Lecture Notes in Measure Theory

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Introduction

This is a set of lecture notes which I took for reviewing stuff that I typed after taking class from *Dr. Sachindranath Jayaraman.* All the typos and errors are of mine.

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§1 Lecture 1 — 10th August 2022 — Review of things done in the previous semester. . .

§1.1 Definitions and Some Results

Definition §1.1.1 (algebra)**.** Let Ω be nonempty set. An algebra $\mathscr{F}$ is a collection of subsets of Ω satisfying the following properties:

1. $\Omega \in \mathscr{F}$,
2. $A \in \mathscr{F} \Rightarrow A^c \in \mathscr{F}$ and
3. $\mathscr{F}$ is closed under finite unions.

It immediately follows from the definition an algebra of sets is closed under taking finite intersections.

Definition §1.1.2 (σ-algebra)**.** Let Ω be nonempty set. A σ-algebra $\mathscr{F}$ is a collection of subsets of Ω satisfying the following properties:

1. $\Omega \in \mathscr{F}$,
2. $A \in \mathscr{F} \Rightarrow A^c \in \mathscr{F}$ and
3. $\mathscr{F}$ is closed under countable unions.

Fact §1.1.3. *Let Ω be a set, $\mathscr{F} \subseteq \mathcal{P}(\Omega)$. $\mathscr{F}$ is an σ-algebra iff $\mathscr{F}$ is an algebra that is continuous from below, that is, if $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathscr{F}$ and $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$ then $\bigcup_n A_n \in \mathscr{F}$.*

Definition §1.1.4 (σ-algebra generated by a subset of power set)**.** Let Ω be a nonempty set. Given an nonempty collection $\mathcal{C}$ of subsets of Ω, the σ-algebra generated by $\mathcal{C}$, $\sigma(\mathcal{C})$ is defined to be the intersection of all σ-algebra containing $\mathcal{C}$. Notationally,

$$\sigma\left(\mathcal{C}\right) = \bigcap \{\sigma - \text{algebra that contains } \mathcal{C}\}$$

Definition §1.1.5 (Borel σ-algebra)**.** If Ω is a topological space then the Borel σ-algebra is the smallestσ-algebra containing the open sets of Ω.

Fact §1.1.6. *If $\Omega = \mathbb{R}^n$ the Borel σ-algebra is generated by*

- $\{(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n) \mid -\infty \leq a_i < b_i \leq b_i \leq +\infty\}$
- $\{(-\infty, a_1) \times (-\infty, a_2) \times \cdots (-\infty, a_n) \mid a_1, a_2, \ldots, a_n \in \mathbb{R}\}$
- $\{(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n) \mid a_i, b_i \in \mathbb{Q}\}$

Definition §1.1.7 (π-system, λ-system)**.** A collection $\mathcal{C}$ of subsets of Ω is called a π-system if $\mathcal{C}$ is closed under finite-$\cap$.

A collection $\mathcal{L}$ of subsets of Ω is called a λ-system if the following hold:

- $\Omega \in \mathcal{L}$,
- $A, B \in \mathcal{L}$ and $A \subset B$ implies $B \setminus A \in \mathcal{L}$
- if $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{L}$ and $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$ then $\bigcup_n A_n \in \mathcal{L}$

Definition §1.1.8. Let $\mathcal{C}$ be a collection of nonempty subsets of a nonempty set Ω. The λ-system generated by $\mathcal{C}$, denoted as $\lambda(\mathcal{C})$ is the intersection of all λ-systems containing $\mathcal{C}$.

