

Functional Analysis - MATH7320

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Chapter 1

Banach Spaces

Textbook : A Course in Functional Analysis, John Conway

Functional analysis is the study of Topological Vector Spaces.

Definition 1.0.1. Let X be a vector space (over \mathbb{R} or \mathbb{C}). A seminorm on X is a map $\|\cdot\| : X \rightarrow [0, \infty)$ such that

- $\|\alpha x\| = |\alpha| \|x\|, \forall \alpha \in \mathbb{F}, \forall x \in X$
- $\|x + y\| \leq \|x\| + \|y\|$

In addition if $\forall x \neq 0, \|x\| \neq 0$, we say $\|\cdot\|$ is a norm on X

Norm induces a metric $d(x, y) = \|x - y\|$

Note. Let X be a normed space. Then the maps

- $+: X \times X \rightarrow X : (x, y) \rightarrow x + y$
- $\cdot : \mathbb{F} \times X \rightarrow X : (\alpha, x) \rightarrow \alpha x$

are continuous.

Hence every normed space is a topological vector space.

Example 1.0.1. \mathbb{F}^n with ℓ_p -norm defined as

$$\|(\alpha_1, \alpha_2, \dots, \alpha_n)\|_p = \left(\sum_{i=1}^n |\alpha_i|^p \right)^{\frac{1}{p}}$$

Example 1.0.2. \mathbb{F}^n with ℓ_∞ -norm defined as

$$\|(\alpha_1, \alpha_2, \dots, \alpha_n)\|_\infty = \max\{|\alpha_i|\}$$

Example 1.0.3. Consider $C_{00} = \{(a_n)_{n \in \mathbb{N}} : a_n \in \mathbb{F}, \forall n \in \mathbb{N}, a_n = 0 \text{ except for finitely many } n \in \mathbb{N}\}$ which can be identified by collection of functions $f : \mathbb{N} \rightarrow \mathbb{F}$ with finite support.

Then

$$\|(a_n)\|_p = \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{\frac{1}{p}}$$

is a norm on C_{00}

Proposition 1.0.1. Let X, Y be normed space, and let $T : X \rightarrow Y$ be linear. Then the following are equivalent.

- T is continuous
- T is continuous on 0
- T is continuous on any point $x \in X$
- $\exists M > 0$ such that $\|T(x)\|_Y \leq M\|x\|_X$ for all $x \in X$

Proof. (1 \implies 2) It is clear that if T is continuous, then it is continuous at 0 from the definition of continuity.

(2 \implies 3) Let $x \in X$ and $\{x_n\}_{n \in \mathbb{N}}$ be any sequence in X that converge to x . Then the sequence $\{y_n = x_n - x\}$ converge to zero by the algebra of limits. By the continuity of T at zero, $\{T(y_n) = T(x_n) - T(x)\}$ converge to 0. Therefore $T(x_n) \rightarrow T(x)$. And this shows T is sequentially continuous at $x \in X$. Since the space is a metric space, sequential continuity is equivalent to continuity.

(4 \implies 2) Let $x \in X$. Then $\|T(0) - T(x)\| = \|T(x)\| \leq M\|x\| = M\|0 - x\|$. Hence T is continuous at 0.

(3 \implies 1)

(2 \implies 4)

□

Example 1.0.4. Let $T : \mathbb{F}^n \rightarrow \mathbb{F}^n$ be defined as $T(\alpha_1, \alpha_2, \dots, \alpha_n) = (\alpha_1, 0, \dots, 0)$. Is T convergent for any norm $\|\cdot\|_1, \|\cdot\|_2$ in the domain and range?

Proof. verify

□

Example 1.0.5. Consider identity function $I : C_{00} \rightarrow C_{00}$. Let the norm in domain be $\|\cdot\|_{\infty}$ and that in range be $\|\cdot\|_1$. Is the function continuous? What if the norms in domain and range are switched?

Proof. verify

□

Note. Let X be a space with two norms $\|\cdot\|_1, \|\cdot\|_2$. When are the two norms topologically equivalent?

When $\exists M, M'$ such that $\|x\|_1 \leq M\|x\|_2$ and $\|x\|_2 \leq M'\|x\|_1$. Equivalently, when the identity map between the two spaces with their respective norms are bi-continuous. (See 4th equivalent statement of previous proposition)

Theorem 1.0.1. *Let X and Y be normed spaces, and $T : X \rightarrow Y$ be linear. Assume X is finite dimensional. Then T is continuous.*

Proof. Since $T(X) \leq Y$ is finite dimensional, we may assume without loss of generality that Y is also finite dimensional and T is onto. Let $\{x_1, x_2, \dots, x_n\}$ be a basis for X . Define another norm on X as follows. For every $x = \sum_{i=1}^n \alpha_i x_i \in X$,

$$\|x\|' = \sum_{i=1}^n |\alpha_i| (\|T(x_i)\| + \|x_i\|)$$

verify that this is a norm. Then for every $x \in X$, we have

$$\|T(x)\| \leq \sum_{i=1}^n |\alpha_i| \|T(x_i)\| \leq \|x\|'$$

Hence T is bound with respect to the norm $\|\cdot\|'$ on X , since all norms are equivalent on X . Therefore T is continuous w.r.t to the original norm on X . \square

Corollary 1.0.1.1. *Let X be a finite dimensional vector space. Then any two norms in X are equivalent.*

Proof. Let $\{e_1, e_2, \dots, e_n\}$ be a basis for X . For each $x = \sum_{i=1}^n \alpha_i e_i \in X$, define

$$\|x\|_\infty = \max\{|\alpha_i|\}$$

Then $\|\cdot\|_\infty$ is a norm and we'll show every norm on X is equivalent to this norm. Let $\|\cdot\|$ be an arbitrary norm on X . For each $x = \sum_{i=1}^n \alpha_i e_i \in X$, we have

$$\begin{aligned} \|x\| &= \left\| \sum_{i=1}^n \alpha_i e_i \right\| \\ &\leq \sum_{i=1}^n |\alpha_i| \|e_i\| \\ &\leq \max\{|\alpha_i|\} \sum_{i=1}^n \|e_i\| \\ &\leq \|x\|_\infty \sum_{i=1}^n \|e_i\| \end{aligned}$$

Therefore the identity map $I : (X, \|\cdot\|_\infty) \rightarrow (X, \|\cdot\|)$ is continuous. Since the set $K = \{x \in X : \|x\|_\infty \leq 1\}$ is compact, K is also compact in $(X, \|\cdot\|)$. Moreover if $V \subset K$ is closed in $\|\cdot\|_\infty$, then V will also be closed in $\|\cdot\|$ being compact. Hence $I|_K$ will be an open map. Now its an easy application of **Proposition 1.0.1**. \square

Corollary 1.0.1.2. • *Every finite dimensional normed space is complete*

- *If X is a normed space and Z is a finite dimensional subspace of X , then Z is the closed in X*

Proof. • Let $(X, \|\cdot\|)$ be a finite dimensional normed space. Let $\|\cdot\|_2$ be the euclidean norm on X . Then by the theorem above there exists $M \in \mathbb{R}$ such that $\frac{1}{M}\|x\|_2 \leq \|x\| \leq M\|x\|_2$ for all $x \in X$.

Let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $(X, \|\cdot\|)$. Then $(x_n)_{n \in \mathbb{N}}$ is also Cauchy in $(X, \|\cdot\|_2)$. Since the latter space is complete, so is $(X, \|\cdot\|)$.

- **verify**

\square

Note. If $T, S : X \rightarrow Y$ are continuous linear maps between normed space, then $T + S$ is also continuous. Also, $\forall \alpha \in \mathbb{F}$, αT is continuous.

Thus the space $B(X, Y)$ of all continuous linear maps from X to Y is a subspace of all linear maps between X and Y .

Definition 1.0.2. Let X, Y be normed spaces. and $T \in B(X, Y)$. We define the operator norm of T as

$$\|T\| = \sup \left\{ \frac{\|T(x)\|}{\|x\|} : x \in X, x \neq 0 \right\}$$

Lemma 1.0.1. *Let $T \in B(X, Y)$. Then the following are equivalent*

- $\|T\|$
- $\sup\{\|T(x)\| : x \in X, \|x\| \leq 1\}$
- $\sup\{\|T(x)\| : x \in X, \|x\| < 1\}$
- $\inf\{M \geq 0 : \|T(x)\| \leq M\|x\|, \forall x \in X\}$

Proof. **verify**

\square

Proposition 1.0.2. *The operator norm is a norm in $B(X, Y)$*

Proof. **verify**

\square

1.1 Hahn Banach Theorem

Lemma 1.1.1. *Let X be a complex normed space. Let $f : X \rightarrow \mathbb{R}$ be an \mathbb{R} -linear functional. Then $g : X \rightarrow \mathbb{C}$ defined as $g(x) = f(x) - if(ix)$ is \mathbb{C} -linear*

Conversely if $g : X \rightarrow \mathbb{C}$ is a \mathbb{C} -linear map, then $f := \Re \circ g : X \rightarrow \mathbb{R}$ is \mathbb{R} -linear.

Moreover $\|f\| = \|g\|$.

Proof. We'll prove that $\|f\| = \|g\|$ and leave the rest for the reader (verify).

Since $|f(x)| \leq |g(x)|$, for all $x \in X$, it is easy to see that $\|f\| \leq \|g\|$. Conversely, $\forall \epsilon > 0, \exists x_o \in X$ with $\|x_o\| = 1$ such that $|g(x_o)| > \|g\| - \epsilon$. If $g(x_o) = re^{i\theta}$, take $\alpha = e^{-i\theta}$. Then $f(\alpha x_o) = \Re(re^{-i\theta}e^{i\theta}) = r = g(\alpha x_o)$. Then $\|f\| \geq |f(\alpha x_o)| = |g(\alpha x_o)| = |\alpha| |g(x_o)| = |g(x_o)| > \|g\| - \epsilon$. Since ϵ is arbitrary, this gives $\|f\| \geq \|g\|$ \square

Theorem 1.1.1 (Hahn-Banach Extension Theorem). *Let X be a normed space over \mathbb{R} , Z be a subspace of X and let $\phi : Z \rightarrow \mathbb{R}$ be a continuous linear functional. Then there exists a linear functional $\psi : X \rightarrow \mathbb{R}$ such that $\psi|_Z = \phi$ and $\|\phi\| = \|\psi\|$.*

Proof. Assume $\|\phi\| = 1$ (If this is not the case, we can always scale the functional down to norm 1). Now we'll extend ϕ from Z to a subspace with one dimension higher than Z , preserving the norm. Let $x_o \in (X \setminus Z)$ and $Y = \text{Span}\{\{x_o\} \cup Z\}$ be the set one dimension higher than Z . Assume ψ is the extension of ϕ to Y . Then ψ will be completely characterized, if we know the value of $\psi(x_o)$. We look to see what real values we can assign $\psi(x_o)$ satisfying our conditions. Let $y = z + x_o \in Y$ where $z \in Z$ (We must be taking an arbitrary element $y = z + \alpha x_o \in Y$, but if we know the image of $y = z + x_o$ for all $z \in Z$ under ψ , then we can get the image of $y = z + \alpha x_o$ for any $\alpha \in \mathbb{R}$ by scaling). Norm preserveness demands that for all $z \in Z$, we must have

$$-\|z + x_o\| \leq \psi(y) = \psi(z) + \psi(x_o) \leq \|z + x_o\|$$

Since ψ agrees with ϕ on Z , this is equivalent to

$$-\phi(z) - \|z + x_o\| \leq \psi(x_o) \leq \|z + x_o\| - \phi(z) \quad (1.1)$$

Moreover since we normalized ϕ to have norm 1, we know ψ must also have norm 1. Then by triangle inequality, we get that for all $a, b \in Y$

$$\psi(a) - \psi(b) = \psi(a - b) \leq \|a - b\| = \|(a + x_o) - (b + x_o)\| \leq \|a + x_o\| + \|b + x_o\|$$

which gives

$$-\psi(b) - \|b + x_o\| \leq \|a + x_o\| - \psi(a)$$

Since this inequality is true for all $a, b \in Y$, taking supremum and infimum over all the possible $a, b \in Y$ preserves the inequality. Hence we get

$$\sup_{b \in Y} \left\{ -\psi(b) - \|b + x_o\| \right\} \leq \inf_{a \in Y} \left\{ \|a + x_o\| - \psi(a) \right\} \quad (1.2)$$

Substituting $a = b = z$ in [Equation 1.2](#) guarantees the existence of $\psi(x_o)$ satisfying [Equation 1.1](#). Hence we get an extension (namely ψ) of ϕ to Y preserving the norm. Since Z was an arbitrary subspace of X , this is true for all such subspaces of X .

Now we will employ Zorn's lemma to get an extension of ϕ from Z to the whole of X . Consider the collection of all linear extensions of ϕ , i.e

$$\mathcal{S} = \left\{ (\psi_Y, Y) : Z \subset Y, \psi_Y|_Z = \phi, \|\psi_Y\| = \|\phi\| \right\}$$

Then we define a partial order in the collection \mathcal{S} as $(\psi_X, X) \leq (\psi_Y, Y)$ if and only if $X \subset Y$ and $\psi_Y|_X = \psi_X$. Now let \mathcal{C} be a chain in \mathcal{S} . Consider the set

$$\tilde{Y}_{\mathcal{C}} = \bigcup_{(\psi_Y, Y) \in \mathcal{C}} Y$$

and the map $\psi_{\tilde{Y}_{\mathcal{C}}} : \tilde{Y}_{\mathcal{C}} \rightarrow \mathbb{R}$ defined as

$$\psi_{\tilde{Y}_{\mathcal{C}}}(x) = \psi_Y(x), \text{ where } x \in Y, \text{ for } (\psi_Y, Y) \in \mathcal{C}$$

To see this map is well defined, assume $x \in X$ and $x \in Y$ for $(\psi_X, X), (\psi_Y, Y) \in \mathcal{C}$. Then either $(\psi_X, X) \leq (\psi_Y, Y)$ or $(\psi_Y, Y) \leq (\psi_X, X)$ since \mathcal{C} is totally ordered. WLOG assume $(\psi_X, X) \leq (\psi_Y, Y)$, then by definition we get that $\psi_Y|_X = \psi_X$. This gives that $\psi_Y(x) = \psi_X(x)$. Hence we get that $\psi_{\tilde{Y}_{\mathcal{C}}}$ is well defined. In a similar fashion we can verify that $\psi_{\tilde{Y}_{\mathcal{C}}}$ is a linear functional.

Now we claim that $(\tilde{Y}_{\mathcal{C}}, \psi_{\tilde{Y}_{\mathcal{C}}})$ is the upper bound of the chain \mathcal{C} . By the definition of \tilde{Y} , we see that there cannot be an element (ψ_Y, Y) in the chain \mathcal{C} , with $\tilde{Y} \subset Y$. Hence the only remaining thing to show is that for all $(\psi_X, X) \in \mathcal{C}$, we have $\psi_{\tilde{Y}_{\mathcal{C}}}|_X = \psi_X$. But this also follows from the definition of the map $\psi_{\tilde{Y}_{\mathcal{C}}}$.

