MATH7049 – Assignment 3
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1.

a)

Model problem:

$$\frac{\partial U}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 U}{\partial x^2}$$

$$\frac{\partial U}{\partial \tau} - \frac{1}{2}\sigma^2 \frac{\partial^2 U}{\partial x^2} = 0$$

Discretising the model problem according to the Crank-Nicolson scheme:

$$\frac{U_{n+1}^j - U_n^j}{\Delta\tau} + \frac{1}{2}\left[-\frac{1}{2}\sigma^2 \frac{U_n^{j+1} - 2U_n^j + U_n^{j-1}}{(\Delta x)^2}\right] + \frac{1}{2}\left[-\frac{1}{2}\sigma^2 \frac{U_{n+1}^{j+1} - 2U_{n+1}^j + U_{n+1}^{j-1}}{(\Delta x)^2}\right] = 0$$

Use $U_n^j = \lambda_k^n e^{\frac{i\pi kj}{J}}, k = -J, \ldots, J, i = \sqrt{-1}, k =$ wave number, $\lambda_k =$ amplification factor:

$$\therefore \frac{\lambda_k^{n+1} e^{\frac{i\pi kj}{J}} - \lambda_k^n e^{\frac{i\pi kj}{J}}}{\Delta\tau} - \frac{1}{4}\sigma^2 \frac{\lambda_k^n e^{\frac{i\pi k(j+1)}{J}} - 2\lambda_k^n e^{\frac{i\pi kj}{J}} + \lambda_k^n e^{\frac{i\pi k(j-1)}{J}}}{(\Delta x)^2}$$
$$- \frac{1}{4}\sigma^2 \frac{\lambda_k^{n+1} e^{\frac{i\pi k(j+1)}{J}} - 2\lambda_k^{n+1} e^{\frac{i\pi kj}{J}} + \lambda_k^{n+1} e^{\frac{i\pi k(j-1)}{J}}}{(\Delta x)^2} = 0$$

Factoring out $\lambda_k^n e^{\frac{i\pi kj}{J}}$:

$$\lambda_k^n e^{\frac{i\pi kj}{J}} \left(\frac{\lambda_k - 1}{\Delta\tau}\right) - \frac{1}{4}\sigma^2 \lambda_k^n e^{\frac{i\pi kj}{J}} \left(\frac{e^{\frac{i\pi k}{J}} - 2 + e^{-\frac{i\pi k}{J}}}{(\Delta x)^2}\right) - \frac{1}{4}\sigma^2 \lambda_k^n e^{\frac{i\pi kj}{J}} \left(\frac{\lambda_k e^{\frac{i\pi k}{J}} - 2\lambda_k + \lambda_k e^{-\frac{i\pi k}{J}}}{(\Delta x)^2}\right) = 0$$

$$\left(\frac{\lambda_k - 1}{\Delta\tau}\right) - \frac{1}{4}\sigma^2 \left(\frac{e^{\frac{i\pi k}{J}} - 2 + e^{-\frac{i\pi k}{J}}}{(\Delta x)^2}\right) - \frac{1}{4}\sigma^2 \lambda_k \left(\frac{e^{\frac{i\pi k}{J}} - 2 + e^{-\frac{i\pi k}{J}}}{(\Delta x)^2}\right) = 0$$

Solving for λ_k:

$$\frac{\lambda_k}{\Delta\tau} - \frac{1}{4}\sigma^2 \lambda_k \left(\frac{e^{\frac{i\pi k}{J}} - 2 + e^{-\frac{i\pi k}{J}}}{(\Delta x)^2}\right) = \frac{1}{\Delta\tau} + \frac{1}{4}\sigma^2 \left(\frac{e^{\frac{i\pi k}{J}} - 2 + e^{-\frac{i\pi k}{J}}}{(\Delta x)^2}\right)$$

$$\lambda_k \left(\frac{1}{\Delta\tau} - \frac{1}{4}\sigma^2 \left(\frac{e^{\frac{i\pi k}{J}} - 2 + e^{-\frac{i\pi k}{J}}}{(\Delta x)^2}\right)\right) = \frac{1}{\Delta\tau} + \frac{1}{4}\sigma^2 \left(\frac{e^{\frac{i\pi k}{J}} - 2 + e^{-\frac{i\pi k}{J}}}{(\Delta x)^2}\right)$$

$$\lambda_k = \frac{\frac{1}{\Delta\tau} + \frac{1}{4}\sigma^2 \left(\frac{e^{\frac{i\pi k}{J}} - 2 + e^{-\frac{i\pi k}{J}}}{(\Delta x)^2}\right)}{\frac{1}{\Delta\tau} - \frac{1}{4}\sigma^2 \left(\frac{e^{\frac{i\pi k}{J}} - 2 + e^{-\frac{i\pi k}{J}}}{(\Delta x)^2}\right)}$$

