MATH7039 – Assignment 3

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1.

See attached excel spreadsheet “Q1.xlsx” for more information.

a)

Q1.														
r	**0.12**													
T	**0.25**													
Δt	**0.125**													
u	**1.1**													
d	**0.9**													
S0	**20**													
K	**21**													
				Suu	**24.2**									
		Su	**22**											
S0	**20**			**Sud**	**19.8**									
		Sd	**18**											
				Sdd	**16.2**									
pu	**(S0e^(rT) - Sd)/(Su - Sd)**			**0.575565**										
pd	**1 - pu**			**0.424435**										
a)														
Solution:							Formulas:							
				Cuu	3.2								Cuu	=MAX(F10-$B$8, 0)
		Cu	1.814388							Cu	=EXP(-$B$2*$B$4)*($E$16*F21+$E$17*F23)			
C0	1.028751			Cud	0		C0	=EXP(-$B$2*$B$4)*($E$16*D22+$E$17*D24)					Cud	=MAX(F12-$B$8, 0)
		Cd	0							Cd	=EXP(-$B$2*$B$4)*($E$16*F23+$E$17*F25)			
				Cdd	0								Cdd	=MAX(F14-$B$8, 0)

Figure 1: Screenshot of calculations for C_0 using given information

b)

Using information from figure 1 (r, T, etc.), the calculated put option price is displayed in figure 2 below:

b)														
Solution							Formulas:							
				Puu	0								Puu	=MAX($B$8-F10, 0)
		Pu	0.501739							Pu	=EXP(-$B$2*$B$4)*($E$16*F29+$E$17*F31)			
P0	1.408107			Pud	1.2		P0	=EXP(-$B$2*$B$4)*($E$16*D30+$E$17*D32)					Pud	=MAX($B$8-F12, 0)
		Pd	2.687351							Pd	=EXP(-$B$2*$B$4)*($E$16*F31+$E$17*F33)			
				Pdd	4.8								Pdd	=MAX($B$8-F14, 0)

Figure 2: Screenshot of calculations for P_0 using given information

c)

c)				
Show	C0 + Kexp(-rT)		=	P0 + S0
LHS				
C0 + Kexp(-rT)		21.40811		
RHS				
P0 + S0		21.40811		
So LHS = RHS				

Figure 3: Screenshot verifying put-call parity holds

Noting that $u = 1.1$ and $d = 0.9$ are constant over time leading to p_u and $p_d = 1 - p_u$ being constant over time and $\Delta t = \frac{T}{2} = 0.125$ since this is a 2-step binomial pricing model, the recursive relationship used to price the call option is:

$$C_t = \begin{cases} e^{-r\Delta t}\left(p_u C_{t+\Delta t}^u + p_d C_{t+\Delta t}^d\right) \; if \; t < T \\ (S_T - K)^+ \; if \; t = T \end{cases}$$

Similarly, the recursive relationship used to price the put option is:

$$P_t = \begin{cases} e^{-r\Delta t}\left(p_u P_{t+\Delta t}^u + p_d P_{t+\Delta t}^d\right) \; if \; t < T \\ (K - S_T)^+ \; if \; t = T \end{cases}$$

2.

a)

Black-Scholes PDE and terminal condition:

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rc$$

$$C(S,T) = (S - K)^+$$

Let $C(S,t)$ be a solution to the Black-Scholes PDE. Relevant substitutions:

$$\tau = T - t$$

$$\rightarrow t = T - \tau$$

$$z = \log(S) + \left(r - \frac{\sigma^2}{2}\right)\tau$$

$$\rightarrow S = e^{z - \left(r - \frac{\sigma^2}{2}\right)\tau}$$

$$U(z,\tau) = e^{r\tau}C\big(S(z,\tau), t(z,\tau)\big)$$

$$U(z,\tau) = e^{r\tau}C\left(e^{z-\left(r-\frac{\sigma^2}{2}\right)\tau}, T - \tau\right)$$

Heat/diffusion PDE and initial condition:

$$\frac{\partial U}{\partial \tau} - \frac{1}{2}\sigma^2 \frac{\partial^2 U}{\partial z^2} = 0$$

$$U(z,0) = f(z)$$

$$\frac{\partial U}{\partial \tau} = \frac{\partial}{\partial \tau}\left(e^{r\tau}C\left(e^{z-\left(r-\frac{\sigma^2}{2}\right)\tau}, T - \tau\right)\right)$$

