3,1,2), \\ -1 & \text{if } (i,j,k) = (3,2,1), (1,3,2), \text{ or } (2,1,3), \\ 0 & \text{else.} \end{cases}$

What 3-linear form $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ does this correspond to?

scratch: $\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) \mapsto 1$

$$\left(\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right)\mapsto 1\ ,\quad \left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right)\mapsto 1$$

$$\left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right)\mapsto -1\ ,\quad \left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right)\mapsto -1$$

$$\left(\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right)\mapsto -1\ ,$$ every other triple of basis elems. maps to 0

Answer: The 3-linear form is the determinant:

$$\phi(\vec{v},\vec{w},\vec{t}) = \det(\vec{v}\,\vec{w}\,\vec{t}).$$

What bilin. fn. from $\mathbb{R}^3\times\mathbb{R}^3$ to $\mathbb{R}^3$ does this define?

$$\tilde{\phi}(\vec{v},\vec{w}) = \phi(\vec{v},\vec{w},\vec{t}) = \det(\vec{t}\,\vec{v}\,\vec{w})$$

$$= t_1(v_2w_3 - v_3w_2) - t_2(v_1w_3 - v_3w_1) + t_3(v_1w_2 - v_2w_1)$$

$$\leftrightarrow \vec{v}\times\vec{w} \quad \text{(cross product).}$$

- (2,1)-tensors:

$$\phi \in V\otimes V\otimes V$$

$$\updownarrow$$

$$\phi : V\times V\times V \to F \qquad \text{(3-linear map)}$$

$$\updownarrow$$

$$\tilde{\phi} : V \to V\otimes V$$

vectors $\mapsto$ plane segments (details...)

In general: (i,j) tensors are j-multilinear fns. from
$V \times V \times \cdots \times V$ (j-times) to $V \otimes \cdots \otimes V$ (i-times).