Rings

Def: A ring is a set R together with binary operations $+$ and $\times$ satisfying:

- $(R,+)$ is an Abelian group. $(0 =$ additive identity$)$
- $\times$ is associative
- $+$ and $\times$ must satisfy distributive laws:

  $\forall a,b,c \in R$,

  $$(a+b)\times c = (a\times c)+(b\times c)$$

  and $$a\times(b+c) = (a\times b)+(a\times c).$$

More defs:

1) If $\times$ is commutative then R is a commutative ring.
2) If $\exists\, 1 \in R$ s.t. $\forall a \in R$, $1a = a1 = a$ then R is a ring with identity.

   (Notation:
   $1 =$ multiplicative identity
   $ab = a\times b$)

Fact: If R is a ring with identity $1=0$ then $R=\{0\}$.

3) If R is a ring with 1 and if $\forall a \in R \setminus \{0\}$, $\exists b \in R \setminus \{0\}$ s.t. $ab = ba = 1$ then R is a <u>division ring</u>.

4) A commutative division ring is called a <u>field</u>.

Exs:

1) Ring w/ 1 element $R = \{0\}$

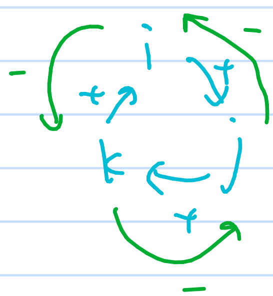
2) <u>Trivial rings</u>: If $(R, +)$ is any Abelian group, define $ab = 0$, $\forall a, b \in R$.

3) $\underset{\text{not a field}}{\mathbb{Z}}, \underbrace{\mathbb{Q}, \mathbb{R}, \mathbb{C}}_{\text{fields}}, \mathbb{Z}/n\mathbb{Z}$

field $\Leftrightarrow$ $n = 1$ or a prime.

4) Example of a non-commutative division ring:

$\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$



5) Function rings:

a) Let X be any set, A any ring, define

$$R = \{f : X \to A\}$$

$$\left.\begin{aligned}(f+g)(x) &= f(x)+g(x)\\ (fg)(x) &= f(x)\,g(x)\end{aligned}\right\}$$ pointwise add. & mult. of fns.

$0_R(x) = 0_A \ , \ \forall x \in X.$

If A has an identity then so does R:

$1_R(x) = 1_A \ , \ \forall x \in X.$

b) $C(\mathbb{R}) = \{$continuous functions from $\mathbb{R} \to \mathbb{R}\}$

with ptwise add. & mult.

$0_{C(\mathbb{R})}(x) = 0 \quad \forall x \in \mathbb{R}, \qquad 1_{C(\mathbb{R})}(x) = 1 \ , \ \forall x \in \mathbb{R}.$

Not a field: Ex: $f(x) = x$ is not invertible in $C(\mathbb{R})$,

because it takes the value 0 at some point.

c) $R = C_c(\mathbb{R}) = \{$cont. fns. from $\mathbb{R} \to \mathbb{R}$ with compact support$\}$,

with ptwise add. and mult.

0 outside of some bounded region

$0_R(x) = 0, \ \forall x \in \mathbb{R}.$

No identity, because $1_R(x) = 1$ doesn't have compact support.

6) Matrix rings: Let R be an arbitrary ring, $n \in \mathbb{N}$, $M_n(R)$ is the collection of square matrices w/ usual rules for matrix add. and mult.

If R is non-trivial and $n \geq 2$ then $M_n(R)$ is non-commutative.

7) Polynomial rings: Assume R is a commutative ring w/ identity. Define

$$R[x] = \{\text{polynomials w/ coeffs in } R\},$$

together w/ usual rules for poly. add. & mult.

Ques: Suppose $f, g \in R[x]$. Is it true that $\deg(fg) = \deg f + \deg g$? It depends on R.

Ex: $R = \mathbb{Z}/4\mathbb{Z}$

$$f(x) = 2x+1, \quad g(x) = 2x-1$$

$$\deg(f) = \deg(g) = 1$$

but $f(x)g(x) = 4x^2 - 1 = -1$

$$\Rightarrow \deg(fg) = 0.$$

Defs:

1) An element $a \in R \setminus \{0\}$ is a <u>zero divisor</u> if $\exists b \in R \setminus \{0\}$ s.t. $ab = 0$.

2) If R has $1 \neq 0$ then a <u>unit</u> $u \in R$ is an element w/ the property that $\exists v \in R$ s.t. $uv = vu = 1$.

   Notation: collection of units $= R^\times$ (group under mult)

3) If R is a commutative ring with $1 \neq 0$, with no zero-divisors, then R is called an <u>integral domain</u>.

4) A <u>subring</u> of R is a subset $S \subseteq R$ which is a ring w/ the same ops.

   Note: A subset $S \subseteq R$ is a subring if $(S, +)$ is an Abelian group and S is closed under mult.