# MATH 6326 - Partial Differential Equations

Joel Sleeba
University of Houston
joelsleeba1@gmail.com

September 4, 2025

# Contents

# Chapter 1

# Derivation of PDEs

## 26/08/2025 1.1 Conservation Laws

These are balanced laws that express the fact that some quantity is balanced throughout a process.

Let $u(x,t)$ be the density of some quantity (mass, heat, energy, bacteria). Given a region $R \subset \mathbb{R}^n$, then

$$\int_R u(x,t)\ dx$$

represents the total amount of quantity inside $R$ at time $t$.

**Definition 1.1.1.** The quantity (represented by $u$) is then locally conserved if it is only gained or lost either through domain boundaries or because of sources or sinks in the domain.

**Definition 1.1.2.** The **flux** of $u(x,t)$ is the direction and rate of flow of this quantity per unit area. It has a magnitude and a direction and is therefore a vector field $\vec{J}$.

*Remark* 1.1.1. Flux is defined so that $\vec{J} \cdot \vec{n} dA$ represent the amount of quantity $u$ flowing across a small area $dA$ per unit time.

*Remark* 1.1.2. $\vec{J}$ has the unit of density times velocity.

Track $u$, area, time on same domain $\Omega$, and assume we have a source term $Q(x,t) > 0$. Conservation of $u$ on a region $R \subset \Omega$ then implies

$$\frac{d}{dt}\int_R u(x,t)\ dt = \int_R \frac{\partial u}{\partial t}\ dx = -\int_{\partial R} \vec{J} \cdot \vec{n}\ dx + \int_R Q(x,t)\ dx$$

Using divergence theorem, we can rewrite this as

$$\int_R \frac{\partial u}{\partial t}\ dx = -\int_R \nabla \cdot \vec{J}\ dx + \int_R Q(x,t)\ dx$$

which gives

$$\int_R \frac{\partial u}{\partial t} + \nabla \cdot \vec{J} - Q(x,t)\ dx = 0$$

Since $R$ can be chosen arbitrarily, and the variables $u, J, Q$ are "nice enough" (sufficiently smooth) this forces

$$\frac{\partial u}{\partial t} + \nabla \cdot \vec{J} - Q = 0 \tag{1.1}$$

for all points in space and time.

*Equation* 1.1 called the continuity equation is the standard form of all conservation equations and depending on the context different $\vec{J}$ will play different roles.

## 1.2 Review of Vector Calculus

**Theorem 1.2.1** (Gauss-Green Theorem)**.** *Suppose* $u \in C^1(\overline{\Omega})$, *and let* $\vec{x} \in \Omega \subset \mathbb{R}^m$ *with* $u_{x_i} := \frac{\partial u}{\partial x_i}$, *then*

$$\int_\Omega u_{x_i}\ dx = \int_{\partial\Omega} u n_i\ dx$$

**Theorem 1.2.2** (Divergence Theorem)**.**

$$\int_\Omega \nabla \cdot \vec{J}\ dx = \int_{\partial\Omega} \vec{J} \cdot \vec{n}\ dx$$

**Theorem 1.2.3** (Green's First Identity)**.**

$$\int_\Omega (u\Delta v + \nabla u \cdot \nabla v)\ dx = \int_{\partial\Omega} u \frac{\partial v}{\partial \vec{n}}\ dS$$

**Theorem 1.2.4** (Green's 2nd identity)**.**

$$\int_\Omega (u\Delta v - v\Delta u)\ dx = \int_\Omega u\frac{\partial v}{\partial \vec{n}} - v\frac{\partial u}{\partial \vec{n}}\ dS$$

*Remark* 1.2.1.

**Theorem 1.2.5.** *Suppose* $f : \Omega \times [a, b] \to \mathbb{C}$ *with* $a < b \in \mathbb{R}$ *and* $f(\cdot, t) : \Omega \to \mathbb{C}$ *is integrable for each* $t \in [a, b]$. *Let* $F(t) = \int_\Omega f(x, t)\ d\mu(x)$, *where* $\mu$ *denote a measure for* $\Omega$. *Suppose* $\frac{\partial f}{\partial t}$ *exists and* $g \in L^1(\mu)$ *such that* $\left|\frac{\partial f}{\partial t}(x, t)\right| < g(x)$, *then* $F$ *is differentiable and*

$$F'(t) = \int_\Omega \frac{\partial f}{\partial t}(x, t)\ d\mu(x)$$

## 1.3 Examples

### 1.3.1 Advection (Convection, Drift)

Advection or drift where a bulk of the quantity is carried along the medium in a given velocity.

Suppose $u(x, t)$ is the density of some quantity that is being transported with velocity $\vec{v}$. Then

$$\vec{J} = \vec{v}u$$

**Example 1.3.1** (Simple transport)**.** Assume $\vec{v}$ is a constant. Then Equation 1.1 becomes

$$\frac{\partial u}{\partial t} + \vec{v}\ \nabla u - Q = 0$$

In case $Q = 0$ and $\Omega \subset \mathbb{R}$, then we get

$$\frac{\partial u}{\partial t} + \vec{v}\ \frac{\partial u}{\partial t} = 0$$

28/08/2025 See Burgers equations and diffusion equation from the notes. Reasons for boundary condition and initial condition.

