CHAPTER 1

Key Definitions

Definition. For a set X, a collection $\mathcal{M} \subseteq \mathcal{P}(X)$ is a **σ-algebra** if it satisfies

1. $X \in \mathcal{M}$
2. If $A \in \mathcal{M}$, then $X \setminus A \in \mathcal{M}$
3. If $(A_n)_{n=1}^{\infty} \in \mathcal{M}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$.

Definition. Let $(X, \mathcal{M})$ be a measurable space and $(Y, \mathcal{T})$ be a topological space. A function

$$f : (X, \mathcal{M}) \to (Y, \mathcal{T})$$

is **measurable** if for every open set $U \in \mathcal{T}$, the preimage $f^{-1}(U)$ is in $\mathcal{M}$. That is,

$$\forall U \in \mathcal{T}, \quad f^{-1}(U) \in \mathcal{M} .$$

Definition. Given a sequence $\{a_n\}_{n=1}^{\infty}$ in $[-\infty, \infty]$, we define the **limit superior** and **limit inferior** as follows:

$$\limsup_{n\to\infty} a_n := \inf_{k\in\mathbb{N}} \left(\sup_{n\geq k} a_n \right), \quad \liminf_{n\to\infty} a_n := \sup_{k\in\mathbb{N}} \left(\inf_{n\geq k} a_n \right).$$

We call $\limsup a_n$ the **upper limit** and $\liminf a_n$ the **lower limit** of the sequence $\{a_n\}$, respectively.

Definition. A function $s : X \to \mathbb{C}$ is called a **simple function** if the image $s(X)$ is a finite set.

Definition. Let $A \in \mathcal{M}$. The **characteristic function** $\chi_A : X \to \mathbb{C}$ is defined by

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{otherwise} \end{cases} .$$

Definition. Let $(X, \mathcal{M})$ be a measurable space. A set function $\mu : \mathcal{M} \to [0, \infty]$ is called **countably additive** if, whenever $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{M}$ is a sequence of disjoint sets, i.e., $A_i \cap A_j = \emptyset$ for all $i \neq j$, we have

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) .$$

Moreover, if $\mu(A) < \infty$ for some $A \in \mathcal{M}$, then μ is called a **positive measure**. The triple $(X, \mathcal{M}, \mu)$ is called a **measure space**.

Definition. Let $s : X \to [0, \infty]$ be a simple measurable function. Then s can be written in the form

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$$

where $\{\alpha_i\}_{i=1}^{n}$ are constants and $\{A_i\}_{i=1}^{n}$ are measurable sets in $\mathcal{M}$, with the property that $A_i \cap A_j = \emptyset$ for all $i \neq j$, and $s(X) = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$.

If $E \in \mathcal{M}$, we define the **integral of** s **over** E by

$$\int_E s \, d\mu := \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E),$$

with the convention that $0 \cdot \infty = 0$.

Definition. Let $f : X \to [0, \infty]$ be a measurable function and let $E \in \mathcal{M}$. The **Lebesgue integral** of f over E is defined by

$$\int_E f \, d\mu := \sup \left\{ \int_E s \, d\mu \ : \ s \text{ is simple, measurable, and } 0 \leq s \leq f \right\}.$$

Definition. A measure μ on a measurable space $(X, \mathcal{M})$ is called **complete** if for every set $E \in \mathcal{M}$ with $\mu(E) = 0$, and for every subset $F \subseteq E$, we have $F \in \mathcal{M}$. That is, all subsets of null sets are measurable.

Key Theorems

Theorem. *If $f : (X, \mathcal{M}) \to [-\infty, \infty]$, then f is measurable if and only if for all $a \in \mathbb{R}$, $f^{-1}((a, \infty]) \in \mathcal{M}$.*

Proof. omitted. □

Theorem. ***[Monotone Convergence Theorem]*** *Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions on $[-\infty, \infty]$ such that*

1. $0 \leq f_1 \leq f_2 \leq \cdots \leq \infty$
2. $\lim_{n \to \infty} f_n(x) = f(x)$ *for all* $x \in X$,

then f is measurable and we have that

$$\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X \lim_{n \to \infty} f_n \, d\mu = \int_X f \, d\mu \, .$$

Proof. For every $c \in (0, 1)$, fix a simple measurable function s, with $0 \leq s \leq f$. Let the following sets be defined by $E_n = \{x \in X : f_n(x) \geq c \cdot s(x)\}$. Then we have $E_1 \subseteq E_2 \subseteq \cdots$ and $\bigcup_{n=1}^{\infty} E_n = X$. So we have

$$\int_X s \, d\mu = \lim_{n \to \infty} \int_{E_n} s \, d\mu \leq \lim_{n \to \infty} \int_{E_n} \frac{1}{c} f_n \, d\mu \, .$$

By taking the supremum over all simple functions "beneath" f, we have

$$\int_X f \, d\mu \leq \lim_{n \to \infty} \int_{E_n} \frac{1}{c} f_n \, d\mu \quad \Longrightarrow \quad c \cdot \int_X f \, d\mu \leq \lim_{n \to \infty} \int_{E_n} f_n \, d\mu$$

and taking the limit as $c \to 1$ gives the desired result. The other inequality is "obvious," concluding the proof. □

Lemma. ***[Fatou's Lemma]*** *Let $f_n : X \to [0, \infty]$ be measurable for each $n \in \mathbb{N}$. Then*

$$\int_X \left(\liminf_{n\to\infty} f_n \right) \, d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu \, .$$

Proof. First, define $g_n(x) = \inf\{f_k(x) : k \geq n\}$. Then $(g_n)_{n=1}^{\infty}$ is a non-decreasing sequence of functions by construction. Then by the Monotone Convergence Theorem:

$$\int_X \liminf_{n\to\infty} f_n \, d\mu = \int_X \lim_{n\to\infty} g_n \, d\mu = \lim_{n\to\infty} \int_X g_n \, d\mu \leq \lim_{n\to\infty} \inf_{m\geq n} \int_X f_m \, d\mu = \liminf_{n\to\infty} \int_X f_n \, d\mu \, . \tag{1}$$

□

Theorem. ***[Dominated Convergence Theorem]*** *For all $n \in \mathbb{N}$, let $f_n : X \to \mathbb{C}$ be measurable. Suppose $\lim_{n\to\infty} f_n(x) =: f(x)$ exists for all $x \in X$ and there exists a $g \in L^1(\mu)$ such that $|f_n(x)| \leq |g(x)|$ for all $x \in X$ and $n \in \mathbb{N}$. Then $f \in L^1(\mu)$,*

$$\lim_{n\to\infty} \int_X |f_n - f| \, d\mu = 0 \quad \textit{and} \quad \lim_{n\to\infty} \int_X f_n \, d\mu = \int_X \lim_{n\to\infty} f_n \, d\mu = \int_X f \, d\mu \, .$$

Proof.

□

Question 1.12

Suppose $f \in L^1(\mu)$. Prove that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $\int_E |f|\, d\mu < \varepsilon$ whenever $\mu(E) < \delta$.

Answer:

Proof. First define $f_n(x) = \min\{\ |f(x)|\ ,\ n\ \}$. Then $\lim\limits_{n\to\infty} f_n(x) = |f(x)|$ almost everywhere from monotonicity, and by the Dominated Convergence Theorem

$$\lim_{n\to\infty} \int f_n \, d\mu = \int |f| \, d\mu \qquad \Longrightarrow \qquad \lim_{n\to\infty} \int (|f| - f_n) \, d\mu = 0 \, .$$

Hence, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\int (|f| - f_n) \, d\mu = \int \big||f| - f_n\big| \, d\mu < \frac{\varepsilon}{2}$$

for all $n \geq N$. Also, by definition there exists a $N' \in \mathbb{N}$ such that

$$\mu(\{x : |f(x)| \geq N'\}) = 0$$

for if no such $N' \in \mathbb{N}$ existed, then $f \notin L^1(\mu)$. Set $M = \max\{N, N'\}$. Now for any measurable set E with $\mu(E) < \delta = \frac{\varepsilon}{2M}$,

$$\int_E |f| \, d\mu \leq \int_E \big||f| - f_n\big| \, d\mu + \int_E |f_n| \, d\mu < \frac{\varepsilon}{2} + \int_E N \, d\mu = \frac{\varepsilon}{2} + N \cdot \mu(E) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \, .$$

$\square$

Question 1.π

Use Fatou's Lemma to prove the Monotone Convergence Theorem.

CHAPTER 2

Key Definitions

Definition. A measurable set E in a measure space is said to have **σ-finite measure** if $E = \bigcup_{n=1}^{\infty} E_n$, where E_n is measurable and $\mu(E_n) < \infty$ for all n. In particular, if X has σ-finite measure, then μ is called **σ-finite**.

Definition. We say the **linear functional** $\Lambda : V \to \mathbb{C}$ is a **positive** linear functional if $f \geq 0$ implies $\Lambda(f) \geq 0$.

Definition. We say that the topological space $(X, \mathcal{T})$ is **locally compact** if for each $p \in X$, there exists a neighborhood U of p such that $\overline{U}$ is compact.

Definition. Let f be a real or extended-real valued function on (X, τ). If $\{x : f(x) > \alpha\} \in \tau$, for all $\alpha \in \mathbb{R}$, then f is said to be **lower semicontinuous**. Likewise, if $\{x : f(x) < \alpha\} \in \tau$, for all $\alpha \in \mathbb{R}$, then f is **upper semicontinuous**.
Definition. Let $(X, \mathcal{T})$ be a topological space, and $f : X \to \mathbb{C}$. The **support** of f is defined as

$$\operatorname{supp}(f) := \{x \in X : f(x) \neq 0\}.$$

We also define

$$C_c(X) := \{f : X \to \mathbb{C} : f \text{ continuous with compact support}\}.$$

Definition. Let $E \in \mathcal{M}$. The measure μ is said to be **outer regular** if

$$\mu(E) = \inf\{\mu(V) : E \subset V,\ V \text{ open}\}.$$

The measure μ is said to be **inner regular** if E is open, or $E \in \mathcal{M}$ with $\mu(E) < \infty$, and

$$\mu(E) = \sup\{\mu(K) : K \subset E,\ K \text{ compact}\}.$$

If every Borel set E is both inner and outer regular, then μ is said to be a **regular measure**.

Key Theorems

Lemma. ***[Urysohn's Lemma]*** *Let $(X, \mathcal{T})$ be a locally compact Hausdorff space. Suppose K is a compact subset and $K \subseteq V \in \mathcal{T}$. Then there exists $f \in C_c(X)$ such that $\chi_K \leq f \leq \chi_V$.*

Theorem. ***[Riesz Representation Theorem]***

Theorem. ***[Lusin's Theorem]*** *Let $f : X \to \mathbb{C}$ be measurable. Define $A = \{x \in X : f(x) \neq 0\}$ and suppose $\mu(A) < \infty$. Then for all $\varepsilon > 0$, there exists $g \in C_c(X)$ such that*

$$\mu(\{x \in X : f(x) \neq g(x)\}) < \varepsilon \quad \textit{and} \quad \sup_{x \in X} |g(x)| \leq \sup_{x \in X} |f(x)| .$$

Question 2.3

Let X be a metric space, with metric ρ. For any nonempty $E \subset X$, define

$$\rho_E(x) = \inf\{\rho(x, y) : y \in E\}$$

Show that ρ_E is a uniformly continuous function on X. If A and B are disjoint nonempty closed subsets of X, examine the relevance of the function

$$f(x) = \frac{\rho_A(x)}{\rho_A(x) + \rho_B(x)}$$

to Urysohn's lemma.

