3) K be split field of x^3-2 over $\mathbb{Q}$.

$$x^3-2 = \prod_{i=0}^{2} (x - 2^{1/3}\zeta_3^i).$$

$$2^{1/3},\ \zeta_3 = \frac{2^{1/3}\zeta_3}{2^{1/3}} \in K \Rightarrow K = \mathbb{Q}(2^{1/3}, \zeta_3) = \mathbb{Q}(2^{1/3})\mathbb{Q}(\zeta_3)$$

Also, $[K:\mathbb{Q}] \leq 3\cdot 2 = 6$, but $2, 3 \mid [K:\mathbb{Q}]$

$$\Rightarrow [K:\mathbb{Q}] = 6$$

$$\Rightarrow |\mathrm{Gal}(K/\mathbb{Q})| = 6.$$

To determine $\mathrm{Gal}(K/\mathbb{Q})$:

$$\min_{\mathbb{Q}}(2^{1/3}) = x^3-2$$

$$\min_{\mathbb{Q}}(\zeta_3) = \Phi_3(x) = x^2+x+1,$$

So any element of $\mathrm{Gal}(K/\mathbb{Q})$ must satisfy

$$2^{1/3} \mapsto \begin{cases} 2^{1/3} \\ 2^{1/3}\zeta_3 \\ 2^{1/3}\zeta_3^2 \end{cases}, \quad \zeta_3 \mapsto \begin{cases} \zeta_3 \\ \zeta_3^2 \end{cases}.$$

Since there are 6 possibilities, all choices must extend to auts.

Let $\sigma, \tau \in \mathrm{Gal}(K/\mathbb{Q})$ be determined by

$$\sigma : \begin{cases} 2^{1/3} \mapsto 2^{1/3}\zeta_3 \\ \zeta_3 \mapsto \zeta_3 \end{cases} , \quad \tau : \begin{cases} 2^{1/3} \mapsto 2^{1/3} \\ \zeta_3 \mapsto \zeta_3^2 \end{cases}$$

Then $|\sigma| = 3$, $|\tau| = 2$,

$$(\sigma\tau)(\sqrt[3]{2}) = \sigma(\tau(2^{1/3})) = \sigma(2^{1/3}) = 2^{1/3}\zeta_3$$

$$(\sigma\tau)(\zeta_3) = \sigma(\tau(\zeta_3)) = \sigma(\zeta_3^2) = (\sigma(\zeta_3))^2 = \zeta_3^2$$

$$(\tau\sigma)(\sqrt[3]{2}) = \tau(\sigma(\sqrt[3]{2})) = \tau(2^{1/3}\zeta_3) = \tau(2^{1/3})\tau(\zeta_3) = 2^{1/3}\zeta_3^2$$

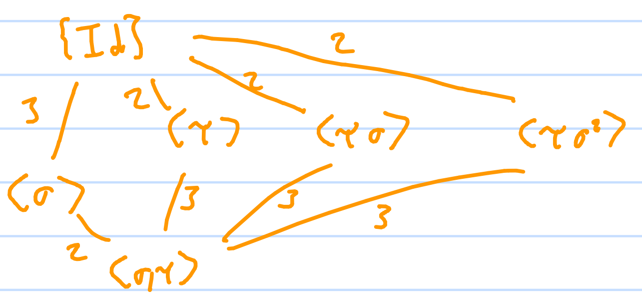
$$(\tau\sigma)(\zeta_3) = \tau(\sigma(\zeta_3)) = \tau(\zeta_3) = \zeta_3^2$$

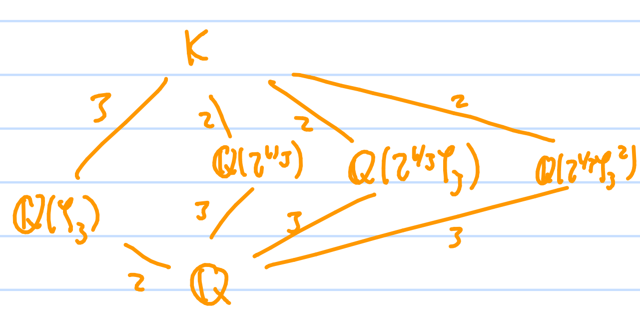
Bonus computations: $\tau\sigma^2 : \begin{cases} 2^{1/3} \xmapsto{\sigma} 2^{1/3}\zeta_3 \xmapsto{\sigma} 2^{1/3}\zeta_3^2 \xmapsto{\tau} 2^{1/3}\zeta_3 \\ \zeta_3 \mapsto \zeta_3 \mapsto \zeta_3 \mapsto \zeta_3^2 \end{cases}$

So $\sigma\tau = \tau\sigma^2$.

$\mathrm{Gal}(K/\mathbb{Q}) \cong S_3$.

