Applied Finite Mathematics

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Contents

Chapter 1

Linear Functions

In this chapter, you will learn to:

1. Graph a linear equation.
2. Find the slope of a line.
3. Determine an equation of a line.
4. Solve linear systems.
5. Do application problems using linear equations.

1.1 Graphing a Linear Equation

In this section, you will learn to:

1. Graph a line when you know its equation.
2. Graph a line when you are given its equation in parametric form.
3. Graph and find equations of vertical and horizontal lines.

1.1.1 Graphing a Line from its Equation

Equations whose graphs are straight lines are called linear equations. The following are some examples of linear equations:

$$\begin{aligned} 2x - 3y &= 6, \\ 3x &= 4y - 7, \\ y &= 2x - 5, \\ 2y &= 3, \\ x - 2 &= 0. \end{aligned}$$

A line is completely determined by two points. Therefore, to graph a linear equation, we need to find the coordinates of two points. This can be accomplished by choosing an arbitrary value for x or y and then solving for the other variable.

Example 1.1.1. *Graph the line* $y = 3x + 2$.

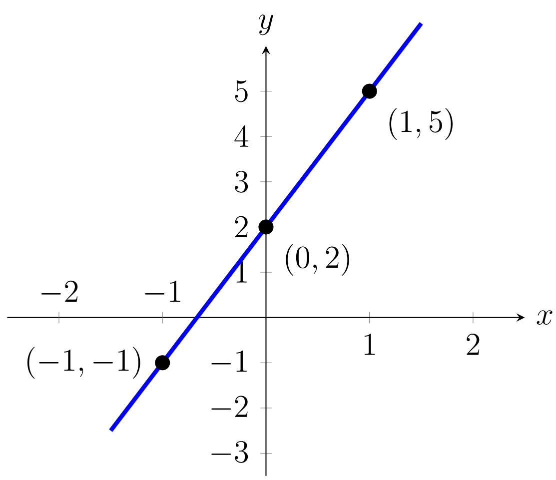
Solution 1.1.1. *We need to find the coordinates of at least two points. We arbitrarily choose* $x = -1$, $x = 0$, *and* $x = 1$.

If $x = -1$, *then* $y = 3(-1) + 2$ *or* $y = -1$. *Therefore,* $(-1, -1)$ *is a point on this line.*

If $x = 0$, *then* $y = 3(0) + 2$ *or* $y = 2$. *Hence the point* $(0, 2)$.

If $x = 1$, *then* $y = 5$, *and we get the point* $(1, 5)$. *Below, the results are summarized, and the line is graphed.*

x	y
-1	-1
0	2
1	5



Example 1.1.2. *Graph the line:* $2x + y = 4$

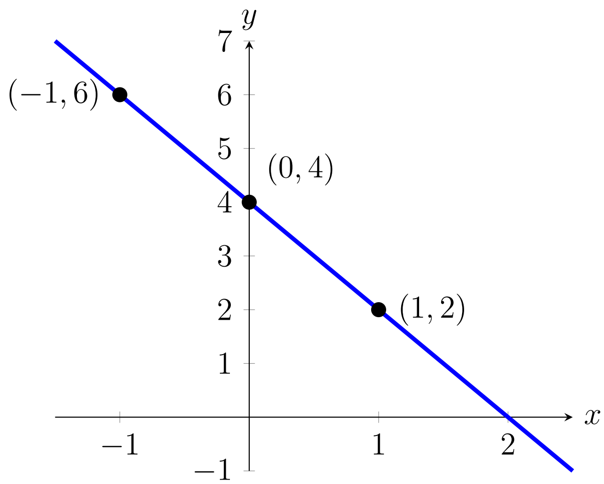
Solution 1.1.2. *Again, we need to find coordinates of at least two points. We arbitrarily choose* $x = -1$*,* $x = 0$*, and* $y = 2$*.*

If $x = -1$*, then* $2(-1) + y = 4$ *which results in* $y = 6$*. Therefore,* $(-1, 6)$ *is a point on this line.*

If $x = 0$*, then* $2(0) + y = 4$*, which results in* $y = 4$*. Hence the point* $(0, 4)$*.*

If $y = 2$*, then* $2x + 2 = 4$*, which yields* $x = 1$*, and gives the point* $(1, 2)$*. The table below shows the points, and the line is graphed.*

x	y
-1	6
0	4
1	2



1.1.2 Intercepts:

The points at which a line crosses the coordinate axes are called the intercepts. When graphing a line by plotting two points, using the intercepts is often preferred because they are easy to find.

- To find the value of the x-intercept, we let $y = 0$.
- To find the value of the y-intercept, we let $x = 0$.

Example 1.1.3. *Find the intercepts of the line: $2x - 3y = 6$, and graph.*

Solution 1.1.3. *To find the x-intercept, let $y = 0$ in the equation, and solve for x.*

$$\begin{aligned} 2x - 3(0) &= 6 \\ 2x &= 6 \\ x &= 3 \end{aligned}$$

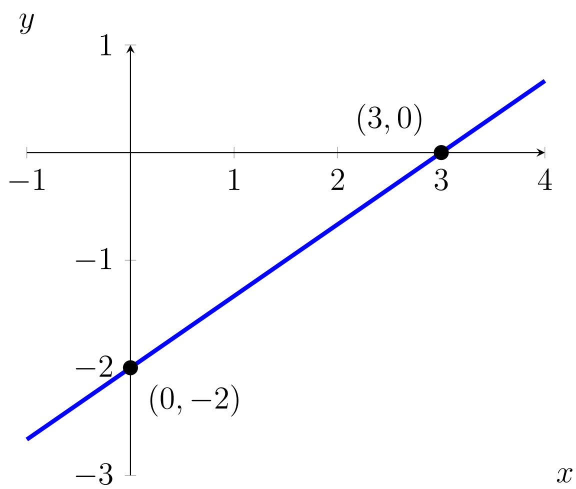
Therefore, the x-intercept is the point $(3, 0)$.

To find the y-intercept, let $x = 0$ in the equation, and solve for y.

$$\begin{aligned} 2(0) - 3y &= 6 \\ 0 - 3y &= 6 \\ -3y &= 6 \\ y &= -2 \end{aligned}$$

Therefore, the y-intercept is the point $(0, -2)$.

To graph the line, plot the points for the x-intercept $(3, 0)$ *and the y-intercept* $(0, -2)$, *and use them to draw the line.*



1.1.3 Graphing a Line from Its Equation in Parametric Form

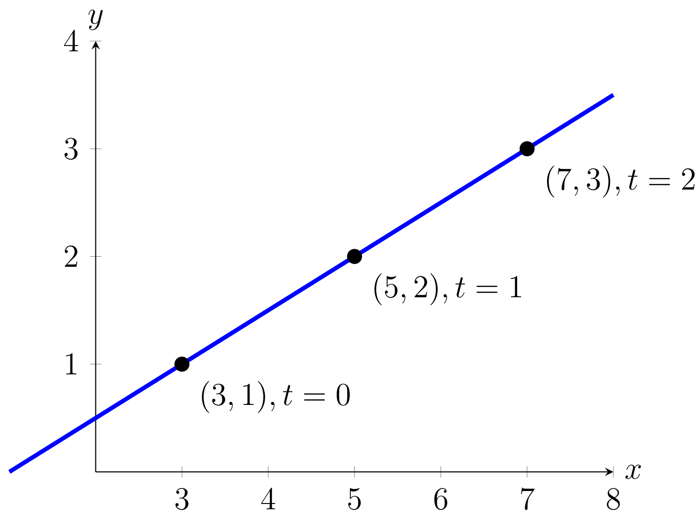
In higher math, equations of lines are sometimes written in parametric form. For example, $x = 3 + 2t, y = 1 + t$. The letter t is called the parameter or the dummy variable.

Parametric lines can be graphed by finding values for x and y by substituting numerical values for t. Plot the points using their (x, y) coordinates and use the points to draw the line.

Example 1.1.4. *Graph the line given by the parametric equations:* $x = 3 + 2t$, $y = 1 + t$

Solution 1.1.4. *Let* $t = 0, 1$ *and* 2*; for each value of* t*, find the corresponding values for* x *and* y*. The results are given in the table below.*

t	x	y
0	3	1
1	5	2
2	7	3



1.1.4 Horizontal and Vertical Lines

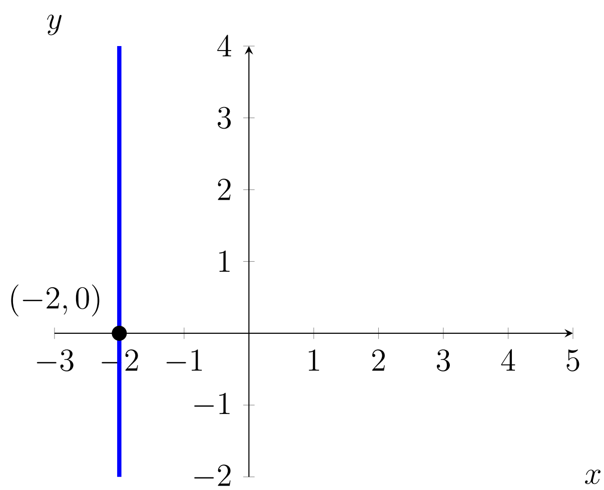
When an equation of a line has only one variable, the resulting graph is a horizontal or a vertical line.

The graph of the line $x = a$, where a is a constant, is a vertical line that passes through the point $(a, 0)$. Every point on this line has the x-coordinate equal to a, regardless of the y-coordinate.

