3b) $\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^3 = V$ (V is an $\mathbb{R}$-v.s.)

b/c of bilin.

- This is generated as an $\mathbb{R}$-v.s. by

$$\{\vec{e}_{i,\mathbb{R}^2} \otimes \vec{e}_{j,\mathbb{R}^3} : 1 \le i \le 2,\ 1 \le j \le 3\}$$

Therefore $\dim_{\mathbb{R}} V \le 2 \cdot 3 = 6$.

- Define

$$\varphi : \mathbb{R}^2 \times \mathbb{R}^3 \to M_{3\times 2}(\mathbb{R})$$

(Kronecker product of $\vec{w}$ and $\vec{v}^t$)

$$(\vec{v}, \vec{w}) \mapsto \vec{w} \cdot \vec{v}^t = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} (v_1\ v_2) = \begin{pmatrix} w_1 v_1 & w_1 v_2 \\ w_2 v_1 & w_2 v_2 \\ w_3 v_1 & w_3 v_2 \end{pmatrix}$$

This is bilin. and onto. As, before, by the univ. prop., $\exists$ $\mathbb{R}$-mod. hom. $\Phi : \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^3 \to M_{3\times 2}(\mathbb{R})$ s.t. $\varphi = \Phi \circ \otimes$.

Therefore $\dim_{\mathbb{R}} V = 6$, so

$$\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^3 \cong \mathbb{R}^6.$$

Props. of tensor products (always assuming R is comm w/ 1_R)

0) If M is generated as an R-mod. by $\mathcal{A}$ and if N is gen. as an R-mod. by $\mathcal{B}$, then $M \otimes_R N$ is gen by $\{m \otimes_R n : m \in \mathcal{A}, n \in \mathcal{B}\}$.

1) $M \otimes_R N \cong N \otimes_R M$

Warning: $m \otimes n \neq n \otimes m$, in general

2) $(M_1 \otimes_R M_2) \otimes_R M_3 \cong M_1 \otimes_R (M_2 \otimes_R M_3)$

3) $(M_1 \oplus M_2) \otimes_R N \cong (M_1 \otimes_R N) \oplus (M_2 \otimes_R N)$

4) If M is free of rank m and N is free of rank n, then $M \otimes_R N \cong R^{mn}$.

More exs:

4a) $\mathbb{Z}/3\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/5\mathbb{Z} = G$

- The simple tensor $1 \otimes 1$ generates G, so G is cyclic.
- Note: $10 = 1 \mod 3$

$$1 \otimes 1 = 10 \otimes 1 = 1 \otimes \overset{\text{''}0}{10} = 0,$$

so $G \cong \{0\}$.

b) $\mathbb{Z}/6\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/10\mathbb{Z} = G$

- The simple tensor $1 \otimes 1$ generates G, so G is cyclic.
- Is there a non-zero bilinear map from $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z}$ into another Abelian group? (After some thought):

$$\varphi : \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \quad (2 = \gcd(6,10))$$

$$(a,b) \mapsto ab \bmod 2$$

This is well-defined. ✓

By the universal property $\varphi((1,1)) = \Phi(1 \otimes 1) = 1 \neq 0$

$\Rightarrow 1 \otimes 1 \neq 0$.

- Note: $2(1 \otimes 1) = 2 \otimes 1 = 20 \otimes 1 = 1 \otimes 20 = 1 \otimes 0 = 0$

$\Rightarrow |1 \otimes 1| \mid 2 \Rightarrow |1 \otimes 1| = 2$

$\Rightarrow G \cong \mathbb{Z}/2\mathbb{Z}$.

c) $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = G$

- generated by $1 \otimes 1$.
- Let $d = (m,n)$ (gcd), and choose $\ell = \left(\frac{n}{d}\right)^{-1} \bmod \left(\frac{m}{d}\right)$

Then $d(1 \otimes 1) = d \otimes 1$

$= n\ell \otimes 1$

$= \ell \otimes n$

$= \ell \otimes 0 = 0$

$\Rightarrow |1 \otimes 1| \mid d$.

Scratch:

$$n\ell = d\left(\ell \cdot \frac{n}{d}\right) = d\left(1 + k\frac{m}{d}\right) = d + km = d \bmod m$$

- Finally, let $\varphi: \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z}$ be def. by $\varphi(a,b) = ab$. This is well-def, and onto. Using the universal property,

  $\forall\ 1 \leq l < d$, $l \bmod d = \varphi(l,1) = \Phi(l \otimes 1)$
  $= \Phi(l(1\otimes 1))$,

  so $l(1\otimes 1) \neq 0 \Rightarrow |1 \otimes 1| = d$

  Conclusion $G \cong \mathbb{Z}/d\mathbb{Z}$.

Tensor prods. of homs:

If $\varphi: M \to M'$, $\phi: N \to N'$ are R-module homs.

then there is a unique R-mod. hom

$$\varphi \otimes \phi : M \otimes_R N \to M' \otimes_R N',$$

satisfying $(\varphi \otimes \phi)(m \otimes n) = \varphi(m) \otimes \phi(n)$, $\forall m \in M, n \in N$.

Ex: $R=F$ (field), $\varphi: M_1 \cong F^{m_1} \xrightarrow{A} M_2 \cong F^{m_2}$

$$\phi: N_1 \cong F^{n_1} \xrightarrow{B} N_2 \cong F^{n_2}$$

Suppose A and B are matrices of the transformations, w.r.t. some ordered bases. Write $A=(a_{ij})$, $B=(b_{ij})$. Then, w.r.t. the bases for $M_1 \otimes N_1$ and $M_2 \otimes N_2$, ordered lexicog. the matrix of $\varphi \otimes \phi$ is the Kronecker product

$$A \otimes B = \left(\begin{array}{c|c|c|c} a_{11}B & a_{12}B & \cdots & a_{1m_1}B \\ \hline a_{21}B & \ddots & & \vdots \\ \hline \vdots & & & \\ \hline a_{m_2 1}B & & \cdots & a_{m_2 m_1}B \end{array}\right).$$