Intermediate fields of $K/\mathbb{Q}$:





Scratch work:

$$(\tau\sigma)(2^{1/3}\zeta_3) = (2^{1/3}\zeta_3^2)(\zeta_3^2) = 2^{1/3}\zeta_3$$

$$(\tau\sigma^2)(2^{1/3}\zeta_3^2) = (2^{1/3}\zeta_3)(\zeta_3^2)^2 = 2^{1/3}\zeta_3^2$$

4) $K =$ spl. field of $x^4 - 2$ over $\mathbb{Q}$

$K = \mathbb{Q}(2^{1/4}, i)$, $[K:\mathbb{Q}] = 8$ (hwk)

$\min_{\mathbb{Q}}(2^{1/4}) = x^4 - 2 = (x - 2^{1/4})(x + 2^{1/4})(x - i2^{1/4})(x + i2^{1/4})$

$\min_{\mathbb{Q}}(i) = x^2 + 1 = (x+i)(x-i)$

Let $\sigma, \tau \in \mathrm{Gal}(K/\mathbb{Q})$ be determined:

$$\sigma : \begin{cases} 2^{1/4} \mapsto 2^{1/4} i \\ i \mapsto i \end{cases}, \qquad \tau : \begin{cases} 2^{1/4} \mapsto 2^{1/4} \\ i \mapsto -i. \end{cases}$$

Now $|\sigma| = 4$, $|\tau| = 2$.

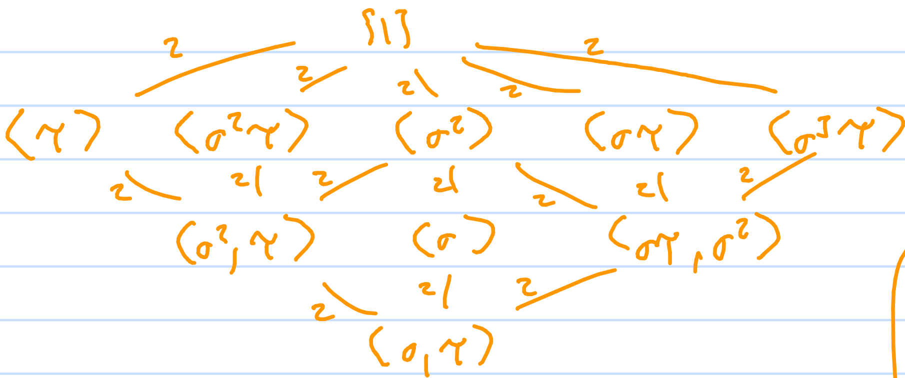
$$\sigma\tau : \begin{cases} 2^{1/4} \overset{\tau}{\longmapsto} 2^{1/4} \overset{\sigma}{\longmapsto} 2^{1/4} i \\ i \longmapsto -i \longmapsto -i \end{cases}$$

$$\tau\sigma : \begin{cases} 2^{1/4} \overset{\sigma}{\longmapsto} 2^{1/4} i \overset{\tau}{\longmapsto} -2^{1/4} i \\ i \longmapsto i \longmapsto -i \end{cases}$$

$$\tau\sigma^3 : \begin{cases} 2^{1/4} \overset{\sigma^2}{\longmapsto} -2^{1/4} \overset{\sigma}{\longmapsto} -2^{1/4} i \overset{\tau}{\longmapsto} 2^{1/4} i \\ i \longmapsto i \overset{\sigma}{\longmapsto} i \longmapsto -i \end{cases}$$

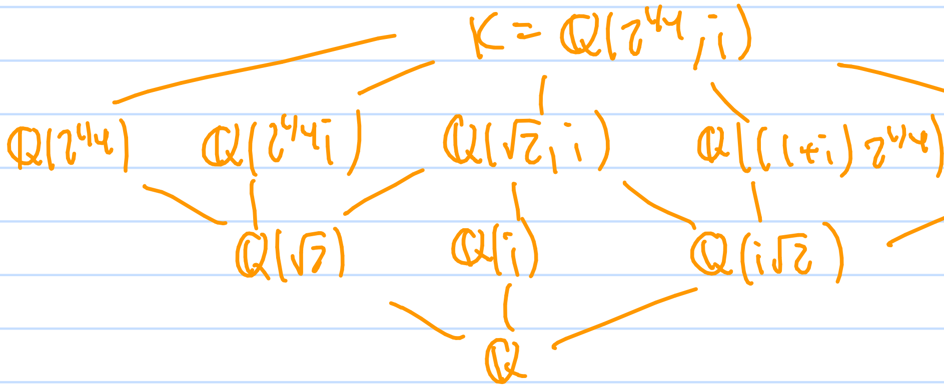
So $\sigma\tau = \tau\sigma^3$, and $\mathrm{Gal}(K/\mathbb{Q}) \cong D_8$.

Lattice of intermediate groups / fields:



$$\sigma: 2^{1/4} \mapsto 2^{1/4} i, \quad i \mapsto i$$

$$\tau: 2^{1/4} \mapsto 2^{1/4}, \quad i \mapsto -i$$



Scratch: $\sigma^2(2^{1/2}) = \sigma(\sigma((2^{1/4})^2)) = \sigma\left((\sigma(2^{1/4}))^2\right) = \sigma(-2^{1/2})$

$= -(\sigma(2^{1/4}))^2 = 2^{1/2}.$