Multiplying numerator and denominator by $\Delta\tau$ and using $\alpha = \frac{1}{2}\sigma^2\frac{\Delta\tau}{(\Delta x)^2}$:

$$\lambda_k = \frac{1 + \frac{1}{4}\sigma^2\frac{\Delta\tau}{(\Delta x)^2}\left(e^{\frac{i\pi k}{J}} - 2 + e^{-\frac{i\pi k}{J}}\right)}{1 - \frac{1}{4}\sigma^2\frac{\Delta\tau}{(\Delta x)^2}\left(e^{\frac{i\pi k}{J}} - 2 + e^{-\frac{i\pi k}{J}}\right)}$$

$$\lambda_k = \frac{1 + \frac{1}{2}\alpha\left(e^{\frac{i\pi k}{J}} - 2 + e^{-\frac{i\pi k}{J}}\right)}{1 - \frac{1}{2}\alpha\left(e^{\frac{i\pi k}{J}} - 2 + e^{-\frac{i\pi k}{J}}\right)}$$

Using Euler's formula for $\cos(\theta) = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right), \theta = \frac{\pi k}{J}$:

$$\lambda_k = \frac{1 + \frac{1}{2}\alpha\left(2\cos\left(\frac{\pi k}{J}\right) - 2\right)}{1 - \frac{1}{2}\alpha\left(2\cos\left(\frac{\pi k}{J}\right) - 2\right)}$$

$$\lambda_k = \frac{1 + \alpha\left(\cos\left(\frac{\pi k}{J}\right) - 1\right)}{1 - \alpha\left(\cos\left(\frac{\pi k}{J}\right) - 1\right)}$$

Using double angle formula for $\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}, 2\theta = \frac{\pi k}{J} \rightarrow \theta = \frac{\pi k}{2J}$:

$$\lambda_k = \frac{1 + \alpha\left(-2\sin^2\left(\frac{\pi k}{2J}\right)\right)}{1 - \alpha\left(-2\sin^2\left(\frac{\pi k}{2J}\right)\right)}$$

$$\therefore \lambda_k^{C-N} = \frac{1 - 2\alpha\sin^2\left(\frac{\pi k}{2J}\right)}{1 + 2\alpha\sin^2\left(\frac{\pi k}{2J}\right)}$$

Since $\Delta\tau, (\Delta x)^2, \sigma^2 > 0 \rightarrow \alpha > 0$ and $-1 \leq \sin\left(\frac{\pi k}{2J}\right) \leq 1 \rightarrow 0 \leq \sin^2\left(\frac{\pi k}{2J}\right) \leq 1$:

$$\left|1 - 2\alpha\sin^2\left(\frac{\pi k}{2J}\right)\right| \leq \left|1 + 2\alpha\sin^2\left(\frac{\pi k}{2J}\right)\right|$$

$$\frac{\left|1 - 2\alpha\sin^2\left(\frac{\pi k}{2J}\right)\right|}{\left|1 + 2\alpha\sin^2\left(\frac{\pi k}{2J}\right)\right|} \leq 1$$

Using $\frac{|z_1|}{|z_2|} = \left|\frac{z_1}{z_2}\right|$:

$$\left|\frac{1 - 2\alpha\sin^2\left(\frac{\pi k}{2J}\right)}{1 + 2\alpha\sin^2\left(\frac{\pi k}{2J}\right)}\right| \leq 1$$

$$\therefore \left|\lambda_k^{C-N}\right| \leq 1$$

b)

$$\alpha = \frac{1}{2}\sigma^2 \frac{\Delta\tau}{(\Delta x)^2}$$

$$\lambda_k^{C-N} = \frac{1 - 2\alpha \sin^2\left(\frac{\pi k}{2J}\right)}{1 + 2\alpha \sin^2\left(\frac{\pi k}{2J}\right)}$$

$$\lambda_k^{Imp} = \frac{1}{1 + 4\alpha \sin^2\left(\frac{\pi k}{2J}\right)}$$

In the Crank-Nicolson scheme, for high-frequency oscillations, as $\alpha \to \infty$:

$$1 - 2\alpha \sin^2\left(\frac{\pi k}{2J}\right) \to -\infty$$

$$1 + 2\alpha \sin^2\left(\frac{\pi k}{2J}\right) \to \infty$$

$$\therefore \frac{1 - 2\alpha \sin^2\left(\frac{\pi k}{2J}\right)}{1 + 2\alpha \sin^2\left(\frac{\pi k}{2J}\right)} \to -\frac{\infty}{\infty}$$

