Gabor filters

Konstantinos G. Derpanis
York University
kosta@cs.yorku.ca

Version 1.3

April 23, 2007

In this note the Gabor filter is reviewed. The Gabor filter was originally introduced by Dennis Gabor (Gabor, 1946). The one-dimensional Gabor filter is defined as the multiplication of a cosine/sine (even/odd) wave with a Gaussian windows (see Fig. 1), as follows,

$$g_e(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \cos(2\pi\omega_0 x) \tag{1}$$

$$g_o(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \sin(2\pi\omega_0 x) \tag{2}$$

where ω_0 defines the centre frequency (i.e., the frequency in which the filter yields the greatest response) and σ the spread of the Gaussian window.

The power spectrum of the Gabor filter is given by the sum of two Gaussians centred at $\pm\omega_0$:

$$\|G(\omega)\| = e^{-2\pi^2\sigma^2(\omega-\omega_0)^2} + e^{-2\pi^2\sigma^2(\omega+\omega_0)^2} \tag{3}$$

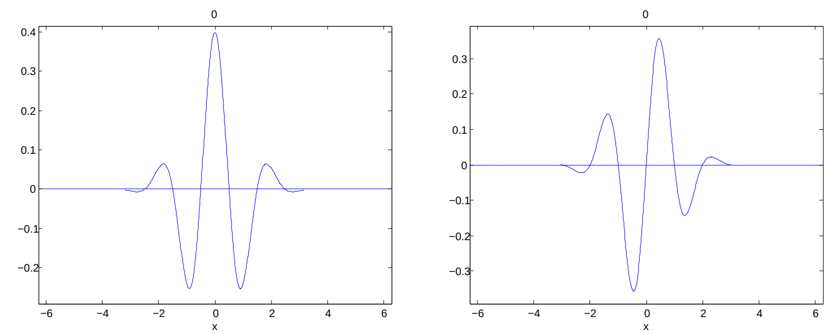
This can be reasoned as follows. The power spectrum of a sine wave are two impulses located at $\pm\omega_0$ and the power spectrum of Gaussian is a (non-normalized) Gaussian. Multiplication in the temporal (spatial) domain is equivalent to convolution in the frequency domain (Oppenheim, Willsky & S.H., 1997).

The *uncertainty principle* states that the product of the spread (i.e., uncertainty) of a signal in the time and frequency domains must exceed or equal a fixed constant,

$$\triangle t \triangle f = c, \tag{4}$$

where c[1] is a constant, $\triangle t$ and $\triangle f$ represent the measure of the spread of the signal in the time and frequency domains, respectively (see Fig. 2). The implication of this

[1]The exact value of the constant c depends on the form of the Fourier transform used.



(a) Gabor cosine (even) (b) Gabor sine (odd)

Figure 1: One-dimensional Gabor filters.

principle is that the accuracy with which one can measure a signal in one domain limits the attainable accuracy of the measurement in the other domain. Gabor (Gabor, 1946) demonstrated that the **complex** Gabor filter given by,

$$
\begin{aligned}
g(x) =& g_e(x) + i g_o(x) \\
=& \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \Big(\cos(2\pi\omega_0 x) + i \sin(2\pi\omega_0 x) \Big) \\
=& \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} e^{i(2\pi\omega_0 x)}
\end{aligned} \tag{5}
$$

attains the optimal (lower bound) compromise between the localization in the time and frequency domains; notice that the Gaussian function is an instance of a Gabor filter with centre frequency $\omega_0 = 0$. Note that the constituent real-valued Gabor elementary functions (i.e., the even and odd part taken separately) do **not** as widely believed minimize the joint uncertainty (Stork & Wilson, 1990). Furthermore, the selection of a different localization measure may result in a different class of "optimal" function altogether (Lerner, 1961; Stork & Wilson, 1990), casting doubt on the primacy of the Gabor function often cited in the literature.

Daugman (Daugman, 1980; Daugman, 1985) extended the Gabor filter to two-dimensions (see Fig. 3), as follows,

$$
g_e(x, y) = \frac{1}{2\pi\sigma_x\sigma_y} e^{-\frac{1}{2}(\frac{x^2}{\sigma_x} + \frac{y^2}{\sigma_y})} \cos(2\pi\omega_{x_0} x + 2\pi\omega_{y_0} y) \tag{6}
$$

$$
g_o(x, y) = \frac{1}{2\pi\sigma_x\sigma_y} e^{-\frac{1}{2}(\frac{x^2}{\sigma_x} + \frac{y^2}{\sigma_y})} \sin(2\pi\omega_{x_0} x + 2\pi\omega_{y_0} y) \tag{7}
$$

where $(\omega_{x_0}, \omega_{y_0})$ defines the centre frequency and (σ_x, σ_y) the (potentially asymmetric) spread of the Gaussian window.