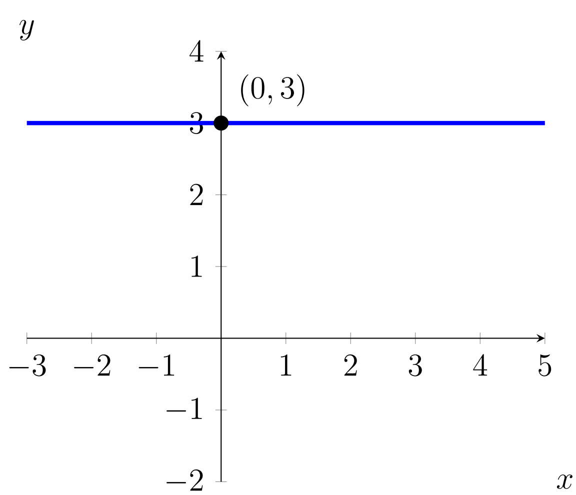
The graph of the line $y = b$, where b is a constant, is a horizontal line that passes through the point $(0, b)$. Every point on this line has the y-coordinate equal to b, regardless of the x-coordinate.

Example 1.1.5. *Graph the lines: $x = -2$, and $y = 3$.*

Solution 1.1.5. *The graph of the line $x = -2$ is a vertical line that has the x-coordinate -2 no matter what the y-coordinate is. The graph is a vertical line passing through point $(-2, 0)$.*



The graph of the line $y = 3$ *is a horizontal line that has the* y*-coordinate* 3 *regardless of what the* x*-coordinate is. Therefore, the graph is a horizontal line that passes through point* $(0, 3)$.



1.2 Slope of a Line

In this section, you will learn to:

1. Find the slope of a line.
2. Graph the line if a point and the slope are given.

In the last section, we learned to graph a line by choosing two points on the line. A graph of a line can also be determined if one point and the "steepness" of the line is known. The number that refers to the steepness or inclination of a line is called the slope of the line. From previous math courses, many of you remember slope as the "rise over run," or "the vertical change over the horizontal change" and have often seen it expressed as:

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

We give a precise definition.

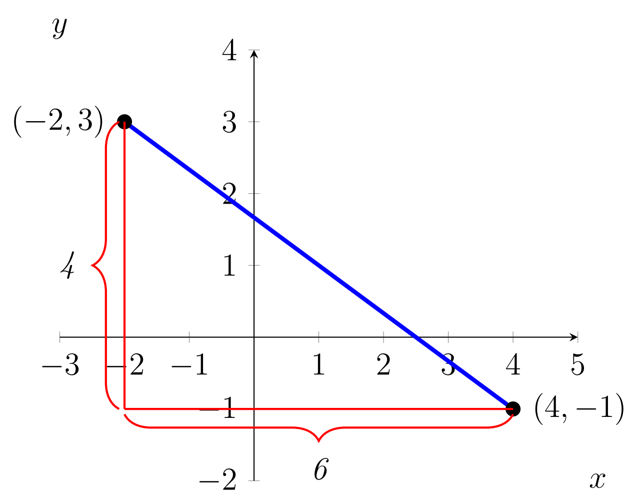
Definition 1.2.1. *If (x_1, y_1) and (x_2, y_2) are two different points on a line, the slope of the line is*

$$\textit{slope} = m = \frac{y_2 - y_1}{x_2 - x_1}$$

Example 1.2.1. *Find the slope of the line passing through points $(-2, 3)$ and $(4, -1)$, and graph the line.*

Solution 1.2.1. *Let $(x_1, y_1) = (-2, 3)$ and $(x_2, y_2) = (4, -1)$, then the slope is*

$$\textit{slope} = m = \frac{-1 - 3}{4 - (-2)} = \frac{-4}{6} = -\frac{2}{3}$$



To give the reader a better understanding, both the vertical change, -4*, and the horizontal change,* 6*, are shown in the above figure.*

When two points are given, it does not matter which point is denoted as (x_1, y_1) and which (x_2, y_2). The value for the slope will be the same.

In Example 1.2.1, if we instead choose $(x_1, y_1) = (4, -1)$ and $(x_2, y_2) = (-2, 3)$, then we will get the same value for the slope as we obtained earlier.

The steps involved are as follows:

$$\mathrm{m} = \frac{3-(-1)}{-2-4} = \frac{4}{-6} = -\frac{2}{3}$$

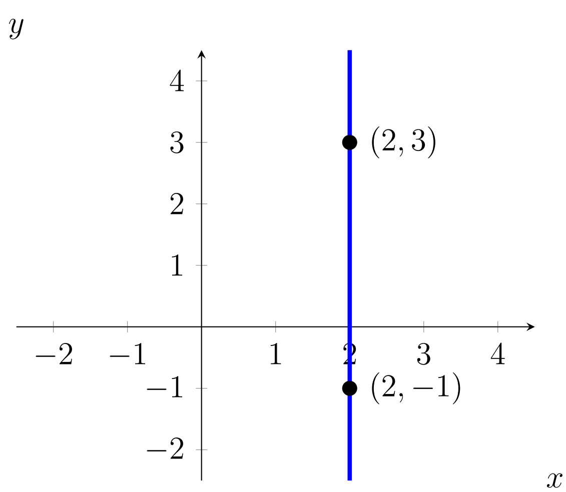
The student should further observe that

- If a line rises when going from left to right, then it has a positive slope. In this situation, as the value of x increases, the value of y also increases.
- If a line falls going from left to right, it has a negative slope; as the value of x increases, the value of y decreases.

Example 1.2.2. *Find the slope of the line that passes through the points* $(2, 3)$ *and* $(2, -1)$*, and graph.*

Solution 1.2.2. *Let* $(x_1, y_1) = (2, 3)$ *and* $(x_2, y_2) = (2, -1)$*, then the slope is*

$$m = \frac{-1-3}{2-2} = \frac{-4}{0} = \textit{undefined.}$$

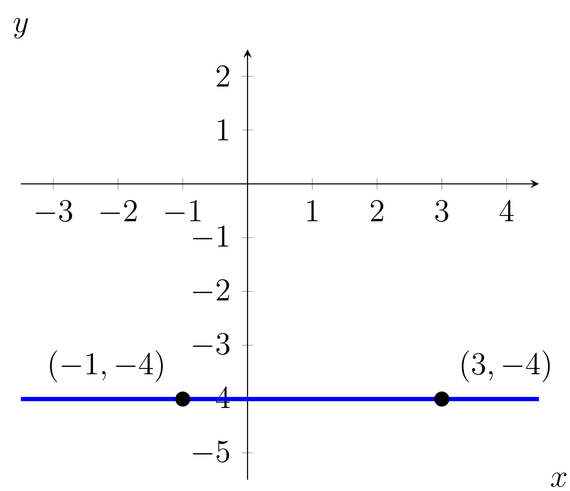


Note 1.2.1. *The slope of a vertical line is undefined.*

Example 1.2.3. *Find the slope of the line that passes through the points* $(-1,-4)$ *and* $(3,-4)$.

Solution 1.2.3. *Let* $(x_1, y_1) = (-1,-4)$ *and* $(x_2, y_2) = (3,-4)$*, then the slope is*

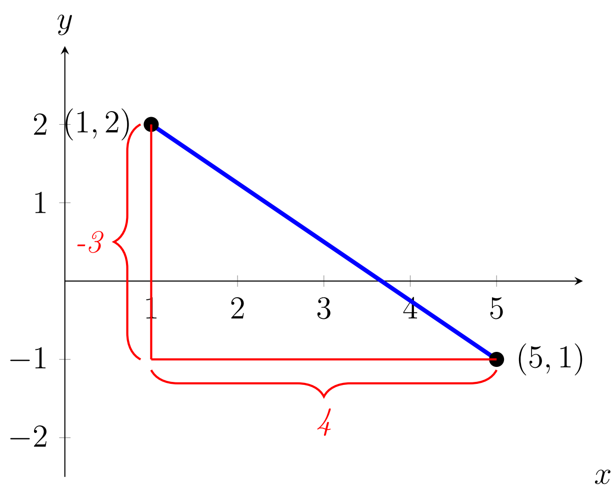
$$m = \frac{-4-(-4)}{3-(-1)} = \frac{0}{4} = 0.$$



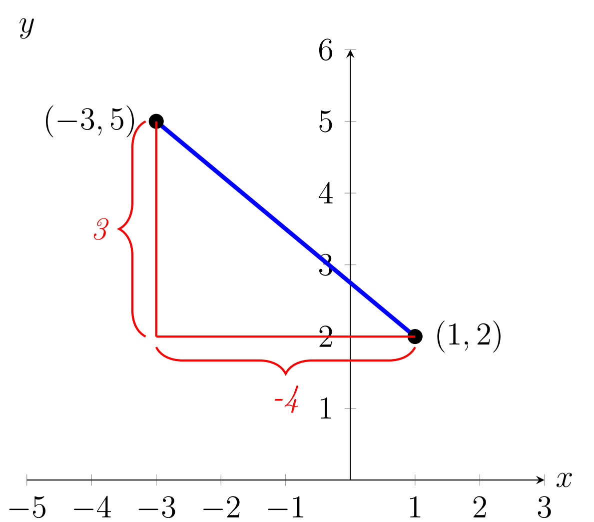
Note 1.2.2. *The slope of a horizontal line is* 0.

Example 1.2.4. *Graph the line that passes through the point* $(1,2)$ *and has a slope of* $-\frac{3}{4}$.

