Exs: (Cont. from last time)

1) $K = \mathbb{Q}(\zeta_5)$, $F = \mathbb{Q}$.

$f(x) = \min_{\mathbb{Q}}(\zeta_5) = \Phi_5(x) = x^4 + x^3 + x^2 + x + 1$,

so $[K:\mathbb{Q}] = 4$.

Also $f(x) = \prod_{a=1}^{4} (x - \zeta_5^a)$, so K is the spl. field of f.

By our thm., K/F is Galois, so $|\mathrm{Aut}(K/F)| = 4$.

Any $\sigma \in \mathrm{Aut}(K/F)$ is uniquely determined by $\sigma(\zeta_5)$, and there are 4 possibilities:

$\sigma(\zeta_5) = \zeta_5^a$, $1 \le a \le 4$.

All of these must occur, since $|\mathrm{Aut}(K/F)| = 4$.

Let $\tau \in \mathrm{Aut}(K/F)$ be determined by $\tau(\zeta_5) = \zeta_5^2$.

Then $\tau^2(\zeta_5) = \tau(\zeta_5^2) = (\tau(\zeta_5))^2 = (\zeta_5^2)^2 = \zeta_5^4$

$\tau^3(\zeta_5) = \tau(\tau^2(\zeta_5)) = \tau(\zeta_5^4) = (\tau(\zeta_5))^4 = (\zeta_5^2)^4 = \zeta_5^3$

$\tau^4(\zeta_5) = \tau(\tau^3(\zeta_5)) = \tau(\zeta_5^3) = \zeta_5$.

The FTGT gives us the correspondence:

$$\begin{array}{ccc} \{id\} & & \mathbb{Q}(\zeta_5) \\ 2| & & 2| \\ H_1 = \langle \tau^2 \rangle & & K_{H_1} \\ 2| & & 2| \\ Gal(K|F) = \langle \tau \rangle \cong C_4 & & \mathbb{Q} \end{array}$$

To figure out what K_{H_1} is:

$$\tau^2(\zeta_5) = \tau(\tau(\zeta_5)) = \tau(\zeta_5^2) = \zeta_5^4 = \zeta_5^{-1}$$

$$\Rightarrow \tau^2(\zeta_5 + \zeta_5^{-1}) = \zeta_5 + \zeta_5^{-1}.$$

Also, $\zeta_5 + \zeta_5^{-1} = \zeta_5 + \zeta_5^4 = -1 - \zeta_5^2 - \zeta_5^3 \notin \mathbb{Q}$, because

$\{1, \zeta_5, \zeta_5^2, \zeta_5^3\}$ is a $\mathbb{Q}$-basis for K.

Therefore $\zeta_5 + \zeta_5^{-1} \in K_{H_1} \setminus \mathbb{Q} \Rightarrow K_{H_1} = \mathbb{Q}(\zeta_5 + \zeta_5^{-1})$.

$([K_{H_1} : \mathbb{Q}] = 2)$

2) K is splitting field of $x^4 - 4$, over $\mathbb{Q}$.

$K|\mathbb{Q}$ is Galois, and $K = \mathbb{Q}(\sqrt{2}, i) \Rightarrow [K:\mathbb{Q}] = 4$.

To determine $Gal(K|\mathbb{Q})$:

$min_{\mathbb{Q}}(\sqrt{2}) = x^2 - 2$,

$min_{\mathbb{Q}}(i) = x^2 + 1$,

$$\begin{array}{c} \mathbb{Q}(\sqrt{2}, i) \\ |2 \quad \leftarrow (i \notin \mathbb{Q}(\sqrt{2})) \\ \mathbb{Q}(\sqrt{2}) \\ |2 \\ \mathbb{Q} \end{array}$$

so any element of $Gal(K|\mathbb{Q})$ must satisfy

$\sqrt{2} \mapsto \pm\sqrt{2}$ and $i \mapsto \pm i$.

There are only 4 possibilities, so they all have to extend to auts. of K which fix $\mathbb{Q}$.

Let $\sigma, \tau \in \mathrm{Gal}(K/\mathbb{Q})$ be def. by:

$$\sigma : \begin{cases} \sqrt{2} \mapsto -\sqrt{2} \\ i \mapsto i \end{cases}, \qquad \tau : \begin{cases} \sqrt{2} \mapsto \sqrt{2} \\ i \mapsto -i \end{cases}$$

Then $\mathrm{Gal}(K/\mathbb{Q}) = \{ id, \sigma, \tau, \sigma\tau \}$.

$$\begin{tikzcd}[column sep=small]
 & \{id\} \arrow[dl, dash, "2"'] \arrow[d, dash, "2"] \arrow[dr, dash, "2"] & \\
H_1 = \langle \sigma \rangle \arrow[dr, dash, "2"'] & H_2 = \langle \tau \rangle \arrow[d, dash, "2"] & H_3 = \langle \sigma\tau \rangle \arrow[dl, dash, "2"] \\
 & \mathrm{Gal}(K/\mathbb{Q}) = \langle \sigma, \tau \rangle &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & \mathbb{Q}(\sqrt{2}, i) \arrow[dl, dash, "2"'] \arrow[d, dash, "2"] \arrow[dr, dash, "2"] & \\
K_{H_1} = \mathbb{Q}(i) \arrow[dr, dash, "2"'] & K_{H_2} = \mathbb{Q}(\sqrt{2}) \arrow[d, dash, "2"] & K_{H_3} = \mathbb{Q}(i\sqrt{2}) \arrow[dl, dash, "2"] \\
 & \mathbb{Q} &
\end{tikzcd}$$

3) K be split field of x^3-2 over $\mathbb{Q}$.

$$x^3-2=\prod_{i=0}^{2}(x-2^{1/3}\zeta_3^i).$$

$$2^{1/3},\ \zeta_3=\frac{2^{1/3}\zeta_3}{2^{1/3}}\in K \Rightarrow K=\mathbb{Q}(2^{1/3},\zeta_3)=\mathbb{Q}(2^{1/3})\mathbb{Q}(\zeta_3)$$

Also, $[K:\mathbb{Q}]\leq 3\cdot 2=6$, but $2,3\mid[K:\mathbb{Q}]$

$$\Rightarrow [K:\mathbb{Q}]=6$$

$$\Rightarrow |Gal(K/\mathbb{Q})|=6.$$

To determine $Gal(K/\mathbb{Q})$:

$$min_{\mathbb{Q}}(2^{1/3})=x^3-2$$

$$min_{\mathbb{Q}}(\zeta_3)=\Phi_3(x)=x^2+x+1,$$

So any element of $Gal(K/\mathbb{Q})$ must satisfy

$$2^{1/3}\mapsto\begin{cases}2^{1/3}\\ 2^{1/3}\zeta_3\\ 2^{1/3}\zeta_3^2\end{cases},\quad \zeta_3\mapsto\begin{cases}\zeta_3\\ \zeta_3^2\end{cases}.$$

Since there are 6 possibilities, all choices must extend to auts. (cont next time...)