Since we have an indeterminate form of type $\frac{\infty}{\infty}$, we need to apply l'Hopital's rule to evaluate this expression as $\alpha \to \infty$. Let:

$$f(\alpha) = 1 - 2\alpha \sin^2\left(\frac{\pi k}{2J}\right)$$

$$g(\alpha) = 1 + 2\alpha \sin^2\left(\frac{\pi k}{2J}\right)$$

$$f'(\alpha) = \frac{\partial f}{\partial \alpha} = -2\sin^2\left(\frac{\pi k}{2J}\right)$$

$$g'(\alpha) = \frac{\partial g}{\partial \alpha} = 2\sin^2\left(\frac{\pi k}{2J}\right)$$

Applying l'Hopital's rule:

$$\lim_{\alpha \to \infty} \frac{f(\alpha)}{g(\alpha)} = \lim_{\alpha \to \infty} \frac{f'(\alpha)}{g'(\alpha)} = \frac{-2\sin^2\left(\frac{\pi k}{2J}\right)}{2\sin^2\left(\frac{\pi k}{2J}\right)} = -1$$

$$\therefore \lambda_k^{C-N} \to -1$$

$$\left|\lambda_k^{C-N}\right| \to 1$$

In the fully implicit scheme, for high-frequency oscillations, as $\alpha \to \infty$:

$$1 + 4\alpha \sin^2\left(\frac{\pi k}{2J}\right) \to \infty$$

$$\frac{1}{1 + 4\alpha \sin^2\left(\frac{\pi k}{2J}\right)} \to 0$$

$$\therefore \lambda_k^{Imp} \to 0$$

$$\left|\lambda_k^{Imp}\right| \to 0$$

Note that high-frequency oscillations above refer to when $k \approx \pm J \to \sin^2\left(\frac{\pi k}{2J}\right) \approx 1$. Evidently, from the above asymptotic analysis, the fully implicit method does a better job of damping high frequency ($k \approx \pm J$) oscillations than the Crank-Nicolson method because $\left|\lambda_k^{Imp}\right| \to 0$ whereas $\left|\lambda_k^{C-N}\right| \to 1$ for large α.

2.

a)

Use linearity boundary conditions instead of Dirichlet boundary conditions because the contract features for an asset-or-nothing call option are more complex relative to a standard European call option which can be solved via the B-S formula to the B-S PDE. Note that this numerical technique works well when the truncated computational domain ($S_{max} - S_{min}$) is large enough. If $S_{max} = S_J$ and $S_{min} = S_{-J}$, then for all $t_n < T$ (not at maturity):

$$C(S_J, t_n) = 2C(S_{J-1}, t_n) - C(S_{J-2}, t_n)$$

$$C(S_{-J}, t_n) = 2C(S_{-J+1}, t_n) - C(S_{-J+2}, t_n)$$

b)

$\boldsymbol{N}$	$\boldsymbol{M = 2J}$	$\boldsymbol{C_{N,M}^{CN}(S_0, t_0)}$	$\boldsymbol{C(S_0, t_0) - C_{N,M}^{CN}(S_0, t_0)}$	**Ratios**
5	100	61.085558	4.957e-02	
10	200	66.507292	-5.372e+00	-0.009
20	400	65.755085	-4.620e+00	1.163
40	800	65.374177	-4.239e+00	1.090
80	1600	65.182540	-4.047e+00	1.047
160	3200	65.086429	-3.951e+00	1.024
True price: $\boldsymbol{C(S_0, t_0) = 61.1351295}$				

Table 1: Convergence test for asset-or-nothing call option using Crank Nicolson finite different scheme.

