9) Suppose $M$ is an $R$-module and $I$ is an ideal of $R$.

If $ax=0$ for all $a\in I$, $x\in M$, then $M$ can be thought of as an $R/I$ module, w/ scal. mult. def. by

$$(r+I)x=rx.$$

Note: If $r+I=s+I$ then $rx-sx=(r-s)x=0$, since $r-s\in I$, so this is well defined.

10) If $(G,+)$ is an Abelian gp. w/ exponent $n$, then it is a $\mathbb{Z}/n\mathbb{Z}$-module.

$(nx=0,\ \forall x\in G)$

Given an $R$-module $M$, a subset $A\subseteq M$ is a <u>generating set</u> for $M$ over $R$ if, for every $x\in M$, $\exists n\in\mathbb{N}$, $r_1,\dots,r_n\in R$, and $x_1,\dots,x_n\in A$ s.t.

$$x=r_1x_1+\dots+r_nx_n.$$

If $M$ can be generated over $R$ by a finite set then it is <u>finitely generated</u> over $R$.

We say that a set $A\subseteq M$ is $R$-linearly independent if, whenever $r_1x_1+\dots+r_nx_n=0$, for distinct $x_1,\dots,x_n\in M$, we have $r_1=\dots=r_n=0$.

Otherwise, $A$ is <u>linearly dependent</u>.

If $M$ contains an $R$-linearly indep. generating set $A$, then $M$ is a <u>free module</u> and $A$ is an <u>$R$-basis</u> for $M$.

A module $M$ is <u>torsion free</u> if, whenever $rx=0$, it must be the case that $r=0$ or $x=0$.

Exs:

(1) Suppose $R$ is an integral domain and that $M$ is an $R$-module which is not torsion free. Then $\exists r\in R\setminus\{0\}$, $x\in M\setminus\{0\}$ s.t. $rx=0$. If $A$ is a generating set. Then $\exists n\in\mathbb{N}$, $r_1,\dots,r_n\in R$, $x_1,\dots,x_n\in M$, s.t.

$$x=r_1x_1+\cdots+r_nx_n.$$

We can assume that no $r_i$'s are $0$, and that all $x_i$'s are distinct. Then $0=rx=(rr_1)x_1+\cdots+(rr_n)x_n$.

Since $R$ is an ID, $rr_i\neq 0$, $\forall 1\leq i\leq n$, so $A$ is linearly dependent.

Conclusion: $M$ is <u>not</u> a free module.

Ex: $R=\mathbb{Z}$, $G$ is any finite Abelian group.

12) What if $R$ is not an ID?

Ex: $R = \mathbb{Z}/6\mathbb{Z}$, $M = (R, +)$.

Then $rx = 0$ for $r = 2, x = 3$, so $M$ is not torsion free, but $A = \{1\}$ is a generating set.

13) $R = \mathbb{Z}$, $M = (\mathbb{Q}, +)$.

Suppose $A$ is a generating set:

Then $|A| > 1$. Let $x_1 = \frac{p_1}{q_1}, x_2 = \frac{p_2}{q_2} \in A \setminus \{0\}$.

Then $(q_1 p_2) x_1 + (-p_1 q_2) x_2 = 0$,

but $q_1 p_2, -p_1 q_2 \neq 0$, so $\mathbb{Q}$ is not a free $\mathbb{Z}$-module.

14) $R = \mathbb{Q}$, $M = (\mathbb{Q}, +)$

Is $M$ a free $R$-module? Yes, because $R$ is a field, and every v.s. has a basis.

For example, $A = \{1\}$ is an $R$-basis for $M$.

If an $M$ is a free $R$-module then any basis for $M$ over $R$ will have the same cardinality, called the <u>rank</u> of $M$ over $R$.

(Warning: not true in general, if $R$ is not assumed to be commutative.)

A few more exs:

15) Quotient modules: If $M$ is an $R$-module, $N$ is a sub $R$-module, then $(M/N, +)$ is an $R$-module with scal. mult. def. by

$$r(x+N) = rx+N.$$

16) Module homomorphisms & isom thms:

Suppose $M$ and $N$ are $R$-modules. An $R$-module homomorphism is a map $\varphi: M \to N$ satisfying:

$$\varphi(x+y) = \varphi(x)+\varphi(y), \quad \forall x,y \in M,$$

$$\varphi(rx) = r\cdot\varphi(x), \quad \forall r \in R, x \in M.$$

All theorems analogous to the 1st-4th isom. thms. hold for modules (see book for details).

1st isom. thm: If $\varphi: M \to N$ is an $R$-module homom., then $M/\ker(\varphi) \cong \varphi(M)$.

(isomorphic as $R$-modules).

(Also: $\ker(\varphi)$ and $\varphi(M)$ are $R$-modules).

Fund. Thm. for Fin. Gen. Modules over a PID:

(invariant factor decomposition):

If $R$ is a PID and $M$ is a finitely generated $R$-module then

$$M \cong R^r \oplus R/(a_1) \oplus \cdots \oplus R/(a_m),$$

where $r \in \{0, 1, \ldots\}$, $a_1, \ldots, a_m \in R \setminus R^\times$, and $a_1 \mid a_2 \mid \cdots \mid a_m$.

This decomp. is unique.

(Elem. div. decomp.):

$$M \cong R^r \oplus R/(p_1^{a_1}) \oplus \cdots \oplus R/(p_k^{a_k}),$$

where $p_1^{a_1}, \ldots, p_k^{a_k}$ are powers of (not nec. dist.) prime elements $p_1, \ldots, p_k \in R$.