Lecture Notes in Measure Theory

Joel Sleeba

forked from Ashish Kujur

Last Updated: August 18, 2022

Introduction

This is a set of lecture notes which I took for reviewing stuff that I typed after taking class from *Dr. Sachindranath Jayaraman*. All the typos and errors are of mine.

Contents

§1 Lecture 1 — 10th August 2022 — Review of things done in the previous semester. . .

§1.1 Definitions and Some Results

Definition §1.1.1 (algebra)**.** Let Ω be nonempty set. An algebra $\mathscr{F}$ is a collection of subsets of Ω satisfying the following properties:

1. $\Omega \in \mathscr{F}$,
2. $A \in \mathscr{F} \implies A^c \in \mathscr{F}$ and
3. $\mathscr{F}$ is closed under finite unions.

Remark §1.1.2. It immediately follows from the definition that an algebra of sets is closed under taking finite intersections. Take compliment of finite intersections and make use of De Morgan's theorem.

Definition §1.1.3 (σ-algebra)**.** Let Ω be nonempty set. A σ-algebra $\mathscr{F}$ is a collection of subsets of Ω satisfying the following properties:

1. $\Omega \in \mathscr{F}$,
2. $A \in \mathscr{F} \implies A^c \in \mathscr{F}$ and
3. $\mathscr{F}$ is closed under countable unions.

Remark §1.1.4. Similar to what we saw in an algebra of sets, the σ-algebra of sets is also closed under countable intersection. The proof is similar to that of the same with algebra of sets.

Proposition §1.1.5. *Let Ω be a set, $\mathscr{F} \subseteq \mathscr{P}(\Omega)$. $\mathscr{F}$ is an σ-algebra iff $\mathscr{F}$ is an algebra that is continuous from below, that is, if $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathscr{F}$ and $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$ then $\bigcup_n A_n \in \mathscr{F}$.*

Proof. ($\implies$) Since $\mathscr{F}$ is closed under countable unions of elements of $\mathscr{F}$ it is also closed under countable unions. To prove this assume $\{A_i\}_{i=1}^N$ is the finite collection of sets, then take $\{B_i\}_{i\in\mathbb{N}}$ where

$$B_i = \begin{cases} A_i, \text{ if } 1 \le i \le N \\ \emptyset, \text{ if } i > N \end{cases}$$

Then $\bigcup_{i=1}^N A_i = \bigcup_{i\in\mathbb{N}} B_i \in \mathscr{F}$, hence $\mathscr{F}$ is an algebra.

Again if $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathscr{F}$ and $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$, then $\bigcup_n A_n \in \mathscr{F}$, since $\mathscr{F}$ being a σ-algebra is closed under countable unions

($\impliedby$)We'll prove an algebra satisfying $\bigcup_n A_n \in \mathscr{F}$, when $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathscr{F}$ and $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$ is a σ-algebra. Assume $\{B_n\}_{n\in\mathbb{N}} \in \mathscr{F}$ is a collection of subsets of Ω.

Define

$$A_n = \bigcup_{i=1}^{n-1} B_i$$

Then $A_n \in \mathscr{F}$ being an algebra and $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$, hence by assumption $\bigcup_n A_n \in \mathscr{F}$, but

$$\bigcup_{i\in\mathbb{N}} A_n = \bigcup_{i\in\mathbb{N}} B_n \in \mathscr{F}$$

Hence $\mathscr{F}$ is a σ-algebra □

Definition §1.1.6 (σ-algebra generated by a subset of power set)**.** Let Ω be a nonempty set. Given an nonempty collection $\mathscr{C}$ of subsets of Ω, the σ-algebra generated by $\mathscr{C}$, $\sigma(\mathscr{C})$ is defined to be the intersection of all σ-algebra containing $\mathscr{C}$. Notationally,

$$\sigma(\mathscr{C}) = \bigcap \{\sigma - \text{algebra that contains } \mathscr{C}\}$$

$\sigma(\mathscr{C})$ is the smalled σ-algebra containing $\mathscr{C}$

Definition §1.1.7 (Borel σ-algebra)**.** If Ω is a topological space then the Borel σ-algebra is the smallest σ-algebra containing the open sets of Ω. *i.e.* by definition Borel σ-algebra is the σ-algebra generated by open sets in Ω.

