Midterm exam: Monday, Oct. 14, in class, 50 mins.

5 questions

Q1] General knowledge of groups.

How to tell when groups are not isomorphic.

Q2] Prove that two groups are isomorphic.

Q3] Prove or disprove the following statement:

(something similar to one of the problems on A4)

Q4] Know props. of homomorphisms.

Q5] Know def. of $C_G(S)$.

This question is based on the notes on internal products.

Hints for A5]

7. Let M and N be normal subgroups of G such that $G = MN$. Prove that $G/(M \cap N) \cong (G/M) \times (G/N)$. [Draw the lattice.]

$$
\begin{tikzcd}[column sep=small]
 & G=MN \arrow[dl, dash] \arrow[dr, dash] & \\
M \arrow[dr, dash] & & N \arrow[dl, dash] \\
 & M\cap N &
\end{tikzcd}
$$

Idea 1:

Find a homom.

$\phi : G \to G/M \times G/M$, show that $\ker\phi = M \cap N$.

Try $\phi(g) = (gM, gN)$.

- ϕ is a hom. ✓
- $\ker\phi = M\cap N$ ✓
- ϕ is onto (the hard part)

Idea 2: By 2nd Isom. thm: $G/M \cong N/M\cap N$

and $G/N \cong M/M\cap N$

"Define" $\phi : G \to N/M\cap N \times M/M\cap N$,

by $\phi(mn) = (n(M\cap N), m(M\cap N))$.

- well-def? N.T.S:

  If $m_1 n_1 = m_2 n_2$ then

  $n_1(M \cap N) = n_2(M \cap N)$ and $m_1(M \cap N) = m_2(M \cap N)$.

- hom?

hard.

Easy: $\ker\phi = M \cap N$, ϕ is onto.

9. Let p be a prime and let G be a group of order $p^a m$, where p does not divide m. Assume P is a subgroup of G of order p^a and N is a normal subgroup of G of order $p^b n$, where p does not divide n. Prove that $|P \cap N| = p^b$ and $|PN/N| = p^{a-b}$. (The subgroup P of G is called a *Sylow p-subgroup* of G. This exercise shows that the intersection of any Sylow p-subgroup of G with a normal subgroup N is a Sylow p-subgroup of N.)

$|G| = p^a m$, $p \nmid m$, $|P| = p^a$, $|N| = p^b n$, $p \nmid n$

$$\begin{tikzcd}[column sep=small]
 & G \arrow[d, dash] & \\
 & PN \arrow[dl, dash] \arrow[dr, dash] & \\
P \arrow[dr, dash] & & N \arrow[dl, dash] \\
 & P\cap N &
\end{tikzcd}$$

$$PN/N \cong P/P\cap N$$

$$\Rightarrow |PN| = \frac{|P||N|}{|P\cap N|} = \frac{p^{a+b} n}{|P\cap N|}$$

use this to deduce that $|P\cap N| = p^b$.

8. Let A be a finite abelian group (written multiplicatively) and let p be a prime. Let

$$A^p = \{a^p \mid a \in A\} \qquad \text{and} \qquad A_p = \{x \mid x^p = 1\}$$

(so A^p and A_p are the image and kernel of the p^{th}-power map, respectively).

(a) Prove that $A/A^p \cong A_p$. [Show that they are both elementary abelian and they have the same order.]

(b) Prove that the number of subgroups of A of order p equals the number of subgroups of A of index p. [Reduce to the case where A is an elementary abelian p-group.]

Part b)

Step 1: · Prove that the # of subgroups of order p in A is equal to the # of subgroups of order p in A_p.

· Prove that # of subgroups of index p in A is equal to the # of subgroups of index p in A/A^p.

By (a), this reduces the problem to the case when A is elementary p-group.

Step 2: Count both quantities, when A is an elem. p-group.