Solution 1.2.4. *The slope equals* $\frac{rise}{run}$. *The fact that the slope is* $\frac{-3}{4}$ *means that for every rise of* -3 *units (fall of 3 units), there is a run of 4 units. So if from the given point* $(1,2)$ *we go down 3 units and go right 4 units, we reach the point* $(5,-1)$. *The graph is obtained by connecting these two points.*



Alternatively, since $\frac{3}{-4}$ *represents the same number, the line can be drawn by starting at the point* $(1, 2)$ *and choosing a rise of 3 units followed by a run of* -4 *units. So from the point* $(1, 2)$*, we go up 3 units and to the left 4 units, thus reaching the point* $(-3, 5)$*, which is also on the same line. See figure below.*

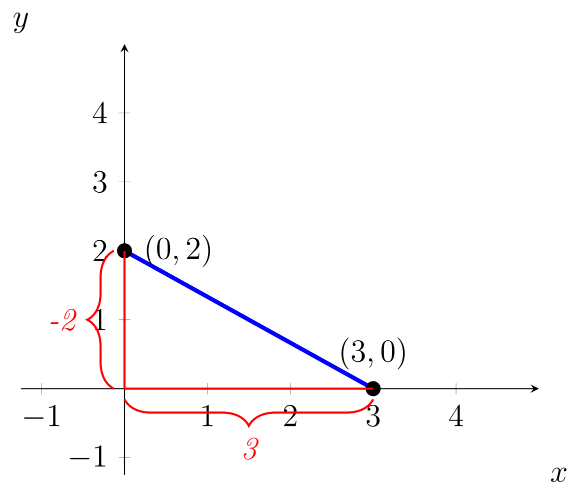


Example 1.2.5. *Find the slope of the line* $2x + 3y = 6$.

Solution 1.2.5. *In order to find the slope of this line, we will choose any two points on this line. Again, the selection of* x *and* y *intercepts seems to be a good choice. The* x*-intercept is* $(3, 0)$*, and the* y*-intercept is* $(0, 2)$*. Therefore,* *the slope is*

$$m = \frac{2-0}{3-0} = \frac{-2}{3}.$$

The graph below shows the line and the x-intercepts and y-intercepts:



Example 1.2.6. *Find the slope of the line $y = 3x + 2$.*

Solution 1.2.6. *We again find two points on the line, say $(0, 2)$ and $(1, 5)$. Therefore, the slope is*

$$m = \frac{5-2}{1-0} = \frac{3}{1} = 3.$$

Look at the slopes and the y-intercepts of the following lines.

Line	Slope	Y-Intercept
$y = 3x + 2$	3	2
$y = -2x + 5$	-2	5
$y = \frac{3}{2}x - 4$	$\frac{3}{2}$	-4

It is no coincidence that when an equation of the line is solved for y, the coefficient of the x term represents the slope, and the constant term represents the y-intercept. In other words, for the line $y = mx + b$, m is the slope, and b is the y-intercept.

Example 1.2.7. *Determine the slope and y-intercept of the line $2x+3y = 6$.*

Solution 1.2.7. *We solve for y:*

$$2x + 3y = 6$$

$$3y = -2x + 6$$
$$y = -\frac{2}{3}x + 2$$

The slope is equal to the coefficient of the x term, which is $-\frac{2}{3}$. The y-intercept is equal to the constant term, which is 2.

1.3 Determining the Equation of a Line

In this section, you will learn to:

1. Find an equation of a line if a point and the slope are given.
2. Find an equation of a line if two points are given.

So far, we were given an equation of a line and were asked to give information about it. For example, we were asked to find points on the line, find its slope, and even find intercepts. Now we are going to reverse the process. That is, we will be given either two points or a point and the slope of a line, and we will be asked to find its equation.

An equation of a line can be written in three forms: the slope-intercept form, the point-slope form, or the standard form. We will discuss each of them in this section.

A line is completely determined by two points or by a point and slope. The information we are given about a particular line will influence which form of the equation is most convenient to use. Once we know any form of the equation of a line, it is easy to re-express the equation in the other forms if needed.

The Slope-Intercept Form of a Line: $y = mx + b$

In the last section, we learned that the equation of a line whose slope $= m$ and y-intercept $= b$ is $y = mx + b$. This is called the slope-intercept form of the line and is the most commonly used form.

Example 1.3.1. *Find an equation of a line whose slope is 5, and y-intercept is 3.*

Solution 1.3.1. *Since the slope is $m = 5$, and the y-intercept is $b = 3$, the equation is $y = 5x + 3$.*

Example 1.3.2. *Find the equation of the line that passes through the point* $(2, 7)$ *and has slope* 3.

Solution 1.3.2. *Since* $m = 3$, *the partial equation is* $y = 3x + b$. *Now* b *can be determined by substituting the point* $(2, 7)$ *in the equation* $y = 3x + b$.

$7 = 3(2) + b$

$b = 1$

Therefore, the equation is $y = 3x + 1$.

Example 1.3.3. *Find an equation of the line that passes through the points* $(-1, 2)$ *and* $(1, 8)$.

Solution 1.3.3. $m = \frac{8-2}{1-(-1)} = \frac{6}{2} = 3$.

So the partial equation is $y = 3x + b$. *We can use either of the two points* $(-1, 2)$ *or* $(1, 8)$ *to find* b. *Substituting* $(-1, 2)$ *gives*

$2 = 3(-1) + b$

$5 = b$

So the equation is $y = 3x + 5$.

Example 1.3.4. *Find an equation of the line that has x-intercept* 3, *and y-intercept* 4.

Solution 1.3.4. *The x-intercept* $= 3$, *and y-intercept* $= 4$ *correspond to the points* $(3, 0)$ *and* $(0, 4)$, *respectively.*

$m = \frac{4-0}{0-3} = \frac{-4}{3}$

We are told the y-intercept is 4*; thus* $b = 4$.

Therefore, the equation is $y = -\frac{4}{3}x + 4$.

The Point-Slope Form of a Line: $y - y1 = m(x - x1)$

The point-slope form is useful when we know two points on the line and want to find the equation of the line.

Let L be a line with slope m, and known to contain a specific point $(x1, y1)$. If (x, y) is any other point on the line L, then the definition of a slope leads us to the point-slope form or point-slope formula.

The slope is $\frac{y-y1}{x-x1} = m$

Multiplying both sides by $(x - x1)$ gives the point-slope form:

$y - y1 = m(x - x1)$

Example 1.3.5. *Find the point-slope form of the equation of a line that has slope* 1.5 *and passes through the point* $(12, 4)$*.*

Solution 1.3.5. *Substituting the point* $(x1, y1) = (12, 4)$ *and* $m = 1.5$ *in the point-slope formula, we get*

$y - 4 = 1.5(x - 12)$

The student may be tempted to simplify this into the slope-intercept form $y = mx + b$*. But since the problem specifically requests point-slope form, we will not simplify it.*

The Standard Form of a Line: $Ax + By = C$

Another useful form of the equation of a line is the standard form.

If we know the equation of a line in point-slope form, $y - y1 = m(x - x1)$, or if we know the equation of the line in slope-intercept form $y = mx + b$, we can simplify the formula to have all terms for the x and y variables on one side of the equation, and the constant on the other side of the equation.

The result is referred to as the standard form of the line: $Ax + By = C$.

Example 1.3.6. *Using the point-slope formula, find the standard form of an equation of the line that passes through the point (2, 3) and has slope* $-\frac{3}{5}$*.*

Solution 1.3.6. *Substituting the point* $(2, 3)$ *and* $m = -\frac{3}{5}$ *in the point-slope formula, we get*

$$y - 3 = -\frac{3}{5}(x - 2).$$

Multiplying both sides by 5 gives us

$$5(y - 3) = -3(x - 2),$$

$$5y - 15 = -3x + 6,$$

$$3x + 5y = 21 \textit{ Standard Form.}$$

Example 1.3.7. *Find the standard form of the line that passes through the points (1, -2) and (4, 0).*

Solution 1.3.7. *First, we find the slope:* $m = \frac{0-(-2)}{4-1} = \frac{2}{3}$.

Then, the point-slope form is: $y - (-2) = \frac{2}{3}(x - 1)$.

Multiplying both sides by 3 gives us

$$3(y + 2) = 2(x - 1),$$

$$3y + 6 = 2x - 2,$$

$$-2x + 3y = -8,$$

$$2x - 3y = 8 \text{ Standard Form.}$$

Example 1.3.8. *Write the equation* $y = -\frac{2}{3}x + 3$ *in the standard form.*

Solution 1.3.8. *Multiplying both sides of the equation by 3, we get*

$$3y = -2x + 9,$$

$$2x + 3y = 9 \text{ Standard Form.}$$

Example 1.3.9. *Write the equation* $3x - 4y = 10$ *in the slope-intercept form.*

Solution 1.3.9. *Solving for* y*, we get*

$$-4y = -3x + 10,$$

$$y = \frac{3}{4}x - \frac{5}{2} \text{ Slope Intercept Form.}$$

Example 1.3.10. *Find the slope of the following lines, by inspection.*

1. $3x - 5y = 10$
2. $2x + 7y = 20$
3. $4x - 3y = 8$

Solution 1.3.10.

1. *For* $3x - 5y = 10$*, we have* $A = 3$ *and* $B = -5$*, therefore,* $m = -\frac{A}{B} = -\frac{3}{-5} = \frac{3}{5}$.
2. *For* $2x + 7y = 20$*, we have* $A = 2$ *and* $B = 7$*, therefore,* $m = -\frac{A}{B} = -\frac{2}{7}$.

3. *For* $4x - 3y = 8$*, we have* $A = 4$ *and* $B = -3$*, therefore,* $m = -\frac{A}{B} = -\frac{4}{-3} = \frac{4}{3}$.

Example 1.3.11. *Find an equation of the line that passes through* $(2, 3)$ *and has slope* $-\frac{4}{5}$.

