Prop: If H, K are groups and $\varphi: K \to \mathrm{Aut}(H)$ is a homom. then:

i) $H \rtimes K$ is a group

ii) $H \trianglelefteq H \rtimes K$ (this is the reason for the notation $\rtimes$)

iii) $\forall h \in H, k \in K$, $(1,k)(h,1)(1,k)^{-1} = (\varphi_k(h), 1)$.

Pf: i) Associativity:

$$(h_1,k_1)\big((h_2,k_2)(h_3,k_3)\big)$$
$$= (h_1,k_1)(h_2\varphi_{k_2}(h_3), k_2k_3)$$
$$= (h_1\underline{\varphi_{k_1}(h_2\varphi_{k_2}(h_3))}, \underline{k_1k_2k_3})$$

$$\big((h_1,k_1)(h_2,k_2)\big)(h_3,k_3)$$
$$= (h_1\varphi_{k_1}(h_2), k_1k_2)(h_3,k_3)$$
$$= (h_1\underline{\varphi_{k_1}(h_2)\varphi_{k_1k_2}(h_3)}, \underline{k_1k_2k_3}) \checkmark$$

$$\varphi_{k_1}(h_2\varphi_{k_2}(h_3)) = \varphi_{k_1}(h_2)\varphi_{k_1}(\varphi_{k_2}(h_3))$$ ($\varphi_{k_1} \in \mathrm{Aut}(H)$, so it is a homom. from H to H)

$$= \varphi_{k_1}(h_2)\varphi_{k_1k_2}(h_3) \checkmark$$ (φ is a homom from K to $\mathrm{Aut}(H)$)

ii) Identity: $\forall (h,k)$, $(1,1)(h,k) = (1\varphi_1(h), 1\cdot k) = (h,k)$ ✓

(identity) — identity aut.

iii) Inverses: Suppose $(h,k) \in H \rtimes K$. N.T.S. $\exists (h_1,k_1)$ s.t.

$(h_1,k_1)(h,k) = (h_1\varphi_{k_1}(h), k_1k) = (1,1)$.

Take $k_1 = k^{-1}$, then $h_1 = (\varphi_{k^{-1}}(h))^{-1} = \varphi_{k^{-1}}(h^{-1}) = \varphi_k^{-1}(h^{-1})$.

ii) Note: $H \leftrightarrow \{(h,1) : h \in H\} \subseteq H \rtimes K$

$\forall (h_0,1), (h,k)$,

$$(h,k)(h_0,1)(h,k)^{-1}$$
$$= (h,k)(h_0,1)(\varphi_k^{-1}(h^{-1}), k^{-1})$$
$$= (h\varphi_k(h_0), k)(\varphi_k^{-1}(h^{-1}), k^{-1})$$
$$= (h\varphi_k(h_0)\varphi_k(\varphi_k^{-1}(h^{-1})), kk^{-1})$$
$$= (h\varphi_k(h_0)h^{-1}, 1) \in H. \checkmark$$

iii) $(1,k)(h,1)(1,k)^{-1} = (\varphi_k(h),1)$ (special case of computation above). ∎

Prop: Suppose H, K are groups and $\varphi : K \to \mathrm{Aut}(H)$ is a homom.

Then T.F.A.E.:

i) The map $H \rtimes K \to H \times K$ is an isomorphism.

$$(h,k) \mapsto (h,k),$$

ii) The homom. φ is trivial,

iii) $K \trianglelefteq H \rtimes K$.

Pf: ii) $\Rightarrow$ i): Follows from def. of $H \rtimes K$

i) $\Rightarrow$ iii): Easy, since $K \trianglelefteq H \times K$

thinking of them as subgroups of $H \rtimes K$.

iii) $\Rightarrow$ i): If $K \trianglelefteq H \rtimes K$ then since $H \cap K = \{(e,e)\}$,

we have $H \rtimes K = HK \cong H \times K$. (recog. thm. for direct products)

i) $\Rightarrow$ ii) : Let $\psi: H \rtimes K \to H \times K$.
$(h,k) \mapsto (h,k)$

Then $\forall h \in H, k \in K$,

$$(1,k)(h,1) = (\varphi_k(h), k)$$ (computation in direct product)

$\big\downarrow$ apply ψ

$$(h,k) = (1,k)(h,1) = (\varphi_k(h), k)$$

computation on direct product

$\Rightarrow \forall h \in H, k \in K, \ \varphi_k(h) = h$

$\Rightarrow \varphi : K \to \mathrm{Aut}(H)$ is the identity map. $\blacksquare$

Cor: If φ is not the identity map then $H \rtimes_\varphi K$ is non-Abelian.

Exs: 1) $H = C_3 = \langle x \rangle$, $K = C_2 = \langle y \rangle$

$\varphi : K \to \mathrm{Aut}(H)$ defined by $\varphi_y(x) = x^{-1}$.

(this is an aut. of H ... check)

this is a homom.

Then $H \rtimes K \cong S_3 \cong D_6$.

(group of order 6, non-Abel. b/c φ is non-trivial)

2) Suppose $n = pq$, $p < q$ prime and $p \mid q-1$ then there is a non-Abel. group of order n.