Fact §1.1.8. *If $\Omega = \mathbb{R}^n$ the Borel σ-algebra is generated by*

- $\{(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n) \mid -\infty \le a_i < b_i \le +\infty\}$
- $\{(-\infty, a_1) \times (-\infty, a_2) \times \cdots (-\infty, a_n) \mid a_1, a_2, \dots, a_n \in \mathbb{R}\}$
- $\left\{(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n) \mid a_i, b_i \in \mathbb{Q}\right\}$

Definition §1.1.9 (π-system, λ-system)**.** A collection $\mathscr{C}$ of subsets of Ω is called a π-system if $\mathscr{C}$ is closed under finite intersections.

A collection $\mathscr{L}$ of subsets of Ω is called a λ-system if the following hold:

- $\Omega \in \mathscr{L}$,
- $A, B \in \mathscr{L}$ and $A \subset B$ implies $B \setminus A \in \mathscr{L}$
- if $\{A_n\}_{n \in \mathbb{N}} \subset \mathscr{L}$ and $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$ then $\bigcup_n A_n \in \mathscr{L}$

Definition §1.1.10. Let $\mathscr{C}$ be a collection of nonempty subsets of a nonempty set Ω. The λ-system generated by $\mathscr{C}$, denoted as $\lambda(\mathscr{C})$ is the intersection of all λ-systems containing $\mathscr{C}$.

§1.2 Dynkin's pi-lambda theorem; Measures and their properties

Theorem §1.2.1 (Dynkin $\pi - \lambda$ theorem)**.** *If $\mathscr{C}$ is a π-system of a nonempty set Ω then $\lambda(\mathscr{C}) = \sigma(\mathscr{C})$. Equivalently, if $\mathscr{L}$ is a λ-system that contains $\mathscr{C}$ then $\lambda(\mathscr{C}) \subset \mathscr{L}$.*

Definition §1.2.2. Let $\mathscr{F}$ be a σ-algebra of subsets of Ω. A extended real valued function μ on $\mathscr{F}$ is called a *measure* if the following hold:

1. $\mu(A) \ge 0$ for all $A \in \mathscr{F}$
2. $\mu(\emptyset) = 0$
3. If $\{A_n\}_{n \in \mathbb{N}} \subset \mathscr{F}$ such that $\bigcup A_n \in \mathscr{F}$ and $A_n \cap A_m = \emptyset$ for all $m \ne n$ then $\mu(\bigcup_i A_i) = \sum_i \mu(A_i)$

Example §1.2.3 (Some examples of measures)**.**

1. Let $\Omega \ne \emptyset$, $\mathscr{F} = \mathscr{P}(\Omega)$. We define μ on $\mathscr{F}$ by $\mu(A)$ is the number of elements of A if A is finite and $\mu = +\infty$ if A contains infinitely many elements. Then μ is a measure on $\mathscr{F}$ called the counting measure on $\mathscr{F}$.
2. Let $\Omega = [0,1]$ and $\mathscr{F} = \mathscr{B}(\Omega)$. Let $\left\{p_n\right\}$ be a sequence of numbers in $[0,1]$ such that $\sum p_i = 1$. Define $\mu(A) = \sum_{i \in \mathbb{N}} p_i \delta_{p_i}(A)$. Then μ is a measure on $\mathscr{F}$.
3. Let F be a non-decreasing right-continuous function on $\mathbb{R}$. Define μ_F to be Lebesgue-Stieljes measure induced by F. Recall that $\mu_F((a,b]) = F(b) - F(a)$. Then μ_F is an example of σ-finite Radon measure on the Borel σ-algebra on $\mathbb{R}$.

