A couple other useful facts that follow from Kronecker++ :

1a) Suppose K/F is a field extension, $f \in F[x]$ irred.,
$\alpha, \beta \in K$ roots of $f(x)$.

Then $F(\alpha) \cong F(\beta)$.

1b) Suppose $\tau : F_1 \to F_2$ is a field isom.,
$f_1 \in F_1[x]$ irred., and $f_2 = \tau(f_1) \in F_2[x]$

(τ extends by linearity to a ring isom.
$\tau : F_1[x] \to F_2[x]$. Note: f_1 irred $\Leftrightarrow$ $\tau(f_1)$ is irred.)

If α_1 is a root of f_1 in an ext. of F_1, and
if α_2 is a root of f_2 in an ext. of F_2,
then $F_1(\alpha_1) \cong F_2(\alpha_2)$.

More defs. and exs:

- Let F be a field. The <u>characteristic</u> of F is the smallest $n \in \mathbb{N}$ s.t. $n1_F = 0$, or 0 if no such n exists. notation: $\operatorname{char} F$

  Exercise: $\operatorname{char} F = 0$ or p for some prime p.

  The <u>prime subfield</u> of F is the smallest subfield contained in F.

  - If $\operatorname{char} F = 0$ then the prime subfield is $\cong \mathbb{Q}$.
  - If $\operatorname{char} F = p$ then the prime subfield is $\cong \mathbb{Z}/p\mathbb{Z}$.

- Simple extensions & primitive elements:

  Suppose K/F is a field ext. If $\exists\, \alpha \in K$ s.t. $K = F(\alpha)$ then K is a <u>simple extension</u> and α is a <u>primitive element</u> for K/F.

- Suppose K/F is a field extension. If $\alpha \in K$ is not algebraic over F then α is called <u>transcendental</u> over F.

Exs: 1) $K = \mathbb{R}$, $F = \mathbb{Q}$.

a) Set set of all polys. $\mathbb{Q}[x]$ is countable, and it follows that the set of elems. of $\mathbb{R}$ which are algebraic over $\mathbb{Q}$ is countable. But $\mathbb{R}$ is uncountable, so there uncountably many elems. of $\mathbb{R}$ which are trans. over $\mathbb{Q}$.

b) π and e are trans. over $\mathbb{Q}$ (hard)

2a) F a field,

field of fractions of $F[t]$

$$K = F(t) = \left\{ \frac{f(t)}{g(t)} : f, g \in F[t],\ g \neq 0 \right\}$$

field of rational functions in t over F.

The element $t \in K$ is transcendental over F.

b) Let $f(x) = x^2 - t \in K[x]$.

This is irreducible: Note that the ideal (t) is a prime ideal in $F[t]$. The polynomial $f(x)$ is Eisenstein at (t), so it's irreducible in $F[t][x]$. By Gauss's lemma, it is irred. over $F(t)[x]$.

Since f is irred., let θ $(\theta = "\sqrt{t}")$ be a root in some ext. of K.

Then $K(\theta) \cong K[x]/(f) = \{a(t) + b(t)\theta : a, b \in K\}$,

so $[K(\theta) : K] = 2$.

More about algebraic extensions:

Thm: Given K/F the collection of all elems. of K which are algebraic over F is a subfield of K.

Pf: Let $\alpha, \beta \in K$ be algebraic over F.

Then $\alpha \pm \beta$, $\alpha\beta$, and α/β (if $\beta \neq 0$) all lie in $F(\alpha, \beta)$.

$F(\alpha,\beta) = (F(\alpha))(\beta)$

But $\underbrace{[F(\alpha,\beta):F] = [F(\alpha)(\beta):F(\alpha)][F(\alpha):F]}_{<\infty}$

$$\begin{array}{c} F(\alpha,\beta) \\ \mid {\scriptstyle <\infty} \\ F(\alpha) \\ \mid {\scriptstyle <\infty} \\ F \end{array}$$

$\Rightarrow$ all of these numbers are algebraic. $\square$