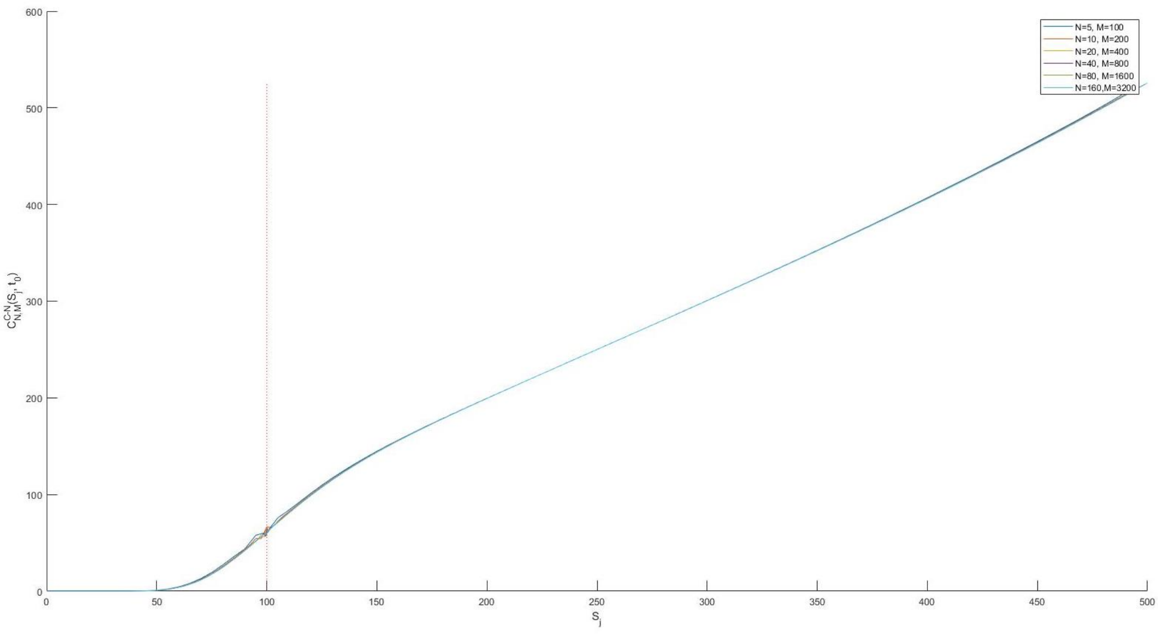

Figure 1: Plot of $C^{CN}_{N,M}(S_j, t_0), j = 0, \dots, M$ for each (N, M) pair in Table 1.

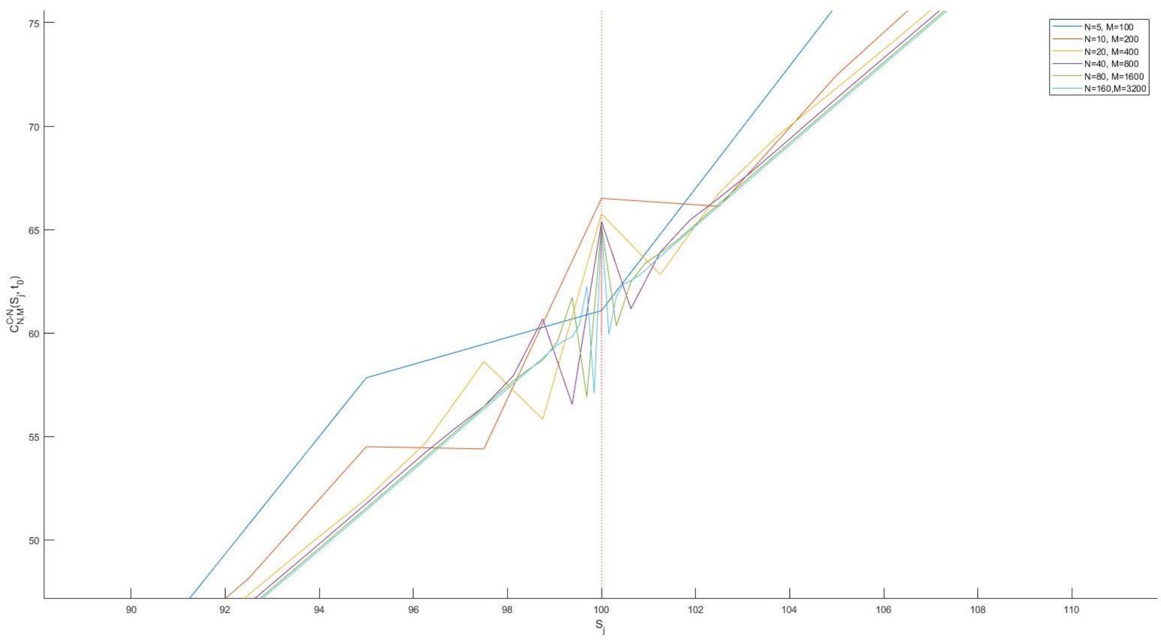

Figure 2: Zoomed in plot around strike price K of $C^{CN}_{N,M}(S_j, t_0), j = 0, \dots, M$ for each (N, M) pair in Table 1.

From table 1 above, it is evident that the Crank-Nicolson scheme used to price the asset-or-nothing call option is not converging because the Ratios column consists predominantly of 1s. This suggests that there is neither second-order convergence in time nor in space which is contradictory to what is expected when using this scheme.

From Q1.b), we know that a fully implicit scheme is preferred to a Crank-Nicolson scheme for large values of α. In the workspace section of $cn\_fm.m$, this is indeed true as we can see that the $alpha$ variable which stores different α for different (N, M) pairs increases from 3.7134 to 39.2787, so $|\lambda_k^{C-N}| \to 1$ whereas $|\lambda_k^{Imp}| \to 0$. While stability still holds in that the results don't blow up because neither $\lambda_k > 1$, the result fails to converge simply because there aren't enough time steps for the Crank-Nicolson scheme to display convergence. Indeed, if we instead try setting $N\_list = [50,100,200, \dots ,1600] = [50 .* 2 .\hat{} [0{:}5]]$ in $cn\_fm.m$ and use N from here, the updated table would show some convergence since the Ratios column would change to 2s (simply change N_list in the code to see for yourself). Note that this still isn't strictly second-order in both time and space (would expect 4s instead). This also explains the abrupt behaviour seen around the point of discontinuity at $S_j = K = 100$ in figure 2.

