3a) Suppose V is an n-dim F-v.s. and K/F is a field extension

Special case of 2b from before:

$$K \otimes_F V \cong K^n.$$

b) If V is an n-dim $\mathbb{R}$-v.s. then

$$\mathbb{C} \otimes_{\mathbb{R}} V \cong \mathbb{C}^n$$ (complexification of V)

Appendix: Other ways to think about tensor products of v.s.'s:

Suppose V and W are F-v.s.'s, $\dim V = m$, $\dim W = n$,

$\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_n\}$ bases.

Our way so far of thinking about $V \otimes_F W$:

Then $V \otimes_F W$ is a vector space over F of dim mn

with basis $\{v_i \otimes w_j : 1 \le i \le m, 1 \le j \le n\}$.

Generic element: $\sum_{i,j} a_{ij} (v_i \otimes w_j)$. (unique)

Another way: Can identify $V \otimes_F W$ with the space of multilinear maps from $V \times W$ to F (this space is a v.s. over F):

multilinear map: $f : V \times W \to F$

$$\updownarrow$$

elems of $V \otimes_F W$: $\sum_{i,j} f(v_i, w_j)(v_i \otimes w_j)$

* This allows us to think about elems. of $V \otimes_F W$ as multilinear maps, and this leads to different ways of thinking about what tensor products represent.

Exs: $V \otimes_F V \otimes_F \cdots \otimes_F V$ (n-times) ($\dim_F V < \infty$)

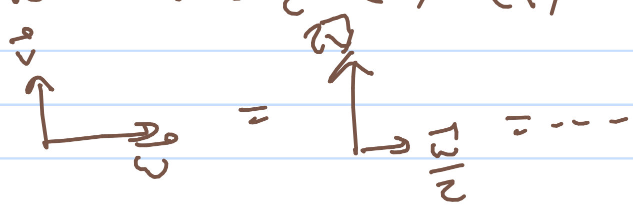
$n=0$) empty product $\Leftrightarrow$ field F of scalars

$n=1$) • vectors (type $(1,0)$)

• 1-forms (linear functionals) (type $(0,1)$)

$n=2$) • simple tensors $\Leftrightarrow$ "plane segments" (type $(2,0)$)

$$v \otimes w = 2v \otimes \frac{w}{2} = (av) \otimes \left(\frac{w}{a}\right) = \cdots$$



• bilinear forms (type $(0,2)$)

• type $(1,1)$:

Suppose $\phi \in V \otimes_F V$.

Think of this as a bilinear map

$$\phi : V \times V \to F.$$

Define $T_\phi : V \to V^*$ (lin. functionals on V)

$$T_\phi(v) = (\phi_v : x \mapsto \phi(v,x)).$$

The fact that ϕ is bilin. guarantees that T_ϕ is linear. Since $V^* \cong V$ this allows us allows us to think of elems of $V \otimes V$ as <u>linear operators on V</u>. This process is also reversible, so

elems of $V \otimes V$ $\Longleftrightarrow$ linear operators on V

$n=3$)
- simple tensors $\Longleftrightarrow$ "space segments" (type $(3,0)$)
- 3-multilinear forms (type $(0,3)$)
- $(1,2)$-tensors