Lecture Notes in Measure Theory

Joel Sleeba
forked from Ashish Kujur

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Introduction

This is a set of lecture notes which I took for reviewing stuff that I typed after taking class from *Dr. Sachindranath Jayaraman*. All the typos and errors are of mine.

Contents

§1 Lecture 1 — 10th August 2022 — Review of things done in the previous semester...

§1.1 Definitions and Some Results

Definition §1.1.1 (algebra)**.** Let Ω be nonempty set. An algebra $\mathscr{F}$ is a collection of subsets of Ω satisfying the following properties:

1. $\Omega \in \mathscr{F}$,
2. $A \in \mathscr{F} \implies A^c \in \mathscr{F}$ and
3. $\mathscr{F}$ is closed under finite unions.

Remark §1.1.2. It immediately follows from the definition that an algebra of sets is closed under taking finite intersections. Take compliment of finite intersections and make use of De Morgan's theorem.

Definition §1.1.3 (σ-algebra)**.** Let Ω be nonempty set. A σ-algebra $\mathscr{F}$ is a collection of subsets of Ω satisfying the following properties:

1. $\Omega \in \mathscr{F}$,
2. $A \in \mathscr{F} \implies A^c \in \mathscr{F}$ and
3. $\mathscr{F}$ is closed under countable unions.

Remark §1.1.4. Similar to what we saw in an algebra of sets, the σ-algebra of sets is also closed under countable intersection. The proof is similar to that of the same with algebra of sets.

Proposition §1.1.5. *Let Ω be a set, $\mathscr{F} \subseteq \mathscr{P}(\Omega)$. $\mathscr{F}$ is an σ-algebra iff $\mathscr{F}$ is an algebra that is continuous from below, that is, if $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathscr{F}$ and $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$ then $\bigcup_n A_n \in \mathscr{F}$.*

Proof. ($\implies$) Since $\mathscr{F}$ is closed under countable unions of elements of $\mathscr{F}$ it is also closed under countable unions. To prove this assume $\{A_i\}_{i=1}^N$ is the finite collection of sets, then take $\{B_i\}_{i\in\mathbb{N}}$ where

$$B_i = \begin{cases} A_i, \text{ if } 1 \le i \le N \\ \emptyset, \text{ if } i > N \end{cases}$$

Then $\bigcup_{i=1}^N A_i = \bigcup_{i\in\mathbb{N}} B_i \in \mathscr{F}$, hence $\mathscr{F}$ is an algebra.

Again if $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathscr{F}$ and $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$, then $\bigcup_n A_n \in \mathscr{F}$, since $\mathscr{F}$ being a σ-algebra is closed under countable unions

($\impliedby$)We'll prove an algebra satisfying $\bigcup_n A_n \in \mathscr{F}$, when $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathscr{F}$ and $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$ is a σ-algebra. Assume $\{B_n\}_{n\in\mathbb{N}} \in \mathscr{F}$ is a collection of subsets of Ω.

Define

$$A_n = \bigcup_{i=1}^{n-1} B_i$$

Then $A_n \in \mathscr{F}$ being an algebra and $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$, hence by assumption $\bigcup_n A_n \in \mathscr{F}$, but

$$\bigcup_{i\in\mathbb{N}} A_n = \bigcup_{i\in\mathbb{N}} B_n \in \mathscr{F}$$

Hence $\mathscr{F}$ is a σ-algebra □

Definition §1.1.6 (σ-algebra generated by a subset of power set)**.** Let Ω be a nonempty set. Given an nonempty collection $\mathscr{C}$ of subsets of Ω, the σ-algebra generated by $\mathscr{C}$, $\sigma(\mathscr{C})$ is defined to be the intersection of all σ-algebra containing $\mathscr{C}$. Notationally,

$$\sigma(\mathscr{C}) = \bigcap \{\sigma - \text{algebra that contains } \mathscr{C}\}$$

$\sigma(\mathscr{C})$ is the smalled σ-algebra containing $\mathscr{C}$

Definition §1.1.7 (Borel σ-algebra)**.** If Ω is a topological space then the Borel σ-algebra is the smallest σ-algebra containing the open sets of Ω. *i.e.* by definition Borel σ-algebra is the σ-algebra generated by open sets in Ω.

