

MATH 6320

Theory of Functions of a Real Variable
Fall 2024

First name: _____ Last name: _____

Points:

Assignment 7, due Thursday, October 31, 11:30am

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Find an example of a sequence of Riemann integrable functions $(f_n)_{n=1}^{\infty}$, $f_n : [0; 1] \rightarrow [0; 1]$ so that $\lim_{n \rightarrow \infty} \int_{[0;1]} f_n dm = 0$ and $g = \sup_{n \in \mathbb{N}} f_n$ is not in $L^1(m)$, where m is the Lebesgue measure on $[0; 1]$. Next, explain how you can modify your example to obtain the same property for a sequence of continuous functions $(f_n)_{n=1}^{\infty}$. Hint: Start with g and let $f_n = \chi_{I_n} g$, where each I_n is a closed interval.

Problem 2

Let $(X; \mathcal{M}; \mu)$ be a measure space with $\mu(X) < \infty$. Let $f : X \rightarrow [0; \infty)$ be bounded, measurable. Prove that

$$\int_X f d\mu = \inf_{j=1}^{\infty} \sum_{j=1}^{\infty} \mu(A_j) \sup_{x \in A_j} f(x) : A_j \text{'s partition } X$$

where it is assumed that each partition contains only measurable $A_j \in \mathcal{M}$. Also show that this identity can fail if the assumption that f bounded is replaced by $f \in L^1(\mu)$.

Problem 3

Let V be an open set in \mathbb{R} and μ a Borel measure on \mathbb{R} such that $\mu(\mathbb{R}) < \infty$. Show that $f(x) = \mu(x+V)$ defines a lower semicontinuous function f on \mathbb{R} .