MATH7091 – Assignment 4
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1.

If $\{X_t\}_{t\in[0,T]}$ is a $\left(\mathbb{P},\{\mathcal{F}_t\}_{t\in[0,T]}\right)$-martingale, then $\forall\ 0 \le s \le t \le T$:

$\mathbb{E}_{\mathbb{P}}[X_t|\mathcal{F}_s] = X_s$

Let $X_{t+dt} = X_t + dX_t$, we want $\mathbb{E}_{\mathbb{P}}[X_{t+dt}|\mathcal{F}_t] = X_t$:

$\mathbb{E}_{\mathbb{P}}[X_{t+dt}|\mathcal{F}_t] = \mathbb{E}_{\mathbb{P}}[X_t + dX_t|\mathcal{F}_t]$

$\mathbb{E}_{\mathbb{P}}[X_{t+dt}|\mathcal{F}_t] = \mathbb{E}_{\mathbb{P}}[X_t + \mu_t dt + \sigma_t dW_t|\mathcal{F}_t]$

$\mathbb{E}_{\mathbb{P}}[X_{t+dt}|\mathcal{F}_t] = \mathbb{E}_{\mathbb{P}}[X_t|\mathcal{F}_t] + \mathbb{E}_{\mathbb{P}}[\mu_t dt|\mathcal{F}_t] + \mathbb{E}_{\mathbb{P}}[\sigma_t dW_t|\mathcal{F}_t], \qquad \text{(linearity)}$

$\mathbb{E}_{\mathbb{P}}[X_{t+dt}|\mathcal{F}_t] = X_t + \mu_t dt + \sigma_t \mathbb{E}_{\mathbb{P}}[dW_t|\mathcal{F}_t], \qquad (X_t, \mu_t, dt, \sigma_t \text{ are } \mathcal{F}_t\text{-measurable})$

$\mathbb{E}_{\mathbb{P}}[X_{t+dt}|\mathcal{F}_t] = X_t + \mu_t dt + \sigma_t \mathbb{E}_{\mathbb{P}}[dW_t], \qquad \text{(BM has independent increments)}$

$\mathbb{E}_{\mathbb{P}}[X_{t+dt}|\mathcal{F}_t] = X_t + \mu_t dt + \sigma_t \cdot 0, \qquad \left(dW_t \sim N(0, dt)\right)$

$\mathbb{E}_{\mathbb{P}}[X_{t+dt}|\mathcal{F}_t] = X_t + \mu_t dt$

Thus, $\mathbb{E}_{\mathbb{P}}[X_{t+dt}|\mathcal{F}_t] = X_t$ if and only if $\mu_t = 0\ \forall\ t \in [0,T]$ almost surely.

2.

a)

$dS_t = \mu S_t dt + \sigma S_t dW_t$

$dB_t = r_t B_t dt$

Need to first find dynamics of $\left\{\frac{S_t}{B_t}\right\}_{t\in[0,T]}$ under $\mathbb{P}$ before performing change of measure using Girsanov's theorem.

First find dynamics of $\left\{\frac{1}{B_t}\right\}_{t\in[0,T]}$:

$f(x) = \dfrac{1}{x}, \qquad f'(x) = -\dfrac{1}{x^2}, \qquad f''(x) = \dfrac{2}{x^3}$

$$\therefore d\left(\frac{1}{B_t}\right) = \frac{\partial f}{\partial x}(B_t)dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(B_t)(dB_t)^2$$

$$d\left(\frac{1}{B_t}\right) = -\frac{1}{B_t^2}(r_t B_t dt) + \frac{1}{2}\cdot\frac{2}{B_t^3}(r_t B_t dt)^2$$

$$d\left(\frac{1}{B_t}\right) = -\frac{r_t}{B_t}dt + \frac{1}{2}\cdot\frac{2}{B_t^3}\cdot 0, \qquad \text{(by Ito rules)}$$

$$d\left(\frac{1}{B_t}\right) = -\frac{r_t}{B_t}dt$$

Using Ito's product rule:

$$d\left(\frac{S_t}{B_t}\right) = S_t d\left(\frac{1}{B_t}\right) + \frac{1}{B_t}dS_t + (dS_t)\left(d\left(\frac{1}{B_t}\right)\right)$$

$$d\left(\frac{S_t}{B_t}\right) = S_t\left(-\frac{r_t}{B_t}dt\right) + \frac{1}{B_t}(\mu S_t dt + \sigma S_t dW_t) + (\mu S_t dt + \sigma S_t dW_t)\left(-\frac{r_t}{B_t}dt\right)$$

$$d\left(\frac{S_t}{B_t}\right) = -r_t\frac{S_t}{B_t}dt + \mu\frac{S_t}{B_t}dt + \sigma\frac{S_t}{B_t}dW_t + 0, \qquad \text{(by Ito rules)}$$

$$d\left(\frac{S_t}{B_t}\right) = (\mu - r_t)\frac{S_t}{B_t}dt + \sigma\frac{S_t}{B_t}dW_t$$

Thus, $\left\{\frac{S_t}{B_t}\right\}_{t\in[0,T]}$ is clearly not a martingale under $\mathbb{P}$ because the dt-term has a non-zero coefficient. Taking $\alpha_t = \frac{\mu - r_t}{\sigma}$ in Girsanov's theorem, we have:

$$\widetilde{W}_t = W_t + \int_0^t \alpha_u \, d_u$$

$$\rightarrow d\widetilde{W}_t = dW_t + \alpha_t dt$$

Therefore, under $\mathbb{Q}$, we have:

$$d\left(\frac{S_t}{B_t}\right) = (\mu - r_t)\frac{S_t}{B_t}dt + \sigma\frac{S_t}{B_t}dW_t$$

$$d\left(\frac{S_t}{B_t}\right) = (\mu - r_t)\frac{S_t}{B_t}dt + \sigma\frac{S_t}{B_t}\left(d\widetilde{W}_t - \alpha_t dt\right)$$

$$d\left(\frac{S_t}{B_t}\right) = (\mu - r_t)\frac{S_t}{B_t}dt + \sigma\frac{S_t}{B_t}d\widetilde{W}_t - \sigma\left(\frac{\mu - r_t}{\sigma}\right)\frac{S_t}{B_t}dt$$

$$d\left(\frac{S_t}{B_t}\right) = (\mu - r_t)\frac{S_t}{B_t}dt - (\mu - r_t)\frac{S_t}{B_t}dt + \sigma\frac{S_t}{B_t}d\widetilde{W}_t$$

$$d\left(\frac{S_t}{B_t}\right) = \sigma\frac{S_t}{B_t}d\widetilde{W}_t$$

Thus, $\left\{\frac{S_t}{B_t}\right\}_{t\in[0,T]}$ is clearly a martingale under $\mathbb{Q}$ because the dt-term has a zero coefficient.