Fact §1.1.8. *If $\Omega = \mathbb{R}^n$ the Borel σ-algebra is generated by*

- $\{(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n) \mid -\infty \le a_i < b_i \le +\infty\}$
- $\{(-\infty, a_1) \times (-\infty, a_2) \times \cdots (-\infty, a_n) \mid a_1, a_2, \ldots, a_n \in \mathbb{R}\}$
- $\left\{(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n) \mid a_i, b_i \in \mathbb{Q}\right\}$

Definition §1.1.9 (π-system, λ-system)**.** A collection $\mathscr{C}$ of subsets of Ω is called a π-system if $\mathscr{C}$ is closed under finite intersections.

A collection $\mathscr{L}$ of subsets of Ω is called a λ-system if the following hold:

- $\Omega \in \mathscr{L}$,
- $A, B \in \mathscr{L}$ and $A \subset B$ implies $B \setminus A \in \mathscr{L}$
- if $\{A_n\}_{n\in\mathbb{N}} \subset \mathscr{L}$ and $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$ then $\bigcup_n A_n \in \mathscr{L}$

Definition §1.1.10. Let $\mathscr{C}$ be a collection of nonempty subsets of a nonempty set Ω. The λ-system generated by $\mathscr{C}$, denoted as $\lambda(\mathscr{C})$ is the intersection of all λ-systems containing $\mathscr{C}$.

§1.2 Dynkin's pi-lambda theorem; Measures and their properties

Theorem §1.2.1 (Dynkin $\pi - \lambda$ theorem)**.** *If $\mathscr{C}$ is a π-system of a nonempty set Ω then $\lambda(\mathscr{C}) = \sigma(\mathscr{C})$. Equivalently, if $\mathscr{L}$ is a λ-system that contains $\mathscr{C}$ then $\lambda(\mathscr{C}) \subset \mathscr{L}$.*

Definition §1.2.2. Let $\mathscr{F}$ be a σ-algebra of subsets of Ω. A extended real valued function μ on $\mathscr{F}$ is called a *measure* if the following hold:

1. $\mu(A) \geq 0$ for all $A \in \mathscr{F}$
2. $\mu(\emptyset) = 0$
3. If $\{A_n\}_{n\in\mathbb{N}} \subset \mathscr{F}$ such that $\bigcup A_n \in \mathscr{F}$ and $A_n \cap A_m = \emptyset$ for all $m \neq n$ then $\mu(\bigcup_i A_i) = \sum_i \mu(A_i)$

Example §1.2.3 (Some examples of measures)**.**

1. Let $\Omega \neq \emptyset$, $\mathscr{F} = \mathscr{P}(\Omega)$. We define μ on $\mathscr{F}$ by $\mu(A)$ is the number of elements of A if A is finite and $\mu = +\infty$ if A contains infinitely many elements. Then μ is a measure on $\mathscr{F}$ called the counting measure on $\mathscr{F}$.
2. Let $\Omega = [0,1]$ and $\mathscr{F} = \mathscr{B}(\Omega)$. Let $\{p_n\}$ be a sequence of numbers in $[0,1]$ such that $\sum p_i = 1$. Define $\mu(A) = \sum_{i\in\mathbb{N}} p_i \delta_{p_i}(A)$. Then μ is a measure on $\mathscr{F}$.
3. Let F be a non-decreasing right-continuous function on $\mathbb{R}$. Define μ_F to be Lebesgue-Stieljes measure induced by F. Recall that $\mu_F((a,b]) = F(b) - F(a)$. Then μ_F is an example of σ-finite Radon measure on the Borel σ-algebra on $\mathbb{R}$.

Theorem §1.2.4. *Let $\mathscr{F}$ be a σ-algebra on a nonempty set Ω. Let $\mu : \mathscr{F} \to [0,\infty]$ be a function. μ is a measure on $\mathscr{F}$ iff*

1. *μ is finitely additive (that is, if $A, B \in \mathscr{F}$ such that $A \cap B = \emptyset$ then $\mu(A \cup B) = \mu(A) + \mu(B)$) and*
2. *μ is continuous from below (that is, if $\{A_n\}$ is nondecreasing sequence of elements from $\mathscr{F}$ then $\mu(\bigcup(A_i)) = \lim_{n\to\infty} \mu(A_n)$).*

§2 Lecture 2 — 12th August 2022 — Properties of Measures and Definition of semialgebra

Proof of Theorem **??**. Let $\mu : \mathscr{F} \to [0,\infty]$ be a function.

