

# Lecture Notes in Measure Theory

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## Introduction

This is a set of lecture notes which I took for reviewing stuff that I typed after taking class from *Dr. Sachindranath Jayaraman*. All the typos and errors are of mine. The pictures that make here (if any) will be hand drawn and I will appreciate it if someone who is knowledgeable in Tikz will help me digitizing my rough hand-drawn pictures.

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## §1 Lecture 1 — 9th August 2022 — Basic Definitions

### §1.1 Algebras and Sigma-algebras

**Definition §1.1.1 (Algebra).** Let  $X$  be any arbitrary set. A collection  $\mathcal{A} \subseteq \mathcal{P}(X)$  is called an algebra if

1.  $X \in \mathcal{A}$
2.  $\forall A \in \mathcal{A}, X \setminus A \in \mathcal{A}$
3.  $\forall \mathcal{F} \subseteq \mathcal{A} (\mathcal{F} \text{ finite} \Rightarrow \bigcup \mathcal{F} \in \mathcal{A})$
4.  $\forall \mathcal{F} \subseteq \mathcal{A} (\mathcal{F} \text{ finite} \Rightarrow \bigcap \mathcal{F} \in \mathcal{A})$

**Definition §1.1.2 ( $\sigma$ -Algebra).** Let  $X$  be any arbitrary set. A collection  $\mathcal{A} \subseteq \mathcal{P}(X)$  is called an algebra if

1.  $X \in \mathcal{A}$

2.  $\forall A \in \mathcal{A}, X \setminus A \in \mathcal{A}$
3.  $\forall \{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{A}, \bigcup_{i \in \mathbb{N}} A_i \in \mathcal{A}$
4.  $\forall \{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{A}, (\mathcal{F} \text{ finite} \Rightarrow \bigcap_{i \in \mathbb{N}} A_i \in \mathcal{A})$

**Example §1.1.3** (Some families of sets that are algebras or  $\sigma$ -algebras, and some that are not). Here's a list of examples:

1. Let  $X$  be any set. Let  $\mathcal{A} = \mathcal{P}(X)$ . Then  $\mathcal{A}$  is a  $\sigma$ -algebra on  $X$ .
2. Let  $X$  be any set. Let  $\mathcal{A}$  be the collection of all subsets  $A$  of  $X$  such that  $A$  or  $A^c$  is countable. Then  $\mathcal{A}$  is  $\sigma$ -algebra.

**Proposition §1.1.4.** *Let  $X$  be any set. The intersection of an arbitrary nonempty collection of  $\sigma$ -algebras on  $X$  is a  $\sigma$ -algebra on  $X$ .*

## §1.2 Measures

**Definition §1.2.1.** A function  $\mu : \mathcal{A} \rightarrow [0, +\infty]$  is said to be *measure* if it satisfies the following two properties:

1.  $\mu(\emptyset) = 0$
2.  $\mu(\bigcup_{n \in \mathbb{N}} A_n) = \sum_{n \in \mathbb{N}} \mu(A_n)$

## §2 Lecture 2

### §2.1 Lebesgue Integral