Lecture Notes in Commutative Algebra

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Introduction

This is a set of lecture notes which I took for reviewing stuff that I typed after taking class from *Dr. Viji Z Thomas*. All the typos and errors are of mine.

Contents

§1 Lecture 1 — 10th August 2022 — Local Rings, Semilocal rings, Chinese Remainder Theorem

We will be assuming the following things before proceeding in the course:

- A ring A is a commutative ring with unity.
- Existence of maximal ideals in a commutative ring with unity (this follows immediately from Zorn's Lemma)
- Definition of ring morphism.
- Definition of prime and maximal ideals and the facts that
  - P is a prime ideal of A iff A/P is an integral domain and
  - M is a maximal ideal of A iff A/P is a field

§1.1 Basic Definitions — Local Rings, Semilocal rings and few other results

Definition §1.1.1 (local ring)**.** Let A be a ring. A is said to be a *local ring* if A has a unique maximal ideal M. A local ring is often denoted by (A, M).

Definition §1.1.2 (semilocal ring)**.** Let A be a ring. A is said to be *semilocal ring* if A has only fintiely many maximal ideals.

How does one come up with a semilocal ring with exactly m maximal ideals? Here's an example:

Example §1.1.3 (A ring with m distinct maximal ideals)**.** Let $A = \mathbb{Z}/n\mathbb{Z}$. It is fairly easy to show that all the ideals of A are of the form $\left(\overline{k}\right)$ where $k \in \mathbb{N}$ and $k \mid n$ and also that if $k, j \mid n$ and $\left(\overline{k}\right) \subset \left(\overline{j}\right)$ iff $j \mid k$. (See Sepanski Exercise 3.47 and 3.48) Now let $p_1, p_2, \ldots, p_m$ be m distinct primes. Define $n = p_1 p_2 \cdots p_m$. It is easy to see from the aforementioned facts that $A = \mathbb{Z}/n\mathbb{Z}$ has m distinct maximal ideals.

Example §1.1.4 (A standard example of a local ring?)**.** Let A be a ring, M be a maximal ideal of A and $n \in \mathbb{N}$. Observe that M^n is a ideal of A (See Sepanski Exercise 3.51). We claim that A/M^n has only prime ideal namely M/M^n. Let $\mathscr{P}$ be a prime ideal of A/M^n. Then by the correspondence theorem, $\mathscr{P} = P/M^n$ where P is a prime ideal of A containing M^n. Then $P \supset M^n$ which further implies that $P \supset M$ since $M^n \supset M$. Since M is a maximal ideal, we have that $P = M$. This completes the proof of the claim. Also, note that since every maximal ideal is prime, we have that A/M^n is a local ring.

Fact §1.1.5. *Let A be ring, B be an integral domain, $f : A \to B$ be a ring morphism and Q be a prime ideal of B. Then* $\ker(f)$ *is a prime ideal of A.*

Proof of the fact. Suppose that $ab \in \ker(f)$. Then $f(ab) = 0$ which further implies $f(a)f(b) = 0$ and hence $a \in \ker(f)$ or $b \in \ker(f)$ since B is an integral domain. □

Lemma §1.1.6. *Let A, B be rings, $f : A \to B$ be a ring morphism and Q be a prime ideal in B. Then $f^{-1}(Q)$ is a prime ideal of A.*

Proof. Let $p : B \to B/Q$ be the canonical homomorphism. Consider the map $p \circ f : A \to B/Q$. We show that $\ker(p \circ f) = f^{-1}(Q)$. The lemma will follows from fact §1.1.5, if we show that $\ker(p \circ f) = f^{-1}(Q)$ as B/Q is an integral domain. So consider the following:

$$\begin{aligned} x \in \ker\left(p \circ f\right) &\Leftrightarrow p(f(x)) = Q \\ &\Leftrightarrow f(x) + Q = Q \\ &\Leftrightarrow f(x) \in Q \\ &\Leftrightarrow x \in f^{-1}(Q) \end{aligned}$$

□

Lemma §1.1.7. *Let A be a ring, let I, J be ideals of A and P be a prime ideal of A. If $P \supset IJ$ then either $P \supset I$ or $P \supset J$.*

Proof. Suppose that $P \not\supset J$. Then there is some $i \in I \setminus P$. We show that $J \subset P$. Let $j \in P$. Then $ij \in IJ$ and hence $ij \in P$. Since P is a prime ideal, we must have that either $i \in P$ or $j \in P$. But the former is not possible by assumption, therefore, $j \in P$. Since j was arbitrary, the proof is complete. □

Remark §1.1.8. Let A be a ring, I be any ideal of A. Then there is a maximal ideal M of A containing A. The proof of this remark is fairly straightforward. Consider the ring A/I. Since every ring has a maximal ideal, so there must be some maximal ideal $\mathscr{M}$ of A/I. By the correspondence theorem, $\mathscr{M} = M/I$ for some ideal M of A. This ideal M of A must be maximal again by the correspondence theorem and this completes the proof of the remark.

