

# Lecture Notes in Measure Theory

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## Introduction

This is a set of lecture notes which I took for reviewing stuff that I typed after taking class from *Dr. Sachindranath Jayaraman*. All the typos and errors are of mine.

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## §1 Lecture 1 — 10th August 2022 — Review of things done in the previous semester...

### §1.1 Definitions and Some Results

**Definition §1.1.1** (algebra). Let  $\Omega$  be nonempty set. An algebra  $\mathcal{F}$  is a collection of subsets of  $\Omega$  satisfying the following properties:

1.  $\Omega \in \mathcal{F}$ ,
2.  $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$  and
3.  $\mathcal{F}$  is closed under finite unions.

It immediately follows from the definition an algebra of sets is closed under taking finite intersections.

**Definition §1.1.2** ( $\sigma$ -algebra). Let  $\Omega$  be nonempty set. A  $\sigma$ -algebra  $\mathcal{F}$  is a collection of subsets of  $\Omega$  satisfying the following properties:

1.  $\Omega \in \mathcal{F}$ ,
2.  $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$  and
3.  $\mathcal{F}$  is closed under countable unions.

**Fact §1.1.3.** Let  $\Omega$  be a set,  $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ .  $\mathcal{F}$  is an  $\sigma$ -algebra iff  $\mathcal{F}$  is an algebra that is continuous from below, that is, if  $\{A_n\}_{n \in \mathbb{N}} \subseteq \mathcal{F}$  and  $A_n \subset A_{n+1}$  for all  $n \in \mathbb{N}$  then  $\bigcup_n A_n \in \mathcal{F}$ .

**Definition §1.1.4** ( $\sigma$ -algebra generated by a subset of power set). Let  $\Omega$  be a nonempty set. Given an nonempty collection  $\mathcal{C}$  of subsets of  $\Omega$ , the  $\sigma$ -algebra generated by  $\mathcal{C}$ ,  $\sigma(\mathcal{C})$  is defined to be the intersection of all  $\sigma$ -algebra containing  $\mathcal{C}$ . Notationally,

$$\sigma(\mathcal{C}) = \bigcap \{\sigma\text{-algebra that contains } \mathcal{C}\}$$

**Definition §1.1.5** (Borel  $\sigma$ -algebra). If  $\Omega$  is a topological space then the Borel  $\sigma$ -algebra is the smallest  $\sigma$ -algebra containing the open sets of  $\Omega$ .

**Fact §1.1.6.** If  $\Omega = \mathbb{R}^n$  the Borel  $\sigma$ -algebra is generated by

- $\{(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n) \mid -\infty \leq a_i < b_i \leq b_i \leq +\infty\}$
- $\{(-\infty, a_1) \times (-\infty, a_2) \times \cdots \times (-\infty, a_n) \mid a_1, a_2, \dots, a_n \in \mathbb{R}\}$
- $\{(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n) \mid a_i, b_i \in \mathbb{Q}\}$

**Definition §1.1.7** ( $\pi$ -system,  $\lambda$ -system). A collection  $\mathcal{C}$  of subsets of  $\Omega$  is called a  $\pi$ -system if  $\mathcal{C}$  is closed under finite- $\cap$ .

A collection  $\mathcal{L}$  of subsets of  $\Omega$  is called a  $\lambda$ -system if the following hold:

- $\Omega \in \mathcal{L}$ ,
- $A, B \in \mathcal{L}$  and  $A \subset B$  implies  $B \setminus A \in \mathcal{L}$
- if  $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{L}$  and  $A_n \subset A_{n+1}$  for all  $n \in \mathbb{N}$  then  $\bigcup_n A_n \in \mathcal{L}$

**Definition §1.1.8.** Let  $\mathcal{C}$  be a collection of nonempty subsets of a nonempty set  $\Omega$ . The  $\lambda$ -system generated by  $\mathcal{C}$ , denoted as  $\lambda(\mathcal{C})$  is the intersection of all  $\lambda$ -systems containing  $\mathcal{C}$ .