Theorem §1.2.4. *Let $\mathscr{F}$ be a σ-algebra on a nonempty set Ω. Let $\mu : \mathscr{F} \to [0, \infty]$ be a function. μ is a measure on $\mathscr{F}$ iff*

1. *μ is finitely additive (that is, if $A, B \in \mathscr{F}$ such that $A \cap B = \emptyset$ then $\mu(A \cup B) = \mu(A) + \mu(B)$) and*
2. *μ is continuous from below (that is, if $\{A_n\}$ is nondecreasing sequence of elements from $\mathscr{F}$ then $\mu(\bigcup(A_i)) = \lim_{n \to \infty} \mu(A_n)$).*

§2 Lecture 2 — 12th August 2022 — Properties of Measures and Definition of semialgebra

Proof of Theorem §1.2.4. Let $\mu : \mathscr{F} \to [0,\infty]$ be a function.

($\Longrightarrow$) Suppose that μ is a measure. We first show that μ is finitely additive. Let $A, B \in \mathscr{F}$ and suppose that $A \cap B = \emptyset$. Let $A_1 = A$ and $A_2 = B$ and $A_n = \emptyset$ for all $n \geq 3$. Then $\mu(A \cup B) = \mu(\cup_{i\in\mathbb{N}} A_i) = \sum_{i\in\mathbb{N}} \mu(A_i) = \mu(A) + \mu(B)$ as $\mu(\emptyset) = 0$.

We now prove that μ is continuous from below. Let $\{A_n\}$ be a nondecreasing sequence of elements from $\mathscr{F}$. *i.e.* $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$. Define $B_1 = A_1$ and $B_n = A_n \setminus A_{n-1}$ for each $n \geq 2$. Clearly, $\bigcup_n B_n = \bigcup_n A_n$ and $B_n \bigcap B_m = \emptyset$ for all $m \neq n$.

$$\begin{aligned}
\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) &= \mu\left(\bigcup_{n\in\mathbb{N}} B_n\right) \\
&= \sum_{n\in\mathbb{N}} \mu(B_n) && \text{(by the property of measure on } B_n\text{)} \\
&= \lim_{n\to\infty} \sum_{m=1}^{n} \mu(B_m) \\
&= \lim_{n\to\infty} \left[\mu(A_1) + \sum_{m=2}^{n} \left(\mu(A_m) - \mu(A_{m-1})\right)\right] && \text{(by the definiton of } B_m\text{)} \\
&= \lim_{n\to\infty} \mu(A_n) && \text{(telescopic sum)}
\end{aligned}$$

($\Longleftarrow$) Now suppose that μ is finitely additive and continuous from below. We intend to prove that μ is a measure. It is clear that from finite additivity that $\mu(\emptyset) = 0$. If not $\mu(\emptyset) = \mu(\emptyset) + \mu(\emptyset)$, which will give a contradiction for any nonzero value for $\mu(\emptyset)$. Let $\{A_n\}$ be a sequence of elments from $\mathscr{F}$. Define $B_n = \bigcup_{i=1}^{n} A_i$ for all $n \in \mathbb{N}$. Clearly, $B_n \nearrow \bigcup_{i\in\mathbb{N}} A_i$. Clearly, $\{B_n\}$ is an nondecreasing sequence of elements from $\mathscr{F}$. Then

$$\begin{aligned}
\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) &= \mu\left(\bigcup_{i\in\mathbb{N}} B_n\right) \\
&= \lim_{n\to\infty} \mu(B_n) && \text{(using continuity from below)} \\
&= \lim_{n\to\infty} \mu\left(\bigcup_{j=1}^{n} A_j\right) \\
&= \lim_{n\to\infty} \sum_{j=1}^{n} \mu(A_j) && \text{(finite additivity)} \\
&= \sum_{j=1}^{\infty} \mu(A_j)
\end{aligned}$$

Hence μ is a measure on $\mathscr{F}$ □

§2.1 Properties of Measures

Theorem §2.1.1. *Let μ be a measure on a σ-algebra $\mathscr{F}$. Then*

(1) μ *is monotone.*

(2) the inclusion-exclusion formula holds.