b)

Since $dB_t = r_t B_t dt$, we have under both $\mathbb{P}$ and $\mathbb{Q}$:

$$B_t = e^{\int_0^t r_u\, d_u}$$

Using the following facts:

- Fact 1 – by the Martingale Representation theorem, $\exists$ a replicating portfolio $\{\Theta_t\}_{t\in[0,T]}$ under the risk-neutral measure $\mathbb{Q}$ whose value process $\{V_t\}_{t\in[0,T]}$ replicates the financial contract payoff $\{C_t\}_{t\in[0,T]}$.
- Fact 2 – from **a)**, $\left\{\frac{S_t}{B_t}\right\} = \left\{e^{-\int_0^t r_u\, d_u} S_t\right\}$ is a $\left(\mathbb{Q}, \{\mathcal{F}_t\}_{t\in[0,T]}\right)$-martingale so the discounted price of any portfolio of (S, B) is a $\left(\mathbb{Q}, \{\mathcal{F}_t\}_{t\in[0,T]}\right)$-martingale.
- Fact 3 – it can be shown that $\mathbb{P}$ and $\mathbb{Q}$ from Girsanov's theorem are equivalent.

Then $\exists$ a replicating portfolio under $\mathbb{Q}$ whose discounted value $\left\{\frac{V_t}{B_t}\right\} = \left\{e^{-\int_0^t r_u\, d_u} V_t\right\}$ is a $\mathbb{Q}$-martingale:

$$e^{-\int_0^t r_u\, d_u} C_t = e^{-\int_0^t r_u\, d_u} V_t$$

$$e^{-\int_0^t r_u\, d_u} C_t = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T r_u\, d_u} V_T \middle| \mathcal{F}_t\right], \qquad \left(\left\{e^{-\int_0^t r_u\, d_u} V_t\right\} \text{ is a } \mathbb{Q}\text{-martingale}\right)$$

$$e^{-\int_0^t r_u\, d_u} C_t = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T r_u\, d_u} C_T \middle| \mathcal{F}_t\right], \qquad (V_T = C_T \text{ at maturity})$$

$$\rightarrow C_t = e^{\int_0^t r_u\, d_u}\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T r_u\, d_u} C_T \middle| \mathcal{F}_t\right]$$

$$C_t = \mathbb{E}_{\mathbb{Q}}\left[e^{\int_0^t r_u\, d_u} e^{-\int_0^T r_u\, d_u} C_T \middle| \mathcal{F}_t\right], \qquad \left(e^{\int_0^t r_u\, d_u} \text{ is } \mathcal{F}_t\text{-measurable}\right)$$

$$C_t = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r_u\, d_u} C_T \middle| \mathcal{F}_t\right]$$

3.

a)

Let C_t denote the option price at time t with the payoff at maturity given by $C_T = (Y_T - K)^+$. We have:

$$C_t = e^{-R_{disc}(T-t)}\mathbb{E}_{\mathbb{Q}}[C_T|\mathcal{F}_t]$$

$$C_t = e^{-R_{disc}(T-t)}\mathbb{E}_{\mathbb{Q}}[(Y_T - K)^+|\mathcal{F}_t]$$

$$C_t = e^{-R_{disc}(T-t)}\mathbb{E}_{\mathbb{Q}}\left[(Y_T - K)\mathbb{I}_{\{Y_T>K\}}\middle|\mathcal{F}_t\right]$$

$$C_t = e^{-R_{disc}(T-t)}\left[\mathbb{E}_{\mathbb{Q}}\left[Y_T\mathbb{I}_{\{Y_T>K\}}\middle|\mathcal{F}_t\right] - \mathbb{E}_{\mathbb{Q}}\left[K\mathbb{I}_{\{Y_T>K\}}\middle|\mathcal{F}_t\right]\right], \qquad \text{(linearity)}$$

$$C_t = e^{-R_{disc}(T-t)}\left[\mathbb{E}_{\mathbb{Q}}\left[Y_T\mathbb{I}_{\{Y_T>K\}}\middle|\mathcal{F}_t\right] - K\mathbb{E}_{\mathbb{Q}}\left[\mathbb{I}_{\{Y_T>K\}}\middle|\mathcal{F}_t\right]\right]$$

$$C_t = e^{-R_{disc}(T-t)}\left[\mathbb{E}_{\mathbb{Q}}\left[Y_T\mathbb{I}_{\{Y_T>K\}}\middle|\mathcal{F}_t\right] - K\mathbb{Q}(Y_T > K|\mathcal{F}_t)\right], \qquad \left(\mathbb{E}_{\mathbb{Q}}[\mathbb{I}_A] = \mathbb{Q}(A)\right)$$

