Introduction to C* algebras

(No proofs, just the ideas)

Joel Sleeba

IISER Thiruvananthapuram

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Things I learned during summer

Definition

C* algebra

1. A unital algebra (vector space + multiplication) $\mathcal{A}$
   - $(ab)c = a(bc)$
   - $(a + \lambda b)c = ac + \lambda bc$
   - $a(b + \lambda c) = ab + \lambda ac$
   - $1_{\mathcal{A}} a = a = a 1_{\mathcal{A}}$
2. with an involution $*$
   - $(\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^*$
   - $(ab)^* = b^* a^*$
   - $a^{**} = a$
3. and a complete norm $\|\cdot\|$ that satisfy
   - $\|ab\| \leq \|a\|\|b\|$ (submultiplicativity)
   - $\|a^*\| = \|a\|$
   - $\|a^* a\| = \|a\|^2$

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1. $\mathbb{C}$ with standard multiplication, conjugation, and standard norm.
2. B(X), complex valued bounded functions on X, with pointwise multiplication, conjugation, and supremum norm.
3. $B(\mathcal{H})$ for a Hilbert space $\mathcal{H}$ with composition, adjoint, and operator norm.

Examples

C* algebra

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3. $B(\mathcal{H})$ for a Hilbert space $\mathcal{H}$ with composition, adjoint, and operator norm.

Definition (Invertible elements of $\mathcal{A}$)

An element $a \in \mathcal{A}$ is called invertible if there is an element $z \in \mathcal{A}$ such that $az = 1_{\mathcal{A}} = za$. We denote the collection of invertible elements of $\mathcal{A}$ by $G(\mathcal{A})$

Definition (Spectrum of an element)

We define the spectum of an element $a \in \mathcal{A}$ as the collection

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid 1_A\lambda - a \notin G(\mathcal{A})\}$$

and the **spectral radius** of a as

$$r(a) = \sup\{|\lambda| \mid \lambda \in \sigma(a)\}$$

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Properties of spectrum

C* algebra

- $G(\mathcal{A})$ is an open in $\mathcal{A}$
- $r(a) \leq \|a\|$
- $\sigma(a)$ is nonempty for all $a \in A$ (Gelfand)
- $\sigma(a)$ is closed compact in $\mathbb{C}$
- $r(a) = \lim \|a^n\|^{\frac{1}{n}}$ (Beurling)
- If every nonzero element in $\mathcal{A}$ is invertible, then $\mathcal{A} = \mathbb{C}1_A$ (Gelfand - Mazur)

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Definition

Let $\mathcal{A}$ be a C* algebra, an element $a \in \mathcal{A}$ is called:

- **self adjoint / hermitian** if $a^* = a$
- **normal** if $a^*a = aa^*$
- **unitary** if $a^*a = 1_A = aa^*$
- **positive** if a is hermitian and $\sigma(a) \subset \mathbb{R}^+$
- **projection** if $a^2 = a$

We can easily show that if

- a is hermitian, then $\sigma(a) \subset \mathbb{R}$
- a is unitary, then $\sigma(a) \subset \mathbb{T}$

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Some Properties

C* algebra

We will now use $\mathcal{A}_{sa}$ and $\mathcal{A}^+$ to denote hermitian and positive elements in $\mathcal{A}$ respectively

- Every $a \in \mathcal{A}$ can be written as $a = b + ic$ where $b, c \in \mathcal{A}_{sa}$ (Compare this with decomposition of a matrix to symmetric and skew-symmetric pairs)
- If $a, b \in \mathcal{A}^+$, then $a + b, \alpha a \in \mathcal{A}^+$ for $\alpha \geq 0$
- $\mathcal{A}^+ = \{a^*a \mid a \in \mathcal{A}\}$
- $a \leq b$ if $b - a \in \mathcal{A}^+$ defines a partial order in $\mathcal{A}_{sa}$

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Definition (Homomorphisms between C* algebras)

An involutive multiplicative bounded linear map between C* algebras is called a homomorphism.

Let $\phi : \mathcal{A} \to \mathcal{B}$ be a linear map between C* algebras $\mathcal{A}, \mathcal{B}$.

- $\phi \in B(\mathcal{A}, \mathcal{B})$ (bounded)
- $\phi(ab) = \phi(a)\phi(b)$ (multiplicative)
- $\phi(a^*) = \phi(a)^*$ (involutive)

Some properties:

- If $\phi : \mathcal{A} \to \mathcal{B}$, then ϕ is norm decreasing. ($\|\phi(a)\| \leq \|a\|$)
- Every injective $*$-homomorphisms are isometric.