($\Longrightarrow$) Suppose that μ is a measure. We first show that μ is finitely additive. Let $A, B \in \mathscr{F}$ and suppose that $A \cap B = \emptyset$. Let $A_1 = A$ and $A_2 = B$ and $A_n = \emptyset$ for all $n \geq 3$. Then $\mu(A \cup B) = \mu(\cup_{i\in\mathbb{N}} A_i) = \sum_{i\in\mathbb{N}} \mu(A_i) = \mu(A) + \mu(B)$ as $\mu(\emptyset) = 0$.

We now prove that μ is continuous from below. Let $\{A_n\}$ be a nondecreasing sequence of elements from $\mathscr{F}$. *i.e.* $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$. Define $B_1 = A_1$ and $B_n = A_n \setminus A_{n-1}$ for each $n \geq 2$. Clearly, $\bigcup_n B_n = \bigcup_n A_n$ and $B_n \bigcap B_m = \emptyset$ for all $m \neq n$.

$$\begin{aligned}
\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) &= \mu\left(\bigcup_{n\in\mathbb{N}} B_n\right) \\
&= \sum_{n\in\mathbb{N}} \mu(B_n) && \text{(by the property of measure on } B_n\text{)} \\
&= \lim_{n\to\infty} \sum_{m=1}^{n} \mu(B_m) \\
&= \lim_{n\to\infty} \left[\mu(A_1) + \sum_{m=2}^{n} \big(\mu(A_m) - \mu(A_{m-1})\big)\right] && \text{(by the definiton of } B_m\text{)} \\
&= \lim_{n\to\infty} \mu(A_n) && \text{(telescopic sum)}
\end{aligned}$$

($\Longleftarrow$) Now suppose that μ is finitely additive and continuous from below. We intend to prove that μ is a measure. It is clear that from finite additivity that $\mu(\emptyset) = 0$. If not $\mu(\emptyset) = \mu(\emptyset) + \mu(\emptyset)$, which will give a contradiction for any nonzero value for $\mu(\emptyset)$. Let $\{A_n\}$ be a sequence of elments from $\mathscr{F}$. Define $B_n = \bigcup_{i=1}^{n} A_i$ for all $n \in \mathbb{N}$. Clearly, $B_n \nearrow \bigcup_{i\in\mathbb{N}} A_i$. Clearly, $\{B_n\}$ is an nondecreasing sequence of elements from $\mathscr{F}$. Then

$$\begin{aligned}
\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) &= \mu\left(\bigcup_{i\in\mathbb{N}} B_n\right) \\
&= \lim_{n\to\infty} \mu(B_n) && \text{(using continuity from below)} \\
&= \lim_{n\to\infty} \mu\left(\bigcup_{j=1}^{n} A_j\right) \\
&= \lim_{n\to\infty} \sum_{j=1}^{n} \mu(A_j) && \text{(finite additivity)} \\
&= \sum_{j=1}^{\infty} \mu(A_j)
\end{aligned}$$

Hence μ is a measure on $\mathscr{F}$ □

§2.1 Properties of Measures

Theorem §2.1.1. *Let μ be a measure on a σ-algebra $\mathscr{F}$. Then*

(1) μ is monotone.

(2) the inclusion-exclusion formula holds.

(3) μ is finitely subadditive. i.e. $\mu(A_1 \cup A_2 \cup \ldots \cup A_n) \leq \sum_{i=1}^{k} \mu(A_k)$ for $A_1, A_2, \ldots, A_k \in \mathscr{F}$.

(4) μ is continuous from above, that is, if $(A_n) \subset \mathscr{F}$ such that $A_n \supset A_{n+1}$ for all $n \in \mathbb{N}$ and $\mu(A_{n_0}) < +\infty$ for some n_0 then $\lim_{n\to\infty} \mu(A_n) = \mu(\bigcap_n A_n)$.

(5) μ is countably subadditive, that is, if $\{A_n\}_{n\in\mathbb{N}} \subset \mathscr{F}$ then $\mu(\cup_n A_n) \leq \sum_{i=1}^{\infty} \mu(A_n)$

Proof. (1) We show that μ is monotone. Let $A \subset B$ be elements of $\mathscr{F}$. Then $B = A \cup B \setminus A$. Hence $\mu(B) = \mu(A) + \mu(B \setminus A)$. Since $\mu(B \setminus A) \geq 0$, we have that $\mu(B) \geq \mu(A)$.