Let $1) = \mathbb{E}_{\mathbb{Q}}\left[Y_T\mathbb{I}_{\{Y_T>K\}}\middle|\mathcal{F}_t\right]$ and $2) = \mathbb{Q}(Y_T > K|\mathcal{F}_t)$. Solving for 2) first, under $\mathbb{Q}$ and conditional on $\mathcal{F}_t$, we have:

$$\log(Y_T) \sim N\left(\log(Y_t) + \left(R_{grow} - \frac{s^2}{2}\right)(T-t), s^2(T-t)\right)$$

$$\therefore\ \mathbb{Q}(Y_T > K|\mathcal{F}_t)$$

$$= \mathbb{Q}(\log(Y_T) > \log(K)\,|\mathcal{F}_t)$$

$$= Q\left(\frac{\log(Y_T) - \log(Y_t) - \left(R_{grow} - \frac{s^2}{2}\right)(T-t)}{\sqrt{s^2(T-t)}} > \frac{\log(K) - \log(Y_t) - \left(R_{grow} - \frac{s^2}{2}\right)(T-t)}{\sqrt{s^2(T-t)}}\right),$$

$$\left(\text{normalising to } Z\sim N(0,1)\right)$$

$$= Q\left(\frac{\log\left(\frac{Y_T}{Y_t}\right) - \left(R_{grow} - \frac{s^2}{2}\right)(T-t)}{s\sqrt{T-t}} > \frac{\log\left(\frac{K}{Y_t}\right) - \left(R_{grow} - \frac{s^2}{2}\right)(T-t)}{s\sqrt{T-t}}\right)$$

$$= N\left(-\frac{\log\left(\frac{K}{Y_t}\right) - \left(R_{grow} - \frac{s^2}{2}\right)(T-t)}{s\sqrt{T-t}}\right), \qquad \left(\text{by symmetry of the normal distribution, } 1 - N(x) = N(-x)\right)$$

$$= N\left(\frac{\log\left(\left(\frac{K}{Y_t}\right)^{-1}\right) + \left(R_{grow} - \frac{s^2}{2}\right)(T-t)}{s\sqrt{T-t}}\right)$$

$$= N\left(\frac{\log\left(\frac{Y_t}{K}\right) + \left(R_{grow} - \frac{s^2}{2}\right)(T-t)}{s\sqrt{T-t}}\right)$$

$$= N\left(\frac{\log\left(\frac{Y_t}{K}\right) + \left(R_{grow} + \frac{s^2}{2}\right)(T-t)}{s\sqrt{T-t}} - s\sqrt{T-t}\right)$$

$$= N\left(d_1 - s\sqrt{T-t}\right)$$

$$= N(d_2)$$

Now solving for 1):

$$\mathbb{E}_{\mathbb{Q}}\left[Y_T \mathbb{I}_{\{Y_T>K\}} \middle| \mathcal{F}_t\right]$$

$$= e^{R_{grow}(T-t)} Y_t \mathbb{E}_{\mathbb{Q}}\left[\frac{Y_T}{e^{R_{grow}(T-t)} Y_t} \mathbb{I}_{\{Y_T>K\}} \middle| \mathcal{F}_t\right], \qquad \left(e^{R_{grow}(T-t)} Y_t \text{ is } \mathcal{F}_t\text{-measurable}\right)$$

$$= F_t \mathbb{E}_{\mathbb{Q}}\left[\frac{Y_T}{F_t} \mathbb{I}_{\{Y_T>K\}} \middle| \mathcal{F}_t\right], \qquad \left(F_t = e^{R_{grow}(T-t)} Y_t\right)$$

$$= F_t \mathbb{E}_{\mathbb{Q}}\left[e^{\log\left(\frac{Y_T}{F_t}\right)} \mathbb{I}_{\left\{\log\left(\frac{Y_T}{F_t}\right) > \log\left(\frac{K}{F_t}\right)\right\}} \middle| \mathcal{F}_t\right]$$

Since under $\mathbb{Q}$ conditional on $\mathcal{F}_t$:

$$\log(Y_T) \sim N\left(\log(Y_t) + \left(R_{grow} - \frac{s^2}{2}\right)(T-t), s^2(T-t)\right)$$

$$\log(Y_T) \sim N\left(\log\left(e^{R_{grow}(T-t)} Y_t\right) - \frac{s^2}{2}(T-t), s^2(T-t)\right)$$

$$\rightarrow \log(Y_T) - \log\left(e^{R_{grow}(T-t)} Y_t\right) \sim N\left(-\frac{s^2}{2}(T-t), s^2(T-t)\right)$$

$$\therefore \log\left(\frac{Y_T}{F_t}\right) \sim N\left(-\frac{s^2}{2}(T-t), s^2(T-t)\right)$$

Let $y = \log\left(\frac{Y_T}{F_t}\right)$. Continuing:

$$\mathbb{E}_{\mathbb{Q}}\left[Y_T \mathbb{I}_{\{Y_T > K\}} \middle| \mathcal{F}_t\right]$$

$$= F_t \int_{\log\left(\frac{K}{F_t}\right)}^{\infty} e^y \cdot \frac{1}{\sqrt{2\pi} \cdot s\sqrt{T-t}} e^{-\frac{1}{2}\frac{\left(y+\frac{s^2}{2}(T-t)\right)^2}{s^2(T-t)}} dy$$

$$= F_t \int_{\log\left(\frac{K}{F_t}\right)}^{\infty} \frac{1}{\sqrt{2\pi} \cdot s\sqrt{T-t}} e^{-\frac{1}{2}\frac{\left(y+\frac{s^2}{2}(T-t)\right)^2}{s^2(T-t)}+y} dy$$

Observe that:

$$-\frac{1}{2}\frac{\left(y+\frac{s^2}{2}(T-t)\right)^2}{s^2(T-t)} + y$$

$$= \frac{-\left(y^2 + ys^2(T-t) + \frac{s^4}{4}(T-t)^2\right) + 2ys^2(T-t)}{2s^2(T-t)}$$

$$= \frac{-\left(y^2 - ys^2(T-t) + \frac{s^4}{4}(T-t)^2\right)}{2s^2(T-t)}$$

$$= -\frac{1}{2}\frac{\left(y-\frac{s^2}{2}(T-t)\right)^2}{s^2(T-t)}$$

Continuing:

$$\mathbb{E}_{\mathbb{Q}}\left[Y_T \mathbb{I}_{\{Y_T > K\}} \middle| \mathcal{F}_t\right]$$

$$= F_t \int_{\log\left(\frac{K}{F_t}\right)}^{\infty} \frac{1}{\sqrt{2\pi} \cdot s\sqrt{T-t}} e^{-\frac{1}{2}\frac{\left(y-\frac{s^2}{2}(T-t)\right)^2}{s^2(T-t)}} dy$$