Question 2.11

Let μ be a regular Borel measure on a compact Hausdorff space X and assume $\mu(X) = 1$. Prove that there is a compact set $K \subset X$ (the carrier or support of μ) such that $\mu(K) = 1$ but $\mu(H) < 1$ for every proper compact subset H of K. *Hint:* Let K be the intersection of all compact K_α with $\mu(K_\alpha) = 1$. Show that every open set V which contains K also contains some K_α. Regularity of μ is needed. Show that K^c is the largest open set in X whose measure is 0.

Answer:

Let $K = \bigcap_\alpha K_\alpha$ where each K_α is compacrt and such that $\mu(K_\alpha) = 1$. By regularity,

$$\mu(K) = \inf\{\mu(V) : V \text{ is open and } K \subset V\}.$$

If such a V is open, then V^c is closed and because X is a compact Hausdorff space, V^c is compact. Then because $K \subset V$, we have that $V^c \subset K^c$, so the set $\{K_\alpha^c\}$ is an open cover of V^c. By compactness of V^c, there exists an finite subcovering $\{K_{\alpha_i}^n\}_{i=1}^n$. For the finite set we have:

$$\mu(V^c) \leq \mu\left(\bigcup_{i=1}^n K_{\alpha_i}^c\right) \leq \sum_{i=1}^n \mu(K_{\alpha_i}^c) = 0$$

since $\mu(K_\alpha^c) = 0$ for all α by assumption. Then for all such open sets V, we have that $\mu(V) = 1$ implying $\mu(K) = 1$.

Now suppose there exists a compact set H such that $H \subset K$. Then because X is a compact Hausdorff space, H is closed. Define U to be the open set such that $U = H^c$. From $H \subset K$, we have $K^c \subset U$. If $\mu(U) = 0$, then $\mu(U) = \mu(U \cup K^c) = 0$. This would force it to be the case that

$$\mu\left((U \cup K^c)^c\right) = \mu(H \cap K) = \mu(H) = 1 \ .$$

However, if H was compact, such that $H \subset K$, and $\mu(H) = 1$, then that would contradict the construction of K, so it must be the case that $\mu(U) > 0$. Therefore K^c is the largest open set in X whose measure is 0, and we can compute that

$$1 = \mu(X) = \mu(H \cup U) = \mu(H) + \mu(U) \implies 1 > 1 - \mu(U) = \mu(H) \ .$$

CHAPTER 3

Key Definitions

Definition. A function $\varphi : (a, b) \to \mathbb{R}$ is said to be **convex** if for all $x, y \in (a, b)$ and $\lambda \in [0, 1]$, we have

$$\varphi((1-\lambda) \cdot x + \lambda \cdot y) \leq (1-\lambda) \cdot \varphi(x) + \lambda \cdot \varphi(y) \ .$$

Definition. Let $p, q > 0$ such that

$$\frac{1}{p} + \frac{1}{q} = 1 \ .$$

Then p and q are called a pair of **conjugate exponents**.

Definition. Let $0 < p < \infty$ and let $f : X \to \mathbb{C}$ be measurable. Define the **L^p-norm** of f by

$$\|f\|_p := \left(\int_X |f|^p \, d\mu \right)^{1/p} .$$

Define the **L^p-space** of X by

$$L^p(\mu) := \{f : X \to \mathbb{C} \text{ measurable with } \|f\|_p < \infty\}.$$

Definition. Let $f : X \to [0, \infty]$ be measurable. The **essential supremum** of f is defined as

$$\operatorname{ess\,sup}(f) := \inf\{\alpha : \mu(\{x \in X : f(x) > \alpha\}) = 0\}.$$

Definition. If $f : X \to \mathbb{C}$ is measurable, we define

$$\|f\|_\infty := \operatorname{ess\,sup}(|f|).$$

Define

$$L^\infty(\mu) := \{f : X \to \mathbb{C}, \text{ measurable with } \|f\|_\infty < \infty\}.$$

Sometimes we call the members of L^∞ **essentially bounded measurable functions** on X.

Key Theorems

Theorem. ***[Jensen's Inequality]*** *Let μ be a positive measure on $(X, \mathcal{M})$ with $\mu(X) = 1$. Let $f : X \to [-\infty, \infty]$, $f \in L^1(\mu)$, $a < f(x) < b$, for all $x \in X$, and φ be convex on (a, b). Then we have*

$$\varphi\left(\int_X f \, d\mu \right) \leq \int_X \varphi \circ f \, d\mu \ .$$

Theorem. *For $1 \leq p < \infty$, $C_c(X)$ is dense in $L^p(\mu)$.*

Theorem. ***[Hölder's and Minkowski's Inequality]*** *Let p and q be conjugate exponents, and $1 < p < \infty$. Let X be a measure space, with measure μ. Let $f, g : X \to [0, \infty]$ be measurable. Then*

$$\textbf{\textit{Hölder's:}} \quad \|f \cdot g\|_1 = \int_X f \cdot g \, d\mu \leq \left(\int_X f^p \, d\mu \right)^{1/p} \cdot \left(\int_X g^q \, d\mu \right)^{1/q} = \|f\|_p \cdot \|g\|_q$$

and

$$\textbf{\textit{Minkowski's:}} \quad \|f + g\|_p = \left(\int_X (f + g)^p \, d\mu \right)^{1/p} \leq \left(\int_X f^p \, d\mu \right)^{1/p} + \left(\int_X g^p \, d\mu \right)^{1/p} = \|f\|_p + \|g\|_p \, .$$

Theorem. ***[Egoroff's Theorem]*** *Let $\mu(X) < \infty$, $f_n : X \to \mathbb{C}$ be measurable and $\lim_{n \to \infty} f_n(x) = f(x)$, for μ-almost every $x \in X$. Then given $\varepsilon > 0$, there is a measurable set $E \subset X$ with $\mu(E^c) < \varepsilon$ such that $(f_n)_{n=1}^{\infty}$ converges uniformly on E.*

Proof.

□

Question 3.4

Suppose that f is a complex measurable function on X, μ is a positive measure on X, and

$$\varphi(p) = \int_X |f|^p \, d\mu = \|f\|_p^p \quad \text{with } 0 < p < \infty$$

Let $E = \{p : \varphi(p) < \infty\}$. Assume $\|f\|_\infty > 0$.

(a) If $r < p < s$ and $r, s \in E$, then $p \in E$.

(b) Prove that $\log \varphi$ is convex in the interior of E and that φ is continuous on E.

(c) By (a), E is connected. Is E necessarily open or closed? Can E consist of a single point? Can E be any connected subset of $(0, \infty)$?

(d) If $r < p < s$, prove that $\|f\|_p \leq \max\{\|f\|_r, \|f\|_s\}$. Show that this implies the inclusion $L^r(\mu) \cap L^s(\mu) \subset L^p(\mu)$.

(e) Assume that $\|f\|_r < \infty$ for some $r < \infty$. Prove that $\lim_{p \to \infty} \|f\|_p = \|f\|_\infty$.

CHAPTER 4

Key Definitions

Definition. If $\langle \cdot, \cdot \rangle$ is a sesquilinear, positive semidefinite form on H, then the **seminorm** of $x \in H$ is defined to be

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

Definition. Let H be an inner product space. If H is complete with respect to $\| \cdot \|$, then H is known as a **Hilbert Space**.

Definition. We say that $x, y \in H$ are **orthogonal** if $\langle x, y \rangle = 0$, and we denote it by $x \perp y$. If $S \subset H$, then we define the set $S^{\perp} = \{x \in H : \langle x, y \rangle = 0, \text{ for all } y \in S\}$

Definition. A family $\{u_\alpha\}_{\alpha \in A} \subset H$ is called **orthonormal** if

$$\langle u_\alpha, u_\beta \rangle = 0, \ \forall \alpha \neq \beta, \quad \text{and} \quad \|u_\alpha\| = 1, \ \forall \alpha \in A.$$

If $x \in H$, the complex numbers $\langle x, u_\alpha \rangle$ are called the **Fourier coefficients** of x relative to the set $\{u_\alpha\}$, or coordinate orthogonal projections onto $\text{Span}(u_\alpha : \alpha \in A)$.

Definition. An orthonormal set $\{u_\alpha\}_{\alpha \in A}$ is called **complete**, or an **orthonormal basis**, if for all $h \in H$,

$$\|h\|^2 = \sum_{\alpha \in A} |\langle h, u_\alpha \rangle|^2 .$$

Definition. The complex numbers $\frac{1}{\sqrt{2\pi}} \langle f, e_n \rangle$ are precisely the **Fourier coefficients** of f.

Definition. Let $(H, \langle \cdot, \cdot \rangle_H)$ and $(K, \langle \cdot, \cdot \rangle_K)$ be Hilbert spaces. A linear map $\Lambda : H \to K$ is called an **isometry** if for all $h, g \in H$,

$$\langle \Lambda(h), \Lambda(g) \rangle_K = \langle h, g \rangle_H .$$

In addition, if Λ is surjective, we say Λ is a **unitary**. If such a Λ exists, then H and K are **isomorphic**. In that case, Λ is called a **Hilbert space isometric isomorphism**.

Key Theorems

Theorem. *Every nontrivial Hilbert space H has an orthonormal basis.*

Proof. Involves Zorn's Lemma XD.

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$\square$

Theorem. ***[Riesz Representation Theorem on Hilbert Space]*** *Let $\Lambda : H \to \mathbb{C}$ be a continuous linear functional. Then there is a unique $y \in H$ such that $\Lambda(x) = \langle x, y\rangle$ for every $x \in H$.*

Proof. First suppose $\Lambda \not\equiv 0$ and let $M = \{x \in H : \Lambda(x) = 0\} = \ker(\Lambda)$. Since M is the preimage image of a closed set under a continuous mapping, it is closed. We also have that $\Lambda \not\equiv 0$ implies $M \subset H$. Hence $M^\perp \neq \{0\}$.

Suppose $u, v \in M^\perp$. By linearity of Λ we have that if $u \neq 0$ and $v \neq 0$, then $\Lambda(u) \neq 0$ and $\Lambda(v) \neq 0$ and

$$\Lambda\left(\frac{u}{\Lambda(u)} - \frac{v}{\Lambda(v)}\right) = \frac{1}{\Lambda(u)} \cdot \Lambda(u) - \frac{1}{\Lambda(v)} \cdot \Lambda(v) = 0 \quad \Longrightarrow \quad \left(\frac{u}{\Lambda(u)} - \frac{v}{\Lambda(v)}\right) \in M$$

On the other hand, $M^\perp$ is subspace of H, so $\left(\frac{u}{\Lambda(u)} - \frac{v}{\Lambda(v)}\right) \in M^\perp$. These two facts together implies that

$$\left(\frac{u}{\Lambda(u)} - \frac{v}{\Lambda(v)}\right) = 0 \quad \Longrightarrow \quad \frac{u}{\Lambda(u)} = \frac{v}{\Lambda(v)} \quad \Longrightarrow \quad u = \frac{\Lambda(u)}{\Lambda(v)} \cdot v$$

and from the fact that u was arbitrary, we have that $M^\perp = \operatorname{span}(v)$.

