MATH 6321 - Theory of Functions of a Real Variable
Homework III

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1. **Solution:**

   (a) Let

   $$A^M = \{x \in X \;:\; |f_n(x)| < M,\ \forall n \in \mathbb{N}\}$$

   Since for all $x \in X$, $f_n(x) \to f(x) \in \mathbb{C}$, there is some $M \in \mathbb{N}$ such that $|f_n(x)| < M$ for all $n \in \mathbb{N}$. Then $x \in A^M$. Thus we see that

   $$X = \bigcup_{M \in \mathbb{N}} A^M$$

   Then by the Baire category theorem, we get that for one of $M_0 \in \mathbb{N}$, $\overline{A^{M_0}}$ has a nonempty interior. Let $V \subset \overline{A^{M_0}}$ be a non-empty open set. We'll show that V is the set we're looking for.

   Let $x \in V$, and $n \in \mathbb{N}$ be given. Then by the continuity of f_n, there exists an $y \in A^{M_0} \cap V$ such that $|f_n(x) - f_n(y)| < 1$. Thus

   $$|f_n(x)| < |f_n(y)| + 1 < M_0 + 1$$

   Since $n \in \mathbb{N}$ was arbitrary, we get that this holds for all $n \in \mathbb{N}$. Hence V is the non-empty open set and $M_0 + 1$ is the integer we need.

   (b) Let $\varepsilon > 0$ be given, and

   $$A^N = \left\{x \in X \;:\; |f_n(x) - f_m(x)| \leq \frac{\varepsilon}{2},\ \forall n, m \geq N\right\}$$

Since $f_n(x) \to f(x)$ pointwise, we get that

$$X = \bigcup_{N \in \mathbb{N}} A^N$$

Moreover, since each

$$A^N = \bigcap_{n,m > N \in \mathbb{N}} (f_n - f_m)^{-1}(\bar{B}_{\frac{\varepsilon}{2}}(0))$$

we get that each A^N is closed being the intersection of closed sets. Now by the same argument as for the previous part, there exist a non-empty open set $W \subset \overline{A^{N_0}} = A^{N_0}$ for some $N_0 \in \mathbb{N}$. We'll show that W is the non-empty open set and N_0 is the integer we need.

Let $x \in W$ and $n \geq N_0$. Since $f_n(x) \to f(x)$, there exists an $m \geq N_0$ such that

$$|f_m(x) - f(x)| < \frac{\varepsilon}{2}$$

Then for that m, we get that

$$\begin{aligned}
|f_n(x) - f(x)| &\leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

where $|f_n(x) - f_m(x)| \leq \frac{\varepsilon}{2}$ since $x \in W \subset A^{N_0}$ and $n, m \geq N_0$. Hence we are done.

2. **Solution:**

(a) We'll first show that $\|M_f\| \leq \|f\|_\infty$. To see this, notice that for any $g \in L^2(\mu)$, we have

$$\|fg\|_2^2 = \int |fg|^2 \, d\mu \leq \int \|f\|_\infty^2 |g|^2 \, d\mu = \|f\|_\infty^2 \|g\|_2^2$$

Thus $\|M_f g\|_2 = \|fg\|_2 \leq \|f\|_\infty \|g\|_2$. Now it follows from the definition of the operator norm, that $\|M_f\| \leq \|f\|_\infty$.

Definition 0.1. A measure μ on (X, Σ) is called seminfinite whenever $\mu(A) = \infty$, there is a $\Sigma \ni B \subset A$ such that $0 < \mu(B) < \infty$.

We'll show that seminiteness of the measure μ is sufficient for $\|M_f\| = \|f\|_\infty$ for all $f \in L^\infty(\mu)$. Let $f \in L^\infty(\mu)$ be given. Let

$$A_n = \left\{ x \in X \ : \ \|f\|_\infty^2 < |f(x)|^2 + \frac{1}{n} \right\}$$

By the definition of the essential supremum, we get that $\mu(A_n) > 0$ for each $n \in \mathbb{N}$. Let $B_n \subset A_n$ be a measurable set such that $0 < \mu(B_n) < \infty$. Now, let $g_n(x) = \frac{1}{\mu(B_n)} \chi_{B_n}$, clearly $\|g_n\|_2 = 1$, and

$$\begin{aligned}
\|M_f(g_n)\|_2^2 &= \int |f g_n|^2 \, d\mu \\
&= \frac{1}{\mu(B_n)} \int_{B_n} |f|^2 \, d\mu \\
&\geq \frac{1}{\mu(B_n)} \int_{B_n} \|f\|_\infty^2 - \frac{1}{n} \, d\mu \\
&= \frac{1}{\mu(B_n)} \int_{B_n} \|f\|_\infty^2 \, d\mu \ - \ \frac{1}{\mu(B_n)} \int_{B_n} \frac{1}{n} \, d\mu \\
&= \|f\|_\infty^2 - \frac{1}{n}
\end{aligned}$$

Hence

$$\sqrt{\|f\|_\infty^2 - \frac{1}{n}} \ \leq \ \|M_f(g_n)\|_2 \ \leq \ \|f\|_\infty \|g\|_2 = \|f\|_\infty$$

Thus, as $n \to \infty$, we see that $\|M_f(g_n)\|_2 \to \|f\|_\infty$. Hence $\|M_f\| = \|f\|_\infty$. Since $f \in L^\infty(\mu)$ was arbitrary, this holds for every $f \in L^\infty(\mu)$.

(b) We claim that the M_f is an onto function from $L^2(\mu) \to L^2(\mu)$ if

$$0 \ \notin \ R_f := \{ w \in \mathbb{C} \ : \ \mu(f^{-1}(B_\varepsilon(w))) > 0, \ \forall \varepsilon > 0 \}$$

the essential range of f. If $0 \notin R_f$, then there exist $\varepsilon > 0$ such that $\mu(f^{-1}(B_\varepsilon(0))) = 0$. That implies $|f(x)| \geq \varepsilon$ almost everywhere. Then $\frac{1}{|f(x)|} < \frac{1}{\varepsilon}$ almost everywhere. Hence $\frac{1}{f} \in L^\infty(\mu)$. Then by the first part, for any $g \in L^2(\mu)$,

$$\frac{g}{f} \in L^2(\mu)$$

Hence $M_f(\frac{g}{f}) = g$, which shows that M_f is onto.