Tensor of modules

* Suppose for this section that R is a comm. ring w/ 1_R.
  Note: story is slightly more complicated otherwise

Motivation, part 1:

Suppose M and N are R-modules, and that we want to construct an R-module $M \otimes_R N$ with the following prop:

(Cartesian product)

$\exists$ a map $\otimes : M \times N \to M \otimes_R N$
$(m,n) \mapsto m \otimes n$

which is bilinear:

- $\forall m_1, m_2 \in M, n \in N$

$$(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$$

- $\forall m \in M, n_1, n_2 \in N,$

$$m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2.$$

- $\forall r \in R, m \in M, n \in N$

$$(rm) \otimes n = r(m \otimes n) = m \otimes (rn).$$

Is this contrived? No, objects like this arise naturally in:

quantum physics
electromagnetism
gravitation
differential topology ...

How to construct objects like this:

1) Start w/ the free Abelian gp. (i.e. free $\mathbb{Z}$-module) generated by the set of all elems. of the form $m \otimes n$, $m \in M$, $n \in N$.

2) Quotient by the subgroup generated by all relations of the form

$$(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$$

$$m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2$$

$$(rm) \otimes n = m \times (rn).$$

Notes:

- The resulting quotient group is denoted $M \otimes_R N$, the <u>tensor product</u> over R of M with N.

  Notation: $m \otimes n$ will denote the <u>coset</u> of $m \otimes n$ in this quotient.

  $M \otimes_R N$ is an R-module, with scal. mult. def by

$$r(m \otimes n) = (rm) \otimes n = m \otimes rn,$$

  and extended linearly to all of $M \otimes_R N$.

- $\forall m \in M, n \in N$, the coset $m \otimes n$ is called a simple tensor. Every elem. of $M \otimes_R N$ can be written (non-uniquely in general) as a sum of simple tensors $\sum_{i=1}^{n} (m_i \otimes n_i)$.

Thm.: $M \otimes_R N$ is an R-module, w/ scal mult. defined as above. The map $\otimes$ is R-bilinear. Furthermore, if L is any R-module and $\varphi: M \times N \to L$ is an R-bilin. map then $\exists$ a unique R-mod. hom. $\Phi: M \otimes_R N \to L$ s.t. the following diagram commutes:

$$\begin{tikzcd} M \times N \arrow[r, "\otimes"] \arrow[dr, "\varphi"'] & M \otimes_R N \arrow[d, "\Phi"] \\ & L \end{tikzcd}.$$

Universal property of $M \otimes_R N$.

Note: Since $\varphi(M \times N) \cong \Phi(M \otimes_R N) / \ker \Phi$, (1st. Isom. Thm for R-modules + a little thought)

the universal property shows that $M \otimes_R N$ is the largest (up to isom.) R-module which can be generated from the image of an R-bilin. map from $M \times N$ into another R-module.

Exs:

1) $\forall m \in M, n \in N$, $m \otimes 0 = 0 \otimes n = 0$.

Pf: $m \otimes 0 = m \otimes (0+0) = m\otimes 0 + m \otimes 0$

$\Rightarrow m \otimes 0 = 0$

Same for $0 \otimes n$. ∎

2) $R = \mathbb{Z}$, $M = \mathbb{Q}$, $G =$ finite Abelian gp.

What can we say about $\mathbb{Q} \otimes_{\mathbb{Z}} G$?

let $n = |G|$. Then $\forall r \in \mathbb{Q}$, $g \in G$,

$$r \otimes g = \left(n \cdot \frac{r}{n}\right) \otimes g = \left(\frac{r}{n}\right) \otimes (ng) = \left(\frac{r}{n}\right) \otimes 0 = 0.$$

R-mod. structure +bilinearity

Conclusion:

$$\mathbb{Q} \otimes_{\mathbb{Z}} G = 0.$$

3a) $\mathbb{R} \otimes_{\mathbb{R}} \mathbb{R}$ (note: $\mathbb{R}$ is a 1-dim v.s. over $\mathbb{R}$)

another note: $\mathbb{R} \otimes_{\mathbb{R}} \mathbb{R}$ is also an $\mathbb{R}$-v.s.

- $\forall r, s \in \mathbb{R}$, $r \otimes s = rs(1 \otimes 1)$, so $1 \otimes 1$ generates $\mathbb{R} \otimes_{\mathbb{R}} \mathbb{R}$ as an $\mathbb{R}$-module $\Rightarrow \dim \leq 1$.

- Define $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $\varphi(r,s) = rs$.

  This is bilinear and surjective, so $\exists$ a (unique) R-mod. hom. $\Phi : \mathbb{R} \otimes_{\mathbb{R}} \mathbb{R} \to \mathbb{R}$, which has the univ. prop.

  Φ is surj. $\Rightarrow \dim(\mathbb{R} \otimes_{\mathbb{R}} \mathbb{R}) \geq 1$, so $\mathbb{R} \otimes_{\mathbb{R}} \mathbb{R} \cong \mathbb{R}$.

3b) $\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^3 = V$ (V is an $\mathbb{R}$-v.s.)

b/c of bilin.

- This is generated as an $\mathbb{R}$-v.s. by

$$\{\vec{e}_{i,\mathbb{R}^2} \otimes \vec{e}_{j,\mathbb{R}^3} : 1 \le i \le 2,\ 1 \le j \le 3\}$$

Ex: let $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2$, $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \in \mathbb{R}^3$

Then $\vec{v} \otimes \vec{w}$

$$= \left(v_1 \vec{e}_{1,\mathbb{R}^2} + v_2 \vec{e}_{2,\mathbb{R}^2}\right) \otimes \left(w_1 \vec{e}_{1,\mathbb{R}^3} + w_2 \vec{e}_{2,\mathbb{R}^3} + w_3 \vec{e}_{3,\mathbb{R}^3}\right)$$

$$= v_1 w_1 \left(\vec{e}_{1,\mathbb{R}^2} \otimes \vec{e}_{1,\mathbb{R}^3}\right) + v_1 w_2 \left(\vec{e}_{1,\mathbb{R}^2} \otimes \vec{e}_{2,\mathbb{R}^3}\right) + v_1 w_3 \left(\vec{e}_{1,\mathbb{R}^2} \otimes \vec{e}_{3,\mathbb{R}^3}\right)$$

$$+ v_2 w_1 \left(\vec{e}_{2,\mathbb{R}^2} \otimes \vec{e}_{1,\mathbb{R}^3}\right) + v_2 w_2 \left(\vec{e}_{2,\mathbb{R}^2} \otimes \vec{e}_{2,\mathbb{R}^3}\right) + v_2 w_3 \left(\vec{e}_{2,\mathbb{R}^2} \otimes \vec{e}_{3,\mathbb{R}^3}\right)$$

Therefore $\dim_{\mathbb{R}} V \le 2 \cdot 3 = 6$

... cont. next time