STAT7301 – Assignment 2
Joel Thomas 44793203

1.

Want to show $\mathbb{E}[\bar{X}] \neq \theta$

$$\mathbb{E}[\bar{X}] = \mathbb{E}\left[\frac{1}{n}\sum_{j=1}^{n} X_j\right]$$

$$\mathbb{E}[\bar{X}] = \frac{1}{n}\sum_{j=1}^{n} \mathbb{E}\left[X_j\right], \qquad \text{(linearity)}$$

$$\mathbb{E}[\bar{X}] = \frac{1}{n} \cdot n\mathbb{E}[X], \qquad \left(X_j \sim^{iid} Poi_0(\theta), j = 1, \dots, n\right)$$

$$\mathbb{E}[\bar{X}] = \mathbb{E}[X]$$

$$\mathbb{E}[\bar{X}] = \sum_{x=1}^{\infty} x\mathbb{P}(X = x)$$

$$\mathbb{E}[\bar{X}] = \sum_{x=1}^{\infty} x \cdot \frac{e^{-\theta}\theta^x}{(1 - e^{-\theta})x!}$$

$$\mathbb{E}[\bar{X}] = \frac{e^{-\theta}}{1 - e^{-\theta}}\sum_{x=1}^{\infty} \frac{\theta^x}{(x-1)!}$$

$$\mathbb{E}[\bar{X}] = \frac{\theta e^{-\theta}}{1 - e^{-\theta}}\sum_{x=1}^{\infty} \frac{\theta^{x-1}}{(x-1)!}$$

$$\mathbb{E}[\bar{X}] = \frac{\theta e^{-\theta}}{1 - e^{-\theta}}\sum_{x=0}^{\infty} \frac{\theta^{x}}{x!}$$

$$\mathbb{E}[\bar{X}] = \frac{\theta e^{-\theta}}{1 - e^{-\theta}} \cdot e^{\theta}, \qquad \left(\sum_{x=0}^{\infty} \frac{\theta^{x}}{x!} = \text{power series for } e^{\theta}\right)$$

$$\therefore\ \mathbb{E}[\bar{X}] = \frac{\theta}{1 - e^{-\theta}} \neq \theta$$

$$bias(\bar{X}) = \mathbb{E}[\bar{X}] - \theta$$

$$bias(\bar{X}) = \frac{\theta}{1-e^{-\theta}} - \theta$$

$$bias(\bar{X}) = \frac{\theta - \theta + \theta e^{-\theta}}{1-e^{-\theta}}$$

$$bias(\bar{X}) = \frac{\theta e^{-\theta}}{1-e^{-\theta}} = \frac{\theta}{e^{\theta}-1}$$

2.

Deriving the likelihood function:

$$L(\theta) = f(x_1, \dots, x_n; \theta)$$

$$L(\theta) = \prod_{j=1}^{n} f(x_j; \theta), \qquad (X_j \sim^{iid} Poi_0(\theta), j = 1, \dots, n)$$

$$L(\theta) = \prod_{j=1}^{n} \frac{e^{-\theta}\theta^{x_j}}{(1-e^{-\theta})x_j!}$$

$$L(\theta) = \frac{e^{-n\theta}\theta^{\sum_{j=1}^{n} x_j}}{(1-e^{-\theta})^n \prod_{j=1}^{n} x_j!}$$

$$\therefore \log L(\theta) = \log\left(\frac{e^{-n\theta}\theta^{\sum_{j=1}^{n} x_j}}{(1-e^{-\theta})^n \prod_{j=1}^{n} x_j!}\right)$$

$$\log L(\theta) = \log\left(e^{-n\theta}\theta^{\sum_{j=1}^{n} x_j}\right) - \log\left((1-e^{-\theta})^n \prod_{j=1}^{n} x_j!\right)$$

$$\log L(\theta) = \log\left(e^{-n\theta}\right) + \log\left(\theta^{\sum_{j=1}^{n} x_j}\right) - \log\left((1-e^{-\theta})^n\right) - \log\left(\prod_{j=1}^{n} x_j!\right)$$

$$\log L(\theta) = -n\theta + \sum_{j=1}^{n} x_j \cdot \log(\theta) - n\log(1-e^{-\theta}) - \sum_{j=1}^{n} \log(x_j!)$$

$$\log L(\theta) = -n\theta + \log(\theta)\sum_{j=1}^{n} x_j - n\log\left(1 - e^{-\theta}\right) - \sum_{j=1}^{n} \log\left(x_j!\right)$$

$$\log\left(x_j!\right) = \log\left(1 \cdot 2 \cdot \ldots \cdot x_j\right)$$

$$\log\left(x_j!\right) = \sum_{i=1}^{x_j} \log(i)$$

$$\therefore \log L(\theta) = -n\theta + \log(\theta)\sum_{j=1}^{n} x_j - n\log\left(1 - e^{-\theta}\right) - \sum_{j=1}^{n}\sum_{i=1}^{x_j} \log(i)$$