§1.2 Dynkin's pi-lambda theorem; Measures and their properties

Theorem §1.2.1 (Dynkin $\pi - \lambda$ theorem)**.** *If $\mathcal{C}$ is a π-system of a nonempty set Ω then $\lambda(\mathcal{C}) = \sigma(\mathcal{C})$. Equivalently, if $\mathcal{L}$ is a λ-system that contains $\mathcal{C}$ then $\mathcal{L} \supset \lambda(\mathcal{C})$.*

Definition §1.2.2. Let $\mathcal{F}$ be a σ-algebra of subsets of Ω. A extended real valued function μ on $\mathcal{F}$ is called a *measure* if the following hold:

1. $\mu(A) \geq 0$ for all $A \in \mathcal{F}$,
2. $\mu(\emptyset) = 0$
3. If $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$ such that $\bigcup A_n \in \mathcal{F}$ and $A_n \cap A_m = \emptyset$ for all $m \neq n$ then $\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_i \mu(A_i)$

Example §1.2.3 (Some examples of measures)**.** 1. Let $\Omega \neq \emptyset$, $\mathcal{F} = \mathcal{P}(\Omega)$. We define μ on $\mathcal{F}$ by $\mu(A)$ is the number of elements of A if A is finite and $\mu = +\infty$ if A contains infinitely many elements. Then μ is a measure on $\mathcal{F}$.

2. Let $\Omega = [0,1]$ and $\mathcal{F} = \mathcal{B}([0,1])$. Let $\{p_n\}$ be a sequence of numbers in $[0,1]$ such that $\sum p_i = 1$. Define $\mu(A) = \sum_{i\in\mathbb{N}} p_i \delta_{p_i}(A)$. Then μ is a measure on $\mathcal{F}$.

3. Let F be a non-decreasing right-continuous function on $\mathbb{R}$. Define μ_F to be Lebesgue-Stieljes measure induced by F. Recall that $\mu_F((a,b]) = b - a$. Then μ_F is an example of σ-finite Radon measure on the Borel σ-algebra on $\mathbb{R}$.

Theorem §1.2.4. *Let $\mathcal{F}$ be a σ-algebra on a nonempty set Ω. Let $\mu : \mathcal{F} \to [0,\infty]$ be a function. μ is a measure on $\mathcal{F}$ iff*

1. *μ is finitely additive (that is, if $A, B \in \mathcal{F}$ such that $A \cap B = \emptyset$ then $\mu(A \cup B) = \mu(A) + \mu(B)$) and*
2. *μ is continuous from below (that is, if $\{A_n\}$ is nondecreasing sequence of elements from $\mathcal{F}$ then $\mu\left(\bigcup (A_i)\right) = \lim_{n\to\infty} \mu(A_n)$).*

§2 Lecture 2 — 12th August 2022 — Properties of Measures and Definition of semialgebra

Proof of Theorem §1.2.4. Let $\mu : \mathcal{F} \to [0,\infty]$ be a function.

($\Rightarrow$) Suppose that μ is a measure. We first show that μ is finitely additive. Let $A, B \in \mathcal{F}$ and suppose that $A \cap B = \emptyset$. Let $A_1 = A$ and $A_2 = B$ and $A_n = \emptyset$ for all $n \geq 3$. Then $\mu(A \cup B) = \mu(\cup_{i\in\mathbb{N}} A_i) = \sum_{i\in\mathbb{N}} \mu(A_i) = \mu(A) + \mu(B)$ as $\mu(\emptyset) = 0$.

We now prove that μ is continuous from below. Let $\{A_n\}$ be anondecreasing sequence of elements from $\mathcal{F}$. Define $B_1 = A_1$ and $B_n = A_n \backslash A_{n-1}$ for each $n \geq 2$. Clearly, $\cup_n B_n = \cup_n A_n$

and $B_n \cap B_m = \emptyset$ for all $m \neq n$.

$$\begin{aligned}\mu\left(\cup_n A_n\right) &= \mu\left(\cup_n B_n\right) \\ &= \sum_{n \in N} \mu\left(B_n\right) \\ &= \lim_m \left[\mu(A_1) + \sum_{n=2}^{m} \left(\mu\left(A_n\right) - \mu\left(A_{n-1}\right)\right)\right] \\ &= \lim_m \mu(A_m)\end{aligned}$$