c)

N	$M = 2J$	$C_{N,M}^{Imp}(S_0, t_0)$	$C(S_0, t_0) - C_{N,M}^{Imp}(S_0, t_0)$	Ratios
5	100	64.555562	-3.420e+00	
10	200	62.756504	-1.621e+00	2.110
20	400	61.926614	-7.915e-01	2.049
40	800	61.526364	-3.912e-01	2.023
80	1600	61.329653	-1.945e-01	2.011
160	3200	61.232122	-9.699e-02	2.006
True price: $C(S_0, t_0) = $ **61.1351295**				

Table 2: Convergence test for asset-or-nothing call option using fully implicit finite different scheme.

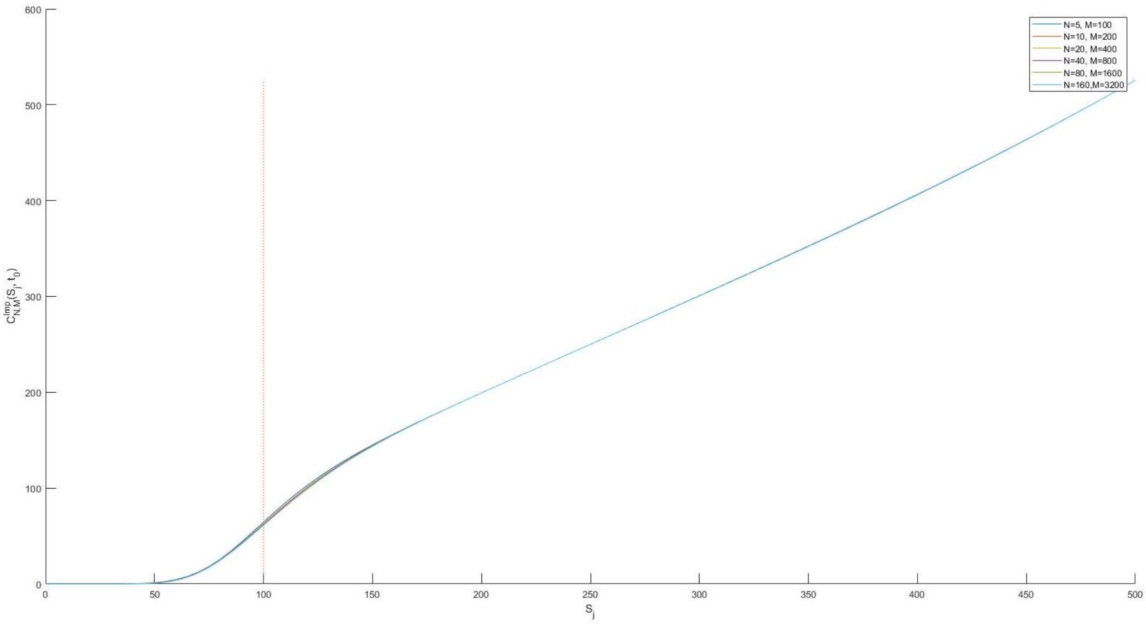

Figure 3: Plot of $C_{N,M}^{Imp}(S_j, t_0), j = 0, \dots, M$ for each (N, M) pair in Table 2.