Solving the likelihood equation to find the ML estimate of $\theta, \hat{\theta}$:

$$\frac{\partial \log L(\theta)}{\partial \theta} = -n + \frac{\sum_{j=1}^{n} x_j}{\theta} - n \cdot \frac{e^{-\theta}}{1 - e^{-\theta}} = 0$$

$$0 = -n + \frac{n\bar{x}}{\theta} - \frac{ne^{-\theta}}{1 - e^{-\theta}}$$

$$0 = -1 + \frac{\bar{x}}{\theta} - \frac{e^{-\theta}}{1 - e^{-\theta}}$$

$$0 = \frac{\bar{x}}{\theta} + \frac{-1 + e^{-\theta} - e^{-\theta}}{1 - e^{-\theta}}$$

$$\frac{\bar{x}}{\theta} = \frac{1}{1 - e^{-\theta}}$$

$$\therefore \hat{\theta} = \bar{x}\left(1 - e^{-\hat{\theta}}\right)$$

3.

Using method of moments, want to solve $\mathbb{E}[X] = \mu = \frac{1}{n}\sum_{j=1}^{n} x_j$ to obtain the moment estimate of $\theta, \hat{\theta}$. Using $\mathbb{E}[X]$ derived in **i)**, we have:

$$\frac{\theta}{1 - e^{-\theta}} = \frac{1}{n}\sum_{j=1}^{n} x_j = \bar{x}$$

$$\therefore \hat{\theta} = \bar{x}\left(1 - e^{-\hat{\theta}}\right)$$

Thus, the ML estimate is the same as the moment estimate.

4.

Denote Fisher's expected information by $\mathbb{f}(\theta)$.

$$\mathbb{f}(\theta) = \mathbb{E}\left[\left(\frac{\partial \log L(\theta)}{\partial \theta}\right)^2\right]$$

$$I(\theta) = -\frac{\partial^2 \log L(\theta)}{\partial \theta^2}$$

Since we have univariate data $(x_1, \dots, x_n)$ and it can be shown that the likelihood function for the observed data belongs to the 1-parameter regular exponential family with natural parameter $c(\theta)$ given by a scalar (see **vii)**), we have that $\mathbb{E}[T(X_1, \dots, X_n)] = T(x_1, \dots, x_n)$ at $\theta = \hat{\theta}$ (ML estimate of θ) where T is a sufficient statistic for θ (Tutorial 2 Q1.iii)). If we were to evaluate $\mathbb{f}(\theta)$ and $I(\theta)$ analytically by working with its 1-parameter regular exponential family form, it would be evident that the only difference between the two would be a $\mathbb{E}[T(X_1, \dots, X_n)]$ term in $\mathbb{f}(\theta)$ relative to a $T(x_1, \dots, x_n)$ in $I(\theta)$. Since $\mathbb{E}[T(X_1, \dots, X_n)] = T(x_1, \dots, x_n)$ at $\theta = \hat{\theta}$, it is clear that $\mathbb{f}(\hat{\theta}) = I(\hat{\theta})$ i.e. estimated Fisher information is equal to the observed information.

5.

$$\frac{\partial \log L(\theta)}{\partial \theta} = -n + \frac{\sum_{j=1}^{n} x_j}{\theta} - n \cdot \frac{e^{-\theta}}{1 - e^{-\theta}}, \qquad \text{(iii))}$$

$$\frac{\partial \log L(\theta)}{\partial \theta} = \frac{\sum_{j=1}^{n} x_j}{\theta} - \frac{n}{1 - e^{-\theta}}$$