Since \mathcal{C} was taken to be an arbitrary chain in the collection \mathcal{S} , we get that every chain in \mathcal{S} has an upper bound. Then by Zorn's lemma, the collection \mathcal{S} has a maximal element (ψ, Y) . We claim that in this maximal element, $Y = X$. If not, we can extend ψ to a space one dimension above Y like we did in the beginning contradicting the maximality of (ψ, Y) . Hence the maximal element is (ψ, X) . This by definition of the collection \mathcal{S} , is the required extension for (ϕ, Z) . \square

Remark 1.1.1. Note that in the proof above, we only used the scaling property and triangle inequality of the norm, hence we can relax the condition for norm and replace it with a seminorm, without messing up the proof.

Theorem 1.1.2 (Hahn-Banach Extension Theorem for \mathbb{C}). *Same statement of Theorem 1.1.1 with only the field changed to \mathbb{C} .*

Proof. Consider X as a normed linear space over \mathbb{R} . Let $f = \Re \circ \phi : Z \rightarrow \mathbb{R}$ and apply Theorem 1.1.1 on f to get a real linear functional $\tilde{f} : X \rightarrow \mathbb{R}$ with the required properties. Now we claim that $\tilde{\phi}$ defined as $\tilde{\phi}(x) = \tilde{f}(x) - i\tilde{f}(ix)$ is the required extension.

First we show $\tilde{\phi}|_Z = \phi$. To see this first we notice that if ϕ can be written as $\phi(x) = f(x) + ig(x)$ where f, g are real valued functionals, then since $-\phi(x) = i\phi(ix) = if(ix) - g(ix)$. Hence $0 = \phi(x) - \phi(x) = (f(x) - g(ix)) + i(g(x) + f(ix))$. Since real part and imaginary part must be equal to 0, we get that $g(x) = -f(ix)$. Therefore we get $\phi(x) = f(x) - if(ix)$. Now we get $\tilde{\phi}|_Z = \phi$ immediately since $\tilde{f}|_Z = f$. To finish the proof, we also have to show that $\|\phi\| = \|\tilde{\phi}\|$. But this follows easily from Lemma 1.1.1 as $\|\phi\| = \|f\| = \|\tilde{f}\| = \|\tilde{\phi}\|$. \square

Remark 1.1.2. It is quite natural to be confused about the well defineness of the expression $f(ix)$ when we are considering X as a normed linear space over \mathbb{R} in the beginning of the proof. But note that since X initially was a complex normed linear space, viewing it as a space over \mathbb{R} doesn't change or remove any elements from the space. Hence $ix \in X$ even though X is viewed as a real normed linear space.

Definition 1.1.1. A sublinear map is a function $\rho : X \rightarrow \mathbb{R}$ with the properties

- $\rho(rx) = r\rho(x), \forall r \in \mathbb{R}$
- $\rho(x + y) \leq \rho(x) + \rho(y)$

Definition 1.1.2. Let X be a normed space. Then the dual of X , denoted by X^* , is the space $B(X, \mathbb{F})$

Lemma 1.1.2. *Let X be a normed space and $x \in X$. Then $\exists f \in X^*$ such that*

$$\|f\| = 1 \text{ and } f(x) = \|x\|$$

Proof. Let $Z = \text{Span}\{x\}$. Define $g : Z \rightarrow \mathbb{F}$ as $g(\alpha x) = \alpha\|x\|$. Then $\|g\| = 1$. By the Hahn Banach theorem, g has an extension f which preserve the norm and extends g to X . \square

Corollary 1.1.2.1. *Let X be a normed space and $x \in X$, then we have*

$$\|x\| = \sup\{|f(x)| : f \in X^*, \|f\| \leq 1\}$$

Proof. If f is any linear functional with $\|f\| \leq 1$, then $|f(x)| \leq \|f\|\|x\| = \|x\|$. Hence $\|x\| \leq \sup\{|f(x)| : f \in X^*, \|f\| \leq 1\}$. Now let f_x be the functional we get from [Lemma 1.1.2](#). Then $f_x \in X^*$ and $\|f_x\| = 1$ with $f_x(x) = |f(x)| = \|x\|$. Hence we get that the inequality is actually an equality, and this proves the corollary. \square

Definition 1.1.3. For every $x \in X$, define a linear map $\hat{x} : X^* \rightarrow \mathbb{F}$ by $\hat{x}(f) = f(x)$

Theorem 1.1.3. For every $x \in X$, $\hat{x} \in (X^*)^*$. The map $\rho : x \rightarrow \hat{x}$ is an isometric linear map.

Proof. The fact that \hat{x} is linear and bounded and the map $X \ni x \rightarrow \hat{x} \in X^{**}$ is linear follows from the definition of $f + g$ and λf .

By definition and [Corollary 1.1.2.1](#)

$$\begin{aligned}\|\hat{x}\| &= \sup\{|\hat{x}(f)| : f \in X^*, \|f\| \leq 1\} \\ &= \sup\{|f(x)| : f \in X^*, \|f\| \leq 1\} \\ &= \|x\|\end{aligned}$$

\square

Definition 1.1.4. A normed space X is said to be reflexive if the map $\rho : X \rightarrow X^{**} : x \rightarrow \hat{x}$ is surjective. (This is a stronger condition than $X \equiv X^{**}$)

Theorem 1.1.4. There are isometric isomorphisms between

- $(\mathbf{c}_0)^*$ and ℓ^1
-
- $(\ell^1)^*$ and ℓ^∞

Proof. • Let $(x_n) \in \ell^1$. Then consider the map $\phi_{(x_n)} : \mathbf{c}_0 \rightarrow \mathbb{F}$ defined as

$$\phi_{(x_n)} : (y_n) \rightarrow \sum_{n \in \mathbb{N}} x_n y_n$$

We claim that $\phi_{(x_n)}$ is a continuous linear functional. But first we should see that the sum is well defined. Since $y_n \rightarrow 0$, there is an $N \in \mathbb{N}$ such that $|y_n| < 1$ for all $n \geq N$. Since

$$\left| \sum_{i=N}^{\infty} x_i y_i \right| \leq \sum_{i=N}^{\infty} |x_i| |y_i| \leq \|(x_n)\|_1$$

we see that the sum is well defined and the map makes sense. Also since $(y_n) + (z_n) = (y_n + z_n) \in \mathbf{c}_0$ whenever $(y_n), (z_n) \in \mathbf{c}_0$, we get that

$$\sum_{n \in \mathbb{N}} x_n(y_n + z_n) = \sum_{n \in \mathbb{N}} x_n y_n + \sum_{n \in \mathbb{N}} x_n z_n$$

which shows the linearity of the map $\phi_{(x_n)}$.

Now we show that $\|\phi_{(x_n)}\| = \|(x_n)\|_1$. We immediately see that for $(y_n) \in c_0$ with $\|(y_n)\|_{\sup} = \sup_{n \in \mathbb{N}} y_n = 1$,

$$|\phi_{(x_n)}((y_n))| = \left| \sum_{n \in \mathbb{N}} x_n y_n \right| \leq \|(y_n)\|_{\sup} \left(\sum_{n \in \mathbb{N}} |x_n| \right) \leq \|(x_n)\|_1$$

which gives $\|\phi_{(x_n)}\| \leq \|(x_n)\|_1$. Now let $\theta_j \in [0, 2\pi)$ such that $|x_j| = e^{i\theta_j} x_j$. Now consider the sequence $s_m \in \mathbf{c}_0$ defined as $s_m = \sum_{j=1}^m e^{i\theta_j} e_j$, where e_j is the sequence with j th entry 1 and the rest of the entries 0. Since $(x_n) \in \ell_1$, for all $\epsilon \geq 0$ there exists an $N_\epsilon \in \mathbb{N}$ such that

$$\sum_{i=N_\epsilon+1}^{\infty} |x_i| < \epsilon$$

Then since

$$|\phi_{(x_n)}(s_{N_\epsilon})| = \left| \sum_{n=1}^{N_\epsilon} e^{i\theta_j} x_n \right| = \sum_{i=1}^{N_\epsilon} |x_n| = \|(x_n)\| - \sum_{i=N_\epsilon+1}^{\infty} |x_n| \geq \|(x_n)\| - \epsilon$$

and $\epsilon > 0$ was arbitrary, we get that $\|\phi_{(x_n)}\| = \|(x_n)\|$

Hence we see that the map $(x_n) \rightarrow \phi_{(x_n)}$ is an isometric linear map. Now for surjectivity, let $\phi \in \mathbf{c}_0^*$. We claim that the sequence $(y_n) = (\phi(e_n)) \in \ell^1$ and $\phi = \phi_{(y_n)}$. Let $\theta_j \in [0, 2\pi)$ such that $e^{i\theta_j} y_j = |y_j|$. Then for any $N \in \mathbb{N}$, we have

$$\begin{aligned} \sum_{j=1}^N |\phi(e_j)| &= \sum_{j=1}^N e^{i\theta_j} \phi(e_j) \\ &= \phi \left(\sum_{j=1}^N e^{i\theta_j} e_j \right) \\ &\leq \|\phi\| \left\| \sum_{j=1}^N e^{i\theta_j} e_j \right\| \\ &= \|\phi\| \end{aligned}$$

Since this is true for all $N \in \mathbb{N}$, taking the limits as $N \rightarrow \infty$, the inequality is preserved and we get that $(y_n) \in \ell^1$. Moreover $\phi = \phi_{(y_n)}$ follows from the definition of $\phi_{(x_n)}$. Hence we get that $\mathbf{c}_0^* \cong^{\text{iso}} \ell^1$.

•

- The proof of this will be extremely similar to what we attempted before when we proved $\mathbf{c}_0^* \cong^{\text{iso}} \ell^1$. Let $(x_n) \in \ell^\infty$. Then consider the map $\phi_{(x_n)} : \mathbf{c}_0 \rightarrow \mathbb{C}$ defined as

$$\phi_{(x_n)} : (y_n) \rightarrow \sum_{n \in \mathbb{N}} x_n y_n$$

By a similar way as we did in the above equivalence we see that $\phi_{(x_n)}$ is linear. Moreover since

$$\left| \sum_{n \in \mathbb{N}} x_n y_n \right| \leq \|(x_n)\|_\infty \left| \sum_{n \in \mathbb{N}} y_n \right| = \|(x_n)\|_\infty \|(y_n)\|_1$$

we see that $\|\phi_{(x_n)}\| \leq \|(x_n)\|_\infty$. To get the reverse inequality, Let $\|(x_n)\|_\infty = M$, then for any $\epsilon > 0$, there exist some x_k in the sequence (x_n) such that $|x_k - M| < \epsilon$. Now consider the sequence $e_k \in \ell^1$ with k th entry 1 and all the rest of them 0. We get that

$$|\phi_{(x_n)}(e_k)| = |x_k| \geq \|(x_n)\|_\infty - \epsilon$$

Since ϵ was arbitrary, we get that $\|\phi_{(x_n)}\| = \|(x_n)\|_\infty$. Hence the map $(x_n) \rightarrow \phi_{(x_n)}$ is an isometry. To show that it is indeed a bijection, assume $\phi \in (\ell^1)^*$, then consider the sequence $y_n = \phi(e_n)$. Since ϕ is continuous, it is bounded above by $\|\phi\|$ and we get that $y_n \leq \|\phi\|$. Therefore $(y_n) \in \ell^\infty$. Moreover we can verify like above that $\phi = \phi_{(y_n)}$ from the definition of $\phi_{(y_n)}$. Hence we get $(\ell^1)^* \cong^{\text{iso}} \ell^\infty$. □

Theorem 1.1.5. Let $1 < p < \infty$, and $q \in \mathbb{R}$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then $(\ell^p)^* \cong \ell^q$

Proof. Let $(a_n) \in \ell^p, (b_n) \in \ell^q$, then $\sum_{n \in \mathbb{N}} a_n \bar{b}_n$ is the map to check for isometric isomorphism. Use Holder's inequality as needed. verify □

1.2 Banach Limits

Theorem 1.2.1. There exists $\phi \in (\ell^\infty)^*$ satisfying the following

$$1 \ \forall (a_n) \in \ell^\infty \text{ with } a_n \geq 0 \text{ for all } n \in \mathbb{N}, \phi((a_n)) \geq 0$$

2 If (a_n) is convergent, then $\phi((a_n)) = \lim_{n \rightarrow \infty} a_n$

3 If $(a_n) \in \ell^\infty$ and $b_n = a_{n+1}$, then $\phi((b_n)) = \phi((a_n))$

Moreover such ϕ is called a Banach limit.

Proof. Let $S : \ell^\infty(\mathbb{R}) \rightarrow \ell^\infty(\mathbb{R})$ and $T = I - S$ where I is the identity map. Also let $V = \text{Range}(T) + c$ where $c \in \mathbf{c}$, the set of convergent sequences.

Define $\phi : V \rightarrow \mathbb{R}$, $\phi(a_n - a_{n+1} + x_n) = \lim_{n \rightarrow \infty} x_n$.

- Claim 1: ϕ is well defined

- Claim 2: $\|\phi\| = 1$

Assuming the claims, by Hahn Banach, ϕ extends to $\tilde{\phi} \in \ell^\infty(\mathbb{R})$ with $\|\tilde{\phi}\| = 1$. Then by the last lemma we get $\tilde{\phi}((y_n)) \geq 0$ for all $(y_n) \in \ell^\infty(\mathbb{R})$ with $y_n \geq 0$ \square

Proof of Claim 1. Suppose that $(a_n) \in \ell^\infty$ is a sequence such that $a_n - a_{n+1}$ converges, say $a_n \rightarrow a_{n+1} \rightarrow \alpha$. If $\alpha > 0$, then $\exists N \in \mathbb{N}$ such that for all $n > N$, $a_n - a_{n+1} > \frac{\alpha}{2}$. So $a_N > \frac{\alpha}{2} + a_{N+1} > \dots > k\frac{\alpha}{2} + a_{N+k}$. So for all $k \in \mathbb{N}$, $a_N - a_{N+k} = k\frac{\alpha}{2} \rightarrow \infty$ contradicting our assumption that $a_n - a_{n+1}$ converges.

Now assume that $(a_n), (b_n) \in \ell^\infty(\mathbb{R})$ with $(x_n), (y_n) \in \mathbf{c}$ such that $a_n - a_{n+1} + x_n = b_n - b_{n+1} + y_n$. Then $(a_n - b_n) - (a_{n+1} - b_{n+1}) = y_n - x_n$. Then since RHS is a convergent limit, LHS must be convergent, which we get from above that it must converge to zero. Then $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n$ \square

Proof Claim 2. To complete the proof, define $\Psi : \ell^\infty \rightarrow \mathbb{C}$ by $\Psi((a_n + ib_n)) = \tilde{\phi}(a_n) + i\tilde{\phi}(b_n)$ **verify** \square

Corollary 1.2.1.1. ℓ^1 is not reflexive

Proof. Let $\phi \in (\ell^\infty)^*$ be a Banach limit. FTOC, assume $\exists f = (\alpha_n) \in \ell^1$ such that

$$\phi((a_n)) = \sum_{i=1}^{\infty} a_n \overline{\alpha_n}$$

Then for all $m \in \mathbb{N}$, $\overline{\alpha_m} = \phi(\delta_m) = 0$, where $\delta_m = (0, 0, \dots, 1, 0, 0, \dots)$. But this contradicts since we assumed $\phi \neq 0$ by the Hahn Banach extension from c_0 \square

Lemma 1.2.1. Let $\psi \in (\ell^\infty)^*$. then the following are equivalent.