## §1.2 Dynkin's pi-lambda theorem; Measures and their properties

**Theorem §1.2.1** (Dynkin  $\pi - \lambda$  theorem). *If  $\mathcal{C}$  is a  $\pi$ -system of a nonempty set  $\Omega$  then  $\lambda(\mathcal{C}) = \sigma(\mathcal{C})$ . Equivalently, if  $\mathcal{L}$  is a  $\lambda$ -system that contains  $\mathcal{C}$  then  $\mathcal{L} \supset \lambda(\mathcal{C})$ .*

**Definition §1.2.2.** Let  $\mathcal{F}$  be a  $\sigma$ -algebra of subsets of  $\Omega$ . A extended real valued function  $\mu$  on  $\mathcal{F}$  is called a *measure* if the following hold:

1.  $\mu(A) \geq 0$  for all  $A \in \mathcal{F}$ ,
2.  $\mu(\emptyset) = 0$
3. If  $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{F}$  such that  $\bigcup A_n \in \mathcal{F}$  and  $A_n \cap A_m = \emptyset$  for all  $m \neq n$  then  $\mu(\bigcup_{i \in \mathbb{N}} A_i) = \sum_i \mu(A_i)$

**Example §1.2.3** (Some examples of measures). 1. Let  $\Omega \neq \emptyset$ ,  $\mathcal{F} = \mathcal{P}(\Omega)$ . We define  $\mu$  on  $\mathcal{F}$  by  $\mu(A)$  is the number of elements of  $A$  if  $A$  is finite and  $\mu = +\infty$  if  $A$  contains infinitely many elements. Then  $\mu$  is a measure on  $\mathcal{F}$ .

2. Let  $\Omega = [0, 1]$  and  $\mathcal{F} = \mathcal{B}([0, 1])$ . Let  $\{p_n\}$  be a sequence of numbers in  $[0, 1]$  such that  $\sum p_i = 1$ . Define  $\mu(A) = \sum_{i \in \mathbb{N}} p_i \delta_{p_i}(A)$ . Then  $\mu$  is a measure on  $\mathcal{F}$ .

3. Let  $F$  be a non-decreasing right-continuous function on  $\mathbb{R}$ . Define  $\mu_F$  to be Lebesgue-Stieljes measure induced by  $F$ . Recall that  $\mu_F((a, b]) = b - a$ . Then  $\mu_F$  is an example of  $\sigma$ -finite Radon measure on the Borel  $\sigma$ -algebra on  $\mathbb{R}$ .

**Theorem §1.2.4.** *Let  $\mathcal{F}$  be a  $\sigma$ -algebra on a nonempty set  $\Omega$ . Let  $\mu : \mathcal{F} \rightarrow [0, \infty]$  be a function.  $\mu$  is a measure on  $\mathcal{F}$  iff*

1.  $\mu$  is finitely additive (that is, if  $A, B \in \mathcal{F}$  such that  $A \cap B = \emptyset$  then  $\mu(A \cup B) = \mu(A) + \mu(B)$ ) and
2.  $\mu$  is continuous from below (that is, if  $\{A_n\}$  is nondecreasing sequence of elements from  $\mathcal{F}$  then  $\mu(\bigcup (A_i)) = \lim_{n \rightarrow \infty} \mu(A_n)$ ).

## §2 Lecture 2 — 12th August 2022 — Properties of Measures and Definition of semialgebra

*Proof of Theorem §1.2.4.* Let  $\mu : \mathcal{F} \rightarrow [0, \infty]$  be a function.

( $\Rightarrow$ ) Suppose that  $\mu$  is a measure. We first show that  $\mu$  is finitely additive. Let  $A, B \in \mathcal{F}$  and suppose that  $A \cap B = \emptyset$ . Let  $A_1 = A$  and  $A_2 = B$  and  $A_n = \emptyset$  for all  $n \geq 3$ . Then  $\mu(A \cup B) = \mu(\bigcup_{i \in \mathbb{N}} A_i) = \sum_{i \in \mathbb{N}} \mu(A_i) = \mu(A) + \mu(B)$  as  $\mu(\emptyset) = 0$ .