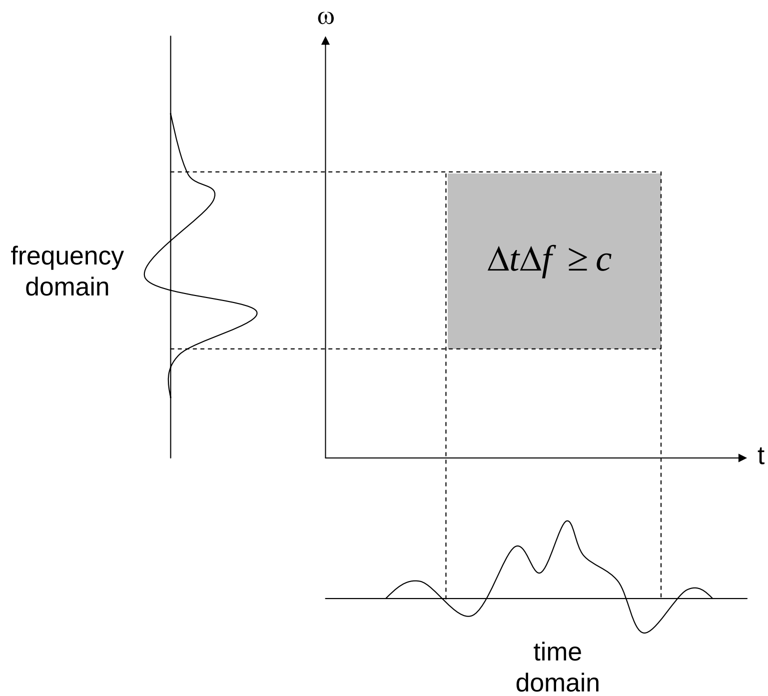


Figure 2: Joint localization of a signal in time and frequency domains.

For the purpose of extracting optical flow, Heeger (Heeger, 1987) utilized the three-dimensional (space-time) Gabor filter,

$$g_e(x, y, t) = \frac{1}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_t} e^{-\frac{1}{2}(\frac{x^2}{\sigma_x}+\frac{y^2}{\sigma_y}+\frac{t^2}{\sigma_t})} \cos(2\pi\omega_{x_0}x + 2\pi\omega_{y_0}y + 2\pi\omega_{t_0}t) \tag{8}$$

$$g_o(x, y, t) = \frac{1}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_t} e^{-\frac{1}{2}(\frac{x^2}{\sigma_x}+\frac{y^2}{\sigma_y}+\frac{t^2}{\sigma_t})} \sin(2\pi\omega_{x_0}x + 2\pi\omega_{y_0}y + 2\pi\omega_{t_0}t) \tag{9}$$

where $(\omega_{x_0}, \omega_{y_0}, \omega_{t_0})$ defines the centre frequency and $(\sigma_x, \sigma_y, \sigma_t)$ defines the (potentially asymmetric) spread of the Gaussian window.

Heeger (Heeger, 1987) demonstrated that the three-dimensional (similarly for the two-dimensional case) Gabor filter can be built from one-dimensional separable components. Considering the two-dimensional Gabor filter, let k be the size of the two-dimensional convolution kernel and n be the size of an image (in pixels). The complexity of the non-separable convolution of the Gabor filter is reduced from $O(k^2n^2)$ to $O(kn^2)$.

An application of Gabor filters is in local time-frequency analysis of signals, specifically, a fixed windowed *Fourier transform*, referred to as the *Gabor transform*. A difficulty with the Gabor transform is that it is linearly independent but highly non-orthogonal and as such cannot be easily inverted. As a result of non-orthogonality, the functions, $r_i(n)$, used for reconstructing the discrete signal, $f(n)$, are highly distinct from the Gabor functions used to recover the coefficients of the representation (i.e., the analysis step), where the coefficients indicate how much of its corresponding reconstruction filter, $r_i(n)$, is to be added, formally,

$$f(n) = \sum_i c_i r_i(n) \qquad \textit{reconstruction step} \tag{10}$$

where,

$$c_i = \sum_n f(n) g_i(n) \qquad \textit{analysis step.} \tag{11}$$

If the Gabor transform were indeed orthogonal (such as the *Fourier transform*), $g_i(n) = r_i(n)$.

