# Solutions to Bartles' Measure Theory and Lebesgue Measure

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# Contents

# Part I

# The Elements of Measure Theory

# Chapter 2

# Measurable Functions

**I.** Give an example of a function $f$ on $X$ to $\mathbb{R}$ which is not $X$ measurable but $f^2$ and $|f|$ are.
*Solution.* Let $X = \mathbb{R}$ with the $\sigma$-algebra $\mathcal{X} = \{\emptyset, \mathbb{R}, \{1\}, \mathbb{R} \setminus \{1\}\}$. Now let $f$ be defined as

$$f(x) = \begin{cases} 1, & x \in (0, \infty) \\ -1, & \text{otherwise} \end{cases}$$

Then $f^2 = |f| = \mathbf{1}$ the constant function is measurable, but $f^{-1}((-2, -1)) = \{-1\} \notin \mathcal{X}$. Hence $f$ is not measurable. $\square$

**S.** Let $f$ be a complex valued function defined on a measurable space $(X, \mathcal{X})$. Show that $f$ is $X$-measurable if and only if

$$E_{a,b,c,d} := \{x \in X \ : \ a < \Re(f)(x) < b, c < \Im(f)(x) < d\} \in \mathcal{X}$$

for all real numbers $a, b, c, d$.

More generally show that $f$ is $X$-measurable if and only if $f^{-1}(G) \in \mathcal{X}$ for every open set $G$ in the complex plane.
*Solution.* Let $f = f_1 + if_2$. Then $f$ is $X$-measurable if and only if both $f_1$ and $f_2$ are $X$-measurable. Hence we get $E_{a,b,c,d} = f_1^{-1}(a, b) \cap f_2^{-1}(c, d) \in \mathcal{X}$.

Conversely if $E_{a,b,c,d} \in \mathcal{X}$ for each real numbers $a, b, c, d$, then we can show that $f_1, f_2$ are $X$-measurable by working with $E_{a,b,-\infty,\infty}$ and $E_{-\infty,\infty,c,d}$. (Note that here we'll violate the requirement that $a, b, c, d$ are reals, but we can modify this to work with reals by choosing large enough values instead of infinities).

Now for the general case, note that $\mathbb{R}^2$(product topology) and $\mathbb{C}$ are topologically equivalent. Since arbitrary open set $U \in \mathbb{R}$ can be written as a countable union of disjoint open intervals, an open set $G \in \mathbb{R}^2$ can be written as countable union of disjoint open rectangles. Now the above result using $E_{a,b,c,d}$ gives the equivalence. $\square$

**T.** Show that sums, products, and limits of complex-valued measurable functions are measurable.

*Solution.* Use the algebra of real measurable functions. $\square$

**U.** Show that a function $f : X \to \mathbb{R}$(or to $\bar{\mathbb{R}}$) is $X$-measurable if and only if the collection $A_\alpha = f^{-1}(\alpha, \infty) \in \mathcal{X}$ for each $\alpha \in \mathbb{Q}$. Equivalently $B_\alpha = f^{-1}(-\infty, \alpha) \in \mathcal{X}$

*Solution.* Follows from the fact that the collection $(\alpha, \infty)$ and $(-\infty, \alpha)$ where $\alpha \in \mathbb{Q}$ independently generate the Borel sigma algebra for $\mathbb{R}$. $\square$

**V.** See the definition for monotone classes from the problem. Show that for any nonempty collection of subsets of $X$, there is a smallest monotone class containing $A$.

*Solution.* Clearly the $\sigma$-algebra generated by $A$ is a monotone class containing $A$. Now take the intersection of all the monotone classes containing $A$. This is a monotone class. (Verify). $\square$

# Chapter 3

# Measures

**A.** If $\mu$ is a measure on $X$ and $E \in \mathcal{X}$, then define $\lambda(A) = \mu(A \cap E)$ for all $A \in \mathcal{X}$. Show that $\lambda$ is a measure.

*Solution.* Since $\mu$ is a measure, $\mu \geq 0$ and therefore it follows that $\lambda \geq 0$. Hence we just need to verify countable subadditivity of $\lambda$. But this follows easily as

$$(\bigcup_{i \in \mathbb{N}} A_i) \cap E = \bigcup_{i \in \mathbb{N}} A_i \cap E$$

$\square$

**B.** If $\mu_1, \mu_2, \ldots \mu_n$ are measures on $X$ on $\mathcal{X}$ and $a_1, a_2, \ldots a_n$ be positive real numbers. Show that the function $\lambda$ defined on $\mathcal{X}$ as

$$\lambda(E) = \sum_{i=1}^{n} a_i \mu_i(E)$$

is a measure on $X$.

