MATH7049 – PRESENTATION

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Content

- Introduction to the fully implicit finite different scheme
- Consistency
- Stability
- Application – option on underlying with continuous dividends

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Introduction

- Given a general untransformed BS PDE:

$$\frac{\partial C}{\partial \tau} + f(S,\tau)\frac{\partial^2 C}{\partial S^2} + g(S,\tau)\frac{\partial C}{\partial S} + h(S,\tau)C = 0$$

- The implicit FD scheme is given by discretising the above PDE at (S_j, τ_n) as follows:

$$\frac{C_{n+1}^{j} - C_{n}^{j}}{\Delta\tau} + f_{n+1}^{j}\frac{C_{n+1}^{j+1} - 2C_{n+1}^{j} + C_{n+1}^{j-1}}{(\Delta S)^2} + g_{n+1}^{j}\frac{C_{n+1}^{j+1} - C_{n+1}^{j-1}}{2\Delta S} + h_{n+1}^{j}C_{n+1}^{j} = 0$$

where

$$f_n^j = f(S_j, \tau_n), \qquad g_n^j = g(S_j, \tau_n), \qquad h_n^j = h(S_j, \tau_n)$$

Notes

- Compared to the fully explicit FD scheme:

$$\frac{C_{n+1}^{j} - C_{n}^{j}}{\Delta\tau} + f_{n}^{j}\frac{C_{n}^{j+1} - 2C_{n}^{j} + C_{n}^{j-1}}{(\Delta S)^{2}} + g_{n}^{j}\frac{C_{n}^{j+1} - C_{n}^{j-1}}{2\Delta S} + h_{n}^{j}C_{n}^{j} = 0$$

- Since using τ instead of usual t, solve the computational grid from left to right (as opposed to right to left).
- Know $C_n \rightarrow$ want C_{n+1}, harder than explicit FD scheme because requires solving a linear system of $2J - 1$ equations in $2J - 1$ unknowns.

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Consistency

- Mathematically prove implicit scheme has accuracy $O(\Delta\tau) + O(\Delta S)^2$ using Taylor analysis
- Local truncation error:

$$L(S,\tau) = \frac{C(S,\tau+\Delta\tau) - C(S,\tau)}{\Delta\tau} + f(S,\tau+\Delta\tau)\frac{C(S+\Delta S,\tau+\Delta\tau) - 2C(S,\tau+\Delta\tau) + C(S-\Delta S,\tau+\Delta\tau)}{(\Delta S)^2} + g(S,\tau+\Delta\tau)\frac{C(S+\Delta S,\tau+\Delta\tau) - C(S-\Delta S,\tau+\Delta\tau)}{2\Delta S} + h(S,\tau+\Delta\tau)C(S,\tau+\Delta\tau)$$

- Taylor expansion of PDE solution C about $(S,\tau+\Delta\tau)$:

$$C(S,\tau+\Delta\tau) = C$$

$$C(S,\tau+\Delta\tau-\Delta\tau) = C(S,\tau) = C - (\Delta\tau)C_\tau + (\Delta\tau)^2 C_{\tau\tau} + O(\Delta\tau)^3$$

$$C(S\pm\Delta S,\tau+\Delta\tau) = C \pm (\Delta S)C_S + (\Delta S)^2 C_{SS} \pm (\Delta S)^3 C_{SSS} + (\Delta S)^4 C_{SSSS} \pm (\Delta S)^5 C_{SSSSS} + O(\Delta S)^6$$

Consistency continued

- Hey, wait a minute! This is starting to get really ugly, why's that?
  - Because we're using the untransformed BS PDE to prove these properties.
  - Using the model problem (heat/diffusion PDE) makes things much cleaner.

- Let's split each of the expressions in $L(S,\tau)$ on the previous slide into 4 terms
- (1) $\frac{C(S,\tau+\Delta\tau)-C(S,\tau)}{\Delta\tau}$
- (2) $f(S,\tau+\Delta\tau)\frac{C(S+\Delta S,\tau+\Delta\tau)-2C(S,\tau+\Delta\tau)+C(S-\Delta S,\tau+\Delta\tau)}{(\Delta S)^2}$
- (3) $g(S,\tau+\Delta\tau)\frac{C(S+\Delta S,\tau+\Delta\tau)-C(S-\Delta S,\tau+\Delta\tau)}{2\Delta S}$
- (4) $h(S,\tau+\Delta\tau)C(S,\tau+\Delta\tau)$

