

Stacked basis Theorem: If R is a PID, M is an R -module which is free of rank n , and $N \subseteq M$ is a sub-module, then:

- i) $\exists 0 \leq m \leq n$ s.t. N is a free R -module of rank m .
- ii) \exists a basis $x_1, \dots, x_n \in M$, $a_1, \dots, a_m \in R \setminus \{0\}$ s.t.
 $a_1 x_1, a_2 x_2, \dots, a_m x_m$ is a basis for N
 and $a_1 | a_2 | \dots | a_m$.

More about free modules:

Thm: \forall set A , \exists a free R -module $F(A)$ on A which satisfies the property that if M is any R -module and if $\varphi: A \rightarrow M$ is a map then there is a unique R -module homom. $\Phi: F(A) \rightarrow M$ s.t. the following diagram commutes:

$$\begin{array}{ccc}
 A & \xrightarrow{\text{inclusion}} & F(A) \\
 \searrow \varphi & & \downarrow \Phi \\
 & & M
 \end{array}$$

Pf: If $A = \emptyset$ let $F(A) = \{0\}$.

If $A \neq \emptyset$, let $F(A) = \left\{ f: A \rightarrow R : f(a) = 0 \text{ for all but finitely many } a \in A \right\}$.

• Turn $F(A)$ into an Abelian gp. by ptwise addition:

$$(f+g)(a) = f(a) + g(a) \quad \forall a \in A, f, g \in F(A).$$

• Define scal. mult. by componentwise mult:

$$(r \cdot f)(a) = r \cdot f(a), \quad \forall r \in R, f \in F(A), a \in A.$$

Then $F(A)$ is an R -module. We can identify A with a subset of $F(A)$ by:

$$a \mapsto f_a, \quad \text{where} \quad f_a(b) = \begin{cases} 1 & \text{if } b=a \\ 0 & \text{else} \end{cases}.$$

Another fact:

Every elem of $F(A)$ has an expansion of the form $r_1 f_{a_1} + r_2 f_{a_2} + \dots + r_n f_{a_n}$, for

some $r_1, \dots, r_n \in R$, $a_1, \dots, a_n \in A$. Also

$\{f_a : a \in A\}$ is lin. ind., so $F(A)$ is free on A .

Suppose $\varphi: A \rightarrow M$ is as in the statement.

Define $\Phi: F(A) \rightarrow M$ by

$$\Phi(r_1 f_{a_1} + r_2 f_{a_2} + \dots + r_n f_{a_n})$$

$$= r_1 \varphi(a_1) + r_2 \varphi(a_2) + \dots + r_n \varphi(a_n).$$

(This is well-def.)

Then: $\cdot \Phi$ is an R -mod. hom.

- $\cdot \Phi$ makes the diagram commute, and it is the only way to do so because a hom. from $F(A)$ to M is uniquely determined by $\{\Phi(f_a) : a \in A\}$, and we must have $\Phi(f_a) = \varphi(a)$. \square

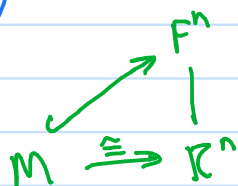
Cors: i) $F(\{a_1, \dots, a_n\}) \cong R^n = R \oplus \dots \oplus R$ (n times)

ii) Any free module w/ basis A is isom to $F(A)$.

Generalizing our def. of rank from last time: The rank of an R -module is the maximum # (or cardinality) of lin. ind. vecs.

Lem: If R is an ID and M is a free R -module of rank n then any $n+1$ elems. of M are linearly dependent.

Pf:



F = field of fractions of R

By the diagram above, you can identify M with a subset of F^n . Under this identification, suppose $x_1, \dots, x_{n+1} \in M$. Then, since F^n is an n -dim. vec.sp.

over F , x_1, \dots, x_{n+1} are F -lin. dependent.

This implies that $\exists \frac{p_i}{q_i} \in F$, $1 \leq i \leq n+1$, $p_i, q_i \in R$, $q_i \neq 0$,

not all 0, s.t. $\frac{p_1}{q_1} x_1 + \dots + \frac{p_{n+1}}{q_{n+1}} x_{n+1} = 0$.

Let $q = q_1 \dots q_{n+1}$. Then $\frac{q p_i}{q_i} \in R$, $\forall i$, at least one of these is not 0, and

$$\left(\frac{q p_1}{q_1} \right) x_1 + \dots + \left(\frac{q p_{n+1}}{q_{n+1}} \right) x_{n+1} = 0$$

$\Rightarrow x_1, \dots, x_{n+1}$ is R -linearly dep. \square

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Pf. of stacked basis thm for the case when R is an ED:

$$M \cong R^n \xrightarrow{\text{lemma}} \text{rank of } N \text{ is } m \leq n.$$

If x_1, \dots, x_n any basis for M and if y_1, \dots, y_m is any generating set for N then

$$\vec{y} = A \vec{x} \text{ for some } A \in M_{m \times n}(R).$$

Goal: Use elem. row & col. ops. to pick a basis and gen. set

$$\text{for which } \vec{y} = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & \ddots & & i \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & a_m \end{pmatrix} \vec{x},$$

where $a_1 | a_2 | \dots | a_m$ (next time).