Apply change of variables, let $z = \frac{y - \frac{s^2}{2}(T-t)}{s\sqrt{T-t}}$:

$$\frac{dz}{dy} = \frac{1}{s\sqrt{T-t}} \rightarrow dy = s\sqrt{T-t}dz$$

$$y = \infty \rightarrow z = \infty$$

$$y = \log\left(\frac{K}{F_t}\right) \rightarrow z = \frac{\log\left(\frac{K}{F_t}\right) - \frac{s^2}{2}(T-t)}{s\sqrt{T-t}}$$

Continuing:

$$\mathbb{E}_{\mathbb{Q}}\left[Y_T \mathbb{I}_{\{Y_T > K\}} \middle| \mathcal{F}_t\right]$$

$$= F_t \int_{\frac{\log\left(\frac{K}{F_t}\right) - \frac{s^2}{2}(T-t)}{s\sqrt{T-t}}}^{\infty} \frac{1}{\sqrt{2\pi} \cdot s\sqrt{T-t}} e^{-\frac{1}{2}z^2} \, s\sqrt{T-t}dz$$

$$= F_t \int_{\frac{\log\left(\frac{K}{F_t}\right) - \frac{s^2}{2}(T-t)}{s\sqrt{T-t}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \, dz$$

$$= F_t N\left(-\frac{\log\left(\frac{K}{F_t}\right) - \frac{s^2}{2}(T-t)}{s\sqrt{T-t}}\right), \qquad \left(\text{by symmetry of the normal distribution, } 1 - N(x) = N(-x)\right)$$

$$= F_t N\left(\frac{\log\left(\left(\frac{K}{e^{R_{grow}(T-t)}Y_t}\right)^{-1}\right) + \frac{s^2}{2}(T-t)}{s\sqrt{T-t}}\right)$$

$$= F_t N\left(\frac{\log\left(\frac{e^{R_{grow}(T-t)}Y_t}{K}\right) + \frac{s^2}{2}(T-t)}{s\sqrt{T-t}}\right)$$

$$= F_t N\left(\frac{\log\left(\frac{Y_t}{K}\right) + \left(R_{grow} + \frac{s^2}{2}\right)(T-t)}{s\sqrt{T-t}}\right)$$

$$= F_t N(d_1)$$

Combining results from 1) and 2):

$$C_t = e^{-R_{disc}(T-t)}\mathbb{E}_{\mathbb{Q}}[(Y_T - K)^+ | \mathcal{F}_t]$$

$$C_t = e^{-R_{disc}(T-t)}\left[\mathbb{E}_{\mathbb{Q}}\left[Y_T \mathbb{I}_{\{Y_T > K\}} \middle| \mathcal{F}_t\right] - K\mathbb{Q}(Y_T > K | \mathcal{F}_t)\right]$$

$$C_t = e^{-R_{disc}(T-t)}[F_t N(d_1) - KN(d_2)]$$

$$C_t = C^{BS}\left(Y_t, t, K, T, R_{grow}, R_{disc}, s\right)$$

b)

i)

$Y_T \sim N(\mu, \sigma^2)$, $\mu \neq 0$, $\sigma \neq 1$ in general.

$$C_T = (K - Y_T)^+$$

$$\therefore C_t = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}[C_T | \mathcal{F}_t]$$

$$C_t = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}[(K - Y_T)^+ | \mathcal{F}_t]$$

$$C_t = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left[(K - Y_T)\mathbb{I}_{\{K > Y_T\}} \middle| \mathcal{F}_t\right]$$

$$C_t = e^{-r(T-t)}\left[\mathbb{E}_{\mathbb{Q}}\left[K\mathbb{I}_{\{Y_T < K\}} \middle| \mathcal{F}_t\right] - \mathbb{E}_{\mathbb{Q}}\left[Y_T \mathbb{I}_{\{Y_T < K\}} \middle| \mathcal{F}_t\right]\right], \qquad \text{(linearity)}$$

$$C_t = e^{-r(T-t)}\left[K\mathbb{E}_{\mathbb{Q}}\left[\mathbb{I}_{\{Y_T < K\}} \middle| \mathcal{F}_t\right] - \mathbb{E}_{\mathbb{Q}}\left[Y_T \mathbb{I}_{\{Y_T < K\}} \middle| \mathcal{F}_t\right]\right]$$

$$C_t = e^{-r(T-t)}\left[K\mathbb{Q}(Y_T < K | \mathcal{F}_t) - \mathbb{E}_{\mathbb{Q}}\left[Y_T \mathbb{I}_{\{Y_T < K\}} \middle| \mathcal{F}_t\right]\right], \qquad \left(\mathbb{E}_{\mathbb{Q}}[\mathbb{I}_A] = \mathbb{Q}(A)\right)$$

Let 1) $= \mathbb{Q}(Y_T < K | \mathcal{F}_t)$ and 2) $= \mathbb{E}_{\mathbb{Q}}\left[Y_T \mathbb{I}_{\{Y_T < K\}} \middle| \mathcal{F}_t\right]$. Solving for 1) first, under $\mathbb{Q}$ and conditional on $\mathcal{F}_t$, we have:

$$Y_T \sim N(\mu, \sigma^2)$$

$$\therefore \mathbb{Q}(Y_T < K | \mathcal{F}_t) = \mathbb{Q}\left(\frac{Y_T - \mu}{\sigma} < \frac{K - \mu}{\sigma} \middle| \mathcal{F}_t\right), \qquad \left(\text{normalising to } Z \sim N(0,1)\right)$$

$$\mathbb{Q}(Y_T < K | \mathcal{F}_t) = N\left(\frac{K - \mu}{\sigma}\right)$$

Now solving for 2):