Now let $x \in H$. Since M is closed, there exists vectors $w \in M$ and $w' \in M^\perp$ such that $x = w + w'$. Further because $M^\perp = \operatorname{span}(v)$, there exists $\alpha \in \mathbb{C}$ such that $w' = \frac{\alpha \cdot v}{\Lambda(v)}$. Therefore we have

$$\begin{aligned}
\Lambda(x) = \Lambda(w) + \Lambda(w') = 0 + \alpha = \alpha\frac{\langle v, v\rangle}{\|v\|^2} &= \langle w, 0\rangle + \left\langle \alpha v, \frac{\overline{\Lambda(v)} \cdot v}{\overline{\Lambda(v)} \cdot \|v\|^2} \right\rangle \\
&= \langle w, 0\rangle + \left\langle \frac{\alpha \cdot v}{\Lambda(v)}, \frac{\overline{\Lambda(v)} \cdot v}{\|v\|^2} \right\rangle \\
&= \langle w, 0\rangle + \left\langle w', \frac{\overline{\Lambda(v)} \cdot v}{\|v\|^2} \right\rangle \\
&= \left\langle x, \frac{\overline{\Lambda(v)} \cdot v}{\|v\|^2} \right\rangle
\end{aligned}$$

Hence, if $y = \frac{\overline{\Lambda(v)} \cdot v}{\|v\|^2}$, then we have the desired vector.

$\square$

Question 4.1

If M is a closed subspace of H, prove that $M = (M^\perp)^\perp$. Is there a similar true statement for subspaces M which are not necessarily closed?

Answer:

Proof. Let $x \in M$. Since $M^\perp = \{y \in H : \langle y, x\rangle = 0 \text{ for all } x \in M\}$, $\langle x, y\rangle = 0$ for all $y \in M^\perp$ implying $x \in (M^\perp)^\perp$. Conversely, suppose $x \in (M^\perp)^\perp$. Because M is closed, by **Theorem 4.12**, $x = u + v$ for some $u \in M$ and $v \in M^\perp$. Since $x \in (M^\perp)^\perp$, we have $0 = \langle x, v\rangle = \langle u + v, v\rangle = \langle u, v\rangle + \langle v, v\rangle = \langle v, v\rangle$ implying $v = 0$. Hence $x = u \in M$, so $M = (M^\perp)^\perp$. □

Question 4.5

If $M = \{x : Lx = 0\}$, where L is a bounded linear functional on H, prove that $M^\perp$ is a vector space of dimension 1.

Answer:

Proof.

□

Question (HW212)

Let f be a continuous, 2π-periodic function on $\mathbb{R}$, and α an irrational number. Show that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(2\pi n\alpha) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, dt.$$

Hint: First examine the special case $f(t) = e^{ikt}$ with $k \in \mathbb{Z}$.

Answer:

Following the hint, let $f(t) = e^{ikt}$ with $k \in \mathbb{Z}$. From calculus we see that when $k = 0$,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} 1 = 1 \qquad \text{and} \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} 1\, dt = 1\ .$$

Now suppose that $k \neq 0$ and let $\alpha \in \mathbb{R}\setminus\mathbb{Q}$. Then the computation of the integral yields:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikt}\, dt = \frac{1}{2\pi i k}\cdot e^{ikt}\Bigg|_{-\pi}^{\pi} = 0\ .$$

And using the fact that $|e^{i\theta}| \leq 1$ for all θ gives

$$\lim_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N}(e^{ik2\pi\alpha})^n\right| = \lim_{N\to\infty}\left|\frac{1}{N}\cdot\frac{e^{ik2\pi\alpha}\left(1-e^{ik2\pi\alpha N}\right)}{1-e^{ik2\pi\alpha}}\right| \leq \lim_{N\to\infty}\left|\frac{1}{N}\cdot\frac{2}{1-e^{ik2\pi\alpha}}\right| = 0\ .$$

Therefore the equality holds in this special case. By linearity of sums and integrals, the equality holds for any trigonometric polynomial.

Now we use the fact that $\{e^{ikt} : k \in \mathbb{Z}\}$ forms an orthonormal basis for continuous functions on $\mathbb{R}$. By **Theorem 4.25**, for all $\varepsilon > 0$, there exists a trigonometric polynomial $P(t)$ such that $|f(t) - P(t)| < \frac{\varepsilon}{3}$. All together we have that for all $\varepsilon > 0$, there exists sufficiently large $N \in \mathbb{N}$ where

$$\begin{aligned}
&\left|\frac{1}{N}\sum_{n=1}^{N} f(2\pi n\alpha) - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, dt\right| \\
&= \left|\frac{1}{N}\sum_{n=1}^{N} f(2\pi n\alpha) - \frac{1}{N}\sum_{n=1}^{N} P(2\pi n\alpha) + \frac{1}{N}\sum_{n=1}^{N} P(2\pi n\alpha) - \frac{1}{2\pi}\int_{-\pi}^{\pi} P(t)\, dt + \frac{1}{2\pi}\int_{-\pi}^{\pi} P(t)\, dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, dt\right| \\
&\leq \frac{1}{N}\left|\sum_{n=1}^{N} f(2\pi n\alpha) - \sum_{n=1}^{N} P(2\pi n\alpha)\right| + \left|\frac{1}{N}\sum_{n=1}^{N} P(2\pi n\alpha) - \frac{1}{2\pi}\int_{-\pi}^{\pi} P(t)\, dt\right| + \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} P(t)\, dt - \int_{-\pi}^{\pi} f(t)\, dt\right| \\
&< \frac{1}{N}\left|\sum_{n=1}^{N}\frac{\varepsilon}{3}\right| + \left|\frac{\varepsilon}{3}\right| + \frac{1}{2\pi}\left|\int_{-\pi}^{\pi}\frac{\varepsilon}{3}\, dt\right| = \varepsilon\ .
\end{aligned}$$

Therefore the equality holds for any such f.

Question 4.3

Show that $L^p(T)$ is separable if $1 \leq p < \infty$, but $L^\infty(T)$ is not separable.

Question 4.7

Suppose $(a_n)_{n=1}^\infty$ is a sequence of positive numbers such that $\sum a_n b_n < \infty$ whenever $b_n \geq 0$ and $\sum b_n^2 < \infty$. Prove that $\sum a_n^2 < \infty$.

Answer:

Proof.

$\square$

CHAPTER 5

Key Definitions

Definition. A complex vector space $(X, \|\cdot\|)$, with a **norm** $\|\cdot\| : X \to [0, \infty)$, is a **normed vector space** if it satisfies the following: for all $x, y \in X$,

1. $\|x + y\| \leq \|x\| + \|y\|$
2. $\|\alpha x\| = |\alpha|\|x\|$, for all $\alpha \in \mathbb{C}$
3. If $\|x\| = 0$, then $x = 0$

Definition. If $\{x_n\}$ is a sequence in $(X, \|\cdot\|)$, the series $\sum_{n=1}^{\infty} x_n$ is said to **converge to *x*** if for some $x \in X$, the partial sums satisfy $\lim\limits_{N\to\infty} \sum_{n=1}^{N} x_n = x$ The series is called **absolutely convergent** if $\sum_{n=1}^{\infty} \|x_n\| < \infty$

Definition. A **Banach Space** is a normed vector space that is complete in the metric topology induced by the norm.

Definition. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed vector spaces. Let $\Lambda : X \to Y$ be a linear map. We define the **operator norm** of Λ by

$$\|\Lambda\| = \sup\{\|\Lambda(x)\|_Y : \|x\| \leq 1\} .$$

If $\|\Lambda\| < \infty$, then we say that Λ is bounded.

Definition. A set $E \subset X$ is called **nowhere dense** if $\overline{E}$ does not contain any open set in X. A countable union of such sets E is called a **set of the first category**. Otherwise, it is a **set of the second category**.

Definition. Let X be a normed vector space. We define the **dual space** of X as

$$X^* = \{\Lambda : X \to \mathbb{C} \mid \Lambda \text{ is a bounded linear functional } \}$$

Key Theorems

Theorem. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed vector spaces, and $\Lambda : X \to Y$ be linear. Then the following are equivalent:*

1. *Λ is bounded*
2. *Λ is continuous*
3. *Λ is continuous at some $x_0 \in X$.*

Theorem. ***[Baire's Category Theorem]*** *Let (X, d) be a complete metric space, and U_n be an open dense subset of X for all $n \in \mathbb{N}$. Then $\bigcap_{n=1}^{\infty} U_n$ is also dense in X.*

Theorem. ***[Banach-Steinhaus Theorem]*** *Let $(X, \|\cdot\|_X)$ be a Banach space and $(Y, \|\cdot\|_Y)$ be a normed vector space. Let $\{\Lambda_\alpha : \alpha \in A\}$ be a collection of bounded linear maps from X to Y. Then either*

1. ***Bounded Uniformly****: There is an $M > 0$ such that for all $\alpha \in A$, $\|\Lambda_\alpha\| \leq M$*
2. ***Everything Blows Up****: There is a G_δ-set $S \subset X$, such that for all $x \in S$, $\sup_{\alpha \in A} \|\Lambda_\alpha\|_Y = \infty$.*

Theorem. ***[Open Mapping Theorem]*** *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. If $\Lambda : X \to Y$ is a surjective bounded linear map, then Λ is an open map.*

Theorem. ***[Closed Graph Theorem]*** *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. Let $\Lambda : X \to Y$ be linear. Then Λ is bounded if and only if the set $\mathcal{G}(\Lambda) = \{(x, \Lambda(x)) : x \in X\}$ is closed in $X \times Y$.*

Theorem. ***[Hanh-Banach]*** *Let $(X, \|\cdot\|)$ be a normed vector space. Let M be a subspace of X, and $\lambda : M \to \mathbb{C}$ be a bounded linear functional. Then, λ can be extended to a bounded linear functional $\Lambda : X \to \mathbb{C}$ such that $\Lambda|_M = \lambda$ and $\|\Lambda\| = \|\lambda\|$.*

Question 5.8

Let X be a normed linear space, and let X^* be its dual space with the norm

$$\|f\| = \sup\{|f(x)| : \|x\| \leq 1\}$$

(a) Prove that X^* is a Banach space.

(b) Prove that the mapping $f \mapsto f(x)$ is, for each $x \in X$, a bounded linear functional on X^*, of norm $\|x\|$.

(c) Prove that the sequence $(\|x_n\|)_{n=1}^{\infty}$ is bounded if $(x_n)_{n=1}^{\infty}$ is a sequence in X such that $(f(x_n))_{n=1}^{\infty}$ is bounded for every $f \in X^*$.

Question 5.9

Let c_0, l^1, and l^∞ be the Banach spaces consisting of all complex sequences $x = (\xi_n)_{n=1}^\infty$, defined as follows:

$$x \in l^1 \text{ if and only if } \|x\|_1 = \sum |\xi_i| < \infty$$
$$x \in l^\infty \text{ if and only if } \|x\|_\infty = \sup |\xi_i| < \infty$$

and c_0 is the subspace of l^∞ consisting of all $x \in l^\infty$ for which $\lim_{i\to\infty} \xi_i = 0$.

