MATH7091 – Assignment 3
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1.

a)

$$I_t^{(n)} = \sum_{i=0}^{n-1} X_{t_i}\left(W_{t_{i+1}} - W_{t_i}\right)$$

$$\mathbb{E}\left[I_t^{(n)}\right] = \mathbb{E}\left[\sum_{i=0}^{n-1} X_{t_i}\left(W_{t_{i+1}} - W_{t_i}\right)\right]$$

$$\mathbb{E}\left[I_t^{(n)}\right] = \sum_{i=0}^{n-1} \mathbb{E}\left[X_{t_i}\left(W_{t_{i+1}} - W_{t_i}\right)\right], \qquad \text{(linearity)}$$

$$\mathbb{E}\left[I_t^{(n)}\right] = \sum_{i=0}^{n-1} \mathbb{E}\left[\mathbb{E}\left[X_{t_i}\left(W_{t_{i+1}} - W_{t_i}\right)\middle|\mathcal{F}_{t_i}\right]\middle|\mathcal{F}_{t_0}\right], \qquad \text{(iterated conditioning)}$$

$$\mathbb{E}\left[I_t^{(n)}\right] = \sum_{i=0}^{n-1} \mathbb{E}\left[X_{t_i}\mathbb{E}\left[W_{t_{i+1}} - W_{t_i}\middle|\mathcal{F}_{t_i}\right]\right], \qquad \left(X_{t_i} \text{ is } \mathcal{F}_{t_i}\text{-measurable}, \mathcal{F}_{t_0} \text{ is trivial } \sigma\text{-algebra}\right)$$

$$\mathbb{E}\left[I_t^{(n)}\right] = \sum_{i=0}^{n-1} \mathbb{E}\left[X_{t_i}\mathbb{E}\left[W_{t_{i+1}} - W_{t_i}\right]\right], \qquad \text{(independent increments)}$$

$$\mathbb{E}\left[I_t^{(n)}\right] = \sum_{i=0}^{n-1} \mathbb{E}\left[X_{t_i} \cdot 0\right], \qquad \left(W_{t_{i+1}} - W_{t_i} \sim N(0, t_{i+1} - t_i)\right)$$

$$\mathbb{E}\left[I_t^{(n)}\right] = \sum_{i=0}^{n-1} 0 = 0$$

$$\rightarrow \mathbb{E}\left[I_t^{(n)}\right]^2 = 0$$

$$\mathbb{E}\left[\left(I_t^{(n)}\right)^2\right] = \mathbb{E}\left[\left(\sum_{i=0}^{n-1} X_{t_i}\left(W_{t_{i+1}} - W_{t_i}\right)\right)^2\right]$$

$$\mathbb{E}\left[\left(I_t^{(n)}\right)^2\right] = \mathbb{E}\left[\sum_{i=0}^{n-1}\left(X_{t_i}\left(W_{t_{i+1}} - W_{t_i}\right)\right)^2 + \sum_{i=0}^{n-1}\sum_{j=0}^{i-1} X_{t_i}\left(W_{t_{i+1}} - W_{t_i}\right) \cdot X_{t_j}\left(W_{t_{j+1}} - W_{t_j}\right)\right]$$

$$\mathbb{E}\left[\left(I_t^{(n)}\right)^2\right] = \sum_{i=0}^{n-1}\mathbb{E}\left[\left(X_{t_i}\left(W_{t_{i+1}} - W_{t_i}\right)\right)^2\right] + \sum_{i=0}^{n-1}\sum_{j=0}^{i-1}\mathbb{E}\left[X_{t_i}\left(W_{t_{i+1}} - W_{t_i}\right) \cdot X_{t_j}\left(W_{t_{j+1}} - W_{t_j}\right)\right], \qquad \text{(linearity)}$$

Evaluating the first expression in $\mathbb{E}\left[\left(I_t^{(n)}\right)^2\right]$:

$$\sum_{i=0}^{n-1}\mathbb{E}\left[\left(X_{t_i}\left(W_{t_{i+1}} - W_{t_i}\right)\right)^2\right] = \sum_{i=0}^{n-1}\mathbb{E}\left[X_{t_i}^2\left(W_{t_{i+1}} - W_{t_i}\right)^2\right]$$

$$\sum_{i=0}^{n-1}\mathbb{E}\left[\left(X_{t_i}\left(W_{t_{i+1}} - W_{t_i}\right)\right)^2\right] = \sum_{i=0}^{n-1}\mathbb{E}\left[\mathbb{E}\left[X_{t_i}^2\left(W_{t_{i+1}} - W_{t_i}\right)^2\middle|\mathcal{F}_{t_i}\right]|\mathcal{F}_{t_0}\right], \qquad \text{(iterated conditioning)}$$

$$\sum_{i=0}^{n-1}\mathbb{E}\left[\left(X_{t_i}\left(W_{t_{i+1}} - W_{t_i}\right)\right)^2\right] = \sum_{i=0}^{n-1}\mathbb{E}\left[X_{t_i}^2\mathbb{E}\left[\left(W_{t_{i+1}} - W_{t_i}\right)^2\middle|\mathcal{F}_{t_i}\right]\right],$$

$$\left(X_{t_i}^2 \text{ is } \mathcal{F}_{t_i}\text{-measurable}, \mathcal{F}_{t_0} \text{ is trivial } \sigma\text{-algebra}\right)$$

$$\sum_{i=0}^{n-1}\mathbb{E}\left[\left(X_{t_i}\left(W_{t_{i+1}} - W_{t_i}\right)\right)^2\right] = \sum_{i=0}^{n-1}\mathbb{E}\left[X_{t_i}^2\mathbb{E}\left[\left(W_{t_{i+1}} - W_{t_i}\right)^2\right]\right], \qquad \text{(independent increments)}$$

$$\sum_{i=0}^{n-1}\mathbb{E}\left[\left(X_{t_i}\left(W_{t_{i+1}} - W_{t_i}\right)\right)^2\right] = \sum_{i=0}^{n-1}\mathbb{E}\left[X_{t_i}^2 \cdot (t_{i+1} - t_i)\right], \qquad \left(W_{t_{i+1}} - W_{t_i} \sim N(0, t_{i+1} - t_i)\right)$$

$$\sum_{i=0}^{n-1}\mathbb{E}\left[\left(X_{t_i}\left(W_{t_{i+1}} - W_{t_i}\right)\right)^2\right] = \mathbb{E}\left[\sum_{i=0}^{n-1} X_{t_i}^2 \cdot (t_{i+1} - t_i)\right], \qquad \text{(linearity)}$$

$$\sum_{i=0}^{n-1}\mathbb{E}\left[\left(X_{t_i}\left(W_{t_{i+1}} - W_{t_i}\right)\right)^2\right] = \mathbb{E}\left[\int_0^t X_s^2\, ds\right], \qquad \text{(by definition of the Riemann integral)}$$

