Department of Mathematics University of Houston

MATH 6321

Theory of Functions of a Real Variable
Spring 2025

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 8, due Thursday, April 3, 10am

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let $1 \leq p < \infty$ and q be the conjugate exponent. Let μ be a σ-finite measure on X. Show that if $g : X \to \mathbb{C}$ is measurable and for each $f \in L^p(\mu)$, we have $fg \in L^1(\mu)$, then $g \in L^q(\mu)$. Hint: Consider $X = \cup_{j=1}^{\infty} X_j$ with $\mu(X_j) < \infty$ and $E_n = \{x \in X_n : |g(x)| \leq n\}$ to turn g into 'nicer' functions $g_n = g\chi_{E_n}$.

Problem 2

Let $(X, \mathcal{M}, \mu)$ be a measure space. A set $\Phi \subset L^1(\mu)$ is called *uniformly integrable* if for each $\epsilon > 0$, there is $\delta > 0$ such that if $E \in \mathcal{M}$ satsifies $\mu(E) < \delta$, then for each $f \in \Phi$, $\left|\int_E f d\mu\right| < \epsilon$. Show that if $\mu(X) < \infty$, $f_n \to f$ μ-a.e., $\{f_n : n \in \mathbb{N}\}$ is uniformly integrable, and $|f(x)| < \infty$ for almost every $x \in X$, then $f \in L^1(\mu)$ and $\|f_n - f\|_1 \to 0$. Hint: Use Egoroff and Fatou, splitting f_n into positive and negative real and imaginary parts.

Problem 3

Let μ be a finite measure on X, $(f_n)_{n=1}^{\infty}$ a sequence of functions in $L^1(\mu)$ that converges almost everywhere to a function $f : X \to \mathbb{C}$, and for some $p > 1$, there is $C \geq 0$ such that $\|f_n\|_p^p \leq C$. Show that then $f \in L^1(\mu)$ and $\|f_n - f\|_1 \to 0$.