# Lecture Notes in Measure Theory

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## Introduction

This is a set of lecture notes which I took for reviewing stuff that I typed after taking class from *Dr. Sachindranath Jayaraman*. All the typos and errors are of mine.

## Contents

## §1    Lecture 1 — 10th August 2022 — Review of things done in the previous semester…

### §1.1    Definitions and Some Results

**Definition §1.1.1** (algebra)**.** Let $\Omega$ be nonempty set. An algebra $\mathscr{F}$ is a collection of subsets of $\Omega$ satisfying the following properties:

1. $\Omega \in \mathscr{F}$,

2. $A \in \mathscr{F} \Rightarrow A^c \in \mathscr{F}$ and

3. $\mathscr{F}$ is closed under finite unions.

It immediately follows from the definition an algebra of sets is closed under taking finite intersections.

**Definition §1.1.2** ($\sigma$-algebra)**.** Let $\Omega$ be nonempty set. A $\sigma$-algebra $\mathscr{F}$ is a collection of subsets of $\Omega$ satisfying the following properties:

1. $\Omega \in \mathscr{F}$,

2. $A \in \mathscr{F} \Rightarrow A^c \in \mathscr{F}$ and

3. $\mathscr{F}$ is closed under countable unions.

**Fact §1.1.3.** *Let $\Omega$ be a set, $\mathscr{F} \subseteq \mathscr{P}(\Omega)$. $\mathscr{F}$ is an $\sigma$-algebra iff $\mathscr{F}$ is an algebra that is continuous from below, that is, if $\{A_n\}_{n \in \mathbb{N}} \subseteq \mathscr{F}$ and $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$ then $\bigcup_n A_n \in \mathscr{F}$.*

**Definition §1.1.4** ($\sigma$-algebra generated by a subset of power set)**.** Let $\Omega$ be a nonempty set. Given an nonempty collection $\mathscr{C}$ of subsets of $\Omega$, the $\sigma$-algebra generated by $\mathscr{C}$, $\sigma(\mathscr{C})$ is defined to be the intersection of all $\sigma$-algebra containing $\mathscr{C}$. Notationally,

$$\sigma(\mathscr{C}) = \bigcap \{\sigma - \text{algebra that contains } \mathscr{C}\}$$

**Definition §1.1.5** (Borel $\sigma$-algebra)**.** If $\Omega$ is a topological space then the Borel $\sigma$-algebra is the smallest$\sigma$-algebra containing the open sets of $\Omega$.

**Fact §1.1.6.** *If $\Omega = \mathbb{R}^n$ the Borel $\sigma$-algebra is generated by*

- $\{(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n) \mid -\infty \le a_i < b_i \le b_i \le +\infty\}$

- $\{(-\infty, a_1) \times (-\infty, a_2) \times \cdots (-\infty, a_n) \mid a_1, a_2, \ldots, a_n \in \mathbb{R}\}$

- $\left\{(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n) \mid a_i, b_i \in \mathbb{Q}\right\}$

**Definition §1.1.7** ($\pi$-system, $\lambda$-system)**.** A collection $\mathscr{C}$ of subsets of $\Omega$ is called a $\pi$-system if $\mathscr{C}$ is closed under finite-$\cap$.

A collection $\mathscr{L}$ of subsets of $\Omega$ is called a $\lambda$-system if the following hold:

- $\Omega \in \mathscr{L}$,

- $A, B \in \mathscr{L}$ and $A \subset B$ implies $B \setminus A \in \mathscr{L}$

- if $\{A_n\}_{n \in \mathbb{N}} \subset \mathscr{L}$ and $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$ then $\bigcup_n A_n \in \mathscr{L}$

**Definition §1.1.8.** Let $\mathscr{C}$ be a collection of nonempty subsets of a nonempty set $\Omega$. The $\lambda$-system generated by $\mathscr{C}$, denoted as $\lambda(\mathscr{C})$ is the intersection of all $\lambda$-systems containing $\mathscr{C}$.

## §1.2 Dynkin's pi-lambda theorem; Measures and their properties

**Theorem §1.2.1** (Dynkin $\pi - \lambda$ theorem)**.** *If $\mathscr{C}$ is a $\pi$-system of a nonempty set $\Omega$ then $\lambda(\mathscr{C}) = \sigma(\mathscr{C})$. Equivalently, if $\mathscr{L}$ is a $\lambda$-system that contains $\mathscr{C}$ then $\mathscr{L} \supset \lambda(\mathscr{C})$.*

**Definition §1.2.2.** Let $\mathscr{F}$ be a $\sigma$-algebra of subsets of $\Omega$. A extended real valued function $\mu$ on $\mathscr{F}$ is called a *measure* if the following hold:

1. $\mu(A) \ge 0$ for all $A \in \mathscr{F}$,

2. $\mu(\emptyset) = 0$

3. If $\{A_n\}_{n\in\mathbb{N}} \subset \mathscr{F}$ such that $\bigcup A_n \in \mathscr{F}$ and $A_n \cap A_m = \emptyset$ for all $m \neq n$ then $\mu(\bigcup_{i\in\mathbb{N}} A_i) = \sum_i \mu(A_i)$

**Example §1.2.3** (Some examples of measures).     1. Let $\Omega \neq \emptyset$, $\mathscr{F} = \mathscr{P}(\Omega)$. We define $\mu$ on $\mathscr{F}$ by $\mu(A)$ is the number of elements of $A$ if $A$ is finite and $\mu = +\infty$ if $A$ contains infinitely many elements. Then $\mu$ is a measure on $\mathscr{F}$.

2. Let $\Omega = [0,1]$ and $\mathscr{F} = \mathscr{B}([0,1])$. Let $\{p_n\}$ be a sequence of numbers in $[0,1]$ such that $\sum p_i = 1$. Define $\mu(A) = \sum_{i\in\mathbb{N}} p_i \delta_{p_i}(A)$. Then $\mu$ is a measure on $\mathscr{F}$.

3. Let $F$ be a non-decreasing right-continuous function on $\mathbb{R}$. Define $\mu_F$ to be Lebesgue-Stieljes measure induced by $F$. Recall that $\mu_F((a,b]) = b - a$. Then $\mu_F$ is an example of $\sigma$-finite Radon measure on the Borel $\sigma$-algebra on $\mathbb{R}$.

**Theorem §1.2.4.** *Let $\mathscr{F}$ be a $\sigma$-algebra on a nonempty set $\Omega$. Let $\mu : \mathscr{F} \to [0,\infty]$ be a function. $\mu$ is a measure on $\mathscr{F}$ iff*

1. *$\mu$ is finitely additive (that is, if $A, B \in \mathscr{F}$ such that $A \cap B = \emptyset$ then $\mu(A \cup B) = \mu(A) + \mu(B)$) and*

2. *$\mu$ is continuous from below (that is, if $\{A_n\}$ is nondecreasing sequence of elements from $\mathscr{F}$ then $\mu(\bigcup(A_i)) = \lim_{n\to\infty} \mu(A_n)$).*

# §2    Lecture 2 — 12th August 2022 — Properties of Measures and Definition of semialgebra

*Proof of Theorem §1.2.4.* Let $\mu : \mathscr{F} \to [0,\infty]$ be a function.

($\Rightarrow$) Suppose that $\mu$ is a measure. We first show that $\mu$ is finitely additive. Let $A, B \in \mathscr{F}$ and suppose that $A \cap B = \emptyset$. Let $A_1 = A$ and $A_2 = B$ and $A_n = \emptyset$ for all $n \geq 3$. Then $\mu(A \cup B) = \mu(\cup_{i\in\mathbb{N}} A_i) = \sum_{i\in\mathbb{N}} \mu(A_i) = \mu(A) + \mu(B)$ as $\mu(\emptyset) = 0$.

We now prove that $\mu$ is continuous from below. Let $\{A_n\}$ be a nondecreasing sequence of elements from $\mathscr{F}$. Define $B_1 = A_1$ and $B_n = A_n \setminus A_{n-1}$ for each $n \geq 2$. Clearly, $\cup_n B_n = \cup_n A_n$ and $B_n \cap B_m = \emptyset$ for all $m \neq n$.

$$
\begin{aligned}
\mu(\cup_n A_n) &= \mu(\cup_n B_n) \\
&= \sum_{n\in N} \mu(B_n) \\
&= \lim_m \left[ \mu(A_1) + \sum_{n=2}^{m} \big(\mu(A_n) - \mu(A_{n-1})\big) \right] \\
&= \lim_m \mu(A_m)
\end{aligned}
$$

($\Leftarrow$) Now suppose that $\mu$ is finitely additive and continuous from below. We intend to prove that $\mu$ is a measure. It is clear that from finite additivity that $\mu(\emptyset) = 0$. Let $\{A_n\}$ be a sequence

of elements from $\mathscr{F}$. Define $B_n = \bigcup_{i=1}^{n} A_i$ for all $n \in \mathbb{N}$. Clearly, $B_n \nearrow \bigcup_{k=1}^{\infty} A_k$. Clearly, $\{B_n\}$ is an nondecreasing sequence of elements from $\mathscr{F}$. Then

$$
\begin{aligned}
\mu\left(\cup_{i=1}^{\infty} A_i\right) &= \mu\left(\cup_{i=1}^{\infty} B_n\right) \\
&= \lim_{n \to \infty} \mu\left(B_n\right) && \text{(using continuity from below)} \\
&= \lim_{n \to \infty} \mu\left(\cup_{j=1}^{n} A_j\right) \\
&= \lim_{n \to \infty} \sum_{j=1}^{n} \mu\left(A_j\right) && \text{(finite additivity)} \\
&= \sum_{j=1}^{\infty} \mu(A_j)
\end{aligned}
$$