Solution 1.3.11. *Since the slope of the line is* $-\frac{4}{5}$*, we know that the left side of the equation is* $4x + 5y$*, and the partial equation is going to be*

$$4x + 5y = c.$$

Of course, c *can easily be found by substituting for* x *and* y.

$$4(2) + 5(3) = c,$$

$$8 + 15 = c,$$

$$23 = c.$$

The desired equation is

$$4x + 5y = 23.$$

If you use this method often enough, you can do these problems very quickly. We summarize the forms for equations of a line below:

Summary 1.3.1: E

quations of Lines

- Slope-Intercept form: $y = mx + b$, where m is the slope and b is the y-intercept.
- Point-Slope form: $y - y_1 = m(x - x_1)$, where m is the slope and (x_1, y_1) is a point on the line.
- Standard form: $Ax + By = C$.
- Horizontal Line: $y = b$, where b is the y-intercept.
- Vertical Line: $x = a$, where a is the x-intercept.

1.4 Applications

In this section, you will learn to use linear functions to model real-world applications.

Now that we have learned to determine equations of lines, we get to apply these ideas in a variety of real-life situations. Read the problem carefully. Highlight important information. Keep track of which values correspond to the independent variable (x) and which correspond to the dependent variable (y).

Example 1.4.1. *A taxi service charges \$0.50 per mile plus a \$5 flat fee. What will be the cost of traveling 20 miles? What will be cost of traveling x miles?*

Solution 1.4.1. *Let x be the distance traveled, in miles, and y be the cost in dollars.*

The cost of traveling 20 miles is $y = (0.50)(20) + 5 = 10 + 5 = 15$ dollars.

The cost of traveling x miles is $y = (0.50)(x) + 5 = 0.50x + 5$ dollars.

In this problem, \$0.50 per mile is referred to as the variable cost, and the flat charge \$5 as the fixed cost. Now if we look at our cost equation $y = 0.50x+5$, we can see that the variable cost corresponds to the slope and the fixed cost to the y-intercept.

Example 1.4.2. *The variable cost to manufacture a product is \$10 per item and the fixed cost \$2500. If x represents the number of items manufactured and y represents the total cost, write the cost function.*

Solution 1.4.2. *The variable cost of \$10 per item tells us that $m = 10$. The fixed cost represents the y-intercept, so $b = 2500$. Therefore, the cost equation is $y = 10x + 2500$.*

Example 1.4.3. *It costs \$750 to manufacture 25 items, and \$1000 to manufacture 50 items. Assuming a linear relationship holds, find the cost equation, and use this function to predict the cost of 100 items.*

Solution 1.4.3. *Let x be the number of items manufactured, and let y be the cost.*

Solving this problem is equivalent to finding an equation of a line that passes through the points $(25, 750)$ and $(50, 1000)$.

$m = \frac{1000-750}{50-25} = 10$

Therefore, the partial equation is $y = 10x + b$.

By substituting one of the points in the equation, we get $b = 500$.

Therefore, the cost equation is $y = 10x + 500$.

To find the cost of 100 items, substitute $x = 100$ *in the equation* $y = 10x+500$.

So the cost $= y = 10(100) + 500 = 1500$.

It costs $1500 to manufacture 100 items.

Example 1.4.4. *The freezing temperature of water in Celsius is 0 degrees, and in Fahrenheit, it's 32 degrees. The boiling temperatures of water in Celsius and Fahrenheit are 100 degrees and 212 degrees, respectively. Write a conversion equation from Celsius to Fahrenheit and use this equation to convert 30 degrees Celsius into Fahrenheit.*

Solution 1.4.4. *Let's look at what is given:*

Celsius	*Fahrenheit*
0	*32*
100	*212*

Solving this problem is equivalent to finding an equation of a line that passes through the points $(0, 32)$ *and* $(100, 212)$. *Since we are finding a linear relationship, we are looking for an equation* $y = mx + b$, *or in this case,* $F = mC + b$, *where* C *represents the temperature in Celsius, and* F *represents the temperature in Fahrenheit.*

The slope $m = \frac{212-32}{100-0} = 95$.

The equation is $F = 95C + b$.

Substituting the point $(0, 32)$, *we get* $F = 95C + 32$.

To convert 30 degrees Celsius into Fahrenheit, substitute $C = 30$ *in the equation:*

$$F = 95C + 32$$

$$F = 95(30) + 32 = 86$$

Example 1.4.5. *The population of Canada in the year 1980 was 24.5 million, and in the year 2010, it was 34 million. The population of Canada over that time period can be approximately modeled by a linear function. Let* x

represent time as the number of years after 1980, and let y represent the size of the population.

a. Write the linear function that gives a relationship between the time and the population.

b. Assuming the population continues to grow linearly in the future, use this equation to predict the population of Canada in the year 2025.

Solution 1.4.5. *The problem can be made easier by using 1980 as the base year, which means we choose the year 1980 as the year zero. This will make the year 2010 correspond to year 30. Now, let's look at the information we have:*

Year	*Population*
0 (1980)	*24.5 million*
30 (2010)	*34 million*

a. Solving this problem is equivalent to finding an equation of a line that passes through the points (0, 24.5) and (30, 34). We use these two points to find the slope:

$$m = \frac{34 - 24.5}{30 - 0} = \frac{9.5}{30} = 0.32$$

The y-intercept occurs when $x = 0$*, so* $b = 24.5$.

So, the equation relating time (x) and population (y) is:

$$y = 0.32x + 24.5$$

b. Now, to predict the population in the year 2025, we let $x = 2025 - 1980 = 45$*:*

$$y = 0.32x + 24.5$$

$$y = 0.32(45) + 24.5 = 38.9$$

In the year 2025, we predict that the population of Canada will be 38.9 million people.

Note that we assumed the population trend will continue to be linear. Therefore, if population trends change and this assumption does not continue to be true in the future, this prediction may not be accurate.

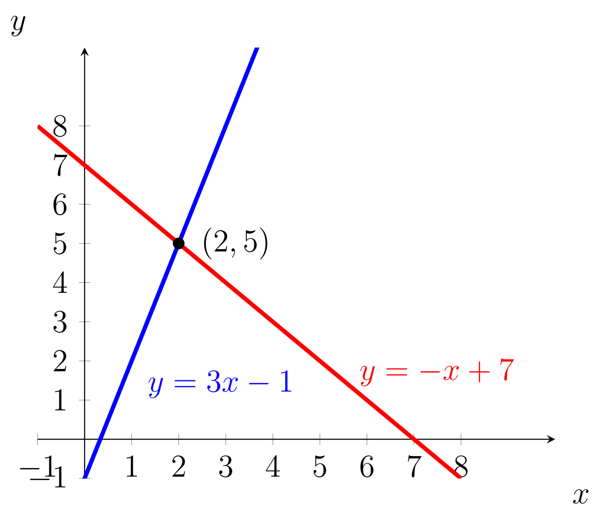
1.5 More Applications

1.5.1 Finding the Point of Intersection of Two Lines

In this section, we will do application problems that involve the intersection of lines. Therefore, before we proceed any further, we will first learn how to find the intersection of two lines.

Example 1.5.1. *Find the intersection of the line $y = 3x - 1$ and the line $y = -x + 7$.*

Solution 1.5.1. *We graph both lines on the same axes, as shown below, and read the solution $(2, 5)$.*



Finding an intersection of two lines graphically is not always easy or practical; therefore, we will now learn to solve these problems algebraically.

At the point where two lines intersect, the x and y values for both lines are the same. So in order to find the intersection, we either let the x-values or the y-values equal.

If we were to solve the above example algebraically, it will be easier to let the

y-values equal. Since $y = 3x - 1$ for the first line, and $y = -x + 7$ for the second line, by letting the y-values equal, we get:

$$\begin{aligned} 3x - 1 &= -x + 7 \\ 4x &= 8 \\ x &= 2 \end{aligned}$$

By substituting $x = 2$ in any of the two equations, we obtain $y = 5$. Hence, the solution is $(2, 5)$.

1.5.2 Solving Systems of Equations: The Elimination Method

A common algebraic method used to solve systems of equations is called the elimination method. The objective is to eliminate one of the two variables by adding the left and right sides of the equations together. Once one variable is eliminated, we have an equation with only one variable that can be solved. Finally, by substituting the value of the variable that has been found in one of the original equations, we can get the value of the other variable.

Example 1.5.2. *Find the intersection of the lines $2x + y = 7$ and $3x - y = 3$ by the elimination method.*

Solution 1.5.2. *We add the left and right sides of the two equations:*

$$\begin{aligned} 2x + y &= 7 \\ 3x - y &= 3 \\ 5x &= 10 \\ x &= 2 \end{aligned}$$

Now we substitute $x = 2$ into any of the two equations and solve for y:

$$\begin{aligned} 2(2) + y &= 7 \\ 4 + y &= 7 \\ y &= 3 \end{aligned}$$

Therefore, the solution is $(2, 3)$.

Example 1.5.3. *Solve the system of equations* $x + 2y = 3$ *and* $2x + 3y = 4$ *by the elimination method.*

Solution 1.5.3. *If we add the two equations directly, none of the variables are eliminated. However, the variable* x *can be eliminated by multiplying the first equation by* -2 *and leaving the second equation unchanged:*

$$\begin{aligned} -2x - 4y &= -6 \\ 2x + 3y &= 4 \\ -y &= -2 \\ y &= 2 \end{aligned}$$

Substituting $y = 2$ *into* $x + 2y = 3$*, we get:*

$$\begin{aligned} x + 2(2) &= 3 \\ x + 4 &= 3 \\ x &= -1 \end{aligned}$$

Therefore, the solution is $(-1, 2)$.

Example 1.5.4. *Solve the system of equations* $3x - 4y = 5$ *and* $4x - 5y = 6$.

Solution 1.5.4. *This time, we multiply the first equation by* -4 *and the second by* 3 *before adding (the choice of numbers is not unique):*

$$\begin{aligned} -12x + 16y &= -20 \\ 12x - 15y &= 18 \\ y &= -2 \end{aligned}$$

By substituting $y = -2$ into any one of the equations, we get:

$$\begin{aligned} 3x - 4(-2) &= 5 \\ 3x + 8 &= 5 \\ 3x &= -3 \\ x &= -1 \end{aligned}$$

Hence, the solution is $(-1, -2)$.

