Lecture Notes in Finite Frames

Joel Sleeba

forked from Ashish Kujur

Last Updated: August 23, 2022

Introduction

This is a set of lecture notes which I took for reviewing stuff that I typed after taking class from *Dr P Devaraj*. All the typos and errors are of mine.

Contents

§1 Lecture 1 — 10th August, 2022 — Hilbert Spaces & Frames

We start by reviewing the elementary notions from Linear Algebra.

§1.1 Inner Product Spaces

Definition §1.1.1. A vector space V over a field F ($\mathbb{R}$ or $\mathbb{C}$) is called an *inner product space* if there exists a function $\langle \cdot, \cdot \rangle : V \times V \to F$ satisfying the following:

1. $\langle x, x \rangle \geq 0$ for all $x \in V$.

2. $\langle x, x \rangle = 0$ iff $x = 0$.

3. (linear in the first argument) for all $x, y, z \in V$, $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

4. (conjugate) for all $x, y \in V$, $\langle x, y\rangle = \overline{\langle y, x\rangle}$

Definition §1.1.2. A norm on a vector space V over $\mathbb{R}$ or $\mathbb{C}$ is a function $\|\cdot\| : V \to [0, +\infty)$ satisfying

1. for all $x \in V$, $\|x\| = 0 \Leftrightarrow x = 0$
2. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$
3. $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in V$ and $\alpha \in F$

It is easy to check that if V is an inner product space then $\|\cdot\|$ defined by $\|x\| = \sqrt{\langle x, x\rangle}$ is a norm on V. To verify the triangle inequality, use Cauchy Schwarz inequality.

Definition §1.1.3. A vector space together with a norm is called a normed linear space.

Note that every normed linear space $(V, \|\cdot\|)$ is a metric space. The metric is given by $d(x, y) = \|x - y\|$ for all $x, y \in V$.

§1.2 Hilbert Spaces & Frames

Definition §1.2.1. An inner product space which is complete wrt the induced norm is called Hilbert Space.

We will only be considering finite dimensional Hilbert spaces in this course!

Example §1.2.2. 1. $\mathbb{R}^n$ with the usual inner product is a Hilbert Space.

2. $\mathbb{C}^n$ with the usual inner product is a Hilbert Space.

Definition §1.2.3. A sequence $\{f_n\}$ in H is called a frame for H if there exists positive constants A and B such that

$$A\|f\|^2 \leq \sum_i |\langle f, f_i\rangle|^2 \leq B\|f\|^2 \tag{1}$$

for all $f \in H$.

Remark §1.2.4. It is possible to have that a frame in a finite dimensional Hilbert space consisting of infinitely many elements. However, it is rather artificial to have infinite number of frame elements in a finite dimensional space. We therefore consider only frames with finite number of elements.

§2 Lecture 2 — 12th August, 2022 — A hell lot of definitions! (and some examples)

§2.1 Some definition and remarks on the definition of Frames

1. In 1, A, B are called the frame bounds.
2. The infimum (corr. supremum) over all the upper (corr. lower) frame bounds is called the optimal upper (corr. lower) frame bound.
3. The optimal frame bounds are also frame bounds! We can verify this in this fashion: Let β be the optimal upper frame bound. Note that in equation (1) holds trivially for $f = 0$. So, we need to only consider the case when $f \neq 0$. Let B be any upper frame bound and f be any nonzero vector. Then $\sum_i \|\langle f, f_i\rangle\|^2 \leq B\|f\|^2$. Since $f \neq 0$ by our choice, $\sum_i \|\langle f, f_i\rangle\|^2/\|f\|^2 \leq B$. Since B is arbitrary, we have that $\sum_i \|\langle f, f_i\rangle\|^2/\|f\|^2 \leq \beta$. Now since f was arbitrary, out result follows. The almost same proof works for optimal lower bound as well!
4. If $A = B$ in equation 1 then frame is called a *tight frame*.
5. If $A = B = 1$ in equation 1 then the frame is called a *Parseval frame* (as it will then satisfy the Parseval's identity).
6. A frame $\{f_i\}$ is called *equiangular* if there is a constant C such that $|\langle f, f_i\rangle| = C$ for all $i \neq j$.
7. A frame $\{f_i\}_{i\in I}$ is called *equal norm frame* if there is a constant C such that $\|f_i\| = C$ for all $i \in I$.
8. A frame $\{f_i\}_{i\in I}$ is called a *exact frame* if for any $j \in I$, $\{f_i\}_{i\in I\setminus\{j\}}$ is no longer a frame!
9. Let $\{f_i\}_{i\in I}$ be a frame and $x \in H$. Then the values $\{\langle x, f_i\rangle\}_{i\in I}$ are called the *frame coefficients* of x.
10. A sequence $\{f_i\}_{i=1}^N$ is called a *Bessel sequence* if there is a positive constant B such that $\sum_{i=1}^N |\langle f, f_i\rangle| \leq B\|f\|^2$ for all $f \in H$.

§2.2 Examples of Frames

In the following examples, let $\{e_1, e_2, \ldots, e_n\}$ be orthonormal basis for H.

1. Consider the list $\{e_1/\sqrt{2}, e_1/\sqrt{2}, e_2/\sqrt{2}, e_2/\sqrt{2}, \ldots, e_n/\sqrt{2}, e_n/\sqrt{2}\}$. Then

$$\begin{aligned}\sum_{i=1}^N |\langle f, f_i\rangle|^2 &= \frac{1}{2}\sum_{i=1}^n |\langle f, e_i\rangle| + \frac{1}{2}\sum_{i=1}^n |\langle f, e_i\rangle| \\ &= \sum_{i=1}^n |\langle f, e_i\rangle|^2 = \|f\|\end{aligned}$$

The last equality holds because $\{e_i\}_{i=1}^n$ is an orthonormal basis. So the aforementioned list of vectors is a Parseval frame.

2. Consider the list $\{e_1, e_1, e_2, \dots, e_n\}$. Then

$$\begin{aligned}\sum_{i=1}^{N} |\langle f, f_i\rangle|^2 &= |\langle f, e_1\rangle|^2 + \sum_{i=1}^{n} |\langle f, e_i\rangle|^2 \\ &\leq \|f\|^2 + \|f\|^2 \\ &= 2\|f\|^2\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{N} |\langle f, f_i\rangle|^2 &= |\langle f, e_1\rangle|^2 + \sum_{i=1}^{n} |\langle f, e_i\rangle|^2 \\ &\geq \|f\|^2\end{aligned}$$

Thus, $\{e_1, e_1, e_2, \dots, e_n\}$ is a frame for H with frame bounds 1 and 2. In fact the frame bounds are optimal, consider $f = e_1$ and $f = e_2$ and note the frame bounds are actually achieved!

3. Consider $\{e_1, e_1, e_2, e_2, \dots, e_n, e_n\}$. Then this is a tight frame bound with bound 2.

§2.3 Properties of frames in finite dimensional Hilbert spaces

Note that the following lemma only holds for finite dimensional Hilbert space H.

Lemma §2.3.1. *Let $\{f_i\}_{i\in I}$ be a family of vectors in H. Then*

(1) If $\{f_i\}_{i\in I}$ is an orthonormal basis, then $\{f_i\}_{i\in I}$ is a Parseval frame but the converse may not be true!

(2) $\{f_i\}_{i\in I}$ is a frame for H iff span $\{f_i\} = H$

(3) If $\{f_i\}$ is a unit norm Parseval frame iff $\{f_i\}_{i\in I}$ is an orthonormal basis for H.

(4) $\{f_i\}$ is exact then $\{f_i\}$ is linearly independent.