Lecture Notes in Partial Differential Equations

Ashish Kujur

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Introduction

This is a set of lecture notes which I took for reviewing stuff that I typed after taking class from *Dr. K.R. Arun.* This course used the textbook *Partial Differential Equations* by L.C. Evans.

Contents

§1 Lecture 1 — 11th August, 2022 — Definition, Classifications & Examples of PDEs

§1.1 Notations

- Let $\mathbb{N}_0$ be defined to be the set $\mathbb{N} \cup \{0\}$. For any $N \in \mathbb{N}$, an element of $\mathbb{N}_0^N$ to be an *multiindex*. If $\alpha \in \mathbb{N}_0^N$ then $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ for some $\alpha_i \in \mathbb{N}_0$.
- For any $x \in \mathbb{R}^N$ and $N \in \mathbb{N}$, we define $x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}$.
- Given any multiindex $\alpha \in \mathbb{N}_0^N$, we define $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_N$.
- Given any multiindex α, we define

$$D^\alpha := \frac{\partial^{|\alpha|}}{\partial {x_1}^{\alpha_1} \cdots \partial {x_N}^{\alpha_N}} = \partial {x_1}^{\alpha_1} \cdots \partial {x_N}^{\alpha_N}$$

- For any $k \in \mathbb{N}$, denote $D^k = \left\{D^\alpha : |\alpha| = k\right\}$
- We will denote $\Omega \subset \mathbb{R}^N$ to be an open subset.

§1.2 Definition, Classification & Examples

Definition §1.2.1 (Partial Differential Equation)**.** Let Ω be an open subset of $\mathbb{R}^N$. An expression of the form

$$F\left(D^k u(x), D^{k-1} u(x), \ldots, Du(x), x\right) = 0 \qquad (x \in \Omega) \tag{1}$$

is called a kth order PDE for the unknown function $u : \Omega \to \mathbb{R}$. One may assume $F : \mathbb{R}^{N^k} \times \mathbb{R}^{N^{k-1}} \times \cdot \times \mathbb{R}^N \times \Omega \to \mathbb{R}$ is a given smooth function.

§1.2.1 Classifications of PDE

(i) The PDE (1) is called *linear* if it has the form

$$\sum_{0 \le |\alpha| \le k} a_\alpha(x) D^\alpha u = f$$

for some functions a_α, f. The linear PDE is homogeneous if $f = 0$.

(ii) The PDE (1) is called *semilinear* if it has the form

$$\sum_{|\alpha| = k} a_\alpha(x) D^\alpha u + a_0\left(D^{k-1} u, \ldots, Du, u, x\right) = 0$$

(iii) The PDE (1) is called *quasilinear* if it has the form

$$\sum_{|\alpha = k} a_\alpha(D^{k-1} u, \ldots, Du, u, x) D^\alpha u + a_0\left(D^{k-1} u, \ldots, Du, u, x\right) = 0$$

(iv) The PDE (1) is called *nonlinear* if the PDE has a nonlinear dependence on the highest order derivative.

Definition §1.2.2 (System of PDE)**.** An expression of the form $\mathbf{F}\left(D^k(\mathbf{u}), D^{k-1}(\mathbf{u})), \ldots, D(\mathbf{u}), \mathbf{u}, x\right) = \mathbf{0}$ is called a kth order system of PDE, where $\mathbf{u} : \Omega \to \mathbb{R}^m$ is the unknown, $\mathbf{u} = \left(u^1, u^2, \ldots, u^n\right)$ and $\mathbf{F} : \mathbb{R}^{mN^k} \times \mathbb{R}^{mN^{k-1}} \times \cdots \mathbb{R}^{mN} \times \mathbb{R}^m \times U \to \mathbb{R}^m$ is given.

§1.2.2 Examples of PDEs

1. Linear Equations

   Laplace Equation $\Delta u = \sum_{i=1}^N \partial_{x_i{}^2} u = 0$

   (linear, second order)

   Linear Transport Equation $\partial_t u + \sum_{i=1}^N \partial_{x_i} u = 0$

   (linear, first order)

   Schrödinger's Equation $i\partial_t u + \Delta u = 0$

   (linear, second order)

   Linear System : Maxwell's Equations

$$\begin{aligned} \partial_t E &= \text{curl } B \\ \partial_t B &= -\text{curl } E \\ \text{div } E &= \text{div } B = 0 \end{aligned}$$

2. Nonlinear equations

   Inviscid Burgers' equation $\partial_t u + u\partial_x u = 0$

   Eikonal equation $|Du| = 1$

   Nonlinear system: Navier-Stokes Equations

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{u} \cdot D\mathbf{u} - \Delta \mathbf{u} &= -Dp \\ \text{div } \mathbf{u} &= 0 \end{aligned}$$

Definition §1.2.3 (Well posed)**.** A PDE is said to be *well posed* if

(Existence) it has at least one solution,

(Uniqueness) it has at most one solution and

(Stability) the solution depends continuously on the data given in the problem.

Definition §1.2.4. A *classical solution* of the k-th order PDE is a function $u \in C^k(\Omega)$ which satisfies the equation pointwise

$$F\left(D^k u(x), D^{k-1} u(x), \ldots, Du(x), u(x), x\right) = 0$$

for all $x \in \Omega$.

Remark §1.2.5. A classical solution may not always exist. For instance, the inviscid Burgers' equation does not have a solution.

The course is divided into three parts:

(a) Representation Formulae for solutions

(b) Linear PDE theory

(c) Nonlinear PDE theory

§1.3 Transport Equation

The PDE

$$\partial_t u + b \cdot Du = 0 \text{ in } \mathbb{R}^n \times (0, \infty)$$

where $t \in (0, \infty), x \in \mathbb{R}^n$ are the independent variables, $u = u(t, x)$ is the dependent variable and $b = (b_1, b_2, \ldots, b_n)$ and $Du = \left(\partial_{x_1} u, \ldots, \partial_{x_2} u\right)$ is the gradient.