1.5.3 Supply, Demand, and the Equilibrium Market Price

In a free market economy, the supply curve for a commodity is the number of items of a product that can be made available at different prices, and the demand curve is the number of items the consumer will buy at different prices. As the price of a product increases, its demand decreases, and supply increases. On the other hand, as the price decreases, the demand increases, and supply decreases. The equilibrium price is reached when the demand equals the supply.

Example 1.5.5. *The supply curve for a product is given by $y = 3.5x - 14$, and the demand curve for the same product is given by $y = -2.5x + 34$, where x is the price and y is the number of items produced. Find the following:*

1. *How many items will be supplied at a price of $10?*
2. *How many items will be demanded at a price of $10?*
3. *Determine the equilibrium price.*
4. *How many items will be produced at the equilibrium price?*

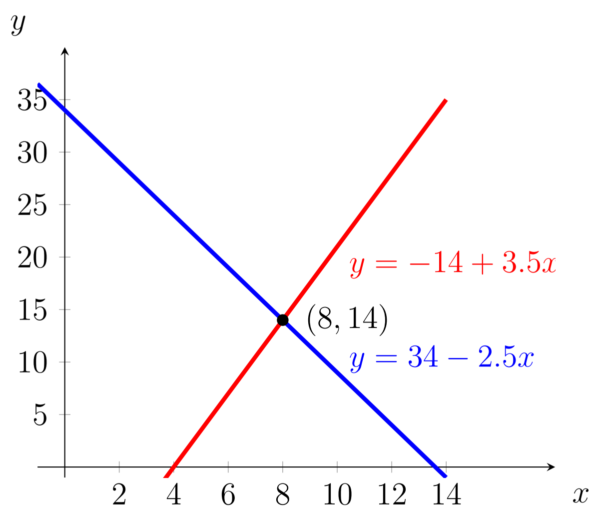
Solution 1.5.5.

1. *To find the number of items supplied at a price of $10, we substitute $x = 10$ into the supply equation $y = 3.5x - 14$. Therefore, $y = 3.5(10) - 14 = 21$ items will be supplied.*
2. *To find the number of items demanded at a price of $10, we substitute $x = 10$ into the demand equation $y = -2.5x + 34$. Therefore, $y = -2.5(10) + 34 = 9$ items will be demanded.*

3. *To determine the equilibrium price, we set the supply equal to the demand:*
$$3.5x - 14 = -2.5x + 34$$
Solving for x:
$$6x = 48$$
$$x = 8$$
So, the equilibrium price is $x = 8$.

4. *To find how many items will be produced at the equilibrium price, we substitute $x = 8$ into either the supply or the demand equation. Using the supply equation, we get $y = 3.5(8) - 14 = 14$ items will be produced.*

The graph shows the intersection of the supply and demand functions and their point of intersection, $(8, 14)$.



1.5.4 Break-Even Point

In a business, profit is generated by selling products. If a company sells x number of items at a price P, then the revenue R is the price multiplied by the number of items sold: $R = P \cdot x$. The production costs C are the sum of the variable costs and the fixed costs, often written as $C = mx + b$, where x is the number of items manufactured.

- The slope m is called the marginal cost and represents the cost to produce one additional item or unit.
- The variable cost, mx, depends on how much is being produced.
- The fixed cost b is constant and does not change regardless of production quantity.

Profit is equal to revenue minus cost: $Profit = R - C$. A company makes a profit if the revenue is greater than the cost, and there is a loss if the cost is greater than the revenue. The point on the graph where the revenue equals the cost is called the break-even point, and at this point, the profit is 0.

Example 1.5.6. *If the revenue function of a product is $R = 5x$ and the cost function is $C = 3x + 12$, find the following:*

1. *If 4 items are produced, what will the revenue be?*
2. *What is the cost of producing 4 items?*
3. *How many items should be produced to break even?*
4. *What will be the revenue and cost at the break-even point?*

Solution 1.5.6.

1. *To find the revenue when 4 items are produced, we substitute $x = 4$ in the revenue equation $R = 5x$, and the answer is $R = 20$.*
2. *To find the cost of producing 4 items, we substitute $x = 4$ in the cost equation $C = 3x + 12$, and the answer is $C = 24$.*
3. *To determine the number of items required to break even, we set the revenue equal to the cost:*

   $$5x = 3x + 12$$

   Solving for x:

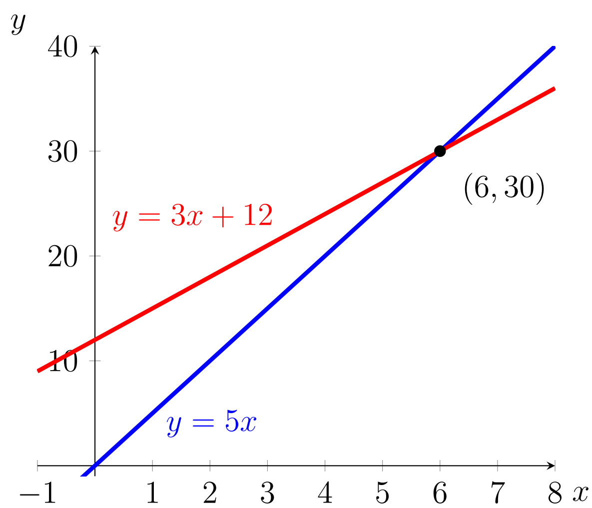
   $$2x = 12$$

   $$x = 6$$

   So, 6 items should be produced to break even.
4. *At the break-even point, when $x = 6$, we can substitute $x = 6$ in either the revenue or the cost equation to find that both revenue and cost are*

equal to 30*. Therefore, the revenue and cost at the break-even point are both* 30*.*

The graph below shows the intersection of the revenue and cost functions and their point of intersection, $(6, 30)$*.*



Chapter 2

Matrices

In this chapter, you will learn to:

1. Do matrix operations.
2. Solve linear systems using the Gauss-Jordan method.
3. Solve linear systems using the matrix inverse method.
4. Do application problems

2.1 Introduction to Matrices

In this section, you will learn to:

1. Add and subtract matrices.
2. Multiply a matrix by a scalar.
3. Multiply two matrices.

A matrix is a 2-dimensional array of numbers arranged in rows and columns. Matrices provide a method of organizing, storing, and working with mathematical information. They have numerous applications and uses in the real world.

(TODO: fix references here)Matrices are particularly useful in working with models based on systems of linear equations, which we'll explore in sections

2.2, 2.3, and 2.4 of this chapter. They are also used in encryption (section 2.5) and economic modeling (section 2.6).

Furthermore, matrices play a crucial role in optimization problems in (TODO: fix references here) Chapter 4, such as maximizing profit or revenue and minimizing costs. They are used in business for scheduling, routing transportation and shipments, and managing inventory. Matrices are applicable in various fields where data organization and problem-solving are essential.

The use of matrices has expanded with the increase in available data across different domains. They are fundamental tools for organizing data and solving problems in science fields like physics, chemistry, biology, genetics, meteorology, and economics. In computer science, matrix mathematics is foundational for animation in movies and video games.

Moreover, matrices are used in analyzing network diagrams, such as social media connections on platforms like Facebook, LinkedIn, etc. The mathematics of network diagrams falls under "graph theory" and relies on matrices to organize information in graphs that depict connections and associations in a network.

A matrix is a rectangular array of numbers. Matrices are useful in organizing and manipulating large amounts of data. In order to get some idea of what matrices are all about, we will look at the following example.

Example 2.1.1. *Fine Furniture Company makes chairs and tables at its San Jose, Hayward, and Oakland factories. The total production, in hundreds, from the three factories for the years 2014 and 2015 is listed in the table below.*

	2014	*2015*		
	CHAIRS	*TABLES*	*CHAIRS*	*TABLES*
SAN JOSE	30	18	36	20
HAYWARD	20	12	24	18
OAKLAND	16	10	20	12

1. *Represent the production for the years 2014 and 2015 as the matrices A and B.*

2. *Find the difference in sales between the years 2014 and 2015.*

3. *The company predicts that in the year 2020 the production at these factories will be double that of the year 2014. What will the production be for the year 2020?*

Solution 2.1.1. 1. *The matrices are as follows:*

$$A = \begin{bmatrix} 30 & 18 \\ 20 & 12 \\ 16 & 10 \end{bmatrix} \qquad B = \begin{bmatrix} 36 & 20 \\ 24 & 18 \\ 20 & 12 \end{bmatrix}$$

2. *We are looking for the matrix $B-A$. When two matrices have the same number of rows and columns, they can be added or subtracted entry by entry. Therefore, we get:*

$$B - A = \begin{bmatrix} 36-30 & 20-18 \\ 24-20 & 18-12 \\ 20-16 & 12-10 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 4 & 6 \\ 4 & 2 \end{bmatrix}$$

3. *We would like a matrix that is twice the matrix of 2014, i.e., $2A$. Whenever a matrix is multiplied by a number, each entry is multiplied by the number.*

$$2A = 2\begin{bmatrix} 30 & 18 \\ 20 & 12 \\ 16 & 10 \end{bmatrix} = \begin{bmatrix} 60 & 36 \\ 40 & 24 \\ 32 & 20 \end{bmatrix}$$

2.1.1 Vocabulary

Before we go any further, we need to familiarize ourselves with some terms that are associated with matrices.

The numbers in a matrix are called the entries or the elements of a matrix.