Verifying expected value of score statistic is zero:

$$\mathbb{E}\left[\frac{\partial \log L(\theta)}{\partial \theta}\right] = \mathbb{E}\left[\frac{\sum_{j=1}^{n} x_j}{\theta} - \frac{n}{1 - e^{-\theta}}\right]$$

$$\mathbb{E}\left[\frac{\partial \log L(\theta)}{\partial \theta}\right] = \frac{1}{\theta}\sum_{j=1}^{n} \mathbb{E}[x_j] - \frac{n}{1 - e^{-\theta}}, \qquad \text{(linearity)}$$

$$\mathbb{E}\left[\frac{\partial \log L(\theta)}{\partial \theta}\right] = \frac{n}{\theta}\mathbb{E}[X] - \frac{n}{1 - e^{-\theta}}, \qquad \left(X_j \sim^{iid} Poi_0(\theta), j = 1, \dots, n\right)$$

$$\mathbb{E}\left[\frac{\partial \log L(\theta)}{\partial \theta}\right] = \frac{n}{\theta} \cdot \frac{\theta}{1 - e^{-\theta}} - \frac{n}{1 - e^{-\theta}}, \qquad \text{(i))}$$

$$\mathbb{E}\left[\frac{\partial \log L(\theta)}{\partial \theta}\right] = \frac{n}{1 - e^{-\theta}} - \frac{n}{1 - e^{-\theta}}$$

$$\mathbb{E}\left[\frac{\partial \log L(\theta)}{\partial \theta}\right] = 0$$

Evaluating Fisher's expected information using the above information:

$$\therefore \mathbb{f}(\theta) = \mathbb{E}\left[\left(\frac{\partial \log L(\theta)}{\partial \theta}\right)^2\right] = Var\left(\frac{\partial \log L(\theta)}{\partial \theta}\right)$$

$$\mathbb{f}(\theta) = Var\left(\frac{\sum_{j=1}^{n} x_j}{\theta} - \frac{n}{1-e^{-\theta}}\right)$$

$$\mathbb{f}(\theta) = \frac{1}{\theta^2} Var\left(\sum_{j=1}^{n} x_j\right), \qquad (\text{properties of } Var(.))$$

$$\mathbb{f}(\theta) = \frac{n}{\theta^2} Var(X), \qquad (X_j \sim^{iid} Poi_0(\theta), j = 1, \dots, n)$$

Since it is too difficult to evaluate $\mathbb{E}[X^2]$ in $Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$, we shall instead first find the PGF (probability generating function) of X denoted by $G_X(z)$ and then use $Var(X) = G_X''(1) + G_X'(1) - \left(G_X'(1)\right)^2$:

$$G_X(z) = \mathbb{E}[z^X]$$

$$G_X(z) = \sum_{x=1}^{\infty} z^x \mathbb{P}(X = x)$$

$$G_X(z) = \sum_{x=1}^{\infty} z^x \cdot \frac{e^{-\theta}\theta^x}{(1-e^{-\theta})x!}$$

$$G_X(z) = \frac{e^{-\theta}}{1-e^{-\theta}} \sum_{x=1}^{\infty} \frac{(z\theta)^x}{x!}$$

$$G_X(z) = \frac{e^{-\theta}}{1-e^{-\theta}} \left(-1 + 1 + \sum_{x=1}^{\infty} \frac{(z\theta)^x}{x!}\right)$$

$$G_X(z) = \frac{e^{-\theta}}{1-e^{-\theta}} \left(-1 + \sum_{x=0}^{\infty} \frac{(z\theta)^x}{x!}\right)$$

$$G_X(z) = \frac{e^{-\theta}}{1-e^{-\theta}} (-1 + e^{z\theta}), \qquad \left(\sum_{x=0}^{\infty} \frac{(z\theta)^x}{x!} = \text{power series for } e^{z\theta}\right)$$

$$G_X(z) = \frac{e^{\theta(z-1)} - e^{-\theta}}{1-e^{-\theta}}$$

We know that $G_X'(1) = \mathbb{E}[X] = \frac{\theta}{1-e^{-\theta}}$, so it remains to find $G_X''(z)$ and then find $G_X''(1)$:

$$G_X'(z) = \frac{\theta e^{\theta(z-1)}}{1-e^{-\theta}}$$

$$G_X''(z) = \frac{\theta^2 e^{\theta(z-1)}}{1-e^{-\theta}}$$

$$\therefore G_X''(1) = \frac{\theta^2 e^{\theta(1-1)}}{1-e^{-\theta}}$$

$$G_X''(1) = \frac{\theta^2 e^0}{1-e^{-\theta}} = \frac{\theta^2}{1-e^{-\theta}}$$

$$\therefore Var(X) = G_X''(1) + G_X'(1) - \left(G_X'(1)\right)^2$$

$$Var(X) = \frac{\theta^2}{1-e^{-\theta}} + \frac{\theta}{1-e^{-\theta}} - \left(\frac{\theta}{1-e^{-\theta}}\right)^2$$

$$Var(X) = \frac{\theta+\theta^2}{1-e^{-\theta}} - \frac{\theta^2}{(1-e^{-\theta})^2}$$