1 $\|\psi\| = \psi((1, 1, 1, \dots))$

2 If $(a_n) \in \ell^\infty$ with $a_n \geq 0, \forall n \in \mathbb{N}$. Then $\psi((a_n)) \geq 0$

Proof. (1 \implies 2) FTSOC assume $\exists(a_n) \in \ell^\infty$, $\psi((a_n)) < 0$. WLOG, assume $|a_n| \leq 1$ for all $n \in \mathbb{N}$. let $b_n = 1 - a_n$. Then $0 \leq b_n \leq 1$ and

$$\psi((b_n)) > \psi((1, 1, 1, \dots)) - \psi((a_n)) > \psi((1, 1, 1, \dots))$$

So

$$\|\psi\| \geq |\psi((b_n))| \geq \psi((1, 1, \dots))$$

(2 \implies 1) Let $(a_n) \in \ell^\infty$ with $|a_n| \leq 1$, then $0 \leq 1 - a_n$. So $\psi((1 - a_n)) \geq 0$ and therefore $\psi((1, 1, 1, \dots)) \geq \psi((a_n))$. Similarly $\psi((-a_n)) \leq \psi((1, 1, 1, \dots))$ which gives $|\psi((a_n))| \leq \psi((1, 1, 1, \dots))$ \square

1.3 Quotient Spaces

Definition 1.3.1. Let X be a normed space and $Y \leq X$ be a closed subspace. For every $x \in X$, define

$$\|x + Y\| = \inf\{\|x + y\| : y \in Y\}$$

Lemma 1.3.1. This defines a norm on $\frac{X}{Y}$. If X is complete, then $\frac{X}{Y}$ is complete.

Proof. Obviously, $\|x + Y\| \geq 0$ for all $x \in X$, and $\|x + z + Y\| \leq \|x + Y\| + \|y + Y\|$. Similarly, we can also show homogeneity.

Now assume $x \in X$ is such that $\|x + Y\| = 0$. Then there is a sequence $(y_n) \in Y$ such that $\|x - y_n\| \rightarrow 0$, that is $y_n \rightarrow x$. Since Y is closed, we get $x \in Y$.

To show the second part of the lemma, consider the sequence $(x_n + Y) \in X/Y$ such that $\sum_{n \in \mathbb{N}} \|x_n - Y\| < \infty$. For each $n \in \mathbb{N}$, choose $y_n \in Y$ such that

$$\|x_n + y_n\| \leq \|x_n + Y\| + \frac{1}{2^n}$$

Then $\sum_{n \in \mathbb{N}} \|x_n + y_n\| \leq \infty$. Since X is complete, the sequence $\sum_{n \in \mathbb{N}} x_n + y_n$ converges to say $z \in X$. Then

$$\begin{aligned} \|(z + Y) - \sum_{n=k}^n (x_n + Y)\| &= \left\| \left(z - \sum_{n=k}^n x_n \right) + Y \right\| \\ &= \left\| \left(z - \sum_{i=1}^k (x_n + y_n) \right) + Y \right\| \\ &\leq \left\| \left(z - \sum_{i=1}^k (x_n + y_n) \right) \right\| \end{aligned}$$

which converges to 0 as $k \rightarrow \infty$ \square

This is also
a direct
consequence of
open
mapping
theorem

Lemma 1.3.2. *The canonical map, $q : X \rightarrow \frac{X}{Y}$ is a continuous open map. A subset $E \subset X/Y$ is open iff $q^{-1}(E) \subset X$ is open.*

Proof. Since $\|x + Y\| \leq \|x\|$, for all $x \in X$, we see that the map q is a contraction. Thus for all open $E \subset X/Y$, we get $q^{-1}(E)$ is open.

Conversely, assume that $A \subset X$ is open. Let $x \in A$ and $r > 0$ such that $B_r(x) \subset A$. Let $z \in X$ such that $\|q(x) - q(z)\| < r$. So, $\|(x - z) + Y\| < r$. Then $\exists y \in Y$ such that $\|x - z - y\| < r$. So $z + y \in B_r(x)$, $q(z + y) = q(z) \in q(B_r(x))$. So $B_r(q(x)) \subset q(B_r(x)) \subset q(A)$. Thus $q(A)$ is open. \square

1.4 Open Mapping Theorem and Closed Graph Theorem

Theorem 1.4.1 (Open Mapping Theorem). *Let X and Y be Banach spaces and $T : X \rightarrow Y$ be a surjective bounded linear map. Then T is an open map i.e $T(E)$ is open in Y i.e if $E \subset X$, then $T(E)$ is open in Y .*

Steps of proof. See Prof. Blecher's Notes on Functional Analysis

Use Baire category theorem to show that $\overline{T(B_r(0))}$ has a non-empty interior.

- Then use linearity of T to show that $0 \in \overline{T(B_{2r}(0))}$.

\square

Proof. Since Y is complete, by the Baire category theorem it is of the second category. Let $r \geq 0$, then

$$\begin{aligned} Y &= T(X) \\ &= T\left(\bigcup_{n=1}^{\infty} B_{nr}(0)\right) \\ &= T\left(\bigcup_{n=1}^{\infty} nB_r(0)\right) \\ &= \bigcup_{n=1}^{\infty} n\overline{T(B_r(0))} \end{aligned}$$

Then by BCT, there exist some $n \in \mathbb{N}$ such that $\text{int}(\overline{T(B_r(0))}) \neq \emptyset$. Let $y_0 \in \text{int}(\overline{T(B_r(0))})$. So there exists $\epsilon > 0$, such that $B_\epsilon(y_0) \subset \overline{T(B_r(0))}$. Let $w \in B_Y(0, \epsilon)$. Then $y_0 + w \in B_Y(y_0, \epsilon)$, and $\exists(x_n) \subset B_X(0, r)$ such that $T(x_n) \rightarrow$

$y_0 + w$. Also $\exists(z_n) \in B_X(0, r)$ such that $T(z_n) \rightarrow y_0$. Then $T(x_n - z_n) \rightarrow w$, so $w \in \overline{T(B_X(0, 2r))}$. Since w was an arbitrary element in $B_Y(0, \epsilon)$ we see that $B_Y(0, \epsilon) \subset \overline{T(B_X(0, 2r))}$. So $0 \in \text{int}(\overline{T(B_X(0, s))})$ for $s > 0$.

Now fix $t > 0$ and let $y_0 \in \overline{T(B_X(0, t))}$. By the above there exists $\epsilon > 0$ such that $B_Y(0, \epsilon) \subset \overline{T(B_X(0, \frac{t}{2}))}$. Then $(y_0 + B_Y(0, \epsilon)) \cap \overline{T(B_X(0, \frac{t}{2}))} \neq \emptyset$. So $\exists x \in B_t(0)$ such that $T(x_1) = y_0 - y_1$ where $y_1 \in B_Y(0, \epsilon) \subset \overline{T(B_X(0, \frac{t}{2}))}$.

Similarly $\exists y_2 \in \overline{T(B_{\frac{t}{4}}(0))}$ and $x_2 \in B_{\frac{t}{2}}(0)$ such that $T(x_2) = y_1 - y_2$. Thus inductively we can choose $y_{n+1} \in \overline{T(B_{\frac{t}{2^{n+1}}}(0))}$ and $x_{n+1} \in B_{\frac{t}{2^n}}(0)$ such that $T(x_{n+1}) = y_n - y_{n+1}$. Now since we constructed nicely, $\sum_{i=0}^{\infty} x_n$ converge. **verify**. Moreover for all $N \in \mathbb{N}$, we have

$$\sum_{n=1}^N T(x_n) = y_0 - y_N$$

Also notice that $y_n \rightarrow 0$. Hence $y_0 = \lim_{N \rightarrow \infty} (y_0 - y_N) = \lim_{N \rightarrow \infty} \sum_{n=1}^N T(x_n) = \lim_{N \rightarrow \infty} T(\sum_{n=1}^N x_n) = T(x) \in \overline{T(B_{2t}(0))}$. So

$$\overline{T(B_t(0))} \subset \overline{T(B_{2t}(0))}$$

Now to complete the proof, let E be an open subset of X . Let $x_0 \in E$ be such that $y_0 = T(x_0)$. Let $\epsilon > 0$ be such that $x_0 + B_\epsilon(0) = B_\epsilon(x_0) \subset E$. So $y_0 + T(B_\epsilon(0)) = T(B_\epsilon(x_0)) \subset T(E)$. By the above $\exists \delta > 0$ such that $B_\delta(0) \subset \overline{T(B_\epsilon(0))}$ \square

Find examples where this fails if we slack the conditions

Corollary 1.4.1.1. *Let X and Y be Banach spaces and $T : X \rightarrow Y$ be a bijective bounded linear map. Then $T^{-1} : Y \rightarrow X$ is bounded.*

Proof. By open mapping theorem, T is a bi-continuous map. Now use **Proposition 1.0.1** \square

Theorem 1.4.2 (Closed Graph Theorem). *Let X and Y be Banach spaces and $T : X \rightarrow Y$ be a linear map. Then T is bounded if and only if graph of T , defined as $g(T) = \{(x, T(x)) : x \in X\}$ is closed in the product topology of $X \times Y$.*

Proof. Define the norm $\|(x, y)\| = \|x\|_X + \|y\|_Y$ on $X \times Y$. Notice that $(x_n, y_n) \rightarrow (x, y)$ if and only if $x_n \rightarrow x$, and $y_n \rightarrow y$. Hence $(X \times Y, \|\cdot\|)$ is a Banach space.

Assume T is continuous. Then if $(x_n, T(x_n))$ is Cauchy in $g(T)$ then x_n and $T(x_n)$ must be cauchy in X and Y respectively. By the completeness of the spaces X and Y , we get $x_n \rightarrow x \in X$ and $T(x_n) \rightarrow y \in Y$. Moreover by continuity of

T , we get $T(x_n) \rightarrow T(x)$. Since the Banach space is Hausdorff, we get $y = x$ and that $(x_n, T(x_n)) \rightarrow (x, T(x)) \in g(T)$ making it closed.

Conversely, define $S : X \rightarrow g(T)$ as $S(x) = (x, T(x))$. S is linear and bijective. Assume $g(T)$ is closed, hence a Banach space. Observe that $S^{-1} : g(T) \rightarrow X$ is bounded (contractive). By the open mapping theorem, S is bounded. Assume $x_n \rightarrow z$. So $S(x_n) \rightarrow S(z)$. Then $(x_n, T(x_n)) \rightarrow (z, T(z))$, which gives $T(x_n) \rightarrow T(z)$. \square

Chapter 2

Topological Vector Spaces

Example 2.0.1. Let X be a vector space and let $f : X \rightarrow \mathbb{C}$ be a linear map. Define $\phi : X \rightarrow \mathbb{R}^+ := \phi(x) = |f(x)|$. Then ϕ is a seminorm.

2.1 Weak and Weak * Topologies

Remark 2.1.1. Let X be a TVS and $A \subset X^*$. We denote by $\sigma(X, A)$, the topology on X defined by A . (initial topology). Recall that $\sigma(X, A)$ is Hausdorff if and only if A separate points of X .

$\sigma(X, X^*)$ is called the weak topology on X .

Also recall that $X \hookrightarrow X^{**}$ by the evaluation maps. Hence we can view X as a subset X^{**} . And with this identification, we call $\sigma(X^*, X)$ the weak * topology on X^*

Definition 2.1.1. Let S be any set. Let I be a directed set. A net in S indexed by Λ is a function $f : \Lambda \rightarrow S$. We denote the net by $(x_\lambda)_{\lambda \in \Lambda}$.

In addition if S is a topological space, we say a net $(x_\lambda)_{\lambda \in \Lambda}$ converges to a point $x \in S$ if for all open set U in S with $x \in U$, there exists an $\lambda_0 \in \Lambda$ such that for all $\lambda \geq \lambda_0$ we have $x_\lambda \in U$.

Remark 2.1.2. By definition, a basis of open neighborhoods of a point $x_0 \in X$ in $\sigma(X, A)$ is given

$$\bigcup_{\substack{p_1, p_2, \dots, p_n \in A \\ \epsilon_1, \epsilon_2, \dots, \epsilon_n > 0}} \bigcap_{i=1}^n \{z \in X \mid p_i(z - x_0) < \epsilon_i\}$$

So the basis in a weak topology is

$$\bigcup_{\substack{f_1, f_2, \dots, f_n \in X^* \\ \epsilon_1, \epsilon_2, \dots, \epsilon_n > 0}} (x_0) = \{z \in X \mid |f_k(z - x_0)| < \epsilon_k, \text{ for all } k = 1, 2, \dots, n\}$$

See how nets become the topological equivalent of sequences in metric spaces. Find what property of the metric space makes it enough to be indexed by a countable totally ordered set for openness.

Example 2.1.1. Let S be a topological space, $E \subset S$, $x_0 \in \overline{E}$. Then there is a net $(x_\alpha) \subset E$ such that $x_\alpha \rightarrow x_0$.

Proof. Consider the collection \mathcal{T} of all open sets of S that contain x_0 . Order \mathcal{T} by the reverse inclusion. That is $A \leq B$ if $B \subset A$. This makes \mathcal{T} , a directed set. Now each $\lambda \in \mathcal{T}$ has a nonempty intersection with E being an open set containing the limit point x_0 of E . For each $\lambda \in \mathcal{T}$ choose an $x_\lambda \in \lambda \cap E$. Then we claim that the net $(x_\lambda) \rightarrow x_0$. \square

Example 2.1.2. Let $X = \ell^1$. Then $X^* = \ell^\infty$. Then the weak * topology on ℓ^∞ is given by the pointwise convergence of a net $(f_\alpha) \subset \ell^\infty$ converges to $f \in \ell^\infty$ if and only if $f_\alpha(n) \rightarrow f(n)$ for all $n \in \mathbb{N}$

Theorem 2.1.1 (Tychonoff). *The product of compact sets is compact.*

Corollary 2.1.1.1. *Let X be a compact Hausdorff space. Then for any set S , the set $\{\phi : S \rightarrow X\} = X^S$ is compact wrt to pointwise convergence.*

Theorem 2.1.2 (Banach-Alaoglu Theorem). *Let X be a normed space. Then the closed unit ball $\overline{B_{X^*}}(0, 1) = \{f \in X^* : \|f\| \leq 1\} = E$ is weak * compact.*

Proof. Let \bar{B} be the closed unit ball of X . Then by Tychonoff theorem, $\bar{D}^{\bar{B}}$ is compact. Define $\phi : E \rightarrow \bar{D}^{\bar{B}}$ as $\phi(f)(x) := f(x)$. Observe that ϕ is injective. Also observe that ϕ is continuous (weak * in LHS, and pointwise in RHS).

Next we show that the image of ϕ a closed subset of $\bar{D}^{\bar{B}}$, hence compact. Let f_i be a net in E such that $\phi(f_i) \rightarrow \psi$ pointwise for some $\psi \in \bar{D}^{\bar{B}}$.

Define $g : X \rightarrow C$ as $g(x) = \alpha\psi(\frac{x}{\alpha})$ where $\|x\| \leq \alpha$. For this to be well defined we must have $\|x\|\psi(\frac{x}{\|x\|}) = \alpha\psi(\frac{x}{\alpha})$ for any $\alpha > \|x\|$. But we get this since ψ is a pointwise limit of linear functionals. Moreover we get that g is linear for the same reason. Thus $\psi = \phi(g)$ and so $\phi(E)$ is closed.

