Isomorphism theorems for rings:

1st) If $\phi : R \to S$ is a ring homom. then $\ker\phi$ is an ideal of R and $R/\ker\phi \cong \phi(R)$.

(isomorphic as a ring)

Also: If $I \subseteq R$ is an ideal, then it is the kernel of the ring homom. $\pi : R \to R/I$.

$a \mapsto a + I$

2nd) Suppose $A \subseteq R$ and $B \subseteq R$ is an ideal. Then

$A+B = \{a+b : a \in A, b \in B\}$ is a subring of R (since B is an ideal),

$A \cap B$ is an ideal of A, and

$$A+B/B \cong A/A\cap B.$$

(ring isom.)

$$\begin{tikzcd}
& R \arrow[d, dash] & \\
& A+B \arrow[dl, dash] \arrow[dr, dash] & \\
A \arrow[dr, dash] & & B \arrow[dl, dash] \\
& A \cap B &
\end{tikzcd}$$

3rd) Suppose $I, J \subseteq R$ are ideals, $I \subseteq J$.

Then J/I is an ideal of R/I and

$$(R/I)/(J/I) \cong R/J.$$

(ring isom.)

$$\begin{tikzcd}
R \arrow[d, dash] \\
J \arrow[d, dash] \\
I
\end{tikzcd}$$

4th) If $I \subseteq R$ is an ideal then the correspondence $A \Leftrightarrow A/I$ is an inclusion preserving bijection between the set of all subrings of A that contain I and the set of subrings of R/I. Also, A is and ideal of R if and only if A/I is an ideal of R/I.

Notation related to ideals:

Suppose $I, J \subseteq R$ are ideals:

1) $I+J = \{a+b : a \in I, b \in J\}$

This is an subring:

- It is a group under ✓
- $\forall a_1, a_2 \in I, \; b_1, b_2 \in J$,

$$(a_1+b_1)(a_2+b_2) = (a_1a_2 + \underbrace{a_1b_2 + b_1a_2}_{\in I \cap J}) + b_1b_2 \in I+J.$$

since they are ideals

It's an ideal: $\forall r \in R, \; a \in I, \; b \in J$

$$r(a+b) = \underbrace{ra}_{\in I} + \underbrace{rb}_{\in J} \in I+J$$

, since they are ideals

$(a+b)r \in I+J$. ✓

2) IJ is the set of all finite sums of elements of the form ab, $a \in I$, $b \in J$.

3) I^n = collection of all finite sums of elems. of the form $a_1 a_2 \cdots a_n$, $a_i \in A$.

4) If $A \subseteq R$ the ideal <u>generated</u> by A, denoted (A), is the smallest ideal containing A.

If I is an ideal and $I = (\{a\})$ for some $a \in R$, then I is a <u>principal ideal</u>.

5) Suppose R has $1 \neq 0$. Then the <u>left ideal</u> generated by $A \subseteq R$ is

$$RA = \{r_1 a_1 + \cdots + r_n a_n : n \in \mathbb{N},\ r_1, \ldots, r_n \in R, \text{ and } a_1, \ldots, a_n \in A\}$$

The <u>right ideal</u> generated by $A \subseteq R$ is

$$AR = \{a_1 r_1 + \cdots + a_n r_n : n \in \mathbb{N},\ r_1, \ldots, r_n \in R, \text{ and } a_1, \ldots, a_n \in A\}$$

The ideal generated by $A \subseteq R$ is

$$(A) = RAR = \{r_1 a_1 s_1 + r_2 a_2 s_2 + \cdots + r_n a_n s_n : n \in \mathbb{N},\ r_1, \ldots, r_n \in R,\ s_1, \ldots, s_n \in R,\ a_1, \ldots, a_n \in A\}$$

Special case: If R is commutative and $A = \{a\}$,

$(A) = (a) = \{ra : r \in R\}$.

Also, if R is commutative then

$b \in (a) \iff (b) \subseteq (a)$

$\iff b = ra$ for some $r \in R$.

In this case, <u>a divides b</u>, or

that <u>b is a multiple a</u>.

("to contain is to divide")

6) An ideal $P \subseteq R$ is <u>prime</u> if $P \neq R$ and if whenever $ab \in P$, it must be the case that $a \in P$ or $b \in P$.