We now prove that  $\mu$  is continuous from below. Let  $\{A_n\}$  be an nondecreasing sequence of elements from  $\mathcal{F}$ . Define  $B_1 = A_1$  and  $B_n = A_n \setminus A_{n-1}$  for each  $n \geq 2$ . Clearly,  $\bigcup_n B_n = \bigcup_n A_n$

and  $B_n \cap B_m = \emptyset$  for all  $m \neq n$ .

$$\begin{aligned}
\mu(\cup_n A_n) &= \mu(\cup_n B_n) \\
&= \sum_{n \in \mathbb{N}} \mu(B_n) \\
&= \lim_m \left[ \mu(A_1) + \sum_{n=2}^m (\mu(A_n) - \mu(A_{n-1})) \right] \\
&= \lim_m \mu(A_m)
\end{aligned}$$

( $\Leftarrow$ ) Now suppose that  $\mu$  is finitely additive and continuous from below. We intend to prove that  $\mu$  is a measure. It is clear that from finite additivity that  $\mu(\emptyset) = 0$ . Let  $\{A_n\}$  be a sequence of elements from  $\mathcal{F}$ . Define  $B_n = \bigcup_{i=1}^n A_i$  for all  $n \in \mathbb{N}$ . Clearly,  $B_n \nearrow \bigcup_{k=1}^{\infty} A_k$ . Clearly,  $\{B_n\}$  is a nondecreasing sequence of elements from  $\mathcal{F}$ . Then

$$\begin{aligned}
\mu(\cup_{i=1}^{\infty} A_i) &= \mu(\cup_{i=1}^{\infty} B_n) \\
&= \lim_{n \rightarrow \infty} \mu(B_n) && \text{(using continuity from below)} \\
&= \lim_{n \rightarrow \infty} \mu(\cup_{j=1}^n A_j) \\
&= \lim_{n \rightarrow \infty} \sum_{j=1}^n \mu(A_j) && \text{(finite additivity)} \\
&= \sum_{j=1}^{\infty} \mu(A_j)
\end{aligned}$$

□

## §2.1 Properties of Measures

**Theorem §2.1.1.** *Let  $\mu$  be a measure on a  $\sigma$ -algebra  $\mathcal{F}$ . Then*

- (1)  $\mu$  is monotone,
- (2)  $\mu$  is finitely additive, that is,  $\mu(A_1 \cup A_2 \cup \dots \cup A_n) \leq \sum_{i=1}^n \mu(A_i)$  for  $A_1, A_2, \dots, A_n \in \mathcal{F}$ .
- (3) the inclusion-exclusion formula holds,
- (4)  $\mu$  is continuous from above, that is, if  $(A_n) \subset \mathcal{F}$  such that  $A_n \supset A_{n+1}$  for all  $n \in \mathbb{N}$  and  $\mu(A_{n_0}) < +\infty$  for some  $n_0$  then  $\lim \mu(A_n) = \mu(\cap_n A_n)$  and
- (5)  $\mu$  is countably subadditive, that is, if  $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{F}$  then  $\mu(\cup_n A_n) \leq \sum_{i=1}^{\infty} \mu(A_i)$

*Proof.* We show that  $\mu$  is monotone. Let  $A \subset B$  be elements of  $\mathcal{F}$ . Then  $B = A \cup B \setminus A$ . Hence  $\mu(B) = \mu(A) + \mu(B \setminus A)$ . Since  $\mu(B \setminus A) \geq 0$ , we have that  $\mu(B) \geq \mu(A)$ .

Now, we prove that the inclusion exclusion formula holds for  $\mu$ . Let  $A, B \in \mathcal{F}$ . If both  $\mu(A) = +\infty$  and  $\mu(B) = +\infty$  then there is nothing to prove. So, assume wlog that  $\mu(A) < \infty$ . Then  $\mu(A \cap B) < \infty$ . Then

$$\begin{aligned}\mu(A \cup B) + \mu(A \cap B) &= \mu(A) + \mu(B \setminus A) + \mu(B \cap A) \\ &= \mu(A) + (\mu(B \setminus A) + \mu(B \cap A)) \\ &= \mu(A) + \mu(B)\end{aligned}$$

Now, finite subadditivity follows immediately from inclusion-exclusion formula and induction.

We now prove that  $\mu$  is continuous from below. Let  $\{A_n\}$  be a sequence of decreasing sequence sets with  $\mu(A_{n_0}) < +\infty$  for some  $n_0$ . Then we have that  $\mu(A_1) \leq \mu(A_{n_0}) < +\infty$ . Define  $B_n = A_1 \setminus A_n$  and  $B = A_1 \setminus \cap_n A_n$ . It is easy to see that  $B_n \uparrow B$  (draw pictures!). From Theorem §1.2.4 (continuity from below), we have that  $\lim \mu(B_n) = \mu(B)$ .