($\Leftarrow$) Now suppose that μ is finitely additive and continuous from below. We intend to prove that μ is a measure. It is clear that from finite additivity that $\mu\left(\emptyset\right) = 0$. Let $\{A_n\}$ be a sequence of elments from $\mathcal{F}$. Define $B_n = \bigcup_{i=1}^{n} A_i$ for all $n \in \mathbb{N}$. Clearly, $B_n \nearrow \cup_{k=1}^{\infty} A_k$. Clearly, $\{B_n\}$ is an nondecreasing sequence of elements from $\mathcal{F}$. Then

$$\begin{aligned}\mu\left(\cup_{i=1}^{\infty} A_i\right) &= \mu\left(\cup_{i=1}^{\infty} B_n\right) \\ &= \lim_{n\to\infty} \mu\left(B_n\right) && \text{(using continuity from below)} \\ &= \lim_{n\to\infty} \mu\left(\cup_{j=1}^{n} A_j\right) \\ &= \lim_{n\to\infty} \sum_{j=1}^{n} \mu\left(A_j\right) && \text{(finite additivity)} \\ &= \sum_{j=1}^{\infty} \mu(A_j)\end{aligned}$$

□

§2.1 Properties of Measures

Theorem §2.1.1. *Let μ be a measure on a σ-algebra $\mathcal{F}$. Then*

(1) μ is monotone,

(2) μ is finitely additive, that is, $\mu\left(A_1 \cup A_2 \cup \ldots \cup A_n\right) \leq \sum_{i=1}^{k} \mu(A_k)$ for $A_1, A_2, \ldots, A_k \in \mathcal{F}$.

(3) the inclusion-exclusion formula holds,

(4) μ is continuous from above, that is, if $(A_n) \subset \mathcal{F}$ such that $A_n \supset A_{n+1}$ for all $n \in \mathbb{N}$ and $\mu(A_{n_0}) < +\infty$ for some n_0 then $\lim \mu(A_n) = \mu\left(\cap_n A_n\right)$ and

(5) μ is countably subadditive, that is, if $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$ then $\mu\left(\cup_n A_n\right) \leq \sum_{i=1}^{\infty} \mu\left(A_n\right)$

Proof. We show that μ is monotone. Let $A \subset B$ be elements of $\mathcal{F}$. Then $B = A \cup B \setminus A$. Hence $\mu(B) = \mu(A) + \mu(B \setminus A)$. Since $\mu(B \setminus A) \geq 0$, we have that $\mu(B) \geq \mu(A)$.

Now, we prove that the inclusion exclusion formula holds for μ. Let $A, B \in \mathcal{F}$. If both $\mu(A) = +\infty$ and $\mu(B) = +\infty$ then there is nothing to prove. So, assume wlog that $\mu(A) < \infty$. Then $\mu(A \cap B) < \infty$. Then

$$\begin{aligned}\mu(A \cup B) + \mu(A \cap B) &= \mu(A) + \mu(B \setminus A) + \mu(B \cap A) \\ &= \mu(A) + (\mu(B \setminus A) + \mu(B \cap A)) \\ &= \mu(A) + \mu(B)\end{aligned}$$

Now, finite subadditivity follows immediately from inclusion-exclusion formula and induction.

We now prove that μ is continuous from below. Let $\{A_n\}$ be a sequence of decreasing sequence sets with $\mu(A_{n_0}) < +\infty$ for some n_0. Then we have that $\mu(A_1) \leq \mu(A_{n_0}) < +\infty$. Define $B_n = A_1 \setminus A_n$ and $B = A_1 \setminus \cap_n A_n$. It is easy to see that $B_n \uparrow B$ (draw pictures!). From Theorem §1.2.4 (continuity from below), we have that $\lim \mu(B_n) = \mu(B)$.