Consistency continued

- Dealing with (1) $\frac{C(S,\tau+\Delta\tau)-C(S,\tau)}{\Delta\tau}$:

$$= \frac{C - C + (\Delta\tau)C_\tau - (\Delta\tau)^2 C_{\tau\tau} + O(\Delta\tau)^3}{\Delta\tau}$$
$$= C_\tau - (\Delta\tau)C_{\tau\tau} + O(\Delta\tau)^2$$

- Dealing with (2) $f(S,\tau+\Delta\tau)\frac{C(S+\Delta S,\tau+\Delta\tau)-2C(S,\tau+\Delta\tau)+C(S-\Delta S,\tau+\Delta\tau)}{(\Delta S)^2}$:

$$= f(S,\tau+\Delta\tau)\frac{C + (\Delta S)C_S + (\Delta S)^2 C_{SS} + (\Delta S)^3 C_{SSS} + (\Delta S)^4 C_{SSSS} + (\Delta S)^5 C_{SSSSS} + O(\Delta S)^6 - 2C + C - (\Delta S)C_S + (\Delta S)^2 C_{SS} - (\Delta S)^3 C_{SSS} + (\Delta S)^4 C_{SSSS} - (\Delta S)^5 C_{SSSSS} + O(\Delta S)^6}{(\Delta S)^2}$$

$$= f(S,\tau+\Delta\tau)\frac{(\Delta S)^2 C_{SS} + \frac{(\Delta S)^4}{12} C_{SSSS} + O(\Delta S)^6}{(\Delta S)^2}$$
$$= f(S,\tau+\Delta\tau)\left(C_{SS} + \frac{(\Delta S)^2}{12} C_{SSSS} + O(\Delta S)^4\right)$$

Consistency continued

- Dealing with (3) $g(S,\tau+\Delta\tau)\frac{C(S+\Delta S,\tau+\Delta\tau)-C(S-\Delta S,\tau+\Delta\tau)}{2\Delta S}$:

$$= g(S,\tau+\Delta\tau)\frac{C+(\Delta S)C_S+(\Delta S)^2C_{SS}+(\Delta S)^3C_{SSS}+(\Delta S)^4C_{SSSS}+(\Delta S)^5C_{SSSSS}+O(\Delta S)^6-C+(\Delta S)C_S-(\Delta S)^2C_{SS}+(\Delta S)^3C_{SSS}-(\Delta S)^4C_{SSSS}+(\Delta S)^5C_{SSSSS}+O(\Delta S)^6}{2(\Delta S)}$$

$$= 2g(S,\tau+\Delta\tau)\frac{(\Delta S)C_S+\frac{(\Delta S)^3}{6}C_{SSS}+\frac{(\Delta S)^5}{120}C_{SSSSS}+O(\Delta S)^6}{2(\Delta S)}$$

$$= g(S,\tau+\Delta\tau)\left(C_S+\frac{(\Delta S)^2}{6}C_{SSS}+\frac{(\Delta S)^4}{120}C_{SSSSS}+O(\Delta S)^5\right)$$

- Dealing with (4) $h(S,\tau+\Delta\tau)C(S,\tau+\Delta\tau)$:

$$= h(S,\tau+\Delta\tau)C$$

Consistency continued

- Combining expressions:

$$L(S,\tau) = (1) + (2) + (3) + (4)$$

$$L(S,\tau) = C_\tau - (\Delta\tau)C_{\tau\tau} + O(\Delta\tau)^2 + f(S,\tau+\Delta\tau)\left(C_{SS} + \frac{(\Delta S)^2}{12}C_{SSSS} + O(\Delta S)^4\right) + g(S,\tau+\Delta\tau)\left(C_S + \frac{(\Delta S)^2}{6}C_{SSS} + \frac{(\Delta S)^4}{120}C_{SSSSS} + O(\Delta S)^5\right) + h(S,\tau+\Delta\tau)C$$

$$L(S,\tau) = C_\tau + f(S,\tau+\Delta\tau)C_{SS} + g(S,\tau)C_S + h(S,\tau)C - (\Delta\tau)C_{\tau\tau} + (\Delta S)^2\left[\frac{g(S,\tau+\Delta\tau)}{6}C_{SSS} + \frac{f(S,\tau+\Delta\tau)}{12}C_{SSSS}\right] + \frac{(\Delta S)^4 g(S,\tau+\Delta\tau)}{120}C_{SSSSS} + O(\Delta\tau)^2 + O(\Delta S)^4$$

$$L(S,\tau) = -(\Delta\tau)C_{\tau\tau} + (\Delta S)^2\left[\frac{g(S,\tau+\Delta\tau)}{6}C_{SSS} + \frac{f(S,\tau+\Delta\tau)}{12}C_{SSSS}\right] + \frac{(\Delta S)^4 g(S,\tau+\Delta\tau)}{120}C_{SSSSS} + O(\Delta\tau)^2 + O(\Delta S)^4$$

$$\therefore L(S,\tau) = O(\Delta\tau) + O(\Delta S)^2 \text{ as } \Delta\tau, \Delta S \to 0$$

- So the implicit FD scheme is accurate to first order in $\Delta\tau$ and second order in ΔS.
- The rate of convergence is given by $O(\Delta\tau) + O(\Delta S)^2$ (provided stability holds).