Whenever we talk about a matrix, we need to know its size or dimension. The dimension of a matrix is the number of rows and columns it has. When we say a matrix is a "3 by 4 matrix," we are saying that it has 3 rows and 4 columns. The rows are always mentioned first, and the columns second. This means that a 3×4 matrix does not have the same dimension as a 4×3 matrix.

$$A = \begin{bmatrix} 1 & 4 & -2 & 0 \\ 3 & -1 & 7 & 9 \\ 6 & 2 & 0 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 9 & 8 \\ -3 & 0 & 1 \\ 6 & 5 & -2 \\ -4 & 7 & 8 \end{bmatrix}$$

Matrix A has dimensions 3×4 — Matrix B has dimensions 4×3

A matrix that has the same number of rows as columns is called a square matrix. A matrix with all entries zero is called a zero matrix. A square matrix with 1's along the main diagonal and zeros everywhere else, is called an identity matrix. When a square matrix is multiplied by an identity matrix of same size, the matrix remains the same.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix I is a 3×3 identity matrix

A matrix with only one row is called a row matrix or a row vector, and a matrix with only one column is called a column matrix or a column vector. Two matrices are equal if they have the same size and the corresponding entries are equal. We can perform arithmetic operations with matrices. Next we will define and give examples illustrating the operations of matrix addition and subtraction, scalar multiplication, and matrix multiplication. Note that matrix multiplication is quite different from what you would intuitively expect, so pay careful attention to the explanation. Note also that the ability to perform matrix operations depends on the matrices involved being compatible in size, or dimensions, for that operation. The definition of compatible dimensions is different for different operations, so note the requirements carefully for each.

2.1.2 Matrix Addition and Subtraction

If two matrices have the same size, they can be added or subtracted. The operations are performed on corresponding entries.

Example 2.1.2. *Given the matrices A, B, C, and D below:*

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 1 \\ 5 & 0 & 3 \end{bmatrix}, \ B = \begin{bmatrix} 2 & -1 & 3 \\ 2 & 4 & 2 \\ 3 & 6 & 1 \end{bmatrix}, \ C = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}, \ D = \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix}$$

Find, if possible:

1. $A + B$
2. $C - D$
3. $A + D$

Solution 2.1.2.

- *We add each element of A to the corresponding entry of B:*

$$A + B = \begin{bmatrix} 3 & 1 & 7 \\ 4 & 7 & 3 \\ 8 & 6 & 4 \end{bmatrix}$$

- *We perform the subtraction entry by entry for $C - D$:*

$$C - D = \begin{bmatrix} 6 \\ 5 \\ -1 \end{bmatrix}$$

- *The sum $A + D$ cannot be found because the two matrices have different sizes. Two matrices can only be added or subtracted if they have the same dimension.*

2.1.3 Multiplying a Matrix by a Scalar

If a matrix is multiplied by a scalar, each entry is multiplied by that scalar.

Example 2.1.3. *Given the matrix A and C in the previous example, find $2A$ and $-3C$.*

Solution 2.1.3.

- *To find $2A$, we multiply each entry of matrix A by 2:*

$$2A = \begin{bmatrix} 2 & 4 & 8 \\ 4 & 6 & 2 \\ 10 & 0 & 6 \end{bmatrix}$$

- *To find $-3C$, we multiply each entry of C by -3:*

$$-3C = \begin{bmatrix} -12 \\ -6 \\ -9 \end{bmatrix}$$

2.1.4 Multiplication of Two Matrices

To multiply a matrix by another is not as easy as the addition, subtraction, or scalar multiplication of matrices. Because of its wide use in application problems, it is important that we learn it well. Therefore, we will try to learn the process in a step by step manner.

Example 2.1.4. *Given $A = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, find the product AB.*

Solution 2.1.4. *The product is a 1×1 matrix whose entry is obtained by multiplying the corresponding entries and then forming the sum:*

$$AB = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 2a + 3b + 4c$$

Note that AB is a 1×1 matrix, and its only entry is $2a + 3b + 4c$.

Example 2.1.5. *Given $A = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$, find the product AB.*

Solution 2.1.5. *Again, we multiply the corresponding entries and add:*

$$AB = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} = (2 \cdot 5) + (3 \cdot 6) + (4 \cdot 7) = 10 + 18 + 28 = 56$$

Example 2.1.6. *Given $A = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 3 \\ 6 & 4 \\ 7 & 5 \end{bmatrix}$, find the product AB.*

Solution 2.1.6. *We know how to multiply a row matrix by a column matrix. To find the product AB, in this example, we will multiply the row matrix A to both the first and second columns of matrix B, resulting in a 1×2 matrix:*

$$AB = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 6 & 4 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} (2 \cdot 5) + (3 \cdot 6) + (4 \cdot 7) & (2 \cdot 3) + (3 \cdot 4) + (4 \cdot 5) \end{bmatrix} = \begin{bmatrix} 56 & 38 \end{bmatrix}$$

We multiplied a 1×3 matrix by a matrix whose size is 3×2. So unlike addition and subtraction, it is possible to multiply two matrices with different dimensions if the number of entries in the rows of the first matrix is the same as the number of entries in the columns of the second matrix.

Example 2.1.7. *Given $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 3 \\ 6 & 4 \\ 7 & 5 \end{bmatrix}$, find the product AB.*

Solution 2.1.7. *This time we are multiplying two rows of matrix A with two columns of matrix B. Since the number of entries in each row of A is the same as the number of entries in each column of B, the product is possible. We do exactly what we did in the last example. The only difference is that matrix A has one more row.*

We multiply the first row of matrix A with the two columns of B, one at a time, and then repeat the process with the second row of A. We get:

$$AB = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 6 & 4 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} (2 \cdot 5 + 3 \cdot 6 + 4 \cdot 7) & (2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5) \\ (1 \cdot 5 + 2 \cdot 6 + 3 \cdot 7) & (1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5) \end{bmatrix}$$

$$AB = \begin{bmatrix} 56 & 38 \\ 38 & 26 \end{bmatrix}$$

Example 2.1.8. *Given matrices $E = \begin{bmatrix} 1 & 2 \\ 4 & 2 \\ 3 & 1 \end{bmatrix}$, $F = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$, $G = \begin{bmatrix} 4 & 1 \end{bmatrix}$, and $H = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$, find the following products if possible:*

1. *EF*

2. *FE*

3. *FH*

4. *GH*

5. *HG*

Solution 2.1.8. 1. *To find EF, we multiply the rows of E with the columns of F. The result is:*

$$EF = \begin{bmatrix} 1 & 2 \\ 4 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} (1 \cdot 2 + 2 \cdot 3) & (1 \cdot -1 + 2 \cdot 2) \\ (4 \cdot 2 + 2 \cdot 3) & (4 \cdot -1 + 2 \cdot 2) \\ (3 \cdot 2 + 1 \cdot 3) & (3 \cdot -1 + 1 \cdot 2) \end{bmatrix} = \begin{bmatrix} 8 & 3 \\ 14 & 0 \\ 9 & -1 \end{bmatrix}$$

2. *Product FE is not possible because F has two entries in each row, while E has three entries in each column.*

3. $FH = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} (2 \cdot -3 + -1 \cdot -1) \\ (3 \cdot -3 + 2 \cdot -1) \end{bmatrix} = \begin{bmatrix} -5 \\ -11 \end{bmatrix}$

4. $GH = \begin{bmatrix} 4 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \end{bmatrix} = (4 \cdot -3 + 1 \cdot -1) = -13$

5. $HG = \begin{bmatrix} -3 \\ -1 \end{bmatrix} \begin{bmatrix} 4 & 1 \end{bmatrix} = \begin{bmatrix} (-3 \cdot 4 & -3 \cdot 1) \\ (-1 \cdot 4 & -1 \cdot 1) \end{bmatrix} = \begin{bmatrix} -12 & -3 \\ -4 & -1 \end{bmatrix}$

We summarize some important properties of matrix multiplication that we observed in the previous examples.

- For the product AB to exist, the number of columns of matrix A must equal the number of rows of matrix B.
- If matrix A has dimensions $m \times n$ and matrix B has dimensions $n \times p$, then the product AB will have dimensions $m \times p$.
- Matrix multiplication is not commutative; that is, in general, AB does not equal BA.

Example 2.1.9. *Given matrices* $R = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 5 \\ 2 & 3 & 1 \end{bmatrix}$, $S = \begin{bmatrix} 0 & -1 & 2 \\ 3 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$, *and* $T = \begin{bmatrix} -2 & 3 & 0 \\ -3 & 2 & 2 \\ -1 & 1 & 0 \end{bmatrix}$, *find* $2RS - 3ST$.