(a) Show that if Λ is a bounded linear functional on c_0, then there is a $y \in l^1$ such that for each $x \in c_0$, $\Lambda(x) = \sum_{n=1}^\infty x_n y_n$.

(b) In the same sense $(l^1)^* = l^\infty$.

(c) Every $y \in l^1$ induces a bounded linear functional on l^∞. However, this does not give all of $(l^\infty)^*$, since $(l^\infty)^*$ contains nontrivial functionals that vanish on all of c_0.

(d) c_0 and l^1 are separable but l^∞ is not.

Answer:

(a) Let $B = \{e_n\}$ denote the canonical basis for c_0. Then any vector $x \in c_0$ can be written as $x = \sum_{n=1}^\infty x_n e_n$, where $x_n \in \mathbb{C}$ for each $n \in \mathbb{N}$. Then by linearity of Λ, we have $\Lambda(x) = \sum_{n=1}^\infty x_n \Lambda(e_n)$. Then our candidate sequence is $y_n = \Lambda(e_n)$ for all $n \in \mathbb{N}$. If $\sum_{n=1}^\infty |\Lambda(e_n)| < \infty$, then we're done. Since $\Lambda : c_0 \to \mathbb{C}$, for any $e_m \in B$, there exists $\theta_m \in [0, 2\pi)$ such that $e^{i\theta_m} \cdot \Lambda(e_m) \in \mathbb{R}^+$. Therefore $\sum_{n=1}^\infty |\Lambda(e_n)| = \sum_{n=1}^\infty e^{i\theta_n} \cdot \Lambda(e_n)$. Now, let $N < \infty$ and consider the following:

$$\sum_{n=1}^N |\Lambda(e_n)| = \sum_{n=1}^N e^{i\theta_n} \cdot \Lambda(e_n) = \Lambda\left(\sum_{n=1}^N e^{i\theta_n} \cdot e_n\right) \leq \|\Lambda\| \cdot \left\|\sum_{n=1}^N e^{i\theta_n} \cdot e_n\right\|_\infty .$$

Since the supremum norm on the right hand side is bounded above by 1, for fixed N, $\sum_{n=1}^N |\Lambda(e_n)|$ is bounded by $\|\Lambda\|$. Using the fact that Λ is a bounded linear functional, $\|\Lambda\| < \infty$. Because this is true for any $N \in \mathbb{N}$, taking the limit $N \to \infty$, we have that $\sum_{n=1}^\infty |\Lambda(e_n)| < \infty$.

Question 5.10

If $\sum \alpha_i \xi_i$ converges for every sequence $(\xi)_{n=1}^{\infty}$ such that $\lim_{i \to \infty} \xi_i = 0$, prove that $\sum |\alpha_i| < \infty$.

Answer:

Proof.

$\square$

CHAPTER 6

Key Definitions

Definition. A **measurable partition** of $E \in \mathcal{M}$ is a sequence $(E_n)_{n=1}^{\infty} \subset \mathcal{M}$ such that $E_i \cap E_j = \phi$ for all $i \neq j$ and $\bigcup_{n=1}^{\infty} E_n = E$.

Definition. Let $\mathcal{M}$ be a σ-algebra. A **complex measure** μ on $\mathcal{M}$ is a set function $\mu : \mathcal{M} \to \mathbb{C}$ such that

$$\mu(E) = \sum_{n=1}^{\infty} \mu(E_n)$$

for each measurable partition $(E_n)_{n=1}^{\infty}$.

Definition. The **total variation** of μ is a set function $|\mu| : \mathcal{M} \to \mathbb{R}$ defined by

$$|\mu|(E) = \sup \left\{ \sum_{n=1}^{\infty} |\mu(E_n)| : (E_n)_{n=1}^{\infty} \text{ is a partition of } E \right\} .$$

Definition. Define $\mathbb{M}(\mathcal{M})$ to be the set of all complex measures on $(X, \mathcal{M})$. For all $\mu, \nu \in \mathbb{M}(\mathcal{M})$ and $\alpha \in \mathbb{C}$, define $\mu + \alpha\nu : \mathbb{M}(\mathcal{M}) \to \mathbb{C}$ by

$$(\mu + \alpha\nu)(E) = \mu(E) + \alpha\nu(E).$$

Hence, $\mathbb{M}(\mathcal{M})$ is a complex vector space. Moreover, define $\|\mu\| = |\mu|(X)$. Then, $\mathcal{M}(\mathcal{M})$ is a normed vector space.

Definition. Let μ be a real measure on X. Define

$$\mu^+ = \frac{1}{2}(|\mu| + \mu) \quad \text{and} \quad \mu^- = \frac{1}{2}(|\mu| - \mu).$$

Both μ^+ and μ^- are positive real measures. They are called the **positive** and **negative variations** of μ, respectively. This representation is also called the **Jordan decomposition**.

Definition. Let μ be a positive measure, and λ be any measure on $(X, \mathcal{M})$. Let $A, B \in \mathcal{M}$.

1) λ is said to be **absolutely continuous** with respect to μ if $\lambda(E) = 0 \Rightarrow \mu(E) = 0$, for all $E \in \mathcal{M}$. We write this as $\mu \ll \lambda$.

2) λ is said to be **concentrated on** A if $\lambda(E) = \lambda(E \cap A)$, for all $E \in \mathcal{M}$.

3) Suppose $A \cap B = \emptyset$ and λ_1, λ_2 are measures on $\mathcal{M}$. If λ_1 is concentrated on A and λ_2 is concentrated on B, then we say λ_1 and λ_2 are **mutually disjoint**, and denote this by $\lambda_1 \perp \lambda_2$.

Definition. A complex Borel measure μ on X is said to be **regular** if $|\mu|$ is regular on X. Denote

$$\mathcal{M}(X) := \{\mu : \text{regular complex Borel measure on } X\}.$$

Key Theorems

Theorem. *If μ is a complex measure on X, then $|\mu|(X) < \infty$.*

Theorem. ***[Lebesgue-Radon-Nikodym Theorem]*** *Let μ be a positive σ-finite measure and λ be a complex measure on $(X, \mathcal{M})$. Then we have*

1. *There exists a unique pair of complex measures λ_a and λ_s such that*

$$\lambda = \lambda_a + \lambda_s \ , \quad \lambda_a \ll \mu \ , \quad \text{and} \quad \lambda_s \perp \mu \ .$$

2. *There is a unique $h \in L^1(\mu)$ such that*

$$\lambda_a(E) = \int_E h \, d\mu \ , \qquad \text{for all } E \in \mathcal{M} \ .$$

*We call the pair (λ_a, λ_s) the **Lebesgue decomposition** of λ relative to μ. Also, we call the function $h \in L^1(\mu)$ the **Radon-Nikodym derivative** of λ_a with respect to μ. Also, $d\lambda_a = h \, d\mu$ or $h = \frac{d\mu}{d\lambda_a}$.*

Theorem. ***[Riesz Representation on Complex Measures Theorem]*** *Let μ be a complex measure on $(X, \mathcal{M})$. Then there is a measurable function h such that $|h(x)| = 1$ for all $x \in X$ and $d\mu = h \, d|\mu|$.*

Theorem. ***[L^p-Isometry]*** *Let $1 \leq p < \infty$, q be its conjugate exponent, and μ be a σ-finite positive measure on $(X, \mathcal{M})$. Then for all bounded linear functionals $\Lambda \in L^p(\mu)^*$, there is a unique $g \in L^q(\mu)$ such that for each $f \in L^p(\mu)$,*

$$\Lambda(f) = \int_X fg \, d\mu \ .$$

Moreover, $\|\Lambda\| = \|g\|_q$. Hence, $L^q(\mu) \cong L^p(\mu)^*$.

Theorem. ***[Hanh Decomposition Theorem]*** *Let μ be a real measure on $(X, \mathcal{M})$. Then there are $A, B \in \mathcal{M}$, $A \cup B = X$, $A \cap B = \phi$, such that $\mu^+(E) = \mu(E \cap A)$ and $\mu^-(E) = E \cap B$, for all $E \in \mathcal{M}$.*

Proof. Because μ is a complex measure, there exists a measurable function h such that $|h| = 1$ and $d\mu = h\ d|\mu|$. Since μ is specifically a real measure $h = \pm 1$. Define the sets

$$A = \{x \in X : h(x) = 1\} \quad \text{and} \quad B = \{x \in X : h(x) = -1\}$$

and note that $\frac{1}{2}(1+h)(x) = \begin{cases} 1 & x = a \\ 0 & x = b \end{cases}$; thus, for all $E \in \mathcal{M}$

$$\begin{aligned} \mu^+(E) = \frac{1}{2}(|\mu|(E) + \mu(E)) &= \frac{1}{2}\left(\int_E d|\mu| + \int_E d\mu\right) \\ &= \frac{1}{2}\left(\int_E d|\mu| + \int_E h\ d|\mu|\right) \\ &= \int_E \frac{1}{2}(1+h)\ d|\mu| \\ &= \int_{A \cap E} h\ d|\mu| \\ &= \int_{A \cap E} d\mu \\ &= \mu(A \cap E)\ . \end{aligned}$$

$\square$

Question 6.4

Suppose $1 \leq p \leq \infty$, and q is the exponent conjugate to p. Suppose μ is a positive σ-finite measure and g is a measurable function such that $fg \in L^1(\mu)$ for every $f \in L^p(\mu)$. Prove that $g \in L^q(\mu)$.

Answer:

Since μ is σ-finite, we can write X as a countable union of measurable sets X_n such that $\mu(X_n) < \infty$ for all $n \in \mathbb{N}$. Consider the characteristic function χ_E of a measurable set $E \subset X$ with $\mu(E) < \infty$. Since $\chi_E \in L^p(\mu)$, the assumption implies that $\chi_E g \in L^1(\mu)$, meaning

$$\int_E |g| \, d\mu < \infty.$$

Thus, g is integrable over any finite-measure subset of X.

