Exs:

1) $R = \mathbb{Z}$

- ideals:

  Additive subgroups of $(\mathbb{Z}, +)$:

  $\{0\}$, $\{nk : k \in \mathbb{Z}\}$, $n \in \mathbb{N}$.

  All of these are ideals: (0), (n), $n \in \mathbb{N}$

- prime ideals: $\{0\}$, (p), p a prime #.

- $I = m\mathbb{Z}$, $J = n\mathbb{Z}$

  $$IJ = \{\underbrace{a_1b_1 + a_2b_2 + \cdots + a_kb_k}_{\equiv 0 \bmod mn} : a_i \in m\mathbb{Z}, b_i \in n\mathbb{Z}\}$$

  $\subseteq mn\mathbb{Z}$.

  Also $mn\mathbb{Z} \subseteq IJ$, so $IJ = mn\mathbb{Z}$.

  Comment: note that for any ideals I, J in a ring R, it is always the case that

  $IJ \subseteq I \cap J$. However it can also happen that $IJ \neq I \cap J$.

  Ex: $R = \mathbb{Z}$, $I = J = 2\mathbb{Z}$. Then

  $IJ = 4\mathbb{Z}$, but $I \cap J = 2\mathbb{Z}$.

- $I = m\mathbb{Z}$, $J = n\mathbb{Z}$,

  $I + J = \{ a + b : m \mid a,\ n \mid b \} = d\mathbb{Z}$, where $d = (m, n)$.

  (follows from Bezout's lemma)

2) $R = \mathbb{Z}[x]$

- $I = (x^2) = \{ x^2 f(x) : f(x) \in \mathbb{Z}[x] \}$

  $R/I = \{ g(x) + I : g(x) \in \mathbb{Z}[x] \}$.

  Complete collection of distinct reps:

  $R/I = \{ a_0 + a_1 x + I : a_0, a_0 \in \mathbb{Z} \}$.

  Multiplication in R/I:

  $$x(x+1) = \overset{\swarrow \in I}{x^2} + x = x \mod I$$

  formally:

  $$(x + I)(x + 1 + I) = x(x+1) + I$$

  $$= x^2 + x + I = x + I.$$

- $I = \{ f(x) \in R : 2 \mid f(0) \}$. (This is an ideal)

  $I^2 = \{ f_1 g_1 + \cdots + f_k g_k : f_i, g_i \in I \}$.

  Note $x^2 + 4 = x \cdot x + 2 \cdot 2 \in I^2$, but

  $$x^2 + 4 \neq fg \text{ for any } f, g \in I.$$

Note: This shows that when computing IJ, in general, you must consider finite sums of products of elems. of I and J.

3a) $R = \mathbb{F}_2[x]$ $\quad (\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z})$

$I = (x^2+x+1) = \{ f(x)(x^2+x+1) : f(x) \in \mathbb{F}_2[x] \}$

$R/I = \{ a_0 + a_1 x : a_0, a_1 \in \mathbb{F}_2 \}$

Notes:

- Additive structure:

$$(R/I, +) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

- Multiplicative structure:

	0	1	x	1+x
0	0	0	0	0
1	0	1	x	1+x
x	0	x	1+x	1
1+x	0	1+x	1	x

$|(R/I)^\times| = 3$

$(R/I)^\times = \langle x \rangle$

Scratch:

$I = (x^2+x+1)$

$x^2 = (x^2+x+1) - (x+1)$

$= x+1+I$

$x(1+x) = x^2+x+1 - 1$

$= 1+I$

$(1+x)^2 = x^2+1 = x^2+x+1 - x$

$= x+I.$

Conclusion: R/I is a field of order 4.

$(R/I \cong \mathbb{F}_4)$

b) $R = \mathbb{F}_2[x]$, $I = (x^2+1)$ (note: $x^2+1 = (x+1)^2$ in R)

$R/I = \{a_0 + a_1 x : a_0, a_1 \in \mathbb{F}_2\}$

- Additive structure:

$$(R/I, +) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

- Multiplicative structure:

	0	1	x	1+x
0	0	0	0	0
1	0	1	x	1+x
x	0	x	1	1+x
1+x	0	1+x	1+x	0

Scratch:
$x(1+x) = x^2+x$
$= x^2+1+x-1$

$\Rightarrow$ 1+x is a zero divisor.

Moral: x^2+1 not irred. over $R \Rightarrow R/I$ not a field.