References

Daugman, J. (1980). Two-dimensional analysis of cortical receptive field profiles. *Vision Research*, *20*, 846–856.

Daugman, J. (1985). Uncertainty relation for resolution in space, spatial frequency, and orientation optimized by two-dimensional visual cortical filters. *Journal of the Optical Society of America-A*, *2(7)*, 1160–1169.

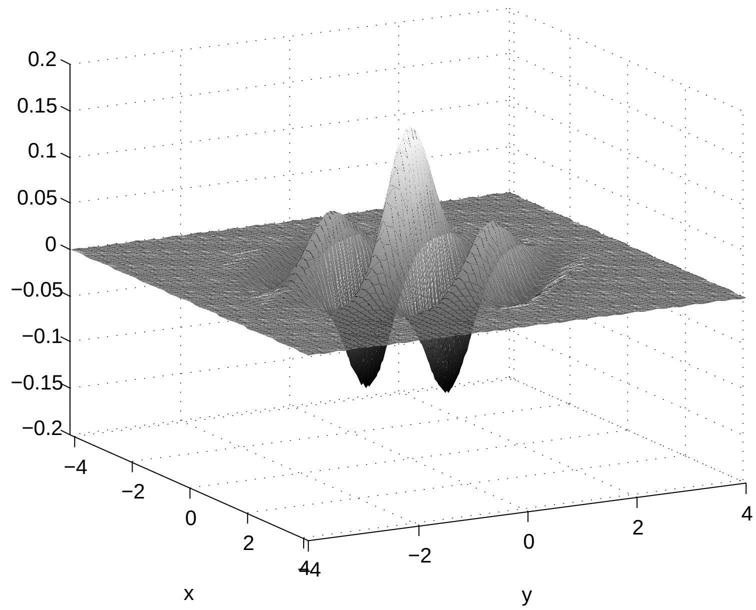
Gabor, D. (1946). Theory of communication. *Journal of the Institute of Electrical Engineers*, *93*, 429–457.

Heeger, D. (1987). Model for the extraction of image flow. *Journal of the Optical Society of America-A*, *2(2)*, 1455–1471.

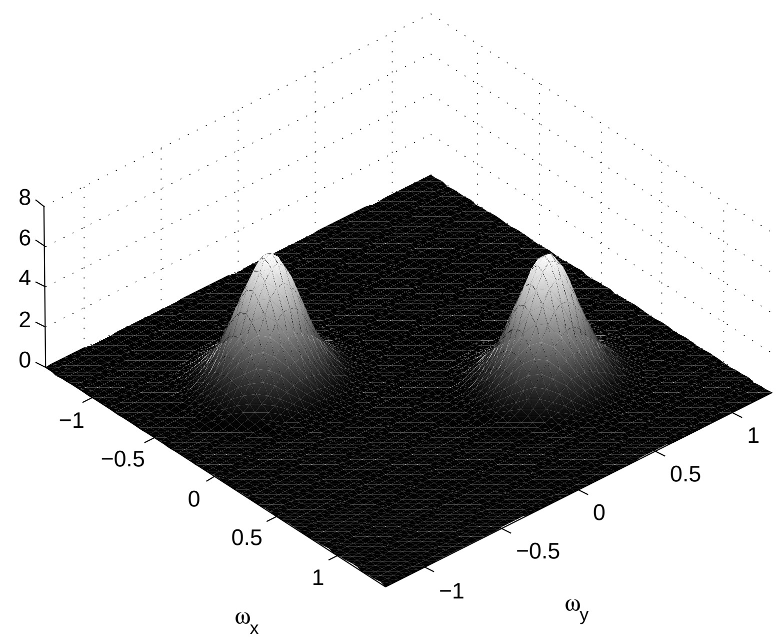
Lerner, R. (1961). Representation of signals. In E. Baghdady (Ed.), *Lectures on Communication System Theory* chapter 10, (pp. 203–242). Mc-Graw-Hill.

Oppenheim, A., Willsky, A. & S.H., N. (1997). *Signals and Systems*. Upper Saddle River, NJ: Prentice Hall.

Stork, D. & Wilson, H. (1990). Do Gabor functions provide appropriate descriptions of visual cortical receptive fields? *Journal of the Optical Society of America-A*, *7(9)*, 1362–1373.



(a) Gabor cosine (even)



(b) Gabor power spectrum

Figure 3: Two-dimensional Gabor filters.