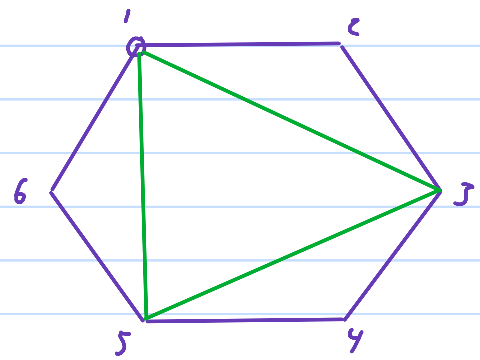
Ex. 1) Use this to show that $D_{12} \cong D_6 \times C_2$.




Pf: $H = \langle s, r^2 \rangle$, $K = \langle r^3 \rangle$.

$H = \{e, r^2, r^4, s, sr^2, sr^4\} \cong D_6$,

$K = \{e, r^3\} \cong C_2$.

$H \cap K = \{e\} \Rightarrow |HK| = \frac{|H||K|}{|H \cap K|} = 12 \Rightarrow HK = D_{12}$

(check)

$H, K \trianglelefteq G \Rightarrow HK \cong H \times K$.

2) $G = D_{12}$, $H = \langle r \rangle$, $K = \langle s \rangle$

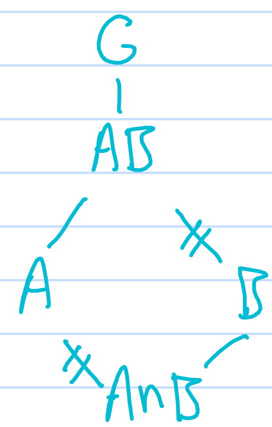
$|H| = 6$, $|K| = 2$, $H \cap K = \{e\} \Rightarrow HK = D_{12}$

But $H \cong C_6$, $K \cong C_2$, so $D_{12} = HK \not\cong H \times K$.

Why? $H \trianglelefteq G$, but $K \not\trianglelefteq G$.

## More isomorphism theorems:

**2nd isomorphism theorem (diamond thm):**




Suppose $A, B \leq G$, $A \subseteq N_G(B)$, then $AB \leq G$, $B \trianglelefteq AB$, $A \cap B \trianglelefteq A$, and

$$\boxed{AB/B \cong A/A\cap B.}$$

Pf: First, note that:

- $A \subseteq N_G(B) \Rightarrow AB \leq G$
- $A \subseteq N_G(B)$ and $B \subseteq N_G(B) \Rightarrow AB \subseteq N_G(B) \Rightarrow B \trianglelefteq AB$.

Define $\phi : A \to AB/B$ by $\phi(a) = aB$.

- $\phi$ is a homom: Suppose $a, a' \in A$. Then

$$\phi(aa') \underset{\substack{\uparrow \\ \text{def. of } \phi}}{=} (aa')B \underset{\substack{\uparrow \\ \text{well-def. of mult. of cosets.}}}{=} (aB)(a'B) = \phi(a)\phi(a').$$

- $\phi$ is surj: Let $(ab)B \in AB/B$. Then

$$\phi(a) = aB = (ab)B.$$

- $\ker \phi = \{a \in A : \phi(a) = eB\}$

$$= \{a \in A : aB = B\} = A \cap B.$$

By the 1st isom. thm., $A \cap B \trianglelefteq A$, and

$$A/A\cap B \cong AB/B. \quad \blacksquare$$

3rd isom. thm. (quotient thm.):

If $H, K \trianglelefteq G$, $H \leq K$, then $H \trianglelefteq K$,

$K/H \trianglelefteq G/H$, and

$$\boxed{\frac{(G/H)}{(K/H)} \cong G/K .}$$

$$\begin{tikzcd} G \arrow[d, dash] \\ K \arrow[d, dash] \\ H \end{tikzcd}$$

Pf: First, since $H \trianglelefteq G$, it follows immed. that $H \trianglelefteq K$.

Next define $\phi : G/H \to G/K$ by $\phi(gH) = gK$

- $\phi$ is well defined:

  Suppose $g, g' \in G$, $gH = g'H$. Then $g^{-1}g' \in H \subseteq K$

  $\Rightarrow g'K = gK$.

- $\phi$ is a homom.
- $\phi$ is surj.
- $\ker \phi = \{ gH \in G/H : \phi(gH) = K \}$

  $= \{ gH \in G/H : gK = K \}$

  $= \{ gH \in G/H : g \in K \} = K/H$.

By the 1st isom. thm.,

$$K/H \trianglelefteq G/H \text{ and } \frac{G/H}{K/H} \cong G/K. \ \blacksquare$$

Ex: 
$$\begin{tikzcd} G = \mathbb{Z} \arrow[d, dash] \\ K = 2\mathbb{Z} \arrow[d, dash] \\ H = 6\mathbb{Z} \end{tikzcd}$$

$$\left(\mathbb{Z}/6\mathbb{Z}\right) \Big/ \left(2\mathbb{Z}/6\mathbb{Z}\right) \cong \mathbb{Z}/2\mathbb{Z}$$

4th isom. thm (lattice thm): If $N \trianglelefteq G$ then there is an inclusion preserving bijection between subgroups $A \leq G$ which contain $N$, and subgroups of $G/N$. This bijection also preserves indices of respective subgroups.

Exs:

1) $G = Q_8$, $N = \{\pm 1\}$

$$\begin{tikzcd} & G = Q_8 \arrow[dl, equal] \arrow[d, equal] \arrow[dr, equal] & \\ \langle i \rangle \arrow[dr, equal] & \langle j \rangle \arrow[d, equal] & \langle k \rangle \arrow[dl, equal] \\ & N = \langle -1 \rangle \arrow[d, dash] & \\ & \{1\} & \end{tikzcd}$$

$$\begin{tikzcd} & G/N \arrow[dl, equal] \arrow[d, equal] \arrow[dr, equal] & \\ H_1 \arrow[dr, equal] & H_2 \arrow[d, equal] & H_3 \arrow[dl, equal] \\ & \{eN\} & \end{tikzcd}$$