$$\therefore \mathbb{f}(\theta) = \frac{n}{\theta^2} \cdot \left(\frac{\theta+\theta^2}{1-e^{-\theta}} - \frac{\theta^2}{(1-e^{-\theta})^2}\right)$$

$$\mathbb{f}(\theta) = n\left(\frac{1+\theta}{\theta(1-e^{-\theta})} - \frac{1}{(1-e^{-\theta})^2}\right)$$

$$\rightarrow \mathbb{f}(\hat{\theta}) = n\left(\frac{1+\bar{x}\left(1-e^{-\hat{\theta}}\right)}{\bar{x}\left(1-e^{-\hat{\theta}}\right)\cdot\left(1-e^{-\hat{\theta}}\right)} - \frac{1}{\left(1-e^{-\hat{\theta}}\right)^2}\right)$$

$$\mathbb{f}(\hat{\theta}) = n\left(\frac{1+\bar{x}\left(1-e^{-\hat{\theta}}\right)}{\bar{x}\left(1-e^{-\hat{\theta}}\right)^2} - \frac{1}{\left(1-e^{-\hat{\theta}}\right)^2}\right)$$

$$\mathbb{f}(\hat{\theta}) = \frac{n}{\left(1-e^{-\hat{\theta}}\right)^2}\left(\frac{1}{\bar{x}} + 1 - e^{-\hat{\theta}} - 1\right)$$

$$\mathbb{f}(\hat{\theta}) = \frac{n}{\left(1-e^{-\hat{\theta}}\right)^2}\left(\frac{1}{\bar{x}} - e^{-\hat{\theta}}\right)$$

Evaluating $I(\theta)$:

$$I(\theta) = -\frac{\partial^2 \log L(\theta)}{\partial \theta^2}$$

$$I(\theta) = -\frac{\partial}{\partial \theta}\left(\frac{\sum_{j=1}^{n} x_j}{\theta} - \frac{n}{1 - e^{-\theta}}\right)$$

$$I(\theta) = -\left(-\frac{n\bar{x}}{\theta^2} - \frac{-ne^{-\theta}}{(1 - e^{-\theta})^2}\right)$$

$$I(\theta) = n\left(\frac{\bar{x}}{\theta^2} - \frac{e^{-\theta}}{(1 - e^{-\theta})^2}\right)$$

$$\rightarrow I(\hat{\theta}) = n\left(\frac{\bar{x}}{\left(\bar{x}\left(1 - e^{-\hat{\theta}}\right)\right)^2} - \frac{e^{-\hat{\theta}}}{\left(1 - e^{-\hat{\theta}}\right)^2}\right)$$

$$I(\hat{\theta}) = n\left(\frac{1}{\bar{x}\left(1 - e^{-\hat{\theta}}\right)^2} - \frac{e^{-\hat{\theta}}}{\left(1 - e^{-\hat{\theta}}\right)^2}\right)$$

$$I(\hat{\theta}) = \frac{n}{\left(1 - e^{-\hat{\theta}}\right)^2}\left(\frac{1}{\bar{x}} - e^{-\hat{\theta}}\right) = \mathbb{I}(\hat{\theta})$$

6.

$$f(k;\theta) = \mathbb{P}(X = k)$$

$$f(k;\theta) = \frac{e^{-\theta}\theta^k}{(1 - e^{-\theta})k!}$$

$$f(k;\theta) = \frac{\theta^k}{(e^{\theta} - 1)k!}$$

To obtain ML estimate of $f(k;\theta)$, set $\frac{\partial f(k;\theta)}{\partial \theta} = 0$ and solve for θ:

$$\frac{\partial f(k;\theta)}{\partial \theta} = \frac{1}{k!}\left(\frac{k\theta^{k-1}\left(e^{\theta} - 1\right) - \theta^k e^{\theta}}{(e^{\theta} - 1)^2}\right) = 0$$

$$\frac{k\theta^{k-1}}{e^{\theta} - 1} - \frac{\theta^k e^{\theta}}{(e^{\theta} - 1)^2} = 0$$

$$\frac{k\theta^{k-1}}{e^{\theta} - 1} = \frac{\theta^k e^{\theta}}{(e^{\theta} - 1)^2}$$

$$ke^{-\theta}\left(e^{\theta}-1\right)=\theta^{k+1-k}$$

$$\therefore\ \hat{\theta}=k\left(1-e^{-\hat{\theta}}\right)$$

$$\therefore\ f\left(k;\hat{\theta}\right)=\frac{e^{-\hat{\theta}}\hat{\theta}^{k}}{\left(1-e^{-\hat{\theta}}\right)k!}$$

$$f\left(k;\hat{\theta}\right)=\frac{e^{-\hat{\theta}}\left(k\left(1-e^{-\hat{\theta}}\right)\right)^{k}}{\left(1-e^{-\hat{\theta}}\right)k!}$$

$$f\left(k;\hat{\theta}\right)=\frac{e^{-\hat{\theta}}\left(k\left(1-e^{-\hat{\theta}}\right)\right)^{k-1}}{(k-1)!}$$

7.