Solution 2.1.9. *Solution: To find $2RS - 3ST$, we first compute the products RS and ST:*

$$RS = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 5 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 2 \\ 3 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (1 \cdot 0 + 0 \cdot 3 + 2 \cdot 4) & (1 \cdot -1 + 0 \cdot 1 + 2 \cdot 2) & (1 \cdot 2 + 0 \cdot 0 + 2 \cdot 1) \\ (2 \cdot 0 + 1 \cdot 3 + 5 \cdot 4) & (2 \cdot -1 + 1 \cdot 1 + 5 \cdot 2) & (2 \cdot 2 + 1 \cdot 0 + 5 \cdot 1) \\ (2 \cdot 0 + 3 \cdot 3 + 1 \cdot 4) & (2 \cdot -1 + 3 \cdot 1 + 1 \cdot 2) & (2 \cdot 2 + 3 \cdot 0 + 1 \cdot 1) \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 3 & 4 \\ 23 & 9 & 9 \\ 13 & 3 & 5 \end{bmatrix}$$

Next, we compute ST:

$$ST = \begin{bmatrix} 0 & -1 & 2 \\ 3 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 3 & 0 \\ -3 & 2 & 2 \\ -1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (0 \cdot -2 + -1 \cdot -3 + 2 \cdot -1) & (0 \cdot 3 + -1 \cdot 2 + 2 \cdot 1) & (0 \cdot 0 + -1 \cdot 2 + 2 \cdot 0) \\ (3 \cdot -2 + 1 \cdot -3 + 0 \cdot -1) & (3 \cdot 3 + 1 \cdot 2 + 0 \cdot 1) & (3 \cdot 0 + 1 \cdot 1 + 0 \cdot 0) \\ (4 \cdot -2 + 2 \cdot -3 + 1 \cdot -1) & (4 \cdot 3 + 2 \cdot 2 + 1 \cdot 1) & (4 \cdot 0 + 2 \cdot 1 + 1 \cdot 0) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -2 \\ -9 & 11 & 2 \\ -15 & 17 & 4 \end{bmatrix}$$

Now we can find $2RS - 3ST$:

$$2RS - 3ST = 2 \cdot \begin{bmatrix} 8 & 3 & 4 \\ 23 & 9 & 9 \\ 13 & 3 & 5 \end{bmatrix} - 3 \cdot \begin{bmatrix} 1 & 0 & -2 \\ -9 & 11 & 2 \\ -15 & 17 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 6 & 8 \\ 46 & 18 & 18 \\ 26 & 6 & 10 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 6 \\ -27 & 33 & 6 \\ -45 & 51 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 6 & 14 \\ 73 & -15 & 12 \\ 71 & -45 & -2 \end{bmatrix}$$

The result of $2RS - 3ST$ *is a matrix with dimensions* 3×3.

Example 2.1.10. *Given matrix* $F = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$, *find* F^2.

Solution 2.1.10. F^2 *is found by multiplying matrix* F *by itself, using matrix multiplication.*

$$F^2 = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 2 + (-1) \cdot 3 & 2 \cdot (-1) + (-1) \cdot 2 \\ 3 \cdot 2 + 2 \cdot 3 & 3 \cdot (-1) + 2 \cdot 2 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 12 & 1 \end{bmatrix}$$

Note that F^2 *is not found by squaring each entry of matrix* F. *The process of raising a matrix to a power, such as finding* F^2, *is only possible if the matrix is a square matrix.*

2.1.5 Systems of Linear Equations

Using matrices to represent a system of linear equations is a powerful technique that allows for efficient solving of such systems. In this method, we define matrices as follows:

- Matrix A represents the coefficients of the variables in the system and is called the coefficient matrix.
- Matrix X is a column matrix that contains the variables of the system.
- Matrix B is a column matrix that contains the constants of the system.

By defining these matrices, we can represent a system of linear equations as the matrix equation $AX = B$, where A, X, and B are matrices. This representation simplifies the process of solving linear systems and allows us to apply matrix operations to find the solution.

In the next sections, we will delve deeper into how to use matrices to solve linear systems and explore various methods and techniques for efficient computation and analysis. Matrix representation is widely used in mathematical modeling, engineering, economics, and various other fields where systems of linear equations arise.

Example 2.1.11. *Verify that the system of two linear equations with two unknowns:*

$$\begin{aligned} ax + by &= h \\ cx + dy &= k \end{aligned}$$

can be written as $AX = B$, where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad B = \begin{bmatrix} h \\ k \end{bmatrix}.$$

Solution 2.1.11. *If we multiply the matrices A and X, we get*

$$AX = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}.$$

If $AX = B$, then

$$\begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} = \begin{bmatrix} h \\ k \end{bmatrix}.$$

If two matrices are equal, then their corresponding entries are equal. It follows that

$$\begin{aligned} ax + by &= h \\ cx + dy &= k \end{aligned}$$

Example 2.1.12. *Express the following system as a matrix equation in the form $AX = B$.*

$$\begin{aligned} 2x + 3y - 4z &= 5 \\ 3x + 4y - 5z &= 6 \\ 5x - 6z &= 7 \end{aligned}$$

Solution 2.1.12. *This system of equations can be expressed in the form $AX = B$ as shown below.*

$$\begin{bmatrix} 2 & 3 & -4 \\ 3 & 4 & -5 \\ 5 & 0 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$$

2.2 Systems of Linear Equations; Gauss-Jordan Method

In this section you will learn to

1. Represent a system of linear equations as an augmented matrix
2. Solve the system using elementary row operations.

In this section, we learn to solve systems of linear equations using a process called the Gauss-Jordan method. The process begins by first expressing the system as a matrix, and then reducing it to an equivalent system by simple row operations. The process is continued until the solution is obvious from the matrix. The matrix that represents the system is called the augmented matrix, and the arithmetic manipulation that is used to move from a system to a reduced equivalent system is called a row operation.

Example 2.2.1. *Write the following system as an augmented matrix.*

$$\begin{aligned} 2x + 3y - 4z &= 5 \\ 3x + 4y - 5z &= -6 \\ 4x + 5y - 6z &= 7 \end{aligned}$$

Solution 2.2.1. *We express the above information in matrix form. Since a system is entirely determined by its coefficient matrix and by its matrix of constant terms, the augmented matrix will include only the coefficient matrix and the constant matrix. So the augmented matrix we get is as follows:*

$$\left[\begin{array}{ccc|c} 2 & 3 & -4 & 5 \\ 3 & 4 & -5 & -6 \\ 4 & 5 & -6 & 7 \end{array}\right]$$

In the last section, we expressed the system of equations as $AX = B$, where A represented the coefficient matrix, and B the matrix of constant terms. As an augmented matrix, we write the matrix as $[A|B]$. It is clear that all of the information is maintained in this matrix form, and only the letters x, y, and z are missing. A student may choose to write x, y, and z on top of the first three columns to help ease the transition.

Example 2.2.2. *For the following augmented matrix, write the system of equations it represents.*

$$\left[\begin{array}{ccc|c} 1 & 3 & -5 & 2 \\ 2 & 0 & -3 & -5 \\ 3 & 2 & -3 & -1 \end{array}\right]$$

Solution 2.2.2. *The system is readily obtained as below.*

$$\begin{aligned} x + 3y - 5z &= 2 \\ 2x - 3z &= -5 \\ 3x + 2y - 3z &= -1 \end{aligned}$$

Once a system is expressed as an augmented matrix, the Gauss-Jordan method reduces the system into a series of equivalent systems by using the row operations. This row reduction continues until the system is expressed in what is called the reduced row echelon form. The reduced row echelon form of the coefficient matrix has 1's along the main diagonal and zeros elsewhere. The solution is readily obtained from this form.

The method is not much different form the algebraic operations we employed in the elimination method in the first chapter. The basic difference is that it is algorithmic in nature, and, therefore, can easily be programmed on a computer.

We will next solve a system of two equations with two unknowns, using the elimination method, and then show that the method is analogous to the Gauss-Jordan method.

Example 2.2.3. *Solve the following system by the elimination method.*

$$\begin{aligned} x + 3y &= 7 \\ 3x + 4y &= 11 \end{aligned}$$

Solution 2.2.3. *We multiply the first equation by -3 and add it to the second equation.*

$$\begin{aligned} -3x - 9y &= -21 \\ 3x + 4y &= 11 \end{aligned}$$

This transforms our original system into an equivalent system:

$$\begin{aligned} x + 3y &= 7 \\ -5y &= -10 \end{aligned}$$

Dividing the second equation by -5, we get the next equivalent system.

$$\begin{aligned} x + 3y &= 7 \\ y &= 2 \end{aligned}$$

Multiplying the second equation by -3 and adding it to the first, we get

$$\begin{aligned} x &= 1 \\ y &= 2 \end{aligned}$$

Example 2.2.4. *Solve the following system from Example 3 by the Gauss-Jordan method, and show the similarities in both methods by writing the equations next to the matrices.*

$$\begin{aligned} x + 3y &= 7 \\ 3x + 4y &= 11 \end{aligned}$$

Solution 2.2.4. *The augmented matrix for the system is as follows.*

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 3 & 4 & 11 \end{array}\right] \quad \Leftrightarrow \quad \begin{aligned} x + 3y &= 7 \\ 3x + 4y &= 11 \end{aligned}$$

We multiply the first row by -3 and add it to the second row.

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & -5 & -10 \end{array}\right] \quad \Leftrightarrow \quad \begin{aligned} x + 3y &= 7 \\ -5y &= -10 \end{aligned}$$

Dividing the second row by -5, we get,

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & 1 & 2 \end{array}\right] \quad \Leftrightarrow \quad \begin{aligned} x + 3y &= 7 \\ y &= 2 \end{aligned}$$

Finally, we multiply the second row by -3 and add to the first row, and we get,

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right] \quad \Leftrightarrow \quad \begin{aligned} x &= 1 \\ y &= 2 \end{aligned}$$

2.2.1 Row Operations in Gauss-Jordan Method

The Gauss-Jordan method employs three fundamental row operations:

1. Any two rows in the augmented matrix may be interchanged.
2. Any row may be multiplied by a non-zero constant.
3. A constant multiple of a row may be added to another row.

One can easily see that these three row operations may make the system look different, but they do not change the solution of the system.

Example of Row Interchange

Consider the system of equations with two unknowns:

$$\begin{aligned} x + 3y &= 7 \\ 3x + 4y &= 11 \end{aligned}$$

If we interchange the rows, we get:

$$\begin{aligned} 3x + 4y &= 11 \\ x + 3y &= 7 \end{aligned}$$

Clearly, this system has the same solution as the original.

Example of Multiplying a Row by a Constant

Consider the system again:

$$\begin{aligned} x + 3y &= 7 \\ 3x + 4y &= 11 \end{aligned}$$

Multiplying the first row by -3, we get:

$$\begin{aligned} -3x - 9y &= -21 \\ 3x + 4y &= 11 \end{aligned}$$

Once again, this new system has the same solution as the original.