Using the product and multivariate chain rule:

$$\frac{\partial U}{\partial \tau} = re^{r\tau}C + e^{r\tau}\left(\frac{\partial C}{\partial S}\frac{\partial S}{\partial \tau} + \frac{\partial C}{\partial t}\frac{\partial t}{\partial \tau}\right)$$

$$\frac{\partial S}{\partial \tau} = -\left(r - \frac{\sigma^2}{2}\right)e^{z-\left(r-\frac{\sigma^2}{2}\right)\tau}$$

$$\frac{\partial t}{\partial \tau} = -1$$

$$\therefore \frac{\partial U}{\partial \tau} = re^{r\tau}C + e^{r\tau}\left(-\left(r - \frac{\sigma^2}{2}\right)e^{z-\left(r-\frac{\sigma^2}{2}\right)\tau}\frac{\partial C}{\partial S} - \frac{\partial C}{\partial t}\right)$$

$$\frac{\partial U}{\partial \tau} = re^{r\tau}C + e^{r\tau}\left(\left(\frac{\sigma^2}{2} - r\right)e^{z-\left(r-\frac{\sigma^2}{2}\right)\tau}\frac{\partial C}{\partial S} - \frac{\partial C}{\partial t}\right)$$

$$\frac{\partial U}{\partial \tau} = e^{r\tau}\left[rC + \left(\frac{\sigma^2}{2} - r\right)e^{z-\left(r-\frac{\sigma^2}{2}\right)\tau}\frac{\partial C}{\partial S} - \frac{\partial C}{\partial t}\right]$$

$$\frac{\partial U}{\partial z} = \frac{\partial}{\partial z}\left(e^{r\tau}C\left(e^{z-\left(r-\frac{\sigma^2}{2}\right)\tau}, T - \tau\right)\right)$$

Using the multivariate chain rule:

$$\frac{\partial U}{\partial z} = e^{r\tau}\left(\frac{\partial C}{\partial S}\frac{\partial S}{\partial z} + \frac{\partial C}{\partial t}\frac{\partial t}{\partial z}\right)$$

$$\frac{\partial S}{\partial z} = e^{z-\left(r-\frac{\sigma^2}{2}\right)\tau}$$

$$\frac{\partial t}{\partial z} = 0$$

$$\therefore \frac{\partial U}{\partial z} = e^{r\tau}\left(e^{z-\left(r-\frac{\sigma^2}{2}\right)\tau}\frac{\partial C}{\partial S} + 0\right)$$

$$\frac{\partial U}{\partial z} = e^{r\tau}\left(e^{z-\left(r-\frac{\sigma^2}{2}\right)\tau}\frac{\partial C}{\partial S}\right)$$

$$\frac{\partial^2 U}{\partial z^2} = \frac{\partial}{\partial z}\left(e^{r\tau}\left(e^{z-\left(r-\frac{\sigma^2}{2}\right)\tau}\frac{\partial C}{\partial S}\right)\right)$$

$$\frac{\partial^2 U}{\partial z^2} = e^{r\tau}\left[\frac{\partial}{\partial z}\left(e^{z-\left(r-\frac{\sigma^2}{2}\right)\tau}\frac{\partial C}{\partial S}\right)\right]$$

Using the product and multivariate chain rule:

$$\frac{\partial^2 U}{\partial z^2} = e^{r\tau}\left[e^{z-\left(r-\frac{\sigma^2}{2}\right)\tau}\frac{\partial C}{\partial S} + e^{z-\left(r-\frac{\sigma^2}{2}\right)\tau}\left(\frac{\partial^2 C}{\partial S^2}\frac{\partial S}{\partial z} + \frac{\partial C}{\partial t \partial S}\frac{\partial t}{\partial z}\right)\right]$$

$$\frac{\partial^2 U}{\partial z^2} = e^{r\tau}\left[e^{z-\left(r-\frac{\sigma^2}{2}\right)\tau}\frac{\partial C}{\partial S} + e^{z-\left(r-\frac{\sigma^2}{2}\right)\tau}\left(e^{z-\left(r-\frac{\sigma^2}{2}\right)\tau}\frac{\partial^2 C}{\partial S^2} + 0\right)\right]$$

$$\frac{\partial^2 U}{\partial z^2} = e^{r\tau}\left[e^{z-\left(r-\frac{\sigma^2}{2}\right)\tau}\frac{\partial C}{\partial S} + e^{2\left(z-\left(r-\frac{\sigma^2}{2}\right)\tau\right)}\frac{\partial^2 C}{\partial S^2}\right]$$

