Stacked basis theorem: If R is a PID, M is an R-module which is free of rank n, and $N \subseteq M$ is a sub-module, then:

i) $\exists\, 0 \le m \le n$ s.t. N is a free R-module of rank m.

ii) $\exists$ a basis $x_1, \dots, x_n \in M$, $a_1, \dots, a_m \in R \setminus \{0\}$ s.t.

$a_1 x_1, a_2 x_2, \dots, a_n x_n$ is a basis for N

and $a_1 \mid a_2 \mid \cdots \mid a_m$.

More about free modules:

(comm. ring w/ iden.)

Thm: $\forall$ set A, $\exists$ a free R-module $F(A)$ on A which satisfies the property that if M is any R-module and if $\varphi : A \to M$ is a map then there is a unique R-module homom. $\Phi : F(A) \to M$ s.t. the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, hook, "\text{inclusion}"] \arrow[dr, "\varphi"'] & F(A) \arrow[d, "\Phi"] \\ & M \end{tikzcd}.$$

Pf: If $A = \emptyset$ let $F(A) = \{0\}$.

If $A \neq \emptyset$, let $F(A) = \left\{ f : A \to R \;:\; f(a) = 0 \text{ for all but finitely many } a \in A \right\}$.

- Turn $F(A)$ into an Abelian gp. by ptwise addition:

$$(f+g)(a) = f(a)+g(a) \quad \forall a\in A,\ f,g\in F(A).$$

- Define scal. mult. by componentwise mult.:

$$(r\cdot f)(a) = r\cdot f(a), \quad \forall r\in R,\ f\in F(A),\ a\in A.$$

Then $F(A)$ is an R-module. We can identify A with a subset of $F(A)$ by:

$$a \mapsto f_a, \quad \text{where} \quad f_a(b) = \begin{cases} 1 & \text{if } b=a \\ 0 & \text{else} \end{cases}.$$

Another fact:

Every elem of $F(A)$ has an expansion of the form $r_1 f_{a_1} + r_2 f_{a_2} + \cdots + r_n f_{a_n}$, for some $r_1,\dots,r_n\in R$, $a_1,\dots,a_n\in A$. Also $\{f_a : a\in A\}$ is lin. ind., so $F(A)$ is free on A.

Suppose $\varphi: A\to M$ is as in the statement.

Define $\Phi: F(A)\to M$ by

$$\Phi(r_1 f_{a_1} + r_2 f_{a_2} + \cdots + r_n f_{a_n}) = r_1\varphi(a_1) + r_2\varphi(a_2) + \cdots + r_n\varphi(a_n).$$

(this is well-defi)

Then: • Φ is an R-mod. hom.

• Φ makes the diagram commute, and it is the only way to do so because and hom. from $F(A)$ to M is uniquely determined by $\{\Phi(f_a) : a \in A\}$, and we must have $\Phi(f_a) = \varphi(a)$. $\boxtimes$

Cors: i) $F(\{a_1, \dots, a_n\}) \cong R^n = R \oplus \cdots \oplus R$ (n-times)

ii) Any free module w/ basis A is isom to $F(A)$.

Generalizing our def. of rank from last time: The <u>rank</u> of an R-module is the maximum # (or cardinality) of lin. ind. vecs.

Lem: If R is an ID and M is a free R-module of rank n then any $n+1$ elms. of M are linearly dependent.

Pf:

$$\begin{tikzcd} & F^n \\ M \arrow[ur] \arrow[r, "\cong"] & R^n \arrow[u, dash] \end{tikzcd}$$

F = field of fractions of R

By the diagram above, you can identify M with a subset of F^n. Under this identification, suppose $x_1, \dots, x_{n+1} \in M$. Then, since F^n is an n-dim. vec. sp.

over F, $x_1, \dots, x_{n+1}$ are F-lin. dependent.

This implies that $\exists \frac{p_i}{q_i} \in F$, $1 \le i \le n+1$, $p_i, q_i \in R$, $q_i \ne 0$,

not all 0, s.t. $\frac{p_1}{q_1} x_1 + \cdots + \frac{p_{n+1}}{q_{n+1}} x_{n+1} = 0$.

Let $q = q_1 \cdots q_{n+1}$. Then $\frac{q p_i}{q_i} \in R$, $\forall i$, at least one of these is not 0, and

$$\left(q \frac{p_1}{q_1}\right) x_1 + \cdots + \left(q \frac{p_{n+1}}{q_{n+1}}\right) x_{n+1} = 0$$

$\Rightarrow$ $x_1, \dots, x_{n+1}$ is R-linearly dep. $\blacksquare$

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$$a_1 x_1, a_2 x_2, \dots, a_n x_n \text{ is a basis for } N$$

and $a_1 | a_2 | \cdots | a_m$.

Pf. of stacked basis thm for the case when R is an ED:

$$M \cong R^n \xRightarrow{\text{(lemma)}} \text{rank of } N \text{ is } m \le n.$$

If $x_1, \dots, x_n$ any basis for M and if $y_1, \dots, y_m$ is any generating set for N then

$$\vec{y} = A\vec{x} \text{ for some } A \in M_{m \times n}(R).$$

Goal: Use elem. row & col. ops. to pick a basis and gen. set for which
$$\vec{y} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & a_m \end{pmatrix} \vec{x},$$

where $a_1 | a_2 | \cdots | a_m$. ... (next time).