It only remains to show that the inverse of ϕ is continuous. **verify**. \square

Proposition 2.1.1. *The closed unit ball of a normed space Y is compact w.r.t the norm topology if and only if Y is finite dimensional.*

Proof. **verify** \square

Theorem 2.1.3. *Let X be a normed space. Then E is weak * metrizable iff X is separable.*

*complete
this proof*

Proof. Assume X is separable. Let $\{x_n : n \in \mathbb{N}\}$ be a dense subset of X . For every $f, g \in E$, define $d(f, g) := \sum_{n \in \mathbb{N}} \frac{1}{2^n} |f(\frac{x_n}{\|x_n\|}) - g(\frac{x_n}{\|x_n\|})|$. Check that d is a metric.

Now assume $f_i \rightarrow f$ weakly in E . Then $f_i(x_n) \rightarrow f(x_n)$ for all $n \in \mathbb{N}$ and $d(f_i, f) \rightarrow 0$ (**verify**).

Assume E is metrizable. Then $\exists \{U_n : n \in \mathbb{N}\}$ of weak $*$ open neighborhoods of 0 such that $\cap_{n=1}^{\infty} U_n = \{0\}$. So, for each $n \in \mathbb{N}$, there exists a finite set $A_n \in X$ and $\epsilon > 0$ such that the (subbasis sets) $\{f \in E : |f| \leq \epsilon \forall x \in A_n \subset U_n\}$. Now let $A = \cup_{n=1}^{\infty} A_n$. Let $\phi \in E$ such that $\phi(x) = 0$ for all $x \in A$. \square

Definition 2.1.2. Let X and Y be normed spaces and $T \in B(X, Y)$. The adjoint of T , denoted by $T^* \in B(Y^*, X^*)$, is the map $T^* : f \rightarrow f \circ T$

Proposition 2.1.2. $\|T\| = \|T^*\|$

Proof. $|T^*(f)| \leq \|f \circ T\| \leq \|T\| \|f\|$ implies $\|T^*\| \leq \|T\|$

$$\begin{aligned} \|T^*\| &= \sup\{\|T^*(\phi)\| : \phi \in Y^*, \|\phi\| \leq 1\} \\ &= \sup\{|\phi(T(x))| : \phi \in Y^*, x \in X, \|\phi\| \leq 1, \|x\| \leq 1\} \\ &= \|T\| \end{aligned}$$

Note that the last equality is a consequence of HBT since it guarantees the existence of $\phi_y \in Y^*$ with $\|\phi_y\| \leq 1$ and $\phi_y(y) = |y|$. \square

Lemma 2.1.1. For any $T \in B(X, Y)$, $T^* : Y^* \rightarrow X^*$ is weak $*$ continuous (in both spaces)

Proof. Let $\phi_i \rightarrow \phi$ weakly in Y^* . Then by definition for all $y \in Y$, $\phi_i(y) \rightarrow \phi(y)$. Then for $x \in X$, $T^*(\phi_i)(x) = \phi_i(T(x)) \rightarrow \phi(T(x)) = (T^*(\phi))(x)$ which shows the continuity of T^* . \square

Lemma 2.1.2. For any normed space X , $i_X(X)$ is weak $*$ dense in X^{**} .

Proof. **verify** \square

Example 2.1.3. Is i_{X^*} weak $*$ - weak $*$ continuous.

2.2 Locally Convex Topological Vector Spaces

Lemma 2.2.1. Let X be a normed space and $x_1, x_2, \dots, x_n \in X$ and $\epsilon_1, \epsilon_2, \dots, \epsilon_n \geq 0$. Then the set

$$\bigcup_{x_1, x_2, \dots, x_n, \epsilon_1, \epsilon_2, \dots, \epsilon_n} (\phi)$$

is convex. Moreover any topological vector spaces with the topology induced by a family of seminorms is locally convex.

Proof. **verify** \square

Review
convexity
arguments
in
Arveson's
'Subalgebras of C^*
Algebras'

Definition 2.2.1. Let X be a vector space and $E \subset X$ be a convex subset. An element $a \in E$ is called an extreme point of E if whenever $x, y \in E$, $0 \leq t \leq 1$ with $a = tx + (1 - t)y$, then $x = y = a$.

Example 2.2.1. Let $\bar{D} = \{\alpha \in \mathbb{C} : |\alpha| \leq 1\}$. Then \bar{D} is convex with $\text{Ext}(\bar{D}) = S^1$

Theorem 2.2.1 (Krein-Milman Theorem). *Let X be a locally convex space, and let K be a compact convex subset of X . Then the $\text{Ext}(K) \neq \emptyset$ and indeed $K = \overline{\text{co}}(\text{Ext}(K))$*

Definition 2.2.2. Let V be a vector space and $S \subset V$. The convex hull of S is defined as

$$\text{co}(S) = \left\{ \sum_{i=1}^n t_i x_i \mid 0 \leq t \leq 1, \sum t_i = 1, x_i \in S \right\}$$

Lemma 2.2.2. *Let K be a convex set. $x_0 \in \text{Ext}(K)$ if and only if $K \setminus \{x_0\}$ is convex.*

Proof. If $K \setminus \{x_0\}$ is not convex, since K is convex, x_0 can be written as the convex combination of elements in $K \setminus \{x_0\}$ which makes $x_0 \notin \text{Ext}(K_0)$. Conversely if $x_0 \in \text{Ext}(K)$, then x_0 cannot be written as the convex combination of elements of K . Hence $K \setminus \{x_0\}$ is closed under convex combinations, making in convex. \square

Theorem 2.2.2 (Krein-Milman Theorem). *Let X be a locally convex space, and let K be a compact convex subset of X . Then the $\text{Ext}(K) \neq \emptyset$ and indeed $K = \overline{\text{co}}(\text{Ext}(K))$*

Proof. We first prove that the $\text{Ext}(K) \neq \emptyset$. Note that $K \setminus \{x_0\}$ is a relatively open subset of K since $\{x_0\}$ is closed and $K \setminus \{x_0\} = \{x_0\}^c$ relative to K .

Now let \mathcal{A} be the collection of all relatively open convex proper subsets of K . Note that $\emptyset \in \mathcal{A}$, therefore \mathcal{A} is nonempty. Equip \mathcal{A} with the partial order defined by the set inclusion. Let \mathcal{C} be a chain in \mathcal{A} and $F_{\mathcal{C}} = \cup_{C \in \mathcal{C}} C$. $F_{\mathcal{C}}$ is relatively open being the union of relatively open subsets of K . To see that $F_{\mathcal{C}}$ is convex, let $x, y \in F_{\mathcal{C}}$. Then since \mathcal{C} is a chain, there exist a $C \in \mathcal{C}$ such that $x, y \in C$. Then by the convexity of C , $tx + (1 - t)y \in C \subset F_{\mathcal{C}}$ for all $t \in [0, 1]$.

We claim that $F_{\mathcal{C}}$ is a proper subset of K . For the sake of contradiction, assume $F_{\mathcal{C}} = K$. Since K is compact and C is open in K for all $C \in \mathcal{C}$, there are finitely many $C_1 \subset C_2 \subset \dots \subset C_k \in \mathcal{C}$ which cover K (i.e $K = \cup_{n=1}^k C_n$). Hence we get $C_k = K$, which is absurd since C_k must be a proper subset of K . Hence $F_{\mathcal{C}} \in \mathcal{A}$ and thus every chain must have an upper bound in \mathcal{A} . Now by Zorn's lemma, \mathcal{A} has a maximal element K_0 .

Since K is a connected space (path connected by a straight line, being convex), we know that the only clopen subsets are \emptyset and K . Since we know that K_0 is open

Zorn's
Lemma to
find a
maximal
proper
open
convex
subset of K

Construct-
ing and
open
convex
subset
containing
 K_0

being in \mathcal{A} , we see that $K_0 \neq K$ and $K_0 \neq \emptyset$. Therefore $\overline{K_0} \neq K_0$. Let $x_0 \in \overline{K_0} \setminus K_0$, $y_0 \in K_0$ and $0 < t < 1$. Define $\varphi_{t,y_0} : K \rightarrow K$ such that $\varphi_{t,y_0}(z) = ty_0 + (1-t)z$. Then φ_{t,y_0} is $(1-t)$ Lipschitz continuous relative to K and thus $\varphi_{t,y_0}^{-1}(K_0)$ is open in K . By the convexity of K_0 , we get $K_0 \subset \phi_{t,y_0}^{-1}(K_0)$.

Also $\varphi_{t,y_0}^{-1}(K_0)$ is convex. Let $a, b \in \phi_{t,y_0}^{-1}(K_0)$. Then $ty_0 + (1-t)a, ty_0 + (1-t)b \in K_0$. By the convexity of K_0 we get $r(ty_0 + (1-t)a) + (1-r)(ty_0 + (1-t)b) = ty_0 + (1-t)(ra + (1-r)b) = \phi_{t,y_0}(ra + (1-r)b) \in K_0$ for all $r \in [0, 1]$. Thus $ra + (1-r)b \in \phi_{t,y_0}^{-1}(K_0)$ for all $r \in [0, 1]$. Hence we get $\phi_{t,y_0}^{-1}(K_0)$ is convex.

We claim, $x_0 \in \varphi_{t,y_0}^{-1}(K_0)$, then the maximality of K_0 will force $\phi_{t,y_0}^{-1}(K_0) = K$. Let U be a convex neighborhood of $0 \in X$ containing $-x$ for all $x \in U$ (just take $-U$ and intersect with U) such that $y_0 + E \subset K_0$, where $E = K \cap U$. Let $w = \varphi_{t,y_0}(x_0)$. Since $x_0 \in \overline{K_0}$, for any $r > 0$, there exists $x_r \in K_0$ such that $x_r \in (x_0 + rE) \cap K_0 \neq \emptyset$. In particular, let $r = \frac{t}{1-t}$. Then by linearity, we get $(x_0 + \frac{t}{1-t}E) \cap K_0 = (\frac{t}{1-t})E \cap (K_0 - x_0) \neq \emptyset$. Choose z in the above set. Then

$$y_0 - \left(\frac{1-t}{t}\right)z \in y_0 + E \subset K_0$$

and $x_0 + z \in K_0$. Since K_0 is convex,

$$t\left(y_0 - \frac{(1-t)}{t}z\right) + (1-t)(x_0 + z) = \phi_{t,y_0}(x_0) \in K_0$$

Thus $\phi_{t,y_0}^{-1}(K_0) = K$.

Now we claim that $K = K_0 \cup \{x_0\}$. For the sake of contradiction assume $\exists p \in K$ such that $p \notin K_0 \cup \{x_0\}$. Since the space is Hausdorff and locally convex, x_0 has an open convex neighborhood E in X such that $p \notin E$. Let $E' = E \cap K$, $a \in K_0, b \in E'$ and $0 < r < 1$. Then since $\phi_{t,y_0}(K) = K_0$ for all $t \in [0, 1]$, $y_0 \in K_0$, we get $\phi_{r,a}(b) = ra + (1-r)b \in K_0$. So $K_0 \cup E'$ is convex (Sine we know that K_0, E' are convex, we only need to worry about $rx + (1-r)y$ for $x \in K_0, y \in E'$. But $\phi_{r,x}$ takes care of that). $K_0 \cup E'$ is also open in K . Hence by maximality, we get $K_0 \cup E' = K$. But this is a contradiction since $p \notin K_0 \cup E'$. Thus by [Lemma 2.2.2](#), we see that $x_0 \in \text{Ext}(K)$.

Next we prove $K = \overline{\text{co}}(\text{Ext}(K))$. Let $P = \overline{\text{co}}(\text{Ext}(K))$ and for the sake of contradiction assume $P \neq K$. Let $x_0 \in K \setminus P$. Now by the geometric Hahn-Banach separation theorem, we get that there is a continuous linear functional $\phi : X \rightarrow \mathbb{R}$ and a number $\alpha, \epsilon \in \mathbb{R}$ such that

$$\Re \phi(x_0) \leq \alpha < \alpha + \epsilon \leq \Re \phi(p), \quad \forall p \in P$$

□

I can't
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the choice
of z

Lemma 2.2.3. *Let K_1, K_2 be compact convex subsets of a locally compact TVS X . Then*

$$\overline{\text{co}}(K_1 \cup K_2) = (\text{co})(K_1 \cup K_2)$$

Proof. **verify.** We'll show that $\text{co}(K_1 \cup K_2)$ is compact and hence closed. Let $x = \alpha_1 a_1, \alpha_2 a_2, \dots, \alpha_n a_n + \beta_1 b_1, \beta_2 b_2, \dots, \beta_m b_m \in \text{co}(K_1 \cup K_2)$, where $\sum_{i=1}^n \alpha_i + \sum_{i=1}^m \beta_i = 1$. Then

$$x = \left(\sum_{i=1}^n \alpha_i \right) \underbrace{\left(\sum_{i=1}^n \left(\frac{\alpha_i}{\sum_{i=1}^n \alpha_i} \right) a_i \right)}_{\in K_1} + \left(\sum_{i=1}^m \beta_i \right) \underbrace{\left(\sum_{i=1}^m \left(\frac{\beta_i}{\sum_{i=1}^m \beta_i} \right) b_i \right)}_{\in K_2}$$

Hence every element $x \in \text{co}(K_1 \cup K_2)$, can be written as $x = ta + (1-t)b$ where $a \in K_1, b \in K_2$.

Now let $x_\lambda = t_\lambda a_\lambda + (1-t_\lambda)b_\lambda$ be a net in $\text{co}(K_1 \cup K_2)$, for $\lambda \in \Lambda$, $a_\lambda \in K_1, b_\lambda \in K_2$. Since (a_λ) is a net in the compact set K_1 , there is a subnet a_σ for $\sigma \in \Sigma \subseteq \Lambda$, such that $a_\sigma \rightarrow a \in K_1$. By similar reasoning b_σ has a convergent subnet b_π for $\pi \in \Pi \subseteq \Sigma$, such that $b_\pi \rightarrow b \in K_2$. Again t_π is a net in the compact space $[0, 1]$, hence it has a convergent subnet t_ω for $\omega \in \Omega \subseteq \Pi$ such that $t_\omega \rightarrow t$ in $[0, 1]$.

Now consider the subnet $x_\omega = t_\omega a_\omega + (1-t_\omega)b_\omega$ of x_λ . Since $\Omega \subseteq \Pi \subseteq \Sigma$, $t_\omega \rightarrow t, b_\omega \rightarrow b$ and $a_\omega \rightarrow a$. Therefore by the continuity of the scalar product and addition in the TVS, we get $x_\omega \rightarrow ta + (1-t)b \in \text{co}(K_1 \cup K_2)$. Hence we get $\text{co}(K_1 \cup K_2)$ is compact. \square

Theorem 2.2.3 (Inverse Krein-Milman). *Let K be a compact convex subset of a locally convex topological vector space X . Let $A \subset K$ be a closed subset of K . If $K = \overline{\text{co}}(A)$, then $\text{Ext}(K) \subset A$.*

Note that $\text{Prob}[0, 1]$, the collection of probability measures identified as a subspace of a $C([0, 1])^*$ is convex, weak * compact with $\text{Ext}(K) = \{\delta_x : x \in [0, 1]\}$

Proof. FSTOC, assume $\exists x_0 \in \text{Ext}(K), x_0 \notin A$. Since A is compact, $\exists y_1, y_2, \dots, y_n \in A$ and an open convex neighborhood B of 0 such that

$$A \subset \cup_{i=1}^n (y_i + B)$$

and $x_0 \notin y_i + \overline{B}$ for all $i = 1, 2, \dots, n$. Let $B_i = (y_i + \overline{B}) \cap K$. Then B_i is a compact convex subset of K for each i . Hence by the previous lemma, we get

$$\text{co}(B_1 \cup B_2 \cup \dots \cup B_n) = \overline{\text{co}}(B_1 \cup B_2 \cup \dots \cup B_n) \supset \overline{\text{co}}(A) = K$$

Thus $\exists b_i \in B_i$ and $0 \leq t_i \leq 1$, $\sum_{i=1}^n t_i = 1$ such that

$$x_0 = t_n b_1 + t_n b_2 + \dots + t_n b_n$$

I struggle
at finding
the contra-
diction

Since $x_0 \in \text{Ext}(K)$, this forces $x_0 = b_j$ for some $1 \leq j \leq n$. This contradicts the assumption that $x_0 \notin y_i + \overline{B}$. Hence $x_0 \in A$. \square

Note that in the following attempt to prove the theorem, it is not obvious why U is convex. If we try to argue using arguments to the proof of separating a compact set and a point using open sets in a Hausdorff space, we will eventually need to show that the finite open subcover of A sits inside a closed convex set that does not contain x_0 , which again is not obvious.

