Functional Analysis II - MATH7321

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Contents

Chapter 1

Banach Algebras

14/01/2025 Let $\mathcal{H}$ be a separable Hilbert space, and $T \in \mathcal{K}(\mathcal{H})$ be self adjoint, and X be the orthonormal basis for $\mathcal{H}$ consisting of the eigenvectors of T. Then $\mathcal{H} \cong \ell^2(X)$ and then T can be identified with a multiplication operator M_f with $f \in \ell^\infty(X)$ such that $f(x) = \lambda_x$, where λ_x is the eigenvalue for the eigenvector, $x \in X$.

If $T \in B(\mathcal{H})$ is normal, then $\operatorname{span}\{T^m T^{*n} \; : \; m, n \in \mathbb{N}\}$ is an involutive algebra.

1.1 Preliminaries

Definition 1.1.1. An algebra over $\mathbb{C}$(or $\mathbb{R}$) is a vector space over $\mathbb{C}$(or $\mathbb{R}$) with a product that is associative and distributive.

product is continuous A normed algebra is an algebra $\mathcal{A}$ equipped with a norm such that $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in \mathcal{A}$.

If the norm is complete in a normed algebra, we call it a Banach algebra.

An algebra is unital if has a neutral element 1 with respect to the multiplication. It is called commutative if $ab = ba$ for all $a, b \in \mathcal{A}$

An involution on an algebra is a map $* : \mathcal{A} \to \mathcal{A}$ such that

(1) $(\alpha a + b)^* = \overline{\alpha} a^* + b^*$

(2) $(ab)^* = b^* a^*$

(3) $(a^*)^* = a$

for all $a, b \in \mathcal{A}$ and $\alpha \in \mathbb{C}$

A normed $*$-algebra is a normed algebra with an involution $*$ such that $\|a^*\| = \|a\|$.

Definition 1.1.2. Let $\mathcal{A}$ be a unital Banach algebra. An element $a \in \mathcal{A}$ is invertible if there exists $b \in \mathcal{A}$ such that $ab = ba = 1$.

The spectrum of a is the set

$$\sigma(a) = \{\lambda \in \mathbb{C} \ : \ \lambda 1 - a \text{ is not invertible } \}$$

Lemma 1.1.1. *In any unital Banach algebra, the set of invertible elements is open.*

Proof. If $\|x\| < 1$, then

$$\frac{1}{1-x} = \sum_{n \in \mathbb{N}} x^n$$

or equivalently if $\|1 - x\| < 1$, then

$$\frac{1}{x} = \sum_{n \in \mathbb{N}} (1-x)^n$$

Now let x_0 be invertible in the unital Banach algebra and let $x \in B(x_0, \frac{1}{\|x_0^{-1}\|})$. Then $\|xx_0^{-1} - 1\| \leq \|x - x_0\|\|x_0^{-1}\| < 1$ and similarly $\|x_0^{-1}x - 1\| < 1$. This shows that $xx_0^{-1}, x_0^{-1}x$ are invertible with inverses (assumed) y, z respectively. Thus $xx_0^{-1}y = 1 = zx_0^{-1}x$. verify. $\square$

Definition 1.1.3. The spectral radius of $a \in \mathcal{A}$ is defined to be

$$r(a) = \sup\{|\lambda| \ : \ \lambda \in \sigma(a)\}$$

Lemma 1.1.2. *For all $a \in \mathcal{A}$,*

$$r(a) \leq \|a\|$$

Theorem 1.1.1. *Let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}$. Then $\sigma(a)$ is a non-empty compact subset of $\mathbb{C}$.*

Proof. proof incoming for non-empty. See last semester notes for proof of compactness. $\square$

Lemma 1.1.3. *Let X be a normed space. Then there exist a compact set K and an isometric linear map $X \to C(K)$.*

Proof. embed X to X^{**}, with K being the weak * closed unit ball of X^* $\square$

Definition 1.1.4. Let $\mathcal{A}$ be a commutative Banach algebra. The spectrum of $\mathcal{A}$ is the set $\text{sp}(\mathcal{A}) = \{\tau \in \mathcal{A}^* \ : \ \tau(ab) = \tau(a)\tau(b), a, b \in \mathcal{A}\}$.

Lemma 1.1.4. *For every $\tau \in sp(\mathcal{A})$,*

1. $\tau(1) = 1$
2. $\|\tau\| = 1$
3. *$sp(\mathcal{A})$ is a weak * closed subset of $\mathcal{A}^*$ (Hence compact)*

Proof. (1) Since $\tau(a) = \tau(a \cdot \mathbf{1}) = \tau(a)\tau(\mathbf{1})$ for all $a \in \mathcal{A}$, it is immediate that $\tau(\mathbf{1}) = 1 \in \mathbb{C}$.

(2) Since $\tau(\mathbf{1}) = 1$, it is clear that $\|\tau\| \geq 1$. The other inequality follows from the fact that $\tau(a) \in \sigma(a)$ (see that $a - \tau(a)\mathbf{1} \in \mathrm{Ker}(\tau)$) and thus $|\tau(a)| \leq r(a) \leq \|a\|$.

(3) Let $\tau, \phi \in \mathrm{sp}(A)$. Then for all $\hat{a} \in \hat{\mathcal{A}}$ (image of $\mathcal{A}$ in $\mathcal{A}^{**}$ under the natural inclusion), we have

$$|\hat{a}(\tau - \phi)| = |\hat{a}(\tau) - \hat{a}(\phi)| = |\tau(a) - \phi(a)| = |(\tau - \phi)(a)| \leq \|a\||\tau - \phi|$$

Thus if τ_α is any weak * Cauchy net, $\tau_\alpha(a)$ converges for all $a \in \mathcal{A}$. Let $\tau(a) = \lim_\alpha \tau_\alpha(a)$. Then all we have left is to prove that $\tau \in \mathrm{sp}(\mathcal{A})$. But since $\tau_(\alpha) \to \tau(a)$ and $\tau_\alpha(b) \to \tau(b)$, by the algebra of limits in $\mathbb{C}$, we get $\tau(ab) = \tau(a)\tau(b)$ proving $\tau \in \mathrm{sp}(\mathcal{A})$.

Thus we see that $\mathrm{sp}(\mathcal{A})$ is a weak * closed subspace of the unit ball of $\mathcal{A}^*$ which is closed by the Banach Alaoglu theorem.

□

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Definition 1.1.5. An ideal $I \subsetneq \mathcal{A}$ is called maximal if for any ideal $I \subset J$, then either $J = I$ or $J = \mathcal{A}$

Proposition 1.1.1. *Recall that R/I is a field if and only if I is a maximal ideal of the ring R.*

Remark 1.1.1. While we consider ideals of an algebra, since the algebra has a vector space structure, we demand the ideal to be subspace with respect to the underlying linear operations

Lemma 1.1.5. *If $\mathcal{A}$ is a (unital) Banach algebra, and I is a closed ideal, then $\mathcal{A}/I$ is a (unital) Banach algebra.*

Proof. Since I is an ideal, the ring structure of A/I is well defined. Moreover since $\mathcal{A}$ is a Banach space and I is a closed subspace, we know that $\mathcal{A}/I$ is a Banach space. Thus, we just need to verify the norm inequality

$$\|ab + I\| \leq \|a + I\|\|b + I\|$$

By the definition of the quotient norm for any $\varepsilon > 0$, there exist a $i_a, i_b \in I$ such that

$$\|a + i_a\| \leq \|a + I\| + \varepsilon, \quad \|b + i_b\| \leq \|b + I\| + \varepsilon$$

Then,

$$\begin{aligned}
\|ab + I\| &\leq \|ab + i_a b + i_b a + i_a i b\| \\
&= \|(a + i_a)(b + i_b)\| \\
&\leq \|a + i_a\|\|b + i_b\| \\
&\leq (\|a + I\| + \varepsilon)(\|b + I\| + \varepsilon) \\
&\leq \|a + I\|\|b + I\| + \varepsilon\|b + I\| + \varepsilon\|a + I\| + \varepsilon^2
\end{aligned}$$

Since ε was chosen arbitrarily, this gives our result. □

Lemma 1.1.6. *In a unital Banach algebra, every maximal ideal is closed.*

Proof. Take a maximal ideal, take its closure, then it must be either the ideal itself or the whole of the algebra. If it contains the whole of the algebra, then the unital element must be there. Then the original ideal must contain invertible elements by the openness of the set of invertible elements. This will make the original ideal, the whole of the algebra, which is a contradiction. □

Lemma 1.1.7. *If $\mathcal{A}$ is a Banach algebra that is a division ring (If every non-zero element has an inverse), then $\mathcal{A} = \mathbb{C}$.*

Proof. Let $0 \neq a \in \mathcal{A}$. Let $\lambda \in \sigma(a)$. Then $\lambda 1 - a$ is not invertible. Hence $\lambda 1 - a = 0$. So $a = \lambda 1$. Hence $\mathcal{A} = \mathbb{C}$. □

Corollary 1.1.1.1. *Let $\mathcal{A}$ be a unital commutative Banach algebra, and I be a maximal ideal. Then $\mathcal{A}/I = \mathbb{C}$.*

Lemma 1.1.8. *For every $a \in \mathcal{A}$, a commutative unital Banach algebra,*

$$\sigma(a) = \{\tau(a) \ : \ \tau \in sp(\mathcal{A})\}$$

Proof. Let $\tau \in \mathrm{sp}(\mathcal{A})$, then $\tau(\tau(a)1 - a) = 0$ and therefore $\tau(a)1 - a \in \mathrm{Ker}(\tau)$, hence it is not invertible. Hence $\tau(a) \in \sigma(a)$.

Conversely, let $\lambda \in \sigma(a)$, then $\lambda 1 - a$ is not invertible, thus the ideal $I = \langle \lambda 1 - a \rangle$ is a proper ideal, since $r(\lambda 1 - a)$ will not be invertible for any $r \in \mathcal{A}$. So $1 \notin \langle \lambda 1 - a \rangle$. By Zorn's lemma, $\langle \lambda 1 - a \rangle$ is contained in a maximal ideal I_λ.

Define $\tau : \mathcal{A} \to \mathbb{C} = \mathcal{A}/I_\lambda := x \mapsto x + I_\lambda$. Then $\tau \in \mathrm{sp}(\mathcal{A})$, and $\tau(a) = a + I_\lambda = \lambda + I_\lambda$ since $\lambda 1 - a \in I_\lambda$. Now by the identification of A/I_λ with $\mathbb{C}$ as in Lemma 1.1.7, we see that $\tau(a) = \lambda$. □

Definition 1.1.6. Let $\mathcal{A}$ be a commutative Banach algebra. Define

$$\Phi : \mathcal{A} \to C(\mathrm{sp}(\mathcal{A})) := \Phi(a)(\tau) = \tau(a)$$

for all $a \in \mathcal{A}, \tau \in \mathrm{sp}(\mathcal{A})$. The map Φ is called the **Gelfand transform**.

