Lecture Notes in Analysis on Manifolds

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Introduction

This is a set of lecture notes which I took for reviewing stuff that I typed after taking class from *Dr. Shrihari Sridharan*. All the typos and errors are of mine. The pictures that make here will be hand drawn and I will appreciate it if someone who is knowledgeable in Tikz will help me digitizing my rough hand-drawn pictures.

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§1 Lecture 1 — 10th August 2022 — Review of Real Analysis

§1.1 Metric Spaces

Definition §1.1.1 (Metric Space)**.** A set X along with a function $d : X \times X \to \mathbb{R}^+$ is called a metric space if d satisfies the following properties

1. $d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, y) \leq d(x, z) + d(z, y)$ where $x, y, z \in Z$

Suppose (X, d) is a metric space and $Y \subset X$, Then (Y, d) is also a metric space

Definition §1.1.2 (Open Ball)**.** We define $B_d(x_0, \epsilon)$, the open ball of radius ϵ about $x_0 \in X$ as

$$B_d\left(x_0, \epsilon\right) = \{x \in X | d\left(x, x_0\right) < \epsilon\}$$

Moreover if the metric d is known we'll write $B_\epsilon(x)$ instead of $B_d(x, \epsilon)$

Definition §1.1.3 (Open Set)**.** Let (X, d) be a metric space. Then a subset $U \subset X$ is open if for all $x \in U$ there exists an $\epsilon > 0$ such that $B_d(x, \epsilon) \subset U$

Fact §1.1.4. *For $x_1, x_2 \in X$ if $B_{\epsilon_1}(x_1) \subset B_{\epsilon_2}(x_2)$, then $d(x_1, x_2) \leq \epsilon$*

Fact §1.1.5. *Let $A_1 \subset B_{\epsilon_1}(x)$ and $A_2 \subset B_{\epsilon_2}(x)$, then $A_1 \cup A_2 \subset B_{max(\epsilon_1, \epsilon_2)}(x)$*

Fact §1.1.6. *Let $\{U_\alpha \mid \alpha \in I\}$ be a collection of open sets in X. Then $\cup_\alpha U_\alpha$ is open*

Fact §1.1.7. *Let (X, d) be a metric space and let $Y \subset X$. Let A be open about $d\mid_Y$, the restriction of d to Y. Then there exist an open set U in X such that $A = U \cap Y$*

Fact §1.1.8. *Finite intersection of open sets is open*

Fact §1.1.9. *$A \subset X$ is closed $\iff$ A^c is open*

§1.2 Norms

We'll define and work mainly with 2 different norms in $\mathbb{R}^n$. Euclidean norm and Supremum norm

Definition §1.2.1 (Euclidean Norm)**.** Let $x = (x_1, x_2, \dots x_n) \in \mathbb{R}^n$, then we define the euclidean norm of x, denoted by $\|x\|$, to be

$$\|x\| = \sqrt{\sum_{i=1}^{n} (x_i^2)}$$

Definition §1.2.2 (Supremum Norm)**.** Let $x = (x_1, x_2, \dots x_n) \in \mathbb{R}^n$, then we define the supremum norm of x, denoted by $|x|$, to be

$$|x| = \max\{x_i \mid 1 \leq i \leq n\}$$

Fact §1.2.3. *The norms we defined above will satisfy*

$$|x-y| \ \le \ \|x-y\| \ \le \ \sqrt{n}|x-y|$$

Fact §1.2.4. *Given any norm $\|\cdot\|$ in X, it induces a metric in X defined as $d(x,y) = \|x-y\|$. This metric is called the metric induced by the norm $\|\cdot\|$*

Fact §1.2.5. *A subset $U \subset \mathbb{R}^n$ is open about euclidean metric iff it is open about the supremum metric*

§1.3 Continuous Functions and properties

Definition §1.3.1 (Function continuous at a point)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. Then a function $f: X \to Y$ is continuous at a point $x_0 \in X$ if for all $\epsilon > 0$ there exist a $\delta > 0$ such that $d_X(x, x_0) < \delta \implies d_Y(y, y_0) < \epsilon$

Definition §1.3.2 (Function continuous on X)**.** We say f is continuous on X if f is continuous at x for all $x \in X$

Definition §1.3.3 (Topological Definition of continuity)**.** $f: X \to Y$ is continuous iff given any open ball $U \in Y$, $f^{-1}(U)$ is open in X

§1.4 Metric topology

Definition §1.4.1 (Limit Point)**.** Let (X, d) be a metric space. x_0 is a limit point of $Y \subset X$ if for all $\epsilon > 0$, $B_\epsilon(x_0) \cap Y$ is an infinite set

Definition §1.4.2 (Closure of a set)**.** If $A \subset X$, then the closure of A, $\bar{A}$ is defined as the union of A with the limit points of A

Definition §1.4.3 (Interior of a Set)**.** If $A \subset X$, then the interior of A, $\mathrm{Int}(A)$ is defined as $(\bar{A}^c)^c$

Definition §1.4.4 (Exterior of a set)**.** If $A \subset X$, then the exterior of A, $\mathrm{Ext}(A)$ is defined as the interior of A^c

Definition §1.4.5 (Boundary of a set)**.** If $A \subset X$, then the boundary of A is defined as $X \setminus (\mathrm{Int}(A) \cup \mathrm{Ext}(A))$

Fact §1.4.6. *The interior of A, Int(A) is an open set*

§1.5 Compact Sets

Definition §1.5.1 (Cover of a set)**.** A collection $\{U_\alpha \subset X \mid \alpha \in I\}$ is a cover of A if $A \subset \cup_{\alpha \in I} U_\alpha$

Theorem §1.5.2 (Heine Borel Theorem)**.** *$A \subset \mathbb{R}^n$ is compact iff every open cover of A has a finite subcover*

Fact §1.5.3. *Let $(X, d_X), (Y, d_Y)$ be metric spaces and $f: X \to Y$ be continuous. If $A \subset X$ is compact then $f(A)$ is also compact.*

Fact §1.5.4. *Let $f: X \to \mathbb{R}$ be a continuous function and $A \subset X$ be compact. Then f attains a maximum in A*

Fact §1.5.5. *Let $f: X \to Y$ be continuous and $A \subset X$ be compact. Then f is uniformly continuous on A*

§1.6 Connected Sets

Definition §1.6.1 (Connected metric space)**.** A metric space X is said to be connected if X cannot be written as a disjoint union of 2 open sets

Fact §1.6.2. *Let $f : X \to Y$ be a continuous function. Then, if X is a connected space, $f(X)$ is also connected in Y*

Fact §1.6.3. *Let $f : X \to \mathbb{R}$ be a continuous function and X be connected. If $a, b \in f(X)$ then for all $r \in \mathbb{R}$ such that $a < r < b$, $r \in f(X)$*

Prove all the statements given as facts. This requires only an exposure to real analysis