Proof. FTSOC, assume that $\exists x_0 \in \text{Ext}(K) \setminus A$. Since the TVS is Hausdorff, there exist convex open sets U, V such that $A \subset U, x_0 \in V$, and $U \cap V = \emptyset$. Moreover we claim that $\overline{U} \cap V = \emptyset$. Otherwise if $x \in \overline{U} \cap V$, then for any net $(x_\lambda) \in V$ that converge to x , by the definition of convergence $x_{\lambda_n} \in U$ for all λ_n greater than some λ_N . This would contradict the assumption that $U \cap V = \emptyset$. Hence we see that $A \subset \overline{U}$, and therefore $\overline{\text{co}}(A) \subset \overline{U}$. But this would again contradict the fact that $\overline{\text{co}}(A) = K$ since $x_0 \notin \overline{U}$. \square

Why is U
convex?

Example 2.2.2. Let X be an infinite dimensional normed space. Then the set $A = \{x \in X : \|x\| = 1\}$ is norm closed. But the weak closure of A is the set

$$\overline{A}^w = \overline{\{x \in X : \|x\| = 1\}}^w = \{y \in X : \|y\| \leq 1\}$$

Hence A is an example of a norm closed set, which is not weak closed.

Theorem 2.2.4. Let X be a normed space and let $K \subset X$ be convex subset of X . Then the norm and the weak closure of K coincide.

Proof. Since norm topology is stronger than weak topology, we get $\overline{K}^{\|\cdot\|} \subset \overline{K}^w$. Let $x \in X$ such that $x \notin \overline{K}^{\|\cdot\|}$. Now since $\{x\}, K$ are convex and compact, by Hahn-Banach separation theorem, there is a $f \in X^*, s \in \mathbb{R}, \epsilon > 0$ such that

$$|f(x)| \leq s < s + \epsilon \leq |f(y)|, \quad \forall y \in \overline{K}^{\|\cdot\|}$$

Since the set $\{z \in X : |f(z)| \geq s + \epsilon\}$ is weakly closed, and contains K , it must contain \overline{K}^w . Hence $x \notin \overline{K}^w$. \square

Corollary 2.2.4.1. Let X be a normed space, and $(x_i)_{i \in I}$ be a net in X such that $x_i \rightarrow x$ weakly in X . Then there exists a net $(y_j)_{j \in J}$ of finite convex combinations of $\{x_i : i \in I\}$ such that $y_j \rightarrow x$ in norm.

Proof. **verify** □

Proposition 2.2.1. *If K is a convex subset of a LCTVS. Then \overline{K} is also convex.*

Proof. **verify** □

Proposition 2.2.2. *Show that c_0 is weakly closed and weak $*$ dense in ℓ_∞ .*

Proof. **verify** □

Theorem 2.2.5 (Krein-Smulian Theorem). *Let X be a Banach space, and let C be a convex subset of X^* . Then C is weak $*$ closed if and only if $C \cap \{f \in X^* : \|f\| \leq r\}$ is weak $*$ closed for all $r \in \mathbb{R}^+$.*

*This should even work if we just take $n \in \mathbb{N}$. **verify**.*

Corollary 2.2.5.1. *Let Z be a subspace of X^* . Then Z is weak $*$ closed if and only if $\{\phi \in Z : \|\phi\| \leq 1\}$ is closed.*

Corollary 2.2.5.2. *Let X be a separable Banach space. Then a convex subset Z of X^* is weak $*$ closed if and only if it is weak $*$ sequentially closed.*

Proof. Since X is separable, for every $r > 0$, the set $\{f \in X^* : \|f\| \leq r\}$ is weak $*$ metrizable. Thus $Z \cap \{f \in X^* : \|f\| \leq 1\}$ is weak $*$ closed iff it is weak $*$ sequentially closed. □

Corollary 2.2.5.3. *Let X be a separable Banach space, and $\phi \in X^{**}$. Then ϕ is weak $*$ continuous if and only if ϕ is sequentially continuous i.e $f_n \rightarrow f$ weak $*$ in X^* implies $\phi(f_n) \rightarrow \phi(f)$.*

Proof. Assume ϕ is sequentially weak $*$ continuous. Let $C = \text{Ker}(\phi)$ be a subspace of X^* . Let $g_n \in C$ be a sequence such that $g_n \rightarrow g \in X^*$. Then by assumption, $0 = \phi(g_n) \rightarrow \phi(g)$ implies $\phi(g) = 0$ and therefore $g \in C$. This shows that C is weak $*$ sequentially closed, hence weak $*$ closed by the separability of X . Hence ϕ is weak $*$ continuous. □

Example 2.2.3. Let $S = \{n\delta_n : n \in \mathbb{N}\} \subset \ell^\infty$. We show that $0 \in \overline{S}^{w*}$

Proof. Let $f \in \ell^1$. Then the set $\{n \in \mathbb{N} : |f(n)| < \epsilon/n\}$ is infinite. (Otherwise this would contradict $f \in \ell^1$). Thus $\exists N \in \mathbb{N}$ such that $N|f_i(N)| < \epsilon$ for all $i = 1, 2, \dots, m$. And therefore

$$N\delta_n \in \bigcup_{f_1, f_2, \dots, f_N, \epsilon} (0)$$

□

Definition 2.2.3. A subset S of a vector space V is called balanced if $\forall s \in S, \alpha \in \mathbb{F}$ with $|\alpha| \leq 1, \alpha s \in S$.

Lemma 2.2.4. Let X be a topological vector space, then every open neighborhood of 0 contains a balanced open neighborhood of 0.

Proof. **verify** □

Lemma 2.2.5. All n -dimensional topological vector spaces are isomorphic as topological vector spaces.

Proof. For the case $n = 1$, and $\mathbb{F} = \mathbb{C}$.

Assume τ is a topology on \mathbb{C} that turns it into a topological vector space. Now think of $i : \mathbb{C} \rightarrow (\mathbb{C}, \tau) := x \rightarrow x$ as the composition of $\mathbb{C} \rightarrow \mathbb{C} \times (\mathbb{C}, \tau) := x \rightarrow (x, 1)$ and $\mathbb{C} \times (\mathbb{C}, \tau) \rightarrow (\mathbb{C}, \tau) := (x, y) \rightarrow xy$. Then we see that i is the composition of these maps which are continuous by the definition of the product topology and the TVS. Hence, i is continuous.

To show that i^{-1} is continuous, consider the annulus $A = \{\alpha \in \mathbb{C} : 1 \leq |\alpha| \leq 2\}$. Then since A is compact in the usual topology and $i(A) = A$ is a continuous image, we get that A is open in τ . Hence $A^c \ni 0$ is open and by the lemma above has a balanced open neighborhood of 0 in it. **(Show that this is actually an open disk)**. □

Theorem 2.2.6. Let X be a normed space. Then the closed unit ball of X is compact in norm topology if and only if X is finite dimensional.

Proof. Suppose X is infinite dimensional normed space and let \bar{B} be the closed unit ball. Let $x_1 \in \bar{B}$ and let $Y_1 = \text{span}\{x_1\}$. Then Y_1 is a closed subspace of X . Since X is a non-zero normed space, let $x_2 \in X$ such that $\|x_2 + Y_1\| = \frac{1}{2}$. Repeat the construction in the proof of Reisz lemma. □

Lemma 2.2.6. Let X be a normed space. Then $i(X)$ is weak * dense in X^{**}

Proof. Let $C = \bar{B}^{w*}$, where B is the closed unit ball. Then C is compact convex. FSTOC, assume $\exists \phi \in X^{**}$ such that $\|\phi\| \leq 1, \phi \notin C$. Then by HBT, there is a $f \in X^{**}$ and $r \in \mathbb{R}, \epsilon > 0$ such that $\Re f(y) \leq r < r + \epsilon \leq \Re f(\phi)$ for all $y \in i(X)$. This implies $\|f\| \leq r$, hence $|f(\phi)| = |\phi(f)| \leq \|\phi\| \|f\| < r$ which gives a contradiction. □

Show that
 $\Re f(x) \leq r\|x\|$ imply
 $\|f\| \leq r$

Theorem 2.2.7. Let X be a Banach space. Then the closed unit ball \bar{B} is weakly compact if and only if X is reflexive.

Proof. If X is reflexive, the weak and weak * topology coincides and the Banach Alaouglu gives the proof. **verify**

Assume \bar{B} is weakly compact. Observe that then the map $i : X \rightarrow X^{**}$ is continuous when we equip X with weak topology and X^{**} with weak topology. Thus $i(\bar{B})$ is weak * compact. Moreover $i(\bar{B})$ is weak * dense in the closed unit ball of X^{**} . Hence the result. \square

Chapter 3

Hilbert Spaces

Definition 3.0.1. Recall that a complex inner product on a complex vector space is a map

$$\langle \cdot, \cdot \rangle : X \times X \rightarrow \mathbb{C}$$

such that

- (1) $\langle x, x \rangle \geq 0$ for all $x \in X$
- (2) $\langle x, y \rangle = \overline{\langle y, x \rangle}$
- (3) $\langle \alpha x + z, y \rangle = \alpha \langle x, y \rangle + \langle z, y \rangle$

Recall the norms induced by the inner product and the Cauchy-Schwarz inequality.

Definition 3.0.2. Complete inner product spaces are called Hilbert spaces

Proposition 3.0.1. *Let X be an inner product space. Then the inner product of X extends to an inner product on the completion (unique metric space completion) of X , turns it into a Hilbert space.*

Definition 3.0.3. If $x, y \in H$, the Hilbert space, we say $x \perp y$ if $\langle x, y \rangle = 0$

Definition 3.0.4. Given a set $S \subset H$, $S^\perp = \{y \in H : \langle x, y \rangle = 0\}$

Proposition 3.0.2. *Let H, K be Hilbert spaces and $T : H \rightarrow K$ be linear. Then the following are equivalent.*

- (1) T is isometry
- (2) $\langle Tx, Ty \rangle = \langle x, y \rangle$ for all $x, y \in H$

Proof. See Homework-5

□

Proposition 3.0.3. For all $x, y \in H$, a Hilbert space, then

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

Example 3.0.1. Show that c_{00} under the usual inner product is not complete and its completion is $\ell^2(\mathbb{N})$.

Proof. Consider the sequence $x_n = (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, \dots)$. Then clearly $x_n \in c_{00}$. Moreover $x_n \rightarrow x = (1, \frac{1}{2}, \dots, \frac{1}{n}, \frac{1}{n+1}, \dots)$ in the norm by the inner product. But $x \notin c_{00}$. But $x \in \ell^2(\mathbb{N})$. Moreover, the same process can be used to approximate any element in $\ell^2(\mathbb{N})$ by elements in c_{00} . Thus we see that c_{00} is dense in $\ell^2(\mathbb{N})$ \square

Example 3.0.2. $L^2(\mathbb{R}, \mu)$ with

$$\langle f, g \rangle = \int_{\mathbb{R}} f \bar{g} \, d\mu$$

is a Hilbert space.

Example 3.0.3. Let J be any set $\ell^2(J) = \{f : J \rightarrow \mathbb{C} : \sum_{j \in J} |f(j)|^2 < \infty\}$ with the usual inner product is a Hilbert space.

Definition 3.0.5. An orthonormal basis for H is a maximal orthonormal set.

Theorem 3.0.1. Let H be a Hilbert space and J be an orthonormal basis for H . Then there exists a bijective linear isometry $T : H \rightarrow \ell^2(J)$.

Theorem 3.0.2. Let \mathcal{H} be a Hilbert space and let C be a non-empty closed convex subset of \mathcal{H} . Then there exist a unique vector $x \in C$ such that $\|x\| \leq \|\eta\|$ for all $\eta \in C$.

construction of the proof is a bit tricky *Proof.* Let $d = \inf\{\|\eta\| : \eta \in C\}$ and choose a sequence $\eta_n \in C$ such that $\|\eta_n\| \rightarrow d$. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $\|\eta_n\|^2 < d^2 + \epsilon$ for all $n \geq N$. Then for all $m, n \geq N$, we have

$$\|\eta_n - \eta_m\|^2 = 2(\|\eta_n\|^2 + \|\eta_m\|^2) - 4\|\frac{1}{2}(\eta_n + \eta_m)\|^2 \leq 4(d^2 + \epsilon) - 4d^2 = 4\epsilon$$

(Note that since C is convex, $\frac{1}{2}\eta_n + \frac{1}{2}\eta_m \in C$ and therefore, by the definition of d , we get $\|\frac{1}{2}(\eta_n + \eta_m)\| \geq d$.) Hence the sequence η_n is Cauchy and hence convergent since the space is complete. Let $\eta = \lim_{n \rightarrow \infty} \eta_n$. Since C is closed $\eta \in C$ and clearly $\|\eta\| = d$.

To see the uniqueness, assume $\alpha \in C$, and $\|\alpha\| = d$. Then

$$\begin{aligned} \|\eta - \alpha\|^2 &= 2(\|\eta\|^2 + \|\alpha\|^2) - 4\|\frac{1}{2}(\eta + \alpha)\|^2 \\ &\leq 4d^2 - 4d^2 = 0 \end{aligned}$$

Verify the second inequality. \square

Corollary 3.0.2.1. *Let $\eta \in \mathcal{H}$ and C be as before. Then there exist a unique vector $x \in C$ such that $d(\eta, C) = \|x - \eta\|$*

Proof. Apply above theorem to $C' = C - \eta$. Since C is closed and convex, so will be its translation $C - \eta$. \square

Proposition 3.0.4 (Pythagoras Theorem). *Let $x, y \in \mathcal{H}$ an inner product space, and $x \perp y$, then $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.*

Lemma 3.0.1. *Let \mathcal{H} be a Hilbert space and K be a nontrivial closed subspace. Let $\eta \in \mathcal{H}$. Then $\xi \in K$ satisfy $\|\xi - \eta\| = d(\eta, K)$ iff $\xi - \eta \perp K$.*

Proof. Let $\xi - \eta \perp K$ and $k \in K$. Then $\eta - k = (\eta - \xi) + (\xi - k)$ and by Pythagoras theorem, we get

$$\|\eta - k\|^2 = \|\eta - \xi\|^2 + \|\xi - k\|^2 \geq \|\eta - \xi\|^2$$

Thus we see that $\|\eta - \xi\| = d(\eta, K)$.

