CTA200 – Assignment 3 Writeup

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Question 1: Iteration in the Complex Plane

To explore the boundary between stability and divergence in a simple iterative system, I investigated the behavior of the recurrence relation:

$$z_{i+1} = z_i^2 + c, \quad z_0 = 0$$

for each complex number $c = x + iy$ in the region $-2 < x < 2$, $-2 < y < 2$. For each c, I applied this recurrence up to a fixed number of iterations (max_iter = 100) and recorded whether the absolute value of z exceeded 2. This allows us to visualize which starting values c cause the iteration to remain bounded and which cause it to diverge.

I implemented the iteration in a separate Python module and evaluated it across a uniform grid in the complex plane. In the first figure, I show a binary image where black points represent values of c that cause divergence within 100 iterations, and white points represent bounded trajectories.

To capture more detail, I then generated a second image where each point is colored according to the number of iterations it took for the sequence to diverge. This "escape-time" plot gives a smooth gradient and highlights the intricate boundary between stability and chaos.

These figures illustrate the sensitivity of this system to initial conditions and demonstrate fractal structure in a seemingly simple quadratic iteration.

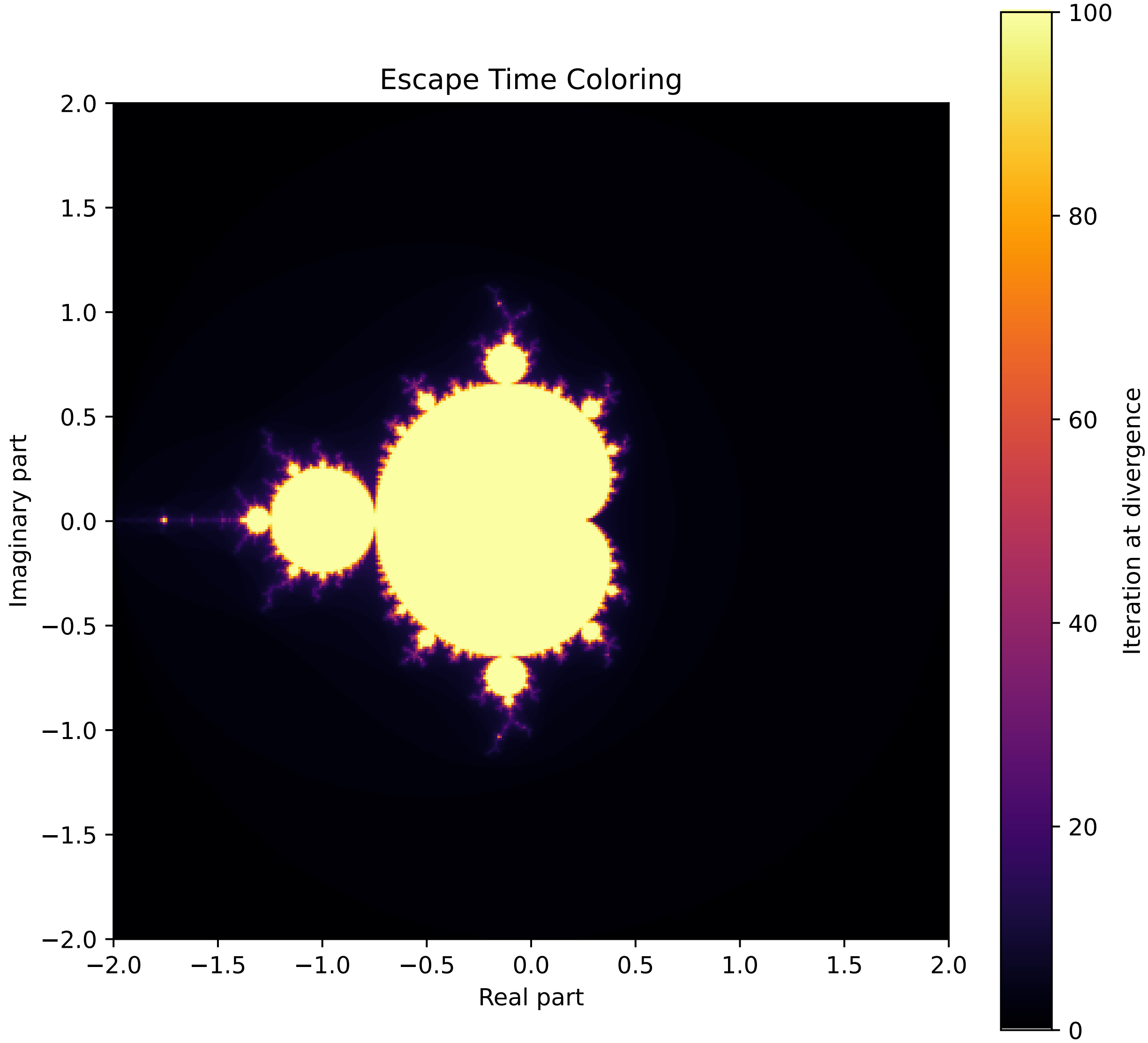

Figure 1: Escape-time coloring of complex iteration. Brighter colors indicate faster divergence.

