

Midterm exam: Monday, Oct. 14, in class, 50 mins.

5 questions

Q1] General knowledge of groups.

How to tell when groups are not isomorphic.

Q2] Prove that two groups are isomorphic.

Q3] Prove or disprove the following statement:

(something similar to one of the problems on A4)

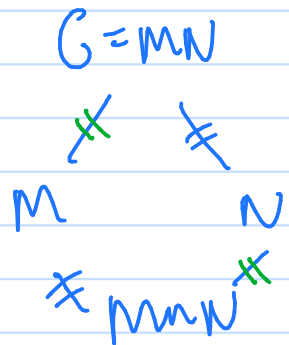
Q4] Know props. of homomorphisms.

Q5] Know def. of $C_G(S)$.

This question is based on the notes on internal products.

Hints for A5]

7. Let M and N be normal subgroups of G such that $G = MN$. Prove that $G/(M \cap N) \cong (G/M) \times (G/N)$. [Draw the lattice.]



Idea 1:

Find a homom.

$\phi: G \rightarrow G/M \times G/N$, show that $\ker \phi = MN$.

Try $\phi(g) = (gM, gN)$.

- ϕ is a hom. ✓

- $\ker \phi = MN$ ✓

- ϕ is onto (the hard part)

Idea 2: By 2nd Isom. thm: $G/M \cong N/MN$

and $G/N \cong M/MN$

"Define" $\phi: G \rightarrow N/MN \times M/MN$,

by $\phi(mn) = (n(MN), m(MN))$.

• well-def? N.T.S:

If $m_1 n_1 = m_2 n_2$ then

$$n_1 (M \cap N) = n_2 (M \cap N) \text{ and } m_1 (M \cap N) = m_2 (M \cap N).$$

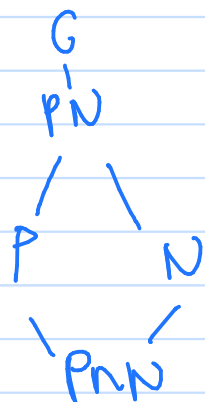
• hom?

hard.

Gap: $\ker \phi = M \cap N$, ϕ is onto.

9. Let p be a prime and let G be a group of order $p^a m$, where p does not divide m . Assume P is a subgroup of G of order p^a and N is a normal subgroup of G of order $p^b n$, where p does not divide n . Prove that $|P \cap N| = p^b$ and $|PN/N| = p^{a-b}$. (The subgroup P of G is called a Sylow p -subgroup of G . This exercise shows that the intersection of any Sylow p -subgroup of G with a normal subgroup N is a Sylow p -subgroup of N .)

$$|G| = p^a m, p \nmid m, |P| = p^a, |N| = p^b n, p \nmid n$$



$$PN/N \cong P/P \cap N$$

$$\Rightarrow |PN| = \frac{|P| |N|}{|P \cap N|} = \frac{p^{a+b} n}{|P \cap N|}$$

use this to deduce that $|P \cap N| = p^b$.

8. Let A be a finite abelian group (written multiplicatively) and let p be a prime. Let

$$A^p = \{a^p \mid a \in A\} \quad \text{and} \quad A_p = \{x \mid x^p = 1\}$$

(so A^p and A_p are the image and kernel of the p^{th} -power map, respectively).

(a) Prove that $A/A^p \cong A_p$. [Show that they are both elementary abelian and they have the same order.]

(b) Prove that the number of subgroups of A of order p equals the number of subgroups of A of index p . [Reduce to the case where A is an elementary abelian p -group.]

Part b)

Step 1: Prove that the # of subgroups of order p in A is equal to the # of subgroups of order p in A_p .

• Prove that # of subgroups of index p in A is equal to the # of subgroups of index p in A/A^p .

By (a), this reduces the problem to the case when A is elementary p -group.

Step 2: Count both quantities, when A is an elem. p -group.