(2) Now, we prove that the inclusion exclusion formula holds for μ. Let $A, B \in \mathscr{F}$. If both $\mu(A) = +\infty$ and $\mu(B) = +\infty$ then there is nothing to prove. So, assume without loss of generality that $\mu(A) < \infty$. Then $\mu(A \cap B) < \infty$. Then

$$\begin{aligned}
\mu(A \cup B) + \mu(A \cap B) &= \mu(A) + \mu(B \setminus A) + \mu(B \cap A) \\
&= \mu(A) + (\mu(B \setminus A) + \mu(B \cap A)) \\
&= \mu(A) + \mu(B)
\end{aligned}$$

(3) Now, finite subadditivity follows immediately from inclusion-exclusion formula and induction.

(4) We now prove that μ is continuous from above. Let $\{A_n\}$ be a sequence of decreasing sequence sets with $\mu(A_1) < \infty$. Then we have that $\mu(A_n) \leq \mu(A_1) < \infty$. Define $B_n = A_1 \setminus A_n$ and $B = A_1 \setminus \bigcap_n A_n$. It is easy to see that $B_n \nearrow B$ (draw pictures!). Now by definition of B_n, $\mu(B_n) = \mu(A_1) - \mu(A_n)$.
Also,

$$\begin{aligned}
\mu(B) &= \lim_{n\to\infty} \mu(B_n) && \text{(By continuity from below, See ??)} \\
&= \mu(A_1) - \lim_{n\to\infty} \mu(A_n) && \text{(by definiton of } B_n)
\end{aligned}$$

Again by definiton of B $\mu(B) = \mu(A_1) - \mu(\bigcap_n A_n)$. Equating this with the above equation we see that $\lim_{n\to\infty} \mu(A_n) = \mu(\bigcap_n A_n)$.

(5) We now prove that μ is countably subadditive. Let $\{A_n\}$ be a sequence of elements from $\mathscr{F}$. Define $B_k = \bigcup_{n=1}^{k} A_n$. Clearly $B_k \nearrow \bigcup_n A_n$. Then,

$$\begin{aligned}
\mu\left(\bigcup_n A_n\right) &= \mu\left(\bigcup_n B_n\right) \\
&= \lim_{k\to\infty} \mu(B_k) && \text{(by continuity from below, See ??)} \\
&\leq \lim_{k\to\infty} \big(\mu(A_1) + \mu(A_2) + \ldots + \mu(A_k)\big) && \text{(by finite subadditivity)} \\
&= \sum_{k=1}^{\infty} \mu(A_k)
\end{aligned}$$

□

Definition §2.1.2. A collection $\mathscr{C}$ of subsets of Ω is called a *semialgebra* if

- $\mathscr{C}$ is closed under finite intersections
- if $A \in \mathscr{C}$ then there exists some $B_1, B_2, \ldots, B_n \in \mathscr{C}$, pairwise disjoint, such that $A^c = \bigcup_{i=1}^n B_i$.

Exercise §2.1.3. *Find a general formulation of the inclusion-exclusion principle for measures.*

§3 Lecture 3 — From semialgebra to algebra, measurable sets...

Remark §3.0.1. If $\mathscr{C}$ is a nonempty semialgebra on Ω then $\emptyset \in \Omega$.

This remark can be verified as follows: Since $\mathscr{C}$ is nonempty, let $A \in \mathscr{C}$. Since $\mathscr{C}$ is a semialgebra, there are elements $B_i \in \mathscr{C} (1 \leq i \leq n)$ such that $A^c = \cup_{i=1}^n B_i$ and $B_i \cap B_j = \emptyset$ for $i \neq j$. If $n = 1$, then $A^c \in \mathscr{C}$ and hence $A \cap A^c = \emptyset \in \mathscr{C}$. Now if $n \geq 2$, we have that $B_1 \cap B_2 = \emptyset \in \mathscr{C}$.

Remark §3.0.2. Let $\{\mathscr{A}_i\}_{i \in I}$ be a collection of algebras on Ω. Then it can be easily checked that $\cap_{i \in I} \mathscr{A}_i$ is an algebra on Ω. So if $\mathscr{C}$ is a collection of subsets of Ω then we denote $\mathscr{A}(\mathscr{C})$ to be the smallest algebra generated by $\mathscr{C}$, which is in fact, the intersection of all algebras that contain $\mathscr{C}$.

