Department of Mathematics University of Houston

MATH 6321

Theory of Functions of a Real Variable
Spring 2025

First name: ______________________ **Last name:** ________________________ **Points:**

Assignment 3, due Thursday, February 13, 10am

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous complex-valued functions on a non-empty complete metric space X, and assume for each $x \in X$, the limit $f(x) = \lim_n f_n(x)$ exists.

a. Prove that there exists an open set $V \neq \emptyset$ and a number $M > 0$ such that for each $x \in V$ and $n \in \mathbb{N}$, $|f_n(x)| < M$.

b. Given $\epsilon > 0$, prove that there is an open set $W \neq \emptyset$ and an integer $N > 0$ such that for each $x \in W$ and $n \geq N$, $|f_n(x) - f(x)| < \epsilon$.

Hint: Baire's theorem.

Problem 2

If μ is a measure and $f \in L^\infty(\mu)$, then we can define a multiplication operator

$$M_f : L^2(\mu) \to L^2(\mu), M_f g = fg\,.$$

a. Show that $\|M_f\| \leq \|f\|_\infty$. For which measures is it true that $\|M_f\| = \|f\|_\infty$ holds for each $f \in L^\infty(\mu)$? Hint: regularity.

b. For which $f \in L^\infty(\mu)$ does M_f map $L^2(\mu)$ onto $L^2(\mu)$?