Lecture Notes in Partial Differential Equations

Ashish Kujur

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Introduction

This is a set of lecture notes which I took for reviewing stuff that I typed after taking class from *Dr. K.R. Arun*. This course used the textbook *Partial Differential Equations* by L.C. Evans.

Contents

§1 Lecture 1 — 11th August, 2022 — Definition, Classifications & Examples of PDEs

§1.1 Notations

- Let $\mathbb{N}_0$ be defined to be the set $\mathbb{N} \cup \{0\}$. For any $N \in \mathbb{N}$, an element of $\mathbb{N}_0^N$ to be an *multiindex*. If $\alpha \in \mathbb{N}_0^N$ then $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ for some $\alpha_i \in \mathbb{N}_0$.
- For any $x \in \mathbb{R}^N$ and $N \in \mathbb{N}$, we define $x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}$.
- Given any multiindex $\alpha \in \mathbb{N}_0^N$, we define $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_N$.
- Given any multiindex α, we define

$$D^\alpha := \frac{\partial^{|\alpha|}}{\partial {x_1}^{\alpha_1} \cdots \partial {x_N}^{\alpha_N}} = \partial {x_1}^{\alpha_1} \cdots \partial {x_N}^{\alpha_N}$$

- For any $k \in \mathbb{N}$, denote $D^k = \{D^\alpha : |\alpha| = k\}$
- We will denote $\Omega \subset \mathbb{R}^N$ to be an open subset.

§1.2 Definition, Classification & Examples

Definition §1.2.1 (Partial Differential Equation)**.** Let Ω be an open subset of $\mathbb{R}^N$. An expression of the form

$$F\left(D^k u(x), D^{k-1} u(x), \dots, Du(x), x\right) = 0 \qquad (x \in \Omega) \tag{1}$$

is called a kth order PDE for the unknown function $u : \Omega \to \mathbb{R}$. One may assume $F : \mathbb{R}^{N^k} \times \mathbb{R}^{N^{k-1}} \times \cdot \times \mathbb{R}^N \times \Omega \to \mathbb{R}$ is a given smooth function.

§1.2.1 Classifications of PDE

(i) The PDE (1) is called *linear* if it has the form

$$\sum_{0 \le |\alpha| \le k} a_\alpha(x) D^\alpha u = f$$

for some functions a_α, f. The linear PDE is homogeneous if $f = 0$.

(ii) The PDE (1) is called *semilinear* if it has the form

$$\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u + a_0\left(D^{k-1}u, \dots, Du, u, x\right) = 0$$

(iii) The PDE (1) is called *quasilinear* if it has the form

$$\sum_{|\alpha=k} a_\alpha(D^{k-1}u, \dots, Du, u, x) D^\alpha u + a_0\left(D^{k-1}u, \dots, Du, u, x\right) = 0$$

(iv) The PDE (1) is called *nonlinear* if the PDE has a nonlinear dependence on the highest order derivative.

Definition §1.2.2 (System of PDE)**.** An expression of the form $\mathbf{F}\left(D^k(\mathbf{u}), D^{k-1}(\mathbf{u})), \dots, D(\mathbf{u}), \mathbf{u}, x\right) = \mathbf{0}$ is called a kth order system of PDE, where $\mathbf{u} : \Omega \to \mathbb{R}^m$ is the unknown, $\mathbf{u} = \left(u^1, u^2, \dots, u^n\right)$ and $\mathbf{F} : \mathbb{R}^{mN^k} \times \mathbb{R}^{mN^{k-1}} \times \cdots \mathbb{R}^{mN} \times \mathbb{R}^m \times U \to \mathbb{R}^m$ is given.

§1.2.2 Examples of PDEs

1. Linear Equations

   Laplace Equation $\Delta u = \sum_{i=1}^N \partial_{x_i{}^2} u = 0$

   (linear, second order)

   Linear Transport Equation $\partial_t u + \sum_{i=1}^N \partial_{x_i} u = 0$

   (linear, first order)

Schrödinger's Equation $i\partial_t u + \Delta u = 0$

(linear, second order)

Linear System : Maxwell's Equations

$$\partial_t E = \text{curl } B$$
$$\partial_t B = -\text{curl } E$$
$$\text{div } E = \text{div } B = 0$$

2. Nonlinear equations

   Inviscid Burgers' equation $\partial_t u + u\partial_x u = 0$

   Eikonal equation $|Du| = 1$

   Nonlinear system: Navier-Stokes Equations

$$\partial_t \mathbf{u} + \mathbf{u} \cdot D\mathbf{u} - \Delta\mathbf{u} = -Dp$$
$$\text{div } \mathbf{u} = 0$$

Definition §1.2.3 (Well posed)**.** A PDE is said to be *well posed* if

(Existence) it has at least one solution,

(Uniqueness) it has at most one solution and

(Stability) the solution depends continuously on the data given in the problem.