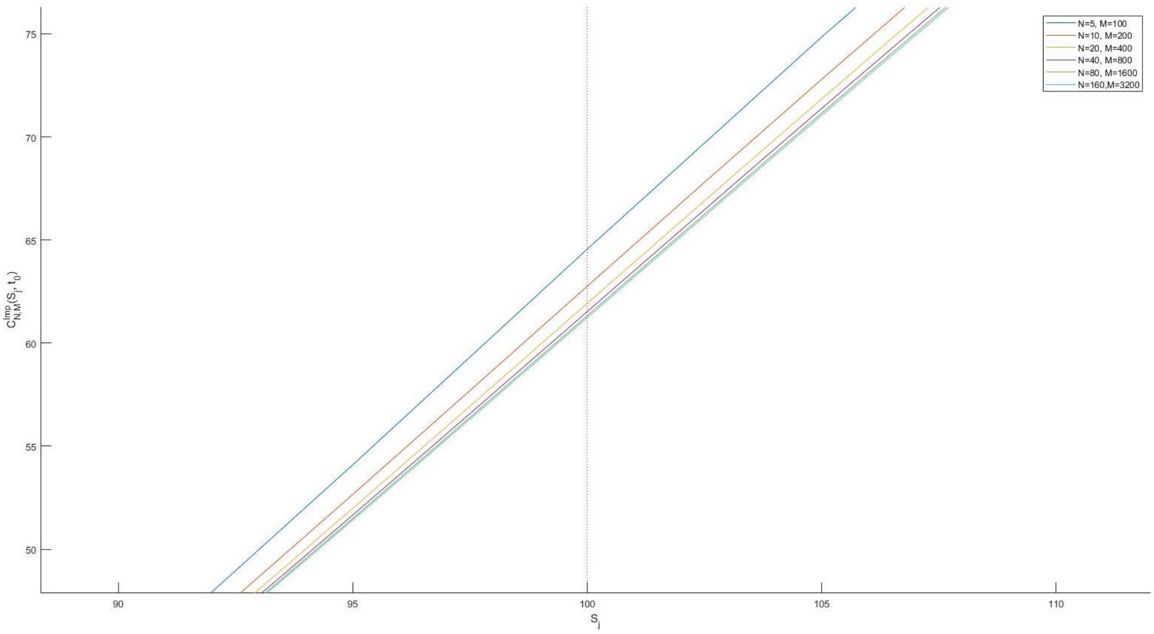

Figure 4: Zoomed in plot around strike price K of $C_{N,M}^{Imp}(S_j, t_0), j = 0, \dots, M$ for each (N, M) pair in Table 2.

Unlike Q2.b), using the fully implicit scheme to price the asset-or-nothing call option yields first-order convergence in time and second-order convergence in space as expected, evident from the approximate 2s seen in the Ratios column in table 2. Following from the argument made earlier in a), this scheme is known to better handle the dampening of high-frequency oscillations. We observe a smooth pattern for each of the different $C_{N,M}^{Imp}(S_j, t_0)$ around the point of discontinuity at $S_j = K = 100$ because the fully-implicit scheme converges faster with a smaller number of required intermediate time steps N. Even though this statement is contradictory to the what should be the faster rate of convergence of the Crank-Nicolson scheme $O(\Delta\tau)^2 + O(\Delta S)^2$, it is due to the special case of having large α or $\Delta\tau \gg \frac{\Delta x^2}{\sigma^2}$.

d)

N	$M = 2J$	$C_{N,M}^{Ran}(S_0, t_0)$	$C(S_0, t_0) - C_{N,M}^{Ran}(S_0, t_0)$	Ratios
5	100	64.555562	-3.420e+00	
10	200	66.507292	-5.372e+00	0.637
20	400	65.755085	-4.620e+00	1.163
40	800	65.374177	-4.239e+00	1.090
80	1600	65.182540	-4.047e+00	1.047
160	3200	65.086429	-3.951e+00	1.024
True price: $C(S_0, t_0) = 61.1351295$				

Table 3: Convergence test for asset-or-nothing call option using Rannacher smoothing.

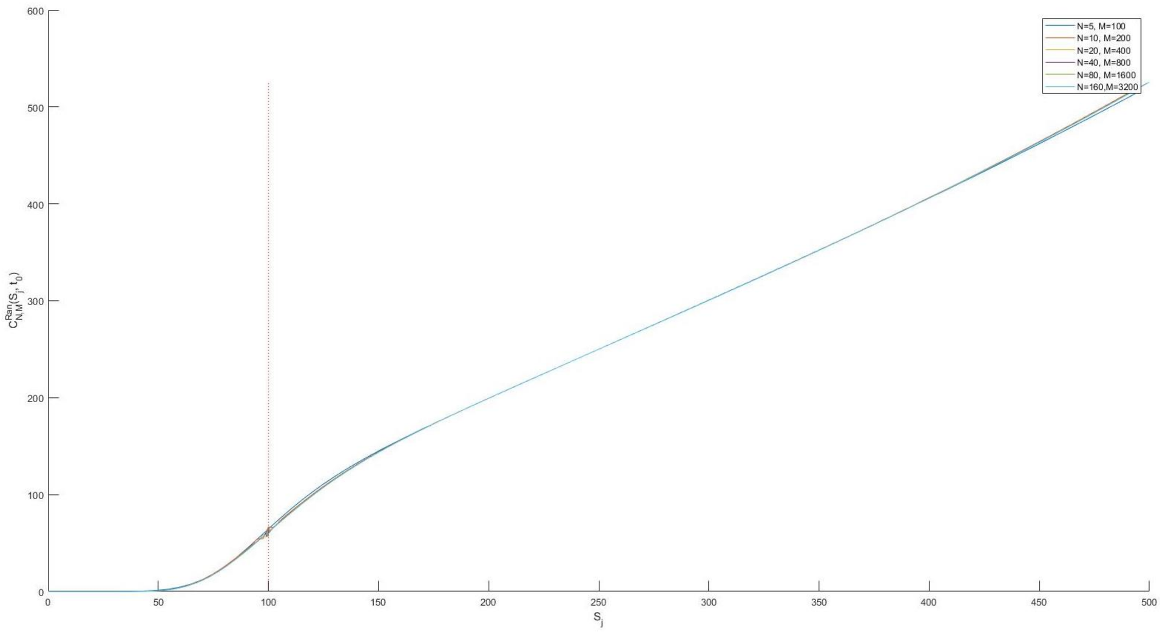

Figure 5: Plot of $C_{N,M}^{Ran}(S_j, t_0), j = 0, \ldots, M$ for each (N, M) pair in Table 3.

