Lecture Notes in Finite Frames

Ashish Kujur

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Introduction

This is a set of lecture notes which I took for reviewing stuff that I typed after taking class from *Dr P Devaraj*. All the typos and errors are of mine.

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§1 Lecture 1 — 10th August, 2022 — Hilbert Spaces & Frames

We start by reviewing the elementary notions from Linear Algebra.

§1.1 Inner Product Spaces

Definition §1.1.1. A vector space V over a field F ($\mathbb{R}$ or $\mathbb{C}$) is called an *inner product space* if there exists a function $\langle\cdot,\cdot\rangle : V \times V \to F$ satisfying the following:

1. $\langle x, x\rangle \geq 0$ for all $x \in V$.
2. $\langle x, x\rangle = 0$ iff $x = 0$.
3. (linear in the first argument) for all $x, y, z \in V$, $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$
4. (conjugate) for all $x, y \in V$, $\langle x, y\rangle = \overline{\langle y, x\rangle}$

Definition §1.1.2. A norm on a vector space V over $\mathbb{R}$ or $\mathbb{C}$ is a function $\|\cdot\| : V \to [0, +\infty)$ satisfying

1. for all $x \in V$, $\|x\| = 0 \Leftrightarrow x = 0$
2. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$

3. $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in V$ and $\alpha \in F$

It is easy to check that if V is an inner product space then $\|\cdot\|$ defined by $\|x\| = \sqrt{\langle x, x\rangle}$ is a norm on V. To verify the triangle inequality, use Cauchy Schwarz inequality.

Definition §1.1.3. A vector space together with a norm is called a normed linear space.

Note that every normed linear space $(V, \|\cdot\|)$ is a metric space. The metric is given by $d(x, y) = \|x - y\|$ for all $x, y \in V$.

§1.2 Hilbert Spaces & Frames

Definition §1.2.1. An inner product space which is complete wrt the induced norm is called Hilbert Space.

We will only be considering finite dimensional Hilbert spaces in this course!

Example §1.2.2.

1. $\mathbb{R}^n$ with the usual inner product is a Hilbert Space.
2. $\mathbb{C}^n$ with the usual inner product is a Hilbert Space.

Definition §1.2.3. A sequence $\{f_n\}$ in H is called a frame for H if there exists positive constants A and B such that

$$A\|f\|^2 \leq \sum_i |\langle f, f_i\rangle|^2 \leq B\|f\|^2$$

for all $f \in H$.

Remark §1.2.4. It is possible to have that a frame in a finite dimensional Hilbert space consisting of infinitely many elements. However, it is rather artificial to have infinite number of frame elements in a finite dimensional space. We therefore consider only frames with finite number of elements.