Evaluating the second expression in $\mathbb{E}\left[\left(I_t^{(n)}\right)^2\right]$:

$$\sum_{i=0}^{n-1}\sum_{j=0}^{i-1}\mathbb{E}\left[X_{t_i}\left(W_{t_{i+1}} - W_{t_i}\right) \cdot X_{t_j}\left(W_{t_{j+1}} - W_{t_j}\right)\right]$$

$$= \sum_{i=0}^{n-1}\sum_{j=0}^{i-1} \mathbb{E}\left[\mathbb{E}\left[X_{t_i}(W_{t_{i+1}} - W_{t_i}) \cdot X_{t_j}\left(W_{t_{j+1}} - W_{t_j}\right)\middle|\mathcal{F}_{t_i}\right]|\mathcal{F}_{t_0}\right], \qquad \text{(iterated conditioning)}$$

$$= \sum_{i=0}^{n-1}\sum_{j=0}^{i-1} \mathbb{E}\left[X_{t_i}X_{t_j}\left(W_{t_{j+1}} - W_{t_j}\right)\mathbb{E}\left[(W_{t_{i+1}} - W_{t_i})|\mathcal{F}_{t_i}\right]\right],$$

$$\left(\text{all terms taken out are } \mathcal{F}_{t_i} - \text{measurable}, \mathcal{F}_{t_0} \text{ is trivial } \sigma\text{-algebra}\right)$$

$$= \sum_{i=0}^{n-1}\sum_{j=0}^{i-1} \mathbb{E}\left[X_{t_i}X_{t_j}\left(W_{t_{j+1}} - W_{t_j}\right)\mathbb{E}\left[(W_{t_{i+1}} - W_{t_i})\right]\right], \qquad \text{(independent increments)}$$

$$= \sum_{i=0}^{n-1}\sum_{j=0}^{i-1} \mathbb{E}\left[X_{t_i}X_{t_j}\left(W_{t_{j+1}} - W_{t_j}\right)\cdot 0\right], \qquad \left(W_{t_{i+1}} - W_{t_i} \sim N(0, t_{i+1} - t_i)\right)$$

$$= \sum_{i=0}^{n-1}\sum_{j=0}^{i-1} 0 = 0$$

$$\therefore \mathbb{E}\left[\left(I_t^{(n)}\right)^2\right] = \mathbb{E}\left[\int_0^t X_s^2\, ds\right] + 0 = \mathbb{E}\left[\int_0^t X_s^2\, ds\right]$$

$$\to Var\left(I_t^{(n)}\right) = \mathbb{E}\left[\left(I_t^{(n)}\right)^2\right] - \mathbb{E}\left[I_t^{(n)}\right]^2$$

$$Var\left(I_t^{(n)}\right) = \mathbb{E}\left[\int_0^t X_s^2\, ds\right]$$

b)

The assertion that the random variable $I_t^{(n)}$ is normally distributed is false. For a counterexample, let $\{X_t\}_{t\in[0,T]} = \{W_t\}_{t\in[0,T]} \in H_T^2$ so that $I_t^{(n)} = \sum_{i=0}^{n-1} W_{t_i}(W_{t_{i+1}} - W_{t_i})$:

$$W_{t_i} \sim N(0, t_i) \to \frac{W_{t_i}}{\sqrt{t_i}} \sim N(0,1)$$

$$W_{t_{i+1}} - W_{t_i} \sim N(0, t_{i+1} - t_i) \to \frac{W_{t_{i+1}} - W_{t_i}}{\sqrt{t_{i+1} - t_i}} \sim N(0,1)$$

$$\therefore \frac{W_{t_i}}{\sqrt{t_i}} \cdot \frac{W_{t_{i+1}} - W_{t_i}}{\sqrt{t_{i+1} - t_i}} = \frac{W_{t_i}\left(W_{t_{i+1}} - W_{t_i}\right)}{\sqrt{t_i(t_{i+1} - t_i)}} \sim \chi_1^2, \qquad \text{(by definition of a } \chi^2 \text{ random variable, 1 degree of freedom)}$$

$$\therefore \sum_{i=0}^{n-1} \frac{W_{t_i}(W_{t_{i+1}} - W_{t_i})}{\sqrt{t_i(t_{i+1} - t_i)}} \sim \chi_n^2, \qquad (n \text{ degrees of freedom})$$

Since we arrived at this result by simply scaling each $W_{t_i}(W_{t_{i+1}} - W_{t_i})$ in the above summation by $\frac{1}{\sqrt{t_i(t_{i+1}-t_i)}}$, we know that each $W_{t_i}(W_{t_{i+1}} - W_{t_i})$ is certainly not normally distributed. It can be proven that each $W_{t_i}(W_{t_{i+1}} - W_{t_i})$ can be written as a linear combination of two non-central chi-squared random variables:

$$W_{t_i}(W_{t_{i+1}} - W_{t_i}) = \frac{1}{4}(W_{t_i} + W_{t_{i+1}} - W_{t_i})^2 - \frac{1}{4}(W_{t_i} - W_{t_{i+1}} + W_{t_i})^2$$

$$W_{t_i}(W_{t_{i+1}} - W_{t_i}) = \frac{W_{t_{i+1}}^2}{4} - \frac{(2W_{t_i} - W_{t_{i+1}})^2}{4}$$

$$W_{t_i}(W_{t_{i+1}} - W_{t_i}) \sim \frac{Var(W_{t_{i+1}})}{4} Q - \frac{Var(2W_{t_i} - W_{t_{i+1}})}{4} R, \qquad (Q, R \sim \chi_1^2 \text{ are non-central and dependent})$$

$$W_{t_i}(W_{t_{i+1}} - W_{t_i}) \sim \frac{Var(W_{t_{i+1}})}{4} Q - \frac{Var(2W_{t_i}) + Var(W_{t_{i+1}}) - 2Cov(2W_{t_i}, W_{t_i+1})}{4} R$$

$$W_{t_i}(W_{t_{i+1}} - W_{t_i}) \sim \frac{Var(W_{t_{i+1}})}{4} Q - \frac{2Var(W_{t_i}) + Var(W_{t_{i+1}}) - 4Cov(W_{t_i}, W_{t_i+1})}{4} R$$

$$W_{t_i}(W_{t_{i+1}} - W_{t_i}) \sim \frac{t_{i+1}}{4} Q - \frac{2t_i + t_{i+1} - 4\min\{t_i, t_{i+1}\}}{4} R$$

$$W_{t_i}(W_{t_{i+1}} - W_{t_i}) \sim \frac{t_{i+1}}{4} Q - \frac{2t_i + t_{i+1} - 4t_i}{4} R$$

$$W_{t_i}(W_{t_{i+1}} - W_{t_i}) \sim \frac{t_{i+1}}{4} Q - \frac{t_{i+1} - 2t_i}{4} R$$