Using the Fisher-Neyman factorisation theorem, we wish to factorise the joint pdf of $X_1,\dots,X_n$ in the form:

$$f(x_1,\dots,x_n;\theta)=h_1(T(x_1,\dots,x_n);\theta)h_2(x_1,\dots,x_n),\qquad \forall\theta\in\Omega$$

Using the likelihood function from **ii)**:

$$f(x_1,\dots,x_n;\theta)=L(\theta)=\frac{e^{-n\theta}\theta^{\sum_{j=1}^{n}x_j}}{(1-e^{-\theta})^{n}\prod_{j=1}^{n}x_j!}$$

$$f(x_1,\dots,x_n;\theta)=\frac{\theta^{\sum_{j=1}^{n}x_j}}{(e^{\theta}-1)^{n}\prod_{j=1}^{n}x_j!}$$

$$f(x_1,\dots,x_n;\theta)=h_1(T(x_1,\dots,x_n);\theta)h_2(x_1,\dots,x_n)$$

Where

$$h_1(T(x_1,\dots,x_n);\theta)=\frac{\theta^{\sum_{j=1}^{n}x_j}}{(e^{\theta}-1)^{n}}$$

$$h_2(x_1,\dots,x_n)=\frac{1}{\prod_{j=1}^{n}x_j!}$$

$$T(x_1,\dots,x_n)=\sum_{j=1}^{n}x_j$$

Since we have successfully obtained the desired form, by the Fisher-Neyman factorisation theorem, T is a sufficient statistic for θ. Furthermore, we wish to show that the joint pdf belongs to the 1-parameter regular exponential family to prove T is also complete sufficient. In other words, we wish to show:

$$f(x_1, \dots, x_n; \theta) = \frac{b(x_1, \dots, x_n)e^{c(\theta)^T T(x_1, \dots, x_n)}}{a(\theta)}$$

Since $c(\theta)$ is a scalar:

$$f(x_1, \dots, x_n; \theta) = \frac{b(x_1, \dots, x_n)e^{c(\theta) T(x_1, \dots, x_n)}}{a(\theta)}$$

Let:

$$a(\theta) = \left(e^\theta - 1\right)^n$$

$$b(\theta) = \frac{1}{\prod_{j=1}^n x_j!}$$

$$c(\theta) = \log(\theta)$$

Then, we have:

$$f(x_1, \dots, x_n; \theta) = \frac{e^{\log\left(\theta^{\sum_{j=1}^n x_j}\right)}}{(e^\theta - 1)^n \prod_{j=1}^n x_j!}$$

$$f(x_1, \dots, x_n; \theta) = \frac{e^{\sum_{j=1}^n x_j \cdot \log(\theta)}}{(e^\theta - 1)^n \prod_{j=1}^n x_j!}$$

$$f(x_1, \dots, x_n; \theta) = \frac{e^{\log(\theta) \sum_{j=1}^n x_j}}{(e^\theta - 1)^n \prod_{j=1}^n x_j!}$$

which is in the desired form. Thus, this form for the joint Zero-Truncated Poisson (ZTP) density pdf $f(x_1, \dots, x_n; \theta)$ for the observed data shows that it belongs to the 1-parameter regular exponential distribution with natural/canonical parameter given by $c(\theta) = \log(\theta)$. Hence, T is also a complete sufficient statistic for θ which in turn implies it is also a minimal sufficient statistic for θ.

8.

An unbiased estimator for the ZTP density at $X = k$ is given by:

$$f(k; n\bar{x}) = f\left(k; \sum_{j=1}^{n} x_j\right)$$

$$f(k; n\bar{x}) = \frac{e^{-n\bar{x}}(n\bar{x})^k}{(1 - e^{-n\bar{x}})k!}$$

It is also an UMVU estimator since we proved in **vii)** that $T(x_1, \ldots, x_n) = n\bar{x} = \sum_{j=1}^{n} x_j$ is a complete sufficient statistic for θ. This was done by firstly successfully factorising the joint pdf into the desired form to use the Fisher-Neyman factorisation theorem to show that T is sufficient for θ and then demonstrating that the joint pdf also belongs to the 1-parameter regular exponential distribution proving that T is also complete sufficient for θ.