Substituting these expressions back into the heat/diffusion PDE:

$$\frac{\partial U}{\partial \tau} - \frac{1}{2}\sigma^2\frac{\partial^2 U}{\partial z^2} = 0$$

$$e^{r\tau}\left[rC + \left(\frac{\sigma^2}{2} - r\right)e^{z-\left(r-\frac{\sigma^2}{2}\right)\tau}\frac{\partial C}{\partial S} - \frac{\partial C}{\partial t}\right] - \frac{1}{2}\sigma^2 e^{r\tau}\left[e^{z-\left(r-\frac{\sigma^2}{2}\right)\tau}\frac{\partial C}{\partial S} + e^{2\left(z-\left(r-\frac{\sigma^2}{2}\right)\tau\right)}\frac{\partial^2 C}{\partial S^2}\right] = 0$$

$$rC + \left(\frac{\sigma^2}{2} - r\right)e^{z-\left(r-\frac{\sigma^2}{2}\right)\tau}\frac{\partial C}{\partial S} - \frac{\partial C}{\partial t} - \frac{1}{2}\sigma^2 e^{z-\left(r-\frac{\sigma^2}{2}\right)\tau}\frac{\partial C}{\partial S} - \frac{1}{2}\sigma^2 e^{2\left(z-\left(r-\frac{\sigma^2}{2}\right)\tau\right)}\frac{\partial^2 C}{\partial S^2} = 0$$

Substituting $S = e^{z-\left(r-\frac{\sigma^2}{2}\right)\tau}$:

$$rC + \left(\frac{\sigma^2}{2} - r\right)S\frac{\partial C}{\partial S} - \frac{\partial C}{\partial t} - \frac{1}{2}\sigma^2 S\frac{\partial C}{\partial S} - \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} = 0$$

$$rc + \frac{1}{2}\sigma^2 S\frac{\partial C}{\partial S} - rS\frac{\partial C}{\partial S} - \frac{\partial C}{\partial t} - \frac{1}{2}\sigma^2 S\frac{\partial C}{\partial S} - \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} = 0$$

$$\therefore \frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} = rc$$

This is the Black-Scholes PDE from earlier and since we let $C(S,t)$ be a solution satisfying this PDE, we now know that $U(z,\tau) = e^{r\tau}C\left(e^{z-\left(r-\frac{\sigma^2}{2}\right)\tau}, T-\tau\right)$ satisfies the heat/diffusion PDE.

b)

The aim in this question is to replace the fundamental solution to the heat/diffusion PDE with one that contains instances of the standard normal distribution CDF $N(x) = \int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx$. The fundamental solution to the heat/diffusion PDE is given by:

$$U(z,\tau) = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2\tau}}e^{-\frac{1}{2}\frac{(z-\zeta)^2}{\sigma^2\tau}}f(\zeta)d\zeta$$

$$(z-\zeta)^2 = (\zeta - z)^2$$

$$\therefore U(z,\tau) = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2\tau}}e^{-\frac{1}{2}\frac{(\zeta-z)^2}{\sigma^2\tau}}f(\zeta)d\zeta$$

Noting that if $\tau = 0, t = T - \tau = T$ and $S = e^{z-\left(r-\frac{\sigma^2}{2}\right)\tau} = e^z$, using the initial condition:

$$U(z,0) = e^{r\cdot 0}C\left(e^{z-\left(r-\frac{\sigma^2}{2}\right)\cdot 0}, T-0\right)$$

$$U(z,0) = C(e^z, T)$$

$$U(z,0) = (e^z - K)^+$$

This is positive iff:

$$e^z - K > 0 \to z > \ln(K)$$

Using this to replace the lower limit of integration ($-\infty$):

$$U(z,\tau) = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2\tau}}e^{-\frac{1}{2}\frac{(\zeta-z)^2}{\sigma^2\tau}}f(\zeta)d\zeta$$

$$f(\zeta) = U(\zeta,0) = \left(e^\zeta - K\right)^+$$

$$\therefore U(z,\tau) = \int_{\ln(K)}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2\tau}}e^{-\frac{1}{2}\frac{(\zeta-z)^2}{\sigma^2\tau}}\left(e^\zeta - K\right)d\zeta$$

$$U(z,\tau) = \int_{\ln(K)}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2\tau}}\left(e^{\zeta-\frac{1}{2}\frac{(\zeta-z)^2}{\sigma^2\tau}} - Ke^{-\frac{1}{2}\frac{(\zeta-z)^2}{\sigma^2\tau}}\right)d\zeta$$

$$U(z,\tau) = \int_{\ln(K)}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2\tau}}e^{\zeta-\frac{1}{2}\frac{(\zeta-z)^2}{\sigma^2\tau}}d\zeta - K\int_{\ln(K)}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2\tau}}e^{-\frac{1}{2}\frac{(\zeta-z)^2}{\sigma^2\tau}}d\zeta$$

Let:

$$U_1 = \int_{\ln(K)}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2\tau}} e^{\zeta - \frac{1}{2}\frac{(\zeta - z)^2}{\sigma^2\tau}} d\zeta$$

$$U_2 = -K \int_{\ln(K)}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2\tau}} e^{-\frac{1}{2}\frac{(\zeta - z)^2}{\sigma^2\tau}} d\zeta$$