*Solution.* Since it is easy to verify that $\lambda(\emptyset) = 0$ and $\lambda(E) \geq 0$ for all $E \in \mathcal{X}$, we will just verify the countable additivity of $\lambda$. Let $A_i$ be a countable collection of disjoint subsets of $X$. Then by the countable additivity of the measures $\mu_i$, we get

$$\mu_j(\bigcup_i A_i) = \sum_i \mu_j(A_i)$$

Hence

$$\lambda(\bigcup_{i \in \mathbb{N}} A_i) = \sum_{j=1}^{n} \mu_j(\bigcup_{i \in \mathbb{N}} A_i) = \sum_{j=1}^{n} \sum_{i \in \mathbb{N}} \mu_j(A_i) = \sum_{i \in \mathbb{N}} \sum_{j=1}^{n} \mu_j(A_i) = \sum_{i \in \mathbb{N}} \lambda(A_i)$$

$\square$

**C.** If $(\mu_n)$ is a sequence of measures on $X$ with $\mu_n(X) = 1$ and if $\lambda$ is defined by

$$\lambda(E) = \sum_{n=1}^{\infty} 2^{-n} \mu_n(E)$$

then $\lambda$ is a measure on $\mathcal{X}$ and $\lambda(X) = 1$

*Solution.* Note that once we prove that $\lambda$ is a measure on $X$, then the fact that $\lambda(X) = 1$ will follow from the definition of $\lambda$. Also the fact that $\lambda(\emptyset) = 0$ and $\lambda(E) \geq 0$ follows easily from the definition. We will verify countable additivity of $\lambda$.

For that let $A_i$ be a countable collection of disjoint elements in $\mathcal{X}$. By the countable additivity of the measures, $\mu_n$, we get that

$$\mu_n(\bigcup_{i \in \mathbb{N}} A_i) = \sum_{i \in \mathbb{N}} \mu_n(A_i)$$

Hence

$$\lambda(\bigcup_{i \in \mathbb{N}} A_i) = \sum_{n=1}^{\infty} 2^{-n} \mu_n(\bigcup_{i \in \mathbb{N}} A_i) = \sum_{n=1}^{\infty} \sum_{i \in \mathbb{N}} 2^{-n} \mu_n(A_i)$$

Since $\mu_i(X) = 1$, we get that $\sum_{i \in \mathbb{N}} \mu_n(A_i) \leq 1$ and therefore the double summation above is absolutely summable enabling us to interchange the order of summation. Hence, we get

$$\lambda(\bigcup_{i \in \mathbb{N}} A_i) = \sum_{i \in \mathbb{N}} \sum_{n=1}^{\infty} 2^{-n} \mu_n(A_i) = \sum_{i \in \mathbb{N}} \lambda(A_i)$$

□

**D.** Let $X = \mathbb{N}$ and let $\mathcal{X}$ be the $\sigma$-algebra of all the subsets of $\mathbb{N}$. If $(a_n)$ is a sequence of non-negative real numbers and if we define $\mu$ by

$$\mu(\emptyset) = 0, \quad \mu(E) = \sum_{n \in E} a_n$$

then $\mu$ is a measure on $\mathbb{N}$. Conversely, every measure on $\mathcal{X}$ is obtained this way.

*Solution.* Once we have a sequence $(a_n)$, it is quite easy to verify that $\mu$ is a measure on $\mathbb{N}$. We'll show that every measure on $\mathbb{N}$ is of this form.

