

## MATH 6320

Theory of Functions of a Real Variable  
Fall 2024

First name: \_\_\_\_\_ Last name: \_\_\_\_\_

Points:

**Assignment 1, due Thursday, August 29, 11:30am**

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

**Problem 1**

Let  $X$  be an uncountable set. Let  $\mathcal{M}$  denote the collection of subsets containing each  $E \subset X$  such that  $E$  is at most countable (including  $E = \emptyset$ ) or  $E^c \subset X$  and  $E$  is at most countable. Is  $\mathcal{M}$  a  $\sigma$ -algebra?

**Problem 2**

Let  $X$  be a measurable space,  $Y$  a set and consider a map  $f : X \rightarrow Y$ . Show that

$$\mathcal{M} := \{E \subset Y : f^{-1}(E) \text{ is a measurable subset of } X\}$$

defines a  $\sigma$ -algebra of subsets of  $Y$ .

**Problem 3**

Let  $\mathcal{M}$  be a  $\sigma$ -algebra on a set  $X$ . Show that if  $\mathcal{M}$  is not finite, then it is not countably infinite.

Hint: Assuming  $\mathcal{M}$  is at most countable, then for each  $x \in X$ , consider  $B_x = \bigcap_{E \in \mathcal{M}: x \in E} E$ , the "smallest" element in  $\mathcal{M}$  containing  $x$ . Show that for any  $x, y \in X$ , either  $B_x = B_y$  or  $B_x \cap B_y = \emptyset$ . Next, observe that for each  $E \in \mathcal{M}$ ,  $E = \bigcup_{x \in E} B_x$ . Use this to argue that if  $\mathcal{M}$  is infinite, then so is  $\{B_x : x \in X\}$ . Choose  $x_1, x_2, \dots \in X$  such that  $A_j = B_{x_j}$  forms a mutually disjoint sequence of non-empty sets in  $\mathcal{M}$ . Next, consider the map  $i : \{0, 1\}^{\mathbb{N}} \rightarrow \mathcal{M}$ ,  $i(a) = \bigcup_{j: a_j = 1} A_j$  that associates with a binary sequence the corresponding union from the sequence of sets. Why is  $i$  one-to-one? You may then quote that the set of binary sequences is uncountable.