

## MATH 6321

Theory of Functions of a Real Variable  
Spring 2025

First name: \_\_\_\_\_ Last name: \_\_\_\_\_

Points:

**Assignment 7, due Thursday, March 28, 10am**

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

**Problem 1**

Let  $\mu$  be the Lebesgue measure on  $[0; 1]$  and  $\nu$  the counting measure on the  $\sigma$ -algebra of Lebesgue-measurable sets. Show that although  $\nu$  and  $\mu$  is bounded, there is no  $h \in L^1(\mu)$  such that  $d\nu = h d\mu$ .

**Problem 2**

Let  $(X; \mathcal{M})$  be a measurable space and  $\mu, \nu$  be finite (positive) measures, and  $\lambda$  be a  $\sigma$ -finite measure on  $\mathcal{M}$ . Show that if  $\mu \ll \lambda$  and  $\nu \ll \lambda$  then the corresponding Radon-Nikodym derivatives satisfy the Leibnitz rule

$$\frac{d(\mu\nu)}{d\lambda} = \frac{d\mu}{d\lambda} \frac{d\nu}{d\lambda}$$

which holds  $\lambda$ -almost everywhere, and if  $\mu \ll \nu$  as well as  $\nu \ll \lambda$ , then

$$\frac{d\mu}{d\lambda} = \frac{d\mu}{d\nu} \frac{d\nu}{d\lambda}^{-1} :$$

holds  $\lambda$ -almost everywhere.

**Problem 3**

Let  $\mu$  be a complex Borel measure on  $[0; 2\pi]$  and let for  $n \in \mathbb{Z}$ ,  $e_n(t) = e^{-int}$  and

$$\hat{\mu}(n) = \int_0^{2\pi} e_n d\mu :$$

Show that if  $\lim_{|n| \rightarrow \infty} \hat{\mu}(n) = 0$ , then  $\lim_{|n| \rightarrow \infty} \hat{\mu}(n) = 0$  as well. Hint: The assumption implies that  $\lim_{|n| \rightarrow \infty} \hat{\mu}(n) = 0$  where  $d\mu = f d\lambda$  for any trigonometric polynomial  $f$ , and hence for any continuous  $f$ , and hence for any bounded  $f$ , and hence for  $f = \sum_{j=-N}^N c_j e_j$ .