The aim is now to convert these integrals into the standard normal CDF. Considering U_1 first:

$$\zeta - \frac{1}{2}\frac{(\zeta - z)^2}{\sigma^2\tau} = \frac{2\sigma^2\tau\zeta - (\zeta - z)^2}{2\sigma^2\tau}$$

$$\zeta - \frac{1}{2}\frac{(\zeta - z)^2}{\sigma^2\tau} = \frac{2\sigma^2\tau\zeta - \zeta^2 + 2z\tau - z^2}{2\sigma^2\tau}$$

$$\zeta - \frac{1}{2}\frac{(\zeta - z)^2}{\sigma^2\tau} = -\frac{\zeta^2 - 2(\sigma^2\tau + z)\zeta + z^2}{2\sigma^2\tau}$$

$$\zeta - \frac{1}{2}\frac{(\zeta - z)^2}{\sigma^2\tau} = -\frac{\zeta^2 - 2(\sigma^2\tau + z)\zeta + (\sigma^2\tau + z)^2 + z^2 - (\sigma^2\tau + z)^2}{2\sigma^2\tau}$$

$$\zeta - \frac{1}{2}\frac{(\zeta - z)^2}{\sigma^2\tau} = -\frac{(\zeta - \sigma^2\tau - z)^2 + z^2 - \sigma^4\tau^2 - 2\sigma^2\tau z - z^2}{2\sigma^2\tau}$$

$$\zeta - \frac{1}{2}\frac{(\zeta - z)^2}{\sigma^2\tau} = -\frac{(\zeta - \sigma^2\tau - z)^2 - \sigma^4\tau^2 - 2\sigma^2\tau z}{2\sigma^2\tau}$$

$$\zeta - \frac{1}{2}\frac{(\zeta - z)^2}{\sigma^2\tau} = \frac{2\sigma^2\tau z + \sigma^4\tau^2 - (\zeta - \sigma^2\tau - z)^2}{2\sigma^2\tau}$$

$$\zeta - \frac{1}{2}\frac{(\zeta - z)^2}{\sigma^2\tau} = z + \frac{\sigma^2\tau}{2} - \frac{1}{2}\frac{(\zeta - \sigma^2\tau - z)^2}{\sigma^2\tau}$$

$$\therefore U_1 = \int_{\ln(K)}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2\tau}} e^{z + \frac{\sigma^2\tau}{2} - \frac{1}{2}\frac{(\zeta - \sigma^2\tau - z)^2}{\sigma^2\tau}} d\zeta$$

$$U_1 = e^{z + \frac{\sigma^2\tau}{2}} \int_{\ln(K)}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2\tau}} e^{-\frac{1}{2}\frac{(\zeta - \sigma^2\tau - z)^2}{\sigma^2\tau}} d\zeta$$

Let:

$$x = \frac{\zeta - \sigma^2\tau - z}{\sigma\sqrt{\tau}}$$

$$\frac{\partial x}{\partial \zeta} = \frac{1}{\sigma\sqrt{\tau}} \rightarrow \partial\zeta = \sigma\sqrt{\tau}\partial x$$

$$\zeta = \ln(K) \rightarrow x = \frac{\ln(K) - \sigma^2\tau - z}{\sigma\sqrt{\tau}}$$

$$\zeta \rightarrow +\infty \rightarrow x \rightarrow +\infty$$

$$\therefore U_1 = e^{z + \frac{\sigma^2\tau}{2}} \int_{\frac{\ln(K) - \sigma^2\tau - z}{\sigma\sqrt{\tau}}}^{\infty} \frac{1}{\sqrt{2\pi}\sigma\sqrt{\tau}} e^{-\frac{1}{2}x^2} \sigma\sqrt{\tau}\, dx$$

$$U_1 = e^{z + \frac{\sigma^2\tau}{2}} \int_{\frac{\ln(K) - \sigma^2\tau - z}{\sigma\sqrt{\tau}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

$$U_1 = e^{z + \frac{\sigma^2\tau}{2}} \int_{\frac{\ln(K) - \sigma^2\tau - z}{\sigma\sqrt{\tau}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

$$U_1 = e^{z+\frac{\sigma^2\tau}{2}} N\left(\frac{z+\sigma^2\tau-\ln(K)}{\sigma\sqrt{\tau}}\right)$$

using $N(x) = N(-x)$

Now considering U_2:

Let:

$$x = \frac{\zeta - z}{\sigma\sqrt{\tau}}$$

$$\frac{\partial x}{\partial \zeta} = \frac{1}{\sigma\sqrt{\tau}} \rightarrow \partial\zeta = \sigma\sqrt{\tau}\partial x$$

$$\zeta = \ln(K) \rightarrow x = \frac{\ln(K) - z}{\sigma\sqrt{\tau}}$$

$$\zeta \rightarrow +\infty \rightarrow x \rightarrow +\infty$$

$$\therefore U_2 = -K\int_{\frac{\ln(K)-z}{\sigma\sqrt{\tau}}}^{\infty} \frac{1}{\sqrt{2\pi}\sigma\sqrt{\tau}} e^{-\frac{1}{2}x^2}\sigma\sqrt{\tau}\,dx$$

$$U_2 = -K\int_{\frac{\ln(K)-z}{\sigma\sqrt{\tau}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\,dx$$

$$U_2 = -KN\left(\frac{z-\ln(K)}{\sigma\sqrt{\tau}}\right)$$

using $N(x) = N(-x)$

$$\therefore U = U_1 + U_2$$

$$U = e^{z+\frac{\sigma^2\tau}{2}} N\left(\frac{z+\sigma^2\tau-\ln(K)}{\sigma\sqrt{\tau}}\right) - KN\left(\frac{z-\ln(K)}{\sigma\sqrt{\tau}}\right)$$

which is expressed in terms of $N(.)$.

c)

To transform the solution $U(z,\tau)$ back to find $C(S,t)$, we make use of the substitutions used for $t(z,\tau)$ and $S(z,\tau)$ used in a):

$$U(z,\tau) = e^{r\tau}C\left(S = e^{z-\left(r-\frac{\sigma^2}{2}\right)\tau}, t = T-\tau\right)$$

$$\therefore C(S,t) = e^{-r\tau}U\left(z = \ln(S) + \left(r - \frac{\sigma^2}{2}\right)\tau, \tau = T-t\right)$$

$$C(S,t) = e^{-r(T-t)}\left[e^{\ln(S)+\left(r-\frac{\sigma^2}{2}\right)(T-t)+\frac{\sigma^2}{2}(T-t)} N\left(\frac{\ln(S)+\left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma^2(T-t)-\ln(K)}{\sigma\sqrt{T-t}}\right) - KN\left(\frac{\ln(S)+\left(r-\frac{\sigma^2}{2}\right)(T-t)-\ln(K)}{\sigma\sqrt{T-t}}\right)\right]$$

$$C(S,t) = e^{-r(T-t)}\left[Se^{r(T-t)}N\left(\frac{\ln\left(\frac{S}{K}\right) + r(T-t) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right) - KN\left(\frac{\ln\left(\frac{S}{K}\right) + r(T-t) - \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right)\right]$$

$$C(S,t) = e^{-r(T-t)}\left[Se^{r(T-t)}N\left(\frac{\ln\left(\frac{S}{K}\right) + r(T-t)}{\sigma\sqrt{T-t}} + \frac{\sigma\sqrt{T-t}}{2}\right) - KN\left(\frac{\ln\left(\frac{S}{K}\right) + r(T-t)}{\sigma\sqrt{T-t}} - \frac{\sigma\sqrt{T-t}}{2}\right)\right]$$

$$C(S,t) = SN\left(\frac{\ln\left(\frac{S}{K}\right) + \ln\left(e^{r(T-t)}\right)}{\sigma\sqrt{T-t}} + \frac{\sigma\sqrt{T-t}}{2}\right) - Ke^{-r(T-t)}N\left(\frac{\ln\left(\frac{S}{K}\right) + \ln\left(e^{r(T-t)}\right)}{\sigma\sqrt{T-t}} - \frac{\sigma\sqrt{T-t}}{2}\right)$$

$$C(S,t) = SN\left(\frac{\ln\left(\frac{Se^{r(T-t)}}{K}\right)}{\sigma\sqrt{T-t}} + \frac{\sigma\sqrt{T-t}}{2}\right) - Ke^{-r(T-t)}N\left(\frac{\ln\left(\frac{Se^{r(T-t)}}{K}\right)}{\sigma\sqrt{T-t}} - \frac{\sigma\sqrt{T-t}}{2}\right)$$

$$\therefore C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2) = C^{BS}(S,t)$$

$$d_{1,2} = d_{+,-} = \frac{\ln\left(\frac{Se^{r(T-t)}}{K}\right)}{\sigma\sqrt{T-t}} \pm \frac{\sigma\sqrt{T-t}}{2}$$

3.

See attached excel spreadsheet "Q3.xlsx" for more information.