Theorem 1.1.2. *Φ is a contractive algebra homomorphism with $\|\Phi(a)\| = r(a)$.*

Proof. That Φ is contractive follows easily from

$$\|\Phi(a)(\tau)\| = \|\tau(a)\| \leq \|a\|$$

since τ is a contraction as proved in Lemma 1.1.4. Linearity and multiplicativity of Φ follows form the fact that every element $\tau \in \mathrm{sp}(\mathcal{A})$ is linear and multiplicative on $\mathcal{A}$. □

Remark 1.1.2 (Maximal ideals of $\mathcal{A}$ and $\mathrm{sp}(\mathcal{A})$). Let $\mathcal{A}$ be unital commutative Banach algebra. Let $\tau \in \mathrm{sp}(\mathcal{A})$, then $\mathrm{Ker}(\tau)$ is a closed ideal of $\mathcal{A}$, and $A/\mathrm{Ker}(\tau) \cong \mathbb{C}$ by the first isomorphism theorem. So $\mathrm{Ker}(\tau)$ is a maximal ideal. The converse of this is also true. Natural map to the quotient space of a maximal ideal (which is now a filed isomorphic to $\mathbb{C}$) gives an element of the $\mathrm{sp}(\mathcal{A})$. Hence $\mathrm{sp}(\mathcal{A})$ can be identified with the maximal ideals of $\mathcal{A}$.

Remark 1.1.3. Suppose $\tau, \tau' \in \mathrm{sp}(\mathcal{A})$, with $\mathrm{Ker}(\tau) = \mathrm{Ker}(\tau')$. Let $a \in \mathcal{A}$, then $\tau(a)1 - a \in \mathrm{Ker}(\tau) = \mathrm{Ker}(\tau')$ implies $\tau(a) = \tau'(a)$ for all $a \in \mathcal{A}$.

Remark 1.1.4. Combining both of the above, we see that $\mathrm{Ker}(\Phi)$ is the intersection of all maximal ideals of $\mathcal{A}$, that is the radical of $\mathcal{A}$.

Theorem 1.1.3. *Let $\mathcal{A}$ be a Banach algebra. Then $\forall a \in \mathcal{A}$, we have*

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$$

Proof. verify □

Corollary 1.1.3.1. *If $\|a^2\| = \|a\|^2$, then $r(a) = \|a\|$ and the Gelfand transform will be halal.*

not sure. Might need C^* algebra structure

Example 1.1.1. Let $T \in B(\mathcal{H})$ be self-adjoint. Let $\mathcal{A} = \overline{\mathrm{span}}\{T^n : n \in \mathbb{N} \cup 0\}$. Then $\mathcal{A}$ is a unital Banach algebra. Moreover we have

$$\|T^2\| = \|T\|^2$$

by the self adjointness of T. Thus the Gelfand transform Φ is isometric on $\mathbb{R}$-$\overline{\mathrm{span}}\{T^n\}$.

Lemma 1.1.9. *Let $\mathcal{A}$ be a unital commutative Banach $*$-algebra such that for all $a \in \mathcal{A}$, $\|a^*a\| = \|a\|^2$. Then for self adjoint $a \in \mathcal{A}$, and $\tau \in sp(\mathcal{A})$, $\tau(a) \in \mathbb{R}$.*

Proof. Let $\tau(a) = \alpha + i\beta$ for $\alpha, \beta \in \mathbb{R}$. $\forall t \in \mathbb{R}$, we have

$$\begin{aligned}
|\tau(a) + it|^2 &= |\tau(a + it1)|^2 \\
&\leq \|a + it\|^2 \\
&= \|(a - it1)(a + it1)\| \\
&= \|a^2 + t^2 I\| \\
&\leq \|a^2\| + t^2 \\
&= \|a\|^2 + t^2
\end{aligned}$$

But $|\tau(a) + it|^2 = |\alpha|^2 + |\beta + t|^2 = |\alpha|^2 + |\beta|^2 + 2|t\beta| + |t|^2$ to begin with. But this gives

$$\alpha^2 + \beta^2 + 2|t\beta| \leq \|a\|^2$$

for all $t \in \mathbb{R}$, which is absurd unless $\beta = 0$. $\square$

Lemma 1.1.10. *Let $\mathcal{A}$ be as in the above lemma, then*

$$\tau(a^*) = \overline{\tau(a)}$$

for all $a \in \mathcal{A}, \tau \in sp(\mathcal{A})$

Proof. Let $a = u + iv$, where $u, v \in \mathcal{A}_{sa}$. Then $a^* = u - iv$, and

$$\tau(a^*) = \tau(u - iv) = \tau(u) - i\tau(v) = \overline{\tau(u) + i\tau(v)} = \overline{\tau(u + iv)} = \overline{\tau(a)}$$

by the above lemma. $\square$

And as a direct consequence, we get $\Phi(a^*) = \overline{\Phi(a)}$.

Theorem 1.1.4. *Let $\mathcal{A}$ be a unital commutative Banach $*$-algebra such that for all $a \in \mathcal{A}$, $\|a^*a\| = \|a\|^2$. Then the Gelfand transform is a bijective isometric $*$-homomorphism.*

Proof. We already have shown that $\Phi : \mathcal{A} \to C(\mathrm{sp}(\mathcal{A}))$ is a contractive algebra homomorphism. And by Theorem 1.1.3 for every self adjoint $a \in \mathcal{A}$, $\|\Phi(a)\| = \|a\|$. As a consequence of Lemma 1.1.10, we get that $\Phi(a^*) = \overline{\Phi(a)}$. Then

$$\begin{aligned}
\|\Phi(a)\|^2 &= \|\Phi(a)^*\Phi(a)\| \\
&= \|\Phi(a^*)\Phi(a)\| \\
&= \|\Phi(a^*a)\| \\
&= \|a^*a\| \quad \text{since } a^*a \text{ is self adjoint} \\
&= \|a\|^2
\end{aligned}$$

and we see that Φ is isometric.

It remains to show that Φ is surjective. First note that $\Phi(\mathcal{A})$ is a unital $*$-subalgebra of $C(\text{sp}(\mathcal{A}))$. Thus by Stone-Weierstrass, we only need to show that $\Phi(\mathcal{A})$ separates points of $\text{sp}(\mathcal{A})$. Let $\tau_1 \neq \tau_2 \in \text{sp}(\mathcal{A})$. Then $\exists a \in \mathcal{A}$ such that $\Phi(a)(\tau_1) = \tau_1(a) \neq \tau_2(a) = \Phi(a)(\tau_2)$. Hence we are done. □

Corollary 1.1.4.1. *Let $T \in B(\mathcal{H})$ be a normal operator, and let $\mathcal{A} = \overline{span}\{T^n T^{*m} : m, n \in \mathbb{N} \cup \{0\}\}$. Then the Gelfand transform $\Phi : \mathcal{A} \to C(sp(\mathcal{A}))$ is halal (preserve all structure).*

Proof. Observe that $\mathcal{A}$ is a Banach $*$ algebra with $\|T^*T\| = \|T\|^2$. Now we are left to prove that the C* identity hold in $\mathcal{A}$. But this is true because $B(\mathcal{H})$ is a C* algebra. □

Lemma 1.1.11. *In the above corollary, the map $\psi : sp(\mathcal{A}) \to \sigma(T) := \tau \mapsto \tau(T)$ is a bijective homeomorphism.*

Proof. Lemma 1.1.8 shows that ψ is a well defined surjection. Assume $\psi(\tau) = \psi(\pi)$ for $\tau, \pi \in \text{sp}(\mathcal{A})$. That is $\tau(T) = \pi(T)$. Then $\tau(T^m) = \pi(T^m)$ and $\tau(T^{*m}) = \pi(T^{*m})$ for all $m \in \mathbb{N} \cup \{0\}$. Moreover since τ, π are multiplicative by the definition, we see that τ, π agree on $\text{span}\{T^n T^{*m} : m, n \in \mathbb{N} \cup \{0\}\}$. Thus $\tau = \pi$. Hence we see that ψ is a bijective map.

Now to prove the continuity, let τ_α be a net weak $*$ converging to $\tau \in \text{sp}(\mathcal{A})$. Then $\tau_\alpha(T) \to \tau(T)$ by definition. Thus we see that ψ is a continuous map. Now since $\text{sp}(\mathcal{A})$ is compact as we proved in Lemma 1.1.4, and $\sigma(T) \subset \mathbb{C}$ is compact, by a general fact in topology, we get that ψ is a homeomorphism. □

Thus for the above algebra $\mathcal{A} = C^*(T)$, we'll use $\text{sp}(\mathcal{A})$ and $\sigma(T)$ interchangeably according our need.

1.2 Continuous Functional Calculus

Theorem 1.2.1 (Continuous functional calculus)**.** *Let $T \in B(\mathcal{H})$ be normal. Then $\exists$ a unique isometric $*$-homomorphism $\Xi : C(\sigma(T)) \to B(\mathcal{H})$ such that for any polynomial $p(z, \bar{z}) \in C(\sigma(T))$, we have*

$$\Xi(p(x)) = p(T, T^*)$$

For every $f \in C(\sigma(T))$, we denote $f(T) := \Xi(f)$ and call this the continuous functional calculus of T.

Proof. Existence of the map Ξ is guaranteed by Corollary 1.1.4.1 and Lemma 1.1.11. For the uniqueness note that the image of Ξ is the algebra in Corollary 1.1.4.1 and that the identity map $\sigma(T) \to \sigma(T)$ must be mapped to $T \in B(\mathcal{H})$. Since the algebra $\mathcal{A}$ is generated by T, the $*$-homomorphism ensure the uniqueness of Ξ. □

Definition 1.2.1. Let $(X, \mathcal{A})$ be a measurable space and $\mathcal{H}$ be a Hilbert space. A **spectral measure** on the triple $(X, \mathcal{A}, \mathcal{H})$ is a map $E : \mathcal{A} \to B(\mathcal{H})$ such that

(1) $E(A)$ is a projection, $\forall A \in \mathcal{A}$

(2) $E(A \cap B) = E(A)E(B), \forall A, B \in \mathcal{A}$

(3) $E(\emptyset) = 0, E(X) = I$

(4) If $\{A_n\}_{n\in\mathbb{N}}$ is a countable family of disjoint measurable sets, then

$$E\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) = \sum_{n\in\mathbb{N}} E(A_n)$$

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Example 1.2.1. Let (X, Σ, μ) be a measure space. Define $E : \Sigma \to B(L^2(X, \mu)) := A \to M_{\chi_A}$. It is easy to see that E satisfies the first 3 properties of a spectral measure. To verify the fourth property, let $A_1, A_2, \ldots \in \Sigma$ be disjoint collection. Then

$$M_{\chi_{\cup_{n=1}^{\infty} A_n}}(f) = \chi_{\cup_{n=1}^{\infty} A_n} f = \sum_{n=1}^{\infty} \chi_{A_n} f = \sum_{n=1}^{\infty} M_{\chi_{A_n}} f$$

shows that the fourth property is also satisfied.

Proposition 1.2.1. *Let E be a spectral measure on $(X, \Sigma, \mathcal{H})$. Then for every $\xi, \eta \in \mathcal{H}$.*

$$E_{\xi,\eta}(A) = \langle E(A)\xi, \eta\rangle$$

defines a finite measure on (X, Σ) with $\|E_{\xi,\eta}\| \leq \|\xi\|\|\eta\|$.