Taking the special case of $n = 2$:

$$\sum_{i=0}^{n-1} W_{t_i}(W_{t_{i+1}} - W_{t_i}) = \sum_{i=0}^{1} W_{t_i}(W_{t_{i+1}} - W_{t_i})$$

$$\sum_{i=0}^{n-1} W_{t_i}(W_{t_{i+1}} - W_{t_i}) = W_{t_0}(W_{t_1} - W_{t_0}) + W_{t_1}(W_{t_2} - W_{t_1})$$

$$\sum_{i=0}^{n-1} W_{t_i}(W_{t_{i+1}} - W_{t_i}) = W_{t_1}(W_{t_2} - W_{t_1}) \sim \frac{t_2}{4} Q - \frac{t_2 - 2t_1}{4} R$$

Hence, in general, the random variable $I_t^{(n)}$ is not normally distributed.

2.

$$Q_T = \lim_{n\to\infty} \sum_{i=0}^{n-1} \left(W_{t_i^*} - W_{t_i}\right)^2, \qquad t_i^* = \frac{t_i + t_{i+1}}{2}$$

$$\mathbb{E}[Q_T] = \mathbb{E}\left[\lim_{n\to\infty} \sum_{i=0}^{n-1} \left(W_{t_i^*} - W_{t_i}\right)^2\right]$$

$$\mathbb{E}[Q_T] = \mathbb{E}\left[\lim_{n\to\infty} \sum_{i=0}^{n-1} \left(W_{t_i^*} - W_{t_i}\right)^2\right]$$

$$\mathbb{E}[Q_T] = \lim_{n\to\infty} \sum_{i=0}^{n-1} \mathbb{E}\left[\left(W_{t_i^*} - W_{t_i}\right)^2\right], \qquad \text{(MCT because as partition gets finer sum increases, linearity)}$$

$$\mathbb{E}[Q_T] = \lim_{n\to\infty} \sum_{i=0}^{n-1} t_i^* - t_i, \qquad \left(W_{t_i^*} - W_{t_i} \sim N(0, t_i^* - t_i)\right)$$

$$\mathbb{E}[Q_T] = \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{t_i + t_{i+1}}{2} - t_i$$

$$\mathbb{E}[Q_T] = \frac{1}{2} \lim_{n\to\infty} \sum_{i=0}^{n-1} t_{i+1} - t_i$$

$$\mathbb{E}[Q_T] = \frac{1}{2}(T - 0) = \frac{T}{2}$$

$$\to \mathbb{E}[Q_T]^2 = \frac{T^2}{4}$$

$$\mathbb{E}[Q_T^2] = \mathbb{E}\left[\left(\lim_{n\to\infty} \sum_{i=0}^{n-1} \left(W_{t_i^*} - W_{t_i}\right)^2\right)^2\right]$$

$$\mathbb{E}[Q_T^2] = \mathbb{E}\left[\lim_{n\to\infty} \left(\sum_{i=0}^{n-1} \left(W_{t_i^*} - W_{t_i}\right)^2\right)^2\right], \qquad \text{(square of a limit = limit of the square)}$$

$$\mathbb{E}[Q_T^2] = \lim_{n\to\infty} \mathbb{E}\left[\left(\sum_{i=0}^{n-1} \left(W_{t_i^*} - W_{t_i}\right)^2\right)^2\right], \qquad \text{(MCT because as partition gets finer squared sum increases)}$$

$$\mathbb{E}[Q_T^2] = \lim_{n\to\infty} \mathbb{E}\left[\sum_{i=0}^{n-1} \left(W_{t_i^*} - W_{t_i}\right)^4 + \sum_{i=0}^{n-1} \sum_{j=0}^{i-1} \left(W_{t_i^*} - W_{t_i}\right)^2 \cdot \left(W_{t_j^*} - W_{t_j}\right)^2\right]$$

$$\mathbb{E}[Q_T^2] = \lim_{n\to\infty}\left[\sum_{i=0}^{n-1}\mathbb{E}\left[\left(W_{t_i^*}-W_{t_i}\right)^4\right]+\sum_{i=0}^{n-1}\sum_{j=0}^{i-1}\mathbb{E}\left[\left(W_{t_i^*}-W_{t_i}\right)^2\cdot\left(W_{t_j^*}-W_{t_j}\right)^2\right]\right], \qquad \text{(linearity)}$$

$$\mathbb{E}\left[\left(W_{t_i^*}-W_{t_i}\right)^4\right] = 3\cdot(t_i^*-t_i)^2, \qquad \text{(For a } N(\mu,\sigma^2) \text{ random variable, fourth central moment is } (\sigma^2)^2=\sigma^4\text{)}$$

$$\mathbb{E}\left[\left(W_{t_i^*}-W_{t_i}\right)^4\right] = 3\left(\frac{t_{i+1}-t_i}{2}\right)^2 = \frac{3}{4}(t_{i+1}-t_i)^2$$

$$\to \sum_{i=0}^{n-1}\mathbb{E}\left[\left(W_{t_i^*}-W_{t_i}\right)^4\right] = \frac{1}{4}\sum_{i=0}^{n-1}3(t_{i+1}-t_i)^2$$

$$\mathbb{E}\left[\left(W_{t_i^*}-W_{t_i}\right)^2\cdot\left(W_{t_j^*}-W_{t_j}\right)^2\right] = \mathbb{E}\left[\left(W_{t_i^*}-W_{t_i}\right)^2\right]\cdot\mathbb{E}\left[\left(W_{t_j^*}-W_{t_j}\right)^2\right], \qquad \text{(independent increments)}$$

$$\mathbb{E}\left[\left(W_{t_i^*}-W_{t_i}\right)^2\cdot\left(W_{t_j^*}-W_{t_j}\right)^2\right] = (t_i^*-t_i)(t_j^*-t_j), \qquad \left(W_{t_i^*}-W_{t_i}\sim N(0,t_i^*-t_i)\right)$$

$$\to \sum_{i=0}^{n-1}\sum_{j=0}^{i-1}\mathbb{E}\left[\left(W_{t_i^*}-W_{t_i}\right)^2\cdot\left(W_{t_j^*}-W_{t_j}\right)^2\right] = \sum_{i=0}^{n-1}\sum_{j=0}^{i-1}(t_i^*-t_i)(t_j^*-t_j)$$

$$\sum_{i=0}^{n-1}\sum_{j=0}^{i-1}\mathbb{E}\left[\left(W_{t_i^*}-W_{t_i}\right)^2\cdot\left(W_{t_j^*}-W_{t_j}\right)^2\right] = \sum_{i=0}^{n-1}(t_i^*-t_i)\sum_{j=0}^{i-1}(t_j^*-t_j)$$