(3) μ *is finitely subadditive. i.e.* $\mu(A_1 \cup A_2 \cup \ldots \cup A_n) \leq \sum_{i=1}^{k} \mu(A_k)$ *for* $A_1, A_2, \ldots, A_k \in \mathscr{F}$.

(4) μ *is continuous from above, that is, if* $(A_n) \subset \mathscr{F}$ *such that* $A_n \supset A_{n+1}$ *for all* $n \in \mathbb{N}$ *and* $\mu(A_{n_0}) < +\infty$ *for some* n_0 *then* $\lim \mu(A_n) = \mu(\bigcap_n A_n)$.

(5) μ *is countably subadditive, that is, if* $\{A_n\}_{n\in\mathbb{N}} \subset \mathscr{F}$ *then* $\mu(\cup_n A_n) \leq \sum_{i=1}^{\infty} \mu(A_n)$

Proof. (1) We show that μ is monotone. Let $A \subset B$ be elements of $\mathscr{F}$. Then $B = A \cup B \setminus A$. Hence $\mu(B) = \mu(A) + \mu(B \setminus A)$. Since $\mu(B \setminus A) \geq 0$, we have that $\mu(B) \geq \mu(A)$.

(2) Now, we prove that the inclusion exclusion formula holds for μ. Let $A, B \in \mathscr{F}$. If both $\mu(A) = +\infty$ and $\mu(B) = +\infty$ then there is nothing to prove. So, assume without loss of generality that $\mu(A) < \infty$. Then $\mu(A \cap B) < \infty$. Then

$$\begin{aligned} \mu(A \cup B) + \mu(A \cap B) &= \mu(A) + \mu(B \setminus A) + \mu(B \cap A) \\ &= \mu(A) + (\mu(B \setminus A) + \mu(B \cap A)) \\ &= \mu(A) + \mu(B) \end{aligned}$$

(3) Now, finite subadditivity follows immediately from inclusion-exclusion formula and induction.

(4) We now prove that μ is continuous from above. Let $\{A_n\}$ be a sequence of decreasing sequence sets with $\mu(A_1) < \infty$. Then we have that $\mu(A_n) \leq \mu(A_1) < \infty$. Define $B_n = A_1 \setminus A_n$ and $B = A_1 \setminus \bigcap_n A_n$. It is easy to see that $B_n \uparrow B$ (draw pictures!). From Theorem §1.2.4 (continuity from below), we have that $\lim \mu(B_n) = \mu(B)$.

Now, observe that $\mu(B_n) = \mu(A_1) - \mu(A_n)$ for each n. So, $\mu(B) = \lim \mu(B_n) = \mu(A_1) - \lim \mu(A_n)$.

Also, we have that $\mu(B) = \mu(A_1) - \mu(\cap_n A_n)$. Hence, we have that $\lim \mu(A_n) = \mu(\cap_n A_n)$.

We now prove that μ is countably subadditive. Let $\{A_n\}$ be a sequence of elements from $\mathscr{F}$. Then $B_k := \cup_{n=1}^{k} A_n \uparrow \cup_n A_n$. By continuity from below, we have that $\mu(\cup_n A_n) = \lim_k \mu(B_k) \leq \lim_k \big(\mu(A_1) + \mu(A_2) + \ldots + \mu(A_k)\big) = \sum_{k=1}^{\infty} \mu(A_k)$. Note that the inequality is due to finite subadditivity.

□

Definition §2.1.2. A collection $\mathscr{C}$ of subsets of Ω is called a *semialgebra* if $\mathscr{C}$ is closed under finite-$\cap$ and if $A \in \mathscr{C}$ then there exists some $B_1, B_2, \ldots, B_n \in \mathscr{C}$, pairwise disjoint, such that $A^c = \cup_{i=1}^{n} B_i$.

Exercise §2.1.3. *Find a general formulation of the inclusion-exclusion principle for measures.*