$\square$

## §2.1   Properties of Measures

**Theorem §2.1.1.** *Let $\mu$ be a measure on a $\sigma$-algebra $\mathscr{F}$. Then*

*(1)  $\mu$ is monotone,*

*(2)  $\mu$ is finitely additive, that is, $\mu\left(A_1 \cup A_2 \cup \ldots \cup A_n\right) \le \sum_{i=1}^{k} \mu(A_k)$ for $A_1, A_2, \ldots, A_k \in \mathscr{F}$.*

*(3)  the inclusion-exclusion formula holds,*

*(4)  $\mu$ is continuous from above, that is, if $(A_n) \subset \mathscr{F}$ such that $A_n \supset A_{n+1}$ for all $n \in \mathbb{N}$ and $\mu(A_{n_0}) < +\infty$ for some $n_0$ then $\lim \mu(A_n) = \mu(\cap_n A_n)$ and*

*(5)  $\mu$ is countably subadditive, that is, if $\{A_n\}_{n \in \mathbb{N}} \subset \mathscr{F}$ then $\mu\left(\cup_n A_n\right) \le \sum_{i=1}^{\infty} \mu\left(A_n\right)$*

*Proof.* We show that $\mu$ is monotone. Let $A \subset B$ be elements of $\mathscr{F}$. Then $B = A \cup B \setminus A$. Hence $\mu(B) = \mu(A) + \mu(B \setminus A)$. Since $\mu(B \setminus A) \ge 0$, we have that $\mu(B) \ge \mu(A)$.

Now, we prove that the inclusion exclusion formula holds for $\mu$. Let $A, B \in \mathscr{F}$. If both $\mu(A) = +\infty$ and $\mu(B) = +\infty$ then there is nothing to prove. So, assume wlog that $\mu(A) < \infty$. Then $\mu(A \cap B) < \infty$. Then

$$
\begin{aligned}
\mu(A \cup B) + \mu(A \cap B) &= \mu(A) + \mu(B \setminus A) + \mu(B \cap A) \\
&= \mu(A) + (\mu(B \setminus A) + \mu(B \cap A)) \\
&= \mu(A) + \mu(B)
\end{aligned}
$$

Now, finite subadditivity follows immediately from inclusion-exclusion formula and induction.

We now prove that $\mu$ is continuous from below. Let $\{A_n\}$ be a sequence of decreasing sequence sets with $\mu(A_{n_0}) < +\infty$ for some $n_0$. Then we have that $\mu(A_1) \le \mu(A_{n_0}) < +\infty$. Define $B_n = A_1 \setminus A_n$ and $B = A_1 \setminus \cap_n A_n$. It is easy to see that $B_n \uparrow B$ (draw pictures!). From Theorem §1.2.4 (continuity from below), we have that $\lim \mu(B_n) = \mu(B)$.

Now, observe that $\mu(B_n) = \mu(A_1) - \mu(A_n)$ for each $n$. So, $\mu(B) = \lim \mu(B_n) = \mu(A_1) - \lim \mu(A_n)$.

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Also, we have that $\mu(B) = \mu(A_1) - \mu(\cap_n A_n)$. Hence, we have that $\lim \mu(A_n) = \mu(\cap_n A_n)$.

We now prove that $\mu$ is countably subadditive. Let $\{A_n\}$ be a sequence of elements from $\mathscr{F}$. Then $B_k := \cup_{n=1}^k A_n \uparrow \cup_n A_n$. By continuity from below, we have that $\mu(\cup_n A_n) = \lim_k \mu(B_k) \leq \lim_k \left( \mu(A_1) + \mu(A_2) + \ldots + \mu(A_k) \right) = \sum_{k=1}^\infty \mu(A_k)$. Note that the inequality is due to finite subadditivity. $\qquad\square$

**Definition §2.1.2.** A collection $\mathscr{C}$ of subsets of $\Omega$ is called a *semialgebra* if $\mathscr{C}$ is closed under finite-$\cap$ and if $A \in \mathscr{C}$ then there exists some $B_1, B_2, \ldots, B_n \in \mathscr{C}$ such that $A^c = \cup_{i=1}^n B_i$.

**Exercise §2.1.3.** *Find a general formulation of the inclusion-exclusion principle for measures.*