Proof. That $E_{\xi,\eta}(\emptyset) = 0$ is evident. Hence we only need to verify the countable disjoint additivity to show that $E_{\xi,\eta}$ is a measure. Let $A_1, A_2, \ldots \in \Sigma$ be a mutually disjoint collection. Then

$$\begin{aligned}
E_{\xi,\eta}\Big(\bigcup_{n=1}^{\infty} A_n\Big) &= \Big\langle E\Big(\bigcup_{n=1}^{\infty} A_n\Big)\xi, \eta\Big\rangle \\
&= \Big\langle \sum_{n=1}^{\infty} E(A_n)\xi, \eta\Big\rangle \\
&= \sum_{n=1}^{\infty} \big\langle E(A_n)\xi, \eta\big\rangle \\
&= \sum_{n=1}^{\infty} E_{\xi,\eta}(A_n)
\end{aligned}$$

Taking the summation outside the inner product is justified by the below lemma. Thus we see that $E_{\xi,\eta}$ is a complex measure. Moreover, since $|\langle E(A)\xi, \eta\rangle| \leq \|\xi\|\|\eta\|$, as $E(A)$ is a projection, we see that the measure is finite (Although this is implicit in complex measures). To see that $\|E_{\xi,\eta}\| \leq \|\xi\|\|\eta\|$, use the definition of bounded variation and the lemma below. □

Lemma 1.2.1. *Let $\{P_i\}_{i\in I}$ be a family of pairwise orthogonal projections on a Hilbert space $\mathcal{H}$. Then there exist a unique projection P on $\mathcal{H}$ such that $\forall \xi, \eta \in \mathcal{H}$,*

$$\sum_{i\in I} \langle P_i \xi, \eta\rangle = \langle P\xi, \eta\rangle$$

This means the convergence of orthogonal projections is not in the norm sense, but rather in the sense above. For example

Example 1.2.2. Consider $P_n = P_{\delta_n}$ in $B(\ell^2(\mathbb{N}))$, and let $Q_n = \sum_{i=1}^n P_i$. Then clearly Q_i doesn't converge in norm, but rather in the above sense to the identity map in $B(\ell^2(\mathbb{N}))$.

I'm not convinced. Need to verify

Lemma 1.2.2 (Reisz representation for sesquilinear forms)**.** *Let $B(\cdot,\cdot)$ be a bounded sesquilinear form on a Hilbert space $\mathcal{H}$. Then there exist a unique $T \in B(\mathcal{H})$ such that for all $\xi, \eta \in \mathcal{H}$,*

$$B(\xi, \eta) = \langle T\xi, \eta\rangle$$

verify if ϕ is a linear functional

Proposition 1.2.2. *Let E be a spectral measure on $(X, \Sigma, \mathcal{H})$, and ϕ be a bounded linear functional on X. Then there exist a unique $T_\phi \in B(\mathcal{H})$ such that for all $\xi, \eta \in \mathcal{H}$*

$$\int_X \phi \; dE_{\xi,\eta} = \langle T_\phi \xi, \eta\rangle$$

and we denote $T_\phi := \int_X \phi \; dE$

Proof. See that the integral is sesquilinear on ξ, η for simple functions. Then use Lemma 1.2.2. □

Note that the set $\mathcal{M}(X)$ of all bounded measurable functions on X, equipped with sup norm is a Banach space. Moreover with pointwise product and complex conjugation, it turns into a commutative C* algebra.

verify if ϕ is a linear functional

Theorem 1.2.2. *Let E be a spectral measure on $(X, \Sigma, \mathcal{H})$ and ϕ be a bounded measurable function. The map*

$$\mathcal{M}(X) \to B(\mathcal{H}) := \phi \mapsto \int_X \phi \; dE$$

is a contractive $*$-homomorphism.

Proof. To show that it is a $*$-homomorphism, observe that $\overline{E_{\xi,\eta}} = E_{\eta,\xi}$. This follows since

$$\overline{E_{\xi,\eta}(A)} = \overline{\langle E(A)\xi, \eta\rangle} = \langle \eta, E(A)\xi\rangle = \langle E(A)\eta, \xi\rangle = E_{\eta,\xi}(A)$$

Now let $T_{\overline{\phi}} \in B(\mathcal{H})$ corresponding to ϕ as in Proposition 1.2.2. Then by definition,

$$\int_X \overline{\phi}\ dE_{\xi,\eta} = \langle T_{\overline{\phi}}\eta, \psi\rangle$$

Now the left integral is equal to

$$\overline{\int_X \phi\ dE_{\eta,\xi}} = \overline{\langle T_\phi \eta, \xi\rangle} = \langle \xi, T_\phi \eta\rangle = \langle T_\phi^* \xi, \eta\rangle$$

Thus we get that $T_{\overline{\phi}} = T_\phi^*$ and hence the map preserve the involution.

To show that the map is multiplicative, we need to show that

$$\int_X \phi\psi\ dE = \int_X \phi\ dE \circ \int_X \psi\ dE$$

Notice that by Proposition 1.2.2, we'll done if we show that for all $\xi, \eta \in \mathcal{H}$,

$$\int_X \phi\psi\ dE_{\xi,\eta} = \int_X \phi\ dE_{(\int_X \psi\ dE)\xi,\eta}$$

But for this, it is enough to show the equivalence of the measures $\psi dE_{\xi,\eta}$ and $E_{(\int_X \psi\ dE)\xi,\eta}$. That is for all $A \in \Sigma$, we need to show that

$$\Big\langle E(A)\Big(\int_X \psi\ dE\Big)\xi, \eta\Big\rangle = \int \psi\chi_A\ dE_{\xi,\eta} \tag{1.1}$$

But again, the left hand side of the inner product is

$$\Big\langle \Big(\int_X \psi\ dE\Big)\xi, E(A)\eta\Big\rangle$$

since $E(A)$ is a projection. Again by Proposition 1.2.2, we see that the above inner product is

$$\int_X \psi\ dE_{\xi,E(A)\eta}$$

Then Equation 1.1 reduces to showing

$$\int_X \psi\ dE_{\xi,E(A)\eta} = \int \psi\chi_A\ dE_{\xi,\eta}$$

Again using the same reasoning, it is enough to show that the measures are the same. That is for any $B \in \Sigma$, we must have

$$E_{\xi,E(A)\eta}(B) = \int_X \chi_A \chi_B \; dE_{\xi,\eta}$$

But this is equivalent to

$$\langle E(A)\xi, E(B)\eta\rangle = \langle E(A)E(B)\xi, \eta\rangle = \langle E(A \cap B)\xi, \eta\rangle = \int_X \chi_{A\cap B} \; dE_{\xi,\eta} = E_{\xi,\eta}(A \cap B)$$

which is true by a property of the Spectral measure. □

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Theorem 1.2.3. *Let X be a compact Hausdorff space and $\mathcal{H}$ be a Hilbert space, and $\Psi : C(X) \to B(\mathcal{H})$ is a $*$-homomorphism (representation). Then there exist a unique spectral measure on $(X, \mathcal{B}, \mathcal{H})$ ($\mathcal{B}$ being the Borel sigma algebra on X), such that*

$$\Psi(f) = \int_X f \; dE$$

Proof. For every $\xi, \eta \in \mathcal{H}$, the map $\nu : C(X) \to \mathbb{C} := f \mapsto \langle \Psi(f)\xi, \eta\rangle$ is a bounded linear functional on $C(X)$, hence by the reisz representation theorem, there exists a unique measure $\mu_{\xi,\eta}$ on X such that

$$\langle \Psi(f)\xi, \eta\rangle = \int f \; d\mu_{\xi,\eta} \quad f \in C(X)$$

Now for any $\xi, \eta \in \mathcal{H}$, the linear map

$$\mathcal{M}(X) \to \mathbb{C} := \phi \mapsto \int \phi \; d\mu_{\xi,\eta}$$

is a bounded sesquilinear form about ξ, η with the operator norm $\|\xi\|\|\eta\|$, which is attained when $\phi = \chi_X$. Hence there exist a unique $T_\phi \in B(\mathcal{H})$ such that

$$\langle T_\phi \xi, \eta\rangle = \int \phi \; d\mu_{\xi,\eta}$$

Now we show that the map $\mathscr{F} : \mathcal{M}(X) \to B(\mathcal{H}) := \phi \mapsto T_\phi$ is a $*$-representation of $\mathcal{M}(X)$. Linearity is obvious, though verify. Argument for $T_{\overline{\phi}} = T_\phi^*$ is as in the last theorem. Observe that for every $f \in C(X)$, $T_f = \Psi(f)$. Take inner product with ξ, η.

verify the rest

To show multiplicativity of the above map, let $f \in C(X), \phi \in \mathcal{M}(X)$. Then $\forall \xi, \eta \in \mathcal{H}$,

$$\begin{aligned}\langle T_{\phi f}(\xi), \eta\rangle &= \int \phi f \; d\mu_{\xi,\eta} \\ &= \int \phi \; d\mu_{\Psi(f)\xi,\eta} \\ &= \langle T_\phi \Psi(f)\xi, \eta\rangle \\ &= \langle T_\phi T_f \xi, \eta\rangle\end{aligned}$$

verify the 2nd step

shows that $\mathscr{F}$ is multiplicative if one of the functions are in $C(X)$. More generally, if $\phi, \psi \in \mathcal{M}(X)$, choose a net $f_i \in C(X)$ such that f_i converges to ϕ weak * in $C(X)^{**} = \mathcal{M}(x)$, and $\|f_i\| \to \|\phi\|$ (This is guaranteed by Goldstein's theorem) and $\|f_i\| \leq \|\phi\|$. Now for any $\xi, \eta \in \mathcal{H}$, we have

$$\langle T_{\phi,\psi}\xi, \eta\rangle = \int \phi\psi \; d\mu_{\xi,\eta} = \lim_i \int \phi f_i \; d\mu_{\xi,\eta} = \lim_i \langle T_{\phi f_i}\xi, \eta\rangle = \lim_i \langle T_\phi T_{f_i}\xi, \eta\rangle$$

which converge to $\langle T_\phi T_\psi \xi, \eta\rangle$. verify. Thus $\mathscr{F}$ is a $*$-representation of $\mathcal{M}(X)$.

Now define $E : \Sigma \to B(\mathcal{H}) := A \mapsto T_{\chi_A}$. Obviously E satisfy all the first three properties of a spectral measure easily.

See also that the map E is countably additive. Let (A_n) be a family of pairwise disjoint measurable sets of X and let $A = \cup_{n=1}^\infty A_n$. Then for every $\xi, \eta \in \mathcal{H}$,

$$\left\langle E\Big(\bigcup_{n\in\mathbb{N}} A_n\Big)\xi, \eta \right\rangle = \int \chi_{\cup_{n=1}^\infty A_n} \; d\mu_{\xi,\eta} = \mu_{\xi,\eta}\Big(\bigcup_{n=1}^\infty A_n\Big) = \sum_{n\in\mathbb{N}} \mu_{\xi,\eta}(A_n)$$

□

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Theorem 1.2.4 (Spectral decomposition)**.** *Let $T \in B(\mathcal{H})$ be a normal operator. Then there exists a unique spectral measure $E \in (\sigma(T), \Sigma, \mathcal{H})$ such that*

$$T = \int_{\sigma(T)} \lambda \; dE(\lambda)$$

Proof. Immediate from what we've proven so far?? verify. . □

See $\sigma(T)$ as $sp(T)$ also

Let X be a compact subset of $\mathbb{C}$. Let μ be a Radon measure on X. Let $\mathcal{H} = L^2(X, \mu)$, and $T \in B(L^2(X, \mu))$ by defining $(T\eta)(x) = x\eta(x)$ almost everywhere. Then $(T^*\eta)(x) = \bar{x}\eta(x)$. Then notice that T is normal. Define spectral measure by $E(A) = M_{\xi_A}$.