$$\sum_{i=0}^{n-1}\sum_{j=0}^{i-1}\mathbb{E}\left[\left(W_{t_i^*}-W_{t_i}\right)^2\cdot\left(W_{t_j^*}-W_{t_j}\right)^2\right] = \frac{1}{4}\sum_{i=0}^{n-1}(t_{i+1}-t_i)\sum_{j=0}^{i-1}(t_{j+1}-t_j)$$

$$\sum_{i=0}^{n-1}\sum_{j=0}^{i-1}\mathbb{E}\left[\left(W_{t_i^*}-W_{t_i}\right)^2\cdot\left(W_{t_j^*}-W_{t_j}\right)^2\right] = \frac{1}{4}\sum_{i=0}^{n-1}(t_{i+1}-t_i)(t_i-t_0)$$

$$\sum_{i=0}^{n-1}\sum_{j=0}^{i-1}\mathbb{E}\left[\left(W_{t_i^*}-W_{t_i}\right)^2\cdot\left(W_{t_j^*}-W_{t_j}\right)^2\right] = \frac{1}{4}\sum_{i=0}^{n-1}t_i(t_{i+1}-t_i)$$

$$\therefore\ \mathbb{E}[Q_T^2] = \lim_{n\to\infty}\left[\frac{1}{4}\sum_{i=0}^{n-1}3(t_{i+1}-t_i)^2+\frac{1}{4}\sum_{i=0}^{n-1}t_i(t_{i+1}-t_i)\right]$$

$$\mathbb{E}[Q_T^2] = \frac{1}{4}\lim_{n\to\infty}\sum_{i=0}^{n-1}3(t_{i+1}-t_i)^2+t_i(t_{i+1}-t_i)$$

$$\mathbb{E}[Q_T^2] = \frac{1}{4} \lim_{n \to \infty} \sum_{i=0}^{n-1} 3t_{i+1}^2 - 6t_{i+1}t_i + 3t_i^2 + t_i t_{i+1} - t_i^2$$

$$\mathbb{E}[Q_T^2] = \frac{1}{4} \lim_{n \to \infty} \sum_{i=0}^{n-1} 3t_{i+1}^2 - 5t_{i+1}t_i + 2t_i^2$$

$$\mathbb{E}[Q_T^2] = \frac{1}{4} \lim_{n \to \infty} \sum_{i=0}^{n-1} (3t_{i+1} - 2t_i)(t_{i+1} - t_i)$$

$$\mathbb{E}[Q_T^2] = \frac{1}{4} \cdot T^2 = \frac{T^2}{4}$$

$$\therefore Var(Q_T) = \mathbb{E}[Q_T^2] - \mathbb{E}[Q_T]^2$$

$$Var(Q_T) = \frac{T^2}{4} - \frac{T^2}{4} = 0$$

$$\therefore Q_T = \mathbb{E}[Q_T] = \frac{T}{2}$$

3.

a)

$$f(x) = \frac{1}{2}x^2, \qquad \frac{\partial f}{\partial x} = x, \qquad \frac{\partial^2 f}{\partial x^2} = 1, \qquad \frac{\partial^m f}{\partial x^m} = 0 \; \forall \; m > 2$$

Applying Taylor expansion to $f(W_{i+1})$ about W_i:

$$f(W_{i+1}) = f(W_i) + \frac{\partial f(W_i)}{\partial x}(W_{i+1} - W_i) + \frac{1}{2!}\frac{\partial f^2(W_i)}{\partial x^2}(W_{i+1} - W_i)^2 + \frac{1}{3!}\frac{\partial f^3(W_i)}{\partial x^3}(W_{i+1} - W_i)^3 + \cdots$$

$$f(W_{i+1}) - f(W_i) = W_i \cdot (W_{i+1} - W_i) + \frac{1}{2} \cdot 1 \cdot (W_{i+1} - W_i)^2 + \frac{1}{6} \cdot 0 \cdot (W_{i+1} - W_i)^3 + \cdots,$$
$$\left(\frac{\partial^m f}{\partial x^m} = 0 \; \forall \; m > 2\right)$$

$$f(W_{i+1}) - f(W_i) = W_i(W_{i+1} - W_i) + \frac{1}{2}(W_{i+1} - W_i)^2$$

$$f(W_{i+1}) - f(W_i) = (W_{i+1} - W_i)\left(W_i + \frac{1}{2}W_{i+1} - \frac{1}{2}W_i\right)$$

$$f(W_{i+1}) - f(W_i) = \frac{1}{2}(W_{i+1} - W_i)(W_{i+1} + W_i)$$

Subtract and add $W_i(W_{i+1} - W_i)$ on RHS:

$$f(W_{i+1}) - f(W_i) = \frac{1}{2}(W_{i+1} - W_i)(W_{i+1} + W_i) - W_i(W_{i+1} - W_i) + W_i(W_{i+1} - W_i)$$

$$f(W_{i+1}) - f(W_i) = \frac{1}{2}(W_{i+1} - W_i)(W_{i+1} + W_i - 2W_i) + W_i(W_{i+1} - W_i)$$

$$f(W_{i+1}) - f(W_i) = \frac{1}{2}(W_{i+1} - W_i)(W_{i+1} - W_i) + W_i(W_{i+1} - W_i)$$

$$f(W_{i+1}) - f(W_i) = \frac{1}{2}(W_{i+1} - W_i)^2 + W_i(W_{i+1} - W_i)$$

$$\rightarrow f(W_T) - f(W_0) = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(W_{i+1}) - f(W_i)$$

$$f(W_T) - f(W_0) = \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{2}(W_{i+1} - W_i)^2 + W_i(W_{i+1} - W_i)$$

$$f(W_T) - f(W_0) = \frac{1}{2}\lim_{n\to\infty} \sum_{i=0}^{n-1} (W_{i+1} - W_i)^2 + \lim_{n\to\infty} \sum_{i=0}^{n-1} W_i(W_{i+1} - W_i)$$

$$f(W_T) - f(W_0) = \frac{1}{2}\lim_{n\to\infty} \sum_{i=0}^{n-1} (W_{i+1} - W_i)^2 + \lim_{n\to\infty} \sum_{i=0}^{n-1} W_i(W_{i+1} - W_i)$$

$$f(W_T) - f(W_0) = \frac{1}{2}[W, W]_T + \int_0^T W_t \, dW_t \,, \qquad \text{(by definition of quadratic variation and Ito integral)}$$

$$f(W_T) - f(W_0) = \frac{T}{2} + \int_0^T W_t \, dW_t$$

$$f(W_T) - f(W_0) = \int_0^T W_t \, dW_t + \frac{1}{2}\int_0^T 1 \, dt$$

$$\therefore f(W_T) - f(W_0) = \int_0^T \frac{\partial f}{\partial x}(W_t) \, dW_t + \frac{1}{2}\int_0^T \frac{\partial^2 f}{\partial x^2}(W_t) \, dt \,, \qquad \left(\frac{\partial f}{\partial x}(W_t) = W_t, \qquad \frac{\partial^2 f}{\partial x^2}(W_t) = 1\right)$$

b)