Now, define the functional $T : L^p(\mu) \to \mathbb{C}$ by

$$T(f) = \int_X fg \, d\mu.$$

For any $f_1, f_2 \in L^p(\mu)$ and scalars $\alpha, \beta \in \mathbb{C}$, we have

$$T(\alpha f_1 + \beta f_2) = \int_X (\alpha f_1 + \beta f_2) g \, d\mu = \alpha \int_X f_1 g \, d\mu + \beta \int_X f_2 g \, d\mu = \alpha T(f_1) + \beta T(f_2).$$

Thus, T is linear. Next, we show that T is bounded. Since μ is σ-finite, we can write $X = \bigcup_{n=1}^{\infty} X_n$ where X_n is a nested sequence of sets with increasing but finite measure. Define

$$f_n = \frac{|g|^{q-1}}{\|g\|_{q,n}^{q-1}} \chi_{X_n}, \quad \text{where} \quad \|g\|_{q,n} = \left(\int_{X_n} |g|^q d\mu \right)^{\frac{1}{q}}.$$

Then $f_n \in L^p(\mu)$ and satisfies $\|f_n\|_p \leq 1$. Evaluating $T(f_n)$, we get

$$|T(f_n)| = \left| \int_X f_n g \, d\mu \right| = \left| \int_X \frac{|g|^{q-1}}{\|g\|_{q,n}^{q-1}} \chi_{X_n} \cdot g \, d\mu \right| \leq \int_{X_n} \frac{|g|^{q-1}}{\|g\|_{q,n}^{q-1}} \cdot |g| \, d\mu = \frac{1}{\|g\|_{q,n}^{q-1}} \int_{X_n} |g|^q \, d\mu = \|g\|_{q,n}.$$

On the other hand, by assumption,

$$|T(f)| = \left| \int_X fg \, d\mu \right| \leq \int_X |fg| \, d\mu < \infty$$

for all $f \in L^p(\mu)$, namely all f such that $\|f\|_p \leq 1$, so T is bounded, and there exists a constant $C > 0$ such that

$$\|g\|_{q,n} = |T(f_n)| \leq C \|f_n\|_p < \infty.$$

Now because the X_n are nested, taking the supremum over n, we obtain $\sup\limits_{n \in \mathbb{N}} \|g\|_{q,n} = \|g\|_q < \infty$.

Question 6.5

Suppose X consists of two points a and b; define $\mu(\{a\}) = 1$, $\mu(\{b\}) = \mu(X) = \infty$, and $\mu(\phi) = 0$. Is it true, for this μ, that $L^\infty(\mu)$ is the dual space of $L^1(\mu)$?

Answer:

Firstly, define the following functions:

$$f_1(x) = \begin{cases} 1 & x = a \\ 0 & x = b \end{cases} \qquad\qquad f_2(x) = \begin{cases} 0 & x = a \\ 1 & x = b \end{cases}$$

If $f \in L^1(\mu)$, then we must have that

$$\int_X |f| \, d\mu = \int_{\{a\}} |f| \, d\mu + \int_{\{b\}} |f| \, d\mu = |f(a)| \cdot \mu(\{a\}) + |f(b)| \cdot \mu(\{b\}) < \infty,$$

but since $\mu(\{b\}) = \infty$, $f \in L^1(\mu)$ implies $|f(b)| = 0$. Therefore if $f \in L^1(\mu)$, then $f(x) = c_1 \cdot f_1(x)$ for some $c_1 \in \mathbb{C}$. Hence L^1 can be thought of as the span of f_1, which is a 1-dimensional vector space. It then follows that because $L^1(\mu)$ has finite dimension, then $\dim L^1(\mu) = \dim (L^1(\mu))^* = 1$.

On the other hand, if $g \in L^\infty(\mu)$, then we must have that

$$\sup_{x \in X} |g(x)| < \infty \qquad \Longrightarrow \qquad |g(a)|, |g(b)| \in \mathbb{C},$$

so $g(x) = c_2 \cdot f_1(x) + c_3 \cdot f_2(x)$ for some $c_2, c_3 \in \mathbb{C}$. Hence $\dim L^\infty(\mu) \neq \dim (L^1(\mu))^*$, so no isomorphism can exist between the two spaces.

Question 6.10

Let $(X, \mathcal{M}, \mu)$ be a positive measure space. Call a set $\Phi \subset L^1(\mu)$ uniformly integrable if to each $\varepsilon > 0$ corresponds a $\delta > 0$ such that

$$\left| \int_E f \, d\mu \right| < \varepsilon$$

whenever $f \in \Phi$ and $\mu(E) < \delta$.

(a) Prove that every finite subset of $L^1(\mu)$ is uniformly integrable.

(b) Prove the following convergence theorem of Vitali: If $\mu(X) < \infty$, $(f_n)_{n=1}^{\infty}$ in uniformly integrable, $\lim_{n\to\infty} f_n(x) = f(x)$ a.e., and $|f(x)| < \infty$ a.e., then $f \in L^1(\mu)$ and

$$\lim_{n\to\infty} \int_X |f_n - f| \, d\mu = 0.$$

Question 6.13

Let $\mathcal{M}$ be the collection of all subsets of $[0,1]$ such that either E or $[0,1] \setminus E$ is at most countable. Let μ be the counting measure on this σ-algebra $\mathcal{M}$. If $g(x) = x$ for $0 \le x \le 1$, show that g is not $\mathcal{M}$-measurable, although the mapping

$$\Lambda : f \mapsto \int fg \, d\mu$$

makes sense for every $f \in L^1(\mu)$ and defines a bounded linear functional on $L^1(\mu)$. Thus $(L^1)^* \neq L^\infty$ in this situation.

Answer:

Firstly, $(0, \frac{1}{2}) \subset [0,1]$ is an open set with respect to the usual topology on $\mathbb{R}$, but $g^{-1}((0,\frac{1}{2})) = (0,\frac{1}{2})$ is not countable, nor is its complement $((0,\frac{1}{2}))^c = \{0\} \cup [\frac{1}{2}, 1]$. Therefore $g^{-1}((0,\frac{1}{2})) \notin \mathcal{M}$, so g is not $\mathcal{M}$-measurable.

Now, let $f \in L^1(\mu)$ and define the sets

$$F_n = \left\{ x \in [0,1] : |f(x)| \ge \frac{1}{n} \right\} \quad \text{and} \quad F = \bigcup_{n=1}^{\infty} F_n = \{x \in [0,1] : |f(x)| > 0\}.$$

Since $f \in L^1(\mu)$, $\int_I |f| \, d\mu < \infty$, but we also have:

$$\infty > \int_{[0,1]} |f| \, d\mu \ge \int_{E_n} |f| \, d\mu \ge \int_{E_n} \frac{1}{n} \, d\mu = \frac{1}{n} \cdot \mu(E_n)$$

for all $n \in \mathbb{N}$. Hence, for all $n \in \mathbb{N}$, $\mu(E_n) < \infty$, and because μ is the counting measure, E is at most countable. Therefore, if $f \in L^1(\mu)$ is such that $\|f\|_1 = 1$, then

$$\|\Lambda(f)\| = \left| \int fg \, d\mu \right| = \left| \int xf(x) \, d\mu \right| = \left| \sum_{x \in E} xf(x) \right| \le \sum_{x \in E} |xf(x)| \le \sum_{x \in E} |f(x)| = \|f\|_1 < \infty$$

Thus Λ is a bounded linear functional on $L^1(\mu)$.

CHAPTER 7

Key Definitions

Definition. A function $f : \mathbb{R} \to \mathbb{C}$ is **differentiable** at $x_0 \in \mathbb{R}$ if there exists $A(x_0) \in \mathbb{C}$ such that for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in (a, b)$ we have:

$$\left| \frac{f(b) - f(a)}{b - a} - A(x_0) \right| < \varepsilon$$

whenever $|b - a| < \delta$. If $A(x_0)$ exists, we denote it by $f'(x_0)$.

Definition. The **symmetric derivative** of μ at x is defined to be

$$(D\mu)(x) = \lim_{r \to 0} \frac{\mu(B(x, r))}{m(B(x, r))}$$

where m is the Lebesgue measure. If it exists for all x we simply denote it by $D\mu$.

Definition. The **maximal function** of μ is the function $M\mu : \mathbb{R}^k \to [0, \infty]$ defined by

$$(M\mu)(x) = \sup_{r>0} \frac{|\mu|(B(x, r))}{m(B(x, r))} .$$

We see that $M\mu$ always exists since its range includes infinity.

Definition. The $f : \mathbb{R}^k \to \mathbb{C}$. Define the **maximal function of *f*** to be

$$(Mf)(x) = \sup_{r>0} \frac{1}{m(B(x, r))} \cdot \int_{B(x,r)} |f| \, dm .$$

Definition. If $f \in L^1(\mathbb{R}^k)$, we say that $x_0 \in \mathbb{R}^k$ is a **Lebesgue point** of f if

$$\lim_{r \to 0} \frac{1}{m(B(x_0, r))} \cdot \int_{B(x_0,r)} |f(x) - f(x_0)| \, dm(x) = 0 .$$

Definition. A function $f : [a, b] \to \mathbb{C}$ is call **absolutely continuous** if for any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $((\alpha_i, \beta_i))_{i=1}^n$ is a finite collection of disjoint intervals in I with $\sum_{i=1}^n (\beta_i - \alpha_i) < \delta$ we have

$$\sum_{i=1}^n |f(\beta_i) - f(\alpha_i)| < \varepsilon .$$

Definition. Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous. The **total variation function** $F : [a, b] \to [0, \infty)$ is defined by

$$F(x) := \sup_{\{t_i\}_{i=0}^N} \sum_{i=1}^N |f(t_i) - f(t_{i-1})|,$$

where $\{t_i\}_{i=0}^N$ is any finite partition of $[a, x]$ with $a = t_0 < t_1 < \cdots < t_N = x$.

Key Theorems

Theorem. ***[Lebesgue Differentiation Theorem]*** *If $f \in L^1(\mathbb{R}^k)$, then m-almost every $x \in \mathbb{R}^k$ is a Lebesgue point of f.*

Theorem. *Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be continuous and non-decreasing. Then the following are equivalent:*

1. *f is absolutely continuous*
2. *f maps sets of measure zero to sets of measure zero*
3. *f is differentiable m-almost everywhere on I, $f' \in L^1(I)$, and $f(x) - f(a) = \int_a^x f' \, dm$.*

Theorem. ***[Fundamental Theorem of Calculus]*** *If $f : I \to \mathbb{C}$ is absolutely continuous, then f' exists m-almost everywhere, $f' \in L^1(I)$, and for all $x \in I$ we have:*

$$f(x) - f(a) = \int_{[a,b]} f' \, dm \ .$$

Question 7.1

Show that $|f(x)| \leq (Mf)(x)$ at every Lebesgue point of f if $f \in L^1(\mathbb{R}^k)$.

Answer:

Let $f \in L^1(\mathbb{R})$. Then by definition

$$Mf(x) = \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} |f(y)|\, dy \qquad \text{and} \qquad \lim_{r\to 0} \frac{1}{2r} \int_{x-r}^{x+r} |f(y) - f(x)|\, dy = 0$$

So for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $0 < r < \delta$, we have

$$\frac{1}{2r} \int_{x-r}^{x+r} |f(y) - f(x)|\, dy < \varepsilon.$$

It follows that

$$\left| \frac{1}{2r} \int_{x-r}^{x+r} f(y)\, dy - f(x) \right| = \left| \frac{1}{2r} \int_{x-r}^{x+r} f(y) - f(x)\ dy \right| \leq \frac{1}{2r} \int_{x-r}^{x+r} |f(y) - f(x)|\, dy < \varepsilon.$$

Therefore, by the reverse triangle inequality

$$\varepsilon > \left| \frac{1}{2r} \int_{x-r}^{x+r} f(y)\, dy - f(x) \right| \geq \left| \left| \frac{1}{2r} \int_{x-r}^{x+r} f(y)\ dy \right| - |f(x)| \right| \implies \left| \frac{1}{2r} \int_{x-r}^{x+r} f(y)\, dy \right| > |f(x)| - \varepsilon.$$

On the other hand

$$\left| \frac{1}{2r} \int_{x-r}^{x+r} f(y)\, dy \right| \leq \frac{1}{2r} \int_{x-r}^{x+r} |f(y)|\, dy.$$

Combining the two inequalities, we get

$$\frac{1}{2r} \int_{x-r}^{x+r} |f(y)|\, dy \geq |f(x)| - \varepsilon.$$

Taking the supremum over all $r > 0$, and letting $\varepsilon \to 0$, we conclude that

$$Mf(x) \geq |f(x)|.$$

Question 7.10

If $f \in \text{Lip } 1$ on $[a, b]$, prove that f is absolutely continuous and that $f' \in L^\infty$.