Let $\mu$ be a measure on $\mathbb{N}$ and $n \in \mathbb{N}$, then $\{n\} \in \mathcal{X}$ and therefore denote $\mu(\{n\})$ by $a_n$. Then we see that if $E$ is any subset of $X$, then

$$\mu(E) = \sum_{n \in E} a_n$$

$\square$

**F.** Let $X = \mathbb{N}$ and let $\mathcal{X}$ be the family of all subsets of $\mathbb{N}$. Define a function $\mu$ on $\mathcal{X}$ as $\mu(E) = 0$ if $E$ is finite and $\mu(E) = +\infty$ if $E$ is infinite. Is $\mu$ a measure on $\mathcal{X}$?
*Solution.* No. Countable union of disjoint finite sets can be infinite. $\square$

**G.** Is $\mu$ a measure if $\mu(E) = +\infty$ for all $E \in \mathcal{X}$ in exercise **F**
*Solution.* No. $\mu(\emptyset) \neq 0$ $\square$

**H.** Show that if the finiteness condition $\mu(F_1) < +\infty$ is dropped, then there exist a decreasing sequence $F_n$ with $\mu(\cap_{i\in\mathbb{N}} F_i) \neq \lim_{i\to\infty} \mu(F_i)$
*Solution.* Let $F_i = \mathbb{R} \setminus [-i, i]$. Then it satisfies all the above. $\square$

**I.** A Let $(X, \mathcal{X}, \mu)$ be a measure space and let $(E_n)$ be a sequence in $\mathcal{X}$. Show that

$$\mu(\liminf E_n) \leq \liminf \mu(E_n)$$

*Solution.* From the previous chapter we know that

$$\liminf E_n = \bigcup_{n\in\mathbb{N}} \bigcap_{m>n} E_m$$

Since the measure is continuous from below, we get that

$$\mu(\liminf E_n) = \lim \mu\Big(\bigcap_{m>n} E_m\Big)$$

Now since $\cap_{m\geq n} E_m \subset E_m$ for all $m \geq n$, we get that

$$\mu\Big(\bigcap_{m>n} E_m\Big) \leq \inf_{m\geq n} \mu(E_m)$$

Substituting this to the above equation gives us our required result. $\square$

**J.** Let $(X, \mathcal{X}, \mu)$ be a measure space and let $(E_n)$ be a sequence in $\mathcal{X}$. Show that

$$\limsup \mu(E_n) \leq \mu(\limsup E_n)$$

*Solution.* Follow similar reasoning as above using the continuity of the measure from above when $\mu(E_1) < \infty$. Note that $E_m \subset \cup_{m\geq n} E_m$ for all $m \geq n$, and therefore

$$\sup_{m\geq n} \mu(E_m) \leq \mu\Big(\bigcup_{m>n} E_m\Big)$$

which will give us our required inequality. $\square$

**L* (Completion of a measure).** Let $(X, \mathcal{X}, \mu)$ be a measure space and $\mathcal{Z}$ be the collection of measure zero sets in $\mathcal{X}$. Let $\mathcal{X}'$ be the family of all subsets of $X$ of the form

$$(E \cup Z_1) \setminus Z_2, \quad E \in \mathcal{X},\ Z_1, Z_2 \text{are subsets of some elements of} \mathcal{Z}$$

Show that $A \in \mathcal{X}'$ if and only if $A = E \cup Z$ where $E \in \mathcal{X}$ and $Z$ is a subset of a set in $\mathcal{Z}$. Also show that $\mathcal{X}'$ forms a $\sigma$-algebra of $X$.
*Solution.* Assume $A = E \cup Z$ where $E \in \mathcal{X}$ and $Z$ is a subset of some set $Z_0 \in \mathcal{Z}$. Then $A = (E \cup Z) \setminus \emptyset$ gives one direction of our proof.

Conversely, if $A = (E \cup Z_1) \setminus Z_2$ with $Z_1 \subset Z_1' \in \mathcal{Z}$ and $Z_2 \subset Z_2' \in \mathcal{Z}$. Consider $A' = E \setminus Z_2' \in \mathcal{X}$. We can verify $A' \subset A$ and that $Z = A \setminus A'$ is a subset of measure zero set $Z_1' \cup Z_2'$. Hence $A = A' \cup Z$ as we need.

