Lecture Notes in Measure Theory

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Introduction

This is a set of lecture notes which I took for reviewing stuff that I typed after taking class from *Dr. Sachindranath Jayaraman*. All the typos and errors are of mine.

Contents

§1 Lecture 1 — 10th August 2022 — Review of things done in the previous semester...

§1.1 Definitions and Some Results

Definition §1.1.1 (algebra)**.** Let Ω be nonempty set. An algebra $\mathscr{F}$ is a collection of subsets of Ω satisfying the following properties:

1. $\Omega \in \mathscr{F}$,
2. $A \in \mathscr{F} \Rightarrow A^c \in \mathscr{F}$ and
3. $\mathscr{F}$ is closed under finite unions.

It immediately follows from the definition an algebra of sets is closed under taking finite intersections.

Definition §1.1.2 (σ-algebra)**.** Let Ω be nonempty set. A σ-algebra $\mathscr{F}$ is a collection of subsets of Ω satisfying the following properties:

1. $\Omega \in \mathscr{F}$,
2. $A \in \mathscr{F} \Rightarrow A^c \in \mathscr{F}$ and

3. $\mathscr{F}$ is closed under countable unions.

Fact §1.1.3. *Let Ω be a set, $\mathscr{F} \subseteq \mathscr{P}(\Omega)$. $\mathscr{F}$ is an σ-algebra iff $\mathscr{F}$ is an algebra that is continuous from below, that is, if $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathscr{F}$ and $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$ then $\bigcap_n A_n \in \mathscr{F}$.*

Definition §1.1.4 (σ-algebra generated by a subset of power set)**.** Let Ω be a nonempty set. Given an nonempty collection $\mathscr{C}$ of subsets of Ω, the σ-algebra generated by $\mathscr{C}$, $\sigma(\mathscr{C})$ is defined to be the intersection of all σ-algebra containing $\mathscr{C}$. Notationally,

$$\sigma(\mathscr{C}) = \bigcap \{\sigma - \text{algebra that contains } \mathscr{C}\}$$

Definition §1.1.5 (Borel σ-algebra)**.** If Ω is a topological space then the Borel σ-algebra is the smallestσ-algebra containing the open sets of Ω.

Fact §1.1.6. *If $\Omega = \mathbb{R}^n$ the Borel σ-algebra is generated by*

- $\{(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n) \mid -\infty \leq a_i < b_i \leq b_i \leq +\infty\}$
- $\{(-\infty, a_1) \times (-\infty, a_2) \times \cdots (-\infty, a_n) \mid a_1, a_2, \ldots, a_n \in \mathbb{R}\}$
- $\left\{(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n) \mid a_i, b_i \in \mathbb{Q}\right\}$

Definition §1.1.7 (π-system, λ-system)**.** A collection $\mathscr{C}$ of subsets of Ω is called a π-system if $\mathscr{C}$ is closed under finite-$\cap$.

A collection $\mathscr{L}$ of subsets of Ω is called a λ-system if the following hold:

- $\Omega \in \mathscr{L}$,
- $A, B \in \mathscr{L}$ and $A \subset B$ implies $B \setminus A \in \mathscr{L}$
- if $\{A_n\}_{n\in\mathbb{N}} \subset \mathscr{L}$ then $\bigcup_n A_n \in \mathscr{L}$

Definition §1.1.8. Let $\mathscr{C}$ be a collection of nonempty subsets of a nonempty set Ω. The λ-system generated by $\mathscr{C}$, denoted as $\lambda(\mathscr{C})$ is the intersection of all λ-systems containing $\mathscr{C}$.

§1.2 Dynkin's pi-lambda theorem; Measures and their properties

Theorem §1.2.1 (Dynkin $\pi - \lambda$ theorem)**.** *If $\mathscr{C}$ is a π-system of a nonempty set Ω then $\lambda(\mathscr{C}) = \sigma(\mathscr{C})$. Equivalently, if $\mathscr{L}$ is a λ-system that contains $\mathscr{C}$ then $\mathscr{L} \supset \lambda(\mathscr{C})$.*

Definition §1.2.2. Let $\mathscr{F}$ be a σ-algebra of subsets of Ω. A extended real valued function μ on $\mathscr{F}$ is called a *measure* if the following hold:

1. $\mu(A) \geq 0$ for all $A \in \mathscr{F}$,
2. $\mu(\emptyset) = 0$

3. If $\{A_n\}_{n\in\mathbb{N}} \subset \mathscr{F}$ such that $\bigcup A_n \in \mathscr{F}$ and $A_n \cap A_m = \emptyset$ for all $m \neq n$ then $\mu(\bigcup_{i\in\mathbb{N}} A_i) = \sum_i \mu(A_i)$

Example §1.2.3 (Some examples of measures)**.** 1. Let $\Omega \neq \emptyset$, $\mathscr{F} = \mathscr{P}(\Omega)$. We define μ on $\mathscr{F}$ by $\mu(A)$ is the number of elements of A if A is finite and $\mu = +\infty$ if A contains infinitely many elements. Then μ is a measure on $\mathscr{F}$.

2. Let $\Omega = [0,1]$ and $\mathscr{F} = \mathscr{B}([0,1])$. Let $\{p_n\}$ be a sequence of numbers in $[0,1]$ such that $\sum p_i = 1$. Define $\mu(A) = \sum_{i\in\mathbb{N}} p_i \delta_{p_i}(A)$. Then μ is a measure on $\mathscr{F}$.

3. Let F be a non-decreasing right-continuous function on $\mathbb{R}$. Define μ_F to be Lebesgue-Stieljes measure induced by F. Recall that $\mu_F((a,b]) = b - a$. Then μ_F is an example of σ-finite Radon measure on the Borel σ-algebra on $\mathbb{R}$.

Theorem §1.2.4. *Let $\mathscr{F}$ be a σ-algebra on a nonempty set Ω. Let $\mu : \mathscr{F} \to \overline{\mathbb{R}}$ be a function. μ is a measure on $\mathscr{F}$ iff*

1. *μ is finitely additive (that is, if $A, B \in \mathscr{F}$ such that $A \cap B = \emptyset$ then $\mu(A \cup B) = \mu(A) + \mu(B)$) and*

2. *μ is continuous from below (that is, if $\{A_n\}$ is nondecreasing sequence of elements from $\mathscr{F}$ then $\mu(\bigcup(A_i)) = \lim_{n\to\infty} \mu(A_n)$).*