Now, observe that $\mu(B_n) = \mu(A_1) - \mu(A_n)$ for each n. So, $\mu(B) = \lim \mu(B_n) = \mu(A_1) - \lim \mu(A_n)$.

Also, we have that $\mu(B) = \mu(A_1) - \mu(\cap_n A_n)$. Hence, we have that $\lim \mu(A_n) = \mu\left(\cap_n A_n\right)$.

We now prove that μ is countably subadditive. Let $\{A_n\}$ be a sequence of elements from $\mathcal{F}$. Then $B_k := \cup_{n=1}^k A_n \uparrow \cup_n A_n$. By continuity from below, we have that $\mu\left(\cup_n A_n\right) = \lim_k \mu(B_k) \leq \lim_k \left(\mu(A_1) + \mu(A_2) + \ldots + \mu(A_k)\right) = \sum_{k=1}^\infty \mu(A_k)$. Note that the inequality is due to finite subadditivity. $\square$

Definition §2.1.2. A collection $\mathcal{C}$ of subsets of Ω is called a *semialgebra* if $\mathcal{C}$ is closed under finite-$\cap$ and if $A \in \mathcal{C}$ then there exists some $B_1, B_2, \ldots, B_n \in \mathcal{C}$, pairwise disjoint, such that $A^c = \cup_{i=1}^n B_i$.

Exercise §2.1.3. *Find a general formulation of the inclusion-exclusion principle for measures.*

§3 Lecture 3 — From semialgebra to algebra, measurable sets...

Remark §3.0.1. If $\mathcal{C}$ is a nonempty semialgebra on Ω then $\emptyset \in \mathcal{C}$.

This remark can be verified as follows: Since $\mathcal{C}$ is nonempty, let $A \in \mathcal{C}$. Since $\mathcal{C}$ is a semialgebra, there are elements $B_i \in \mathcal{C} (1 \leq i \leq n)$ such that $A^c = \bigcup_{i=1}^n B_i$ and $B_i \cap B_j = \emptyset$ for $i \neq j$. If $n = 1$, then $A^c \in \mathcal{C}$ and hence $A \cap A^c = \emptyset \in \mathcal{C}$. Now if $n \geq 2$, we have that $B_1 \cap B_2 = \emptyset \in \mathcal{C}$.

Remark §3.0.2. Let $\{\mathcal{A}_i\}_{i \in I}$ be a collection of algebras on Ω. Then it can be easily checked that $\cap_{i \in I} \mathcal{A}_i$ is an algebra on Ω. So if $\mathcal{C}$ is a collection of subsets of Ω then we denote $\mathcal{A}(\mathcal{C})$ to be the smallest algebra generated by $\mathcal{C}$, which is in fact, the intersection of all algebras that contain $\mathcal{C}$.

Definition §3.0.3 (Measure on a semi-algebra)**.** A nonnegative set function μ on a semialgebra $\mathcal{C}$ of subsets of Ω is called a *measure on* $\mathcal{C}$ if

(i) $\mu(\emptyset) = 0$

(ii) μ is countably additive, that is, if $(A_n) \subset \mathcal{C}$, $A_i \cap A_j = \emptyset$ and $\bigcap_{n\in\mathbb{N}} A_n \in \mathcal{C}$ then

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n)$$

§3.1 Extension of measures from semialgebras to algebras

Let $\mathcal{C}$ be a semialgebra on Ω. Define $\mathscr{A}$ be the collection of all finite unions of elements of $\mathcal{C}$. Then in Question 6 from Assigment 1, we showed that $\mathscr{A} = \mathcal{A}(\mathcal{C})$, that is, $\mathscr{A}$ is the smallest algebra containing $\mathcal{C}$. The following lemma has much more to say though:

Lemma §3.1.1. *Let $\mathcal{C}$ be a semialgebra of Ω. Let*

$$\mathcal{F}(\mathcal{C}) := \left\{A \subset X : A = \bigcup_{i=1}^{n} B_i \text{ for some } B_i \in \mathcal{C}, B_i \cap B_j = \emptyset \text{ for } i \neq j\right\}$$

Then $\mathcal{F}(\mathcal{C}) = \mathcal{A}(\mathcal{C})$.