To prove $\mathcal{X}'$ is a sigma algebra, first we observe that $\mathcal{X} \subset \mathcal{X}'$ by taking $Z_1 = Z_2 = \emptyset$. Then we just need to verify $\mathcal{X}'$ is closed under countable union of disjoint subsets. Let $A_i = E_i \cup Z_i \in \mathcal{X}'$ be a countable collection of disjoint sets for $E_i \in \mathcal{X}$ and $Z_i \subset Z_i' \in \mathcal{Z}$. Hence $E_i$ is again a collection of disjoint sets. Moreover

$$\bigcup_{i \in \mathbb{N}} A_i = \bigcup_{i \in \mathbb{N}} E_i \ \cup \ \bigcup_{i \in \mathbb{N}} Z_i = E \cup Z \ \in \mathcal{Z}'$$

where $E \in \mathcal{X}$ and $Z \subset \cup_{i \in \mathbb{N}} Z_i' \in \mathcal{Z}$ $\square$

**M.** With the notations in exercise **L**, let $\mu'$ be defined on $\mathcal{X}'$ by

$$\mu'(E \cup Z) = \mu(E)$$

where $E \in \mathcal{X}$ and $Z$ is a subset of a set in $\mathcal{Z}$. Show that $\mu'$ is a well defined measure on $\mathcal{X}'$ that agree with $\mu$ on $\mathcal{X}$.
*Solution.* To show that $\mu'$ is a well defined function on $\mathcal{X}'$, let $E \cup Z_1 = F \cup Z_2$ for $E, F \in \mathcal{X}$ and $Z_1 \subset Z_1' \in \mathcal{Z}, Z_2 \subset Z_2' \in \mathcal{Z}$. We need to show $\mu(E) = \mu(F)$. But

$$\mu(E) = \mu'(E \cup Z_1) = \mu'(F \cup Z_2) = \mu(F)$$

gives our well defineness. Now it is easy to see that $\mu'$ agree with $\mu$ on $\mathcal{X}$, since we can just take $Z = \emptyset$ $\square$

**N.** Let $(X, \mathcal{X}, \mu)$ be a measure space and $(X, \mathcal{X}', \mu')$ be its completion. Let $f$ be an $X'$ measurable function from $X \to \bar{\mathbb{R}}$. Show that there exists an $X$ measurable function $g$ which agrees almost everywhere with $f$.

*Solution.* Consider the collection of sets $A_r = \{x \in X \ : \ f(x) > r\}$ for $r \in \mathbb{Q}$. Since $f$ is $\mathcal{X}'$ measurable, $A_r \in \mathcal{X}'$ and by previous exercise $A_r = E_r \cup Z_r$ for each $r \in \mathbb{Q}$ where $E_r \in \mathcal{X}$ and $Z_r \subset Z'_r \in \mathcal{Z}$. Now let $Z = \cup_{r\in\mathbb{Q}} Z'_r$ and define

$$g(x) = \begin{cases} f(x), & x \notin Z \\ 0, & x \in Z \end{cases} \tag{3.1}$$

Since $Z$ is a measure zero set, it is clear that $g$ agrees with $f$ almost everywhere in $X$. We just need to prove that $g$ is $\mathcal{X}$ measurable. Consider $F_r = g^{-1}(r, \infty)$. $F_r = E_r$ if $r \geq 0$ and $E_r \cup Z$ otherwise. Since both $E_r, Z \in \mathcal{X}$, we get $g$ is $\mathcal{X}$ measurable. □

**O.** Show that if $(X, \mathcal{X})$ is a measurable space with a charge $\mu$, then $\mu$ is again continuous from above and below. That is if $E_n$ is an increasing sequence and $F_n$ is a decreasing sequence with $\mu(F_1) < \infty$, then

$$\mu\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \lim \mu(E_n), \quad \mu\Big(\bigcap_{n=1}^{\infty} F_n\Big) = \lim \mu(F_n)$$

*Solution.* Disjointifying $E_n$ as $E'_1 = E_1, E'_m = E_m \setminus E_{m-1}$ lets us use the countable disjoint additivity of the charge $\mu$. Since $E_m$ is an increasing sequence, $E_m = \cup_{i=1}^m E_i = \cup_{i=1}^m E'_i$ for all $m \in \mathbb{N}$, and we get

$$\mu\Big(\bigcup_{i\in\mathbb{N}} E_m\Big) = \mu\Big(\bigcup_{i\in\mathbb{N}} E'_m\Big) = \sum_{i\in\mathbb{N}} \mu(E'_m)$$

Now since $E'_m = E_m \setminus E_{m-1}$, using the telescoping sums we get the RHS of the above equation to be $\lim \mu(E_m)$

The argument is similar for the continuity from above. □

**P**.** Let $\mu$ be a charge on $\mathcal{X}$ and $\pi$ be defined for $E \in \mathcal{X}$ by

$$\pi(E) = \sup\{\mu(A) \ : \ A \subset E, \ A \in \mathcal{X}\}$$

Then show that $\pi$ is a measure on $\mathcal{X}$

*Solution.* $\pi(\emptyset) = 0$ and that $\pi(E) \geq 0$ for all $E \in \mathcal{X}$ follows from the fact that $\mu(\emptyset) = 0$ and $\emptyset \subset E$ for all $E \subset X$. Now we only need to show countable additivity of $\pi$ for countable disjoint subsets $E_i$.