$\mathbb{E}_{\mathbb{Q}}\left[Y_T \mathbb{I}_{\{Y_T<K\}} \middle| \mathcal{F}_t\right]$

$$= \int_{-\infty}^{K} y \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}}\, dy$$

$$= \int_{-\infty}^{K} (y-\mu) \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}}\, dy + \int_{-\infty}^{K} \mu \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}}\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{K} \frac{y-\mu}{\sigma} e^{-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}}\, dy + \mu \int_{-\infty}^{K} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}}\, dy$$

Observe that:

$$\frac{d}{dy}\left(-e^{-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}}\right) = \frac{y-\mu}{\sigma} e^{-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}}$$

Continuing:

$\mathbb{E}_{\mathbb{Q}}\left[Y_T \mathbb{I}_{\{Y_T<K\}} \middle| \mathcal{F}_t\right]$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{K} \frac{d}{dy}\left(-e^{-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}}\right) dy + \mu \int_{-\infty}^{K} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}}\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{K} \frac{d}{dy}\left(-e^{-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}}\right) dy + \mu \int_{-\infty}^{\frac{K-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz,$$

$$\left(\text{apply change of variables to second integral, let } z = \frac{y-\mu}{\sigma}\right)$$

$$= \frac{1}{\sqrt{2\pi}} \left[-e^{-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}}\right]_{y=-\infty}^{y=K} + \mu N\left(\frac{K-\mu}{\sigma}\right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(-e^{-\frac{1}{2}\frac{(K-\mu)^2}{\sigma^2}} + 0\right) + \mu N\left(\frac{K-\mu}{\sigma}\right)$$

$$= -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(K-\mu)^2}{\sigma^2}} + \mu N\left(\frac{K-\mu}{\sigma}\right)$$

Combining expressions 1) and 2):

$$C_t = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}[C_T|\mathcal{F}_t]$$

$$C_t = e^{-r(T-t)}\left[K\mathbb{Q}(Y_T < K|\mathcal{F}_t) - \mathbb{E}_{\mathbb{Q}}\left[Y_T\mathbb{I}_{\{Y_T<K\}}\middle|\mathcal{F}_t\right]\right]$$

$$C_t = e^{-r(T-t)}\left[KN\left(\frac{K-\mu}{\sigma}\right) - \left(-\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\frac{(K-\mu)^2}{\sigma^2}} + \mu N\left(\frac{K-\mu}{\sigma}\right)\right)\right]$$

$$C_t = e^{-r(T-t)}\left[(K-\mu)N\left(\frac{K-\mu}{\sigma}\right) + \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\frac{(K-\mu)^2}{\sigma^2}}\right]$$

$$\therefore C_0 = e^{-rT}\left[(K-\mu)N\left(\frac{K-\mu}{\sigma}\right) + \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\frac{(K-\mu)^2}{\sigma^2}}\right]$$

ii)

We wish to show that $X_T \sim N(\mu, \sigma^2)$ for some $\mu \neq 0, \sigma^2 \neq 1$ in general. We also know that $\{\widetilde{W}_t\}_{t\in[0,T]}$ is a $\mathbb{Q}$-BM.

$$dX_t = (\alpha - \beta X_t)dt + \gamma d\widetilde{W}_t$$

Note that this SDE has the same form as the Vasicek interest rate SDE which was covered in Tutorial 8 Q7. Hence, citing the result from Tutorial 8 Q7.d) but replacing r_t, r_0, σ, dW_s appearing in the Vasicek interest rate SDE by $X_t, X_0, \gamma, d\widetilde{W}_s$ appearing in the SDE for X_t, the solution for X_t is given by:

$$X_t = e^{-\beta t}X_0 + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + \gamma e^{-\beta t}\int_0^t e^{\beta s}\, d\widetilde{W}_s$$

Which is of the form $constant + constant \times Ito\ integral$. We wish to find the distribution of the Ito integral:

$$\int_0^t e^{\beta s}\, d\widetilde{W}_s = \lim_{n\to\infty}\sum_{i=1}^{n} e^{\beta t_i}\left(\widetilde{W}_{t_{i+1}} - \widetilde{W}_{t_i}\right), \qquad \text{(by definition of the Ito integral)}$$

Note that each:

$$\widetilde{W}_{t_{i+1}} - \widetilde{W}_{t_i} \sim N(0, t_{i+1} - t_i)$$

$$\rightarrow e^{\beta t_i}\left(\widetilde{W}_{t_{i+1}} - \widetilde{W}_{t_i}\right) \sim N\left(0, (t_{i+1} - t_i)e^{2\beta t_i}\right)$$

Since each $\left(\widetilde{W}_{t_{i+1}} - \widetilde{W}_{t_i}\right)$ is an independent increment, using the fact that the probabilistic limit of a sum of independent normal random variables is normally distributed, we arrive at:

$$\int_0^t e^{\beta s}\, d\widetilde{W}_s \sim N(\mu, \sigma^2)$$

We just need to find the mean and variance parameters:

$$\mathbb{E}_{\mathbb{Q}}\left[\int_0^t e^{\beta s}\, d\widetilde{W}_s\right] = 0, \text{(using the property that } \mathbb{E}[.] \text{ of any Ito integral is 0, see L6.32)}$$

$$\rightarrow \mathbb{E}_{\mathbb{Q}}\left[\int_0^t e^{\beta s}\, d\widetilde{W}_s\right]^2 = 0$$

$$Var_{\mathbb{Q}}\left(\int_0^t e^{\beta s}\, d\widetilde{W}_s\right) = \mathbb{E}_{\mathbb{Q}}\left[\left(\int_0^t e^{\beta s}\, d\widetilde{W}_s\right)^2\right] - \mathbb{E}_{\mathbb{Q}}\left[\int_0^t e^{\beta s}\, d\widetilde{W}_s\right]^2$$