Applying Taylor expansion to $f(W_{i+1})$ about W_i:

$$f(W_{i+1}) = f(W_i) + \frac{\partial f(W_i)}{\partial x}(W_{i+1} - W_i) + \frac{1}{2!}\frac{\partial f^2(W_i)}{\partial x^2}(W_{i+1} - W_i)^2 + \frac{1}{3!}\frac{\partial f^3(W_i)}{\partial x^3}(W_{i+1} - W_i)^3 + \cdots$$

$$f(W_{i+1}) - f(W_i) = \frac{\partial f(W_i)}{\partial x}(W_{i+1} - W_i) + \frac{1}{2}\frac{\partial f^2(W_i)}{\partial x^2}(W_{i+1} - W_i)^2 + \frac{1}{6}\frac{\partial f^3(W_i)}{\partial x^3}(W_{i+1} - W_i)^3 + \cdots$$

$$\rightarrow f(W_T) - f(W_0) = \lim_{n\to\infty}\sum_{i=0}^{n-1} f(W_{i+1}) - f(W_i)$$

$$f(W_T) - f(W_0) = \lim_{n\to\infty}\sum_{i=0}^{n-1} \frac{\partial f(W_i)}{\partial x}(W_{i+1} - W_i) + \frac{1}{2}\frac{\partial f^2(W_i)}{\partial x^2}(W_{i+1} - W_i)^2 + \frac{1}{6}\frac{\partial f^3(W_i)}{\partial x^3}(W_{i+1} - W_i)^3 + \cdots$$

$$\begin{aligned} f(W_T) - f(W_0) \\ &= \lim_{n\to\infty}\sum_{i=0}^{n-1} \frac{\partial f(W_i)}{\partial x}(W_{i+1} - W_i) + \lim_{n\to\infty}\sum_{i=0}^{n-1} \frac{1}{2}\frac{\partial f^2(W_i)}{\partial x^2}(W_{i+1} - W_i)^2 \\ &+ \lim_{n\to\infty}\sum_{i=0}^{n-1} \frac{1}{6}\frac{\partial f^3(W_i)}{\partial x^3}(W_{i+1} - W_i)^3 + \cdots \end{aligned}$$

Define $\Delta W_i = W_{i+1} - W_i$:

$$f(W_T) - f(W_0) = \lim_{n\to\infty}\sum_{i=0}^{n-1} \frac{\partial f(W_i)}{\partial x}(W_{i+1} - W_i) + \frac{1}{2}\lim_{n\to\infty}\sum_{i=0}^{n-1} \frac{\partial f^2(W_i)}{\partial x^2}(\Delta W_i)^2 + \frac{1}{6}\lim_{n\to\infty}\sum_{i=0}^{n-1} \frac{\partial f^3(W_i)}{\partial x^3}(\Delta W_i)^3 + \cdots$$

$$f(W_T) - f(W_0) = \lim_{n\to\infty}\sum_{i=0}^{n-1} \frac{\partial f(W_i)}{\partial x}(W_{i+1} - W_i) + \frac{1}{2}\lim_{n\to\infty}\sum_{i=0}^{n-1} \frac{\partial f^2(W_i)}{\partial x^2}\Delta t_i + \frac{1}{6}\lim_{n\to\infty}\sum_{i=0}^{n-1} \frac{\partial f^3(W_i)}{\partial x^3}\cdot 0 + \cdots$$

The above line follows from Ito rules because as the partition gets finer and finer with $||\Pi_n|| \to 0$ or $n \to \infty$, $(\Delta W)^2 \sim \Delta t$ but $(\Delta W)^m \sim (\Delta W)^2(\Delta W)^{m-2} \sim (\Delta t)(\Delta W)^{m-2} \ll \Delta t$ for $m > 2$, so we may drop all these terms. Continuing:

$$f(W_T) - f(W_0) = \lim_{n\to\infty}\sum_{i=0}^{n-1} \frac{\partial f(W_i)}{\partial x}(W_{i+1} - W_i) + \frac{1}{2}\lim_{n\to\infty}\sum_{i=0}^{n-1} \frac{\partial f^2(W_i)}{\partial x^2}\Delta t_i + \frac{1}{6}\lim_{n\to\infty}\sum_{i=0}^{n-1} 0 + \cdots$$

$$f(W_T) - f(W_0) = \lim_{n\to\infty}\sum_{i=0}^{n-1} \frac{\partial f(W_i)}{\partial x}(W_{i+1} - W_i) + \frac{1}{2}\lim_{n\to\infty}\sum_{i=0}^{n-1} \frac{\partial f^2(W_i)}{\partial x^2}(t_{i+1} - t_i)$$

$$f(W_T) - f(W_0) = \int_0^T \frac{\partial f}{\partial x}(W_t)\,dW_t + \frac{1}{2}\int_0^T \frac{\partial^2 f}{\partial x^2}(W_t)\,dt, \qquad \text{(by definition of Ito and Riemann integrals)}$$

4.

a)

Want to find dynamics of ζ_t, let $f(x,t) = -\theta x - \left(r + \frac{\theta^2}{2}\right)t$.

$$\frac{\partial f}{\partial t} = -\left(r + \frac{\theta^2}{2}\right), \qquad \frac{\partial f}{\partial x} = -\theta, \qquad \frac{\partial^2 f}{\partial x^2} = 0$$

Applying 2-D Ito's lemma:

$$df(x,t) = \frac{\partial f(W_t,t)}{\partial t}dt + \frac{\partial f(W_t,t)}{\partial x}dW_t + \frac{1}{2}\frac{\partial^2 f(W_t,t)}{\partial x^2}(dW_t)^2$$

$$df(x,t) = -\left(r + \frac{\theta^2}{2}\right)dt - \theta dW_t + \frac{1}{2}\cdot 0 \cdot dt$$

$$df(x,t) = -\left(r + \frac{\theta^2}{2}\right)dt - \theta dW_t$$

$$\zeta_t = e^{f(W_t,t)}$$

$$\frac{\partial \zeta}{\partial f} = \frac{\partial^2 \zeta}{\partial f^2} = e^{f(x,t)}$$

