

Functional Calculus

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Definition

C* algebra

1. A unital algebra (vector space + multiplication) \mathcal{A}

- $(ab)c = a(bc)$
- $(a + \lambda b)c = ac + \lambda bc$
- $a(b + \lambda c) = ab + \lambda ac$
- $1_{\mathcal{A}}a = a = a1_{\mathcal{A}}$

2. with an **involution** $*$

- $(\alpha a + \beta b)^* = \overline{\alpha}a^* + \overline{\beta}b^*$
- $(ab)^* = b^*a^*$
- $a^{**} = a$

3. and a complete norm $\|\cdot\|$ that satisfy

- $\|ab\| \leq \|a\|\|b\|$ (submultiplicativity)
- $\|a^*\| = \|a\|$
- $\|a^*a\| = \|a\|^2$

Examples

C^* algebra

1. \mathbb{C} with standard multiplication, conjugation, and standard norm.
2. $B(X)$, complex valued bounded functions on X , with pointwise multiplication, conjugation, and supremum norm.
3. $B(\mathcal{H})$ for a Hilbert space \mathcal{H} with composition, adjoint, and operator norm.

Spectrum

C* algebra

Definition (Invertible elements of \mathcal{A})

An element $a \in \mathcal{A}$ is called invertible if there is an element $z \in \mathcal{A}$ such that $az = 1_{\mathcal{A}} = za$. We denote the collection of invertible elements of \mathcal{A} by $G(\mathcal{A})$

Definition (Spectrum of an element)

We define the spectrum of an element $a \in \mathcal{A}$ as the collection

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid 1_{\mathcal{A}}\lambda - a \notin G(\mathcal{A})\}$$

and the **spectral radius** of a as

$$r(a) = \sup\{|\lambda| \mid \lambda \in \sigma(a)\}$$

Properties of spectrum

C^* algebra

- $G(\mathcal{A})$ is an open in \mathcal{A}
- $r(a) \leq \|a\|$
- $\sigma(a)$ is nonempty for all $a \in A$ (Gelfand)
- $\sigma(a)$ is closed compact in \mathbb{C}
- $r(a) = \lim \|a^n\|^{\frac{1}{n}}$ (Beurling)
- If every nonzero element in \mathcal{A} is invertible, then $\mathcal{A} = \mathbb{C}1_A$ (Gelfand - Mazur)

Some special elements

C* algebra

Definition

Let \mathcal{A} be a C* algebra, an element $a \in \mathcal{A}$ is called:

- **self adjoint / hermitian** if $a^* = a$
- **normal** if $a^*a = aa^*$
- **unitary** if $a^*a = 1_A = aa^*$
- **positive** if a is hermitian and $\sigma(a) \subset \mathbb{R}^+$
- **projection** if $a^2 = a$

We can easily show that if

- a is hermitian, then $\sigma(a) \subset \mathbb{R}$
- a is unitary, then $\sigma(a) \subset \mathbb{T}$

Some Properties

C* algebra

We will now use \mathcal{A}_{sa} and \mathcal{A}^+ to denote hermitian and positive elements in \mathcal{A} respectively

- Every $a \in \mathcal{A}$ can be written as $a = b + ic$ where $b, c \in \mathcal{A}_{sa}$

$$b = \frac{a + a^*}{2}, c = \frac{a - a^*}{2i}$$

- If $a, b \in \mathcal{A}^+$, then $a + b, \alpha a \in \mathcal{A}^+$ for $\alpha \geq 0$
- $\mathcal{A}^+ = \{a^*a \mid a \in \mathcal{A}\}$
- $a \leq b$ if $b - a \in \mathcal{A}^+$ defines a partial order in \mathcal{A}_{sa}

Homomorphisms

C^* algebra

Definition (Homomorphisms between C^* algebras)

An involutive multiplicative bounded linear map between C^* algebras is called a homomorphism.

Let $\phi : \mathcal{A} \rightarrow \mathcal{B}$ be a linear map between C^* algebras \mathcal{A}, \mathcal{B} .

- $\phi \in B(\mathcal{A}, \mathcal{B})$ (bounded)
- $\phi(ab) = \phi(a)\phi(b)$ (multiplicative)
- $\phi(a^*) = \phi(a)^*$ (involutive)

Some properties:

- If $\phi : \mathcal{A} \rightarrow \mathcal{B}$, then ϕ is norm decreasing. ($\|\phi(a)\| \leq \|a\|$)
- Every injective $*$ -homomorphisms are isometric.