The formulas for the Black-Scholes option pricing model are:

$$C^{BS}(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$P^{BS}(S,t) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1)$$

$$N(x) = \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

$$d_1 = \frac{\log\left(\frac{Se^{r(T-t)}}{K}\right)}{\sigma\sqrt{T-t}} + \left(\frac{\sigma\sqrt{T-t}}{2}\right)$$

$$d_2 = 1 - d_1$$

Q3.						
r	0.12					
T	0.25					
S	20					
K	21					
σ	0.2					
Since t = 0:						Formulas:
d1	(ln(S/K) + (r + σ^2/2)T)/(σT^(1/2))				-0.1379	=(LN($B$4/$B$5)+($B$2+($B$6^2)/2)*$B$3)/($B$6*$B$3^(1/2))
d2	d1 - σT^(1/2)				-0.2379	=F9-$B$6*$B$3^(1/2)
a)						
Solution:						Formulas:
C(S, 0)	SN(d1) - Kexp(-rT)N(d2)			0.629597		=$B$4*NORM.S.DIST(F9, TRUE)-$B$5*EXP(-$B$2*$B$3)*NORM.S.DIST(F10, TRUE)
b)						
Solution:						Formulas:
P(S, 0)	Kexp(-rT)N(-d2) - SN(-d1)			1.008953		=$B$5*EXP(-$B$2*$B$3)*NORM.S.DIST(-F10, TRUE)-$B$4*NORM.S.DIST(-F9, TRUE)
c)						
Show	C0 + Kexp(-rT)		=	P0 + S0		
LHS						Formulas:
C0 + Kexp(-rT)		21.00895				=E14+$B$5*EXP(-$B$2*$B$3)

Figure 4: Verifying parts a) b) and c) of Q1. using the B-S formulas using $\sigma = 0.2$

We make use of Excel's Goal Seek in order to estimate an appropriate value for σ such that the B-S model matches the binomial model:

d)				
From Q1.				
Binomial	C0	1.028751		
	P0	1.408107		
B-S C0:				
σ	0.300554		Change this cell using Excel's Goal Seek	Formulas:
d1	-0.0499			=(LN($B$4/$B$5)+($B$2+($B$37^2)/2)*$B$3)/($B$37*$B$3^(1/2))
d2	-0.20018			=B38-$B$37*$B$3^(1/2)
C0	1.029014			=$B$4*NORM.S.DIST(B38, TRUE)-$B$5*EXP(-$B$2*$B$3)*NORM.S.DIST(B39, TRUE)
B-S P0:				
σ	0.300513		Change this cell using Excel's Goal Seek	Formulas:
d1	-0.04993			=(LN($B$4/$B$5)+($B$2+($B$43^2)/2)*$B$3)/($B$43*$B$3^(1/2))
d2	-0.20018			=B44-$B$43*$B$3^(1/2)
P0	1.408207			=$B$5*EXP(-$B$2*$B$3)*NORM.S.DIST(-B45, TRUE)-$B$4*NORM.S.DIST(-B44, TRUE)
Show	C0 + Kexp(-rT)		= P0 + S0	
LHS				Formulas:
C0 + Kexp(-rT)		21.40837		=B40+$B$5*EXP(-$B$2*$B$3)
RHS				
P0 + S0		21.40821		=B46+$B$4
So LHS = RHS				

Figure 5: Using Goal Seek to estimate an appropriate value for σ to match the binomial model

4.

See attached excel spreadsheet "Q4.xlsx" for more information.

a)

The mean and standard deviation of the shares are calculated using:

$\mu_i = \mathbb{E}[R_i]$

$$\mu_i = \frac{1}{T}\sum_{t=1}^{T} R_{i,t}$$

where $R_{i,t}$ = random return on the i-th asset at time t.

$$\sigma_i^2 = \mathbb{E}[(R_i - \mathbb{E}[R_i])^2]$$

$$\sigma_i^2 = \mathbb{E}[(R_i - \mu_i)^2]$$

$$\sigma_i^2 = \frac{1}{T}\sum_{t=1}^{T} (R_{i,t} - \mu_i)^2$$

$$\therefore \sigma_i = \sqrt{\frac{1}{T}\sum_{t=1}^{T} (R_{i,t} - \mu_i)^2}$$

In Excel, these are calculated using the AVERAGE and STDEV.P (standard deviation of a population) functions respectively:

a)										
					Formulas of column B only (others identical):					
μi	0.053658	0.065292	0.049515		=AVERAGE(B4:B103)					
σi	0.129754	0.224264	0.227065		=STDEV.P(B4:B103)					

Figure 6: Screenshot of μ_i and σ_i for the 3 shares

b)

The correlation coefficient between any two shares is given by:

$$\rho_{i,j} = \frac{\sigma_{i,j}}{\sigma_i \sigma_j}$$

$$\rho_{i,j} = \frac{\mathbb{E}\left[(R_i - \mathbb{E}[R_i])(R_j - \mathbb{E}[R_j])\right]}{\mathbb{E}[(R_i - \mathbb{E}[R_i])^2] \cdot \mathbb{E}\left[\left(R_j - \mathbb{E}[R_j]\right)^2\right]}$$

$$\rho_{i,j} = \frac{\frac{1}{T}\sum_{t=1}^{T}(R_{i,t} - \mu_i)(R_{j,t} - \mu_j)}{\sqrt{\frac{1}{T}\sum_{t=1}^{T}(R_{i,t} - \mu_i)^2} \cdot \sqrt{\frac{1}{T}\sum_{t=1}^{T}(R_{j,t} - \mu_j)^2}}$$

$$\rho_{i,j} = \frac{\sum_{t=1}^{T}(R_{i,t} - \mu_i)(R_{j,t} - \mu_j)}{\sqrt{\sum_{t=1}^{T}(R_{i,t} - \mu_i)^2 \cdot \sum_{t=1}^{T}(R_{j,t} - \mu_j)^2}}$$