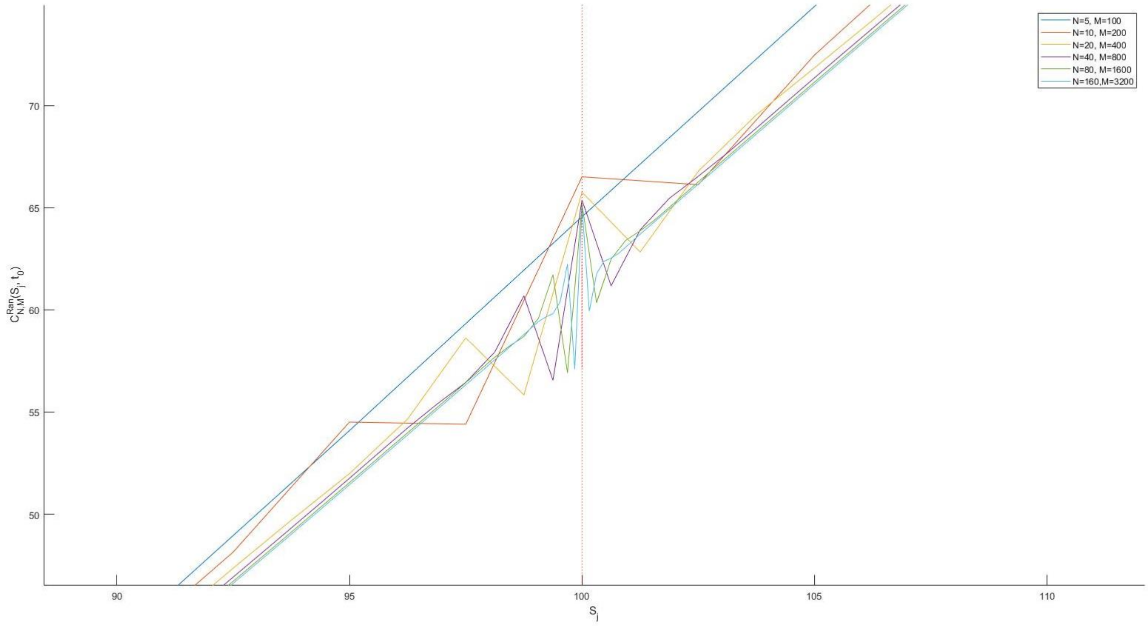

Figure 6: Zoomed in plot around strike price K of $C^{Ran}_{N,M}(S_j, t_0), j = 0, \dots, M$ for each (N, M) pair in Table 3.

The table and figure results after using Rannacher smoothing are almost identical to those from using solely the Crank-Nicolson scheme for pricing the same asset-or-nothing call option. There is no convergence due to the 1s in the Ratios column. The same abrupt pattern is observed in figure 6 around the point of discontinuity at $S_j = K = 100$ for most $C^{Ran}_{N,M}(S_j, t_0)$ except for the pair $(N, M) = (5,100)$ which is clearly quite smooth. Since the implicit scheme is only run for the first 3 time steps before expiration, in this case, the Crank-Nicolson scheme only runs for 2 additional steps and so the final vector u is not changed much leading to this result. However, when the Crank-Nicolson scheme is let to run for more time steps i.e. $N > 5$, because there aren't a sufficient number of time steps for the scheme to show convergence and the Crank-Nicolson scheme dominates the changes made to the payoff vector u, the abrupt pattern emerges once again. Indeed, if we again try setting $N\_list = [50,100,200, \dots ,1600] = [50 .* 2 .\hat{} [0:5]]$ in $rannacher.m$ and use N from here, the updated table would show some convergence since the Ratios column would change to 2s (simply change N_list in the code to see for yourself).

3.

a)

M	Value ($\widehat{C}_M$)	Radius ($\Phi^{-1}\left(1 - \frac{p}{2}\right)\frac{\widehat{\sigma}_M}{\sqrt{M}}$)	Ratios
2000	61.822390	2.755455	
4000	61.603049	1.965120	1.402
8000	60.701831	1.387511	1.416
16000	61.800949	0.974788	1.423
32000	61.413283	0.692086	1.408
64000	61.133881	0.489985	1.412
128000	61.144331	0.346690	1.413

Table 4: Convergence test for asset-or-nothing call option using ordinary Monte Carlo.

