MATH6320 - Theory of Functions of a Real Variable
Assignment 9

Joel Sleeba

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1. **Solution:**

   (a) Let $r < p < s$, where $r, s \in E$. Then by the convexity of $[r, s] \subset \mathbb{R}$, there is a $t \in [0, 1]$ such that $p = tr + (1-t)s$. Then Holder's inequality on $\frac{1}{t}$ and $\frac{1}{(1-t)}$ gives,

$$\begin{aligned}
\int |f|^p \, d\mu &= \int |f|^{tr}|f|^{(1-t)s} \, d\mu \\
&\leq \left(\int |f|^{\frac{tr}{t}} \, dm\right)^t \left(\int |f|^{\frac{(1-t)s}{(1-t)}} \, dm\right)^{1-t} \\
&= \left(\int |f|^r \, dm\right)^t \left(\int |f|^s \, dm\right)^{1-t} \\
&= \|f\|_r^{rt} \|f\|_s^{s(1-t)}
\end{aligned}$$

   Thus we get $\|f\|_p \leq \|f\|_r^{\frac{rt}{p}} \|f\|_s^{\frac{s(1-t)}{p}}$

   For the sake of contradiction, assume that $\|f\|_p > \max\{\|f\|_r, \|f\|_s\}$. Then by the monotonicity of the function $x \to x^k$, where $k > 0$, we get

$$\|f\|_p^{\frac{rt}{p}} > \|f\|_r^{\frac{rt}{p}} \quad \text{and} \quad \|f\|_p^{\frac{s(1-t)}{p}} > \|f\|_s^{\frac{s(1-t)}{p}}$$

   Then we'll get

$$\|f\|_p = \|f\|_p^{\frac{rt}{p}} \|f\|_p^{\frac{s(1-t)}{p}} > \|f\|_r^{\frac{rt}{p}} \|f\|_s^{\frac{s(1-t)}{p}}$$

   contradicting our previous result. Hence we see that $\|f\|_p \leq \max\{\|f\|_r, \|f\|_s\}$

(b) Let $0 < \epsilon$. Consider the set $A_\epsilon = \{x \in X \ : \ \|f\|_\infty < |f(x)| + \epsilon\}$. Then

$$\begin{aligned}\int_X |f|^p \, d\mu &\geq \int_{A_\epsilon} |f|^p \, d\mu \\ &\geq \int_{A_\epsilon} (\|f\|_\infty - \epsilon)^p \, d\mu \\ &= (\|f\|_\infty - \epsilon)^p \mu(A_\epsilon)\end{aligned}$$

Since we are given that $\|f\|_\infty \in (0, \infty]$, there is an $\varepsilon > 0$ such that $\|f\|_\infty > \varepsilon$. Moreover since $\|f\|_r < \infty$, the above inequality forces $\mu(A_\varepsilon) < \infty$. Then taking power $\frac{1}{p}$ to the above inequality, we get

$$\|f\|_p \geq (\|f\|_\infty - \epsilon)\mu(A_\varepsilon)^{\frac{1}{p}}$$

Now taking limits, we get

$$\lim_{p\to\infty} \inf \|f\|_p \geq (\|f\|_\infty - \varepsilon)$$

since $\mu(A_\varepsilon)^{\frac{1}{p}} \to 1$ as $p \to \infty$. Again since $\varepsilon > 0$ was arbitrary, we get

$$\lim_{p\to\infty} \inf \|f\|_p \geq \|f\|_\infty$$

Now to get the other inequality, observe that for $p > r$

$$\begin{aligned}\int |f|^p \, d\mu &= \int |f|^r \, |f|^{p-r} \, d\mu \\ &\leq \|f\|_\infty^{p-r} \int |f|^r \, d\mu\end{aligned}$$

Hence we get

$$\|f\|_p = \Big(\int |f|^p \, d\mu\Big)^{1/p} \leq \|f\|_\infty^{\frac{p-r}{p}} \Big(\int |f|^r \, d\mu\Big)^{\frac{1}{p}} = \|f\|_\infty^{\frac{p-r}{p}} \|f\|_r^{\frac{r}{p}}$$

Thus taking limits, we see that

$$\lim_{p\to\infty} \sup \|f\|_p \leq \|f\|_\infty$$

as $\|f\|_r^{\frac{r}{p}} \to 0$ as $p \to \infty$ since $\|f\|_r < \infty$

Combining both the inequalities, we see

$$\lim_{p\to\infty} \sup \|f\|_p \leq \|f\|_\infty \leq \lim_{p\to\infty} \inf \|f\|_p$$

Thus

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$$

2. **Solution:** Since $f_n \to f$ in $L^p(\mu)$, there is a subsequence f_{n_k} such that $f_{n_k} \to f$ pointwise everywhere. Let $A \subset X$ such that $\mu(A) = 0$ and $f_{n_k}(x) \to f(x)$ for all $x \in A^c$. Let B be the set such that $\mu(B) = 0$ and $f_n(x) \to g(x)$ for all $x \in B^c$. Therefore $f_{n_k}(x) \to g(x)$ for all $x \in B^c$, being a subsequence of f_n. Then for all $x \in (A \cup B)^c$, we have $g(x) = f(x)$ by the uniqueness of the pointwise limit in $\mathbb{C}$. Moreover $\mu(A \cup B) \leq \mu(A) + \mu(B) = 0$. Hence $f = g$ almost everywhere.