Conversely, let $\|\eta - \xi\| = d(\eta, K)$. Then for all $\rho \in K$ and $t > 0$, we have

$$\begin{aligned} \|\eta - \xi\|^2 &\leq \|\eta - (\xi + t\rho)\|^2 \\ &= \|\eta - \xi - t\rho\|^2 \\ &= \|\eta - \xi\|^2 + t^2\|\rho\|^2 - 2t\Re\langle\eta - \xi, \rho\rangle \end{aligned}$$

cool proof
technique

Hence we see that $|2\Re\langle\eta - \xi, \rho\rangle| \leq t\|\rho\|^2$. Since $t > 0$ was arbitrary, limiting it to zero, we get $\Re\langle\eta - \xi, \rho\rangle = 0$. Replacing ρ with $-i\rho$ will give the imaginary part is also zero. \square

Remark 3.0.1. If K is a closed subspace of the Hilbert space \mathcal{H} , then $\mathcal{H} = K \oplus K^\perp$. To see this notice that $K \cap K^\perp = \{0\}$. Now if $x \in \mathcal{H}$, then there is a $k_x \in K$ such that $d(x, K) = \|x - k_x\|$. Moreover, from **Lemma 3.0.1**, we see that $x - k_x \in K^\perp$. Hence $x = k_x + (x - k_x) \in K \oplus K^\perp$. Now replacing K with K^\perp , we see that $(K^\perp)^\perp = K$.

Theorem 3.0.3 (Reisz Representation Theorem). *Let \mathcal{H} be a Hilbert space and $f \in \mathcal{H}^*$. Then there exists a unique $\eta_f \in \mathcal{H}$ such that $f(x) = \langle x, \eta_f \rangle$ for all $x \in \mathcal{H}$. The map $\phi : \mathcal{H}^* \rightarrow \mathcal{H} := f \rightarrow \eta_f$ is conjugate linear isometric bijection.*

Proof. If f is the zero linear functional, $\eta_f = 0$ and we're done. If not let $K = \text{Ker}(f)$. Then K has co-dimension 1. Consider $f|_{K^\perp}$. Let $v \in K^\perp$ such that $f(v) = 1$. Clearly $K^\perp = \text{span}(v)$. And therefore for any $\alpha v \in K^\perp$, $f(\alpha v) = \alpha = \langle \alpha v, \frac{v}{\|v\|^2} \rangle$. Choose $\eta_f = \frac{v}{\|v\|^2}$. Now verify. \square

3.1 Orthogonal Projections

Theorem 3.1.1. *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces, and $T : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a bounded linear map. Then there exists a unique bounded linear map $T^* : \mathcal{H}_2 \rightarrow \mathcal{H}_1$ satisfying $\langle Tx, y \rangle_{\mathcal{H}_2} = \langle x, T^*y \rangle_{\mathcal{H}_1}$ for all $x \in \mathcal{H}_1, y \in \mathcal{H}_2$.*

Proof. For every given $y \in \mathcal{H}_2$ define a linear functional $f^y : \mathcal{H}_1 \rightarrow \mathbb{C}$ as $f^y(x) = \langle Tx, y \rangle$. Since f^y is bounded, $f^y \in \mathcal{H}_1^*$. Hence by Reisz representation, there is a unique $T^*(y) \in \mathcal{H}_1$ such that $\langle Tx, y \rangle = \langle x, T^*y \rangle$.

Uniqueness follows from the fact that in any inner product space X , if $x, y \in X$ such that $\langle x, z \rangle = \langle y, z \rangle$ for all $z \in X$, then $\langle x - y, z \rangle = 0$ for all $z \in X$, in particular $z = x - y$. Then by the positive definiteness of the inner product, we get $x = y$. \square

Theorem 3.1.2. *Let \mathcal{H} be a Hilbert space and K a closed subspace. For every $\eta \in \mathcal{H}$, denote by $P_K(\eta)$, the unique closest vector in K , closest to η . Then*

(1) $P_K : \mathcal{H} \rightarrow \mathcal{H}$ is linear, bounded with $\|P_K\| = 1$ and idempotent.

(2) $P_K^* = P_K$ (self-adjoint)

Proof. (1) Let $\eta_1, \eta_2 \in \mathcal{H}$ and $\alpha \in \mathbb{C}$. Then using [Lemma 3.0.1](#) for all $\xi \in K$, we have

$$\begin{aligned} \langle \alpha\eta_1 + \eta_2 - \alpha P_K(\eta_1) - P_K(\eta_2), \xi \rangle &= \alpha \langle \eta_1 - P_K(\eta_1), \xi \rangle + \langle \eta_2 - P_K(\eta_2), \xi \rangle \\ &= 0 \end{aligned}$$

Thus we get $P_K(\alpha\eta_1 + \eta_2) = \alpha P_K(\eta_1) + P_K(\eta_2)$ again by the same [Lemma 3.0.1](#). Moreover since $P_K(x)$ is a vector in K it the vector in K closest to $P_K(x)$ is $P_K(x)$ itself. Thus we see that $P_K^2(x) = P_K(x)$. Hence idempotent. Moreover since K is closed, we have $X = K \oplus K^\perp$ ([Remark 3.0.1](#)). Thus for all $x \in X$, we can write $x = k_x + k'_x$, where $k_x \in K$ and $k'_x \in K^\perp$. Then $P_K(x) = k_x$ and triangle inequality shows $\|P_K(x)\| = \|k_x\| \leq \|x\|$. Thus $\|P_K\| \leq 1$.

(2) Since $X = K \oplus K^\perp$, let $x = k_x + k'_x$ and $y = k_y + k'_y$. Then

$$\langle P(x), y \rangle = \langle k_x, k_y + k'_y \rangle = \langle k_x, k_y \rangle = \langle k_x + k'_x, k_y \rangle = \langle x, P_K(y) \rangle$$

and the uniqueness of the adjoint proves that $P_K^* = P_K$

\square

Proposition 3.1.1. $I - P_K = P_{K^\perp}$

Proof. Let $x \in X$ and $k \in K$. Then

$$\begin{aligned}
\langle (I - P_k)(x), k \rangle &= \langle x - P_k(x), k \rangle \\
&= \langle x, k \rangle - \langle P_k(x), k \rangle \\
&= \langle x, k \rangle - \langle x, P_k(k) \rangle \\
&= \langle x, k \rangle - \langle x, k \rangle \\
&= 0
\end{aligned}$$

Shows that □

Proposition 3.1.2. *Let K be a closed subspace of \mathcal{H} . Let $E \subset K$ be an orthonormal basis for K . Extend E to an orthonormal basis \tilde{E} for \mathcal{H} . Then*

$$P_K|_E = I_K, \quad P_K|_{\tilde{E} \setminus E} = 0$$

Remark 3.1.1 (Parseval's Inequality). Let \mathcal{H} be a Hilbert space. Let E be an orthonormal set. Then for every vector $\eta \in \mathcal{H}$,

$$\|\eta\|^2 \geq \sum_{e \in E} |\langle \eta, e \rangle|^2$$

Lemma 3.1.1. *Let S be a nonempty subset of \mathcal{H} . Then*

$$(S^\perp)^\perp = \overline{\text{span}}(S)$$

Proof. Notice from the above proposition that $\text{Ker}(P_K) = K^\perp$. Thus $(S^\perp)^\perp = \text{Ker}(P_{S^\perp})$. Since $P_{S^\perp}(s) = 0$ for all $s \in S$, we see that $S \subset \text{Ker}(P_{S^\perp})$. Moreover, since $\text{Ker}(P_{S^\perp})$ is a closed subspace, we see that $\overline{\text{span}}(S) \subset \text{Ker}(P_{S^\perp}) = (S^\perp)^\perp$.

Now see that if $A \subset B$, then $B^\perp \subset A^\perp$. Note that if $\langle x, b \rangle = 0$ for all $b \in B$, then $\langle x, a \rangle = 0$ for all $a \in A$ since $A \subset B$. Thus we see that $B^\perp \subset A^\perp$. Then since $S \subset \overline{\text{span}}(S)$, we see that $\overline{\text{span}}(S)^\perp \subset S^\perp$ and $(S^\perp)^\perp \subset (\overline{\text{span}}(S)^\perp)^\perp$. Thus by [Remark 3.0.1](#), we see that $(S^\perp)^\perp \subset \overline{\text{span}}(S)$. □

Corollary 3.1.2.1. *Let E be an orthonormal subset of \mathcal{H} . Then the following are equivalent.*

- (1) E is an orthonormal basis
- (2) $(E^\perp)^\perp = \mathcal{H}$
- (3) $\overline{\text{span}}(E) = \mathcal{H}$
- (4) $\|\eta\|^2 = \sum_{e \in E} |\langle \eta, e \rangle|^2, \quad \forall \eta \in \mathcal{H}$

Proof. $1 \implies 2 \implies 3$ follows easily from [Lemma 3.1.1](#). To see $3 \implies 4$, since E is orthonormal and $\overline{\text{span}}(E) = \mathcal{H}$, we claim that it will be an orthonormal basis. Because if $y \perp E$, then $y \perp \text{span}(E)$. If $x_n \in \text{span}(E)$ such that $x_n \rightarrow x$, then since $\langle y, x_n \rangle = 0$ for each $n \in \mathbb{N}$, we get

$$|\langle x, y \rangle| = |\langle x - x_n, y \rangle + \langle x_n, y \rangle| = |\langle x - x_n, y \rangle| \leq \|x - x_n\| \|y\|$$

Since $x_n \rightarrow x$, we see that $\langle x, y \rangle = 0$. Thus we see that E is an orthonormal basis. Thus we see that for all $\eta \in \mathcal{H}$,

$$\eta = \sum_{e \in E} \langle \eta, e \rangle e$$

and Parseval's equality follows. □

Theorem 3.1.3. *Let $P \in B(\mathcal{H})$ be an idempotent. Then the following are equivalent.*

- (1) $P = P_K$ for some closed subspace $K \leq \mathcal{H}$
- (2) $P = P^*$
- (3) $\|P\| = 1$

Proof. $1 \implies 2, 3$ is proved in [Theorem 3.1.2](#). To see $2 \implies 1$. Let $K = \text{Im}(P)$. Let $\rho \in K^\perp$. Then for all $\eta \in H$,

$$\langle P(\rho), \eta \rangle = \langle \rho, P^*(\eta) \rangle = \langle \rho, P(\eta) \rangle = 0$$

So $P|_{K^\perp} = 0$. Since $P^2 = P$, $P|_K = I$. Hence $P = P_K$.

To see $3 \implies 1$, assume $P \neq P_K$. Then $\exists \rho \in K^\perp$ such that $P(\rho) \neq 0$. For each $n \in \mathbb{N}$, we have $\|\rho + nP(\rho)\|^2 = \|\rho\|^2 + n^2\|P(\rho)\|^2$ by [Proposition 3.0.4](#). And

$$\|P(\rho + nP(\rho))\|^2 = (n+1)^2\|P(\rho)\|^2$$

*nice
technique*

Hence we have

$$\|P(\rho + nP(\rho))\|^2 - \|\rho + nP(\rho)\|^2 = 2n\|P(\rho)\|^2 + \|P(\rho)\|^2 - \|\rho\|^2$$

So for large n , we have

$$\|P(\rho + nP(\rho))\| > \|\rho + P(\rho)\|$$

so $\|P\| > 1$. □

3.2 Compact Operators

Definition 3.2.1. let \mathcal{X}, \mathcal{Y} be Banach Spaces. A linear map $T : \mathcal{X} \rightarrow \mathcal{Y}$ is called a compact operator if $\overline{T(B_1^{\mathcal{X}})}$ is compact, where $B_1^{\mathcal{X}}$ is the unit ball of \mathcal{X} . We denote by $\mathcal{K}(\mathcal{X}, \mathcal{Y})$, the set of all compact operators.

Example 3.2.1. If either \mathcal{X} or \mathcal{Y} is finite, then every linear map $T : \mathcal{X} \rightarrow \mathcal{Y}$ is compact. If \mathcal{X} be any infinite dimensional Banach space. Then $T = \text{Id} : \mathcal{X} \rightarrow \mathcal{X}$ is not compact. This follows from [Theorem 2.2.6](#).

Definition 3.2.2. $T : \mathcal{X} \rightarrow \mathcal{Y}$ is called a finite rank if the dimension of T is finite. Then the dimension of the image is called the rank of the operator. Let $\mathcal{F}(\mathcal{X}, \mathcal{Y})$ denote finite rank operators.

Lemma 3.2.1. *Every compact operator is bounded.*

Proof. Every compact set is bounded in any metric space. □

Theorem 3.2.1. *Let \mathcal{H} be a Hilbert space. Then $\mathcal{K}(\mathcal{H}) = \overline{\mathcal{F}(\mathcal{H})}^{\|\cdot\|}$*

Proof. It is evident that $\mathcal{F}(\mathcal{H}) \subset \mathcal{K}(\mathcal{H})$. We'll now show that $\overline{\mathcal{F}(\mathcal{H})} \subset \mathcal{K}(\mathcal{H})$. Let T_n be a sequence in $\mathcal{F}(\mathcal{H})$ and $T_n \rightarrow T \in B(\mathcal{H})$ (in norm). We'll show that $\overline{T(B_1^{\mathcal{H}})}$ is closed and totally bounded, which is equivalent to compactness in metric spaces.

Let $\epsilon > 0$ be given. Then there exist some $N \in \mathbb{N}$ such that $\|T_N - T\| < \epsilon$. Moreover

$$\overline{T_N(B_1^{\mathcal{H}})} \subset \bigcup_{\eta \in B_1^{\mathcal{H}}} B_{\epsilon}(T_N(\eta))$$

By the compactness of $\overline{T_N(B_1^{\mathcal{H}})}$ there exist $\eta_1, \eta_2, \dots, \eta_k \in B_1^{\mathcal{H}}$ such that

$$\overline{T_N(B_1^{\mathcal{H}})} \subset \bigcup_{i=1}^k B_{\epsilon}(T_N(\eta_i))$$

Thus for an arbitrary $\eta \in B_1^{\mathcal{H}}$, $\exists i \leq k$ such that $\|T_N(\eta) - T_N(\eta_i)\| < \epsilon$. Then

$$\begin{aligned} \|T(\eta) - T(\eta_i)\| &\leq \|T(\eta) - T_N(\eta)\| + \|T_N(\eta) - T_N(\eta_i)\| + \|T_N(\eta_i) - T(\eta_i)\| \\ &< 3\epsilon \end{aligned}$$

Thus we see that

$$T(B_1^{\mathcal{H}}) \subset \bigcup_{i=1}^k B_{3\epsilon}(T(\eta_i))$$

and hence

$$\overline{T(B_1^{\mathcal{H}})} \subset \bigcup_{i=1}^k \overline{B_{3\epsilon}(T(\eta_i))}$$

which gives total boundedness. Closure of $\overline{T(B_1^{\mathcal{H}})}$ is obvious. Hence we see that $\overline{T(B_1^{\mathcal{H}})}$. Thus T is compact and we see $\overline{\mathcal{F}(\mathcal{H})} \subset \mathcal{K}(\mathcal{H})$.