Now, observe that  $\mu(B_n) = \mu(A_1) - \mu(A_n)$  for each  $n$ . So,  $\mu(B) = \lim \mu(B_n) = \mu(A_1) - \lim \mu(A_n)$ .

Also, we have that  $\mu(B) = \mu(A_1) - \mu(\cap_n A_n)$ . Hence, we have that  $\lim \mu(A_n) = \mu(\cap_n A_n)$ .

We now prove that  $\mu$  is countably subadditive. Let  $\{A_n\}$  be a sequence of elements from  $\mathcal{F}$ . Then  $B_k := \cup_{n=1}^k A_n \uparrow \cup_n A_n$ . By continuity from below, we have that  $\mu(\cup_n A_n) = \lim_k \mu(B_k) \leq \lim_k (\mu(A_1) + \mu(A_2) + \dots + \mu(A_k)) = \sum_{k=1}^{\infty} \mu(A_k)$ . Note that the inequality is due to finite subadditivity.  $\square$

**Definition §2.1.2.** A collection  $\mathcal{C}$  of subsets of  $\Omega$  is called a *semialgebra* if  $\mathcal{C}$  is closed under finite  $\cap$  and if  $A \in \mathcal{C}$  then there exists some  $B_1, B_2, \dots, B_n \in \mathcal{C}$ , pairwise disjoint, such that  $A^c = \cup_{i=1}^n B_i$ .

**Exercise §2.1.3.** Find a general formulation of the inclusion-exclusion principle for measures.

### §3 Lecture 3 — From semialgebra to algebra, measurable sets...

*Remark §3.0.1.* If  $\mathcal{C}$  is a nonempty semialgebra on  $\Omega$  then  $\emptyset \in \mathcal{C}$ .

This remark can be verified as follows: Since  $\mathcal{C}$  is nonempty, let  $A \in \mathcal{C}$ . Since  $\mathcal{C}$  is a semialgebra, there are elements  $B_i \in \mathcal{C} (1 \leq i \leq n)$  such that  $A^c = \cup_{i=1}^n B_i$  and  $B_i \cap B_j = \emptyset$  for  $i \neq j$ . If  $n = 1$ , then  $A^c \in \mathcal{C}$  and hence  $A \cap A^c = \emptyset \in \mathcal{C}$ . Now if  $n \geq 2$ , we have that  $B_1 \cap B_2 = \emptyset \in \mathcal{C}$ .

*Remark §3.0.2.* Let  $\{\mathcal{A}_i\}_{i \in I}$  be a collection of algebras on  $\Omega$ . Then it can be easily checked that  $\bigcap_{i \in I} \mathcal{A}_i$  is an algebra on  $\Omega$ . So if  $\mathcal{C}$  is a collection of subsets of  $\Omega$  then we denote  $\mathcal{A}(\mathcal{C})$  to be the smallest algebra generated by  $\mathcal{C}$ , which is in fact, the intersection of all algebras that contain  $\mathcal{C}$ .

**Definition §3.0.3** (Measure on a semi-algebra). A nonnegative set function  $\mu$  on a semialgebra  $\mathcal{C}$  of subsets of  $\Omega$  is called a *measure on  $\mathcal{C}$*  if

(i)  $\mu(\emptyset) = 0$

(ii)  $\mu$  is countably additive, that is, if  $(A_n) \subset \mathcal{C}$ ,  $A_i \cap A_j = \emptyset$  and  $\bigcap_{n \in \mathbb{N}} A_n \in \mathcal{C}$  then

$$\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \mu(A_n)$$

### §3.1 Extension of measures from semialgebras to algebras

Let  $\mathcal{C}$  be a semialgebra on  $\Omega$ . Define  $\mathcal{A}$  be the collection of all finite unions of elements of  $\mathcal{C}$ . Then in Question 6 from Assignment 1, we showed that  $\mathcal{A} = \mathcal{A}(\mathcal{C})$ , that is,  $\mathcal{A}$  is the smallest algebra containing  $\mathcal{C}$ . The following lemma has much more to say though:

