MATH6321 - Theory of functions of a real variable
Homework 7

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March 27, 2025

1. **Solution:** Let $A \in \mathcal{M}$, the σ-algebra of Lebesgue measurable sets. If $\nu(A) = 0$, then $A = \infty$, since ν is the counting measure. Then $\mu(A) = \mu(\emptyset) = 0$. This shows that $\mu \ll \nu$. Moreover, $\mu([0,1]) = 1$ shows that μ is bounded.

   For the sake of contradiction, assume that $d\mu = hd\nu$ for some $h \in L^1(\nu)$. Then for any measurable function f

   $$\int f \; d\mu = \int fh \; d\nu$$

   Specifically for $f = \chi_{\{a\}}$ for some $a \in [0,1]$, we get that

   $$0 = \mu(\{a\}) = \int \chi_{\{a\}} \; d\mu = \int \chi_{\{a\}} h \; d\nu = h(a)$$

   This implies $h = 0$ everywhere in $[0,1]$ forcing $\mu = 0$, which is a contradiction.

2. **Solution:** Given $\nu \ll \lambda \ll \mu$ and let $g_1 = \frac{d\lambda}{d\mu} \in L^1(\mu), g_2 = \frac{d\nu}{d\lambda} \in L^1(\lambda)$. Then for any ν-measurable f, we have

   $$\int f \; d\nu = \int f g_2 \; d\lambda = \int f g_2 g_1 \; d\mu \tag{1}$$

   Now by the uniqueness criterion in the Radon-Nikodyn theorem, we'll be done if we show that $g_2 g_1 \in L^1(\mu)$. For that, choose

   $$f(x) = \begin{cases} 0, & \text{if } g_2(x)g_1(x) = 0 \\ \frac{\overline{g_2(x)g_1(x)}}{|g_2(x)g_1(x)|}, & \text{otherwise} \end{cases}$$

Then $\|f\|_\infty \leq 1$ and f is measurable (wrt μ, ν, λ). Hence by Equation 1 and the finiteness of the measure ν,

$$
\begin{aligned}
\left| \int |g_2 g_1| \, d\mu \right| &= \left| \int f g_2 g_1 \, d\mu \right| \\
&= \left| \int f \, d\nu \right| \\
&\leq \int |f| \, d\nu \\
&\leq \int \|f\|_\infty \, d\nu \\
&\leq \nu(X) < \infty
\end{aligned}
$$

Thus $g_2 g_1 \in L^1(\mu)$ and we are done.

Now for the special case, when $\mu = \nu$, Equation 1 becomes

$$
\int f \, d\nu = \int f g_2 \, d\lambda = \int f g_2 g_1 \, d\nu
$$

and the uniqueness in the Radon-Nikodyn theorem for $\nu \ll \nu$ forces $g_2 g_1 = 1$ almost everywhere in X. Hence we get that $g_1 = g_2^{-1}$ ν-almost everywhere. Since the measure zero sets of λ and ν agree, we see that $g_1 = g_2^{-1}$ λ-almost everywhere.

3. **Solution:** Given that $\hat{\mu}(n) := \int e^{-int} \, d\mu \to 0$ as $n \to \infty$. Let $f(t) = e^{imt}$ for some $m \in \mathbb{Z}$. Then

$$
\begin{aligned}
\int e^{-int} f \, d\mu &= \int e^{-int} e^{imt} \, d\mu \\
&= \int e^{-i(n-m)t} \, d\mu
\end{aligned}
$$

And clearly $\int e^{-int} f \, d\mu \to 0$ as $n \to \infty$. By linearity of the integral, this is true if f is any trigonometric polynomial.

Now, let $f \in C([0, 2\pi])$. By stone Weierstrass theorem, for any $\varepsilon > 0$, there is a trigonometric polynomial g such that $\|f-g\|_\infty < \frac{\varepsilon}{2|\mu|([0,2\pi])}$. Then

$$\begin{aligned}\left|\int e^{-int}(f-g)\ d\mu\right| &\leq \int |e^{-int}||f-g|\ d\mu \\ &\leq \frac{\varepsilon}{2|\mu|([0,2\pi])}\mu([0,2\pi]) \\ &\leq \frac{\varepsilon}{2}\end{aligned}$$

Thus by the linearity of the intergral

$$\left|\int e^{-int}f\ d\mu\right| \leq \left|\int e^{-int}g\ d\mu\right| + \frac{\varepsilon}{2}$$

Since $g \in C([0,2\pi])$ and we know that $\int e^{-int}g\ d\mu \to 0$, there is a $N \in \mathbb{N}$ such that for all $n > N$,

$$\left|\int e^{-int}g\ d\mu\right| < \frac{\varepsilon}{2}$$

Thus we see that

$$\left|\int e^{-int}f\ d\mu\right| \leq \left|\int e^{-int}g\ d\mu\right| + \frac{\varepsilon}{2} \leq \varepsilon$$

But since $\varepsilon > 0$ was chosen arbitrarily, this shows that

$$\int e^{-int}f\ d\mu \overset{n\to\infty}{\longrightarrow} 0$$

Thus we see that for any continuous function $f \in C([0,2\pi])$, we have

$$\int e^{-int}f\ d\mu \overset{n\to\infty}{\longrightarrow} 0$$

Now, let f be any bounded ($\|f\|_\infty < M$) μ-measurable function. Then by Lusin's theorem, for any $\varepsilon > 0$, there is a continuous function $g \in C([0,1])$ such that

$$\mu(\{f \neq g\}) < \frac{\varepsilon}{4M} \quad \text{and} \quad \|g\|_\infty \leq \|f\|_\infty < M$$

Then

$$\begin{aligned}\Big|\int e^{-int}f\ d\mu - \int e^{-int}g\ d\mu\Big| = \Big|\int e^{-int}(f-g)\ d\mu\Big| &\le \int \chi_{\{f\neq g\}}|f-g|\ d\mu \\ &\le \|f-g\|_\infty \int \chi_{\{f\neq g\}}\ d\mu \\ &= 2M\mu(\{f\neq g\}) \\ &< 2M\frac{\varepsilon}{4M} \\ &= \frac{\varepsilon}{2}\end{aligned}$$

shows that

$$\int e^{-int}f\ d\mu \le \int e^{-int}g\ d\mu + \frac{\varepsilon}{2}$$

Since we know $\int e^{-int}g\ d\mu \to 0$, for appropriate choice of n, we get

$$\int e^{-int}f\ d\mu \le \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, this gives that for any bounded measurable function f,

$$\int e^{-int}\ d\mu \xrightarrow{n\to\infty} 0 \quad \implies \quad \int e^{-int}f\ d\mu \xrightarrow{n\to\infty} 0 \tag{2}$$

By polarization identity $d\mu = hd|\mu|$, where $|h| = 1$ almost everywhere in $[0, 2\pi]$. By Equation 2 for $f = \bar{h}^2$, we get

$$\int e^{-int}\bar{h}^2\ d\mu \to 0$$

Since $|h| = 1$ almost everywhere, $\bar{h}h = 1$ almost everywhere and thus,

$$\overline{\int e^{-int}\bar{h}^2\ d\mu} = \overline{\int e^{-int}\bar{h}\ d|\mu|} = \int e^{int}h\ d|\mu| = \int e^{int}\ d\mu \xrightarrow{n\to\infty} \bar{0} = 0$$