Conversely, let $T \in B(\mathcal{H})$ be compact and $\epsilon > 0$ be given. So $\exists \eta_1, \eta_2, \dots, \eta_m \in B_1(\mathcal{H})$ such that

$$\overline{T(B_1^{\mathcal{H}})} \subset \bigcup_{i=1}^n B_{\epsilon}(T(\eta_i))$$

Assume that \mathcal{H} is separable. Let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis for \mathcal{H} and let $P_n = P_{\text{Span}\{e_1, e_2, \dots, e_n\}}$. Then choose $N > 0$ such that for all $n > N$, $\forall i = 1, 2, \dots, m$,

$$\|(P_n T - T)(\eta_i)\| < \epsilon$$

Then $\forall \xi \in \mathcal{H}$ with $\|\xi\| = 1$. Choose $1 \leq j \leq m$ such that $\|T(\xi) - T(\eta_j)\| < \epsilon$. Then

$$\begin{aligned} \|P_n T(\xi) - T(\xi)\| &\leq \|P_n T(\xi) - P_n T(\eta_j)\| + \|P_n T(\eta_j) - T(\eta_j)\| + \|T(\eta_j) - T(\xi)\| \\ &< 3\epsilon \end{aligned}$$

Which gives that $\|P_n T - T\| < 3\epsilon$ for all $n \in \mathbb{N}$.

The generalization of the case when \mathcal{H} is non-separable, will use the fact that $T(\mathcal{H})$ is separable and using an orthonormal basis for the pre-image of $T(\mathcal{H})$, which will again be separable. \square

Proposition 3.2.1. *Let \mathcal{K} be a separable Hilbert space. Let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis for \mathcal{K} . For each $n \in \mathbb{N}$, let P_n be the projection to the $\text{Span}\{e_1, e_2, \dots, e_n\}$. Then for all $x \in \mathcal{K}$*

$$P_n(x) \rightarrow x$$

pointwise

Proof. Direct application of [Corollary 3.1.2.1\(iv\)](#) \square

Lemma 3.2.2. *If $T \in B(\mathcal{H})$ is compact, then $T(\mathcal{H})$ is separable.*

Proof. For the sake of contradiction, assume that $T(\mathcal{H})$ is not separable. Then $T(\mathcal{H})$ must have an uncountable orthonormal set E . This is because if E is a countable basis, then $\mathbb{Q}E + i\mathbb{Q}E$ will be a countable dense subset of $T(\mathcal{H})$ making it separable. [verify](#) \square

Corollary 3.2.1.1. *The set $\mathcal{K}(\mathcal{H})$ of all compact operators on \mathcal{H} is a closed two-sided ideal in $B(\mathcal{H})$.*

Proof. Use the fact that compact operators are the closure of finite rank operators. □

Corollary 3.2.1.2. *$T \in \mathcal{K}(\mathcal{H})$ implies $T^* \in \mathcal{K}(\mathcal{H})$.*

Proof. Use the fact that T is finite rank implies T^* is finite rank. □

Example 3.2.2. Let

$$T : L^2([0, 1], m) \rightarrow L^2([0, 1], m)$$

be defined as $T(f)(x) = xf(x)$. Prove that $T = T^*$ and that T has no eigenvectors.

Only for Joseph: If $\xi = \eta$ almost everywhere, then show that $f\xi = f\eta$ almost everywhere if $f \in C([0, 1])$

Proof. See homework-5 □

Example 3.2.3. Let α_n be a bounded sequence in \mathbb{C} . Consider $T : \ell^2 \rightarrow \ell^2$, such that

$$T(\delta_n) = \alpha_n \delta_n$$

for all $n \in \mathbb{N}$. Then T is bounded with $\|T\| = \|(\alpha_n)\|_\infty$.

Example 3.2.4. Prove that the T above is compact if and only if $(\alpha_n) \in c_0$. Prove that $T^*(\delta_n) = \overline{\alpha_n} \delta_n$

Proof. See homework-5 □

3.3 Spectral Theorem for Compact Normal Operators

We'll be working to prove the following theorem.

Theorem 3.3.1. *Let \mathcal{H} be separable Hilbert space and $T \in \mathcal{K}(\mathcal{H})$ be normal. i.e $TT^* = T^*T$. Then there exist an orthonormal basis $\{e_n : n \in \mathbb{N}\}$ of \mathcal{H} , and a sequence $(\alpha_n) \in c_0$, such that $T(e_n) = \alpha_n e_n$.*

Lemma 3.3.1. *Let $T \in \mathcal{K}(\mathcal{H})$ be self-adjoint. Then there exist a non-zero $\lambda \in \mathbb{C}$ such that $\text{Ker}(T - \lambda I) \neq \{0\}$.*

Proof. See the first part of the proof of [Theorem 3.3.2](#). □

Definition 3.3.1. Let $T \in B(\mathcal{H})$. A subspace $W \leq H$ is said to be invariant under T if $T(W) \subset W$. We say W reduces T , if $T(W) \subset W$ and $T(W^\perp) \subset W^\perp$

Example 3.3.1. Let $T \in B(\mathcal{H})$. Prove that $W \subset \mathcal{H}$ is invariant under T if and only if $P_W T = T P_W$. Prove that W reduces T if and only if $P_W T = T P_W$ and $P_{W^\perp} T = T P_{W^\perp}$.

Proof. See homework-5 □

Proposition 3.3.1. Let $T \in B(\mathcal{H})$ be self adjoint with $T(\mathcal{M}) \subset \mathcal{M}$, where \mathcal{M} is a closed subspace of \mathcal{H} . Then \mathcal{M} reduces T .

Proof. Let $m \in \mathcal{M}$ and $x \in \mathcal{M}^\perp$. Then

$$\langle T(x), m \rangle = \langle x, T(m) \rangle = 0$$

shows that $T(\mathcal{M}^\perp) \subset \mathcal{M}^\perp$. □

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Lemma 3.3.2. If $T \in B(\mathcal{H})$ is self adjoint. Then every eigenvalue of T is real.

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of T with eigenvector v_λ . Then

$$\lambda \|v\|^2 = \langle \lambda v_\lambda, v_\lambda \rangle = \langle T(v_\lambda), v_\lambda \rangle = \langle v_\lambda, T(v_\lambda) \rangle = \langle v_\lambda, \lambda v_\lambda \rangle = \bar{\lambda} \|v_\lambda\|^2$$

forces $\lambda = \bar{\lambda}$. □

Example 3.3.2. Let $\alpha_n \in c_0$. Define $T : \ell^2 \rightarrow \ell^2$, $T(\delta_n) = \delta_n \delta_n$ for all $n \in \mathbb{N}$. Then the set of eigenvalues of T is precisely the set of α_n s.

Lemma 3.3.3. If $T \in B(\mathcal{H})$ is self adjoint, then $\|T\| = \sup\{|\langle T\xi, \xi \rangle| : \xi \in \mathcal{H}, \|\xi\| \leq 1\}$

Proof. $\langle T\xi, \xi \rangle \leq \|T\|$ inequality is clear from Cauchy-Schwarz and definition of operator norm. To see the converse, let $\alpha, \beta \in \mathcal{H}$ with $\|\alpha\| = \|\beta\| = 1$. Then by the self-adjointness of T , we get

$$\begin{aligned} \langle T(\alpha + \beta), \alpha + \beta \rangle - \langle T(\alpha - \beta), \alpha - \beta \rangle &= 2(\langle T\alpha, \beta \rangle + \langle T\beta, \alpha \rangle) \\ &= 4\Re\langle T\alpha, \beta \rangle \end{aligned}$$

Now with enough rotation on β , we get $\langle T\alpha, \beta \rangle = \Re\langle T\alpha, \beta \rangle$. Moreover, if $A = \sup\{|\langle T\xi, \xi \rangle| : \xi \in \mathcal{H}, \|\xi\| \leq 1\}$, notice that $|\langle Tx, x \rangle| \leq A\|x\|^2$. Then we get

$$\begin{aligned} 4|\langle T\alpha, \beta \rangle| &\leq A\|\alpha + \beta\|^2 + A\|\alpha - \beta\|^2 \\ &= 2A(\|\alpha\|^2 + \|\beta\|^2) \\ &= 4A \end{aligned}$$

Thus we see that $\|T\| \leq A$. □

Lemma 3.3.4. *Let $T \in \mathcal{K}(\mathcal{H})$ and $0 \neq \lambda \in \mathbb{C}$. Then the space*

$$\text{Ker}(T - \lambda I)$$

is finite dimensional.

Proof. For the sake of contradiction, assume that $K = \text{Ker}(T - \lambda I)$ is not finite dimensional. Then $\overline{T(K \cap B_1^{\mathcal{H}})} = \overline{\lambda K \cap B_1^{\mathcal{H}}}$, is a closed non-compact ([Theorem 2.2.6](#)) subspace of $\overline{T(B_1^{\mathcal{H}})}$. This contradicts the compactness of T . \square

Theorem 3.3.2 (Spectral theorem for compact self-adjoint operators). *Let \mathcal{H} be a separable Hilbert space and $T \in \mathcal{K}(\mathcal{H})$ be self-adjoint. Then there is an orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ and a sequence $\{\alpha_n\} \in c_0$ such that*

$$Te_n = \alpha_n e_n$$

for all $n \in \mathbb{N}$.

Proof. We claim that T has an eigenvalue $\lambda \in \mathbb{R}$ ([Lemma 3.3.2](#)) such that $|\lambda| = \|T\|$. By [Lemma 3.3.3](#), $\exists(\xi_n)$ of unit vectors in \mathcal{H} such that

$$|\langle T\xi_n, \xi_n \rangle| \rightarrow \|T\|$$

Hence by taking a subsequence ξ_{n_k} , we get

$$\langle T\xi_{n_k}, \xi_{n_k} \rangle \rightarrow \lambda$$

where $\lambda = \pm\|T\|$. Since T is compact, by passing to a subsequence, we may also assume $T\xi_{n_{k_j}} \rightarrow \eta \in \mathcal{H}$. Then we have

$$\begin{aligned} \|(T - \lambda I)\xi_{n_{k_j}}\|^2 &= \|T\xi_{n_{k_j}}\|^2 + \lambda^2\|\xi_{n_{k_j}}\|^2 - 2\Re\langle T\xi_{n_{k_j}}, \lambda\xi_{n_{k_j}} \rangle \\ &\leq \|T\|^2 + \lambda^2 \end{aligned}$$

Now in the first equality above, since $2\Re\langle T\xi_{n_{k_j}}, \lambda\xi_{n_{k_j}} \rangle \rightarrow 2\lambda^2$ as $n \rightarrow \infty$, we get that $T\xi_{n_{k_j}} - \lambda\xi_{n_{k_j}} \rightarrow 0$ and since $T\xi_{n_{k_j}} \rightarrow \eta$, by the uniqueness of the limit, we get $\lambda\xi_{n_{k_j}} \rightarrow \eta$ and thus applying T to both sides, we get

$$\lambda T\xi_{n_{k_j}} \rightarrow T\eta$$

and hence we get $T\eta = \lambda\eta$. If $\eta = 0$, this will contradict our precious assumption that T is nonzero since $|\langle T\xi_n, \xi_n \rangle| \rightarrow \|T\|$.

Note that the kernel of $T - \lambda I$, H_1 is invariant under T being the eigenspace, and has finite dimension ([Lemma 3.3.4](#)). Moreover, $T(H_1^\perp) \subset H_1^\perp$ by [Proposition 3.3.1](#).

Therefore $T_1 := T|_{H_1^\perp}$ is self adjoint. Since T is compact, its restriction T_1 is compact. Hence $T_1 \in \mathcal{K}(H_1^\perp)$. Applying the claim again it follows that T_1 has an eigenvalue λ_1 such that $|\lambda_1| = \|T_1\|$.

Continuing this construction, we get $|\lambda| \geq |\lambda_1| \geq |\lambda_2| \geq \dots$ such that the $\text{Ker}(T - \lambda_i I) \perp \text{Ker}(T - \lambda_j I)$ for all $i \neq j$.

For each k (where λ_k exist), choose an orthonormal basis E_k for $\text{Ker}(T - \lambda_k I)$ and let $E = \cup E_k$. If \mathcal{H} is finite dimensional, there is nothing new to prove. Hence assume \mathcal{H} is infinite dimensional. If $\text{Ker}(T - \lambda_{n+1} I) = (\text{Ker}(T - \lambda_1 I) \oplus \text{Ker}(T - \lambda_2 I) \oplus \dots \oplus \text{Ker}(T - \lambda_n I))^\perp$, then the above lemma gives $\lambda_{n+1} = 0$. In this case E contains finitely many eigenvectors corresponding to nonzero eigenvalues and infinitely many vectors corresponding to the 0 eigenvalue. Hence we are done in this case. This happens only when T is finite rank.

Otherwise, T has infinitely many eigenvalues and therefore E is an infinite set, hence by [Example 3.2.4](#), the sequence $\lambda_n \rightarrow 0$.

It remains to show that E is an orthonormal basis for \mathcal{H} . Let F be the collection of all eigenvectors of T and let $\mathcal{M} = \overline{\text{span}}(F)$. Then $T(\mathcal{M}) \subset \mathcal{M}$, and so $T(\mathcal{M}^\perp) \subset \mathcal{M}^\perp$ by [Proposition 3.3.1](#). Thus $T|_{\mathcal{M}^\perp}$ has no non-zero eigenvectors. Thus from what we have proved in the first part, \mathcal{M}^\perp must be $\{0\}$. \square

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Lemma 3.3.5. *Every $T \in B(\mathcal{H})$ can be decomposed as a linear combination of two self-adjoint operators.*

Proof.

$$T = \frac{T + T^*}{2} + i \frac{(T - T^*)}{2i}$$

Verify that $\frac{T+T^*}{2}$ and $\frac{T-T^*}{2i}$ are self adjoint. \square

Lemma 3.3.6. *$T \in B(\mathcal{H})$ is normal if and only if $T + T^*$ and $T - T^*$ commutes*

Proof.

$$\begin{aligned} (T + T^*)(T - T^*) &= T^2 + T^*T - TT^* - T^{*2} \\ (T - T^*)(T + T^*) &= T^2 - T^*T + TT^* - T^{*2} \end{aligned}$$

\square

Lemma 3.3.7. *$T \in B(\mathcal{H})$ is normal if and only if $T = T_1 + iT_2$ where $T_1, T_2 \in S(\mathcal{H})$ and $T_1T_2 = T_2T_1$*

Proof. Directly apply above lemma \square

Lemma 3.3.8. *Let $T \in B(\mathcal{H})$, $\lambda \in \mathbb{C}$ be an eigenvalue of T , and let $S \in B(\mathcal{H})$ with $ST = TS$. Then $\text{Ker}(T - \lambda I)$ is invariant under S*

Proof. Let $M = \text{Ker}(T - \lambda I)$ and $m \in M$. Then

$$\begin{aligned} (T - \lambda I)(S(m)) &= TS(m) - \lambda S(m) \\ &= ST(m) - \lambda S(m) \\ &= S(\lambda m) - \lambda S(m) \\ &= \lambda S(m) - \lambda S(m) = 0 \end{aligned}$$

□

Lemma 3.3.9. *Let $T \in B(\mathcal{H})$, be self-adjoint. Let $\lambda_1 \neq \lambda_2 \in \sigma(T)$ and $\xi_1, \xi_2 \in \mathcal{H}$ their corresponding eigenvectors, then $\langle \xi_1, \xi_2 \rangle = 0$*

Proof.

$$0 = \langle T\xi_1, \xi_2 \rangle - \langle \xi_1, T\xi_2 \rangle = \lambda_1 \langle \xi_1, \xi_2 \rangle - \lambda_2 \langle \xi_1, \xi_2 \rangle = (\lambda_1 - \lambda_2) \langle \xi_1, \xi_2 \rangle$$

□

Theorem 3.3.3 (Spectral theorem for compact normal operators). *Let \mathcal{H} be a separable Hilbert space and $T \in \mathcal{K}(\mathcal{H})$ normal. Then there is an orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ and a sequence $(\alpha_n) \in \mathbf{c}_0$ such that*

$$Te_n = \alpha_n e_n$$

for all $n \in \mathbb{N}$.