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Stability

- From the lecture slides, know that the implicit FD scheme is unconditionally stable **but only when applied to the standard untransformed BS PDE where:**

$$f(S,\tau) = -\frac{1}{2}\sigma^2 S^2, \qquad g(S,\tau) = -rS, \qquad h(S,\tau) = r$$

- When we have different choices of $f(S,\tau), g(S,\tau)$ and $h(S,\tau)$, then the implicit FD scheme may become unstable.
  - This means a modified variant of the BS PDE (which may not have a closed form solution).
  - Too difficult to prove for the general case.

Stability via Fourier analysis

- We look for solutions to the FD scheme of the form:

$$C_n^j = \lambda_k^n e^{\frac{i\pi kj}{J}}, k = -J, \dots, J$$

- Substituting this into the discretised untransformed BS PDE:

$$\frac{\lambda_k^{n+1} e^{\frac{i\pi kj}{J}} - \lambda_k^n e^{\frac{i\pi kj}{J}}}{\Delta\tau} + f_{n+1}^j \frac{\lambda_k^{n+1} e^{\frac{i\pi k(j+1)}{J}} - 2\lambda_k^{n+1} e^{\frac{i\pi kj}{J}} + \lambda_k^{n+1} e^{\frac{i\pi k(j-1)}{J}}}{(\Delta S)^2} + g_{n+1}^j \frac{\lambda_k^{n+1} e^{\frac{i\pi k(j+1)}{J}} - \lambda_k^{n+1} e^{\frac{i\pi k(j-1)}{J}}}{2\Delta S} + h_{n+1}^j \lambda_k^{n+1} e^{\frac{i\pi kj}{J}} = 0$$

- Factor out $\lambda_k^n e^{\frac{i\pi kj}{J}}$:

$$\lambda_k^n e^{\frac{i\pi kj}{J}} \left[\frac{\lambda_k - 1}{\Delta\tau} + f_{n+1}^j \lambda_k \frac{e^{\frac{i\pi k}{J}} - 2 + e^{-\frac{i\pi k}{J}}}{(\Delta S)^2} + g_{n+1}^j \lambda_k \frac{e^{\frac{i\pi k}{J}} - e^{-\frac{i\pi k}{J}}}{2\Delta S} + h_{n+1}^j \lambda_k \right] = 0$$

- Since exp() is always > 0:

$$\frac{\lambda_k - 1}{\Delta\tau} + f_{n+1}^j \lambda_k \frac{e^{\frac{i\pi k}{J}} - 2 + e^{-\frac{i\pi k}{J}}}{(\Delta S)^2} + g_{n+1}^j \lambda_k \frac{e^{\frac{i\pi k}{J}} - e^{-\frac{i\pi k}{J}}}{2\Delta S} + h_{n+1}^j \lambda_k = 0$$

Stability via Fourier analysis (continued)

$$\frac{\lambda_k}{\Delta\tau} + f_{n+1}^j \lambda_k \frac{e^{\frac{i\pi k}{J}} - 2 + e^{-\frac{i\pi k}{J}}}{(\Delta S)^2} + g_{n+1}^j \lambda_k \frac{e^{\frac{i\pi k}{J}} - e^{-\frac{i\pi k}{J}}}{2\Delta S} + h_{n+1}^j \lambda_k = \frac{1}{\Delta\tau}$$

Factoring out λ_k:

$$\lambda_k \left[\frac{1}{\Delta\tau} + f_{n+1}^j \frac{e^{\frac{i\pi k}{J}} - 2 + e^{-\frac{i\pi k}{J}}}{(\Delta S)^2} + g_{n+1}^j \frac{e^{\frac{i\pi k}{J}} - e^{-\frac{i\pi k}{J}}}{2\Delta S} + h_{n+1}^j\right] = \frac{1}{\Delta\tau}$$

Use Euler's formula for $\cos(\theta) = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right), \sin(\theta) = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right), \theta = \frac{\pi k}{J}$:

$$\lambda_k \left[\frac{1}{\Delta\tau} + f_{n+1}^j \frac{2\cos\left(\frac{\pi k}{J}\right) - 2}{(\Delta S)^2} + i g_{n+1}^j \frac{\sin\left(\frac{\pi k}{J}\right)}{\Delta S} + h_{n+1}^j\right] = \frac{1}{\Delta\tau}$$