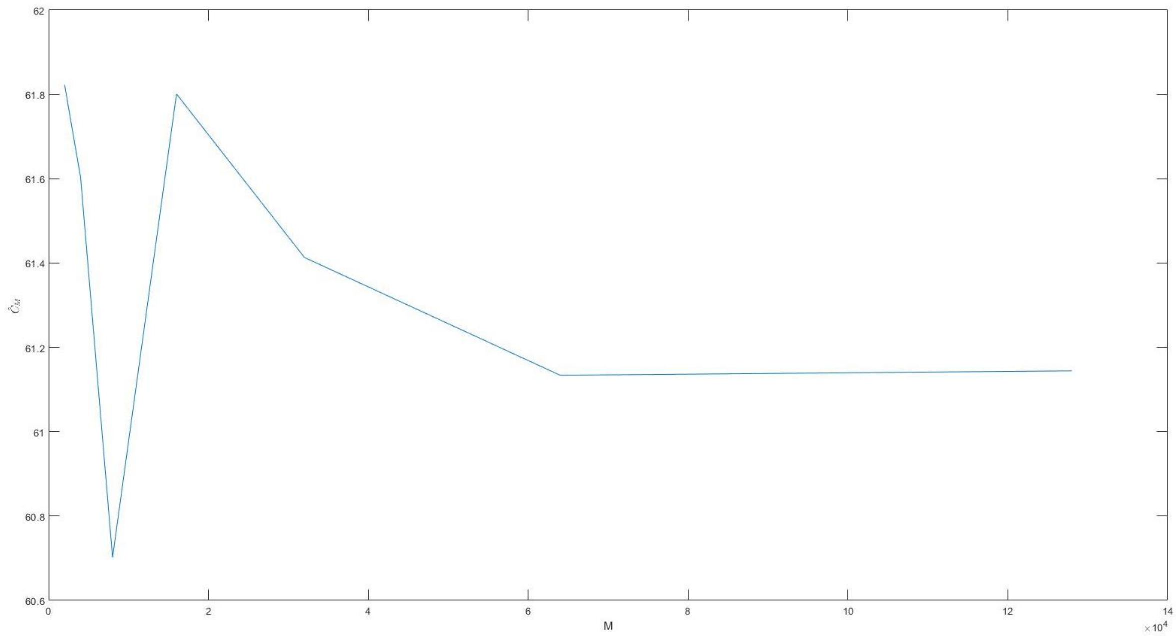

Figure 7: Plot of $\hat{C}_M$ against different M.

From figure 7, it is clear that as $M \uparrow, \hat{C}_M \to \mathbb{E}[\hat{C}_M] = C = 61.1351295$. In fact, this is a numerical proof of the Strong Law of Large Numbers which states that $\mathbb{P}\left(\lim_{M\to\infty} \hat{C}_M = C\right) = 1$ almost surely. From table 4, it is also clear that the confidence interval radius is decreasing i.e. we are becoming increasingly more confident in our estimate of $\hat{C}_M$ as M gets larger. Finally, we note the half-order convergence ($\sqrt{2} \approx 1.414$) of this method as seen in the Ratio column.

b)

For step 1, in order to estimate how large M should be, we use the formula:

$$radius = \Phi^{-1}\left(1 - \frac{p}{2}\right)\frac{\hat{\sigma}_M}{\sqrt{M}}$$

$$0.1 = \Phi^{-1}\left(1 - \frac{0.05}{2}\right)\frac{\hat{\sigma}_M}{\sqrt{M}}$$

$$\to \widehat{M} = \left(\Phi^{-1}(0.975)\frac{\hat{\sigma}_M}{0.1}\right)^2$$

Note that this requires knowing $\hat{\sigma}_M$ in advance. From a), we know that $M = 128000$ is a safe value because its $radius = 0.3467 \gg 0.1$. So, following from L5.11, we can use the stored $\hat{\sigma}_{128000} \approx 63.1912$ as a close enough estimate for $\hat{\sigma}_M$ for much larger M.

$$\therefore \widehat{M} = \left(\Phi^{-1}(0.975)\frac{63.1912}{0.1}\right)^2 \approx 1539415$$

Estimated M	**Value ($\hat{C}_M$)**	**Confidence Interval**	**Radius ($\Phi^{-1}\left(1-\frac{p}{2}\right)\frac{\hat{\sigma}_M}{\sqrt{M}}$)**
1539415	61.151653	[61.051706, 61.251599]	0.099946

Table 5: Convergence test for asset-or-nothing call option using ordinary Monte Carlo with pilot computation to estimate M.

$$|\hat{C}_{1539415} - C| = |61.15163 - 61.1351295| = 0.0165 > 0.0092 = |61.144331 - 61.1351295| = |\hat{C}_{128000} - C|$$

Although the error between the estimate and the exact price is larger, because the radius is smaller i.e. $0.0999 < 0.3467$, we are more confident in this estimate of $\hat{C}_M$ at the 95% confidence level than its previous estimates with smaller M.