**Lemma §3.1.1.** *Let  $\mathcal{C}$  be a semialgebra of  $\Omega$ . Let*

$$\mathcal{F}(\mathcal{C}) := \{A \subset X : A = \bigcup_{i=1}^n B_i \text{ for some } B_i \in \mathcal{C}, B_i \cap B_j = \emptyset \text{ for } i \neq j\}$$

*Then  $\mathcal{F}(\mathcal{C}) = \mathcal{A}(\mathcal{C})$ .*

*Proof.* Clearly,  $\mathcal{F}(\mathcal{C}) \subset \mathcal{A}(\mathcal{C})$  (see the previous paragraph). Since  $\mathcal{A}(\mathcal{C})$  is the smallest algebra generated by  $\mathcal{C}$ , we will be done if we show that  $\mathcal{F}(\mathcal{C})$  is an algebra containing  $\mathcal{C}$ .

Clearly,  $\emptyset \in \mathcal{F}(\mathcal{C})$  as  $\emptyset \in \mathcal{C}$ . The fact that  $\mathcal{F}(\mathcal{C})$  is closed under finite unions is pretty evident. To show that  $\mathcal{F}(\mathcal{C})$  is closed under complement, let  $A \in \mathcal{F}(\mathcal{C})$ . Then there are elements  $\{B_i\}_{1 \leq i \leq n} \subset \mathcal{C}$ , pairwise disjoint, such that  $A = \bigcup_{i=1}^n B_i$ . Then since  $B_i \in \mathcal{C}$ , there exists  $k_i \in \mathbb{N}$  and  $C_{i1}, C_{i2}, \dots, C_{i,k_i} \in \mathcal{C}$  such that  $B_i^c = \bigcup_{j=1}^{k_i} C_{i,j}$ . Then  $A^c = \bigcap_{i=1}^n \bigcup_{j=1}^{k_i} C_{i,j}$ . Interchanging the union and intersection, the result quickly follows.  $\square$

**Theorem §3.1.2.** *Suppose  $\mu$  is a measure on a semialgebra  $\mathcal{C}$  of subsets of  $\Omega$ . Let  $\mathcal{A}$  be the algebra generated by  $\mathcal{C}$ . If  $A \in \mathcal{A}$  has a representation  $A = \bigcup_{i=1}^n B_i$ , where  $B_i \in \mathcal{C}$  and  $B_i \cap B_j = \emptyset$  for  $i \neq j$ , then we define a function  $\bar{\mu}$  on a subset of  $\mathcal{A}$  where the elements  $A \in \mathcal{A}$  have aforementioned representation given by  $\bar{\mu}(A) = \sum_{i=1}^n \mu(B_i)$ . Then*

1.  $\bar{\mu}$  is well defined,
2.  $\bar{\mu}$  is finitely additive and
3.  $\bar{\mu}$  is countably additive.

*Proof.* We first show that  $\bar{\mu}$  is indeed well-defined. Let  $A \in \mathcal{A}$  and suppose that  $A = \bigcup_{i=1}^m B_i$  and  $A = \bigcup_{j=1}^n C_j$  where  $\{B_i\} \subset \mathcal{C}$  and  $\{C_j\} \subset \mathcal{C}$  are pairwise disjoint. Then note that for  $i \in \{1 \leq l \leq n\}$ , we have that  $B_i = B_i \cap A = B_i \cap \left(\bigcup_{j=1}^n C_j\right) = \bigcup_{j=1}^n (B_i \cap C_j)$ . Note that

previous union is a pairwise disjoint union. Hence,

$$\begin{aligned}
\sum_{i=1}^m \mu(B_i) &= \sum_{i=1}^m \sum_{j=1}^n \mu(B_i \cap C_j) \\
&= \sum_{j=1}^n \sum_{i=1}^m \mu(B_i \cap C_j) \\
&= \sum_{j=1}^n \mu(C_j)
\end{aligned}$$

This shows that  $\bar{\mu}$  is well defined.

We show that  $\bar{\mu}$  is countably additive. Let  $\{A_n\}$  be the collection of elements of  $\mathcal{A}$  which can be written as a union of pairwise disjoint elements of  $\mathcal{C}$ ,  $A_n \cap A_m = \emptyset$  for  $n \neq m$  and  $\cup_{n \in \mathbb{N}} A_n \in \mathcal{A}$ .