Question 2: The Lorenz System

The Lorenz system of differential equations models atmospheric convection in a simplified setting, capturing nonlinear dynamics and sensitivity to initial conditions. The equations are:

$$\dot{X} = -\sigma(X - Y)$$
$$\dot{Y} = rX - Y - XZ$$
$$\dot{Z} = -bZ + XY$$

I solved these equations using the Runge-Kutta method via `solve_ivp` in Python's `scipy` library. I used the standard Lorenz parameters ($\sigma = 10$, $r = 28$, $b = \frac{8}{3}$) and initial conditions $[0, 1, 0]$. The system was integrated over a time span of 60 time units with a step size of approximately 0.01.

Figure 1 in Lorenz's original paper shows $Y(t)$ over time. Our numerical reproduction reveals how the system begins with smooth oscillations and transitions into chaotic behavior. Small changes in the trajectory compound quickly, leading to an unpredictable long-term outcome even though the equations are deterministic.

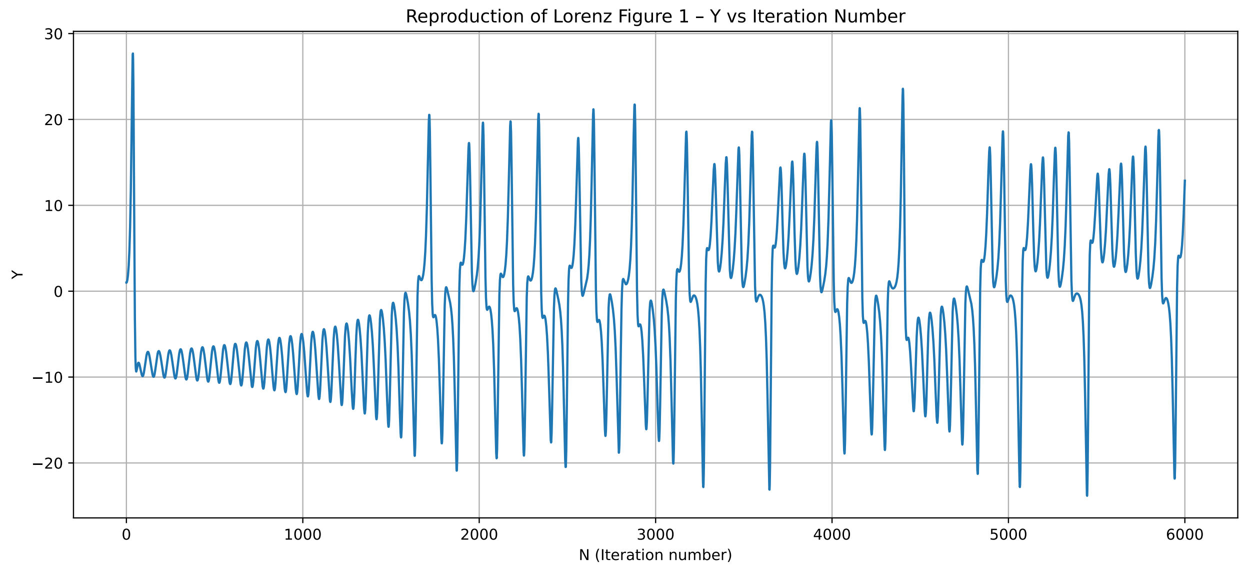

Figure 2: Reproduction of Lorenz Figure 1: Y vs. iteration number $N = t/0.01$.

To recreate Lorenz's Figure 2, I extracted the segment of the trajectory corresponding to iteration numbers 1400 to 1900 (i.e., time $t = 14$ to $t = 19$). I plotted two projections of the attractor: X vs. Y and Y vs. Z. These phase-space diagrams show the "butterfly"-shaped structure for which the Lorenz system is famous, demonstrating how the state moves between two lobes in a chaotic, non-repeating pattern.

