Functional Calculus

Joel Sleeba

October 22, 2023

Definition

C* algebra

1. A unital algebra (vector space + multiplication) $\mathcal{A}$
   - $(ab)c = a(bc)$
   - $(a + \lambda b)c = ac + \lambda bc$
   - $a(b + \lambda c) = ab + \lambda ac$
   - $1_{\mathcal{A}} a = a = a 1_{\mathcal{A}}$
2. with an involution $*$
   - $(\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^*$
   - $(ab)^* = b^* a^*$
   - $a^{**} = a$
3. and a complete norm $\|\cdot\|$ that satisfy
   - $\|ab\| \leq \|a\|\|b\|$ (submultiplicativity)
   - $\|a^*\| = \|a\|$
   - $\|a^* a\| = \|a\|^2$

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Examples

C* algebra

1. $\mathbb{C}$ with standard multiplication, conjugation, and standard norm.
2. B(X), complex valued bounded functions on X, with pointwise multiplication, conjugation, and supremum norm.
3. $B(\mathcal{H})$ for a Hilbert space $\mathcal{H}$ with composition, adjoint, and operator norm.

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Definition (Invertible elements of $\mathcal{A}$)

An element $a \in \mathcal{A}$ is called invertible if there is an element $z \in \mathcal{A}$ such that $az = 1_{\mathcal{A}} = za$. We denote the collection of invertible elements of $\mathcal{A}$ by $G(\mathcal{A})$

Definition (Spectrum of an element)

We define the spectum of an element $a \in \mathcal{A}$ as the collection

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid 1_A\lambda - a \notin G(\mathcal{A})\}$$

and the **spectral radius** of a as

$$r(a) = \sup\{|\lambda| \mid \lambda \in \sigma(a)\}$$

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Properties of spectrum

C* algebra

- $G(\mathcal{A})$ is an open in $\mathcal{A}$
- $r(a) \leq \|a\|$
- $\sigma(a)$ is nonempty for all $a \in A$ (Gelfand)
- $\sigma(a)$ is closed compact in $\mathbb{C}$
- $r(a) = \lim \|a^n\|^{\frac{1}{n}}$ (Beurling)
- If every nonzero element in $\mathcal{A}$ is invertible, then $\mathcal{A} = \mathbb{C}1_A$ (Gelfand - Mazur)

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Let $\mathcal{A}$ be a C* algebra, an element $a \in \mathcal{A}$ is called:

- **self adjoint / hermitian** if $a^* = a$
- **normal** if $a^*a = aa^*$
- **unitary** if $a^*a = 1_A = aa^*$
- **positive** if a is hermitian and $\sigma(a) \subset \mathbb{R}^+$
- **projection** if $a^2 = a$

We can easily show that if

- a is hermitian, then $\sigma(a) \subset \mathbb{R}$
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Some Properties
C* algebra

We will now use $\mathcal{A}_{sa}$ and $\mathcal{A}^+$ to denote hermitian and positive elements in $\mathcal{A}$ respectively

- Every $a \in \mathcal{A}$ can be written as $a = b + ic$ where $b, c \in \mathcal{A}_{sa}$

$$b = \frac{a + a^*}{2}, c = \frac{a - a^*}{2i}$$

- If $a, b \in \mathcal{A}^+$, then $a + b, \alpha a \in \mathcal{A}^+$ for $\alpha \geq 0$
- $\mathcal{A}^+ = \{a^* a \mid a \in \mathcal{A}\}$
- $a \leq b$ if $b - a \in \mathcal{A}^+$ defines a partial order in $\mathcal{A}_{sa}$

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Definition (Homomorphisms between C* algebras)

An involutive multiplicative bounded linear map between C* algebras is called a homomorphism.

Let $\phi : \mathcal{A} \to \mathcal{B}$ be a linear map between C* algebras $\mathcal{A}, \mathcal{B}$.