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If $\mathcal{A}$ is a C* algebra, we define the Gelfand specturm of $\mathcal{A}$ to be the collection of all $*$ homomorphims from $\mathcal{A} \to \mathbb{C}$ and denote it by $\Omega(\mathcal{A})$. Since $\Omega(\mathcal{A}) \subset \mathcal{A}^*$, the dual space of $\mathcal{A}$, we can endow $\Omega(\mathcal{A})$ with the weak * topology from $\mathcal{A}^*$

Note that by the norm decreasing property of the $*$-homomorphisms we see that $\Omega(\mathcal{A})$ is a subset of the closed unit ball of $\mathcal{A}^*$

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Assuming $\mathcal{A}$ to be abelian gives us extra results

1. $\Omega(A)$ to be compact.
2. $\sigma(a) = \Omega(A)a = \{\tau(a) \mid \tau \in \Omega(A)\}$

Definition (Gelfand Transform)

Given any abelian C* algebra $\mathcal{A}$, we define the Gelfand transform of $a \in \mathcal{A}$ as the map

$$\hat{a} : C(\Omega(A)) \to \mathbb{C} := \hat{a}(f) = f(a)$$

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If the C* algebra is abelian, we can represent the abstract C* algebra with a concrete a C* algebra of continuous functions in a compact space. This is given by the Gelfand representation.

Theorem (Gelfand)

For any abelian C algebra $\mathcal{A}$, Gelfand representation, defined as*

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Definition (States of a C* algebra)

Given a C* algebra $\mathcal{A}$, a linear functional $\phi : \mathcal{A} \to \mathbb{C}$ is called a **state** if

1. $\phi(a) \in \mathbb{R}^+$ for all $a \in \mathcal{A}^+$ (positive)
2. $\phi(1_A) = 1$

Definition (Representation of a C* algebra)

Given a C* algebra $\mathcal{A}$ and a Hilbert space H, a map $\pi : \mathcal{A} \to B(H)$ is called a representation if it is a $*$-homomorphism. If π is injective we call the representation **faithful**.

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GNS Constructions

towards a more general representation

Given any representation (H, π) of a C* algebra $\mathcal{A}$ to a Hilbert space H, it can be verified that

$$\phi : \mathcal{A} \to \mathbb{C} := a \to \langle \pi(a)\xi, \xi \rangle$$

is a state for any unit vector $\xi \in H$.

The reverse is also true.

Given any state ϕ of a C* algebra $\mathcal{A}$, we can construct a Hilbert space H_ϕ, a map π_ϕ, and a unit vector $\xi_\phi \in H_\phi$ such that (H_ϕ, π_ϕ) is a representation for $\mathcal{A}$ and

$$\phi(a) = \langle \pi_\phi(a)\xi_\phi, \xi_\phi \rangle_{H_\phi}$$

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Given any state ϕ of a C* algebra $\mathcal{A}$, we can construct a Hilbert space H_ϕ, a map π_ϕ, and a unit vector $\xi_\phi \in H_\phi$ such that (H_ϕ, π_ϕ) is a representation for $\mathcal{A}$ and

$$\phi(a) = \langle \pi_\phi(a)\xi_\phi, \xi_\phi \rangle_{H_\phi}$$

Definition (Universal Representation)

Let $S(\mathcal{A})$ be the collection of all states of a C* algebra $\mathcal{A}$. For any $\phi \in S(\mathcal{A})$ let $(H_\phi, \pi_\phi, \xi_\phi)$ be the corresponding GNS representation, Let

$$H = \bigoplus_{\phi \in S(\mathcal{A})} H_\phi, \qquad \pi = \bigoplus_{\phi \in S(\mathcal{A})} \pi_\phi, \qquad \xi = \bigoplus_{\phi \in S(\mathcal{A})} \xi_\phi$$

Then (H, π) is called the universal representation of the C* algebra $\mathcal{A}$.

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Theorem (Gelfand Naimark Theorem)

Given any C algebra $\mathcal{A}$, its universal representation is faithful.*

Along with the fact that injective $*$-homomorphisms are isometric and image of a C* homomorphism is a C* subalgebra, we see that every C* algebra is isometrically isomorphic to a subalgebra of operators in a Hilbert space.

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For More
https://joelsleeba.github.io/resources/

Thank you for listening!

Joel Sleeba

joelsleeba1@gmail.com
joelsleeba.github.io