But first we'll show that $\pi$ is countably subadditive on disjoint union of sets in $\mathcal{X}$. That is if $E_i$ is a disjoint collection of elements in $\mathcal{X}$, then

$$\pi\Big(\bigcup_{n\in\mathbb{N}} E_i\Big) \leq \sum_{n\in\mathbb{N}} \pi(E_i)$$

For this consider $A \subset \cup_{i=1}^m E_i$ and let $A_i = A \cap \cup_{i=1}^m E_i$. Then by the additivity of the charge $\mu$ ($\mu(A) = \sum_{i=1}^m \mu(A_i)$), we get

$$\pi\Big(\bigcup_{i=1}^m E_i\Big) = \sup\Big\{\mu(A) \ : \ A \subset \bigcup_{i=1}^m E_m\Big\} \le \sum_{i=1}^m \sup\Big\{\mu(A_i) \ : \ A_i \subset E_i\Big\} = \sum_{i=1}^m \pi(E_i)$$

since this is true for all $m \in \mathbb{N}$, taking limits preserves the inequality. Hence we get disjoint subadditivity.

Now we will proceed to prove disjoint additivity. The definition of $\pi(E_i)$ guarantees the existence of a $F_i \subset E_i$ such that $\mu(F_i) \le \pi(E_i) \le \mu(F_i) + 2^{-i}\varepsilon$ for every $\varepsilon > 0$. Now since $\cup_{i\in\mathbb{N}} F_i \subset \cup_{i\in\mathbb{N}} E_i$, the additivity of the charge $\mu$, and the subadditivity of $\pi$, we get

$$\sum_{i\in\mathbb{N}} \mu(F_i) = \mu\Big(\bigcup_{i\in\mathbb{N}} F_i\Big) \le \pi\Big(\bigcup_{i\in\mathbb{N}} E_i\Big) \le \sum_{i\in\mathbb{N}} \pi(E_i) \le \sum_{i\in\mathbb{N}} \mu(F_i) + 2^{-i}\varepsilon = \sum_{i\in\mathbb{N}} \mu(F_i) + \varepsilon$$

Since $\varepsilon$ was arbitrary by choice, limiting it to zero gives us disjoint additivity of $\pi$. Hence we are done. $\square$

*Remark* 3.1. Using the same reasoning in exercise **P**, one can show that

$$\pi^-(E) := \sup\{-\mu(A) \ : \ A \subset E, \ A \in \mathcal{X}\}$$

is again a measure on $\mathcal{X}$. We can also show that $\mu = \pi - \pi^-$. This is known as the Hahn decomposition of charges.

**Q (Total variation measure).** If $\mu$ is a charge on $X$, let $\nu$ be defined for $E \in \mathcal{X}$ by

$$\nu(E) = \sup \sum_{i=1}^n |\mu(A_i)|$$

where the supremum is taken over all finite disjoint collection $A_i$ with $\cup_{i=1}^n A_i = E$. Show that $\nu$ is a measure on $\mathcal{X}$.

*Solution.* It is clear that $\nu(\emptyset) = 0$ and $\nu(E) \ge 0$ for all $E \in \mathcal{X}$. We just need to verify countable disjoint additivity of $\nu$. Like we did in the last exercise, we will prove countable disjoint subadditivity first.

