Lecture Notes in Measure Theory

Ashish Kujur

June 2022

Introduction

This is a set of lecture notes which I took for reviewing stuff that I typed after taking class from Professor *insert name here*. All the typos and errors are of mine. I like to take notes in LaTeXas it motivates me to drag my ass to class. The pictures that make here will be hand drawn and I will appreciate it if someone who is knowledgeable in Tikz will help me digitizing my rough hand-drawn pictures.

Contents

1 Lecture 1 — 9th August 2022 — Basic Definitions

1.1 Algebras and Sigma-algebras

Definition 1.1.1 (Algebra)**.** Let X be any arbitrary set. A collection $\mathscr{A} \subseteq \mathscr{P}(X)$ is called an algebra if

1. $X \in \mathscr{A}$
2. $\forall A \in \mathscr{A}, X \setminus A \in \mathscr{A}$
3. $\forall \mathscr{F} \subseteq \mathscr{A}$ ($\mathscr{F}$ finite $\Rightarrow \bigcup \mathscr{F} \in \mathscr{A}$)
4. $\forall \mathscr{F} \subseteq \mathscr{A}$ ($\mathscr{F}$ finite $\Rightarrow \bigcap \mathscr{F} \in \mathscr{A}$)

Definition 1.1.2 (σ -Algebra)**.** Let X be any arbitrary set. A collection $\mathscr{A} \subseteq \mathscr{P}(X)$ is called an *algebra* if

1. $X \in \mathscr{A}$
2. $\forall A \in \mathscr{A}, X \setminus A \in \mathscr{A}$
3. $\forall \{A_i\}_{i \in \mathbb{N}} \subseteq \mathscr{A}, \bigcup_{i \in \mathbb{N}} A_i \in \mathscr{A}$
4. $\forall \{A_i\}_{i \in \mathbb{N}} \subseteq \mathscr{A}, (\mathscr{F} \text{ finite } \Rightarrow \bigcap_{i \in \mathbb{N}} A_i \in \mathscr{A})$

Example 1.1.3 (Some families of sets that are algebras or σ-algebras , and some that are not)**.** Here's a list of examples:

1. Let X be any set. Let $\mathscr{A} = \mathscr{P}(X)$. Then $\mathscr{A}$ is a σ-algebra on X.
2. Let X be any set. Let $\mathscr{A}$ be the collection of all subsets A of X such that A or A^c is countable. Then $\mathscr{A}$ is σ-algebra.

Proposition 1.1.4. *Let X be any set. The intersection of an arbitrary nonempty collection of σ-algebras on X is a σ-algebra on X.*

1.2 Measures

Definition 1.2.1. A function $\mu : \mathscr{A} \to [0, +\infty]$ is said to be *measure* if it satisfies the following two properties:

1. $\mu(\emptyset) = 0$
2. $\mu(\bigcup_{n \in \mathbb{N}} A_n) = \sum_{n \in \mathbb{N}} \mu(A_n)$

2 Lecture 2

2.1 Lebesgue Integral

Definition 2.1.1 (Function)**.** Let X, Y be two sets. A function $f : X \to Y$ is a object which takes every element $x \in X$ to a unique element $y \in Y$ such that $f(x) = y$.