Definition 1.2.2. Let $\mathcal{A}$ be a $*$-subalgebra of $B(\mathcal{H})$. A vector $\eta \in \mathcal{H}$ is called cyclic for $\mathcal{A}$ iff $\{a\eta : a \in \mathcal{A}\}$ is dense in $\mathcal{H}$.

Definition 1.2.3. Let $\mathcal{A}$ be a $*$-subalgebra of $B(\mathcal{H})$. A vector $\eta \in \mathcal{H}$ is called *separating* for $\mathcal{A}$ iff $a\eta = 0$ for $a \in \mathcal{A}$ implies $a = 0$.

Definition 1.2.4. Let $T \in B(\mathcal{H})$ be normal. We denote $C^*(T) = \overline{\text{span}}\{T^n T^{*m} : m, n \geq 0\}$

Let $\mathcal{A}$ be a *-subalgebra of $B(\mathcal{H})$, and $\eta \in \mathcal{H}$. Define $\langle \cdot, \cdot \rangle' : \mathcal{A} \times \mathcal{A} \to \mathbb{C} := \langle a, b \rangle' := \langle a\eta, b\eta \rangle_{\mathcal{H}}$

Lemma 1.2.3. *Let $\mathcal{A}$ be a $*$-subalgebra of $B(\mathcal{H})$, and let $\eta \in \mathcal{H}$. Then η is cyclic for $\mathcal{A}$ if and only if it is separating for $\mathcal{A}'$, the commutant of $\mathcal{A}$.*

Proof. verify □

If $\mathcal{A}$ is commutative and $\eta \in \mathcal{A}$ is cyclic for $\mathcal{A}$, then $\langle \cdot, \cdot \rangle'$ is an inner product on $\mathcal{A}$.

Denote $\Xi : \mathcal{A} \to \mathcal{H} := a \mapsto a\eta$. Then note that this map is an isometry under the new inner product.

Theorem 1.2.5. *Let $T \in B(\mathcal{H})$ be normal. Assume there exists a cyclic vector $\eta \in \mathcal{H}$ for $C^*(T)$. Then there exists a Radon probability measure $\mu \in Prob(sp(T))$ and a unitary*

$$U : \mathcal{H} \to L^2(sp(T), \mu)$$

such that $UTU^* = M_I$, where I is the identity function on $sp(T)$.

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Lemma 1.2.4. *Let μ, μ' be Radon probability measures on X, such that $\mu \sim \mu'$. Define $U : L^2(X, \mu) \to L^2(X, \mu')$ by sending*

$$U : \xi \to \sqrt{\frac{d\mu}{d\mu'}}\xi$$

Then U is a unitary, and $\forall f \in C(X)$, and

$$UM_fU^* = M'_f$$

where $M_f \in B(L^2(X, \mu))$ and $M'_f \in B(L^2(X, \mu'))$.

Proof. To see that U is an isometry, take the L^2 norms on both of these spaces. verify the rest. □

Lemma 1.2.5. *Let $\mu, \mu' \in Prob(X)$. If $\exists$ unitary $U : L^2(\mu) \to L^2(\mu')$ such that $UM_fU^* = M_f$ for all $f \in C(X)$, then $\mu \sim \mu'$.*

Proof. Let $h = |U1|^2 \in L^1(\mu')$. $\forall f \in C(X)$,

$$\int f \, d\mu = \langle M_f 1, 1\rangle_\mu = \langle U^* M_f' U 1, 1\rangle = \int fh \, d\mu'$$

Thus we get $\mu = h\mu'$. Hence we see that $\mu \sim \mu'$. □

Definition 1.2.5 (Strong operator topology)**.** The strong operator topology is the locally convex topology on $B(\mathcal{H})$ for separable $\mathcal{H}$, defined by the family of seminorms $\{\phi_x \ : \ x \in \mathcal{H}\}$, where $\phi_x(T) := \|Tx\|$. We have $T_i \overset{SOT}{\to} T$ if and only if $\|(T_i - T)x\| \to 0$ for all $x \in \mathcal{H}$. For any $x_1, x_2, \ldots, x_n \in \mathcal{H}$, and $\varepsilon > 0$, $T \in B(\mathcal{H})$, the set

$$\{S \in B(\mathcal{H}) \ : \ \|(T - S)x_i\| < \varepsilon, \ \forall i = 1, 2, \ldots n\}$$

is an open neighborhood of T.

Definition 1.2.6. The weak operator topology is the locally convex topology on $B(\mathcal{H})$ for separable $\mathcal{H}$, defined by the family of seminorms $\{\phi_{x,y} \ : \ x \in \mathcal{H}\}$, where $\phi_{x,y}(T) := |\langle Tx, y\rangle|$. We have $T_i \overset{WOT}{\to} T$ if and only if $|\langle (T_i - T)x, y\rangle| \to 0$ for all $x, y \in \mathcal{H}$. For any $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in \mathcal{H}$, and $\varepsilon > 0$, $T \in B(\mathcal{H})$, the set

$T_i \to T$ in WOT $T_i(x) \overset{w^*}{\to} T(x)$ for all $x \in \mathcal{H}$

$$\{S \in B(\mathcal{H}) \ : \ |\langle (T - S)x_i, y_i\rangle| < \varepsilon, \ \forall i = 1, 2, \ldots n\}$$

is an open neighborhood of T.

Example 1.2.3. Let $\{f_i\} \subset C([0,1]) \ni f$ identified as multiplication operators on $B(L^2([0,1]))$. Then $M_{f_i} \overset{WOT}{\to} M_f$ if and only if $f_i \to f$ in weak * topology on $L^\infty([0,1])$.

Proof. verify with intergrals and the fact with dual space of $L^1([0,1])$ □

Lemma 1.2.6. *For any convex subset $K \subset B(\mathcal{H})$, $\overline{K}^{SOT} = \overline{K}^{WOT}$.*

Proof. Clearly $\overline{K}^{\text{SOT}} \subset \overline{K}^{\text{WOT}}$ since the topologies are relative. Let $T \in \overline{K}^{\text{WOT}}$, $x_1, x_2, \ldots, x_n \in H$ and $\varepsilon > 0$. Let $\mathcal{H}^n = \mathcal{H} \oplus \mathcal{H} \oplus \ldots \oplus \mathcal{H}$, and $T^n \in B(H^n)$ be defined as $T^n = T \otimes C_n$, then n-amplification of T. Similarly, let

$$K^n = \{T^n \ : \ T \in K\}.$$

Then K^n is a convex subset of $B(\mathcal{H}^n)$, and

$$K^n(x_1, x_2, \ldots, x_n) = \{T^n(x_1, x_2, \ldots, x_n) \ : \ T \in K\}$$

is a convex subset of $\mathcal{H}^n$. Thus, the weak and norm closures of the set coincide by what we proved last semester. By assuming, $T^n(x_1, x_2, \ldots, x_n)$ in the weak closure of $K^n(x_1, x_2, \ldots, x_n)$, there is a $T_i \in K$ such that $T_i^n(x_1, x_2, \ldots, x_n) \to T^n(x_1, x_2, \ldots, x_n)$ weakly. $\square$

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Exercise 1.2.1. Explore the correspondance between states on $\ell^\infty(X)$ and the finitely additive probability measures on X.

Theorem 1.2.6. *Let $\mathcal{H}$ be a Hilbert space, and let $f : B(\mathcal{H}) \to \mathbb{C}$ be linear. Then, the following are equivalent.*

(1) f is SOT continuous.

(2) f is WOT continuous.

(3) There exists $\xi_1, \xi_2, \ldots, \xi_n, \eta_1, \eta_2, \ldots, \eta_n \in \mathcal{H}$ such that

$$f(T) = \sum_{i=1}^{n} \langle T\xi_i, \eta_i \rangle$$

Proof. We just need to prove $1 \implies 3$. By a lemma from last semester, there must exist $\xi_1, \xi_2, \ldots, \xi_n \in \mathcal{H}$ such that $|f(T)| \leq \sum_{i=1}^n \|T\xi_i\|$ for all $T \in B(\mathcal{H})$. Let $\mathcal{K} = \overline{\{T\xi_o \ : \ T \in B(\mathcal{H})\}} \subset \mathcal{H}$. Then

$$\phi : \mathcal{K} \to \mathbb{C} := T\xi_0 \to f(T)$$

is a well defined linear functional for $\mathcal{K}$. Now by Reisz representation, we see that $\phi(T\xi_0) = \langle T\xi_0, \eta_0 \rangle$ for some $\eta_0 \in \mathcal{H}$.

But here, for $\xi_1, \xi_2, \ldots, \xi_n$, consider them in $\mathcal{H}^n$ and do the same process to get a $(\eta_1, \eta_2, \ldots, \eta_n) \in \mathcal{H}^n$. $\square$

Theorem 1.2.7. *Let X, Y be normed spaces. For $1 \leq p \leq \infty$, define*

$$\|(x, y)\|_p = \Big(\|x\|^p + \|y\|^p\Big)^{\frac{1}{p}}$$

Then $\|\cdot\|_p$ is a norm extending both norms in X and Y. If X and Y are Banach, then so is $(X \oplus Y, \|\cdot\|_p)$. All these norms are equivalent for all $1 \leq p \leq \infty$.

Proposition 1.2.3. *We have that $(X \oplus_p Y)^* = X^* \oplus_q Y^*$.*

Proof. We first show when $p = 1, q = \infty$. Consider the map

$$\Xi : (X \oplus_1 Y)^* \to X^* \oplus_\infty Y^* := F \to F_X \oplus F_Y$$

once we identify X as $X \oplus 0$ and Y similarly as subspaces of $X \oplus Y$. See that this is a bijective linear map. Now show that the norms are preserved in Ξ. $\square$

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Theorem 1.2.8 (Bicommutant Theorem)**.** *Let $A \subset B(\mathcal{H})$ be a unital $*$-subalgebra, where $\mathcal{H}$ is separable. Then $\overline{\mathcal{A}}^{SOT} = \mathcal{A}''$*

Proof. Let $T \in \overline{\mathcal{A}}^{SOT}$, and $S \in \mathcal{A}'$. Let $(T_i) \in \mathcal{A}$ be a net converging in SOT to T. Then for all $\xi \in \mathcal{H}$,

$$TS\xi = \lim_i T_i S\xi = \lim_i ST_i\xi = S \lim_i T_i\xi = ST\xi$$

shows that $T \in \mathcal{A}''$.