Let $E_j$ be a countable collection of disjoint elements in $\mathcal{X}$. Then for any finite collection $A_i$ with $\cup_{i=1}^n A_i = \cup_{j\in\mathbb{N}} E_j$ let $A_{ij} = A_i \cap E_j$. Then

$$\sum_{i=1}^n |\mu(A_i)| \le \sum_{i=1}^n \sum_{j\in\mathbb{N}} |\mu(A_{ij})| = \sum_{j\in\mathbb{N}} \sum_{i=1}^n |\mu(A_{ij})|$$

where the rearrangement in the summation is justified since all the terms are non-negative. Hence the inequality is preserved when taken supremum over all such partitions $A_i$. This gives the disjoint countable subadditivity of $\nu$.

$$\nu\Big(\bigcup_{j\in\mathbb{N}} E_j\Big) = \sup\sum_{i=i}^{n} |\mu(A_i)| \leq \sum_{j\in\mathbb{N}} \sup\sum_{i=1}^{n} |\mu(A_{ij})| = \sum_{j\in\mathbb{N}} \nu(E_j)$$

Now we will proceed to prove countable disjoint additivity. Consider the disjoint collection $E_j \in \mathcal{X}$. By the definition of $\nu$, for all $E_j$ and $\varepsilon > 0$ we can find a finite collection $A_{ij}$ (the number of them might vary, but finite nevertheless) with $\cup_{i=1}^{n} A_{ij} = E_j$ such that

$$\sum_{i=1}^{n} |\mu(A_{ij})| \ \leq \ \nu(E_j) \ \leq \ 2^{-j}\varepsilon + \sum_{i=1}^{n} |\mu(A_{ij})|$$

Since $\cup_{j\in\mathbb{N}} \cup_{i=1}^{n} A_{ij} = \cup_{j\in\mathbb{N}} E_j$, by the subadditivity of $\nu$ we get,

$$\sum_{i=1}^{n}\sum_{j\in\mathbb{N}} |\mu(A_{ij})| \leq \nu\Big(\bigcup_{j\in\mathbb{N}} E_j\Big) \leq \sum_{j\in\mathbb{N}} \nu(E_j) \leq \sum_{j\in\mathbb{N}} 2^{-j}\varepsilon + \sum_{i=1}^{n}\sum_{j\in\mathbb{N}} |\mu(A_{ij})| \leq \varepsilon + \sum_{i=1}^{n}\sum_{j\in\mathbb{N}} |\mu(A_{ij})|$$

Since $\varepsilon$ was chosen arbitrarily small, it can be limited to 0 giving

$$\nu\Big(\bigcup_{j\in\mathbb{N}} E_j\Big) = \sum_{j\in\mathbb{N}} \nu(E_j)$$

Hence $\nu$ is a measure on $\mathcal{X}$. □

*Remark* 3.2. One can show that the total variation measure $\nu$ defined in the above exercise satisfy $\nu = \pi + \pi^-$ where $\pi, \pi^-$ are as defined in exercise **P** and Remark 3.1

# Chapter 4

# The Integral

**A.** If the simple function $\phi$ in $M^+(X, \mathcal{X})$ is of the form (not necessarily standard representation)

$$\phi = \sum_{k=1}^{m} b_k \chi_{F_k}$$

, then show that

$$\int \phi \, d\mu = \sum_{k=1}^{m} b_k \mu(F_k)$$

*Solution.* If $\phi(x) = \sum_{k=1}^{m} b_k \chi_{F_k}$ is the standard form, the rest will follow from the definition of the integral itself. So without loss of generality, assume $F_1 \cap F_2 \neq \emptyset$ and $\psi = b_1\chi_{F_1} + b_2\chi_{F_2}$. Now disjointifying $F_1$ and $F_2$, if $E_1 = F_1 \setminus F_2, E_2 = F_1 \cap F_2, E_3 = F_2 \setminus F_1$, then the standard form of $\psi$ is $\psi = b_1\chi_{E_1} + (b_1 + b_2)\chi_{E_2} + b_2\chi_{E_3}$. Then by the definition of the integral for simple functions in their standard form, we get

$$\begin{aligned}
\int \psi \, d\mu &= b_1\mu(E_1) + (b_1 + b_2)\mu(E_2) + b_3\mu(E_3) \\
&= b_1(\mu(E_1) + \mu(E_2)) + b_2(\mu(E_2) + \mu(E_3)) \\
&= b_1\mu(F_1) + b_2\mu(F_2)
\end{aligned}$$

Now to prove the general thing, we'll use induction. Assume if $\phi_{m-1} = \sum_{k=1}^{m-1} b_k \chi_{F_k}$ has its integral

$$\int \phi_{m-1} \, d\mu = \sum_{k=1}^{m-1} b_k \mu(F_k)$$

Now consider $\phi_m = \sum_{k=1}^{m} b_k \chi_{F_k}$.

