MATH6321 - Theory of functions of a real variable
Homework 10

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April 24, 2025

1. **Solution:** Let $s_n := \sum_{i=1}^{m} a_i^n \chi_{A_i^n}$ be an increasing sequence of simple functions which converge pointwise to f from below where A_i^n are Lebesgue measurable in $\mathbb{R}$, $A_i^n \cap A_j^n = \emptyset$ if $i \neq j$, and $a_i^n \in [0, \infty]$. Then

$$\begin{aligned}
A(s_n) &= \{(x, y) \in \mathbb{R}^2 \ : \ 0 < y < s_n(x)\} \\
&= \{(x, y) \ : \ 0 < y < \sum_{i=1}^{m} a_i^n \chi_{A_i^n}\} \\
&= \bigcup_{i=1}^{m} \{(x, y) \ : \ x \in A_i^n, 0 < y < a_i^n\} \\
&= \bigcup_{i=1}^{m} A_i^n \times (0, a_i^n)
\end{aligned}$$

Since each A_i^n and $(0, a_i^n)$ are Lebesgue measurable in $\mathbb{R}$, $A_i^n \times (0, a_i^n)$ is Lebesgue measurable in $\mathbb{R}^2$ by the definition of the product σ-algebra. Moreover, since s_n converge pointwise to f from below, if $y < f(x)$, then $y < s_n(x)$ for some $n \in \mathbb{N}$. Hence,

$$A(f) = \bigcup_{n=1}^{\infty} A(s_n)$$

Thus, $A(f)$ is a Lebesgue measurable set in $\mathbb{R}^2$.

To show that $m_2(A(f)) = \int f \ dm$, since

$$A(s_n) = \bigcup_{i=1}^{m} A_i^n \times (0, a_i^n)$$

for s_n defined as before, and since $m_2 = m \times m$,

$$m_2(A(s_n)) = \sum_{i=1}^{m} m(A_i^n)m((0, a_i^n)) = \sum_{i=1}^{m} a_i^n m(A_i^n) = \int s_n \ dm$$

Since s_n is an increasing sequence, $A(s_i) \subset A(s_{i+1})$, and by the continuity of the measure m_2 from below, we get

$$m_2(A(f)) = \lim_n m_2(A(s_n)) = \sup_n \int s_n \ dm = \int f \ dm$$

2. **Solution:** Notice that since $f \in L^1(\mathbb{R})$, by the translation invariance of the Lebesgue measure, for all $x \in \mathbb{R}$, $f_x(y) := f(x-y)$ is also in $L^1(\mathbb{R})$. Then

$$\begin{aligned}
|f * g(x)| &\leq \int |f(x-y)g(y)| \ dm \\
&= \int |f(x-y)|^{\frac{1}{q}} |f(x-y)|^{\frac{1}{p}} |g(y)| \ dm \\
&\leq \Big(\int |f(x-y)| \ dm \Big)^{\frac{1}{q}} \Big(\int |f(x-y)||g(y)|^p \ dm \Big)^{\frac{1}{p}} \\
&= \|f\|_1^{\frac{1}{q}} \Big(\int |f(x-y)||g(y)|^p \ dm \Big)^{\frac{1}{p}}
\end{aligned}$$

Therefore,

$$\int |f * g(x)|^p \ dm \leq \|f\|_1^{\frac{p}{q}} \int \int |f(x-y)||g(y)|^p \ dm(y)dm(x)$$

Since $|f(x-y)||g(y)|^p \geq 0$, by Fubini's theorem,

$$\begin{aligned}
\int |f * g(x)|^p \ dm &\leq \|f\|_1^{\frac{p}{q}} \int \int |f(x-y)||g(y)|^p \ dm(x)dm(y) \\
&= \|f\|_1^{\frac{p}{q}} \int |g(y)|^p \int |f(x-y)| \ dm(x)dm(y) \\
&= \|f\|_1^{\frac{p}{q}} \int |g(y)|^p \|f\|_1 dm(y) \\
&= \|f\|_1^{1+\frac{p}{q}} \|g\|_p^p \\
&= \|f\|_1^p \|g\|_p^p
\end{aligned}$$

Thus taking powers with $\frac{1}{p}$, we get

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p$$

Since the p norm of $f * g$ is finite, $f * g$ is finite almost everywhere.

3. **Solution:** We need to show that

$$\lim_{n\to\infty} \int_0^n \int_0^\infty \frac{\sin(x)}{e^{xt}}\ dt\ dx = \frac{\pi}{2} \tag{1}$$

Since we know that for Riemann integrable functions, the Riemann integral coincides with the Lebesgue integral, we'll use Reimann integration techniques without sweat. Since $\frac{\sin(x)}{x}$ is a continuous function on $(0, \infty)$, $\frac{\sin(x)}{x} \in L^1(0, n)$ for all $n \in \mathbb{N}$. Hence we can interchange the order of integration in Equation 1. Thus, we'll show that

$$\lim_{n\to\infty} \int_0^\infty \int_0^n \frac{\sin(x)}{e^{xt}}\ dx\ dt = \frac{\pi}{2}$$

Using integration by parts, we get

$$\int_0^n \frac{\sin(x)}{x}\ dx = \left(1 - \frac{t\sin(n) - \cos(n)}{e^{nt}}\right) \frac{1}{1+t^2}$$

Since $1 + t < e^{nt}$ for all $n \in \mathbb{N}, t > 0$, we get

$$\left|1 - \frac{t\sin(n) - \cos(n)}{e^{nt}}\right| \leq 1 + \frac{1+t}{e^{nt}} < 2$$

Thus

$$\left|\int_0^n \frac{\sin(x)}{x}\ dx\right| \leq \frac{2}{1+t^2}$$

and since $\frac{2}{1+t} \in L^1(0, \infty)$, by Lebesgue dominated convergence theorem,

$$\begin{aligned}
\lim_{n\to\infty} \int_0^\infty \int_0^n \frac{\sin(x)}{e^{xt}}\ dx\ dt &= \int_0^\infty \lim_{n\to\infty} \int_0^n \frac{\sin(x)}{e^{xt}}\ dx\ dt \\
&= \int_0^\infty \lim_{n\to\infty} \left(1 - \frac{t\sin(n) - \cos(n)}{e^{nt}}\right) \frac{1}{1+t^2} \\
&= \int_0^\infty \frac{1}{1+t^2}\ dx \\
&= \arctan(\infty) - \arctan(0) \\
&= \frac{\pi}{2}
\end{aligned}$$