To see the converse, let $T \in \mathcal{A}''$, fix $\xi \in \mathcal{H}$, and $\varepsilon > 0$ and let $K = \overline{\mathcal{A}\xi}$. Then notice that K is a reducing subspace for all operators in $\mathcal{A}$. verify. Then $P_K T = TP_K$ by our results on reducing subsapces in last semester. Thus $P_K \in \mathcal{A}'$.

Let $\xi_1, \xi_2, \ldots, \xi_n \in \mathcal{H}$, $\varepsilon > 0$ be given. Let

$$\mathcal{K} = \left\{ \begin{bmatrix} A\xi_1 \\ \vdots \\ A\xi_n \end{bmatrix} \ : \ A \in \mathcal{A} \right\}$$

For each $A \in \mathcal{A}$, let $\mathcal{A}^{(n)} = I_n \otimes A \in B(\mathcal{H}^n)$. Let $\mathcal{A}^n = \{A^{(n)} \ : \ A \in \mathcal{A}\}$. Observe that $\mathcal{A}^{(n)}(\mathcal{K}) \subset \mathcal{K}$. Also observe that $\mathcal{K}$ is reducing for all $A^{(n)} \in \mathcal{A}^{(n)}$. So $P_\mathcal{K} \in (\mathcal{A}^{(n)})' = M_n(\mathcal{A}')$. verify the rest. $\square$

Definition 1.2.7. For every $\xi, \eta \in \mathcal{H}$, denote $\omega_{\xi,\eta} \in B(\mathcal{H})^*$ such that

$$\omega_{\xi,\eta}(T) = \langle T\xi, \eta \rangle$$

Theorem 1.2.9. $B(\mathcal{H}) = \overline{span}\{\omega_{\xi,\eta} \ : \ \xi, \eta \in \mathcal{H}\}^*$

Proposition 1.2.4. *For $T \in B(\mathcal{H})$, the following are equivalent.*

(1) $\langle T\xi, \xi \rangle \geq 0, \forall \xi \in \mathcal{H}$

(2) $\exists S \in B(\mathcal{H})$ *such that* $T = S^*S$

(3) $\exists S \geq 0$ *such that* $T = S^2$. *In this case we write* $S = T^{\frac{1}{2}}$.

Proof. 2 $\implies$ 1 is obvious.

Now for the other one, it is clear that T is self adjoint. So we get that $\Xi : C(\mathrm{sp}(T)) \to B(\mathcal{H})$. We claim that $\mathrm{sp}(T) \subset \mathbb{R}^{\geq 0}$, which then completes the proof by the identification of T with the multiplication operator corresponding to the identity function in $C(\mathrm{sp}(T))$.

Let $\lambda < 0$. Then for all $\xi \in \mathcal{H}$,

$$\begin{aligned}\|(T-\lambda I)\xi\|^2 &= \langle (T-\lambda I)\xi, (T-\lambda I)\xi\rangle \\ &= \|T\xi\|^2 - 2\lambda\langle T\xi, \xi\rangle + \lambda^2\|\xi\|^2 \\ &\geq \lambda^2\|\xi\|^2\end{aligned}$$

which shows that $T - \lambda I$ is injective. Then it has a linear left inverse S. Then $S : (T-\lambda I)\xi \mapsto \xi$. Thus we get $\|S\| \leq \frac{1}{\lambda}$. Also $S(T-\lambda I) = I$ implies $(T-\lambda I)S^* = I$, which shows that $T - \lambda I$ is not invertible. Hence $\lambda \notin \sigma(T)$. ☐

Definition 1.2.8. $T \in B(\mathcal{H})$ is called positive if it satisfies any of the above conditions.

Definition 1.2.9. Let $T \in B(\mathcal{H})$. We define $|T| = (T^*T)^{\frac{1}{2}}$

Definition 1.2.10. Given $\xi, \eta \in \mathcal{H}$, we denote $P_{\xi,\eta} : \mathcal{H} \to \mathcal{H} := \rho \mapsto \langle \rho, \eta\rangle\xi$.

Lemma 1.2.7. *$\forall T \in B(\mathcal{H})$, $\exists S \in B(\mathcal{H})$ such that $T = S|T|$.*

Proof. For every $\xi \in \mathcal{H}$, define the map $S_1 : \mathrm{Image}(|T|) \to \mathrm{Image}(T) := |T|\xi \mapsto T\xi$. If $|T|\xi = 0$, then $\langle |T|\xi, |T|\xi\rangle = \langle |T|^2\xi, \xi\rangle = \langle T^*T\xi, \xi\rangle = \|T\xi\|^2 = 0$. Moreover S_1 is linear. Thus the map is well defined.

We have $\|S_1|T|(\xi)\|^2 = \|T\xi\|^2 = \langle T\xi, T\xi\rangle = \langle T^*T\xi, \xi\rangle = \langle |T|\xi, |T|\xi\rangle = \||T|\xi\|^2$. Hence S_1 is an isometry. Thus it extends to an isometry from $\overline{\mathrm{Image}(|T|)}$ onto $\overline{\mathrm{Image}(T)}$. Now define $S \in B(\mathcal{H})$ to be $S = S_1 P_{\overline{\mathrm{Image}(|T|)}}$. Then $S^*S = P_{\overline{\mathrm{Image}(|T|)}}$, and $SS^* = P_{\overline{\mathrm{Image}(T)}}$. ☐

Theorem 1.2.10. *If $T \in \mathcal{K}(\mathcal{H})$, $\exists$ orthonormal basis $\{\xi_n\}, \{\eta_n\}$ for $\mathcal{H}$ and $(\alpha_n) \in \boldsymbol{c}_0$ such that*

$$T = \sum_{n\in\mathbb{N}} \alpha_n P_{\xi_n,\eta_n}$$

Proof. By the lemma, $T = S|T|$. Observe $|T| \in \mathcal{K}(\mathcal{H})$. There exists an orthonormal basis ξ_n and $(\alpha_n) \in \mathbf{c}_0(\mathbb{R})_+$ such that $|T| = \sum_n \alpha_n P_{\xi_n}$. Then $T = S|T| = \sum_n \alpha_n S P_{\xi_n,\eta_n} = \sum_n a_n P_{S\eta_n,\xi_n}$. Work on with previous lemma to show that $S\xi_n$ is orthonormal basis. ☐

Definition 1.2.11. Let $f \in \mathcal{K}(\mathcal{H})^*$. Define $T_f \in B(\mathcal{H})$ to be the unique operator satisfying $\langle T_f\xi, \eta\rangle = f(P_{\xi,\eta})$. Then the map $f \to T_f$ is linear.

Exercise 1.2.2. Show that if $\omega_{\xi,\eta} \in B(\mathcal{H})^*$ such that $\omega_{\xi,\eta}(T) = \langle T\xi, \eta \rangle$, then $T_{\omega_{\xi,\eta}} = P_{\xi,\eta}$.

Solution 1.2.1. *verify*

Theorem 1.2.11. *$T_\phi \in \mathcal{K}(\mathcal{H})$ for every $\phi \in \mathcal{K}(\mathcal{H})^*$*

Proof. First assume $T_\phi \geq 0$. Let $T_\phi = \int_0^\infty \lambda \, dE$ for a spectral measure E. Let $\epsilon > 0$ be given. Let $P = \int_\varepsilon^\infty \, dE$. We'll show that P is finite rank.

For the sake of contradiction, assume otherwise. Let $\{\xi_n\}_{n\in\mathbb{N}}$ be an orthonormal set such that $P\xi_n = \xi_n$ for all $n \in \mathbb{N}$. $\forall N \in \mathbb{N}$, let $S_N = \sum_{n=1}^N P_{\xi_n,\eta_n}$. Then $S_N \in \mathcal{K}(\mathcal{H})$ is a finite rank projection. In particular $\|S_N\| = 1$.

Then $\forall N \in \mathbb{N}$,

$$\begin{aligned}
\|\phi\| &\geq |\phi(S_n)| \\
&= \Big| \sum_{n=1}^N \phi(P_{\xi_n,\xi_n}) \Big| \\
&= \sum_{n=1}^N \langle T_\phi \xi_n, \xi_n \rangle \\
&= \sum_{n=1}^N \int_0^\infty \lambda \, dE_{\xi_n,\xi_n} \\
&\geq \sum_{n=1}^N \int_\varepsilon^\infty \lambda \, dE_{\xi_n,\xi_n} \\
&\geq \varepsilon \sum_{n=1}^N \int_\varepsilon^\infty \, dE_{\xi_n,\xi_n} \\
&= \varepsilon N
\end{aligned}$$

which is absurd, since we can take $N \to \infty$.

Now, we have

$$\begin{aligned}
\|T_\phi - T_\phi P\| &= \Big\| \int_0^\infty \lambda \, dE - \int_\varepsilon^\infty \lambda \, dE \Big\| \\
&= \Big\| \int_0^\varepsilon \lambda \, dE \Big\| \\
&\leq \varepsilon
\end{aligned}$$

Hence we see that $T_\phi \in \mathcal{K}(\mathcal{H})$.

Now for the general case, let $T_\phi = V|T_\phi|$ be the polar decomposition of T_ϕ. Define $\psi \in \mathcal{K}(\mathcal{H})^*$, by

$$\psi(T) := \phi(TV^*)$$

for every $T \in \mathcal{K}(\mathcal{H})$. Then $\langle T_\psi \xi, \eta\rangle = \psi(P_{\xi,\eta}) = \phi(P_{\xi,\eta}V^*) = \phi(P_{\xi,V\eta}) = \langle T_\phi \xi, V\eta\rangle = \langle V^* T_\phi \xi, \eta\rangle = \langle |T_\phi|\xi, \eta\rangle$. So $T_\psi = |T_\phi|$, and by the first part of the proof, $|T_\phi| \in \mathcal{K}(\mathcal{H})$. Thus $T_\phi = V|T_\phi| \in \mathcal{K}(\mathcal{H})$. □

Theorem 1.2.12. *Let $\phi \in \mathcal{K}(\mathcal{H})^*$. Then there are orthonormal basis $\{\xi_i\}, \{\eta_i\}$ for $\mathcal{H}$ and $(\alpha_i) \in \ell^1(\mathbb{R})_+$ such that*

$$\phi = \sum_{i\in\mathbb{N}} \alpha_i \omega_{\xi_i,\eta_i}$$

Proof. For every $T \in B(\mathcal{H})$, observe that $|\sum \alpha_n \omega_{\xi_n,\eta_n}(T)| \leq (\sum \alpha_i)\|T\|$.