Answer:

From $f \in \text{Lip } 1$, we have that for all $s, t \in [a, b]$

$$\frac{|f(s) - f(t)|}{|s-t|} < \infty \quad \Longrightarrow \quad \text{there exists } M > 0 \text{ such that } |f(s) - f(t)| \leq M \cdot |s-t| \; .$$

Then for all $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{M}$ and let $\{(\alpha_i, \beta_i)\}_{i=1}^n$ be a finite collection of disjoint subintervals of $[a, b]$. Whenever we have that $\sum_{i=1}^n |\beta_i - \alpha_i| < \delta$, then

$$\sum_{i=1}^n |f(\beta_i) - f(\alpha_i)| \leq \sum_{i=1}^n M \cdot |\beta_i - \alpha_i| = M \cdot \sum_{i=1}^n |\beta_i - \alpha_i| < M \cdot \delta = \varepsilon \; .$$

Hence f is absolutely continuous, and by a theorem we have $f' \in L^1([a, b])$ and that

$$f(x) - f(a) = \int_a^x f' \, dm \quad \Longrightarrow \quad \int_a^x f' \, dm \leq M \cdot |x - a| \quad \Longrightarrow \quad \frac{1}{|x-a|} \cdot \int_a^x f' \, dm \leq M \; .$$

This holds for any $s, t \in [a, b]$ since $f' \in L^1([s, t])$.

Now by the Lebesgue Differentiation Theorem, almost every $x_0 \in [a, b]$ is a Lebesgue point of f', and at all such points we have

$$f'(x_0) = \lim_{r \to 0} \frac{1}{m(B_r(x_0))} \cdot \int_{B_r(x_0)} f'(x) \, dm \leq M \; .$$

Therefore, $f'(x) \leq M$ for almost every $x \in [a, b]$; thus, $f' \in L^\infty([a, b])$.

Question 7.11

Assume that $1 < p < \infty$, f is absolutely continuous on $[a, b]$, $f' \in L^p$, and $\alpha = \frac{1}{q}$, where q is the exponent conjugate of p. Prove that $f \in \text{Lip } \alpha$.

Question 7.14

Show that the product of two absolutely continuous functions on $[a, b]$ is absolutely continuous. Use this to derive a theorem about integration by parts.

CHAPTER 8

Key Definitions

Definition. An **Algebra** $\mathcal{A} \subset \mathcal{P}(X)$ is a nonempty collection of subsets of X which is closed under finite unions and complements.

Definition. A **monotone class** $\mathcal{M} \subset \mathcal{P}(X)$ is a nonempty collection of subsets in X such that $\mathcal{M}$ is closed under countable increasing unions and countable decreasing intersections. In other words, for each $i \in \mathbb{N}$, $A_i \subset A_{i+1}$, $B_{i+1} \subset B_i$ and $A_i, B_i \in \mathcal{M}$, then $\bigcup_{i=1}^{\infty} A_i$, $\bigcap_{i=1}^{\infty} B_i \in \mathcal{M}$.

Definition. A **measurable rectangle** $E \in X \times Y$ is of the form $A \times B$ where $A \in \mathcal{M}$ and $B \in \mathcal{N}$. An **elementary set** is a finite union of disjoint measurable rectangles. Let $\mathcal{E}$ denote the collection of all elementary sets, and $\mathcal{M} \otimes \mathcal{N}$ the σ-algebra generated by $\mathcal{E}$.

Definition. Let $E \subset X \times Y$. Define the ***x*-section** and ***y*-section**, respectively, by the following:

$$E_x = \{y \in Y : (x, y) \in E\} \quad \text{and} \quad E^y = \{x \in X : (x, y) \in E\}$$

Definition. The **product measure** $\mu \times \nu : (\mathcal{M} \otimes \mathcal{N}) \to [0, \infty]$ is defined by

$$(\mu \times \nu)(Q) = \int_X \nu(Q_x)\, d\mu(x) = \int_Y \mu(Q^y)\, d\nu(y) .$$

Definition. Let f and g be functions on $\mathbb{R}^n$. The **convolution** of f and g is the function $f * g : \mathbb{R}^n \to \mathbb{C}$ defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - t) g(t)\, dt .$$

We denote $f * g$ as the convolution of f and g.

Key Theorems

Theorem. ***[Monotone Class Theorem]*** *If $\mathcal{A}$ is an algebra of a set X, then the monotone class $\mathcal{M}$ generated by $\mathcal{A}$ is precisely the σ-algebra generated by $\mathcal{A}$.*

Theorem. *If $f : (X \times Y) \to \mathbb{C}$ is $\mathcal{M} \otimes \mathcal{N}$-measurable, then $f^y(x) = f(x, y)$ is $\mathcal{M}$-measurable. Likewise, $f_x(y) = f(x, y)$ is $\mathcal{N}$-measurable.*

Theorem. ***[Fubini's Theorem]*** *Let* $(X, \mathcal{M}, \mu)$ *and* $(Y, \mathcal{N}, \nu)$ *be* σ*-finite measure spaces,* f *be a complex* $\mathcal{M} \otimes \mathcal{N}$*-measurable. Then we have:*

1. *If* $f \geq 0$*, then*

$$\int_X \int_Y f_x \, d\nu \, d\mu = \int_{X \times Y} f \, d(\mu \times \nu) = \int_Y \int_X f^y \, d\mu \, d\nu$$

2. *If* $f \in L^1(\mu \times \nu)$*, then for* μ*-almost every* $x \in X$*,* $f_x \in L^1(\nu)$*,* $\int_Y |f_x| \, d\nu < \infty$*, and* $\psi(x) = \int_Y f_x \, d\nu \in L^1(\mu)$*.*

3. *If* f *is complex and*

$$\int_X \left(\int_Y |f_x| \, d\nu \right) \, d\mu \, ,$$

   then $f \in L^1(\mu \times \nu)$ *and* (2) *holds.*

4. *For all* $\mathcal{M} \otimes \mathcal{N}$*-measurable* $f \in L^1(\mu \times \nu)$*,*

$$\int_X \left(\int_Y f_x \, d\nu \right) \, d\mu = \int_{X \times Y} f \, d(\mu \times \nu) = \int_Y \left(\int_X f^y \, d\mu \right) \, d\nu$$

Question 8.4

Suppose $1 \leq p \leq \infty$, $f \in L^1(\mathbb{R})$, and $g \in L^p(\mathbb{R})$.

(a) Imitate the proof of Theorem 8.14 to show that the integral defining $(f * g)(x)$ exists for almost all x, that $f * g \in L^p(\mathbb{R})$ and that $\|f * g\|_p \leq \|f\|_1 \|g\|_p$.

(b) Show that equality can hold in (a) if $p = 1$ and if $p = \infty$, and find the conditions under which this happens.

(c) Assume $1 < p < \infty$, and equality holds in (a). Show that then either $f = 0$ a.e. or $g = 0$ a.e.

(d) Assume $1 \leq p \leq \infty$, $\varepsilon > 0$, and show that there exist $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$ such that

$$\|f * g\|_p > (1 - \varepsilon)\|f\|_1 \|g\|_p.$$

Answer:

(a) Since $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$, both are measurable functions. Hence, the function $h_x(y) = f(x - y)g(y)$ is measurable in y for each fixed x, because it is the product of measurable functions and translation preserves measurability. Moreover, by Hölder's inequality we have:

$$\int |f(x - y)g(y)| \, dy \leq \|f\|_1 \|g\|_p < \infty$$

for almost every x. Thus, $f * g$ is well-defined almost everywhere.

Consider the case when $p = 1$. Then by definition

$$\|f * g\|_1 = \int |f * g| \, dx = \int \left| \int f(x - y) \cdot g(y) \, dy \right| \, dx \leq \iint |f(x - y) \cdot g(y)| \, dy \, dx \, .$$

Then we use the fact that f and g are measurable to invoke Fubini's theorem

$$\begin{aligned} \|f * g\|_1 &\leq \iint |f(x - y) \cdot g(y)| \, dx \, dy \\ &= \int |g(y)| \int |f(x - y)| \, dx \, dy \\ &= \int |g(y)| \cdot \|f\|_1 \, dy \\ &= \|f\|_1 \cdot \|g\|_1 \, . \end{aligned}$$

Now consider the case when $p = \infty$. Then we have:

$$\|f * g\|_\infty = \left|\big(f * g\big)(x)\right| = \left|\int f(x-y)\cdot g(y)\ dy\right| \le \int |f(x-y)\cdot g(y)|\ dy$$

and by measurability of f and g, Hölder's inequality gives

$$\begin{aligned}
\|f * g\|_\infty &\le \int |f(x-y)|\cdot \|g\|_\infty\ dy \\
&= \int |f(x-y)|\ dy \cdot \|g\|_\infty \\
&= \|f\|_1 \cdot \|g\|_\infty\ .
\end{aligned}$$

Finally, we consider the case when $1 < p < \infty$. Let q be the exponent conjugate of p. Knowing that $\left|\big(f * g\big)(x)\right| \le \int |f(x-y)\cdot g(y)|dy$ for each $x \in \mathbb{R}$, we apply Hölder's inequality:

$$\begin{aligned}
\left|\big(f * g\big)(x)\right| &\le \int |f(x-y)|\cdot |g(y)|\ dy \\
&= \int |f(x-y)|^{\frac{1}{p}+\frac{1}{q}}\cdot |g(y)|\ dy \\
&= \int \left(|f(x-y)|^{\frac{1}{q}}\right)\cdot\left(|f(x-y)|^{\frac{1}{p}}\cdot |g(y)|\right)\ dy \\
&\le \left(\int |f(x-y)|\ dy\right)^{1/q}\cdot\left(\int |f(x-y)|\cdot |g(y)|^p\ dy\right)^{1/p}\ .
\end{aligned}$$

With this in hand, consider the following:

$$\begin{aligned}
\|f * g\|_p^p &= \int \left|\big(f * g\big)(x)\right|^p\ dx \\
&\le \int \left(\int |f(x-y)|\ dy\right)^{\frac{p}{q}}\cdot\left(\int |f(x-y)|\cdot |g(y)|^p\ dy\right)\ dx \\
&= \int \|f\|_1^{\frac{p}{q}}\cdot\left(\int |f(x-y)|\cdot |g(y)|^p\ dy\right)\ dx \\
&= \iint \|f\|_1^{\frac{p}{q}}\cdot |f(x-y)|\cdot |g(y)|^p\ dy\ dx\ .
\end{aligned}$$

The integrand in the final integral is measurable by assumption, so we use Fubini's theorem to swap the order of integration.

$$\begin{aligned}
\|f * g\|_p^p &\le \iint \|f\|_1^{\frac{p}{q}}\cdot |f(x-y)|\cdot |g(y)|^p\ dx\ dy \\
&= \int \|f\|_1^{\frac{p}{q}}\cdot |g(y)|^p \int |f(x-y)|\ dx\ dy \\
&= \|f\|_1^{\frac{p}{q}}\cdot \|f\|_1 \cdot \int |g(y)|^p\ dy \\
&= \|f\|_1^{\frac{p}{q}}\cdot \|f\|_1 \cdot \|g\|_p^p\ .
\end{aligned}$$

At last, taking the p^{th} root on both sides of the inequality yields

$$\|f * g\|_p \leq \|f\|_1^{\frac{1}{q}} \cdot \|f\|_1^{\frac{1}{p}} \cdot \|g\|_p = \|f\|_1 \cdot \|g\|_p \,.$$

Because the right hand side in every case is finite by assumption, $f * g \in L^p(\mathbb{R})$.