## 1.4 Boundary Contitions

**Definition 1.4.1.** Dirichlet boundary condition is when the prescribed value of unknown $u$ at the boundary $\partial\Omega$. For example when $u$ is the temperature of the metal rod, and the temperature along the boundary is known.

**Definition 1.4.2.** Neumann boundary condition is when $\nabla \cdot \vec{n} = g(x)$, where $x \in \partial\Omega$. When $g(x) = 0$, we say it is **homogeneus Neumann boundary condition**.

**Definition 1.4.3.** Robin boundary condition is when $u + \nabla \cdot \vec{n} = g(x)$, where $x \in \partial\Omega$.

**Definition 1.4.4.** No flux boundary condition is when $\vec{J} \cdot \vec{n} = 0$, where $x \in \partial\Omega$.

## 1.5 Steady States

A steady state solution is a solution ot

$$\partial_t u + \nabla \cdot J = Q$$

that does not depend on time. So it solves $\nabla \cdot J = Q$. If we have $u$ to be the concentration of chemical species, then $J = -D\nabla u$. So $\nabla \cdot J = Q \implies D\Delta u = Q$ if $D$ is a constant.

## 1.6 Euler's Equation

Let $\vec{u} = \vec{u}(x,t)$ be the velocity of a fluid for $(x,t) \in \Omega \subset \mathbb{R}^3 \times [0,T]$, $T > 0$.

Using conservation of mass $R \subset \Omega \subset \mathbb{R}^3$, where $\rho = \rho(x,t)$ is the density of the fluid.

$$\frac{d}{dt}\int_R \rho \, d\nu = -\int_{\partial R} \vec{J} \cdot \vec{n} \, dA = -\int_{\partial R} \rho\vec{u} \cdot \vec{n} \, dA$$

By divergence theorem, we have

$$\frac{\partial \rho}{\partial t} + \nabla(\rho\vec{u}) = 0$$

Assuming the fluid is incompressible, we have $\rho$ is a constant, hence

$$\nabla \cdot \vec{u} = 0$$

04/09/2025

## 1.7 Euler Equation

Let $\vec{u} = \vec{u}(x,t)$ be the velocity field of a fluid, where $(x,t) \in \Omega \subset \mathbb{R}^3 \times [0,T], T > 0$. Conservation of mass gives us

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\vec{u})$$

If the fluid is incompressible, then $\rho$ is a constant and then $\nabla \cdot \vec{u} = 0$.

By the convervation of momentum,

$$\underbrace{\frac{d}{dt}\int_R \rho\vec{u} \, dx}_{\text{Rate of change of momentum}} = -\underbrace{\int_{\partial R} \rho\vec{u}(\vec{u} \cdot \vec{n}) \, dA}_{\text{momenum flux term}} - \underbrace{\int_{\partial R} P\vec{n} \, dA}_{\text{Forces felt by region } R}$$

where $\rho\vec{u}$ is the momentum density, and hence $\rho\vec{u}(\vec{u})$ is the momentum flux, being the product of density and velocity.

We are additionaly assuming that the fluid is ideal, i.e there's no viscocity. Then using the divergence theorem to switch to volume integrals, we get

$$\int_{\partial R} \rho u^i (\vec{u}\cdot\vec{n})\ dA = \int_R \nabla\cdot(\rho u^i \vec{u})\ dx$$

where

$$\begin{aligned}\nabla\cdot(\rho u^i\vec{u}) &= \langle \partial x, \partial y, \partial z\rangle \cdot u^i \langle \rho u^1, \rho u^2, \rho u^3\rangle \\ &= u^i \nabla\cdot(\rho\vec{u}) + \rho\vec{u}\cdot\nabla u^i\end{aligned}$$

which gives

$$\int_{\partial R} \rho u^i (\vec{u}\cdot\vec{n})\ dA = \int_R \nabla\cdot(\rho u^i \vec{u})\ dx = \int_R (u^i\nabla\cdot(\rho\vec{u}) + \rho\vec{u}\cdot\nabla u^i)\ dx$$

Similarly

$$\begin{aligned}\int_{\partial R} P n^i\ dA &= \int_{\partial R} P\vec{e}_i\vec{n}\ dA \\ &= \int_R \nabla\cdot(P\vec{e}_i)\ dx \\ &= \int_R \frac{\partial P}{\partial x_i}\ dx\end{aligned}$$

Combining things together, we get the conservation of momentum as

$$\int_R \frac{d}{dt}(\rho u^i) + u^i\nabla\cdot\rho\vec{u} + \rho\vec{u}\cdot\nabla u^i + \frac{\partial P}{\partial x_i} = 0$$

This can be written shortly as

$$(\rho\vec{u})_t + \vec{u}\nabla\cdot(\rho\vec{u}) + (\rho\vec{u}\cdot\nabla\vec{u}) + \nabla P = 0$$

If the fluid is incompressible, then $\rho$ is a constant and we get

$$\vec{u}_t + \vec{u}\nabla\cdot\vec{u} + \vec{u}\cdot\nabla\vec{u} + \frac{1}{\rho}\nabla P = 0$$

which can also be written as

$$\vec{u}_t + (\vec{u}\cdot\nabla)\vec{u} + \frac{1}{\rho}\nabla P = 0$$