

Lecture Notes in Partial Differential Equations

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Introduction

This is a set of lecture notes which I took for reviewing stuff that I typed after taking class from *Dr. K.R. Arun*. This course used the textbook *Partial Differential Equations* by L.C. Evans.

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§1 Lecture 1 — 11th August, 2022 — Definition, Classifications & Examples of PDEs

§1.1 Notations

- Let \mathbb{N}_0 be defined to be the set $\mathbb{N} \cup \{0\}$. For any $N \in \mathbb{N}$, an element of \mathbb{N}_0^N to be an *multiindex*. If $\alpha \in \mathbb{N}_0^N$ then $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ for some $\alpha_i \in \mathbb{N}_0$.
- For any $x \in \mathbb{R}^N$ and $N \in \mathbb{N}$, we define $x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}$.
- Given any multiindex $\alpha \in \mathbb{N}_0^N$, we define $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_N$.

- Given any multiindex α , we define

$$D^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N}} = \partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N}$$

- For any $k \in \mathbb{N}$, denote $D^k = \{D^\alpha : |\alpha| = k\}$
- We will denote $\Omega \subset \mathbb{R}^N$ to be an open subset.

§1.2 Definition, Classification & Examples

Definition §1.2.1 (Partial Differential Equation). Let Ω be an open subset of \mathbb{R}^N . An expression of the form

$$F(D^k u(x), D^{k-1} u(x), \dots, Du(x), x) = 0 \quad (x \in \Omega) \quad (1)$$

is called a k th order PDE for the unknown function $u : \Omega \rightarrow \mathbb{R}$. One may assume $F : \mathbb{R}^{N^k} \times \mathbb{R}^{N^{k-1}} \times \dots \times \mathbb{R}^N \times \Omega \rightarrow \mathbb{R}$ is a given smooth function.

§1.2.1 Classifications of PDE

- (i) The PDE (1) is called *linear* if it has the form

$$\sum_{0 \leq |\alpha| \leq k} a_\alpha(x) D^\alpha u = f$$

for some functions a_α, f . The linear PDE is homogeneous if $f = 0$.

- (ii) The PDE (1) is called *semilinear* if it has the form

$$\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u + a_0(D^{k-1} u, \dots, Du, u, x) = 0$$

- (iii) The PDE (1) is called *quasilinear* if it has the form

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1} u, \dots, Du, u, x) D^\alpha u + a_0(D^{k-1} u, \dots, Du, u, x) = 0$$

- (iv) The PDE (1) is called *nonlinear* if the PDE has a nonlinear dependence on the highest order derivative.

Definition §1.2.2 (System of PDE). An expression of the form $\mathbf{F}(D^k(\mathbf{u}), D^{k-1}(\mathbf{u}), \dots, D(\mathbf{u}), \mathbf{u}, x) = \mathbf{0}$ is called a k th order system of PDE, where $\mathbf{u} : \Omega \rightarrow \mathbb{R}^m$ is the unknown, $\mathbf{u} = (u^1, u^2, \dots, u^n)$ and $\mathbf{F} : \mathbb{R}^{mN^k} \times \mathbb{R}^{mN^{k-1}} \times \dots \times \mathbb{R}^{mN} \times \mathbb{R}^m \times U \rightarrow \mathbb{R}^m$ is given.

§1.2.2 Examples of PDEs

1. Linear Equations

Laplace Equation $\Delta u = \sum_{i=1}^N \partial_{x_i}^2 u = 0$

(linear, second order)

Linear Transport Equation $\partial_t u + \sum_{i=1}^N \partial_{x_i} u = 0$

(linear, first order)

Schrödinger's Equation $i\partial_t u + \Delta u = 0$

(linear, second order)

Linear System : Maxwell's Equations

$$\partial_t E = \text{curl } B$$

$$\partial_t B = -\text{curl } E$$

$$\text{div } E = \text{div } B = 0$$

2. Nonlinear equations

Inviscid Burgers' equation $\partial_t u + u\partial_x u = 0$

Eikonal equation $|Du| = 1$

Nonlinear system: Navier-Stokes Equations

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \Delta \mathbf{u} = -\nabla p$$

$$\text{div } \mathbf{u} = 0$$

Definition §1.2.3 (Well posed). A PDE is said to be *well posed* if

(**Existence**) it has at least one solution,

(**Uniqueness**) it has at most one solution and

(**Stability**) the solution depends continuously on the data given in the problem.