OR

use the additivity of the integral of simple functions (Lemma 4.3a) $\square$

**B.** The sum, scalar multiple, and product of simple functions are simple functions.
*Solution.* If $\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$ and $\psi = \sum_{j=1}^{m} b_j \chi_{B_j}$ are simple functions in their standard form, then

$$\phi + \psi = \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} \chi_{C_{ij}} \quad \phi\psi = \sum_{i=1}^{n} \sum_{j=1}^{m} d_{ij} \chi_{D_{ij}}$$

where $c_{ij} = a_i + b_j, d_{ij} = a_i b_j$ and $C_{ij} = A_i \cap B_j = D_{ij}$. Which shows $\phi + \psi$ and $\phi\psi$ are simple functions. Verifying it for scalar multiple is has a similar proof. $\square$

**C.** If $\phi_1, \phi_2$ are simple functions in $M(X, \mathcal{X})$, then

$$\varphi = \sup\{\phi_1, \phi_2\}, \quad \omega = \inf\{\phi_1, \phi_2\}$$

are also simple functions in $M(X, \mathcal{X})$
*Solution.* It is clear from the second chapter that functions $\varphi, \omega$ are measurable being the supremum and infimum of two measurable functions. What is remain to verify is that they are simple, that is they only take a finitely many values in the range. But this follows easily since the range of both $\varphi, \omega$ must be a subset of the union of ranges of $\phi_1$ and $\phi_2$ which are both finite. $\square$

**D.** If $f \in M^+$ and $c > 0$, then the mapping $\phi \to \psi = c\phi$ is a one-to-one mapping between simple functions $\phi \leq f$ and $\psi \leq cf$. Use this fact to show that

$$\int cf \; d\mu = c \int f \; d\mu$$

*Solution.* Assuming that it is a one-to-one map, if $\psi$ is any simple function with $\psi \leq cf$, then $\phi = \frac{1}{c}\psi \leq f$. This works over taking supremum of such simple function and gives the equality of the said integrals.

Now the question remaining is whether it is such a one-to-one map. Yes. $\square$

**E.** Let $f, g \in M^+$ and $\omega \in M^+$ be a simple function such that $\omega \leq f + g$ and let $\phi_n(x) = \sup\{\frac{m}{n}\omega(x) : \text{ for } 0 \leq m \leq n \text{ with } \frac{m}{n}\omega(x) \leq f(x)\}$. Also let $\psi_n(x) = \sup\{(1 - \frac{1}{n})\omega(x) - \phi_n(x), 0\}$. Show that $(1 - \frac{1}{n})\omega \leq \phi_n + \psi_n$ and $\phi_n \leq f, \psi_n \leq g$
*Solution.* From the definition of $\phi_n$, it is clear that $\phi_n \leq f$ (supremum of a finite set is an element of the set). Similarly showing $(1 - \frac{1}{n})\omega \leq \phi_n + \psi_n$ is equivalent to showing $(1 - \frac{1}{n})\omega - \phi_n \leq \psi_n$ which follows directly from the definition of $\psi_n$.

To show $\psi_n \leq g$, assume that for some $x, \phi_n(x) = \frac{k}{n}\omega(x)$. Then by the definition of $\phi_n$, we get that $f(x) \leq \frac{k+1}{n}\omega(x)$. Now $(1 - \frac{1}{n})\omega(x) - \phi_n(x) = \omega(x) - \frac{k+1}{n}\omega(x) \leq f + g - f = g$. Since $(1 - \frac{1}{n})\omega(x) - \phi_n(x) \leq g(x)$ and $g(x) \geq 0$, we get $\psi_n \leq g$. $\square$

**F.** Employ chapter 4 to establish Corollary 4.7(b) (Additivity of integral in positive functions) without using Monotone convergence theorem.
*Solution.* Since we could show

$$\int (f+g)\ d\mu \geq \int f\ d\mu + \int g\ d\mu$$

without using MCT, we will assume this. The idea was whenever $\phi, \psi$ are non-negative simple functions with $\phi \leq f$ and $\psi \leq g$, then $\phi + \psi$ is a non-negative simple function with $\phi + \psi \leq f + g$. Hence we get the equality.

