Solutions to Papa Rudin (3rd ed)

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December 21, 2024

Contents

Chapter 4

Elementary Hilbert Space Theory

Exercise 4.3. *Show that $L^p(\mathbb{T})$ is separable, but $L^\infty(\mathbb{T})$ is not.*

Solution 4.3. From Theorem 4.25, we see that for all $f \in C(\mathbb{T})$ and $\varepsilon > 0$, there is a trigonometric polynomial P such that

$$|f(t) - P(t)| < \varepsilon$$

for all $t \in \mathbb{R}$. Hence we see that $C(\mathbb{T}) \subset \overline{\operatorname{span}}(e^{int})$. Moreover by Theorem 3.14, we know that $C(\mathbb{T})$ is a dense subspace of $L^p(T)$ for all $1 \leq p < \infty$. Thus we see that $L^p(\mathbb{T})$ is separable for all $1 \leq p < \infty$.

That $L^\infty(\mathbb{T})$ is not separable follows from a similar proof as why ℓ^∞ is not separable. Using trigonometric identities, we can show that

$$|e^{int} - e^{imt}| = 2 - 2\cos((n-m)t)$$

Hence $\|e^{int} - e^{imt}\|_\infty = 2$. Now repeat the proof of the non-separability of ℓ^∞.

Exercise 4.6. *If $D = \{u_\alpha\}$ is an orthonormal set for a Hilbert space $\mathcal{H}$, then show that D is a closed bounded set which is not compact. Also show that the set Q of all $x \in \mathcal{H}$ such that*

$$x = \sum_{n\in\mathbb{N}} c_n u_n, \quad |c_n| \leq \frac{1}{n}$$

is compact. More generally, let $\{\delta_n\}$ be a sequence of positive numbers, and S be the set of all $x \in \mathcal{H}$ of the form

$$x = \sum_{n\in\mathbb{N}} c_n u_n, \quad |c_n| \leq \delta_n$$

Prove that S is compact if and only if $\sum_{n\in\mathbb{N}} \delta_n^2 < \infty$.

Solution 4.6. That D is not compact follows form the fact that $\|u_i - u_j\| = 2$ for all $i \neq j$.exercise

Now we'll prove that S is compact if and only if $\sum_{n\in\mathbb{N}} \delta_n^2 < \infty$. Then compactness of Q will easily follow.

Let $\sum_{n\in\mathbb{N}} \delta_n^2 < \infty$ and $\{U_\alpha \ : \ \alpha \in A\}$ open cover for S. That is

$$S \subset \bigcup_\alpha U_\alpha$$

Since $0 \in S$, $0 \in U_{\alpha_0}$ for some $\alpha_0 \in A$. Then there exists an $\varepsilon > 0$ such that $B_\varepsilon(0) \subset U_{\alpha_0}$. Since $\sum_{n\in\mathbb{N}} \delta_n^2 < \infty$, there is an N_ε such that

$$\sum_{n=N_\varepsilon+1}^{\infty} \delta_n^2 < \varepsilon$$

By the triangle inequality and the orthonormality of u_n,

$$\Big\| \sum_{n=N_\varepsilon+1}^{\infty} \delta_n u_n \Big\|^2 = \sum_{n=N_\varepsilon+1}^{\infty} \delta_n^2 < \varepsilon$$

Hence

$$S \cap \overline{\operatorname{span}}\{u_n \ : \ n > N_\varepsilon\} \subset U_{\alpha_0}$$

This seems hard, If you can't still find it refer ProofWiki.

Solution 4.7. Choose

$$c_k = \frac{1}{k\big(\sum_{n\in E_k} a_n^2\big)}$$

Solution 4.8. Notice that any Hilbert space $\mathcal{H}$ is isometrically isomorphic to $\ell^2(I)$, where I is an orthonormal basis to $\mathcal{H}$. The rest follows.

Solution 4.9. First assume $A = [a, b] \subset [0, 2\pi]$ is an interval. Then show that it holds. Now use the fact that every open set in $\mathbb{R}$ is a countable union of disjoint intervals. Then that is true for all open sets using an $\varepsilon/2^i$ argument. Now for any closed set A, use the fact that $A = [0, 2\pi] \setminus U$ for some open set U. And therefore

$$\int_A f_n \, d\mu = \int_{[0,2\pi]} f_n \, d\mu - \int_U f_n \, d\mu$$

both of the which converge to 0. Now the fact that for any borel set A and an $\varepsilon > 0$, there is an open set U and a closed set V such that $V \subset A \subset U$ and $\mu(U \setminus V) < \varepsilon$. Then

$$\int_U f_n \, d\mu - \int_A f_n \, d\mu = \int f_n \chi_{U\setminus A} \, d\mu < \mu(U \setminus A) < \varepsilon$$

Since $\int_U f_n \, d\mu \to 0$, we get that $\int_A f_n \, d\mu \to 0$ as well.

Solution 4.10. Since $1 - 2\sin^2(\alpha) = \cos(2\alpha)$ for all α, we get that

$$\int_E 1 - 2\sin^2(n_k x)\ d\mu = \int_E \cos(2n_k x)\ d\mu$$

which converges to 0 by the previous exercise. verify

Solution 4.11.

$$\left\{ \left(1 + \frac{1}{n}\right) e^{int}\ :\ n \in \mathbb{N} \right\}$$

Solution 4.12. *skipping for now*

Solution 4.13. *skipping for now, but verify.*

Solution 4.14. Use the projection onto the space spanned by $1, x, x^2$. First find an orthonormal basis to the space and then evaluate the projection.

By a similar reasoning, we can find that $g(x) = \sqrt{\frac{7}{2}} x^3$

Solution 4.15.

$$\langle f, g \rangle = \int_0^\infty f(x) g(x) e^{-x}\ dx$$

is an inner product in the real square integrable functions in $[0, \infty)$. Now do the same Gram-Schmidt procedure to $1, x, x^2$ as above question.