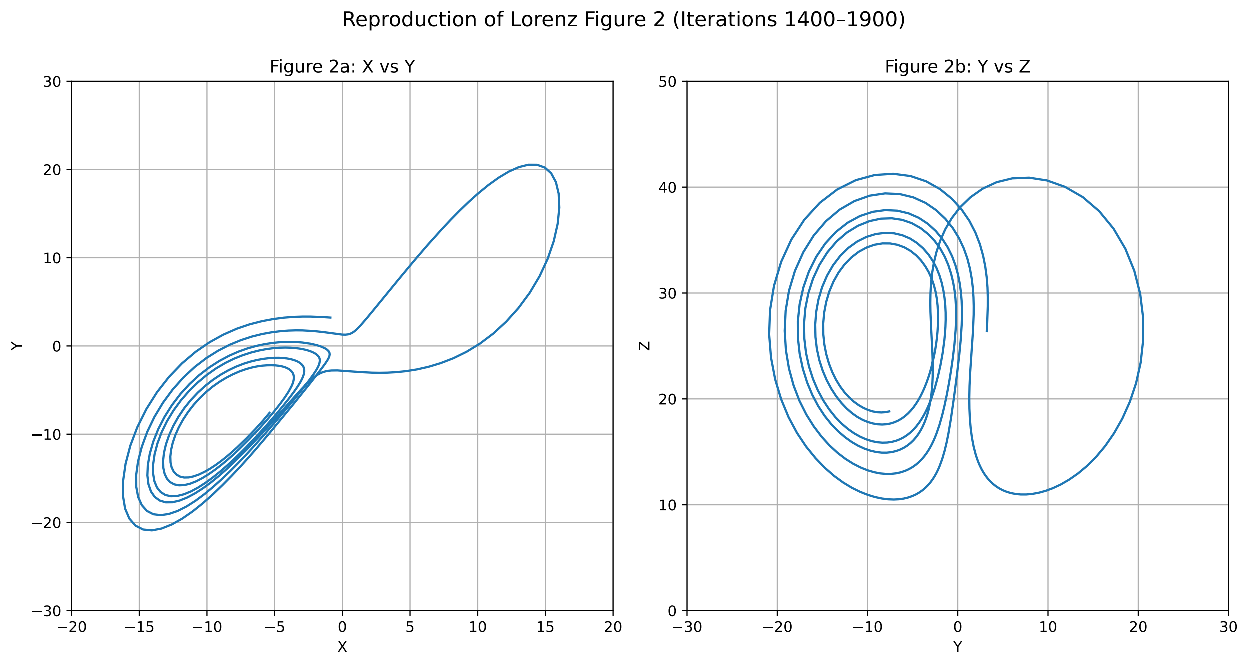

Figure 3: Reproduction of Lorenz Figure 2: X vs Y (left) and Y vs Z (right), for iterations 1400–1900.

Sensitivity to Initial Conditions

To examine the chaotic nature of the Lorenz system, I repeated the simulation using an initial condition that was nearly identical to the original: $[0.0, 1.00000001, 0.0]$. Despite the minuscule difference in Y_0, the trajectories of the two solutions diverged rapidly over time.

I computed the Euclidean distance between the two trajectories at each time point. Figure 4 shows the results, with the Euclidean distance between the two trajectories plotted on a semilog scale using a logarithmic y-axis. This highlights the exponential rate of divergence typical of chaotic systems.. The linear trend in this plot confirms exponential divergence, which is a hallmark of chaos. This finding supports Lorenz's conclusion that deterministic systems can exhibit unpredictable behavior due to extreme sensitivity to initial conditions — the so-called "butterfly effect."

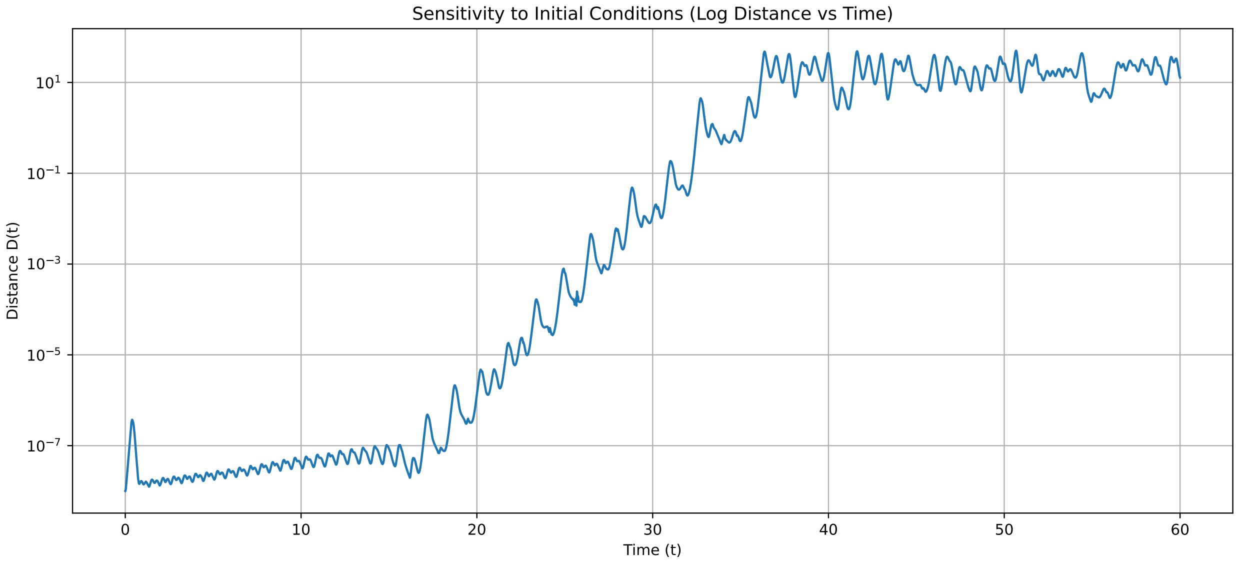

Figure 4: Log plot of distance between two Lorenz trajectories with slightly different initial conditions.