Field extensions

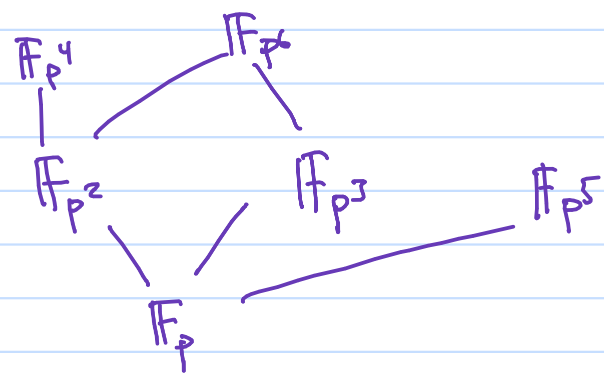
Def: If $F \subseteq K$ are fields then K is an <u>extension</u> of F

Notation: K/F , $\begin{array}{c} K \\ | \\ F \end{array}$.

F is the <u>base field</u> or <u>ground field</u>.

Exs: 1) $\mathbb{R}/\mathbb{Q}$, $\mathbb{C}/\mathbb{R}$

$$\begin{array}{c} \mathbb{C} \\ | \\ \mathbb{R} \\ | \\ \mathbb{Q} \end{array}$$



2a) Suppose $\alpha \in \mathbb{C}$, and define $\mathbb{Q}(\alpha)$ to be the smallest subfield of $\mathbb{C}$ containing $\mathbb{Q} \cup \{\alpha\}$.

(i.e. the intersection of all subfields of $\mathbb{C}$ containing $\mathbb{Q} \cup \{\alpha\}$)

This is called the field obtained by <u>adjoining</u> α to $\mathbb{Q}$, and $\mathbb{Q}(\alpha)$ is a field extension of $\mathbb{Q}$.

2b) More generally suppose K/F is a field ext. and $\alpha_1, \ldots, \alpha_n \in K$. Then $F(\alpha_1, \ldots, \alpha_n)$ is defined to be the smallest subfield of K containing $F \cup \{\alpha_1, \ldots, \alpha_n\}$.

Lemma: If K/F is a field ext., $\alpha_1, \dots, \alpha_n \in K$, then
$$F(\alpha_1, \dots, \alpha_n) = \left(F(\alpha_1, \dots, \alpha_{n-1})\right)(\alpha_n).$$

3) Suppose F is a field, $f \in F[x]$ an irred. poly.
Then (f) is a maximal ideal, so $K = F[x]/(f)$
is a field. The map
$$\phi: F \to K \text{ defined by } \phi(a) = a + (f)$$
is a ring homom. with $\ker(\phi) = \{0\}$, so
$\phi(F)$ is an isom. copy of F, in K.
This allows us to think of K/F as a field
extension

Note: If K/F is a field extension then K can be thought of naturally as an F-vector space (scal. prod: $(f,k) \mapsto fk$). ...