Definition §3.0.3 (Measure on a semi-algebra)**.** A nonnegative set function μ on a semialgebra $\mathscr{C}$ of subsets of Ω is called a *measure on* $\mathscr{C}$ if

(i) $\mu(\emptyset) = 0$

(ii) μ is countably additive, that is, if $(A_n) \subset \mathscr{C}$, $A_i \cap A_j = \emptyset$ and $\cap_{n \in \mathbb{N}} A_n \in \mathscr{C}$ then

$$\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \mu(A_n)$$

§3.1 Extension of measures from semialgebras to algebras

Let $\mathscr{C}$ be a semialgebra on Ω. Define $\mathscr{A}$ be the collection of all finite unions of elements of $\mathscr{C}$. Then in Question 6 from Assigment 1, we showed that $\mathscr{A} = \mathscr{A}(\mathscr{C})$, that is, $\mathscr{A}$ is the smallest algebra containing $\mathscr{C}$. The following lemma has much more to say though:

Lemma §3.1.1. *Let $\mathscr{C}$ be a semialgebra of Ω. Let*

$$\mathscr{F}(\mathscr{C}) := \left\{A \subset X : A = \cup_{i=1}^n B_i \text{ for some } B_i \in \mathscr{C}, B_i \cap B_j = \emptyset \text{ for } i \neq j\right\}$$

Then $\mathscr{F}(\mathscr{C}) = \mathscr{A}(\mathscr{C})$.

Proof. Clearly, $\mathscr{F}(\mathscr{C}) \subset \mathscr{A}(\mathscr{C})$ (see the previous paragraph). Since $\mathscr{A}(\mathscr{C})$ is the smallest algebra generated by $\mathscr{C}$, we will be done if we show that $\mathscr{F}(\mathscr{C})$ is an algebra containing $\mathscr{C}$.

Clearly, $\emptyset \in \mathscr{F}(\mathscr{C})$ as $\emptyset \in \mathscr{C}$. The fact that $\mathscr{F}(\mathscr{C})$ is closed under finite unions is pretty evident. To show that $\mathscr{F}(\mathscr{C})$ is closed under complement, let $A \in \mathscr{F}(\mathscr{C})$. Then there are elements $\{B_i\}_{1 \leq i \leq n} \subset \mathscr{C}$, pairwise disjoint, such that $A = \cup_{i=1}^n B_i$. Then since $B_i \in \mathscr{C}$, there exists $k_i \in \mathbb{N}$ and $C_{i1}, C_{i2}, \ldots, C_{i,k_i}$ such that $B_i^c = \cup_{j=1}^{k_i} C_{i,j}$. Then $A^c = \cap_{i=1}^n \cup_{j=1}^{k_i} C_{i,j}$. Interchanging the union and intersection, the result quickly follows. $\square$

Theorem §3.1.2. *Suppose μ is a measure on a semialgebra $\mathscr{C}$ of subsets of Ω. Let $\mathscr{A}$ be the algebra generated by $\mathscr{C}$. If $A \in \mathscr{A}$ has a representation $A = \cup_{i=1}^{n} B_i$, $B_i \cap B_j = \emptyset$ for $i \neq j$ then we define a function $\overline{\mu}$ on a subset of $\mathscr{A}$ where the elements $A \in \mathscr{A}$ have aforementioned representation given by $\overline{\mu}(A) = \sum_{i=1}^{n} \mu(B_i)$. Then*

1. *$\overline{\mu}$ is well defined,*
2. *$\overline{\mu}$ is finitely additive and*
3. *$\overline{\mu}$ is countably additive.*

Proof. We first show that $\overline{\mu}$ is indeed well-defined. Let $A \in \mathscr{A}$ and suppose that $A = \cup_{i=1}^{m} B_i$ and $A = \cup_{j=1}^{n} C_j$ where $\{B_i\} \subset \mathscr{C}$ are pairwise disjoint and $\{C_j\} \subset \mathscr{C}$ are pairwise disjoint. Then note that for $i \in \{1 \leq l \leq n\}$, we have that $B_i = B_i \cap A = B_i \cap \left(\cup_{j=1}^{n} C_j\right) = \cup_{j=1}^{n}(B_i \cap C_j)$. Note that previous union is a pairwise disjoint union. Hence,

$$\begin{aligned}
\sum_{i=1}^{m} \mu(B_i) &= \sum_{i=1}^{m} \sum_{j=1}^{n} \mu(B_i \cap C_j) \\
&= \sum_{j=1}^{n} \sum_{i=1}^{m} \mu(B_i \cap C_j) \\
&= \sum_{j=1}^{n} \mu(C_j)
\end{aligned}$$

This shows that $\overline{\mu}$ is well defined.

