Hmk 6, 4.3] #28) Suppose G is a non-Abelian group of order pq, $p<q$ primes.

- Prove that G has a non-normal subgroup of index q.
- Prove that there is an injective homomorphism from G into S_q.
- Prove that G is isomorphic to a subgroup of the normalizer in S_q of the q cycle $(1\ 2 \cdots q)$.

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7) Ex: Suppose we color the 4 vertices of a square each w/ one of 3 colors. Two colorings are the same if they are the same up to rigid motions. How many different colorings are there, with this identification?

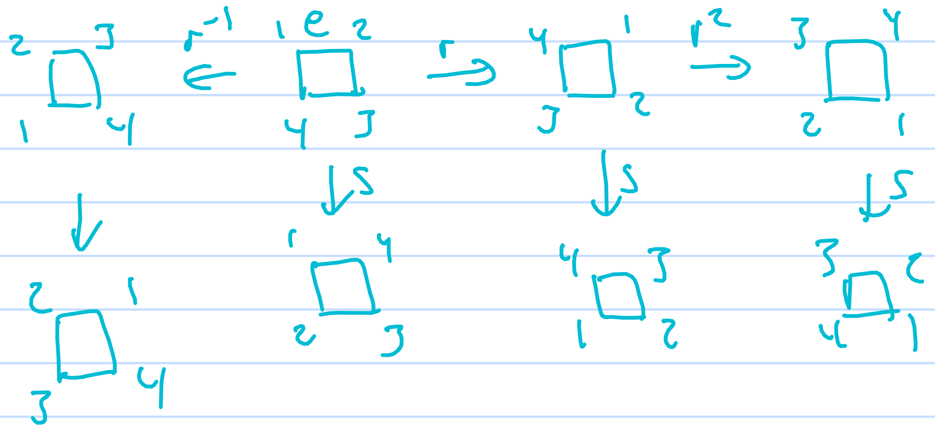
Solution using Burnside's lemma: Finite $G \curvearrowright A$,

$$|A/G| = \frac{1}{|G|} \sum_{g \in G} |A^g|.$$

Let $G = D_8$, $A = \{$all 81 colorings of the vertices of the square using 3 colors$\}$

g	$\lvert A^g \rvert$
e	81
r	3
r^2	9
r^3	3
s	27
sr	9
sr^2	27
sr^3	9

Scratch:



$$|A/G| = \frac{1}{8}(81+3+9+3+27+9+27+9) = \boxed{21}$$

Automorphisms

An automorphism of a group G is a bijective homom. of G to itself. The collection of all automorphisms $\mathrm{Aut}(G)$ is a group under composition of maps.

Prop: If $H \trianglelefteq G$ then:

i) G acts on H by conjugation.

ii) For each $g \in G$, the associated element of S_H is an element of $\mathrm{Aut}(H)$.

iii) The kernel of the associated permutation representation $\varphi: G \to \mathrm{Aut}\, H$ is

$$\ker \varphi = C_G(H).$$

Pf: i) ✓

ii) Fix $g \in G$. Let $\varphi: G \to S_H$ be the perm. rep.

Want to show that $\varphi(g): H \to H$ is a homom.

(already knew that it is a bijection)

$\forall h_1, h_2 \in H$,

$$g \cdot (h_1 h_2) = g h_1 h_2 g^{-1} = (g h_1 g^{-1})(g h_2 g^{-1}) = (g \cdot h_1)(g \cdot h_2). ✓$$

iii) $\forall g \in G$,

$$g \in \ker\varphi \iff \forall h \in H,\ g.h = h$$
$$\iff \forall h \in H,\ gh = hg$$
$$\iff g \in C_G(H). \ \boxed{}$$

Cor: If G is a group and $H \leq G$ then $N_G(H)/C_G(H)$ is isomorphic to a subgroup of $\mathrm{Aut}(H)$.

Pf: Apply the prop. from before, noting that $H \trianglelefteq N_G(H)$, and use the 1st Isom Thm. $\boxed{}$

Def: Let $G \curvearrowright G$ by conjugation and let $\varphi: G \to \mathrm{Aut}(G)$ be the associated perm. rep. Define $\varphi(G) = \mathrm{Inn}(G)$, the group of <u>inner automorphisms</u> of G.

Note: By the prop. (or its corollary) $\mathrm{Inn}(G) \cong G/Z(G)$. $\quad (Z(G) = C_G(G))$

Exs: 1a) Suppose G is Abelian. Then $Z(G)=G$,

so $\mathrm{Inn}(G) \cong G/Z(G) = \{1\}$.

b) Let $G = \mathbb{Z}/n\mathbb{Z}$. Then $\mathrm{Aut}(G) \cong (\mathbb{Z}/n\mathbb{Z})^\times$.

Pf: Any homomorphism $\tau: \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is uniquely determined by $\tau(1)$. Such a homom. will be an isom. $\Leftrightarrow$ $(\tau(1), n) = 1$. Therefore there are $\varphi(n)$ (distinct) autom's.

Furthermore, the map $\gamma: \mathrm{Aut}(\mathbb{Z}/n\mathbb{Z}) \to (\mathbb{Z}/n\mathbb{Z})^\times$ defined by $\gamma(\tau) = \tau(1)$ is an isomorphism:

Let $\tau_1, \tau_2 \in \mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})$. Then

$$\gamma(\tau_1\tau_2) = (\tau_1\tau_2)(1) = \tau_1(\tau_2(1))$$

$$= \tau_1(\underbrace{1+1+\cdots+1}_{\tau_2(1)\text{-times}})$$

$$= \underbrace{\tau_1(1)+\cdots+\tau_1(1)}_{\tau_2(1)\text{-times}} = \tau_1(1)\,\tau_2(1). \quad \ldots \checkmark . \checkmark \blacksquare$$