Gelfand Spectrum

C^* algebra

Definition

If \mathcal{A} is a C^* algebra, we define the Gelfand spectrum of \mathcal{A} to be the collection of all $*$ homomorphisms from $\mathcal{A} \rightarrow \mathbb{C}$ and denote it by $\Omega(\mathcal{A})$. Since $\Omega(\mathcal{A}) \subset \mathcal{A}^*$, the dual space of \mathcal{A} , we can endow $\Omega(\mathcal{A})$ with the weak $*$ topology from \mathcal{A}^*

Note that by the norm decreasing property of the $*$ -homomorphisms we see that $\Omega(\mathcal{A})$ is a subset of the closed unit ball of \mathcal{A}^*

Gelfand Transform

Abelian C^* algebra

Assuming \mathcal{A} to be abelian gives us extra results

1. $\Omega(A)$ to be compact.
2. $\sigma(a) = \Omega(A)a = \{\tau(a) \mid \tau \in \Omega(A)\}$

Definition (Gelfand Transform)

Given any abelian C^* algebra \mathcal{A} , we define the Gelfand transform of $a \in \mathcal{A}$ as the map

$$\hat{a} : C(\Omega(A)) \rightarrow \mathbb{C} := \hat{a}(f) = f(a)$$

Gelfand Representation

Abelian C^* algebra

If the C^* algebra is abelian, we can represent the abstract C^* algebra with a concrete C^* algebra of continuous functions in a compact space. This is given by the Gelfand representation.

Theorem (Gelfand)

For any abelian C^ algebra \mathcal{A} , Gelfand representation, defined as*

$$\mathcal{A} \rightarrow C(\Omega(\mathcal{A})) : a \rightarrow \hat{a}$$

is an isometric $$ -isomorphism.*

Lemma (Theorem 2.1.11, Murphy)

Let \mathcal{B} be a C^ subalgebra of \mathcal{A} . Then if $b \in \mathcal{B}$,*

$$\sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{B}}(a)$$

Functional Calculus

Definition

Motivation: If $p \in \mathbb{C}[z, \bar{z}]$ is a polynomial and $a \in \mathcal{A}$, then $p(a)$ is a well defined element of \mathcal{A} . Can we generalize this to continuous functions?

Theorem

Let a be a normal element in a unital C^ algebra \mathcal{A} . Let $z : \sigma(a) \rightarrow \mathbb{C}$ be the inclusion map. Then there is a unique unital $*$ -homomorphism from $\phi_a : C(\sigma(a)) \rightarrow \mathcal{A}$ such that $\phi_a : z \rightarrow a$. Moreover image of ϕ_a is the C^* algebra generated by a and $1_{\mathcal{A}}$. (Note that this C^* algebra is abelian since $a^*a = aa^*$)*

Functional Calculus

Definition

Definition (Functional calculus)

Let a be a normal element of a unital C^* algebra \mathcal{A} and $f \in C(\sigma(a))$, then we call ϕ_a the functional calculus of a .

Notation: For ease of notation when $a \in \mathcal{A}$ and $f \in C(\sigma(a))$, we will write $f(a)$ instead of $\phi_a(f)$.

Theorem (Spectral Mapping Theorem)

Let $a \in \mathcal{A}$ be normal and $f \in C(\sigma(a))$ then

$$\sigma(f(a)) = f(\sigma(a))$$

Functional Calculus

Proof of Spectral Mapping

Proof.

Let \mathcal{B} be the C^* subalgebra generated by $1_{\mathcal{A}}, a$. Then \mathcal{B} is abelian. Moreover

$$\sigma(f(a)) = \{\tau(f(a)) \mid \tau \in \Omega(B)\} = \{f(\tau(a)) \mid \tau \in \Omega(B)\} = f(\sigma(a))$$

Note that $f(\tau(a)) = \tau(f(a))$ since $f \rightarrow \tau(f(a))$ and $f \rightarrow f(\tau(a))$ are C^* homomorphisms from $C(\sigma(a)) \rightarrow C^*(1, a)$ which agree for $f = 1, z$ and $C(\sigma(a))$ is generated by $1, z$.
Hence it holds true for all $f \in C(\sigma(a))$ □

Functional Calculus

Applications

Lemma

Every self adjoint element can be written as the difference of two positive elements.

Proof.

Consider $f^+, f^- \in C(\sigma(a))$ where $a \in \mathcal{A}_{sa}$. where

$$f^+(x) = \begin{cases} x, & x \geq 0 \\ 0, & x < 0 \end{cases} \quad f^-(x) = \begin{cases} 0, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

See that $a = f^+(a) - f^-(a)$ and that this proves it. □

Functional Calculus

Applications

Theorem (Theorem 2.1.15, Murphy)

Let Γ be compact Hausdorff. For $\gamma \in \Gamma$, let δ_γ be a character for $C(\Gamma)$ given by $f \rightarrow f(\gamma)$. Then the map $\Gamma \rightarrow \Omega(C(\Gamma)) : \gamma \rightarrow \delta_\gamma$ is a homeomorphism.

References



C* Algebras and Operator Theory

Gerald Murphy



Functional Analysis; Spectral Theory

V. S. Sunder



For More

<https://joelsleebea.github.io/resources/>

Thank you for listening!

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