This is achieved using the CORREL function in Excel:

b)								
					Formulas:			
ρ12	-0.56007				=CORREL(B4:B103,C4:C103)			
ρ13	0.683001				=CORREL(B4:B103,D4:D103)			

Figure 7: Screenshot of $\rho_{i,j}$ for $(i,j) = (1,2)$ and $(1,3)$

c)

Using the formulas:

$w_1 = 0.5$

$w_2 = 1 - w_1 = 0.5$

$\mu_p = w_1\mu_1 + w_2\mu_2$

$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{1,2}$

$\sigma_p = \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\rho_{1,2}}$

c)							
w1		0.5					
w2	1-w1	0.5					
					Formulas:		
μp	0.059475				=C116*B107+C117*C107		
σp	0.092918				=(C116^2*B108^2+C117^2*C108^2+2*C116*C117*B108*C108*B112)^(1/2)		

Figure 8: Screenshot of μ_p and σ_p

Diversification benefits occur whenever a portfolio is created where each of the individual assets have < 1 correlation with any of the other assets in the portfolio. Hence, due to the strong negative correlation $\rho_{1,2}$ between shares A_1 and A_2, the portfolio risk σ_p is lower than the risk from owning the individual shares (σ_1 or σ_2) alone.

d)

Using the formulas:

$w_1 = 0.5$

$w_3 = 1 - w_1 = 0.5$

$\mu_p = w_1\mu_1 + w_3\mu_3$

$\sigma_p^2 = w_1^2\sigma_1^2 + w_3^2\sigma_3^2 + 2w_1w_3\sigma_{1,3}$

$\sigma_p = \sqrt{w_1^2\sigma_1^2 + w_3^2\sigma_3^2 + 2w_1w_3\sigma_1\sigma_3\rho_{1,3}}$

d)							
w1		0.5					
w3	1-w1	0.5					
					Formulas:		
μp	0.051586				=C124*B107+C125*D107		
σp	0.164804				=(C124^2*B108^2+C125^2*D108^2+2*C124*C125*B108*D108*B113)^(1/2)		

Figure 8: Screenshot of μ_p and σ_p

Similar to c), diversification benefits occur whenever a portfolio is created where each of the individual assets have < 1 correlation with any of the other assets in the portfolio. However, this time, the portfolio risk σ_p is much higher because the correlation is strongly positive and thus lower diversification benefits are enjoyed in this portfolio.

e)

Need to find w_1 and w_2 and then calculate μ_p and σ_p as in c).

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{1,2}$$

$$\sigma_p^2 = w_1^2\sigma_1^2 + (1-w_1)^2\sigma_2^2 + 2w_1(1-w_1)\sigma_{1,2}$$

$$\sigma_p^2 = w_1^2\sigma_1^2 + (1-w_1)^2\sigma_2^2 + 2w_1\sigma_1\sigma_2\rho_{1,2} - 2w_1^2\sigma_1\sigma_2\rho_{1,2}$$

Portfolio variance minimised by setting first partial derivative w.r.t w_1 equal to 0:

$$\frac{\partial\sigma_p^2}{\partial w_1} = 2w_1\sigma_1^2 - 2(1-w_1)\sigma_2^2 + 2\sigma_1\sigma_2\rho_{1,2} - 4w_1\sigma_1\sigma_2\rho_{1,2}$$

$$0 = 2w_1\sigma_1^2 - 2\sigma_2^2 + 2w_1\sigma_2^2 + 2\sigma_1\sigma_2\rho_{1,2} - 4w_1\sigma_1\sigma_2\rho_{1,2}$$

$$0 = w_1\sigma_1^2 - \sigma_2^2 + w_1\sigma_2^2 + \sigma_1\sigma_2\rho_{1,2} - 2w_1\sigma_1\sigma_2\rho_{1,2}$$

$$\sigma_2^2 - \sigma_1\sigma_2\rho_{1,2} = w_1\left(\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{1,2}\right)$$

$$\therefore w_1 = \frac{\sigma_2^2 - \sigma_1\sigma_2\rho_{1,2}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{1,2}}$$

$$w_2 = 1 - w_1$$

These are the optimal weights to minimise the portfolio variance:

e)						
					Formulas:	
w1	(σ2^2 - σ1σ2ρ1,2)/(σ1^2 + σ2^2 -2σ1σ2ρ1,2)			0.667751	=(C108^2-B108*C108*B112)/(B108^2+C108^2-2*B108*C108*B112)	
w2	1 - w1			0.332249	=1-G133	
μp	0.057523				=G133*B107+G134*C107	
σp	0.076338				=(G133^2*B108^2+G134^2*C108^2+2*G133*G134*B108*C108*B112)^(1/2)	