- $\phi \in B(\mathcal{A}, \mathcal{B})$ (bounded)
- $\phi(ab) = \phi(a)\phi(b)$ (multiplicative)
- $\phi(a^*) = \phi(a)^*$ (involutive)

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- If $\phi : \mathcal{A} \to \mathcal{B}$, then ϕ is norm decreasing. ($\|\phi(a)\| \leq \|a\|$)
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If $\mathcal{A}$ is a C* algebra, we define the Gelfand specturm of $\mathcal{A}$ to be the collection of all $*$ homomorphims from $\mathcal{A} \to \mathbb{C}$ and denote it by $\Omega(\mathcal{A})$. Since $\Omega(\mathcal{A}) \subset \mathcal{A}^*$, the dual space of $\mathcal{A}$, we can endow $\Omega(\mathcal{A})$ with the weak * topology from $\mathcal{A}^*$

Note that by the norm decreasing property of the $*$-homomorphisms we see that $\Omega(\mathcal{A})$ is a subset of the closed unit ball of $\mathcal{A}^*$

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Assuming $\mathcal{A}$ to be abelian gives us extra results

1. $\Omega(A)$ to be compact.
2. $\sigma(a) = \Omega(A)a = \{\tau(a) \mid \tau \in \Omega(A)\}$

Definition (Gelfand Transform)

Given any abelian C* algebra $\mathcal{A}$, we define the Gelfand transform of $a \in \mathcal{A}$ as the map

$$\hat{a} : C(\Omega(A)) \to \mathbb{C} := \hat{a}(f) = f(a)$$

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If the C* algebra is abelian, we can represent the abstract C* algebra with a concrete a C* algebra of continuous functions in a compact space. This is given by the Gelfand representation.

Theorem (Gelfand)

For any abelian C algebra $\mathcal{A}$, Gelfand representation, defined as*

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Before Functional Calculus

Lemma (Theorem 2.1.11, Murphy)

Let $\mathcal{B}$ be a C subalgebra of $\mathcal{A}$. Then if $b \in \mathcal{B}$,*

$$\sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{B}}(a)$$

Motivation: If $p \in \mathbb{C}[z, \bar{z}]$ is a polynomial and $a \in \mathcal{A}$, then $p(a)$ is a well defined element of $\mathcal{A}$. Can we generalize this to continuous functions?

Theorem

Let a be a normal element in a unital C algebra $\mathcal{A}$. Let $z : \sigma(a) \to \mathcal{C}$ be the inclusion map. Then there is a unique unital $*$-homomorphism from $\phi_a : C(\sigma(a)) \to \mathcal{A}$ such that $\phi_a : z \to a$. Moreover image of ϕ_a is the C* algebra generated by a and $1_{\mathcal{A}}$. (Note that this C* algebra is abelian since $a^*a = aa^*$)*

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Definition (Functional calculus)

Let a be a normal element of a unital C* algebra $\mathcal{A}$ and $f \in C(\sigma(a))$, then we call ϕ_a the functional calculus of a.

Notation: For ease of notation when $a \in \mathcal{A}$ and $f \in C(\sigma(a))$, we will write $f(a)$ instead of $\phi_a(f)$.

Theorem (Spectral Mapping Theorem)

Let $a \in \mathcal{A}$ be normal and $f \in C(\sigma(a))$ then

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Theorem (Spectral Mapping Theorem)

Let $a \in \mathcal{A}$ be normal and $f \in C(\sigma(a))$ then

$$\sigma(f(a)) = f(\sigma(a))$$

Proof.

Let $\mathcal{B}$ be the C* subalgebra generated by $1_{\mathcal{A}}, a$. Then $\mathcal{B}$ is abelian. Moreover

$$\sigma(f(a)) = \{\tau(f(a)) \,|\tau \in \Omega(B)\} = \{f(\tau(a)) \,|\tau \in \Omega(B)\} = f(\sigma(a))$$

Note that $f(\tau(a)) = \tau(f(a))$ since it holds for $f = 1, z$ and $C(\sigma(a))$ is generated by $1, z$. $\square$

Lemma

Every self adjoint element can be written as the difference of two positive elements.

Proof.

Consider $f^+, f^- \in C(\sigma(a))$ where $a \in \mathcal{A}_{sa}$. where

$$f^+(x) = \begin{cases} x, & x \geq 0 \\ 0, & x < 0 \end{cases} \quad f^-(x) = \begin{cases} 0, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

See that $a = f^+(a) - f^-(a)$ and that this proves it. $\square$

Theorem (Theorem 2.1.15, Murphy)

Let Γ be compact Hausdorff. For $\gamma \in \Gamma$, let δ_γ be a character for $C(\Gamma)$ given by $f \to f(\gamma)$. Then the map $\Gamma \to \Omega(C(\Gamma)) : \gamma \to \delta_\gamma$ is a homeomorphism.

References

C* Algebras and Operator Theory
Gerald Murphy

Functional Analysis; Spectral Theory
V. S. Sunder

For More
https://joelsleeba.github.io/resources/

Thank you for listening!

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