By the previous theorem, there exists a $\alpha_i \in \mathbf{c}_0$, and orthonormal basis $\{\xi_i\}, \{\eta_i\}$ for $\mathcal{H}$ such that

$$T_\phi = \sum_{i\in\mathbb{N}} \alpha_i P_{\xi_i,\eta_i}$$

Now let $(\beta_i) \in \mathbf{c}_0$ be arbitrary. Let $S = \sum_{i\in\mathbb{N}} \beta_i P_{\eta_i,\xi_i}$. Then $S \in \mathcal{K}(\mathcal{H})$, and $\|S\| \leq \|(\beta_i)\|_\infty$. So,

$$\begin{aligned}
\|\phi\|\|(\beta_i)\|_\infty &\geq |\phi(S)| \\
&= \Big|\phi\Big(\sum_{i\in\mathbb{N}} \beta_i P_{\eta_i,\xi_i}\Big)\Big| \\
&= \Big|\sum_{i\in\mathbb{N}} \beta_i \langle T_\phi \eta_i, \xi_i\rangle\Big| \\
&= \Big|\sum_{i\in\mathbb{N}} \beta_i \Big\langle\Big(\sum_{j\in\mathbb{N}} \alpha_j P_{\xi_j,\eta_j}\Big)\eta_i, \xi_i\Big\rangle\Big| \\
&= \Big|\sum_{i,j\in\mathbb{N}} \beta_i \alpha_j \langle P_{\xi_j,\eta_j}(\eta_i), \xi_i\rangle\Big| \\
&= \Big|\sum_{i\in\mathbb{N}} \beta_i \alpha_i\Big|
\end{aligned}$$

Since (β_i) was chosen arbitrarily, we get that $(\alpha_i) \in \ell^1$. Thus the series $\sum_{i\in\mathbb{N}} \alpha_i \omega_{\xi_i,\eta_i}$ converges in norm in $\mathcal{K}(\mathcal{H})^*$. □

Exercise 1.2.3. Show that every finite rank operator is in the span of $P_{\xi,\eta}$, for $\xi, \eta \in \mathcal{H}$.

Definition 1.2.12. An operator $T \in B(\mathcal{H})$ of the form $T = T_\phi$ for some $\phi \in \mathcal{K}(\mathcal{H})^*$ is called a trace class operator. We denote by $\mathcal{T}(\mathcal{H})$, the space of all trace class operators.

Exercise 1.2.4.

$$\Big\| \sum_{i \in \mathbb{N}} \alpha_i \omega_{\xi_i, \eta_i} \Big\| = \|(\alpha_i)\|_1$$

and $\|T_\phi\| \leq \|\phi\|$

$\mathcal{T}(\mathcal{H})$ is a two-sided ideal in $\mathcal{K}(\mathcal{H})$ 27/02/2025

Showed that $\mathcal{T}(\mathcal{H})^* = B(\mathcal{H})$ and defined the Shatten-p class of operators.

Theorem 1.2.13. *$\mathcal{K}(\mathcal{H})$ is the unique closed two-sided ideal in $B(\mathcal{H})$*

Proof. Let $I \leqslant B(\mathcal{H})$ be a closed two-sided ideal. Then $\mathcal{K}(\mathcal{H}) \leqslant I$. This is because I can find one rank one projection in I, and then multiply it with unitary operators to get all rank one operators. Now taking the closed span gives me all the compact operators.

Let $0 \geq T \in I \notin \mathcal{K}(\mathcal{H})$. (Or take TT^*). Then by spectral decomposition,

$$T = \int_0^\infty \lambda \, dE(\lambda)$$

Then there exist $\epsilon > 0$ such that

$$P = \int_\varepsilon^\infty \, dE(\lambda)$$

is infinite dimensional.

Now the bounded Borel function $g(x) = \chi_{[\varepsilon,\infty]} \frac{1}{x}$ embedded into $B(\mathcal{H})$ when multiplied with T gives P. □

04/03/2025

1.3 Tensor Product of Normed Spaces

1.3.1 Few Facts

Suppose $T : X \to X'$ and $S : Y \to Y'$ are linear. Then there exist a unique linear map $T \otimes S : X \otimes Y \to X' \otimes Y'$, such that

$$(T \otimes S)(x \otimes y) = T(x) \otimes S(y)$$

Lemma 1.3.1. *Let $x_1, x_2, \ldots, x_n \in X$, and $y_1, y_2, \ldots, y_n \in Y$. Then the following are equivalent*

(1) $\sum_{i=1}^n x_i \otimes y_i = 0$

(2) $\sum_{i=1}^n \phi(x_i)\psi(y_i) = 0$ *for all* $\phi \in X^\dagger, \sigma \in Y^\dagger$

(3) $\sum_{i=1}^n \psi(y_i)x_i = 0$

(4) $\sum_{i=1}^n \phi(y_i)x_i = 0$

where $X^\dagger$ is the collection of all linear functionals of X and similarly $Y^\dagger$ for Y.

Proof. Notice that we only need to prove $2 \implies 1$, and the rest is trivial verify. Let $z = \sum_{j=1}^k \sum_{i=1}^m c_{ij}(e_i \otimes d_j)$, where $c_{ij} \in \mathbb{C}$, e_i, d_j are linearly independent in X, Y respectively. For each $1 \leq i_o, j_o \leq m$ by Hahn-Banach theorem, there is a $\phi \in X^*, \psi \in Y^*$ such that $\phi(e_i) = \delta_{i,i_o}, \psi(d_j) = \delta_{j,j_o}$. □

Definition 1.3.1. Define for every $z \in X \otimes Y$,

$$\|z\|_\wedge = \inf\left\{\sum_{i=1}^n \|x_i\|\|y_i\| \;:\; z = \sum_{i=1}^n x_i \otimes y_i\right\}$$

Proof. Scalar product comes out easily. See that tringle inequality might also be satisfied.

Suppose $\|z\|_\wedge = 0$. Let $\varepsilon > 0$ be given. Then there exists $x_1, x_2, \ldots, x_n \in X$, $y_1, y_2, \ldots, y_n \in Y$ such that $z = \sum_{i=1}^n x_i \otimes y_i$, and

$$\sum_{i=1}^n \|x_i\|\|y_i\| < \varepsilon$$

Let $\phi \in X^\dagger$ with $\|\phi\| = 1$. Then

$$\left\|\sum_{i=1}^n \phi(x_i)y_i\right\| < \varepsilon$$

Then similarly for any $\psi \in Y^\dagger$, with $\|\psi\| = 1$, we get

$$\big|(\phi \otimes \psi)(z)\big| = \left\|\sum_{i=1}^n \phi(x_i)\psi(y_i)\right\| < \varepsilon$$

Notice that $\phi \otimes \psi \in (X \otimes Y)^\dagger$ since $\mathbb{C} \otimes \mathbb{C} = \mathbb{C}$. By appropriate rotation on each x_i, y_i, the above inequality becomes

$$\sum_{i=1}^n |\phi(x_i)||\psi(y_i)| < \varepsilon$$

Now by the Lemma 1.3.1, we are done. □

Exercise 1.3.1. Show that

$$\|x \otimes y\|_\wedge = \|x\|\|y\|$$

Observe if $\phi \in X^*, \psi \in Y^*$, $z \in X \otimes Y$. Let $z = \sum_{i=1}^n x_i \otimes y_i$. Then $|\phi \otimes \psi| \leq \|\phi\|\|\psi\| \sum_{i=1}^n |x_i||y_i|$. This shows that $\|\phi \otimes \psi\|_\wedge \leq \|\psi\|\|\phi\|$. 06/03/2025

Exercise 1.3.2. Show that $\|\phi \otimes \psi\|_\wedge = \|\phi\|\|\psi\|$.

Definition 1.3.2. Let X, Y be Banach spaces, then completion of $X \otimes Y$ with respect to $\|\cdot\|_\wedge$ is called the projective tensor product of X and Y, denoted by $X \hat{\otimes} Y$.

Proposition 1.3.1. *Let $T : X \to W, S : Y \to Z$ be bounded linear maps between Banach spaces. Then there is a unique bounded linear map $T \otimes S : X \hat{\otimes} Y \to W \hat{\otimes} Z$ such that $T \otimes S : x \otimes y \to T(x) \otimes T(y)$. Furthermore, $\|T \otimes S\| = \|T\|\|S\|$.*

Definition 1.3.3. If X is a normed space, for every $1 \leq p \leq \infty$, we define the space

$$\ell^p(X) := \{(x_n)_{n\in\mathbb{N}} \ : \ x_n \in X, (\|x_n\|) \in \ell^p(\mathbb{N})\}$$

equipped with the $\|(x_n)\|_p := \|(\|x_n\|)\|_p$ is a normed space which is complete if X is complete.

Theorem 1.3.1. *Let X be a Banach space. Then*

$$\ell^1 \hat{\otimes} X \cong \ell^1(X)$$

Proof. Define the bilinear map $b : \ell^1 \hat{\otimes} X \to \ell^1(X)$ by

$$(b(f \otimes x))(n) := f(n)x$$

There exists a unique linear map $B : \ell^1 \otimes X \to \ell^1(X)$ such that

$$B(f \otimes x) = b(f, x)$$

Let $z \in \ell^1 \otimes X$. Let $z = \sum_{i=1}^n f_i \otimes x_i$ be a representation of z. Then

$$\|B(z)\| \leq \sum_{i=1}^n \|B(f_i \otimes x_i)\| = \sum_{i=1}^m \left(\|x_i\| \sum_{n\in\mathbb{N}} |f_i(n)|\right) = \sum_{i=1}^m \|x_i\|\|f_i\|_1$$

So it follows that

$$\|B(z)\| \leq \|\phi\|_\wedge$$

To prove the reverse inequality, let $\varepsilon > 0$ be given. Let $z = \sum_{i=1}^{n} g_i \otimes x_i$ be a representative such that

$$\sum_{i=1}^{k} \|g_i\|_1 \|x_i\| < \|\phi\|_\wedge + \varepsilon$$

Choose $g_1', g_2', \ldots, g_k' \in c_{00}$ such that $\|g_i - g_i'\| < \varepsilon$, for all $i = 1, 2, \ldots k$. $\square$

18/03/2025 Recall that

Proposition 1.3.2. *Let $T \in B(X, W), S \in B(Y, Z)$. Then $\exists! T \otimes S : X \otimes_\alpha Y \to W \otimes_\alpha Z$ such that $(T \otimes S)(x \otimes y) = T(x) \otimes S(y)$.*

Proof. Let $z = \sum_{i=1}^{n} x_i \otimes y_i$. By our definitions

$$\begin{aligned} \|(T \otimes S)(z)\|_\alpha &= \sup_{\phi \in W_1^\dagger, \psi \in Z_1^\dagger} \left\{ \Big| \sum_{i=1}^{n} \phi(T(x_i))\psi(S(y_i)) \Big| \right\} \\ &\leq \|T\| \|S\| \|z\|_\alpha \end{aligned}$$

Moreover, $\|T \otimes S\| = \|T\| \|S\|$ by taking $x \otimes y$, where $\|T(x)\| \leq \|T\| - \varepsilon$ and similarly for S. $\square$

Exercise 1.3.3. Let $z \in \mathbf{c}_{00} \otimes X$ such that $z = \sum_{i=1}^{n} \delta_{n_i} \otimes x_i$. Then

$$\Big\| \sum_{i=1}^{n} \delta_{n_i} \otimes x_i \Big\|_\alpha = \sup_{\phi \in \ell_1^1, \psi \in X_1^*} \left\{ \Big| \sum_{i=1}^{n} \phi(\delta_{n_i})\psi(x_i) \Big| \right\} \leq \max\{\|x_i\|\}$$

Theorem 1.3.2.

$$\boldsymbol{c}_0 \otimes_\alpha X \cong \boldsymbol{c}_0(X)$$

For $z \in X \otimes Y$, define $B_z : X^* \times Y^* \to \mathbb{C}$ such that $B_z(\phi, \psi) := (\phi \otimes \psi)(z)$.