Stability via Fourier analysis (continued)

- Use Double Angle formula for $\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}, 2\theta = \frac{\pi k}{J} \rightarrow \theta = \frac{\pi k}{2J}$:

$$\lambda_k \left[\frac{1}{\Delta\tau} - 4 f_{n+1}^{j} \frac{\sin^2\left(\frac{\pi k}{2J}\right)}{(\Delta S)^2} + i g_{n+1}^{j} \frac{\sin\left(\frac{\pi k}{J}\right)}{\Delta S} + h_{n+1}^{j} \right] = \frac{1}{\Delta\tau}$$

$$\rightarrow \lambda_k = \frac{1}{\Delta\tau \left[\frac{1}{\Delta\tau} - 4 f_{n+1}^{j} \frac{\sin^2\left(\frac{\pi k}{2J}\right)}{(\Delta S)^2} + i g_{n+1}^{j} \frac{\sin\left(\frac{\pi k}{J}\right)}{\Delta S} + h_{n+1}^{j} \right]}$$

$$\lambda_k = \frac{1}{1 - 4 \frac{\Delta\tau}{(\Delta S)^2} f_{n+1}^{j} \sin^2\left(\frac{\pi k}{2J}\right) + i \frac{\Delta\tau}{\Delta S} g_{n+1}^{j} \sin\left(\frac{\pi k}{J}\right) + \Delta\tau h_{n+1}^{j}}$$

Stability via Fourier analysis (continued)

- Converting from $\frac{1}{a+bi}$ to the form $\frac{a}{a^2+b^2} + \frac{b}{a^2+b^2} i$ by multiplying and dividing with its complex conjugate:

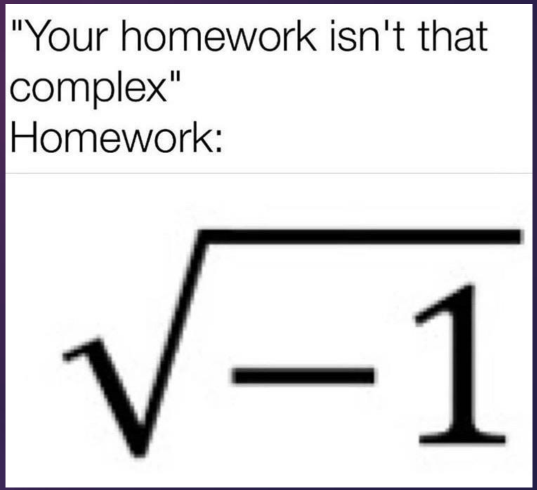
$$\lambda_k = \frac{1 - 4\frac{\Delta\tau}{(\Delta S)^2} f_{n+1}^j \sin^2\left(\frac{\pi k}{2J}\right) + \Delta\tau h_{n+1}^j}{\left(1 - 4\frac{\Delta\tau}{(\Delta S)^2} f_{n+1}^j \sin^2\left(\frac{\pi k}{2J}\right) + \Delta\tau h_{n+1}^j\right)^2 + \left(\frac{\Delta\tau}{\Delta S} g_{n+1}^j \sin\left(\frac{\pi k}{J}\right)\right)^2} + \frac{\frac{\Delta\tau}{\Delta S} g_{n+1}^j \sin\left(\frac{\pi k}{J}\right)}{\left(1 - 4\frac{\Delta\tau}{(\Delta S)^2} f_{n+1}^j \sin^2\left(\frac{\pi k}{2J}\right) + \Delta\tau h_{n+1}^j\right)^2 + \left(\frac{\Delta\tau}{\Delta S} g_{n+1}^j \sin\left(\frac{\pi k}{J}\right)\right)^2} i$$

- Require $|\lambda_k| \leq 1$ for stability to hold and results to not blow up:

$$|\lambda_k| \leq 1 + 0i$$

- Because λ_k is complex, we need to compare real part:

$$|Re\{\lambda_k\}| \leq 1$$



Stability via Fourier analysis (continued)

- Looks like this is getting really complicated!
- Too difficult to find expressions for arbitrary choices of $f(S,\tau), g(S,\tau)$ and $h(S,\tau)$.
- Instead prove the implicit FD scheme is unconditionally stable for the **standard untransformed BS PDE:**

$$f(S,\tau) = -\frac{1}{2}\sigma^2 S^2, \qquad g(S,\tau) = -rS, \qquad h(S,\tau) = r$$