Lemma §1.1.9. *Let A be a ring, I, J, K be ideals of A. Furthermore, assume that I, J are comaximal and I, K are comaximal. Then $I + JK = A$. (Recall that two ideals I, J are said to be comaximal if $I + J = A$.)*

Proof. Suppose that $I + JK \subsetneq A$. Then by Remark §1.1.8, we have that there is some maximal (and hence prime) ideal P containing $I + JK$. Thus, we have that $I \subset P$ and $JK \subset P$.

From $JK \subset P$, we can conclude that $J \subset P$ or $K \subset P$ from Lemma §1.1.7. But in the either case, we have that $I + J \subset P \subsetneq A$. A contradiction and hence $I + JK = A$. □

Example §1.1.10. Let $A = \mathbb{Z}$. Note that the ideal $(3,4)$ generated by 3 and 4 and the ideal $(3,5)$ generated by 3 and 5 are exactly $\mathbb{Z}$. Thus, the ideal $(3,20) = A$ by Lemma §1.1.9.

§1.2 Chinese Remainder Theorem

Theorem §1.2.1 (Chinese Remainder Theorem)**.** *Let A be a ring, $I_1, I_2, \ldots, I_n$ be ideals of A. Consider the canonical map $\varphi : A \to A/I_1 \times A/I_2 \times \cdots A/I_n$ given by $\varphi(x) = (x + I_1, \ldots, x + I_n)$. Then the following holds:*

1. *If I_p, I_q are comaximal for all $1 \le p < q \le n$ then $I_1 I_2 \ldots I_n = I_1 \cap I_2 \cap \ldots \cap I_n$*
2. *φ is injective iff $\ker\varphi = I_1 \cap I_2 \cap \ldots \cap I_n = \{0\}$*
3. *If φ is surjective iff I_m, I_n are comaximal for all $1 \le m < n \le n$*

Proof of (1). We proceed by induction on n. Suppose that $n = 2$. Consider the ideals I_1, I_2 satisfying $I_1 + I_2 = A$. We show that $I_1 I_2 = I_1 \cap I_2$.

It is fairly easy to see that $I_1 I_2 \subset I_1 \cap I_2$. if $i_1 \in I_1$ and $i_2 \in I_2$ then $i_1 i_2 \in I_1$ and $i_1 i_2 \in I_2$ as I_1 and I_2 are both ideals of A. Hence, $i_1 i_2 \in I_1 \cap I_2$. To see the reverse inclusion, we use the comaximality of I_1 and I_2. Since $I_1 + I_2 = A$, $1 = i_1 + i_2$ for some $i_1 \in I_1$ and some $i_2 \in I_2$. Let $c \in I_1 \cap I_2$. Then $c = i_1 c + c i_2$. Clearly $i_1 c \in I_1 I_2$ and $c i_2 \in I_1 I_2$ and hence $c \in I_1 I_2$.

Suppose that (1) holds true for any $n-1$ ideals of A where $n > 2$. Let $I_1, I_2, \ldots, I_n$ be ideals of A. Define $J = I_1 I_2 \cdots I_{n-1}$ and $I = I_n$. We show that $I + J = A$.

It is easy to see that $I + J \subset A$. Now we use that comaximality of I_{n-1} and I_n. By the comaximality, we have $1 = i_{n-1} + i_n$ for some $i_{n-1} \in I_{n-1}$ and some $i_n \in I_n$. Let $a \in A$. Then $a = a i_{n-1} + a i_n$. Clearly, $a i_n \in I_n$ as I_n is an ideal and $a i_{n-1} \in I_{n-1}$. Since $I_{n-1} \subset I$, we are done.

By the $n = 2$, it follows that $IJ = I \cap J$. Now our result follows from the induction hypothesis:

$$\begin{aligned} I_1 \ldots I_{n-1} I_n &= JI \\ &= J \cap I \\ &= I_1 \ldots I_{n-1} \cap I_n \\ &= I_1 \cap \ldots \cap I_{n-1} \cap I_n \end{aligned}$$

Observe that the third equality follows from the induction hypothesis.

□

§2 Lecture 2 — 12th August 2022 —

§2.1 Proof of Chinese Remainder Theorem continued ...