MATH 6321

Theory of Functions of a Real Variable
Spring 2025

First name: ______________________ **Last name:** ________________________ **Points:**

Assignment 5, due Thursday, February 27, 10am

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let $(\Lambda_n)_{n=1}^{\infty}$ be a sequence of bounded linear functionals from a normed vector space X to a Banach space Y and suppose $\sup_{n\in\mathbb{N}} \|\Lambda_n\| = M < \infty$, and assume there is a dense set $E \subset X$ such that for each $x \in E$, $(\Lambda_n x)_{n=1}^{\infty}$ is a convergent sequence. Prove that $(\Lambda_n x)_{n=1}^{\infty}$ converges for each $x \in X$.

Problem 2

Let X be a Banach space with norm $\|\cdot\|$ and $\phi : X \to \mathbb{C}$ a linear functional. Define another norm $\|\cdot\|_\phi$ on X by

$$\|x\|_\phi = \|x\| + |\phi(x)|$$

(no need to prove the norm properties). Show that if X with the norm $\|\cdot\|_\phi$ is also a Banach space, then there is $M \geq 0$ such that for each $x \in X$, $|\phi(x)| \leq M\|x\|$.

Problem 3

Let V be a subspace of a normed vector space X and $y \in X$. Show that $y \in \overline{V}$ if and only if $\phi(y) = 0$ for each bounded linear functional such that $\phi|_V = 0$.