20/03/2025

Definition 1.3.4. A bounded linear map $T : X \to Y$ between normed spaces is called a quotient map if it is surjective and

$$\|y\| = \inf\{\|x\| \ : \ T(x) = y\}$$

for all $y \in Y$

Proposition 1.3.3. *Quotient map between normed spaces extends to a quotient map between their completion.*

Proof. verify $\square$

Theorem 1.3.3. *Let $T : X \to W$ and $S : Y \to Z$ be quotient maps between normed spaces. Then*

$$T\hat{\otimes}S : X\hat{\otimes}Y \to W\hat{\otimes}Z$$

is also a quotient map.

Proof. To see that $T\hat{\otimes}S$ is a surjection, first observe that $T \otimes S : X \otimes Y \to W \otimes Z$ is a surjection. See that $T \otimes S$ is also a quotient map. Now use the above proposition. $\square$

Note. If X is a Banach space, then there is a set I such that there is a quotient map from $L^1(I) \to X$.

Proof. Notice that we can take $I = X_1$, the unit ball of X. $\square$

Proposition 1.3.4. *Let X, Y be Banach spaces. Let I be a set such that $T : \ell^1(I) \to X$ is a quotient map. Then*

$$T\hat{\otimes}Id_Y : \ell^1(I, Y) \to X\hat{\otimes}Y$$

is a quotient map.

Proof. Let $z \in X\hat{\otimes}Y$. verify $\square$

Show that if $z \in X\hat{\otimes}Y$, with $z = \sum_{n\in\mathbb{N}} x_i \otimes y_i$, then $\sum_{n\in\mathbb{N}} \|x_i\|\|y_i\| < \infty$. 25/03/2025

Proposition 1.3.5. *Let X, Y, Z be Banach spaces. Then $B(X\hat{\otimes}Y, Z) \cong Bil(X \times Y, Z)$.*

Proof. Let $\phi \in B(X\hat{\otimes}Y, Z)$. Define $\rho : \mathrm{Bil}(X \times Y, Z)$ such that $\rho(x, y) = \phi(x \otimes y)$. Then the map $\phi \to \rho$ is a linear injective contraction.

Conversely, given a bounded bilinear $\rho : X \times Y \to Z$, let $\phi : X \otimes Y \to Z$ be the respective linear map, defined as $\phi(x \otimes y) = \rho(x, y)$. Let $\omega \in X \otimes Y$. Choose $x_1, x_2, \ldots, x_n \in X, y_1, y_2, \ldots, y_n \in Y$ such that $\omega = \sum_{i=1}^n x_i \otimes y_i$ and $\|\omega\|_\wedge > \sum_{i=1}^n \|x_i\|\|y_i\| - \varepsilon$. Then

$$\begin{aligned} \|\phi(\omega)\| &= \|\sum_{i=1}^n \rho(x_i, y_i)\| \\ &\leq \|\rho\| \sum_{i=1}^n \|x_i\|\|y_i\| \\ &\leq \|\rho\|(\|\omega\|_\wedge + \varepsilon) \end{aligned}$$

$\square$

Corollary 1.3.3.1. $(X\hat{\otimes}Y)^* \cong Bil(X \times Y) \cong B(X, Y^*) \cong B(Y, X^*)$

Proof. Notice that the last three equivalences follow easily since if $\rho : X \times Y \to \mathbb{C}$ is bilinear, then $\tilde{\rho} : X \to Y^*$ defined as $\tilde{\rho}(x)(y) = \rho(x, y)$, and similarly for $Y \to X^*$ □

Proposition 1.3.6. *Let $\sum_{n\in\mathbb{N}} x_n \otimes y_n \in X \otimes Y$ with $\sum_{n\in\mathbb{N}} \|x_i\|\|y_i\| < \infty$ and*

$$\sum_{n\in\mathbb{N}} f(x_n)y_n = 0$$

for all $f \in X^*$. Then

Proof. Notice that all the finite rank operators satisfy this.

Let $T \in B(Y, X^*)$. Assume that there's a net T_j of finite rank operators such that $T_j \to T$ uniformly on compact subsets of Y. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $\sum_{i=N+1}^{\infty} \|x_i\|\|y_i\| < \varepsilon$. □

27/03/2025

Definition 1.3.5. A bounded linear map $T \in B(X, Y)$ is called a nuclear operator if T is in the image of $X^*\hat{\otimes}Y$.

Definition 1.3.6. Let X be a Banach space. We say X has the approximation property, if the identity map on X can be approximated uniformly on compact subsets of X by finite rank operators on X.

Theorem 1.3.4. *If either X^* or Y has the approximation property, then $X^*\hat{\otimes}Y \cong \mathcal{N}(X, Y)$.*

Lemma 1.3.2. *Let X be a Banach space. Then $\mathcal{N}(X)$ is an ideal of $B(X)$.*

01/04/2025

Definition 1.3.7. Let V, W be normed spaces. A crossed norm on $V \otimes W$ is a norm satisfying $\|v \otimes w\| \leq \|v\|\|w\|$.

Definition 1.3.8. Let X, Y be normed spaces. A tensor norm on $X \otimes Y$ is a crossed norm such that $X^* \otimes Y^* \subset (X \otimes Y)^*$, and $\|\phi \otimes \psi\| \leq \|\phi\|\|\psi\|$ for all $\phi \in X^*, \psi \in Y^*$.

Lemma 1.3.3. *If $\|\cdot\|$ is a tensor norm on $X \otimes Y$, then*

$$\|x \otimes y\| = \|x\|\|y\|$$
$$\|\phi \otimes \psi\| = \|\phi\|\|\psi\|$$

for all $x \in X, y \in Y, \phi \in X^*, \psi \in Y^*$.

Proof. Exercise □

Recall that the projective norm is a tensor norm and it is the largest. Also recall that injective norm is a tensor norm, which is the smallest.

Theorem 1.3.5. *If $Z \leqslant X$, then*

$$Z\check{\otimes}Y \leqslant X\check{\otimes}Y$$

Proof. Immediate from the definition of the injective tensor norm. □

Moreover note that $X \otimes Y$ has an injection to $\mathrm{Bil}(X^* \times Y^*)$ as evaluations. Also note that $\mathrm{Bil}(X^* \times Y^*)$ has a norm where $\phi \in \mathrm{Bil}(X^* \times Y^*)$ has norm

$$\|\phi\| = \sup\{|\phi(x,y)| \;:\; \|x\|, \|y\| \leq 1\}$$

Then the push-back norm into $X \otimes Y$ is the injective tensor norm.

If instead $X \otimes Y$ borrows the norm via the injection $X \otimes Y \hookrightarrow \mathrm{Bil}(X \times Y)^*$ again via the obvious injection. This push-back gives the projective tensor norm.

Recall that $(X\hat{\otimes}Y)^* = \mathrm{Bil}(X \times Y)$

Theorem 1.3.6. *Let $\rho \in Bil(X \times Y)$. Then, $\rho \in (X\check{\otimes}Y)^*$ iff there is a radon measure (from Reisz representation) μ in $X_1^* \times Y_1^*$ such that*

$$\rho(x \otimes y) = \int_{X_1^* \times Y_1^*} \phi(x)\psi(y)\; d\mu(\phi, \psi)$$

In this case, we say ρ is an integral Bilinear map.

Proof. Use $z \to f_z$: □

Definition 1.3.9. Let X, Y be Banach spaces. Let α be a tensor norm on $X \otimes Y$. We define the dual norm α^* on $X^* \otimes Y^*$ via the embedding

$$X^* \otimes Y^* \hookrightarrow (X \otimes Y)^*$$

Lemma 1.3.4. *α^* is a tensor norm.*

Definition 1.3.10. Let $\Xi : X \otimes Y \to Y \otimes X$ be the flip map. The flip of α denoted α_f is $\alpha \circ \Xi$.

clearly the projective and injective norms are flip invariant.

Exercise 1.3.4. Let X, Y be finite dimensional normed spaces. Then

(1) $\wedge^* = \vee$ and $\vee^* = \wedge$

(2) $(\alpha^*)^* = \alpha$

Proof. (1)

(2) Let $\rho \in X^* \otimes Y^*$. By definition,

$$\begin{aligned}\|\rho\|_{\wedge^*} &= \sup\{|\rho(z)| \ : \ z \in (X\hat{\otimes}Y)_1\} \\ &\geq \|\rho\|_{\vee} \\ &= \sup\{\rho(x \otimes y) \ : \ x \in X_1, y \in Y_1\}\end{aligned}$$

Let $\varepsilon > 0$, and let $z = \sum_{i=1}^{N} x_i \otimes y_i \in (X \otimes Y)_1$ be such that $\|z\|_{\wedge} > \sum_{i=1}^{n} \|x_i\|\|y_i\| - \varepsilon$ and $\sum_{i=1}^{n} \rho(x_i \otimes y_i) = \rho(z) \geq \|\rho\|_{\wedge^*} - \varepsilon$. Then

$$\sum_{i=1}^{n} \rho(x_i \otimes y_i) \leq \|\rho\|_{\wedge} \sum_{i=1}^{n} \|x_i\|\|y_i\| \leq \|\rho\|_{\wedge}()$$

□

08/04/2025

Lemma 1.3.5. *Let α be a tensor norm on $X \otimes Y$. The canonical algebraic vector space isomorphism between $(X \otimes Y)^\circ \to \{T : X \times Y \to \mathbb{C} \mid T \text{ is bilinear}\}$, maps $(X \otimes Y, \alpha)^\circ$ into bounded bilinear maps.*

Proof. Let $\phi \in (X \otimes Y, \alpha)^\circ$. Let $\rho : X \times Y \to \mathbb{C}$, $\rho(x, y) = \phi(x \otimes y)$. □

Definition 1.3.11. A linear map $T : X^* \to Y$ is called an integral operator if the corresponding bilinear map $\rho : X^* \times Y^* \to \mathbb{C} := (\phi, \psi) \mapsto \psi(T(\phi))$ is integral.

Example 1.3.1. Let $T : \ell^1 \to \mathbf{c}_0$ be the canonical inclusion. Define $\rho : \ell^1 \times \ell^1 \to C$ by $\rho(f, g) = \sum_{n\in\mathbb{N}} f(n)\overline{g(n)}$. Now we look at the measure which makes this an integral bilinear map. Let ν be the normalized lebesgue maeusre on $\mathbb{T}$. Let $\tilde{\nu} := \nu^{\mathbb{N}} \in \text{Prob}(\mathbb{T}^{\mathbb{N}})$

15/04/2025

Definition 1.3.12. Let α be a family of tensor norms defined uniquely for $X \otimes Y$ for any finite dimensional normed spaces X and Y such that if X_1, X_2, Y_1, Y_2 are finite dimensional normed spaces and $T : X_1 \to X_2$ and $S : Y_1 \to Y_2$ are linear maps, then $\|T \otimes S\| \leq \|T\|\|S\|$. In this case we say, α is a *good* family of tensor norms.

Here the word *uniquely* mean if $X_1 \overset{iso}{\cong} X_2$ and $Y_1 \overset{iso}{\cong} Y_2$, then α takes same values on $X_1 \otimes Y_1$ and $X_2 \otimes Y_2$ via the isomorphism.

Examples are projective and injective tensor norms.