To get the reverse inequality we use chapter 4. That is if $\omega$ is a non-negative simple function with $\omega \leq f+g$, then we get corresponding $\phi_n, \psi_n$ with the required properties. Hence in a similar way we get the reverse inequality. (This feels a more organic way to go abt proving it) □

**J(b).** Let $X = \mathbb{R}, \mathcal{X}$ be the Borel measurable sets and $\lambda$ be the Lebesgue measure on $\mathcal{B}$. If $g_n = n\chi_{[1/n,2/n]}$, then the sequence converges to $g = \mathbf{0}$. Is the convergence uniform? Does MCT apply? Does Fatou's lemma apply?
*Solution.* The convergence is not uniform. This is because $\|g_n - g\| = n$ in sup norm. MCT does not apply since $g_n$ are not monotone. But Fatou's lemma is satisfied. □

**K.** If $(X, \mathcal{X}, \mu)$ is a finite measure space, and if $(f_n)$ is a real-valued sequence in $M^+(X, \mathcal{X})$ which converges uniformly to a function $f$, then $f \in M^+(X, \mathcal{X})$ and

$$\int f\ d\mu = \lim \int f_n\ d\mu$$

*Solution.* The fact that $f \in M(X, \mathcal{X})$ follows from the fact that limits of measurable functions are measurable and uniform convergence imply pointwise convergence and a sequence of non-negative real numbers if converges must converge to a non-negative real number.

Now using Fatou's lemma, we get

$$\int f\ d\mu \leq \limsup \int f_n\ d\mu$$

Since $f_n$ converges to $f$ uniformly, for a given $\epsilon \geq 0$ there exists $N_\epsilon \in \mathbb{N}$ such that for all $n \geq N_\epsilon$, we get $|f(x) - f_n(x)| \leq \epsilon$ for all $x \in X$. Therefore $f_n(x) \leq f(x) + \epsilon$ and taking integral gives

$$\int f_n\ d\mu \leq \int f\ d\mu + \epsilon\mu(X)$$

Since this is true for all $\epsilon \geq 0$ and $n \geq N_\epsilon$, we must have

$$\limsup \int f_n \, d\mu \leq \int f \, d\mu$$

We us the fact that $\liminf \int f \, d\mu \leq \limsup \int f \, d\mu$ and combine it with the first inequality to get

$$\int f \, d\mu \leq \liminf \int f_n \, d\mu \leq \limsup \int f \, d\mu \leq \int f \, d\mu$$

which gives our required result. $\square$

**L*.** Let $X$ be a finite closed interval $[a, b] \subset \mathbb{R}$, $\mathcal{X}$ be the collection of Borel sets in $X$ and $\lambda$ be the Lebesgue measure on $X$. If $f$ is a non-negative continuous function on $X$, show that

$$\int f \, d\lambda = \int_a^b f(x) \, dx$$

where the right hand side integral is the Riemann integral of $f$.
*Solution.* We'll begin small by proving this for step functions $f$. Assume $f = \sum_{i=1}^n \alpha_i \chi_{E_i}$ with $a_i \geq 0$ where $E_i = [a_{i-1}, a_i]$ are disjoint partitions of $X$. That is $a = a_0 \leq a_1 \leq \ldots \leq a_n = b$. Then

$$\int f \, d\lambda = \sum_{i=1}^n \alpha_i (a_{i-1} - a_i) = \int_a^b f(x) \, dx$$

Now for a general non-negative continuous function $f$, since it is Riemann integrable the Riemann integral of $f$ is the supremum of the lower Riemann sum. The lower Riemann sum of any partition $a = a_0 \leq \ldots a_n = b$ is the integral of the simple function $\phi = \sum_{i=1}^n m_i \chi_{E_i}$ where $E_i = [a_{i-1}, a_i]$ and $m_i = \inf_{x \in E_i} f(x)$. Hence every partition of $[a, b]$ gives a simple function $\phi \leq f$ where the lower Riemann sum of the partition and measure theoretic integral of $\phi$ are equal. This gives

$$\int_a^b f(x) \, dx \leq \int f \, d\lambda$$

Conversely if $\phi \leq f$ is a simple function of the standard form $\phi = \sum_{i=1}^n a_i \chi_{A_i}$, then (continuity of $f$ should restrict $A_i$s to be the union of intervals. Justify how?)
$\square$