Definition §1.2.4. A *classical solution* of the k -th order PDE is a function $u \in C^k(\Omega)$ which satisfies the equation pointwise

$$F(D^k u(x), D^{k-1} u(x), \dots, Du(x), u(x), x) = 0$$

for all $x \in \Omega$.

Remark §1.2.5. A classical solution may not always exist. For instance, the inviscid Burgers' equation does not have a solution.

The course is divided into three parts:

- (a) Representation Formulae for solutions
- (b) Linear PDE theory
- (c) Nonlinear PDE theory

§1.3 Transport Equation

The PDE

$$\partial_t u + b \cdot Du = 0 \text{ in } \mathbb{R}^n \times (0, \infty)$$

where $t \in (0, \infty)$, $x \in \mathbb{R}^n$ are the independent variables, $u = u(t, x)$ is the dependent variable and $b = (b_1, b_2, \dots, b_n)$ and $Du = (\partial_{x_1} u, \dots, \partial_{x_n} u)$ is the gradient.

§2 Lecture 2 — 16th August, 2022 — Linear Transport Equation

Consider the linear transport equation given by

$$u_t + b \cdot Du = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty)$$

where $b \in \mathbb{R}^n$ is a fixed vector and $u : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$ is the unknown function, $Du = D_x u(u_{x_1}, u_{x_2}, \dots, u_{x_n})$.

Let $(x, t) \in \mathbb{R}^n \times (0, \infty)$ be fixed. Our aim is to obtain a representation of the solution.

Assuming that u is a smooth function, we define the function $z(s) := u(x + bs, t + s)$ where $s \in \mathbb{R}$.

Note that z is the restriction of the function u to the line $L = \{(x + bs, t + s) : s \in \mathbb{R}\}$. Note that the line L passes through (x, t) and is in the direction of the vector $(b, 1)$.

Differentiating z , we have that for all $s \in \mathbb{R}$,

$$z'(s) = b \cdot Du(x + bs, t + s) + u_t(x + bs, t + s) = 0$$

Thus, u is a constant on the line L . If we know the solution at any point on L , the problem is solved. We use the aforementioned result in the following subsection.

§2.1 Solution of an IVP

Let $g : \mathbb{R}^n \rightarrow \mathbb{R}$ where $g = g(x_1, x_2, \dots, x_n)$. We consider the following IVP

$$\begin{aligned} u_t + b \cdot Du &= 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u &= g & \text{in } \mathbb{R}^n \times \{0\} \end{aligned}$$

Just note that the last equation in IVP means $u(x, 0) = g(x)$ for all $x \in \mathbb{R}^n$. From the discussion before the start of the subsection, it suffices to know the solution on the hyperplane $\Gamma = \mathbb{R}^n \times \{0\}$. The line L passes through Γ at the point $(x - bt, 0)$. So,

$$\begin{aligned} u(x, t) &= z(0) \\ &= z(-t) \\ &= u(x - bt, 0) \\ &= g(x - bt) \end{aligned}$$

Note that the first equality is true by definition of z , the second equality is true by z being constant on the line L and the third is again true by definition of z and the last is true by $u = g$ in $\mathbb{R}^n \times \{0\}$.

$$\text{So, } \boxed{u(x, t) = g(x - bt) \quad t \geq 0, x \in \mathbb{R}^n}.$$

Remark §2.1.1. Draw the x versus u sketch when $n = 1$ and taking $t = 0$ and $t = 1$ and notice the solution gets translated or transported and hence the name of the PDE is linear transport equation.

§2.2 Solution of a homogeneous problem

Consider the problem

$$\begin{aligned} u_t + b \cdot Du &= f && \text{in } \mathbb{R}^n \times (0, \infty) \\ u &= g && \text{in } \mathbb{R}^n \times \{0\} \end{aligned}$$

Take any $(x, t) \in \mathbb{R}^n \times (0, \infty)$ and define $z(s) = u(x + sb, t + s)$.

Hence, $z'(s) = u_t(x + sb, t + s) + b \cdot Du(x + sb, t + s) = f(x + sb, t + s)$. So,

$$\begin{aligned} u(x, t) - g(x - tb) &= z(0) - z(-t) \\ &= \int_{-t}^0 z'(s) ds \\ &= \int_{-t}^0 f(x + sb, t + s) ds \\ &= \int_0^t f(x + (s - t)b, s) ds \end{aligned}$$

Note that the last equality is by change of variable. Therefore, we have

$$\boxed{u(x, t) = g(x - tb) + \int_0^t f(x + (s - t)b, s) ds} \quad (x \in \mathbb{R}^n, t \geq 0)$$

is the required solution.