Proof. Clearly, $\mathcal{F}(\mathcal{C}) \subset \mathcal{A}(\mathcal{C})$ (see the previous paragraph). Since $\mathcal{A}(\mathcal{C})$ is the smallest algebra generated by $\mathcal{C}$, we will be done if we show that $\mathcal{F}(\mathcal{C})$ is an algebra containing $\mathcal{C}$.

Clearly, $\emptyset \in \mathcal{F}(\mathcal{C})$ as $\emptyset \in \mathcal{C}$. The fact that $\mathcal{F}(\mathcal{C})$ is closed under finite unions is pretty evident. To show that $\mathcal{F}(\mathcal{C})$ is closed under complement, let $A \in \mathcal{F}(\mathcal{C})$. Then there are elements $\{B_i\}_{1\leq i\leq n} \subset \mathcal{C}$, pairwise disjoint, such that $A = \cup_{i=1}^{n} B_i$. Then since $B_i \in \mathcal{C}$, there exists $k_i \in \mathbb{N}$ and $C_{i1}, C_{i2}, \ldots, C_{i,k_i} \in \mathcal{C}$ such that $B_i^c = \cup_{j=1}^{k_i} C_{i,j}$. Then $A^c = \cap_{i=1}^{n} \cup_{j=1}^{k_i} C_{i,j}$. Interchanging the union and intersection, the result quickly follows. □

Theorem §3.1.2. *Suppose μ is a measure on a semialgebra $\mathcal{C}$ of subsets of Ω. Let $\mathcal{A}$ be the algebra generated by $\mathcal{C}$. If $A \in \mathcal{A}$ has a representation $A = \cup_{i=1}^{n} B_i$, where $B_i \in \mathcal{C}$ and $B_i \cap B_j = \emptyset$ for $i \neq j$, then we define a function $\overline{\mu}$ on a subset of $\mathcal{A}$ where the elements $A \in \mathcal{A}$ have aforementioned representation given by $\overline{\mu}(A) = \sum_{i=1}^{n} \mu(B_i)$. Then*

1. *$\overline{\mu}$ is well defined,*
2. *$\overline{\mu}$ is finitely additive and*
3. *$\overline{\mu}$ is countably additive.*

Proof. We first show that $\overline{\mu}$ is indeed well-defined. Let $A \in \mathcal{A}$ and suppose that $A = \cup_{i=1}^{m} B_i$ and $A = \cup_{j=1}^{n} C_j$ where $\{B_i\} \subset \mathcal{C}$ and $\{C_j\} \subset \mathcal{C}$ are pairwise disjoint. Then note that for

$i \in \{1 \le l \le n\}$, we have that $B_i = B_i \cap A = B_i \cap \left(\cup_{j=1}^n C_j\right) = \cup_{j=1}^n (B_i \cap C_j)$. Note that previous union is a pairwise disjoint union. Hence,

$$\begin{aligned}\sum_{i=1}^m \mu(B_i) &= \sum_{i=1}^m \sum_{j=1}^n \mu(B_i \cap C_j)\\ &= \sum_{j=1}^n \sum_{i=1}^m \mu(B_i \cap C_j)\\ &= \sum_{j=1}^n \mu(C_j)\end{aligned}$$

This shows that $\overline{\mu}$ is well defined.

We show that $\overline{\mu}$ is countably additive. Let $\{A_n\}$ be the collection of elements of $\mathcal{A}$ which can be written as a union of pairwise disjoint elements of $\mathcal{C}$, $A_n \cap A_m = \emptyset$ for $n \neq m$ and $\cup_{n\in\mathbb{N}} A_n \in \mathcal{A}$.