Figure 9: Screenshot of μ_p and σ_p using portfolio variance minimising weights

Using the MV portfolio optimisation approach to minimise portfolio variance, the optimal weights are $w_1 = 0.6678$ and $w_2 = 0.3322$. A portfolio comprised of these weights has lower risk $\sigma_p = 0.076$ than an equally weighted portfolio of the same assets $\sigma_p = 0.093$ (i.e. b)) and the individual assets alone.

5.

See attached excel spreadsheet "Q5.xlsx" for more information.

f)

Using the same formulas from 4.a), in Excel, μ_i and σ_i are calculated using the AVERAGE and STDEV.P (standard deviation of a population) functions respectively:

f)					
					Formulas of column F only (others identical):
μi	0.015101	0.01821	0.012511		=AVERAGE(F5:F104)
σi	0.068516	0.103254	0.101841		=STDEV.P(F5:F104)

Figure 10: Screenshot of μ_i and σ_i for the 3 shares

g)

Using the same formulas from 4.b), $\rho_{i,j}$ is calculated using the CORREL function in Excel:

g)							
					Formulas:		
p12	-0.69329				=CORREL(F5:F104,G5:G104)		
p13	0.75801				=CORREL(F5:F104,H5:H104)		

Figure 11: Screenshot of $\rho_{i,j}$ for $(i,j) = (1,2)$ and $(1,3)$

h)

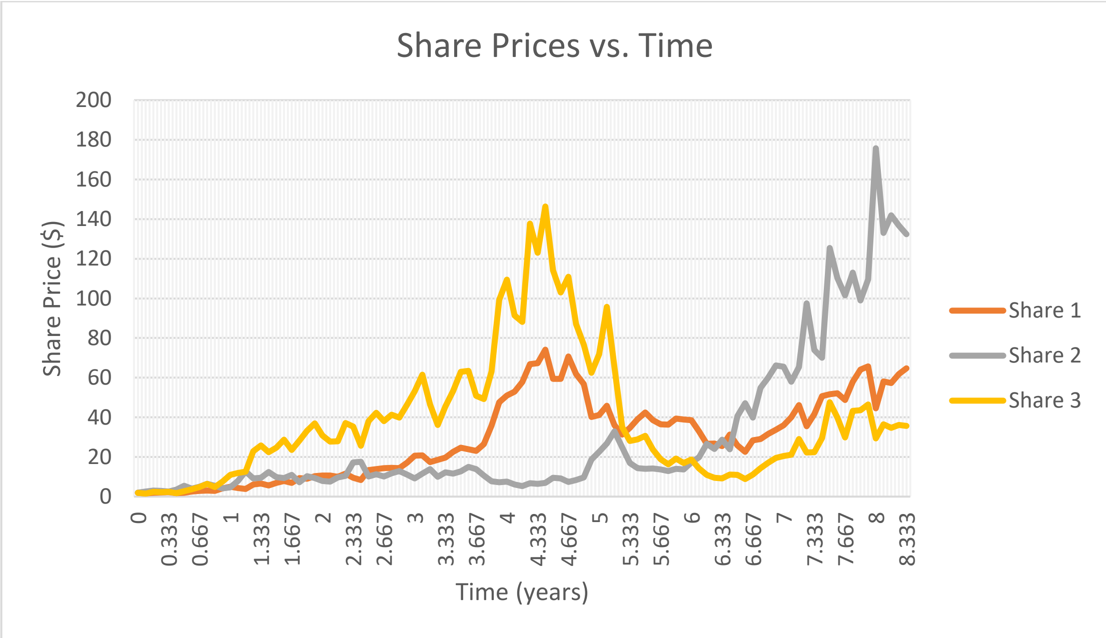

Figure 12: Graph of all 3 share prices over time

Shares 1 and 2 have a strong negative correlation whereas shares 1 and 3 have a strong positive correlation from g). This is very similar to 4.b). From the graph, negative correlation is evident between shares 1 and 2 because shares 1 (orange) and 2 (grey) tend to move in opposite directions over time. Conversely, positive correlation between shares 1 and 3 is evident because shares 1 (orange) and 3 (yellow) tend to move together over time. The time-series analogue for this relationship between shares 1 and 3 is known as cointegration where large deviations from both series do not persist over time.