MATH6321 - Theory of functions of one real variable
Homework I

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1. **Solution:** For the sake of contradiction, assume that $\|(f+g)/2\|_p = 1$. That is $\|f+g\|_p = 2$, while $\|f\|_p = \|g\|_p = 1$. Thus we have an equality in the Minkowski's inequality. We know that equality in Minkowski's inequality for $1 < p < \infty$ occour if and only if $f = \lambda g$ for some scalar $\lambda \in \mathbb{C}$. Thus we see that $f = \lambda g$. Then

$$2 = \|f+g\|_p = \|(1+\lambda)f\|_p = |1+\lambda|\|f\|_p = |1+\lambda|$$

and

$$1 = \|g\|_p = \|\lambda f\|_p = |\lambda|\|f\|_p = |\lambda|$$

The only complex number which satisfy both of them are $\lambda = 1$. But this would give $f = g$, which is a contradiction. Hence we are done.

2. **Solution:** Let f_n be a cauchy sequence in M. Then $f_n \to f$ in $C([0,1])$, since $C([0,1])$ is complete under sup norm. For $\varepsilon > 0$, let $N_\varepsilon \in \mathbb{N}$ such that $\|f_n - f\|_\infty < \varepsilon$ for all $n > N_\varepsilon$. Then

$$\frac{-\varepsilon}{2} \leq \int_0^{\frac{1}{2}} f - f_n \, d\mu \leq \frac{\varepsilon}{2}$$

and similarly

$$\frac{-\varepsilon}{2} \leq \int_{\frac{1}{2}}^1 f - f_n \, d\mu \leq \frac{\varepsilon}{2}$$

Together, they give us

$$\frac{-\varepsilon}{2} + \frac{-\varepsilon}{2} \leq \int_0^{\frac{1}{2}} f - f_n \ dx - \int_{\frac{1}{2}}^1 f - f_n \ dx \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

Thus, we see that

$$\begin{aligned}
\left| \int_0^{\frac{1}{2}} f \ dx - \int_{\frac{1}{2}}^1 f \ dx - 1 \right| &= \left| \Big(\int_0^{\frac{1}{2}} f \ dx - \int_{\frac{1}{2}}^1 f \ dx \Big) - \Big(\int_0^{\frac{1}{2}} f_n \ dx - \int_{\frac{1}{2}}^1 f_n \ dx \Big) \right| \\
&= \left| \int_0^{\frac{1}{2}} f - f_n \ dx - \int_{\frac{1}{2}}^1 f - f_n \ dx \right| \\
&\leq \varepsilon
\end{aligned}$$

But since $\varepsilon > 0$ was chosen arbitrarily, we get that $f \in M$.

Now if $f, g \in M$, and $h = tf + (1-t)g$ for $t \in [0,1]$, then

$$\begin{aligned}
\int_0^{\frac{1}{2}} h(x) \ dx - \int_{\frac{1}{2}}^1 h(x) \ dx &= \int_0^{\frac{1}{2}} tf(x) + (1-t)g(x) \ dx - \int_{\frac{1}{2}}^1 tf(x) - (1-t)g(x) \ dx \\
&= t\Big(\int_0^{\frac{1}{2}} f(x) \ dx - \int_{\frac{1}{2}}^1 f(x) \ dx \Big) + (1-t)\Big(\int_0^{\frac{1}{2}} g(x) \ dx - \int_{\frac{1}{2}}^1 g(x) \ dx \Big) \\
&= t1 + (1-t)1 \\
&= 1
\end{aligned}$$

Thus we get that M is convex.

Now we'll show that if $f \in M$, then $\|f\|_\infty > 1$. Let $f \in C([0,1])$, such that $\|f\|_\infty \leq 1$. Then

$$\int_0^{\frac{1}{2}} f(x) \ dx \leq \Big| \int_0^{\frac{1}{2}} f(x) \ dx \Big| \leq \int_0^{\frac{1}{2}} |f(x)| \ dx \leq \int_0^{\frac{1}{2}} 1 \ dx = \frac{1}{2}$$