Then for each n, there is $k_n \in \mathbb{N}$ and pairwise disjoint elements $B_{n1}, B_{n2}, \ldots, B_{n,k_n} \in \mathcal{C}$ such that $A_n = \cup_{i=1}^{k_n} B_{n,i}$.

We also have by Lemma §3.1.1 that there are some $B_1, \ldots, B_k \in \mathcal{C}$, pairwise disjoint, such that $\bigcup_{n\in\mathbb{N}} A_n = \cup_{i=1}^k B_i$.

Now, for any i, we have that

$$\begin{aligned}B_i &= B_i \cap \left(\bigcup_{n\in\mathbb{N}} A_n\right)\\ &= \bigcup_{n\in\mathbb{N}} (B_i \cap A_n)\\ &= \bigcup_{n\in\mathbb{N}} \left(B_i \cap \bigcup_{j=1}^{k_n} B_{n,j}\right)\\ &= \bigcup_{n\in\mathbb{N}} \bigcup_{j=1}^{k_n} (B_i \cap B_{n,j})\\ &= \bigcup_{\substack{n\in\mathbb{N}\\ j\in\{1,\ldots k_n\}}} (B_i \cap B_{n,j})\end{aligned}$$

Note that the previous union is a pairwise disjoint union. Thus by definition of measure on a semialgebra, we have that for any i

$$\mu(B_i) = \sum_{\substack{n\in\mathbb{N}\\ j\in\{1,\ldots k_n\}}} \mu(B_i \cap B_{n,j})$$

Hence,

$$
\begin{aligned}
\overline{\mu}(A) &= \sum_{i=1}^{k} \mu(B_i) \\
&= \sum_{i=1}^{k} \sum_{\substack{n\in\mathbb{N} \\ j\in\{1,\dots k_n\}}} \mu(B_i \cap B_{n,j}) \\
&= \sum_{\substack{n\in\mathbb{N} \\ j\in\{1,\dots k_n\}}} \sum_{i=1}^{k} \mu(B_i \cap B_{n,j}) \\
&= \sum_{\substack{n\in\mathbb{N} \\ j\in\{1,\dots k_n\}}} \mu\left(\bigcup_{i=1}^{k} (B_i \cap B_{n,j})\right) \\
&= \sum_{\substack{n\in\mathbb{N} \\ j\in\{1,\dots k_n\}}} \mu(B_{n,j}) \\
&= \sum_{n\in\mathbb{N}} \sum_{j=1}^{k_n} \mu(B_{n,j}) \\
&= \sum_{n\in\mathbb{N}} \overline{\mu}\,(A_n)
\end{aligned}
$$

This completes the proof of countable additivity. The proof of finite additivity follows from countable subadditivity. □

§3.2 Outer Measures

Definition §3.2.1. Given a measure μ on a semialgebra $\mathcal{C}$, the *outer measure induced by* μ is the set function μ^* defined on $\mathcal{P}(\Omega)$ as

$$
\mu^*(A) := \inf\left\{\sum_{n=1}^{\infty} \mu(A_n) : \{A_n\}_{n\geq 1} \subset \mathcal{C}, A \subset \bigcup_{n\geq 1} A_n\right\}
$$

We'd like to remark that μ^* is not an overestimate, that is, $\mu^* = \mu$ on $\mathcal{C}$ and $\mu^* = \overline{\mu}$ on $\mathcal{A}$.

To verify this remark, let $C \in \mathcal{C}$. We need to show that $\mu^*(C) = \mu(C)$. Clearly by definition of μ^*, we have that $\mu^*(C) \leq \mu(C)$. (Fill the details!)

Definition §3.2.2. A set A is said to be *μ^*-measurable* if

$$
\mu^*(E) = \mu^*(A \cap E) + \mu^*(A^c \cap E) \text{ for all } E \subset \Omega
$$

The set of all μ^*-measurable sets is denoted by $\mathcal{M}_{\mu^*}$.