Then for each  $n$ , there is  $k_n \in \mathbb{N}$  and pairwise disjoint elements  $B_{n1}, B_{n2}, \dots, B_{n,k_n} \in \mathcal{C}$  such that  $A_n = \cup_{i=1}^{k_n} B_{n,i}$ .

We also have by Lemma §3.1.2 that there are some  $B_1, \dots, B_k \in \mathcal{C}$ , pairwise disjoint, such that  $\cup_{n \in \mathbb{N}} A_n = \cup_{i=1}^k B_i$ .

Now, for any  $i$ , we have that

$$\begin{aligned}
B_i &= B_i \cap \left( \bigcup_{n \in \mathbb{N}} A_n \right) \\
&= \bigcup_{n \in \mathbb{N}} (B_i \cap A_n) \\
&= \bigcup_{n \in \mathbb{N}} \left( B_i \cap \bigcup_{j=1}^{k_n} B_{n,j} \right) \\
&= \bigcup_{n \in \mathbb{N}} \bigcup_{j=1}^{k_n} (B_i \cap B_{n,j}) \\
&= \bigcup_{\substack{n \in \mathbb{N} \\ j \in \{1, \dots, k_n\}}} (B_i \cap B_{n,j})
\end{aligned}$$

Note that the previous union is a pairwise disjoint union. Thus by definition of measure on a semialgebra, we have that for any  $i$

$$\mu(B_i) = \sum_{\substack{n \in \mathbb{N} \\ j \in \{1, \dots, k_n\}}} \mu(B_i \cap B_{n,j})$$

Hence,

$$\begin{aligned}
\bar{\mu}(A) &= \sum_{i=1}^k \mu(B_i) \\
&= \sum_{i=1}^k \sum_{\substack{n \in \mathbb{N} \\ j \in \{1, \dots, k_n\}}} \mu(B_i \cap B_{n,j}) \\
&= \sum_{\substack{n \in \mathbb{N} \\ j \in \{1, \dots, k_n\}}} \sum_{i=1}^k \mu(B_i \cap B_{n,j}) \\
&= \sum_{\substack{n \in \mathbb{N} \\ j \in \{1, \dots, k_n\}}} \mu \left( \bigcup_{i=1}^k (B_i \cap B_{n,j}) \right) \\
&= \sum_{\substack{n \in \mathbb{N} \\ j \in \{1, \dots, k_n\}}} \mu(B_{n,j}) \\
&= \sum_{n \in \mathbb{N}} \sum_{j=1}^{k_n} \mu(B_{n,j}) \\
&= \sum_{n \in \mathbb{N}} \bar{\mu}(A_n)
\end{aligned}$$

This completes the proof of countable additivity. The proof of finite additivity follows from countable subadditivity.  $\square$

### §3.2 Outer Measures

**Definition §3.2.1.** Given a measure  $\mu$  on a semialgebra  $\mathcal{C}$ , the *outer measure induced by  $\mu$*  is the set function  $\mu^*$  defined on  $\mathcal{P}(\Omega)$  as

$$\mu^*(A) := \inf \left\{ \sum_{n=1}^{\infty} \mu(A_n) : \{A_n\}_{n \geq 1} \subset \mathcal{C}, A \subset \bigcup_{n \geq 1} A_n \right\}$$

We'd like to remark that  $\mu^*$  is not an overestimate, that is,  $\mu^* = \mu$  on  $\mathcal{C}$  and  $\mu^* = \bar{\mu}$  on  $\mathcal{A}$ .

To verify this remark, let  $C \in \mathcal{C}$ . We need to show that  $\mu^*(C) = \mu(C)$ . Clearly by definition of  $\mu^*$ , we have that  $\mu^*(C) \leq \mu(C)$ . (Fill the details!)

**Definition §3.2.2.** A set  $A$  is said to be  $\mu^*$ -measurable if

$$\mu^*(E) = \mu^*(A \cap E) + \mu^*(A^c \cap E) \text{ for all } E \subset \Omega$$

The set of all  $\mu^*$ -measurable sets is denoted by  $\mathcal{M}_{\mu^*}$ .