$$Var_{\mathbb{Q}}\left(\int_0^t e^{\beta s}\, d\widetilde{W}_s\right) = \mathbb{E}_{\mathbb{Q}}\left[\left(\int_0^t e^{\beta s}\, d\widetilde{W}_s\right)^2\right]$$

$$Var_{\mathbb{Q}}\left(\int_0^t e^{\beta s}\, d\widetilde{W}_s\right) = \mathbb{E}_{\mathbb{Q}}\left[\int_0^t \left(e^{\beta s}\right)^2 ds\right], \qquad \text{(by Ito isometry)}$$

$$Var_{\mathbb{Q}}\left(\int_0^t e^{\beta s}\, d\widetilde{W}_s\right) = \mathbb{E}_{\mathbb{Q}}\left[\int_0^t e^{2\beta s}\, ds\right]$$

$$Var_{\mathbb{Q}}\left(\int_0^t e^{\beta s}\, d\widetilde{W}_s\right) = \int_0^t e^{2\beta s}\, ds$$

$$Var_{\mathbb{Q}}\left(\int_0^t e^{\beta s}\, d\widetilde{W}_s\right) = \frac{1}{2\beta}\left[e^{2\beta s}\right]_{s=0}^{s=t}$$

$$Var_{\mathbb{Q}}\left(\int_0^t e^{\beta s}\, d\widetilde{W}_s\right) = \frac{1}{2\beta}\left(e^{2\beta t} - 1\right)$$

$$\therefore \int_0^t e^{\beta s}\, d\widetilde{W}_s \sim N\left(0, \frac{1}{2\beta}\left(e^{2\beta t} - 1\right)\right)$$

$$\rightarrow \gamma e^{-\beta t}\int_0^t e^{\beta s}\,d\widetilde{W}_s \sim N\left(0, \gamma^2 e^{-2\beta t}\cdot\frac{1}{2\beta}\left(e^{2\beta t}-1\right)\right)$$

$$\gamma e^{-\beta t}\int_0^t e^{\beta s}\,d\widetilde{W}_s \sim N\left(0, \frac{\gamma^2}{2\beta}\left(1-e^{-2\beta t}\right)\right)$$

$$\rightarrow X_t = e^{-\beta t}X_0 + \frac{\alpha}{\beta}\left(1-e^{-\beta t}\right) + \gamma e^{-\beta t}\int_0^t e^{\beta s}\,d\widetilde{W}_s$$

$$\sim N\left(e^{-\beta t}X_0 + \frac{\alpha}{\beta}\left(1-e^{-\beta t}\right), \frac{\gamma^2}{2\beta}\left(1-e^{-2\beta t}\right)\right)$$

Now using result from **i)**, redefine $C_T = (K - X_T)^+$

$$\rightarrow C_0 = e^{-rT}\left[(K-\mu)N\left(\frac{K-\mu}{\sigma}\right) + \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\frac{(K-\mu)^2}{\sigma^2}}\right]$$

$$\mu = e^{-\beta t}X_0 + \frac{\alpha}{\beta}\left(1-e^{-\beta t}\right)$$

$$\sigma^2 = \frac{\gamma^2}{2\beta}\left(1-e^{-2\beta t}\right)$$

4.

a)

Under $\mathbb{P}$:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

$$dB_t = rB_t dt$$

Want $\left\{\frac{B_t}{S_t}\right\}_{t\in[0,T]}$ to be a martingale, first need to find dynamics of $\left\{\frac{1}{S_t}\right\}_{t\in[0,T]}$:

$$f(x) = \frac{1}{x}, \qquad f'(x) = -\frac{1}{x^2}, \qquad f''(x) = \frac{2}{x^3}$$

$$\therefore d\left(\frac{1}{S_t}\right) = \frac{\partial f}{\partial x}(S_t)dS_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(S_t)(dS_t)^2$$

$$d\left(\frac{1}{S_t}\right) = -\frac{1}{S_t^2}(\mu S_t dt + \sigma S_t dW_t) + \frac{1}{2}\cdot\frac{2}{S_t^3}(\mu S_t dt + \sigma S_t dW_t)^2$$

$$d\left(\frac{1}{S_t}\right) = -\frac{\mu}{S_t}dt - \frac{\sigma}{S_t}dW_t + \frac{1}{S_t^3}\cdot(\sigma^2 S_t^2 (dW_t)^2), \qquad \text{(by Ito rules)}$$

$$d\left(\frac{1}{S_t}\right) = -\frac{\mu}{S_t}dt - \frac{\sigma}{S_t}dW_t + \frac{\sigma^2}{S_t}dt$$

$$d\left(\frac{1}{S_t}\right) = (\sigma^2 - \mu)\frac{1}{S_t}dt - \frac{\sigma}{S_t}dW_t$$

Using Ito's product rule:

$$d\left(\frac{B_t}{S_t}\right) = B_t d\left(\frac{1}{S_t}\right) + \frac{1}{S_t}dB_t + (dB_t)\left(d\left(\frac{1}{S_t}\right)\right)$$

$$d\left(\frac{B_t}{S_t}\right) = B_t\left((\sigma^2 - \mu)\frac{1}{S_t}dt - \frac{\sigma}{S_t}dW_t\right) + \frac{1}{S_t}(rB_t dt) + (rB_t dt)\left((\sigma^2 - \mu)\frac{1}{S_t}dt - \frac{\sigma}{S_t}dW_t\right)$$

$$d\left(\frac{B_t}{S_t}\right) = (\sigma^2 - \mu)\frac{B_t}{S_t}dt - \sigma\frac{B_t}{S_t}dW_t + r\frac{B_t}{S_t}dt + 0, \qquad \text{(by Ito rules)}$$

$$d\left(\frac{B_t}{S_t}\right) = (r + \sigma^2 - \mu)\frac{B_t}{S_t}dt - \sigma\frac{B_t}{S_t}dW_t$$

Observe that the coefficient of the dt-term is non-zero so $\left\{\frac{B_t}{S_t}\right\}_{t\in[0,T]}$ is not a martingale under $\mathbb{P}$.