- Proving relevant bounds on different terms:

$$1 - 4\frac{\Delta\tau}{(\Delta S)^2} f_{n+1}^j \sin^2\left(\frac{\pi k}{2J}\right) + \Delta\tau h_{n+1}^j = 1 + 4\alpha S_j^2 \sin^2\left(\frac{\pi k}{2J}\right) + \Delta\tau r, \qquad \alpha = \frac{1}{2}\sigma^2\frac{\Delta\tau}{(\Delta S)^2}$$

$$1 - 4\frac{\Delta\tau}{(\Delta S)^2} f_{n+1}^j \sin^2\left(\frac{\pi k}{2J}\right) + \Delta\tau h_{n+1}^j \geq 1$$

$$\therefore \left(1 - 4\frac{\Delta\tau}{(\Delta S)^2} f_{n+1}^j \sin^2\left(\frac{\pi k}{2J}\right) + \Delta\tau h_{n+1}^j\right)^2 \geq 1 - 4\frac{\Delta\tau}{(\Delta S)^2} f_{n+1}^j \sin^2\left(\frac{\pi k}{2J}\right) + \Delta\tau h_{n+1}^j \geq 1$$

Stability via Fourier analysis (continued)

$$\frac{\Delta\tau}{\Delta S} g_{n+1}^{j} \sin\left(\frac{\pi k}{J}\right) = -r\frac{\Delta\tau}{\Delta S} S_j \sin\left(\frac{\pi k}{J}\right)$$

$$\therefore \left(\frac{\Delta\tau}{\Delta S} g_{n+1}^{j} \sin\left(\frac{\pi k}{J}\right)\right)^2 \geq 0$$

$$\therefore |Re\{\lambda_k\}| = \left|\frac{1 + 4\alpha S^2 \sin^2\left(\frac{\pi k}{2J}\right) + \Delta\tau r}{\left(1 + 4\alpha S^2 \sin^2\left(\frac{\pi k}{2J}\right) + \Delta\tau r\right)^2 + \left(-r\frac{\Delta\tau}{\Delta S} S \sin\left(\frac{\pi k}{J}\right)\right)^2}\right| \leq 1$$

- So we have proved that $|\lambda_k| \leq 1$ as required.
- So for the standard untransformed BS PDE, having consistency and unconditional stability for the implicit FD scheme implies the scheme is indeed convergent as required.

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Application – continuous dividends

- Wanted to originally test the scheme on an option on futures but issues with code.
- Instead chose option on underlying with continuous dividends
  - Closed form solution exists.
  - Simple modification of standard BS equation/PDE solution.
  - All relevant formulae for this section obtained from Cvitanic & Zapatero (2004), see references.

Application continued

- Skipping intermediate steps to arrive at the modified pricing PDE:

$$\frac{\partial C}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r-q)S\frac{\partial C}{\partial S} - rC = 0$$

$$f(S,\tau) = -\frac{1}{2}\sigma^2 S^2, \qquad g(S,\tau) = -(r-q)S, \qquad h(S,\tau) = r$$

- Only difference is in $g(S,\tau) = -(r-q)S$ where q = the continuous dividend yield.
- Consistency and rate of convergence is still $O(\Delta\tau) + O(\Delta S)^2$ because not dependent on choices of $f(S,\tau), g(S,\tau)$ and $h(S,\tau)$.

Application continued

- Unconditionally stable because:

$$\frac{\Delta\tau}{\Delta S} g_{n+1}^{j} \sin\left(\frac{\pi k}{J}\right) = -(r-q)\frac{\Delta\tau}{\Delta S} S \sin\left(\frac{\pi k}{J}\right)$$

$$\therefore \left(\frac{\Delta\tau}{\Delta S} g_{n+1}^{j} \sin\left(\frac{\pi k}{J}\right)\right)^2 \geq 0$$

- From slide 19, $|Re\{\lambda_k\}| \leq 1$ and so we have $|\lambda_k| \leq 1$ as required.
- Consistency + Stability → Convergence!

Application continued

- Closed form solution for European call option (to compare against numerical solution):

$$C(S,\tau) = Se^{-q\tau}N(d_1) - Ke^{-r\tau}N(d_2)$$

$$d_{1,2} = d_{+,-} = \frac{\log\left(\frac{Se^{(r-q)\tau}}{K}\right)}{\sigma\sqrt{\tau}} \pm \frac{\sigma\sqrt{\tau}}{2}$$

- Run final demonstration in MATLAB.

References

- Jaksa Cvitanic & Fernando Zapatero, 2004, *Introduction to the Economics and Mathematics of Financial Markets*, viewed 28/10/2020.