and by a similar reasoning, we get

$$-\int_{\frac{1}{2}}^1 f(x) \ dx \leq \Big| \int_{\frac{1}{2}}^1 f(x) \ dx \Big| \leq \int_{\frac{1}{2}}^1 |f(x)| \ dx \leq \int_{\frac{1}{2}}^1 1 \ dx = \frac{1}{2}$$

which gives

$$\int_0^{\frac{1}{2}} f(x) \ dx - \int_{\frac{1}{2}}^1 f(x) \ dx \leq 1$$

Thus equality in the above inequalities hold only when

$$\int_0^{\frac{1}{2}} f(x)\ dx = \frac{1}{2} = -\int_1^{\frac{1}{2}} f(x)\ dx$$

Now if $f(x) = u(x) + iv(x)$, where $u(x), v(x)$ are real valued functions, this would imply that

$$\int_0^{\frac{1}{2}} u(x)\ dx = \frac{1}{2} = -\int_1^{\frac{1}{2}} u(x)\ dx$$

and

$$\int_0^{\frac{1}{2}} v(x)\ dx = 0 = \int_1^{\frac{1}{2}} v(x)\ dx$$

This is rather replacing $f(x)$ with $u(x)$, and therefore without loss of generality, we might very well assume that f is a real valued function. Moreover

$$\frac{1}{2} = \int_0^{\frac{1}{2}} f(x)\ dx \leq \int_0^{\frac{1}{2}} |f(x)|\ dx \leq \frac{1}{2}$$

shows that $f = |f|$ almost everywhere in $[0, \frac{1}{2}]$. Again since $\|f\|_\infty \leq 1$, $\chi_[0, \frac{1}{2}] -$ f is a non-negative function in $[0, \frac{1}{2}]$ which satisfy

$$\int_0^{\frac{1}{2}} f - \chi_{[0,\frac{1}{2}]}\ dx = \int_0^{\frac{1}{2}} f\ dx - \int_0^{\frac{1}{2}} \chi_{[0,\frac{1}{2}]}\ dx = \frac{1}{2} - \frac{1}{2} = 0$$

Then by a result we proved before which states that if

$$\int_E f\ d\mu = 0$$

either $\mu(E) = 0$ or $f = 0$ almost everywhere, we get that $f = \chi_{[0,\frac{1}{2}]}$ almost everywhere in $[0, \frac{1}{2}]$. By continuity of f, we see that $f(x) = 1$ for all $x = [0, \frac{1}{2})$. By a similar reasoning we get $f(x) = \frac{-1}{2}$ for all $x \in (\frac{1}{2}, 1]$. But such a continuous function do not exist. Hence we have shown that if $f \in M$, then $\|f\|_\infty > 1$.

Now we'll find a sequence of functions $f_i \in M$ such that $\|f_i\|_\infty \to 1$. Define

$$f_n(x) = \begin{cases} 1 + \frac{1}{n}, & 0 \leq x \leq \frac{1}{2} - \frac{1}{n} \\ (n+1)(\frac{1}{2} - x), & \frac{1}{2} - \frac{1}{n} < x \leq \frac{1}{2} + \frac{1}{n} \\ -1 - \frac{1}{n}, & \frac{1}{2} + \frac{1}{n} < x \leq 1 \end{cases}$$

Then $f_n \in M$ for all $n \geq 2$ and $\|f_n\|_\infty = 1 + \frac{1}{n}$. Hence $\|f_n\|_\infty \to 1$.

3. **Solution:** Let $f \in M$, then

$$1 = \int f \; dm \leq \int |f| \; dm = \|f\|_1$$

Thus the minimal norm of elements of M is 1. Now let $f_n = n\chi_{[0,1/n]}$. Clearly, each $f_n \in L^1([0,1])$ and

$$\|f_n\|_1 = \int |f_n| \; dm = \int f_n \; dm = \int n\chi_{[0,\frac{1}{n}]} \; dm = nm([0,1/n]) = 1$$

Thus f_n is an example of infinitely many elements in M attaining minimal norm.