(b) I think I'm too dumb to do the rest, but this is probably a good question to know.

Question 8.5

Let M be the Banach space of all complex Borel measures on $\mathbb{R}$. The norm in M is $\|\mu\| = |\mu|(\mathbb{R})$. Associate to each Borel set $E \subset \mathbb{R}$ the set

$$E_2 = \{(x, y) \in \mathbb{R}^2 : x + y \in E\}.$$

If $\mu, \lambda \in M$, define their convolution $\mu * \lambda$ to be the set function given by $(\mu * \lambda)(E) = (\mu \times \lambda)(E_2)$ for every Borel set $E \subset \mathbb{R}$; where $\mu \times \lambda$ is as in Definition 8.7.

(a) Prove that $\mu * \lambda \in M$ and that $\|\mu * \lambda\| \leq \|\mu\| \cdot \|\lambda\|$.

(b) Prove that $\mu * \lambda$ is the unique $\nu \in M$ such that

$$\int f \, d\nu = \int \int f(x + y) \, d\mu(x) \, d\lambda(y)$$

for every $f \in C_0(\mathbb{R})$.

(c) Prove that the convolution in M is commutative, associative, and distributive with respect to addition.

(d) Prove the formula

$$(\mu * \lambda)(E) = \int \mu(E - t) \, d\lambda(t)$$

for every $\mu, \lambda \in M$ and every Borel set E. Here $E - t = \{x - t : x \in E\}$.

Question 8.14

Complete the following proof of Hardy's inequality. Suppose $f \geq 0$ on $(0, \infty)$, $f \in L^p$, $1 < p < \infty$, and

$$F(x) = \frac{1}{x} \int_0^x f(t)\, dt.$$

Write $xF(x) = \int_0^x f(t) t^\alpha t^{-\alpha}\, dt$, where $0 < \alpha < \frac{1}{q}$, use Hölder's inequality to get an upper bound for $F(x)^p$, and integrate to obtain

$$\int_0^\infty F^p(x)\, dx \leq (1 - \alpha q)^{1-p} (\alpha p)^{-1} \int_0^\infty f^p(t)\, dt.$$

Show that the best choice of α yields

$$\int_0^\infty F^p(x)\, dx \leq \left(\frac{p}{p-1} \right)^p \int_0^\infty f^p(t)\, dt.$$

CHAPTER 9

Key Definitions

Definition. Let $f, g \in L^1(\mathbb{R})$. The **convolution** of f and g is defined by

$$(f * g)(x) := \int_{\mathbb{R}} f(x-y)g(y)\,dy, \quad x \in \mathbb{R}. \tag{9.1.1}$$

The **Fourier transform** of f is defined by

$$\hat{f}(t) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{-ixt}\,dx, \quad t \in \mathbb{R}. \tag{9.1.2}$$

Key Theorems

Lemma. *Let $f, g \in L^1(\mathbb{R})$. The Fourier transform $\hat{f}$ satisfies the following elementary properties:*

(a) If $g(x) = e^{i\alpha x} f(x)$, then $\hat{g}(t) = \hat{f}(t-\alpha)$.

(b) If $g(x) = f(x-\alpha)$, then $\hat{g}(t) = e^{-i\alpha t}\hat{f}(t)$.

(c) If $f, g \in L^1(\mathbb{R})$, then

$$(f \hat{*} g)(t) = \sqrt{2\pi}\,\hat{f}(t)\hat{g}(t)\ .$$

(d) If $g(x) = f(-x)$, then $\hat{g}(t) = \hat{f}(-t)$.

(e) For $\lambda \neq 0$, if $g(x) = f\left(\frac{x}{\lambda}\right)$, then $\hat{g}(t) = |\lambda|\,\hat{f}(\lambda t)$.

(f) If $g(x) = -ixf(x)$ and $g \in L^1(\mathbb{R})$, then $\hat{f}$ is differentiable and

$$\hat{f}'(t) = \hat{g}(t)\ .$$

Proof. (b) Let $f, g \in L^1(\mathbb{R})$. Define the convolution

$$(f * g)(x) := \int_{\mathbb{R}} f(x-y)g(y)\,dy.$$

Which substituting the definition of convolution into the Fourier transform yields

$$(f \hat{*} g)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x-y)g(y)\,dy \right) e^{-ixt}\,dx.$$

By Fubini's Theorem (justified since $f, g \in L^1(\mathbb{R})$ implies $f * g \in L^1(\mathbb{R})$), and by making the substitution $u = x - y$, so that $x = u + y$ and $dx = du$:

$$\begin{aligned}
(f \hat{*} g)(t) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(y) \left(\int_{\mathbb{R}} f(u)e^{-i(u+y)t}\,du \right) dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(y)e^{-iyt} \left(\int_{\mathbb{R}} f(u)e^{-iut}\,du \right) dy \\
&= \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{R}} f(u)e^{-iut}\,du \right) \left(\int_{\mathbb{R}} g(y)e^{-iyt}\,dy \right) = \sqrt{2\pi} \cdot \hat{f}(t) \cdot \hat{g}(t).
\end{aligned}$$

$\square$

Theorem. *Let $f : \mathbb{R} \to \mathbb{C}$, $y \in \mathbb{R}$, and $1 \le p < \infty$. Define the **translate** of f by*

$$f_y(x) = f(x - y).$$

If $f \in L^p(\mathbb{R})$, then the map $y \mapsto f_y$ is uniformly continuous from $\mathbb{R}$ to $L^p(\mathbb{R})$.

Theorem. *If $f \in L^1(\mathbb{R})$, then $\hat{f} \in C_0(\mathbb{R})$ and $\|\hat{f}\|_\infty \le \frac{1}{\sqrt{2\pi}} \|f\|_1$.*

Theorem. *Let $H(t) = e^{-|t|}$. For $\lambda > 0$, define*

$$h_\lambda(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} H(\lambda t) e^{itx} \, dt = \sqrt{\frac{2}{\pi}} \cdot \frac{\lambda}{\lambda^2 + x^2}.$$

If $f \in L^1(\mathbb{R})$, then

$$(f * h_\lambda)(x) = \int_{\mathbb{R}} \frac{\hat{f}(t)}{\lambda + ix} H(\lambda t) e^{ixt} \, dt.$$

Theorem. *If $g \in L^\infty(\mathbb{R})$ and g is continuous at $x \in \mathbb{R}$, then*

$$\lim_{\lambda \to 0} \frac{1}{\sqrt{2\pi}} (g * h_\lambda)(x) = g(x).$$

Theorem. ***[Inversion Theorem]*** *If $f, \hat{f} \in L^1(\mathbb{R})$, and*

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(t) e^{itx} \, dt,$$

then $g \in C_0(\mathbb{R})$ and $f(x) = g(x)$ for m-almost every $x \in \mathbb{R}$.

Question 9.6

Suppose $f \in L^1$, f is differentiable almost everywhere, and $f' \in L^1$. Does it follow that the Fourier transform of f' is $ti\hat{f}(t)$?

Question 9.8

If p and q are conjugate exponents, $f \in L^p$, $g \in L^q$, and $h = f * g$, prove that h is uniformly continuous. If also $1 < p < \infty$, then $h \in C_0$; show that this fails for some $f \in L^1$ and $g \in L^\infty$.

QUESTION GRAVEYARD

Question 1 (1.1)

Does there exist an infinite σ-algebra which has only countably many members?

Question 1.8

Put $f_n = \chi_E$ if n is odd, $f_n = 1 - \chi_E$ if n is even. What is the relevance of this example to Fatou's lemma?

Question 1.7

Suppose $f_n : X \to [-\infty, \infty]$ is measurable for $n \in \mathbb{N}$, $f_1 \geq f_2 \geq \cdots \geq 0$, $\lim\limits_{n\to\infty} f_n(x) = f(x)$ for every $x \in X$, and $f_1 \in L^1(\mu)$. Prove that

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$$

and show that this conclusion does not follow if the condition "$f_1 \in L^1(\mu)$" is omitted.

Question 3 (1.5)

(a) Suppose that $f : X \to [-\infty, \infty]$ and $g : X \to [\infty, \infty]$ are measurable. Prove that the sets

$$\{x : f(x) < g(x)\} \text{ and } \{x : f(x) = g(x)\}$$

are measurable.

(b) Prove that the set of points at which a sequence of a measurable real-valued function converges to a finite limit is measurable.

Question 2 (1.4)

Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences in $[-\infty, \infty]$. Prove the following:

(a) $\limsup\limits_{n\to\infty}(-a_n) = -\liminf\limits_{n\to\infty} a_n$

(b) $\limsup\limits_{n\to\infty}(a_n + b_n) \leq \limsup\limits_{n\to\infty} a_n + \limsup\limits_{n\to\infty} b_n$

provided none of the sums is of the form $\infty - \infty$.

(c) If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $\liminf\limits_{n\to\infty} a_n \leq \liminf\limits_{n\to\infty} b_n$.

Show by an example that strict inequality can hold in part (b).

Question 2.21

If X is compact and $f : X \to (-\infty, \infty)$ is upper semicontinuous, prove that f attains its maximum at some point of X.

Question 2.22

Suppose that X is a metric space, with metric d, that $f : X \to [0, \infty]$ is lower semicontinuous, and that $f(p) < \infty$ for at least one $p \in X$. For $n \in \mathbb{N}$, $x \in X$, define

$$g_n(x) = \inf\{f(p) + n \cdot d(x, p) : p \in X\}$$

and prove that

1. $|g_n(x) - g_n(y)| \leq n \cdot d(x, y)$,
2. $0 \leq g_1 \leq g_2 \leq \cdots \leq f$,
3. $\lim\limits_{n\to\infty} g_n(x) = f(x)$ for all $x \in X$.

Thus f is the pointwise limit of an increasing sequence of continuous functions. Note that the converse is almost trivial.

Question 2.5

Let E be Cantor's familiar "middle thirds" set. Show that $\mu(E) = 0$, even though E and $\mathbb{R}$ have the same cardinality.

Question 2.7

If $0 < \varepsilon < 1$, construct an open set $E \subset [0, 1]$ which is dense in $[0, 1]$, such that $\mu(E) = \varepsilon$. To say that A is dense in B means that the closure of A contains B.