Applying 1-D Ito's lemma:

$$d\zeta_t = \frac{\partial \zeta(W_t,t)}{\partial f}df(W_t,t) + \frac{1}{2}\frac{\partial^2 \zeta(W_t,t)}{\partial f^2}\left(df(W_t,t)\right)^2$$

$$d\zeta_t = e^{f(W_t,t)}\left(-\left(r + \frac{\theta^2}{2}\right)dt - \theta dW_t\right) + \frac{1}{2}e^{f(W_t,t)}\left(-\left(r + \frac{\theta^2}{2}\right)dt - \theta dW_t\right)^2$$

$$d\zeta_t = \zeta_t\left(-\left(r + \frac{\theta^2}{2}\right)dt - \theta dW_t\right) + \frac{1}{2}\zeta_t\left(\left(r + \frac{\theta^2}{2}\right)dt + \theta dW_t\right)^2$$

$$d\zeta_t = -\left(r + \frac{\theta^2}{2}\right)\zeta_t dt - \theta\zeta_t dW_t + \frac{1}{2}\zeta_t\left(\left(r + \frac{\theta^2}{2}\right)^2(dt)^2 + 2\left(r + \frac{\theta^2}{2}\right)\theta(dt)(dW_t) + \theta^2(dW_t)^2\right)$$

$$d\zeta_t = -\left(r + \frac{\theta^2}{2}\right)\zeta_t dt - \theta\zeta_t dW_t + \frac{1}{2}\theta^2\zeta_t dt, \qquad \text{(by Ito rules, } (dt)^2 = (dt)(dW_t) = 0, d(W_t)^2 = dt\text{)}$$

$$\therefore d\zeta_t = -r\zeta_t dt - \theta\zeta_t dW_t$$

b)

Want to show that dynamics of $\zeta_t V_t$ given by $d(\zeta_t V_t)$ has 0 dt term. Using Ito's product rule:

$$d(\zeta_t V_t) = V_t d\zeta_t + \zeta_t dV_t + d\zeta_t dV_t$$

$$\begin{aligned} d(\zeta_t V_t) = {} & V_t(-r\zeta_t dt - \theta\zeta_t dW_t) + \zeta_t\big((rV_t + a_t(\mu - r)S_t)dt + a_t\sigma S_t W_t\big) \\ & + (-r\zeta_t dt - \theta\zeta_t dW_t)\big((rV_t + a_t(\mu - r)S_t)dt + a_t\sigma S_t W_t\big) \end{aligned}$$

$$\begin{aligned} d(\zeta_t V_t) = {} & -r\zeta_t V_t dt - \theta\zeta_t V_t dW_t + (rV_t + a_t(\mu - r)S_t)\zeta_t dt + a_t\sigma\zeta_t S_t W_t \\ & + (-r\zeta_t(rV_t + a_t(\mu - r)S_t)(dt)^2 - ra_t\zeta_t S_t(dt)(dW_t) - (rV_t + a_t(\mu - r)S_t)\theta\zeta_t(dt)(dW_t) \\ & - a_t\theta\sigma\zeta_t S_t(dW_t)^2) \end{aligned}$$

$$\begin{aligned} d(\zeta_t V_t) = {} & -r\zeta_t V_t dt - \theta\zeta_t V_t dW_t + (rV_t + a_t(\mu - r)S_t)\zeta_t dt + a_t\sigma\zeta_t S_t W_t + (0 - 0 - 0 - a_t\theta\sigma\zeta_t S_t dt), \\ & \text{(by Ito rules)} \end{aligned}$$

$$d(\zeta_t V_t) = -r\zeta_t V_t dt - \theta\zeta_t V_t dW_t + (rV_t + a_t(\mu - r)S_t)\zeta_t dt + a_t\sigma\zeta_t S_t W_t + a_t\frac{(\mu - r)}{\sigma}\sigma\zeta_t S_t dt$$

$$d(\zeta_t V_t) = -r\zeta_t V_t dt - \theta\zeta_t V_t dW_t + r\zeta_t V_t dt + a_t(\mu - r)\zeta_t S_t dt + a_t\sigma\zeta_t S_t W_t + a_t(\mu - r)\zeta_t S_t dt$$

$$d(\zeta_t V_t) = (a_t\sigma S_t - \theta V_t)\zeta_t d(W_t)$$

So $d(\zeta_t V_t)$ has 0 dt term → $\zeta_t V_t$ is a martingale.

c)

We know that $\zeta_t V_t$ is a martingale from **b)**:

$$\rightarrow \mathbb{E}_{\mathbb{P}}[\zeta_T V_T] = \mathbb{E}_{\mathbb{P}}[\zeta_T V_T | \mathcal{F}_0], \qquad \text{(trivial } \sigma\text{-algebra)}$$

$$\mathbb{E}_{\mathbb{P}}[\zeta_T V_T] = \zeta_0 V_0, \qquad \text{(by definition of a martingale)}$$

$$\zeta_t = e^{-\theta W_t - \left(r + \frac{\theta^2}{2}\right)t}$$

$$\rightarrow \zeta_0 = e^{-\theta W_0 - \left(r + \frac{\theta^2}{2}\right)0}$$

$$\zeta_0 = e^{-\theta \cdot 0 - \left(r + \frac{\theta^2}{2}\right) \cdot 0}$$

$$\zeta_0 = e^0 = 1$$

$\therefore \mathbb{E}_{\mathbb{P}}[\zeta_T V_T] = 1 \cdot V_0 = V_0$

Since the investor wants to perfectly replicate X_T by $V_T \to V_T = X_T$ almost surely ($\mathbb{P}(V_T = X_T) = 1$). Then, we have:

$V_0 = \mathbb{E}_{\mathbb{P}}[\zeta_T V_T] = \mathbb{E}_{\mathbb{P}}[\zeta_T X_T]$

5.

Let:

$$d_1 = \frac{\ln\left(\frac{K}{x^\beta}\right) - \beta\left(r - \frac{\sigma^2}{2}\right)(T-t)}{\beta\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln\left(\frac{K}{x^\beta}\right) - \beta\left(r - \frac{\sigma^2}{2}\right)(T-t)}{\beta\sigma\sqrt{T-t}} - \beta\sigma\sqrt{T-t}$$

$$d_2 = d_1 - \beta\sigma\sqrt{T-t}$$

$$\therefore C(x,t) = e^{-r(T-t)}KN(d_1) - x^\beta e^{\left(r - \frac{r}{\beta} + (\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)}N(d_2)$$

Let 1) $= \frac{\partial C}{\partial t}$, 2) $= \frac{\partial C}{\partial x}$, 3) $= \frac{\partial^2 C}{\partial x^2}$