Proof. Let $T \in \mathcal{K}(\mathcal{H})$ be normal. So $\exists S_1, S_2 \in \mathcal{K}(\mathcal{H}) \cap \mathcal{S}(\mathcal{H})$ such that $T = S_1 + iS_2$ and $S_1 S_2 = S_2 S_1$. Then by spectral theorem for self-adjoint operators, we get $\sigma(S_1)$, the set of eigenvalues of S_1 such that

$$H = \bigoplus_{\lambda \in \sigma(S_1)} \text{Ker}(S_1 - \lambda I)$$

where each $\text{Ker}(S_1 - \lambda I)$ is finite dimensional.

Since S_2 commutes with S_1 , $\text{Ker}(S_1 - \lambda I)$ is invariant under S_2 by [Lemma 3.3.8](#), and $S_2|_{\text{Ker}(S_1 - \lambda I)}$ is self-adjoint by [Proposition 3.3.1](#) and compact. Thus, by the first part of the proof, for each $\lambda \in \sigma(S_1)$, we can choose an orthonormal basis E_λ for $\text{Ker}(S_1 - \lambda I)$ consisting of eigenvectors of S_2 .

Observe that if $\xi \in \mathcal{H}$ is such that $S_1 \xi = \lambda \xi$ and $S_2 \xi = \beta \xi$, then

$$T\xi = (\lambda + i\beta)\xi$$

Now let $E = \bigcup_{\lambda \in \sigma(S_1)} E_\lambda$. Then E is an orthonormal basis for \mathcal{H} consisting of eigenvectors of T . □

Lemma 3.3.10. *Let V be a vector space and $f_1, f_2, \dots, f_n : V \rightarrow \mathbb{C}$ linear. Then $f \in \text{span}\{f_1, f_2, \dots, f_n\}$ iff*

$$\bigcap_{k=1}^n \text{Ker}(f_k) \subset \text{Ker}(f)$$

Proof. If $f \in \text{span}\{f_1, f_2, \dots, f_n\}$ and if $x \in \text{Ker}(f_i)$ for each $1 \leq i \leq n$, then clearly $f(x) = 0$. Conversely **verify**. \square

Lemma 3.3.11.

$$(X, \text{weak})^* = X^*$$

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Given T as above, denote by $\sigma(T)$, the set of eigenvalues of T and for each eigenvalue $\lambda \in \sigma(T)$, denote $P_\lambda := P_{\text{Ker}(T - \lambda I)}$. Then spectral theorem gives

$$T = \sum_{\lambda \in \sigma(T)} \lambda P_\lambda$$

Definition 3.3.2. Let Ξ be a non-empty subset of $B(\mathcal{H})$. The commutant of Ξ , is the set $\Xi' = \{S \in B(\mathcal{H}) : ST = TS, \forall T \in \Xi\}$

Exercise 3.3.1. 1. Prove that for any set Ξ' is a closed subspace of $B(\mathcal{H})$.

2. $\Xi' = \text{span}(\Xi)'$

Proof. (1) Let (S_n) be a sequence in Ξ' such that $S_n \rightarrow S$ in $B(\mathcal{H})$. Then by the continuity of the map $m_T : B(\mathcal{H}) \rightarrow B(\mathcal{H}) : R \rightarrow TR$ as $\|m_T(R)\| \leq \|T\| \|R\|$, we see that $S_n T \rightarrow ST$ and similarly by the continuity of $m'_T : S \rightarrow TS$, we see that $TS_n \rightarrow TS$. Then by the algebra of limits, for $T \in \Xi$, we see that

$$0 = S_n T - TS_n \rightarrow ST - TS$$

which forces $S \in \Xi'$.

(2)

\square

3.4 Functional Calculus

Theorem 3.4.1 (Functional calculus for compact normal operators). *Let $T \in B(\mathcal{H})$ be compact normal with spectral decomposition*

$$T = \sum_{\lambda \in \sigma(T)} \lambda P_\lambda.$$

For each $f \in L^\infty(\mathbb{C})$, define $f(T) := \sum_{\lambda \in \sigma(T)} f(\lambda) P_\lambda$. Then

- (1) $\forall f \in \ell^\infty(\mathbb{C}), \|f(T)\| = \sup\{|f(\lambda)| : \lambda \in \sigma(T)\}$
- (2) The map $\ell^\infty(\mathbb{C}) \rightarrow B(\mathcal{H})$ defines as $f \rightarrow f(T)$ is linear.
- (3) $\forall f, g \in \ell^\infty(\mathbb{C}), (fg)(T) = f(T)g(T)$
- (4) $\overline{f}(T) = f(T)^*$
- (5) $\{f(T) : f \in \ell^\infty(\mathbb{C})\} = \{T\}''$

Proof. (1) **exercise**

(2) **exercise**

(3) **exercise**

(4) **exercise**

(5) Let $f \in \ell^\infty(\mathbb{C})$, and let $S \in \{T\}'$. Since $ST = TS$, by **Lemma 3.3.8** for every $\lambda \in \sigma(T)$, the eigenspace $\text{Ker}(T - \lambda I)$ is invariant under S . Hence they are all reducing for S . i.e $SP_\lambda = P_\lambda S$. Thus,

$$\begin{aligned}
 Sf(T) &= S \sum_{\lambda \in \sigma(T)} f(\lambda) P_\lambda \\
 &= \sum_{\lambda \in \sigma(T)} f(\lambda) SP_\lambda \\
 &= \sum_{\lambda \in \sigma(T)} f(\lambda) P_\lambda S \\
 &= f(T)S
 \end{aligned}$$

So we get $f(T) \in \{T\}''$

Let $R \in \{T\}''$. Note that $\forall \lambda \neq \lambda' \in \sigma(T)$, $P_\lambda P_{\lambda'} = 0$ by **Lemma 3.3.9**. Then for all $\gamma \in \sigma(T)$, we get

$$P_\gamma T = P_\gamma \sum_{\lambda \in \sigma(T)} \lambda P_\lambda = \gamma P_\gamma$$

So $P_\gamma \in \{T\}'$. Thus $RP_\gamma = P_\gamma R$ for all $\gamma \in \sigma(T)$.

Fix $\gamma \in \sigma(T)$. Given $A \in B(P_\gamma(\mathcal{H}))$. Let $S := P_\gamma A P_\gamma : \mathcal{H} \rightarrow \mathcal{H}$. Then

$$\begin{aligned}
 ST &= P_\gamma A P_\gamma \sum_{\lambda \in \sigma(T)} \lambda P_\lambda \\
 &= \gamma P_\gamma A P_\gamma \\
 &= \gamma S \\
 &= TS
 \end{aligned}$$

Thus $RS = SR$. This shows that $R|_{P_\gamma(\mathcal{H})}$ commutes with every $A \in B(P_\gamma(\mathcal{H}))$. Hence there exist such a scalar $f(\gamma) \in \mathbb{C}$ such that $R|_{P_\gamma(\mathcal{H})} = f(\gamma)I_{P_\gamma(\mathcal{H})}$. Thus $R = \sum_{\lambda \in \sigma(T)} f(\lambda)P_\lambda$ and $f \in \ell^\infty(\mathbb{C})$.

Corollary 3.4.1.1. *The mapping above restricted to $\sigma(T)$ is an isometric bijective linear multiplicative $*$ -preserving map.*

Definition 3.4.1. Let $T \in B(\mathcal{H})$. Define $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}$

□

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Definition 3.4.2. A Banach algebra \mathcal{A} is a Banach space with a multiplication ring with addition satisfying $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in \mathcal{A}$. We say \mathcal{A} is unital if the ring above is unital with multiplicative identity $1_{\mathcal{A}}$. Units (invertible elements) may also exist similarly.

Definition 3.4.3. Given a Banach algebra \mathcal{A} , and $a \in \mathcal{A}$, the spectrum of a , denoted by

$$\sigma(a) = \{\lambda \in \mathbb{C} : \lambda 1_{\mathcal{A}} - a \text{ is not invertible in } \mathcal{A}\}$$

Example 3.4.1. Let $(\alpha_n) \in \mathbf{c}_0$ and $T : \ell^2 \rightarrow \ell^2$ such that $T((x_n)) = (\alpha_n x_n)$. We claim that

$$\sigma(T) = \{\alpha_n : n \in \mathbb{N}\} \cup \{0\}$$

Proof. Since T is a compact operator (see Homework-5), we must have $0 \in \sigma(T)$. Otherwise if T is invertible, we'll get $I = T \circ T^{-1}$ be also compact, which is a contradiction since ℓ^2 is infinite dimensional. Moreover since $(T - \alpha_n I)(e_n) = 0$, we have $\alpha_n \in \sigma(T)$.

Now assume that $\beta \notin \{\alpha_n : n \in \mathbb{N}\} \cup \{0\}$. Then let $S \in B(\ell^2)$ defined by

$$S(e_n) = \frac{1}{\alpha - \beta} e_n$$

Then **show that indeed $S \in B(\ell^2)$** and $(T - \beta I)S = S(T - \beta I) = I$ □

Example 3.4.2. Let $T \in B(L^2([0, 1]))$ such that $T(f)(x) = xf(x)$ for all $f \in L^2([0, 1])$. Then $\sigma(T) = [0, 1]$.

Proof. First let us see that $0 \in \sigma(T)$. Suppose $S = T^{-1}$ exists. Then $\forall n \in \mathbb{N}$ if $f = S\chi_{[0, 1]}$, then $Tf = \chi_{[0, 1]}$. So $xf(x) = \chi_{[0, 1]}$. But this is absurd, since $\frac{1}{x}\chi_{[0, 1]} \notin L^2([0, 1])$. □

Lemma 3.4.1. *Let $S \in B(\mathcal{H})$ such that $\|S - I\| \leq 1$. Then S is invertible.*

Proof. Since $\|S - I\| \leq 1$,

$$\sum_{n \in \mathbb{N}} \|(S - I)^n\| \leq \sum_{n \in \mathbb{N}} \|S - I\|^n$$

Moreover since $B(\mathcal{H})$ is a Banach space, absolutely convergent sequences converge and this gives that $\sum_{n \in \mathbb{N}} (I - S)^n$ converges. Thus

$$R = \sum_{n=1}^{\infty} (I - S)^n$$

exists. Now for each $N \in \mathbb{N}$, we have

$$\begin{aligned} S \left(\sum_{n=0}^N (I - S)^n \right) &= (I - (I - S)) \left(\sum_{n=1}^N (I - S)^n \right) = \sum_{n=0}^n (I - S)^n - \sum_{n=0}^N (I - S)^{n+1} \\ &= I - (I - S)^{N+1} \end{aligned}$$

which converges to 0 as $n \rightarrow \infty$. □

Corollary 3.4.1.2. *The set of invertible operators is open in $B(\mathcal{H})$.*

Proof. Let S be invertible. Then

$$\|T - S\| = \|S(S^{-1}T - I)\| \leq \|S\| \|S^{-1}T - I\|$$

□

Theorem 3.4.2. *For any $T \in B(\mathcal{H})$, $\sigma(T)$ is a non-empty, compact subspace of \mathbb{C} .*

Proof. Observe that the function $f : \lambda \rightarrow T - \lambda I$ is continuous. Then $f^{-1}(G(\mathcal{A}))$ is open, where $G(\mathcal{A})$ is the collection of all invertible elements of $B(\mathcal{H})$. But $\sigma(T)^c = f^{-1}(G(\mathcal{A}))$. So we see $\sigma(T)$ is closed. Moreover let $\lambda \in \mathbb{C}$ with $\|T\| < |\lambda|$. Then,

$$-T + \lambda I = \lambda \left(\frac{-T}{\lambda} + I \right)$$

Then $\|S - I\| = \left\| \frac{T}{\lambda} \right\| = \frac{\|T\|}{|\lambda|} < 1$ Hence show that $\sigma(T)$ is bounded. □

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Example 3.4.3. Let $T : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$ be defined as $T\delta_1 = 0$ and $T\delta_{n+1} = \delta_n$. Then clearly $(\delta_1, 0)$ is an eigenvector-eigenvalue pair. Moreover for any $\lambda \in \mathbb{C}$ with $|\lambda| < 1$. Then

$$\begin{aligned} T : (1, \lambda, \lambda^2, \lambda^3, \dots) &\rightarrow (\lambda, \lambda^2, \lambda^3, \dots) \\ &= \lambda(1, \lambda, \lambda^2, \lambda^3, \dots) \end{aligned}$$

Thus we see that the open unit disk $\mathbb{D} \subset \sigma(T)$. Moreover since $\|T\| = 1$, we see that $\sigma(T) = \overline{\mathbb{D}}$.

Proposition 3.4.1.

$$\sigma(T^*) = \overline{\sigma(T)}$$

Proof. Use the fact that $(T^{-1})^* = (T^*)^{-1}$ and resolvent. □

Example 3.4.4. Now consider $S : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$, defined as $S\delta_n = \delta_{n+1}$. Then $\sigma(S) = \overline{\mathbb{D}} = \overline{\sigma(T)}$ since $S = T^*$ and $\sigma(S) = \overline{\sigma(T)}$.

Theorem 3.4.3 (Spectral Mapping Theorem). *Let $T \in B(\mathcal{H})$, and let $\phi : \mathbb{C} \rightarrow \mathbb{C}$ be holomorphic in an open neighborhood of $\sigma(T)$. Then, $\phi(\sigma(T)) = \sigma(\phi(T))$.*

Let $T \in B(\mathcal{H})$ be self adjoint. Consider the set

$$\mathcal{A} = \overline{\{p(T) : p(x) \text{ is a polynomial}\}}^{\|\cdot\|}$$

Then observe that \mathcal{A} is a closed algebra of $B(\mathcal{H})$. Then for any $0 \neq f \in \mathcal{A}^*$, assume that $f(ab) = f(a)f(b)$ and $a \in \text{Ker}(f)$ and $b \in \mathcal{A}$, then $ab \in \text{Ker}(f)$. In particular let $b = f(a)I = a \in \mathcal{A}$. Then $f(b) = 0 \implies f(a)I - a \in \text{Ker}(f)$. Since $f \neq 0$, $1 \notin \text{Ker}(f)$ and hence $f(a) \in \sigma(a)$

Conversely, let $\lambda \in \sigma(a)$. Thus $\lambda I - a$ is not invertible. Thus the ideal generated by $\lambda I - a$ is not the whole of \mathcal{A} . Then by Zorn's lemma, there is a maximal ideal \mathcal{M} , which contain $\lambda I - a$.

Exercise 3.4.1. The set $\{0 \neq \phi \in \mathcal{A}^* : \phi(ab) = \phi(a)\phi(b)\}$ is weak * compact.

Proof. **Homework** □

Exercise 3.4.2. Maximal ideals of $B(\mathcal{H})$ are closed

Proof. **Homework.** Show that closure of an ideal is an ideal. □

Define $f : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{M} \cong \mathbb{C}$, defined as $f(a) = a + \mathcal{M}$.

Theorem 3.4.4 (Gelfand Transform).