Question 2.2

Let f be an arbitrary complex function on $\mathbb{R}$, and define

$$\begin{aligned} \varphi(x, \delta) &= \sup\{|f(s) - f(t)| : s, t \in (x - \delta, x + \delta)\}, \\ \varphi(x) &= \inf\{\varphi(x, \delta) : \delta > 0\} \end{aligned}$$

Prove that φ is upper semicontinuous, that f is continuous at a point x if and only if $\varphi(x) = 0$, and hence the set of points of continuity of an arbitrary complex function is a G_δ set.

Question 2.1

Let $(f_n)_{n=1}^{\infty}$ be a sequence of real nonnegative functions on $\mathbb{R}$, and consider the following four statements:

(a) If f_1 and f_2 are upper semicontinuous, then $f_1 + f_2$ is upper semicontinuous.

(b) If f_1 and f_2 are lower semicontinuous, then $f_1 + f_2$ is lower semicontinuous.

(c) If each f_n is upper semicontinuous, then $\sum_{n=1}^{\infty} f_n$ is upper semicontinuous.

(d) If each f_n is lower semicontinuous, then $\sum_{n=1}^{\infty} f_n$ is lower semicontinuous.

Show that three of these are true and that one is false. What happens if the word "nonnegative" is omitted? Is the truth of the statements affected if $\mathbb{R}$ is replaced by a general topological space?

Question 3.1

Prove that the supremum of any collection of convex functions on (a, b) is convex on (a, b) if it is finite and that pointwise limits of sequences of convex functions are convex. What can you say about upper and lower limits of sequences of convex functions?

Question 3.5

Assume, in addition to the hypothesis in the previous exercise, that $\mu(X) = 1$.

(a) Prove that $\|f\|_r \leq \|f\|_s$ if $0 < r < s \leq \infty$.

(b) Under what conditions does it happen that $0 < r < s \leq \infty$ and $\|f\|_r = \|f\|_s < \infty$?

(c) Prove that $L^s(\mu) \subset L^r(\mu)$ if $0 < r < s$. Under what conditions do these two spaces contain the same functions?

(d) Assume that $\|f\|_r < \infty$ for some $r > 0$, and prove that

$$\lim_{p \to \infty} \|f\|_p = \exp \int_X \log |f| \, d\mu$$

if $\exp\{-\infty\}$ is defined to be 0.

Question 3.7

For some measures, the relation $r < s$ implies $L^r(\mu) \subset L^s(\mu)$; for others, the inclusion is reversed; and there are some for which $L^r(\mu)$ does not contain $L^s(\mu)$ if $r \neq s$. Give examples of these situations, and find conditions on μ under which these situations will occur.

Question 3.10

Suppose $(f_n)_{n=1}^{\infty} \in L^p(\mu)$, $\lim_{n\to\infty} \|f_n - f\|_p = 0$, and $\lim_{n\to\infty} f_n = g$. What relation exists between f and g?

Question 3.14a

Suppose $1 < p < \infty$, $f \in L^p((0,\infty))$, relative to the Lebesgue measure, and

$$F(x) = \frac{1}{x}\int_0^x f(t)\, d\mu \quad (0 < x < \infty)$$

Prove Hardy's inequality

$$\|F\|_p \leq \frac{p}{p-1}\|f\|_p$$

which shows that the mapping $f \mapsto F$ carries L^p to L^p.

Question 3.14d

Suppose $1 < p < \infty$, $f \in L^p((0,\infty))$, relative to the Lebesgue measure, and

$$F(x) = \frac{1}{x}\int_0^x f(t)\, d\mu \quad (0 < x < \infty)$$

If $f > 0$ and $f \in L^1$, prove that $F \notin L^1$.

Question 4.2

Let $(x_n)_{n=1}^{\infty}$ be a linearly independent set of vectors in H. Show that the following construction yields an orthonormal set $(u_n)_{n=1}^{\infty}$ such that $\{x_1, x_2, \ldots, x_N\}$ and $\{u_1, u_2, \ldots, u_N\}$ have the same span for all N.

Put $u_1 = \frac{x_1}{\|x_1\|}$. Having $u_1, \ldots, u_{n-1}$ define

$$v_n = x_n - \sum_{i=1}^{n-1} \langle x_n, u_i \rangle u_i, \quad u_n = \frac{v_n}{\|v_n\|}.$$

Note that this leads to a proof of the existence of a maximal orthonormal set in separable Hilbert spaces which makes no appeal to the Hausdorff maximality principle.

Question 4.4

Show that H is separable if and only if H contains a maximal orthonormal system which is at most countable.

Question 4.9

If $A \subset [0, 2\pi]$ and A is measurable, prove that

$$\lim_{n\to\infty} \int_A \cos nx \, dx = \lim_{n\to\infty} \int_A \sin nx \, dx = 0.$$

Question 5.2

Prove that the unit ball (open or closed) is convex in every normed linear space.

Question 5.6

Let f be a bounded linear functional on a subspace M of a Hilbert space H. Prove that f has a unique norm-preserving extension to a bounded linear functional on H, and that extension vanishes on $M^\perp$.

Question 5.11

For $0 < \alpha \le 1$, let Lip α denote the space of all complex functions f on $[a, b]$ for which

$$M_f = \sup_{s \neq t} \frac{|f(s) - f(t)|}{|s-t|^\alpha} < \infty.$$

Prove that Lip α is a Banach space, if $\|f\| = |f(a)| + M_f$ or if $\|f\| = M_f + \sup_{x\in[a,b]} |f(x)|$.

Question 5.16

Suppose that X and Y are Banach, and suppose Λ is a linear mapping of X into Y, with the following property: For every sequence $(x_n)_{n=1}^\infty$ in X for which $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} \Lambda x_n$ exist, it is true that $y = \Lambda x$. Prove that Λ is continuous. Observe that there exist nonlinear mappings (of $\mathbb{R}$ onto $\mathbb{R}$, for instance) whose graph is closed although they are not continuous: $f(x) = \frac{1}{x}$ if $x \neq 0$ and $f(0) = 0$.

Question 5.17

If μ is a positive measure, each $f \in L^\infty(\mu)$ defines a multiplication operator M_f on $L^2(\mu)$ into $L^2(\mu)$, such that $M_f(g) = fg$. Prove that $\|M_f\| \leq \|f\|_\infty$. For which measures μ is it true that $\|M_f\| = \|f\|_\infty$ for all $f \in L^\infty(\mu)$? For which $f \in L^\infty(\mu)$ does M_f map $L^2(\mu)$ onto $L^2(\mu)$?

Question 5.18

Suppose $(\Lambda_n)_{n=1}^\infty$ is a sequence of bounded linear transformations from a normed linear space X to a Banach space Y, suppose that $\|\Lambda_n\| \leq M < \infty$ for all $n \in \mathbb{N}$, and suppose there is a dense set $E \subset X$ such that $(\Lambda_n x)_{n=1}^\infty$ converges for each $x \in E$. Prove that $(\Lambda_n x)_{n=1}^\infty$ converges for each $x \in X$.

Question 6.3

Prove that the vector space $M(X)$ of all complex regular Borel measures on a locally compact Hausdorff space X is a Banach space if $\|\mu\| = |\mu|(X)$. *Hint:* Compare to Question 5.8

Question 7.12

Suppose that $\varphi : [a, b] \to \mathbb{R}$ is nondecreasing.

(a) Show that there is a left-continuous nondecreasing f on $[a, b]$ so that $\{x \in [a, b] : f(x) \neq \varphi(x)\}$ is at most countable. (Left-continuous means: if $a < x \leq b$ and $\varepsilon > 0$, then there is a $\delta > 0$ so that $|f(x) - f(x - t)| < \varepsilon$ whenever $0 < t < \delta$.)

(b) Imitate the proof of Theorem 7.18 to show that there is a positive Borel measure μ on $[a, b]$ for which $f(x) - f(a) = \mu([a, x])$ for $a \leq x \leq b$.

Question 7.23

The definition of Lebesgue points applies to individual integrable functions and not to their equivalence classes (section 3.10). However if $F \in L^1(\mathbb{R}^k)$ is one of these equivalence classes, one may call a point $x \in \mathbb{R}^k$ a Lebesgue point of F if there is a complex number, let us call it $(SF)(x)$, such that

$$\lim_{r \to 0} \frac{1}{m(B_r)} \int_{B(x,r)} |f - (SF)(x)| \, dm = 0$$

for one (hence every) $f \in F$. Define $(SF)(x)$ to be 0 at those points $x \in \mathbb{R}^k$ that are not Lebesgue points of F. Prove the following statement: If $f \in F$, and x is a Lebesgue point of f, then x is also a Lebesgue point of F, and $f(x) = (SF)(x)$. Hence $SF \in F$. Thus S "selects" a member of F that has a maximal set of Lebesgue points.

Question 8.2

Suppose f is a Lebesgue measurable nonnegative real function on $\mathbb{R}$ and $A(f)$ is the ordinate set of f. This is the set of all points $(x, y) \in \mathbb{R}^2$ for which $0 < y < f(x)$.

(a) Is it true that $A(f)$ is Lebesgue measurable, in the two-dimensional sense?

(b) If the answer to (a) is affirmative, is the integral of f over $\mathbb{R}$ equal to the measure of $A(f)$?

(c) Is the graph of f a measurable subset of $\mathbb{R}^2$?

(d) If the answer to (c) is affirmative, is the measure of the graph equal to zero?

Question 8.3

Find an example of a positive continuous function f in the open unit square in $\mathbb{R}^2$, whose integral (relative to the Lebesgue measure) is finite but such that $\varphi(x)$ (in the notation of Theorem 8.8) is infinite for some $x \in (0, 1)$.

Question 8.12

Use Fubini's theorem and the relation

$$\frac{1}{x} = \int_0^\infty e^{-xt}\, dt \quad (x > 0)$$

to prove that

$$\lim_{b \to \infty} \int_0^b \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

Question 8.15

Put $\varphi(t) = 1 - \cos t$ if $0 \le t \le 2\pi$, $\varphi(t) = 0$ for all other real t. For $-\infty < x < \infty$, define

$$f(x) = 1, \quad g(x) = \varphi'(x), \quad h(x) = \int_{-\infty}^x \varphi(t)\, dt.$$

Verify the following statements about convolutions of these functions:

1. $(f * g)(x) = 0$ for all x.

2. $(g * h)(x) = (\varphi * \varphi)(x) > 0$ on $(0, 4\pi)$.

3. Therefore $(f * g) * h = 0$, whereas $f * (g * h)$ is a positive constant.

But convolution is supposedly associative, by Fubini's theorem (Question 8.5c). What went wrong?

Question 9.2

Compute the Fourier transform of the characteristic function of an interval. For $n \in \mathbb{N}$, let g_n be the characteristic function of $[-n, n]$, let h be the characteristic function of $[-1, 1]$, and compute $g_n * h$ explicitly. Show that $g_n * h$ is the Fourier transform of a function $f_n \in L^1$; except for a multiplicative constant,

$$f_n(x) = \frac{\sin x \sin nx}{x^2}.$$

Show that $\lim_{n\to\infty} \|f_n\|_1 = \infty$ and conclude that the mapping $f \mapsto \hat{f}$ maps L^1 into a proper subset of C_0. Show, however, that the range of this mapping is dense in C_0.