Evaluating 1) first:

$$\frac{\partial C}{\partial t} = re^{-r(T-t)}KN(d_1) + e^{-r(T-t)}KN'(d_1)\frac{\partial d_1}{\partial t} + \left(r - \frac{r}{\beta} + (\beta-1)\frac{\sigma^2}{2}\right)\beta x^\beta e^{\left(r - \frac{r}{\beta} + (\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)}N(d_2)$$
$$- x^\beta e^{\left(r - \frac{r}{\beta} + (\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)}N'(d_2)\frac{\partial d_2}{\partial t}, \qquad \text{(product and chain rules)}$$

$$N'(d_1) = f(d_1; \mu = 0, \sigma^2 = 1) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_1^2}, \qquad \text{(standard normal distribution pdf evaluated at } d_1\text{)}$$

$$N'(d_2) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_2^2}$$

$$N'(d_2) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(d_1 - \beta\sigma\sqrt{T-t}\right)^2}, \qquad \left(d_2 = d_1 - \beta\sigma\sqrt{T-t}\right)$$

$$N'(d_2) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(d_1^2 - 2d_1\beta\sigma\sqrt{T-t} + \beta^2\sigma^2(T-t)\right)}$$

$$N'(d_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} e^{d_1 \beta\sigma\sqrt{T-t} - \frac{1}{2}\beta^2\sigma^2(T-t)}$$

$$N'(d_2) = N'(d_1) e^{\frac{\ln\left(\frac{K}{x^\beta}\right) - \beta\left(r - \frac{\sigma^2}{2}\right)(T-t)}{\beta\sigma\sqrt{T-t}} \cdot \beta\sigma\sqrt{T-t} - \frac{1}{2}\beta^2\sigma^2(T-t)}$$

$$N'(d_2) = N'(d_1) e^{\ln\left(\frac{K}{x^\beta}\right)} e^{-\beta\left(r - \frac{\sigma^2}{2}\right)(T-t) - \frac{1}{2}\beta^2\sigma^2(T-t)}$$

$$N'(d_2) = N'(d_1) \frac{K}{x^\beta} e^{-\beta\left(r - \frac{\sigma^2}{2} + \frac{\beta\sigma^2}{2}\right)(T-t)}$$

$$\therefore N'(d_2) = \frac{K}{x^\beta} e^{-\left(r + (\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)} N'(d_1)$$

Also note that:

$$\frac{\partial d_2}{\partial t} = \frac{\partial d_1}{\partial t} - \frac{\partial}{\partial t}\left(\beta\sigma\sqrt{T-t}\right)$$

$$\frac{\partial d_2}{\partial t} = \frac{\partial d_1}{\partial t} + \frac{1}{2}\frac{\beta\sigma}{\sqrt{T-t}}$$

Substituting $N'(d_2)$ and $\frac{\partial d_2}{\partial t}$ into the last expression for $\frac{\partial C}{\partial t}$ above:

$$\begin{aligned}\therefore \frac{\partial C}{\partial t} = {} & r e^{-r(T-t)} K N(d_1) + e^{-r(T-t)} K N'(d_1) \frac{\partial d_1}{\partial t} + \left(r - \frac{r}{\beta} + (\beta-1)\frac{\sigma^2}{2}\right) \beta x^\beta e^{\left(r - \frac{r}{\beta} + (\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)} N(d_2) \\ & - x^\beta e^{\left(r - \frac{r}{\beta} + (\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)} \cdot \frac{K}{x^\beta} e^{-\left(r + (\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)} N'(d_1) \cdot \left(\frac{\partial d_1}{\partial t} + \frac{1}{2}\frac{\beta\sigma}{\sqrt{T-t}}\right)\end{aligned}$$

Simplify $e^{\left(r - \frac{r}{\beta} + (\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)} \cdot e^{-\left(r + (\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)} = e^{\left(-\frac{r}{\beta}\right)\beta(T-t)} = e^{-r(T-t)}$:

$$\begin{aligned}\frac{\partial C}{\partial t} = {} & r e^{-r(T-t)} K N(d_1) + e^{-r(T-t)} K N'(d_1) \frac{\partial d_1}{\partial t} + \left(r - \frac{r}{\beta} + (\beta-1)\frac{\sigma^2}{2}\right) \beta x^\beta e^{\left(r - \frac{r}{\beta} + (\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)} N(d_2) \\ & - e^{-r(T-t)} K N'(d_1) \frac{\partial d_1}{\partial t} - \frac{1}{2}\frac{\beta\sigma}{\sqrt{T-t}} e^{-r(T-t)} K N'(d_1)\end{aligned}$$

$$\therefore \frac{\partial C}{\partial t} = r e^{-r(T-t)} K N(d_1) + \left(r - \frac{r}{\beta} + (\beta-1)\frac{\sigma^2}{2}\right) \beta x^\beta e^{\left(r - \frac{r}{\beta} + (\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)} N(d_2) - \frac{1}{2}\frac{\beta\sigma}{\sqrt{T-t}} e^{-r(T-t)} K N'(d_1)$$

Now evaluating 2):

$$\frac{\partial C}{\partial x} = e^{-r(T-t)} K N'(d_1) \frac{d_1}{\partial x} - \beta x^{\beta-1} e^{\left(r - \frac{r}{\beta} + (\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)} N(d_2) - x^{\beta} e^{\left(r - \frac{r}{\beta} + (\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)} N'(d_2) \frac{d_2}{\partial x},$$

(product and chain rules)

Aside (for use later in simplifying final expressions):

$$\frac{\partial d_1}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\ln\left(\frac{K}{x^{\beta}}\right) - \beta\left(r - \frac{\sigma^2}{2}\right)(T-t)}{\beta\sigma\sqrt{T-t}}\right)$$

$$\frac{\partial d_1}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\ln(K) - \beta\ln(x) - \beta\left(r - \frac{\sigma^2}{2}\right)(T-t)}{\beta\sigma\sqrt{T-t}}\right)$$

$$\frac{\partial d_1}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\ln(K)}{\beta\sigma\sqrt{T-t}} - \frac{\ln(x)}{\sigma\sqrt{T-t}} - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\sqrt{T-t}\right)$$

$$\frac{\partial d_1}{\partial x} = -\frac{1}{x\sigma\sqrt{T-t}}$$

$$\frac{\partial d_2}{\partial x} = \frac{\partial}{\partial x}\left(d_1 - \beta\sigma\sqrt{T-t}\right)$$

$$\frac{\partial d_2}{\partial x} = \frac{\partial d_1}{\partial x} = -\frac{1}{x\sigma\sqrt{T-t}}$$

Substituting $N'(d_2)$ from 1) and $\frac{\partial d_1}{\partial x}, \frac{\partial d_2}{\partial x}$ into the last expression for $\frac{\partial C}{\partial x}$ above:

$$\frac{\partial C}{\partial x} = e^{-r(T-t)} K N'(d_1) \frac{d_1}{\partial x} - \beta x^{\beta-1} e^{\left(r - \frac{r}{\beta} + (\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)} N(d_2) - x^{\beta} e^{\left(r - \frac{r}{\beta} + (\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)}$$
$$\cdot \frac{K}{x^{\beta}} e^{-\left(r + (\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)} N'(d_1) \cdot \frac{d_1}{\partial x}$$

