Modules over commutative rings

Assume that R is a commutative ring w/ identity 1.

An R-module is an Abelian group $(M,+)$ together with a binary operation $\cdot : R\times M \to M$ called scalar multiplication (notation: $r\cdot m = rm$) satisfying:

$(\forall r,s \in R,\ x,y \in M)$

i) $(rs)x = r(sx)$

ii) $1x = x$

iii) $(r+s)x = rx + sx$

iv) $r(x+y) = rx + ry$.

Exs:

1) If F a field then being an F-module is the same as being a vec. sp. over F.

F-modules V $\Longleftrightarrow$ F vector spaces

sub F-modules of V $\Longleftrightarrow$ F-subspaces of V

2) Any Abelian group $(G,+)$ can be thought of as a $\mathbb{Z}$-module: $n \cdot x = x + x + \cdots + x$ (n-times). The $\mathbb{Z}$-module rules don't add any additional structure to G.

$\mathbb{Z}$-modules G $\Longleftrightarrow$ Abelian groups

sub $\mathbb{Z}$-modules of G $\Longleftrightarrow$ subgroups of G

3) Suppose F is a field, let $R = F[x]$.

- Suppose V is an F-vec. sp. and $T: V \to V$ is a lin. trans. Then we can define a bin. oper $F[x] \times V \to V$ by:

$\forall f \in F[x]$, $v \in V$, $f(x) = \sum_{i=0}^{n} a_i x^i$,

$$f \cdot v := \sum_{i=0}^{n} a_i \cdot T^i(v).$$

Check that this rule turns V into an $F[x]$-module.

- Other direction: Suppose V is an $F[x]$ module. Then V is an F-module (F is a subring of $F[x]$), so V is a vector space over F. The map $T: V \to V$ defined by $T(v) = x \cdot v$ is a linear transformation, and it uniquely determines the entire module structure, as above.

$F[x]$-modules V $\Leftrightarrow$ F-vector spaces V, together w/ a linear transformation $T: V \to V$

What are the $F[x]$ submodules $W \subseteq V$?

- $(W,+)$ has to be a subgroup of V.
  ↳ W has to be an F-module $\Rightarrow$ W has to be an F-subspace of V
- W must be closed under mult. by elems. of $F[x]$

  This happens $\Leftrightarrow$ $xW \subseteq W$

  In the language of linear transformations, this is the same as $T(W) \subseteq W$.

$F[x]$-submodules of V $\Leftrightarrow$ T-invariant subspaces $W \subseteq V$.

$(T(W) \subseteq W)$

4) If $n \in \mathbb{N}$ then the direct product of additive groups $R^n = R \oplus R \oplus \cdots \oplus R$ (n-times) (external direct sum).

5) If S is a subring of R then $(R,+)$ can be thought of as an S-module in the natural way.

6) If M is an R-module and $S \subseteq R$ is a subring then M can also be thought of in a natural way as an S-module.

7) If M is an additive subgroup of R, then M will be an R-module (in the natural way) if and only if M is an ideal of R.

8) If I is an ideal of R then the additive group R/I is an R-module, with scal multiplication defined by

$$r(x+I) = rx + I.$$

The fact that I is an ideal guarantees that this is well-defined.