Thm: Suppose $f \in \mathbb{Q}[x]$ is solvable by radicals, and let K be its splitting field over $\mathbb{Q}$. Then $Gal(K/\mathbb{Q})$ is solvable.

f solv. by rads:

$$
\begin{array}{lll}
K \subseteq F_k = F_{k-1}(\alpha_k), & & \alpha_k^{m_k} \in F_{k-1} \\
\quad \vdots & & \quad \vdots \\
F_2 = F_1(\alpha_2), & & \alpha_2^{m_2} \in F_1 \\
\quad | & & \\
F_1 = F_0(\alpha_1), & & \alpha_1^{m_1} \in F_0 = \mathbb{Q} \\
\quad | & & \\
F_0 = \mathbb{Q} & &
\end{array}
$$

Lemma: Suppose $K = F(\alpha)$ for some $\alpha \in K$ with $\alpha^m \in F$. If F contains all mth roots of unity then K/F is Galois and $Gal(K/F)$ is Abelian. (proved last time)

Pf. of thm: Suppose f is solv. by rads. and let K be its spl. field over $\mathbb{Q}$. Then $\exists k \in \mathbb{N}$, $F_0 \subseteq \cdots \subseteq F_k$ s.t. $F_0 = \mathbb{Q}$, $K \subseteq F_k$, $F_i = F_{i-1}(\alpha_i)$, $\alpha_i^{m_i} \in F_{i-1}$.

Define $L_{-1}, L_0, \dots, L_k$ by $L_{-1} = \mathbb{Q}$, $L_0 = \mathbb{Q}(\zeta_{m_1}, \dots, \zeta_{m_k})$, $L_i = L_{i-1}(\alpha_i)$, $1 \le i \le k$.

$$
\begin{array}{l}
L_k \\
| \\
L_{k-1} \\
| \\
\vdots \\
| \\
L_2 \\
| \\
L_1 \\
| \\
L_0 = \mathbb{Q}(\zeta_m),\ m = \operatorname{lcm}(m_1, \dots, m_k) \\
| \\
L_{-1} = \mathbb{Q}
\end{array}
$$

Galois, and Abelian by the lemma.

Galois, w/ Gal. gp $\cong (\mathbb{Z}/m\mathbb{Z})^\times$

Since $|\mathrm{Gal}(L_1/L_0)| = [L_1 : L_0]$ and $|\mathrm{Gal}(L_0/\mathbb{Q})| = [L_0 : \mathbb{Q}]$, and since the map $\mathrm{Gal}(L_1/L_0) \times \mathrm{Gal}(L_0/\mathbb{Q})$ to $\mathrm{Aut}(L_1/\mathbb{Q})$ defined by $(\sigma, \tau) \mapsto \sigma\tau$ ← (needs explanation) is injective, the extension $L_1/\mathbb{Q}$ is Galois. Similarly, each $L_i/\mathbb{Q}$ is Galois.

Let $H_0 = G = \mathrm{Gal}(L_k/\mathbb{Q})$, and $\forall\ 1 \le i \le k+1$,
let H_i be the subgroup of G which fixes L_{i-1}. From FTGT we have:

i) $L_{i-1}/\mathbb{Q}$ Galois $\Rightarrow H_i \trianglelefteq G$, and $\mathrm{Gal}(L_{i-1}/\mathbb{Q}) \cong G/H_i$

ii)
$$G/H_i \left\{ \begin{array}{c} L_{i-1} \\ | \\ L_{i-2} \\ | \\ \mathbb{Q} \end{array} \right. \begin{array}{l} \left.\begin{array}{c} \\ \\ \end{array}\right\} H_{i-1}/H_i \\ \left.\begin{array}{c} \\ \\ \end{array}\right\} G/H_{i-1} \end{array}$$

By the 3rd isom thm.

$H_{i-1}/H_i \cong \mathrm{Gal}(L_{i-1}/L_{i-2})$,

so it's Abelian, by our previous.

Since $H_{k+1} = \{1\}$, this shows that G is solvable.

Finally consider the extension
$$\begin{array}{c} L_k \\ | \\ K \\ | \\ \mathbb{Q} \end{array}$$

The extension $K/\mathbb{Q}$ is Galois (it's the spl. fld. of f).

Therefore (FTGT),

$\mathrm{Gal}(K/\mathbb{Q}) \cong G/G_K$. Since the canonical map $G \to G/G_K$ is a homom., $\mathrm{Gal}(K/\mathbb{Q})$. ∎