Definition §1.2.4. A *classical solution* of the k-th order PDE is a function $u \in C^k(\Omega)$ which satisfies the equation pointwise

$$F\left(D^k u(x), D^{k-1} u(x), \dots, Du(x), u(x), x\right) = 0$$

for all $x \in \Omega$.

Remark §1.2.5. A classical solution may not always exist. For instance, the inviscid Burgers' equation does not have a solution.

The course is divided into three parts:

(a) Representation Formulae for solutions

(b) Linear PDE theory

(c) Nonlinear PDE theory

§1.3 Transport Equation

The PDE

$$\partial_t u + b \cdot Du = 0 \text{ in } \mathbb{R}^n \times (0, \infty)$$

where $t \in (0, \infty)$, $x \in \mathbb{R}^n$ are the independent variables, $u = u(t, x)$ is the dependent variable and $b = (b_1, b_2, \dots, b_n)$ and $Du = \left(\partial_{x_1} u, \dots, \partial_{x_2} u\right)$ is the gradient.

§2 Lecture 2 — 16th August, 2022 — Linear Transport Equation

Consider the linear transport equation given by

$$u_t + b \cdot Du = 0 \qquad \text{in } \mathbb{R}^n \times (0,\infty)$$

where $b \in \mathbb{R}^n$ is a fixed vector and $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ is the unknown function, $Du = D_x u(u_{x_1}, u_{x_2}, \dots, u_{x_n})$.

Let $(x,t) \in \mathbb{R}^n \times (0,\infty)$ be fixed. Our aim is to obtain a representation of the solution.

Assuming that u is a smooth function, we define the function $z(s) := u(x+bs, t+s)$ where $s \in \mathbb{R}$.

Note that z is the restriction of the function u to the line $L = \{(x+bs, t+s) : s \in \mathbb{R}\}$. Note that the line L passes through (x,t) and is in the direction of the vector $(b,1)$.

Differentiating z, we have that for all $s \in \mathbb{R}$,

$$z'(s) = b \cdot Du(x+bs, t+s) + u_t(x+bs, t+s) = 0$$

Thus, u is a constant on the line L. If we know the solution at any point on L, the problem is solved. We use the aforementioned result in the following subsection.

§2.1 Solution of an IVP

Let $g : \mathbb{R}^n \to \mathbb{R}$ where $g = g(x_1, x_2, \dots, x_n)$. We consider the following IVP

$$\begin{aligned} u_t + b \cdot Du &= 0 \quad \text{in } \mathbb{R}^n \times (0,\infty) \\ u &= g \quad \text{in } \mathbb{R}^n \times \{0\} \end{aligned}$$

Just note that the last equation in IVP means $u(x,0) = g(x)$ for all $x \in \mathbb{R}^n$. From the discusssion before the start of the subsection, it suffices to know the solution on the hyperplane $\Gamma = \mathbb{R}^n \times \{0\}$. The line L passes through Γ at the point $(x - tb, 0)$. So,

$$\begin{aligned} u(x,t) &= z(0) \\ &= z(-t) \\ &= u(x-bt, 0) \\ &= g(x-bt) \end{aligned}$$

Note that the first equality is true by definition of z, the second equality is true by z being constant on the line L and the third is again true by definition of z and the last is true by $u = g$ in $\mathbb{R}^n \times \{0\}$.

So, $\boxed{u(x,t) = g(x-bt) \qquad t \geq 0, x \in \mathbb{R}^n}$.

Remark §2.1.1. Draw the x versus u sketch when $n = 1$ and taking $t = 0$ and $t = 1$ and notice the solution gets translated or transported and hence the name of the PDE is linear transport equation.

§2.2 Solution of a homogeneous problem

Consider the problem

$$\begin{aligned} u_t + b \cdot Du &= f \quad \text{in } \mathbb{R}^n \times (0,\infty) \\ u &= g \quad \text{in } \mathbb{R}^n \times \{0\} \end{aligned}$$

Take any $(x,t) \in \mathbb{R}^n \times (0,\infty)$ and define $z(s) = u(x+sb, t+s)$.
Hence, $z'(s) = u_t(x+sb,t+s) + b \cdot Du(x+sb,t+s) = f(x+sb,t+s)$. So,

$$\begin{aligned} u(x,t) - g(x-tb) &= z(0) - z(-t) \\ &= \int_{-t}^{0} z'(s)\,ds \\ &= \int_{-t}^{0} f(x+sb,t+s)\,ds \\ &= \int_{0}^{t} f(x+(s-t)b,s)\,ds \end{aligned}$$

Note that the last equality is by change of variable. Therefore, we have

$$\boxed{u(x,t) = g(x-tb) + \int_0^t f(x+(s-t)b,s)\,ds} \quad (x \in \mathbb{R}^n, t \geq 0)$$

is the required solution.