Exs: 1) Maximal ideals in comm. rings w/ identity $1 \neq 0$ are always prime (the converse is not true in general).

If $M \subseteq R$ is maximal then

R/M is a field $\Rightarrow$ R/M is an ID $\Leftrightarrow$ M is a prime ideal.

2) $R = \mathbb{Z}$, maximal ideals $= (p)$, p prime

prime ideals $= \{0\}$, maximal ideals.

3) $R = \mathbb{Z}[x]$, $I = (x)$.

- Consider $\phi: R \to \mathbb{Z}$, $\phi(f) = f(0)$.

This a surjective ring homom.,

$\ker \phi = I$, so by 1st isom thm,

$R/I \cong \mathbb{Z}$.

Since $\mathbb{Z}$ is an ID, I is a prime ideal.

However $\mathbb{Z}$ is not a field, so I not a maximal ideal.

- What is a maximal ideal it is contained in?

Let $\tau: R \to \mathbb{Z}/2\mathbb{Z}$ be def. by

$\tau(f) = f(0)$. Then τ is a surj. ring hom, and $\ker \tau = (2, x)$.

By 1st Isom. Thm, $R/(2,x) \cong \mathbb{Z}/2\mathbb{Z}$.

Since $\mathbb{Z}/2\mathbb{Z}$ is a field, $(2,x)$ is a maximal ideal, which contains (x).

- Also, note that $J=(2,x)$ is not principal:

  Suppose $J=(f(x))=\{g(x)f(x) : g(x)\in\mathbb{Z}[x]\}$.

  Then $2\in J \Rightarrow \exists\, g(x)\in\mathbb{Z}[x]$ s.t. $2=f(x)g(x)$

  $\Rightarrow \deg f=0$, so $f(x)=m$, for some $m\in\mathbb{Z}$.

  $\Rightarrow f(x)=\pm 2 \cdots \Rightarrow x\notin J$, which is a contradiction.

  So $\mathbb{Z}[x]$ is not a PID.

Fields of fractions

Suppose R is an ID (comm. ring w/ iden $1 \neq 0$ and no zero-divs.).

Then there is a field F, called the field of fractions of R, satisfying:

i) F contains an isomorphic copy of R,

ii) Any field K which contains an isomorphic copy of R also contains an isomorphic copy of F.

How to construct F:

Let $F = \{(a,b) \in R \times R : b \neq 0\}/\sim$,

where $\sim$ is the equiv. rel. defined by:

$(a,b) \sim (a',b') \iff a'b = ab'$.

Define: $\forall (a,b), (c,d) \in F$,

$(a,b) + (c,d) = (ad + bc, bd)$

$(a,b)(c,d) = (ac, bd)$.

(well-defined: doesn't depend on choice of equiv. class)

- $(F, +, \cdot)$ is a field:

$(0,1)$ is the additive identity,

$(1,1)$ is the mult. identity.

If $(a,b) \in F$, $a \neq 0$, then $(a,b)^{-1} = (b,a)$.

i) F contains an isomorphic copy of R:

The map $\varphi: R \to F$ defined by

$\varphi(r) = (r, 1)$ is an injective ring homom.

ii) Suppose K is a field and $\phi: R \to K$ is an injective ring hom., then $\tau: F \to K$ def. by

$\tau(a,b) = \phi(a)\phi(b)^{-1}$ is an injective ring hom.,

so $\tau(F) \cong F$.

Next topic: ED's, PID's, UFD's, ID's.

- ID = Integral Domain = commutative ring w/ iden. $1 \neq 0$ and no zero-divs.
- UFD = Unique Factorization Domain

Defs: ① If R is an ID then a non-zero element $a \in R \setminus R^\times$ <u>irreducible</u> if a is not a product of non-units. Otherwise a is called <u>reducible</u>.

② A nonzero element $a \in R$ is <u>prime</u> if (a) is a prime ideal.