Taking $\alpha_t = \alpha = -\frac{r+\sigma^2-\mu}{\sigma}$ in Girsanov's theorem:

$$\widetilde{W}_t^S = W_t + \int_0^t \alpha_u \, d_u$$

$$\to d\widetilde{W}_t^S = dW_t + \alpha_t dt$$

$$d\widetilde{W}_t^S = dW_t + \alpha dt$$

$$d\widetilde{W}_t^S = dW_t - \left(\frac{r + \sigma^2 - \mu}{\sigma}\right)dt$$

Therefore, under $\mathbb{Q}^S$, we have:

$$d\left(\frac{S_t}{B_t}\right) = (r + \sigma^2 - \mu)\frac{B_t}{S_t}dt - \sigma\frac{B_t}{S_t}dW_t$$

$$d\left(\frac{S_t}{B_t}\right) = (r + \sigma^2 - \mu)\frac{B_t}{S_t}dt - \sigma\frac{B_t}{S_t}\left(d\widetilde{W}_t^S + \left(\frac{r + \sigma^2 - \mu}{\sigma}\right)dt\right)$$

$$d\left(\frac{S_t}{B_t}\right) = (r + \sigma^2 - \mu)\frac{B_t}{S_t}dt - \sigma\frac{B_t}{S_t}d\widetilde{W}_t^S - (r + \sigma^2 - \mu)\frac{B_t}{S_t}dt$$

$$d\left(\frac{S_t}{B_t}\right) = -\sigma\frac{B_t}{S_t}d\widetilde{W}_t^S$$

Observe that the coefficient of the dt-term is zero so $\left\{\frac{B_t}{S_t}\right\}_{t\in[0,T]}$ is clearly a martingale under $\mathbb{Q}^S$. Under $\mathbb{Q}^S$, we also have:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

$$dS_t = \mu S_t dt + \sigma S_t\left(d\widetilde{W}_t^S + \left(\frac{r + \sigma^2 - \mu}{\sigma}\right)dt\right)$$

$$dS_t = \mu S_t dt + \sigma S_t d\widetilde{W}_t^S + (r + \sigma^2 - \mu)S_t dt$$

$$dS_t = (r + \sigma^2)S_t dt + \sigma S_t d\widetilde{W}_t^S$$

$$dB_t = rB_t dt, \qquad \text{(no change)}$$

b)

$$V_t = \mathbf{\Theta}_t \cdot \boldsymbol{X}_t$$

$$V_t = (a_t, b_t) \cdot (S_t, B_t)$$

$$V_t = a_t S_t + b_t B_t$$

Under $\mathbb{P}$, we have:

$$dV_t = a_t dS_t + b_t dB_t$$

Using Ito's product rule:

$$d\left(\frac{V_t}{S_t}\right) = V_t d\left(\frac{1}{S_t}\right) + \frac{1}{S_t} dV_t + (dV_t)\left(d\left(\frac{1}{S_t}\right)\right)$$

$$d\left(\frac{V_t}{S_t}\right) = (a_t S_t + b_t B_t)\left((\sigma^2 - \mu)\frac{1}{S_t}dt - \frac{\sigma}{S_t}dW_t\right) + \frac{1}{S_t}(a_t dS_t + b_t dB_t)$$
$$+ (a_t dS_t + b_t dB_t)\left((\sigma^2 - \mu)\frac{1}{S_t}dt - \frac{\sigma}{S_t}dW_t\right)$$

Let 1) $= (a_t S_t + b_t B_t)\left((\sigma^2 - \mu)\frac{1}{S_t}dt - \frac{\sigma}{S_t}dW_t\right)$, 2) $= \frac{1}{S_t}(a_t dS_t + b_t dB_t)$, 3) $= (a_t dS_t + b_t dB_t)\Big((\sigma^2 -$ $\mu)\frac{1}{S_t}dt - \frac{\sigma}{S_t}dW_t\Big)$. Evaluating 1) first:

$$(a_t S_t + b_t B_t)\left((\sigma^2 - \mu)\frac{1}{S_t}dt - \frac{\sigma}{S_t}dW_t\right)$$

$$= (\sigma^2 - \mu)a_t dt - \sigma a_t dW_t + (\sigma^2 - \mu)b_t \frac{B_t}{S_t}dt - \sigma b_t \frac{B_t}{S_t}dW_t$$

$$= (\sigma^2 - \mu)\left(a_t + b_t \frac{B_t}{S_t}\right)dt - \sigma\left(a_t + b_t \frac{B_t}{S_t}\right)dW_t$$

Now evaluating 2):

$$\frac{1}{S_t}(a_t dS_t + b_t dB_t)$$

$$= \frac{a_t}{S_t}dS_t + \frac{b_t}{S_t}dB_t$$

$$= \frac{a_t}{S_t}(\mu S_t dt + \sigma S_t dW_t) + \frac{b_t}{S_t}(r B_t dt)$$

$$= \mu a_t dt + \sigma a_t dW_t + r b_t \frac{B_t}{S_t}dt$$

$$= \left(\mu a_t + r b_t \frac{B_t}{S_t}\right)dt + \sigma a_t dW_t$$

Finally evaluating 3):

$$(a_t dS_t + b_t dB_t)\left((\sigma^2-\mu)\frac{1}{S_t}dt - \frac{\sigma}{S_t}dW_t\right)$$

$$= (\sigma^2-\mu)a_t\frac{1}{S_t}(dS_t)(dt) - \sigma a_t\frac{1}{S_t}(dS_t)(dW_t) + (\sigma^2-\mu)b_t\frac{1}{S_t}(dB_t)(dt) - \sigma b_t\frac{1}{S_t}(dB_t)(dW_t)$$

$$= 0 - \sigma a_t\frac{1}{S_t}(\mu S_t dt + \sigma S_t dW_t)(dW_t) + 0 - 0, \qquad \text{(by Ito rules)}$$

