Practice Midterm Exam – Math 6320
October, 2024

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1 Problem

Consider an uncountable set X and the collection of singleton sets $\mathcal{F} = \{\{x\} : x \in X\}$. Show that the σ-algebra $\mathcal{M}_{\mathcal{F}}$ generated by $\mathcal{F}$ is identical to

$$\mathcal{M} = \{E \subset X : E \text{ is (at most) countable or } E^c \text{ is (at most) countable}\}.$$

Hint: For part of the proof, it may be useful to recall that $\mathcal{M}_{\mathcal{F}}$ is the smallest σ-algebra containing $\mathcal{F}$, so if $\mathcal{N}$ is a σ-algebra and $\mathcal{F} \subset \mathcal{N}$, then $\mathcal{M}_{\mathcal{F}} \subset \mathcal{N}$.

2 Problem

Let $(X, \mathcal{M}, \mu)$ be a measure space and let $0 < c < \infty$.

(a) Let $f \geq 0$ be a measurable non-negative real-valued function on X such that $\int_E f d\mu \leq c\mu(E)$ for each $E \in \mathcal{M}$. If $\mu(X) < \infty$, prove that $\mu(\{x \in X : f(x) > c\}) = 0$.

(b) Let $g \geq 0$ be a measurable non-negative real-valued function on X such that $\mu(\{x \in X : g(x) > c\}) = 0$. Prove that then $\int_E g d\mu \leq c\mu(E)$ for each $E \in \mathcal{M}$.

3 Problem

Let $(X, \mathcal{M}, \mu)$ be a measure space, let $(f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}$ be two sequences of non-negative real-valued measurable functions and assume $f_n(x) \leq g_n(x)$ for each $n \in \mathbb{N}, x \in X$. Assume as well the point-wise convergence $f_n(x) \to f(x)$ and $g_n(x) \to g(x)$, for each $x \in X$ with non-negative real-valued functions f and g. Assume that all $\int g_n d\mu$ and $\int g d\mu$ are finite, and that $\int g_n d\mu \to \int g d\mu$.

(a) Quote a famous result from class to establish that $\liminf_n \int f_n d\mu \geq \int f d\mu$.

(b) Prove $\limsup_n \int f_n d\mu \leq \int f d\mu$. Hint: consider $h_n = g_n - f_n$.

(c) What can you say about $\lim_n \int f_n \, d\mu$?

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