4. **Solution: Part I: X_m is closed**

We'll first show that the collection $X_m = \{f \in C([0,1]) \; : \; \exists x \in [0,1], \forall y \in [0,1], \; |f(x) - f(y)| \leq m|x-y|\}$ is closed. Let (f_n) be a cauchy sequence in X_m. Since $X_m \subset C([0,1])$ and $C([0,1])$ is closed under the sup norm, $f_n \to f \in C([0,1])$. We'll show that $f \in X_m$. Let $x_n \in [0,1]$ correspond to each f_n such that for all $y \in [0,1]$,

$$|f_n(x_n) - f_n(y)| \leq m|x_n - y|$$

Since $[0,1]$ is compact, x_n has a convergent subsequence x_{n_k} which converge. Let $x_{n_k} \to x_0$ and $\varepsilon > 0$. Since $f_{n_k} \to f$ in the sup norm, by a slight abuse of notation assume $x_n \to x_0$. Let $N_\varepsilon \in \mathbb{N}$ such that for all $n > N_\varepsilon$, we have $\|f - f_n\|_\infty < \varepsilon$. Let $M_\varepsilon \in \mathbb{N}$ such that for all $n > M_\varepsilon$, $|x_0 - x_n| < \varepsilon/m$. Then for $n > N := \max\{N_\varepsilon, M_\varepsilon\}$ and $y \in [0,1]$,

$$\begin{aligned}
|f(x_0) - f(y)| &\leq |f(x_0) - f_n(x_0)| + |f_n(x_0) - f_n(x_n)| + |f_n(x_n) - f_n(y)| + |f_n(y) - f(y)| \\
&< \varepsilon + m|x_n - x_0| + m|x_n - y| + \varepsilon \\
&< \varepsilon + \varepsilon + m|x_n - y| + \varepsilon \\
&= 3\varepsilon + m|x_n - x_0 + x_0 - y| \\
&\leq 3\varepsilon + m|x_n - x_0| + m|x_0 - y| \\
&< 4\varepsilon + m|x_0 - y|
\end{aligned}$$

Since $\varepsilon > 0$ was chosen arbitrarily, this gets that $|f(x_0) - f(y)| \leq m|x_0 - y|$. Thus $f \in X_m$, and we get that X_m is closed. We restate one of of the results we proved, since we'll reuse it in below.

Proposition 0.1. *If $f_n \to f$, uniformly in X_m, and $x_n \in [0,1]$ such that for all $y \in [0,1]$,*

$$|f_n(x_n) - f_n(y)| = m|x_n - y|$$

Then for any convergent subsequence $x_{n_k} \to x_0 \in [0,1]$, we have that for all $y \in [0,1]$

$$|f(x_0) - f(y)| \leq m|x_0 - y|$$

Part II: X_m has empty interior

Now to show that X_m has empty interior, for any $\varepsilon > 0$ we'll find an $h \in B_\varepsilon(f)$ such that $h \notin X_m$. Let $f \in X_m$, and $\varepsilon > 0$ be given. Assume for contradiction that $B_\varepsilon(f) \subset X_m$. Let $h_{p,q}(x) := f(x) + \varepsilon/p \sin(qx)$ for $p, q \in \mathbb{N}$

$$\|f - h_{p,q}\|_\infty = \|\varepsilon/p \sin(qx)\| = \frac{\varepsilon}{p}$$

shows that $h_{p,q} \in B_\varepsilon(f) \subset X_m$. Then there exist $x_{p,q} \in [0,1]$ such that for all $y \in [0,1]$,

$$|f(x_{p,q}) - f(y) + \varepsilon/p(\sin(qx_{p,q}) - \sin(qy))| = |h_{p,q}(x_{p,q}) - h_{p,q}(y)| \leq m|x_{p,q} - y|$$

Then for all $y \in [0,1]$

$$\big||f(x_{p,q}) - f(y)| - \varepsilon/p|\sin(qx_{p,q}) - \sin(qy)|\big| \leq m|x_{p,q} - y|$$

which gives that

$$\varepsilon/p|\sin(qx_{p,q}) - \sin(qy)| \leq |f(x_{p,q}) - f(y)| + m|x_{p,q} - y|$$

by choosing $a = |f(x_{p,q})|, b = 1/p|\sin(qx_{p,q}) - \sin(qy)|$, and $c = m|x_{p,q} - y|$, and using the fact that

$$\begin{aligned} &|a - b| \leq c \\ \implies &-c \leq a - b \leq c \\ \implies &-c - a \leq -b \leq c - a \\ \implies &b \leq a + c \end{aligned}$$