Simplify $e^{\left(r - \frac{r}{\beta} + (\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)} \cdot e^{-\left(r + (\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)} = e^{\left(-\frac{r}{\beta}\right)\beta(T-t)} = e^{-r(T-t)}$:

$$\frac{\partial C}{\partial x} = e^{-r(T-t)} K N'(d_1) \frac{d_1}{\partial x} - \beta x^{\beta-1} e^{\left(r - \frac{r}{\beta} + (\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)} N(d_2) - e^{-r(T-t)} K N'(d_1) \cdot \frac{d_1}{\partial x}$$

$$\therefore \frac{\partial C}{\partial x} = -\beta x^{\beta-1} e^{\left(r - \frac{r}{\beta} + (\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)} N(d_2)$$

Finally evaluating 3):

$$\frac{\partial^2 C}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial C}{\partial x}\right)$$

$$\frac{\partial^2 C}{\partial x^2} = \frac{\partial}{\partial x}\left(-\beta x^{\beta-1} e^{\left(r-\frac{r}{\beta}+(\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)} N(d_2)\right)$$

$$\frac{\partial^2 C}{\partial x^2} = -\beta(\beta-1)x^{\beta-2} e^{\left(r-\frac{r}{\beta}+(\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)} N(d_2) - \beta x^{\beta-1} e^{\left(r-\frac{r}{\beta}+(\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)} N'(d_2)\frac{\partial d_2}{\partial x},$$

$$\text{(product and chain rules)}$$

Substituting $N'(d_2)$ from 1) and $\frac{\partial d_2}{\partial x} = \frac{\partial d_1}{\partial x}$ into the expression for $\frac{\partial^2 C}{\partial x^2}$ above:

$$\frac{\partial^2 C}{\partial x^2} = -\beta(\beta-1)x^{\beta-2} e^{\left(r-\frac{r}{\beta}+(\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)} N(d_2) - \beta x^{\beta-1} e^{\left(r-\frac{r}{\beta}+(\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)} \cdot \frac{K}{x^\beta} e^{-\left(r+(\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)} N'(d_1) \cdot \frac{\partial d_1}{\partial x}$$

Simplify $e^{\left(r-\frac{r}{\beta}+(\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)} \cdot e^{-\left(r+(\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)} = e^{\left(-\frac{r}{\beta}\right)\beta(T-t)} = e^{-r(T-t)}$:

$$\therefore \frac{\partial^2 C}{\partial x^2} = -(\beta-1)\beta x^{\beta-2} e^{\left(r-\frac{r}{\beta}+(\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)} N(d_2) - \beta e^{-r(T-t)}\left(\frac{K}{x}\right) N'(d_1) \cdot \frac{\partial d_1}{\partial x}$$

Recombining the final expressions for (1), (2) and (3) from above to show $C(x,t)$ satisfies the B-S PDE:

$$LHS = \frac{\partial C}{\partial t} + rx\frac{\partial C}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C}{\partial x^2}$$

$$= (1) + rx \cdot (2) + \frac{1}{2}\sigma^2 x^2 \cdot (3)$$

$$\begin{aligned} &= re^{-r(T-t)}KN(d_1) + \left(r-\frac{r}{\beta}+(\beta-1)\frac{\sigma^2}{2}\right)\beta x^\beta e^{\left(r-\frac{r}{\beta}+(\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)} N(d_2) - \frac{1}{2}\frac{\beta\sigma}{\sqrt{T-t}} e^{-r(T-t)}KN'(d_1) \\ &\quad + rx\left(-\beta x^{\beta-1} e^{\left(r-\frac{r}{\beta}+(\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)} N(d_2)\right) \\ &\quad + \frac{1}{2}\sigma^2 x^2\left(-(\beta-1)\beta x^{\beta-2} e^{\left(r-\frac{r}{\beta}+(\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)} N(d_2) - \beta e^{-r(T-t)}\left(\frac{K}{x}\right) N'(d_1) \cdot \frac{\partial d_1}{\partial x}\right) \end{aligned}$$

Use $\frac{\partial d_1}{\partial x} = -\frac{1}{x\sigma\sqrt{T-t}}$ from 1) and now expanding terms:

$$= re^{-r(T-t)}KN(d_1) + \left(r - \frac{r}{\beta} + (\beta-1)\frac{\sigma^2}{2}\right)\beta x^\beta e^{\left(r-\frac{r}{\beta}+(\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)}N(d_2) - \frac{1}{2}\frac{\beta\sigma}{\sqrt{T-t}}e^{-r(T-t)}KN'(d_1)$$
$$- r\beta x^\beta e^{\left(r-\frac{r}{\beta}+(\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)}N(d_2) - (\beta-1)\frac{\sigma^2}{2}\beta x^\beta e^{\left(r-\frac{r}{\beta}+(\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)}N(d_2) - \frac{1}{2}\sigma^2 x^2$$
$$\cdot \beta e^{-r(T-t)}KN'(d_1) \cdot -\frac{1}{x^2\sigma\sqrt{T-t}}$$

Grouping terms by $N(d_1), N(d_2), N'(d_1)$:

$$= re^{-r(T-t)}KN(d_1) + \left(r - \frac{r}{\beta} + (\beta-1)\frac{\sigma^2}{2} - r - (\beta-1)\frac{\sigma^2}{2}\right)\beta x^\beta e^{\left(r-\frac{r}{\beta}+(\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)}N(d_2)$$
$$+ \left(\frac{1}{2}\frac{\beta\sigma}{\sqrt{T-t}} - \frac{1}{2}\frac{\beta\sigma}{\sqrt{T-t}}\right)e^{-r(T-t)}KN'(d_1)$$

$$= re^{-r(T-t)}KN(d_1) + \left(-\frac{r}{\beta}\right)\beta x^\beta e^{\left(r-\frac{r}{\beta}+(\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)}N(d_2) + 0 \cdot e^{-r(T-t)}KN'(d_1)$$

$$= re^{-r(T-t)}KN(d_1) - rx^\beta e^{\left(r-\frac{r}{\beta}+(\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)}N(d_2) + 0$$

$$= r\left(e^{-r(T-t)}KN(d_1) - x^\beta e^{\left(r-\frac{r}{\beta}+(\beta-1)\frac{\sigma^2}{2}\right)\beta(T-t)}N(d_2)\right)$$

$$= rC = RHS$$

Thus, it has been rigorously shown that $C(x,t)$ satisfies the B-S PDE with the given terminal condition $C(x,T) = (K - x^\beta)^+$.