MATH 6321 - Theory of functions on a real variable
Homework 6

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1. **Solution:** Let $f : X \to \mathbb{C}$ be a function. Then

$$\|f\|_1 = |f(a)|\mu(\{a\}) + |f(b)|\mu(\{b\}) = |f(a)| + |f(b)| \cdot \infty$$

and

$$\|f\|_\infty = \max\{|f(a)|, |f(b)|\}$$

Then it is clear that $f \in L^1(X)$ iff $f(b) = 0$ and $|f(a)| < \infty$. Then $\|f\|_1 = |f(a)|$. Thus $L^1(X) = \text{span}\{\delta_a\}$, where $\delta_a : X \to \mathbb{C}$ such that $\delta_a(a) = 1$ and $\delta_a(b) = 0$. If $f \in L^1(X)$, then $f = f(a)\delta_a$. Thus $L^1(X)$ is a one-dimensional vector space.

But $L^\infty(X) = \text{span}\{\delta_a, \delta_b\}$, since any $f \in L^\infty(X)$ can be written as $f = f(a)\delta_a + f(b)\delta_b$. Thus $L^2(X)$ is a two-dimensional vector space.

Since $L^1(X), L^\infty(X)$ are of different dimensions, their dual spaces will also be non-isomorphic.

2. **Solution:** By a warm-up exercise we did in the lecture, $f \in L^1(\mu)$ forces f to be zero except on a countable set. Thus fg is zero everywhere except a countable set. Hence fg is measurable. Moreover, since $|g(x)| < 1$

$$\left|\int fg \, d\mu\right| \le \int |fg| \, d\mu \le \int |f| \, d\mu$$

Thus the map $\Lambda : f \to \int_E f \; d\mu$ is a contraction. That Λ is a linear functional follows form the linearity of the integration. Thus Λ is a bounded linear map.

Now to see that g is not measurable, notice that $g^{-1}([0, \frac{1}{2}]) = [0, \frac{1}{2}]$ is uncountable with it's complement $(\frac{1}{2}, 1]$ also uncountable, while $[0, \frac{1}{2}]$ belongs to the Borel-sigma algebra of $\mathbb{R}$.

3. **Solution:** Recall that $C(I)$ is closed under the sup-norm. Let $f \in L^\infty(m) \setminus C(I)$ with $\|f\|_\infty = 1$. Define $\Lambda : \overline{\text{span}}(C(I) \cup \{f\}) \to \mathbb{C}$ such that $\Lambda|_{C(I)} = 0$ and $\Lambda(f) = 1$ and linearly extending to the whole of $\overline{\text{span}}(C(I) \cup \{f\})$. Clearly $\|\Lambda\| = 1$. Then by Hahn-Banach extension theorem, Λ can be extended to a linear functional, (say Λ by an abuse of notation) to the whole of $L^\infty(I)$ preserving the norm. Then Λ is an example of a linear functional which is non-zero but vanishes on all of $C(I)$.

   For the sake of contradiction, assume that $\Lambda = \Lambda_g$ for some $g \in L^1(m)$. By the density of $C(I)$ in $L^1(I)$, there is a sequence of continuous functions g_n which converge to g in L_1 norm. Then $fg_n \to fg$ in L_1 norm for all $f \in L^\infty(I)$, since

$$\int |fg_n - fg| \; dm \leq \|f\|_\infty \int |g_n - g| \; dm$$

   Specifically for $f = g_m$, we get that

$$\int g_m g_n \; dm \to \int g_m g \; dm = \Lambda(g_m) = 0$$

   This should force $g = 0$ almost everywhere which will give a contradiction.