But since for a fixed $q \in \mathbb{N}$, $h_{p,q}$, as a sequence indexed by p converge uniformly to f, by Proposition 0.1, for a subsequence of $x_{p,q}$ (indexed by p), converging to x_q, we have for all $y \in [0,1]$,

$$|f(x_q) - f(y)| \leq m|x_q - y|$$

Without loss of generality, assume that for any fixed q, $x_{p,q} \to x_q$ as a sequence in p. Then for all $\delta > 0$ there exists an $M_p \in \mathbb{N}$, such that for all $p > M_p$, $|x_{p,q} - x_q| < \frac{\delta}{m}$. Then for $p > M_p$,

$$\begin{aligned}
|f(x_{p,q}) - f(y)| + m|x_{p,q} - y| &\leq |f(x_{p,q}) - f(x_q)| + |f(x_q) - f(y)| + m|x_{p,q} - y| \\
&\leq m|x_{p,q} - x_q| + m|x_q - y| + m|x_{p,q} - y| \\
&< \delta + m|x_q - x_{p,q}| + m|x_{p,q} - y| + m|x_{p,q} - y| \\
&< \delta + \delta + 2m|x_{p,q} - y| \\
&= 2\delta + 2m|x_{p,q} - y|
\end{aligned}$$

Since $\delta > 0$ was arbitrary, this shows that for $p > M_p$, for all $y \in [0,1]$

$$\varepsilon/p|\sin(qx_{p,q}) - \sin(qy)| < 2m|x_{p,q} - y|$$

But we know that for $q > 15 > 4\pi$ (some estimate), either $[x_{p,q} - \frac{2\pi}{q}, x_{p,q}]$ or $[x_{p,q}, x_{p,q} + \frac{2\pi}{q}]$ is a subset of $[0,1]$. Let us call this subset A, then there is $y \in A$ such that $|\sin(qx_{p,q}) - \sin(qy)| > 1$. Moreover for such y, we'll have $|x_{p,q} - y| < \frac{2\pi}{q}$. Thus we get that for a fixed $p > M_p$,

$$\frac{\varepsilon}{p} \leq \varepsilon/p|\sin(qx_{p,q}) - \sin(qy)| < 2m|x_{p,q} - y| \leq \frac{4m\pi}{q}$$

Since this is true for all q, and a fixed ε, p, this gives a contradiction as $q \to \infty$ and $\frac{4m\pi}{q} \to 0$.

Part III: G_δ dense set of nowhere differentiable functions

Since each X_n is closed and has empty interior, each X_n are nowhere dense. Then X_n^c are dense open subsets of $C([0,1])$. Then by the Baire category theorem, we get that

$$X = \bigcap_{n=1}^{\infty} X_n^c$$

is a dense G_δ subset of $C([0,1])$. We'll show that X is precisely the set of nowhere differentiable functions. Let $f \in C([0,1])$ be differentiable at $x \in [0,1]$. Then by the definition of the derivative, there is a $\delta > 0$ such that

$$|x - y| < \delta \implies \left|\frac{f(x) - f(y) - f'(x)(x-y)}{x-y}\right| < 1$$

Then for $y \in B_\delta(x)$,

$$\Big||f(x) - f(y)| - |f'(x)(x-y)|\Big| < |x-y|$$

By taking $a = |f(x) - f(y)|, b = |f'(x)(x-y)|$, and $c = |x-y|$ and using the same reasoning as in part II, we get

$$|f(x) - f(y)| \leq (|f'(x)| + 1)|x - y|$$

Let $\mathbb{N} \ni N_1 \geq |f'(x)| + 1$. If $y \notin B_\delta(x)$, then choose $M = \|f\|_\infty$ and $N_2 \in \mathbb{N}$ such that $2M < N_2\delta$. Then

$$|f(x) - f(y)| \leq 2M < N_2\delta \leq N_2|x - y|$$

Now for $N = \max\{N_1, N_2\}$, we see that for all $y \in [0, 1]$

$$|f(x) - f(y)| \leq N|x - y|$$

and therefore $f \in X_N$. Thus we see that if $f \in X$, then f is not differentiable anywhere in $[0, 1]$. Hence we are done.