3. **Solution:** Let's define a new measure $\nu := |f|^p \mu$ defined as

$$\nu(A) = \int_A |f|^p \, d\mu$$

for all $A \in \mathcal{M}$. Then since $\|f\|_p < \infty$, we get $\nu(X) < \infty$. Thus by Egorov's theorem, for all $\epsilon > 0$ there exist a set $A' \in \mathcal{M}$ such that $\nu(A') < \frac{\epsilon}{2}$ and f_n converges to f uniformly on A'^c.

Now, for $r > 0$, let $A_r = \{x \in X \ : \ |f(x)|^p < r\}$. Since $f \in L^p(\mu)$, and $|f|^p \chi_{A_r^c} \geq r \chi_{A_r^c}$, we get

$$\infty > \int_{A_r^c} |f|^p \, d\mu \geq \int r \chi_{A_r^c} \, d\mu = r\mu(A_r^c)$$

Thus we see that $\mu(A_r^c) < \infty$ for all $r > 0$. Again $f \in L^p(\mu)$ forces f to be finite almost everywhere. Thus $|f|^p \chi_{A_r} \to 0$ almost everywhere. Moreover $|f|^p \chi_{A_r}$ is dominated by $|f|^p \in L^1(\mu)$. Hence by the Lebesgue dominated convergence theorem, we see that

$$\lim_{r \to \infty} \int |f|^p \chi_{A_r} \, d\mu = 0$$

Hence there is a $r_\epsilon > 0$ such that $\int_{A_{r_\epsilon}} |f|^p \, d\mu < \frac{\epsilon}{2}$.

Let $A = A' \cup A_{r_\epsilon}$. Then since $\nu(A') < \epsilon/2$

$$\int_A |f|^p \, d\mu \leq \int_{A'} |f|^p \, d\mu + \int_{A_{r_\epsilon}} |f|^p \, d\mu < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Then for $B = A^c$, since $B \subset A'^c$, we get $f_n \to f$ uniformly on B. Moreover since $B \subset A_{r_\epsilon}^c$, we get $\mu(B) \leq \mu(A_{r_\epsilon}) < \infty$.

Now let's evaluate $\|f_n - f\|_p$. Since $A \cup B = X$, we see that

$$\|f_n - f\|_p^p = \int |f_n - f|^p \, d\mu = \int_A |f_n - f|^p \, d\mu + \int_B |f_n - f|^p \, d\mu \qquad (1)$$

Since $f_n \to f$ uniformly on B, there is an $N_\epsilon \in \mathbb{N}$ such that for all $n \geq N_\epsilon$, $|f_n(x) - f(x)| < \sqrt[p]{\frac{\epsilon}{\mu(B)}}$.

Then for $n \geq N_\epsilon$, we get

$$\int_B |f_n - f|^p \, d\mu \leq \int_B \frac{\epsilon}{\mu(B)} \, d\mu = \epsilon$$

Since $X = A \cup B$, we note that

$$\int_A |f_n|^p \, d\mu = \|f_n\|_p^p - \int_B |f_n|^p \, d\mu$$

Then by Fatou's lemma, and the fact that $\|f_n\| \to \|f\|$ we get

$$\begin{aligned}
\limsup_n \int_A |f_n|^p \, d\mu &= \limsup_n \|f_n\|_p^p - \liminf_n \int_B |f_n|^p \, d\mu \\
&\leq \|f\|_p^p - \int_B \liminf_n |f_n|^p \, d\mu \\
&= \int_X |f|^p \, d\mu - \int_B |f|^p \, d\mu \\
&= \int_{X \setminus B} |f|^p \, d\mu \\
&= \int_A |f|^p \, d\mu
\end{aligned}$$

where $\liminf |f_n|^p = |f|^p$ in B since $f_n \to f$ uniformly on B. Since we know that $\int_A |f|^p \, d\mu < \epsilon$, we see that there is an $M_\epsilon \in \mathbb{N}$ such that for all $n > M_\epsilon$,

$$\int_A |f_n|^p \, d\mu \leq \sup_{m \geq n} \int_A |f_n|^p \, d\mu \leq \int_A |f|^p \, d\mu \leq \epsilon$$

Then for all $n > M_\epsilon$, Minkowski inequality gives

$$\begin{aligned}
\int_A |f_n - f| \, d\mu &\leq \left[\left(\int_A |f_n|^p \, d\mu \right)^{1/p} + \left(\int_A |f|^p \, d\mu \right)^{1/p} \right]^p \\
&\leq (\epsilon^{1/p} + \epsilon^{1/p})^p = 2^p \epsilon
\end{aligned}$$

We note that $2^p \epsilon \to 0$ as $\epsilon \to 0$.

Hence we see from Equation 1 that for $n > \max\{N_\epsilon, M_\epsilon\}$,

$$\|f_n - f\|_p^p < (2^p + 1)\epsilon$$

Since $\epsilon > 0$ was arbitrary, and $(2^p + 1)\epsilon \to 0$ as $\epsilon \to 0$, we see that $\|f_n - f\|_p^p \to 0$. Now by the continuity of the function $x \to x^{1/p}$, we see that $\|f_n - f\| \to 0$.