We show that $\overline{\mu}$ is countably additive. Let $\{A_n\}$ be the collection of elements of $\mathscr{A}$ which can be written as a union of pairwise disjoint elements of $\mathscr{C}$, $A_n \cap A_m = \emptyset$ for $n \neq m$ and $\cup_{n\in\mathbb{N}} A_n \in \mathscr{A}$.

Then for each n, there is $k_n \in \mathbb{N}$ and pairwise disjoint elements $B_{n1}, B_{n2}, \ldots, B_{n,k_n} \in \mathscr{C}$ such that $A_n = \cup_{i=1}^{k_n} B_{n,i}$.

We also have by Lemma **??** that there are some $B_1, \ldots, B_k \in \mathscr{C}$, pairwise disjoint, such that $\cup_{n\in\mathbb{N}} A_n = \cup_{i=1}^{k} B_i$.

Now, for any i, we have that

$$\begin{aligned}
B_i &= B_i \cap \left(\bigcup_{n\in\mathbb{N}} A_n\right) \\
&= \bigcup_{n\in\mathbb{N}} (B_i \cap A_n) \\
&= \bigcup_{n\in\mathbb{N}} \left(B_i \cap \bigcup_{j=1}^{k_n} B_{n,j}\right) \\
&= \bigcup_{n\in\mathbb{N}} \bigcup_{j=1}^{k_n} (B_i \cap B_{n,j}) \\
&= \bigcup_{\substack{n\in\mathbb{N} \\ j\in\{1,\ldots k_n\}}} (B_i \cap B_{n,j})
\end{aligned}$$

Note that the previous union is a pairwise disjoint union. Thus by definition of measure on a semialgebra, we have that for any i

$$\mu(B_i) = \sum_{\substack{n\in\mathbb{N}\\ j\in\{1,\dots k_n\}}} \mu(B_i \cap B_{n,j})$$

Hence,

$$\begin{aligned}
\overline{\mu}(A) &= \sum_{i=1}^{k} \mu(B_i) \\
&= \sum_{i=1}^{k} \sum_{\substack{n\in\mathbb{N}\\ j\in\{1,\dots k_n\}}} \mu(B_i \cap B_{n,j}) \\
&= \sum_{\substack{n\in\mathbb{N}\\ j\in\{1,\dots k_n\}}} \sum_{i=1}^{k} \mu(B_i \cap B_{n,j}) \\
&= \sum_{\substack{n\in\mathbb{N}\\ j\in\{1,\dots k_n\}}} \mu\left(\bigcup_{i=1}^{k} \left(B_i \cap B_{n,j}\right)\right) \\
&= \sum_{\substack{n\in\mathbb{N}\\ j\in\{1,\dots k_n\}}} \mu(B_{n,j}) \\
&= \sum_{n\in\mathbb{N}} \sum_{j=1}^{k_n} \mu(B_{n,j}) \\
&= \sum_{n\in\mathbb{N}} \overline{\mu}(A_n)
\end{aligned}$$

This completes the proof of countable additivity. The proof of finite additivity follows from countable subadditivity. □

§3.2 Outer Measures

Definition §3.2.1. Given a measure μ on a semialgebra $\mathscr{C}$, the *outer measure induced by* μ is the set function μ^* defined on $\mathscr{P}(\Omega)$ as

$$\mu^*(A) := \inf\left\{\sum_{n=1}^{\infty} \mu(A_n) : \{A_n\}_{n\geq 1} \subset \mathscr{C}, A \subset \cup_{n\geq 1} A_n\right\}$$

We'd like to remark that μ^* is not an overestimate, that is, $\mu^* = \mu$ on $\mathscr{C}$ and $\mu^* = \overline{\mu}$ on $\mathscr{A}$.

To verify this remark, let $C \in \mathscr{C}$. We need to show that $\mu^*(C) = \mu(C)$. Clearly by definition of μ^*, we have that $\mu^*(C) \leq \mu(C)$. (Fill the details!)

Definition §3.2.2. A set A is said to be μ^**-measurable* if

$$\mu^*(E) = \mu^*(A \cap E) + \mu^*(A^c \cap E) \text{ for all } E \subset \Omega$$

The set of all μ^*-measurable sets is denoted by $\mathscr{M}_{\mu^*}$.