

## MATH 6320

Theory of Functions of a Real Variable  
Fall 2024

First name: \_\_\_\_\_ Last name: \_\_\_\_\_

Points:

**Assignment 8, due Thursday, November 7, 11:30am**

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

**Problem 1**

Let  $f$  be a measurable complex-valued function on a locally compact Hausdorff space  $X$  with a Borel measure  $\mu$ . Let for  $w \in \mathbb{C}$ ,  $\epsilon > 0$ ,

$$A_{w,\epsilon} = \{x \in X : |f(x) - w| < \epsilon\}$$

and

$$R_f = \{w \in \mathbb{C} : \text{for each } \epsilon > 0; \mu(A_{w,\epsilon}) > 0\}.$$

Show that  $R_f$  is a closed subset of  $\mathbb{C}$ .

**Problem 2**

A step function on  $\mathbb{R}$  is a finite linear combination of characteristic functions of bounded intervals in  $\mathbb{R}$ . Let  $m$  be the Lebesgue measure and let  $f \in L^1(m)$ . Show that for any  $\epsilon > 0$ , there is a step function  $g$  such that  $\int |f - g| dm < \epsilon$ . Hint: First prove this for a function  $f$  which is bounded and for which  $A = \{x \in \mathbb{R} : f(x) \neq 0\}$  has finite measure  $m(A) < 1$ . Then use that for a general  $f$ ,  $f|_B$  has these properties where  $B = \{x \in \mathbb{R} : \frac{1}{n} \leq |f(x)| \leq n\}$  and consider  $n \rightarrow \infty$ .

**Problem 3**

Let  $f \in L^1(m)$ , where  $m$  is the Lebesgue measure on  $\mathbb{R}$ . For  $t \in \mathbb{R}$ , let  $f_t : x \mapsto f(x - t)$ . Show that

$$\lim_{t \rightarrow 0} \int |f_t - f| dm = 0.$$

Hint: Use the preceding problem.