Lecture Notes in Partial Differential Equations

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Last Updated: August 11, 2022

Introduction

This is a set of lecture notes which I took for reviewing stuff that I typed after taking class from *Dr. K.R. Arun*. This course used the textbook *Partial Differential Equations* by L.C. Evans.

Contents

§1 Lecture 1 — 11th August, 2022 — Definition, Classifications & Examples of PDEs

§1.1 Notations

- Let $\mathbb{N}_0$ be defined to be the set $\mathbb{N} \cup \{0\}$. For any $N \in \mathbb{N}$, an element of $\mathbb{N}_0^N$ to be an *multiindex*. If $\alpha \in \mathbb{N}_0^N$ then $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ for some $\alpha_i \in \mathbb{N}_0$.
- For any $x \in \mathbb{R}^N$ and $N \in \mathbb{N}$, we define $x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}$.
- Given any multiindex $\alpha \in \mathbb{N}_0^N$, we define $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_N$.
- Given any multiindex α, we define

$$D^\alpha := \frac{\partial^{|\alpha|}}{\partial {x_1}^{\alpha_1} \cdots \partial {x_N}^{\alpha_N}} = \partial {x_1}^{\alpha_1} \cdots \partial {x_N}^{\alpha_N}$$

- For any $k \in \mathbb{N}$, denote $D^k = \left\{D^\alpha : |\alpha| = k\right\}$
- We will denote $\Omega \subset \mathbb{R}^N$ to be an open subset.

§1.2 Definition, Classification & Examples

Definition §1.2.1 (Partial Differential Equation)**.** Let Ω be an open subset of $\mathbb{R}^N$. An expression of the form

$$F\left(D^k u(x), D^{k-1} u(x), \ldots, Du(x), x\right) = 0 \qquad (x \in \Omega) \tag{1}$$

is called a kth order PDE for the unknown function $u : \Omega \to \mathbb{R}$. One may assume $F : \mathbb{R}^{N^k} \times \mathbb{R}^{N^{k-1}} \times \cdot \times \mathbb{R}^N \times \Omega \to \mathbb{R}$ is a given smooth function.

§1.2.1 Classifications of PDE

(i) The PDE (1) is called *linear* if it has the form

$$\sum_{0 \le |\alpha| \le k} a_\alpha(x) D^\alpha u = f$$

for some functions a_α, f. The linear PDE is homogeneous if $f = 0$.

(ii) The PDE (1) is called *semilinear* if it has the form

$$\sum_{|\alpha| = k} a_\alpha(x) D^\alpha u + a_0\left(D^{k-1} u, \ldots, Du, u, x\right) = 0$$

(iii) The PDE (1) is called *quasilinear* if it has the form

$$\sum_{|\alpha = k} a_\alpha(D^{k-1} u, \ldots, Du, u, x) D^\alpha u + a_0\left(D^{k-1} u, \ldots, Du, u, x\right) = 0$$

(iv) The PDE (1) is called *nonlinear* if the PDE has a nonlinear dependence on the highest order derivative.

§1.2.2 Examples