4) Thm: If $|G| = p^\alpha$, $\alpha \in \mathbb{N}$, p prime. Then $|Z(G)| > 1$.

Pf: Suppose W.L.O.G. that G is non-Abelian. Let $g_1, \dots, g_r$ be representatives for the non-central conjugacy classes. Then by the class eqn:

$$|G| = |Z(G)| + \sum_{i=1}^{r} |G : C_G(g_i)|.$$

Since $g_i \notin Z(G)$, we have $C_G(g_i) \neq G$

$$\Rightarrow |G : C_G(g_i)| = p^{a_i} \text{ for some } a_i \geq 1.$$

Then $|Z(G)| = p^\alpha - \sum_{i=1}^{r} p^{a_i} = 0 \mod p$. ☒

Cor: If $|G| = p^2$ then G is Abelian.

Pf: $|Z(G)| > 1$, and $Z(G) \trianglelefteq G$.

$\Rightarrow |G/Z(G)| = 1$ or p

$\Rightarrow G/Z(G)$ is cyclic

$\Rightarrow G$ is Abelian (previous HW prob) ☒

5) Suppose G is a finite group acting transitively on a finite set A with $|A|>1$.

Then $\exists g \in G$ s.t. $\forall a \in A$, $g \cdot a \neq a$. [such an element g is called a fixed pt. free element]

Pf: First fix $b \in A$. Then $\forall c \in A$, $\exists g \in G$ s.t. $g \cdot b = c$. (G acts transitively)

Then $G_c = \{h \in G : h.c = c\}$

$= \{h \in G : h.(g.b) = g.b\}$

$= \{h \in G : (hg).b = g.b\}$ (assoc. of $G \curvearrowright A$)

$= \{h \in G : (g^{-1}hg).b = b\}$ (assoc. + iden. of $G \curvearrowright A$)

$= \{h \in G : g^{-1}hg \in G_b\}$

$= g G_b g^{-1}$.

Then $\bigcup_{c \in A} G_c = \bigcup_{g \in G} g G_b g^{-1}$.

Note that $G_b \neq G$, since $|A|>1$ and G acts transitively.

Therefore $\bigcup_{g \in G} g G_b g^{-1} \neq G$, by Hmwk. 4.3 #24.

This implies that $\exists g \in \left(\bigcup_{c \in A} G_c\right)^c$ ← complement.

Then $\forall c \in A$, $g.c \neq c$. ∎

6) Burnside's lemma: Suppose $G \curvearrowright A$, $|G|, |A| < \infty$.

Then $|A/G| = \frac{1}{|G|} \cdot \sum_{g \in G} |A^g|$. (# of orbits = average # of fixed pts)

Notation: $A/G = \{ \mathrm{orb}_G(a) : a \in A \}$,

$\forall g \in G$, $A^g = \{ a \in A : g.a = a \}$.

Pf: $\sum_{g \in G} |A^g| = \sum_{a \in A} |G_a|$ $\left(|\mathrm{orb}_G\, a| = |G : G_a| = \frac{|G|}{|G_a|} \right)$

$= \sum_{a \in A} \frac{|G|}{|\mathrm{orb}_G\, a|} = |G| \sum_{a \in A} \frac{1}{|\mathrm{orb}_G\, a|}$

$= |G| \cdot \sum_{O \in A/G} \frac{1}{|O|} \cdot \underbrace{\sum_{a \in O} 1}_{= |O|} = |G| \cdot |A/G|$. ∎

7) Ex: Suppose we color the 4 vertices of a square each w/ one of 3 colors. Two colorings are the same if they are the same up to rigid motions. How many different colorings are there, with this identification?