Exercise 1.3.5. Show that if α is a *good* norm. Then its flip norm α_f and the dual norm α^* are also *good* norms.

Definition 1.3.13. Let α be a *good* family of tensor norms. Let X, Y be normed spaces. Given $z \in X \otimes Y$, define

$$\|z\|_\alpha = \inf\{\|z\|_{E\otimes F,\alpha} \ : \ E, F \text{ are finite dimensional subspaces of } X, Y, z \in E \otimes F\}$$

Lemma 1.3.6. *The above defines a tensor norm on* $X \otimes Y$

Proof. We only need to show triangle inequality and that $\|z\|_\alpha = 0 \implies z = 0$. Triangle inequality follows since if $z_1 + z_2 \subset E \otimes F$ demands $z_1 \in E \otimes F$ and $z_2 \otimes E \otimes F$, shows that the supremums are taken over a smaller set in case of $z_1 + z_2$. Hence triangle inequality satisfies.

To show the other one, we'll show that $\|z\|_\alpha \geq \|z\|_\vee$ and since the injective tensor product preserve injections, this inequality follows in the set where inf giving the inequality outside inf set.

To show that this is a tensor norm, let $x \in X, y \in Y$. Then

$$\begin{aligned}\|x \otimes y\|_\alpha &= \inf\{\|x \otimes y\|_{E\otimes F,\alpha} \ : \ x \otimes y \in E \otimes F\} \\ &= \inf\{\|x \otimes y\|_{E\otimes F,\alpha} \ : \ x \in E, y \in F\} \\ &= \|x\|\|y\|\end{aligned}$$

verify the second line.

Now let $\phi \in X^*, \psi \in Y^*$ and let $z \in (X \otimes Y, \alpha)$. Then $\forall E \overset{f.d}{\leqslant} X, F \overset{f.d}{\leqslant} Y$, with $z \in E \otimes F$, we have

$$\begin{aligned}\Big|(\phi \otimes \psi)(z)\Big| &= \Big|(\phi|_E \otimes \psi|_F)(z)\Big| \\ &\leq \|\phi|_E\|_{E^*}\|\psi|_F\|_{F^*}\|z\|_{(E\otimes F,\alpha)}\end{aligned}$$

□

Lemma 1.3.7. *Let α be a good famliy. $T : X_1 \to X_2$, $T_2 : Y_1 \to Y_2$ be bounded linear maps. Then $T \otimes S$ is bounded and $\|T \otimes S\| \leq \|T\|\|S\|$.*

Lemma 1.3.8. *For any normed spaces X, Y and $z \in X \otimes Y$, $\|z\|_\wedge = \|z\|_\wedge$. That is projective tensor norm is a good norm.*

17/04/2025

Definition 1.3.14. Let $\lambda \geq 1$ be a Banach space. X is said to have λ-approximation property if three exist a net ϕ_i of finite rank maps on X with $\|\phi_i\| \leq \lambda$ such that $\phi_i \to I$ uniformly on compact subsets of X.

In particular if $\lambda = 1$, then we say the space has metric approximation property.

Theorem 1.3.7. *If X, Y be Banach spaces with metric approximation property. Let α be a good family of tensor norms. Then the canonical map*

$$X \otimes^{\alpha} Y \to (X^* \otimes^{\alpha^*} Y^*)^*$$

is isometric. Note here that here α^* comes from the good family of norms in finite dimensional spaces.

Proof. verify that this is a contraction. Notice that if $\rho \in X^* \otimes Y^*$, the algebraic tensor product, then

$$\|\rho\|_{\alpha^*} = \inf\{\|\rho\|_{(E\otimes F, \alpha^*)} \ : \ \rho = \rho|_{E\otimes F}\}$$

Let $z = \sum_{i=1}^n x_i \otimes y_i$. By metric approximation property, there are finite rank maps $\phi : X \to X, \psi : Y \to Y$ such that $\|\phi\|, \|\psi\| \leq 1$, and $\|\phi(x_i) - x_i\| < \varepsilon$, $\|\psi(y_i) - y_i\| < \varepsilon$ for all i. Let $E' = \mathrm{Im}(\phi), F' = \mathrm{Im}(\psi)$. Then

$$\|(\phi \otimes \psi)(z) - z\| = \sum_{i=1}^n \|(\phi(x_i) \otimes \psi(y_i)) - x_i \otimes y_i\| \leq \varepsilon'$$

Now for a general z, since we need to take inf over all such finite dimensional representations, we have $\|z\|_\alpha \leq \|z\|_{E'\otimes^\alpha F'}$.

Choose $\rho \in (E' \otimes^\alpha F')^* = (E')^* \otimes^{\alpha^*} (F')^*$, such that $\|\rho\| = 1$ and $|\rho((\phi \otimes \psi)(z))| = \|\phi \otimes \psi(z)\|_{E'\otimes^\alpha F'}$. Then

$$\begin{aligned}
\|z\|_\alpha &\leq \|(\phi \otimes \psi)(z)\|_{X\otimes^\alpha Y} + \varepsilon \\
&\leq \|(\phi \otimes \psi)(z)\|_{E'\otimes F'} + \varepsilon \\
&= |\langle(\phi \otimes \psi)(z), \rho\rangle| + \varepsilon \\
&= |\langle z, (\phi^* \otimes \psi^*)(\rho)\rangle| + \varepsilon \\
&\leq \|z\|_{(X^*\otimes^{\alpha^*} Y^*)^*}
\end{aligned}$$

□

22/04/2025

1.4 Tensor Products of C* Algebras

Definition 1.4.1. Recall that a C* algebra is a Banach space with a continuous multiplication turning it into an algebra such that $\|ab\| \leq \|a\|\|b\|$ and $\|a^*\| = \|a\|$, and an isometric involution $*$ such that $\|a^*a\| = \|a\|^2$.

Given two algebras $\mathcal{A}, \mathcal{B}$, the algebraic tensor product admits a canonical algebra structure

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb'$$

and same holds if it is a $*$-algebra with

$$(a \otimes b)^* = a^* \otimes b^*$$

Definition 1.4.2. Let $\mathcal{A}, \mathcal{B}$ be two unital C* algebras. By a C* norm on the algebraic tensor product, we mean a norm satisfying

(1) $\|a \otimes b\| \leq \|a\|\|b\|$

(2) $\|z^* z\| \leq \|z\|^2, \ \forall z \in A \otimes B$

Example 1.4.1. Consider $C(X) \otimes C(Y)$, where X, Y are compact Hausdorff spaces. Then $C(X) \otimes C(Y)$ sits inside $C(X \times Y)$ densely (Stone-Weierstrass) via $f_i \otimes g_i \to ((x, y) \to f_i(x) g_i(x))$

Example 1.4.2. Similarly $B(H) \otimes B(K) \to B(H \otimes_2 K)$.

Lemma 1.4.1. *Every $*$-homomorphisms between C^* algebra is contractive*

Proof. Let $\phi : A \to B$ be the $*$-homomorphism. Then for all $a \in \mathcal{A}$,

$$\|a\|^2 = \|a^* a\| = r(a^* a) \geq r(\phi(a^* a)) = r(\phi(a)^* \phi(a)) = \|\phi(a)^* \phi(a)\| = \|\phi(a)\|^2$$

$\square$

Theorem 1.4.1. *Let $(\mathcal{A}, \|\cdot\|)$ be a C^* algebra. Suppose $\|\cdot\|'$ is a pre-C^* norm on $\mathcal{A}$. Then $\|\cdot\| = \|\cdot\|'$.*

Example 1.4.3. Let $\mathcal{A}$ be a unital C* algebra, and $n \in \mathbb{N}$. Then $\exists!$ C* norm on $M_n(\mathbb{C}) \otimes \mathcal{A}$

Definition 1.4.3. Let $\mathcal{A}, \mathcal{B}$ be unital C* algebra. Define $\forall z \in A \otimes B$,

$$\begin{aligned} \|z\|_{\max} &= \sup\{\|z\| \ : \ \|\cdot\| \text{ is a C* norm on } A \otimes B\} \\ &= \sup\{\|\pi(z)\| \ : \ \pi : A \otimes B \to B(H) \text{ is a representation.}\} \end{aligned}$$

note that this norm we defined will have its upper bound to be the projective tensor norm of A, B viewed as Banach spaces.

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Theorem 1.4.2. *Let $\mathcal{A}, \mathcal{B}$ be unital C^* algebras and $\mathcal{H}, \mathcal{H}', \mathcal{K}, \mathcal{K}'$ be Hilbert spaces, and $\pi : \mathcal{A} \to B(\mathcal{H}), \pi' : \mathcal{A} \to B(\mathcal{H}')$, $\sigma : \mathcal{B} \to B(\mathcal{K}), \sigma' : \mathcal{B} \to B(\mathcal{K}')$ be injective $*$-homomorphisms. Then $\forall z \in A \otimes B$,*

$$\|(\pi \otimes \sigma)(z)\|_{B(H \otimes K)} = \|(\pi' \otimes \sigma')(z)\|_{B(H' \otimes K')}$$

Proof. Let P_n be sequence of finite rank projections that converge to I in the strong operator topology. Because the result holds in finite dimensions,

$$\|(I \otimes P_n)((\pi \otimes \sigma)(z))(I \otimes P_n)\|_{B(\mathcal{H} \otimes \mathcal{K})} = \|(I \otimes P_n)((\pi' \otimes \sigma)(z))(I \otimes P_n)\|_{B(\mathcal{H}' \otimes \mathcal{K})}$$

Now taking limits as $n \to \infty$, and repeating the same steps 3 more times, we get our result. □

Definition 1.4.4. The above unique norm is called the minimal tensor product of $\mathcal{A}$ and $\mathcal{B}$, denoted $\otimes_{\min}$.

Theorem 1.4.3. *$\otimes_{min}$ is the smallest tensor norm on $\mathcal{A} \otimes \mathcal{B}$.*

Proof. Assuming $\|\phi \otimes \psi\|_{(A \otimes_{\min} B)} = \|\phi\|\|\psi\|$ for all $\phi \in \mathcal{A}^*$ such that $\phi(a^*a) > 0$ for all $a \in \mathcal{A} \setminus \{0\}$. Similarly for $\mathcal{B}$.

Given $a_1, a_2 \in \mathcal{A}$, define $\langle a_1, a_2 \rangle = \phi(a_2^* a_1)$. This defines an inner product on $\mathcal{A}$. Denote by $\mathcal{H}_\phi$, the Hilbert space completion. Denote by $\Lambda : \mathcal{A} \to \mathcal{H}_\phi$. And do GNS constructions. □

Let $\mathcal{A}_1 \leqslant \mathcal{A}_2 \leqslant \ldots \leqslant \mathcal{A}$ be an increasing sequence of C* algebras where $\mathcal{A}_n \cong M_n(\mathbb{C})$, and $\overline{\cup_{n \in \mathbb{N}} \mathcal{A}_n} = \mathcal{A}$. This implies for all C* algebra B,

$$A \otimes_{\max} = \mathcal{A} \otimes_{\min} \mathcal{B}$$

Discussion about the max and min tensor product of $C_\lambda(\mathbb{F}_2)$ with $C_\rho(\mathbb{F}_2)$ are not the same.