$$= -\sigma a_t\frac{1}{S_t}(\sigma S_t(dW_t)^2)$$

$$= -\sigma a_t\frac{1}{S_t}(\sigma S_t dt)$$

$$= -\sigma^2 a_t dt$$

Combining all three expressions for 1),2) and 3):

$$d\left(\frac{V_t}{S_t}\right) = (\sigma^2-\mu)\left(a_t + b_t\frac{B_t}{S_t}\right)dt - \sigma\left(a_t + b_t\frac{B_t}{S_t}\right)dW_t + \left(\mu a_t + r b_t\frac{B_t}{S_t}\right)dt + \sigma a_t dW_t - \sigma^2 a_t dt$$

$$d\left(\frac{V_t}{S_t}\right) = \left(\sigma^2 a_t + \sigma^2 b_t\frac{B_t}{S_t} - \mu a_t - \mu b_t\frac{B_t}{S_t} + \mu a_t + r b_t\frac{B_t}{S_t} - \sigma^2 a_t\right)dt + \left(-\sigma a_t - \sigma b_t\frac{B_t}{S_t} + \sigma a_t\right)dW_t$$

$$d\left(\frac{V_t}{S_t}\right) = (r+\sigma^2-\mu)b_t\frac{B_t}{S_t}dt - \sigma b_t\frac{B_t}{S_t}dW_t$$

$$d\left(\frac{V_t}{S_t}\right) = b_t\left((r+\sigma^2-\mu)\frac{B_t}{S_t}dt - \sigma\frac{B_t}{S_t}dW_t\right)$$

$$d\left(\frac{V_t}{S_t}\right) = b_t d\left(\frac{B_t}{S_t}\right)$$

Under $\mathbb{Q}^S$:

$$d\left(\frac{V_t}{S_t}\right) = b_t d\left(\frac{B_t}{S_t}\right)$$

$$d\left(\frac{V_t}{S_t}\right) = -\sigma b_t\frac{B_t}{S_t}d\widetilde{W}_t^S$$

$\rightarrow \left\{\frac{V_t}{S_t}\right\}_{t\in[0,T]}$ is also a $\mathbb{Q}^S$-martingale. Recall from **a)** that under $\mathbb{Q}^S$, we have:

$$dS_t = (r + \sigma^2)S_t dt + \sigma S_t d\widetilde{W}_t^S$$

$$dB_t = rB_t dt$$

$\therefore dV_t = a_t dS_t + b_t dB_t = \mathbf{\Theta}_t \cdot d\mathbf{X}_t^S$ still holds and hence the portfolio $\mathbf{\Theta}$ still self-finances under the $\mathbb{P} \rightarrow \mathbb{Q}^S$ measure change. Note that the proportional drift rate for the GBM $\{S_t\}_{t\in[0,T]}$ has changed from μ to $r + \sigma^2$ but volatility stays the same.

c)

Using the following facts:

- Fact 1 – by the Martingale Representation theorem, $\exists$ a replicating portfolio $\{\Theta_t\}_{t\in[0,T]}$ under the risk-neutral measure $\mathbb{Q}^S$ whose value process $\{V_t\}_{t\in[0,T]}$ replicates the financial contract payoff $\{C_t\}_{t\in[0,T]}$.
- Fact 2 – from **a)**, $\left\{\frac{B_t}{S_t}\right\}$ is a $\left(\mathbb{Q}^S, \{\mathcal{F}_t\}_{t\in[0,T]}\right)$-martingale so the S-normalised price of any portfolio of (S, B) is a $\left(\mathbb{Q}, \{\mathcal{F}_t\}_{t\in[0,T]}\right)$-martingale.
- Fact 3 – it can be shown that $\mathbb{P}$ and $\mathbb{Q}$ from Girsanov's theorem are equivalent.

Then $\exists$ a replicating portfolio under $\mathbb{Q}^S$ whose S-normalised value $\left\{\frac{V_t}{S_t}\right\}$ is a $\mathbb{Q}^S$-martingale:

$$\frac{C_t}{S_t} = \frac{V_t}{S_t}$$

$$\frac{C_t}{S_t} = \mathbb{E}_{\mathbb{Q}^S}\left[\frac{V_T}{S_T}\middle|\mathcal{F}_t\right]$$

$$\frac{C_t}{S_t} = \mathbb{E}_{\mathbb{Q}^S}\left[\frac{C_T}{S_T}\middle|\mathcal{F}_t\right]$$

$$\therefore C_t = S_t \mathbb{E}_{\mathbb{Q}^S}\left[\frac{C_T}{S_T}\middle|\mathcal{F}_t\right]$$

d)

$C_T = S_T \log(S_T)$

$\rightarrow \frac{C_T}{S_T} = \log(S_T)$

$\therefore C_0 = S_0 \mathbb{E}_{\mathbb{Q}^S}\left[\frac{C_T}{S_T}\middle|\mathcal{F}_0\right]$

$C_0 = S_0 \mathbb{E}_{\mathbb{Q}^S}[\log(S_T)\,|\mathcal{F}_0]$

Recall that under $\mathbb{Q}^S$:

$dS_t = (r + \sigma^2)S_t dt + \sigma S_t d\widetilde{W}_t^S$

So $\{S_t\}_{t\in[0,T]}$ is still a GBM under $\mathbb{Q}^S$ with drift rate proportional to $r + \sigma^2$ and the same volatility. Hence, under $\mathbb{Q}^S$ and conditional on $\mathcal{F}_0$, we have:

$\log(S_T) \sim N\left(\log(S_0) + \left(r + \frac{\sigma^2}{2}\right)T, \sigma^2 T\right)$

So $\mathbb{E}_{\mathbb{Q}^S}[\log(S_T)\,|\mathcal{F}_0]$ is simply the mean of the normally distributed random variable $\log(S_T)$:

$\rightarrow C_0 = S_0 \mathbb{E}_{\mathbb{Q}^S}[\log(S_T)\,|\mathcal{F}_0]$

$C_0 = \left(\log(S_0) + \left(r + \frac{\sigma^2}{2}\right)T\right)S_0$