Example of Adding a Constant Multiple of One Row to Another

For the system:

$$\begin{aligned} x + 3y &= 7 \\ 3x + 4y &= 11 \end{aligned}$$

If we multiply the first row by -3 and add it to the second row, we get:

$$\begin{aligned} x + 3y &= 7 \\ -5y &= -10 \end{aligned}$$

The solution remains unchanged.

Now that we understand how the three row operations work, it is time to introduce the Gauss-Jordan method to solve systems of linear equations. As mentioned earlier, the Gauss-Jordan method starts out with an augmented matrix, and by a series of row operations ends up with a matrix that is in the reduced row echelon form. A matrix is in the reduced row echelon form if the first nonzero entry in each row is a 1, and the columns containing these 1's have all other entries as zeros. The reduced row echelon form also requires that the leading entry in each row be to the right of the leading entry in the row above it, and the rows containing all zeros be moved down to the bottom. We state the Gauss-Jordan method as follows.

Gauss-Jordan Method Steps

Here are the steps of the Gauss-Jordan method for solving linear systems:

1. Write the augmented matrix.
2. Interchange rows if necessary to obtain a non-zero number in the first row, first column.
3. Use a row operation to get a 1 as the entry in the first row and first column.
4. Use row operations to make all other entries as zeros in column one.
5. Interchange rows if necessary to obtain a nonzero number in the second row, second column. Use a row operation to make this entry 1. Use row operations to make all other entries as zeros in column two.

6. Repeat step 5 for row 3, column 3. Continue moving along the main diagonal until you reach the last row, or until the number is zero.

7. The final matrix is called the reduced row-echelon form.

Example 2.2.5. *Solve the following system by the Gauss-Jordan method:*

$$\begin{array}{rcrcrcr} 2x & + & y & + & 2z & = & 10 \\ x & + & 2y & + & z & = & 8 \\ 3x & + & y & - & z & = & 2 \end{array}$$

Solution 2.2.5. *We write the augmented matrix.*

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 10 \\ 1 & 2 & 1 & 8 \\ 3 & 1 & -1 & 2 \end{array}\right]$$

We want a 1 in row one, column one. This can be obtained by dividing the first row by 2, or interchanging the second row with the first. Interchanging the rows is a better choice because that way we avoid fractions.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 2 & 1 & 2 & 10 \\ 3 & 1 & -1 & 2 \end{array}\right] \quad \textit{we interchanged row 1(R1) and row 2(R2)}$$

We need to make all other entries zeros in column 1. To make the entry (2) a zero in row 2, column 1, we multiply row 1 by -2 and add it to the second row. We get,

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -3 & 0 & -6 \\ 3 & 1 & -1 & 2 \end{array}\right] \quad -2R1 + R2$$

To make the entry (3) a zero in row 3, column 1, we multiply row 1 by -3 and add it to the third row. We get,

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -3 & 0 & -6 \\ 0 & -5 & -4 & -22 \end{array}\right] \quad -3R1 + R3$$

So far we have made a 1 in the left corner and all other entries zeros in that column. Now we move to the next diagonal entry, row 2, column 2. We need *to make this entry(-3) a 1 and make all other entries in this column zeros. To make row 2, column 2 entry a 1, we divide the entire second row by -3.*

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 1 & 0 & 2 \\ 0 & -5 & -4 & -22 \end{array}\right] \quad R2 \cdot \frac{1}{(-3)}$$

Next, we make all other entries zeros in the second column.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -4 & -12 \end{array}\right] \quad -2R2 + R1 \text{ and } 5R2 + R3$$

We make the last diagonal entry a 1, by dividing row 3 by -4.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad R3 \cdot \frac{1}{(-4)}$$

Finally, we make all other entries zeros in column 3.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad -R3 + R1$$

Clearly, the solution reads $x = 1$, $y = 2$, *and* $z = 3$.

Before we leave this section, we mention some terms we may need in the fourth chapter.

The process of obtaining a 1 in a location, and then making all other entries zeros in that column, is called pivoting.

The number that is made a 1 is called the pivot element, and the row that contains the pivot element is called the pivot row.

We often multiply the pivot row by a number and add it to another row to obtain a zero in the latter. The row to which a multiple of pivot row is added is called the target row.

2.3 Systems of Linear Equations – Special Cases

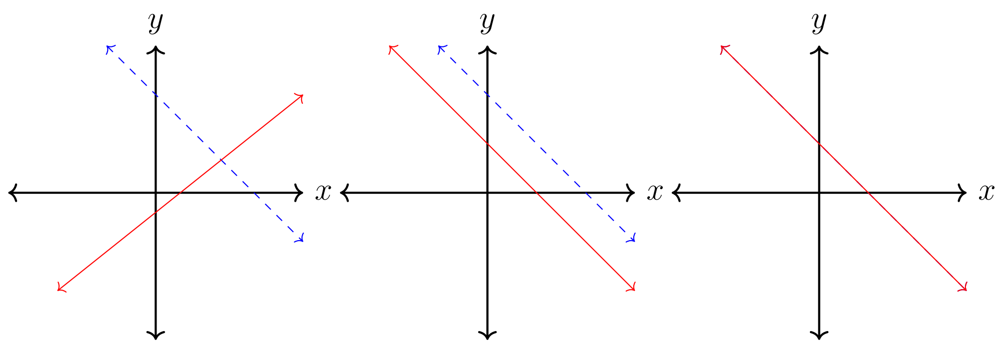
In this section you will learn to:

1. Determine the linear systems that have no solution.
2. Solve the linear systems that have infinitely many solutions.

If we consider the intersection of two lines in a plane, three things can happen.

1. The lines intersect in exactly one point. This is called an independent system.
2. The lines are parallel, so they do not intersect. This is called an inconsistent system.
3. The lines coincide; they intersect at infinitely many points. This is a dependent system.

The figures below show all three cases:



Every system of equations has either one solution, no solution, or infinitely many solutions.

In the last section, we used the Gauss-Jordan method to solve systems that had exactly one solution. In this section, we will determine the systems that have no solution, and solve the systems that have infinitely many solutions.

Example 2.3.1. *Solve the following system of equations using the Gauss-Jordan method:*

$$\begin{aligned} x + y &= 7 \\ x + y &= 9 \end{aligned}$$

Solution 2.3.1. *Let us use the Gauss-Jordan method to solve this system. The augmented matrix is*

$$\left[\begin{array}{cc|c} 1 & 1 & 7 \\ 1 & 1 & 9 \end{array}\right]$$

If we multiply the first row by -1 *and add it to the second row, we get*

$$\left[\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 0 & 2 \end{array}\right]$$

Since 0 *cannot equal* 2*, the last equation cannot be true for any choices of* x *and* y*. Alternatively, it is clear that the two lines are parallel; therefore, they do not intersect.*

In the examples that follow, we are going to start using a calculator to row reduce the augmented matrix, in order to focus on understanding the answer rather than focusing on the process of carrying out the row operations.

Example 2.3.2. *Solve the following system of equations:*

$$\begin{aligned} 2x + 3y - 4z &= 7 \\ 3x + 4y - 2z &= 9 \\ 5x + 7y - 6z &= 20 \end{aligned}$$

Solution 2.3.2. *We represent the system as an augmented matrix:*

$$\left[\begin{array}{ccc|c} 2 & 3 & -4 & 7 \\ 3 & 4 & -2 & 9 \\ 5 & 7 & -6 & 20 \end{array}\right]$$

By obtaining the reduced row-echelon form from a matrix calculator, we get:

$$\left[\begin{array}{ccc|c} 1 & 0 & 10 & 0 \\ 0 & 1 & -8 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The bottom row implies $0x + 0y + 0z = 1$*, which is a contradiction. Thus, the system is inconsistent and has no solution.*

Example 2.3.3. *Solve the following system of equations:*

$$\begin{aligned} x + y &= 7 \\ x + y &= 7 \end{aligned}$$

Solution 2.3.3. *The problem asks for the intersection of two identical lines, meaning the lines coincide and intersect at an infinite number of points.*

A few intersection points are listed as follows: (3, 4), (5, 2), (-1, 8), (-6, 13), etc. However, when a system has an infinite number of solutions, the solution is often expressed in parametric form. This can be done by assigning an arbitrary constant, t, to one of the variables and solving for the remaining variables. If we let $y = t$, then $x = 7 - t$. In other words, all ordered pairs of the form $(7 - t, t)$ satisfy the given system of equations.

Alternatively, solving with the Gauss-Jordan method, we obtain the reduced row-echelon form below, which includes a row of all zeros that can be ignored since it provides no additional information about the values of x and y that solve the system.

$$\left[\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 0 & 0 \end{array}\right]$$

This leaves us with only one equation but two variables. Whenever there are more variables than equations, the solution must be expressed as a parametric solution in terms of an arbitrary constant, as shown above.

Parametric Solution: $x = 7 - t$, $y = t$.

Example 2.3.4. *Solve the following system of equations:*

$$\begin{aligned} &x + y + z = 2 \\ &2x + y - z = 3 \\ &3x + 2y = 5 \end{aligned}$$

Solution 2.3.4. *The augmented matrix and the reduced row-echelon form are given below:*

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & -1 & 3 \\ 3 & 2 & 0 & 5 \end{array}\right] \quad \to \quad \left[\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since the last equation dropped out, we are left with two equations and three variables. This means the system has an infinite number of solutions. We express those solutions in the parametric form by letting the last variable z equal the parameter t.

The first equation reads $x - 2z = 1$, therefore, $x = 1 + 2z$. The second equation reads $y + 3z = 1$, therefore, $y = 1 - 3z$. And now if we let $z = t$, the parametric solution is expressed as follows:

$$\textit{Parametric Solution:} \quad x = 1 + 2t, \quad y = 1 - 3t, \quad z = t.$$

The reader should note that particular solutions, or specific solutions, to the system can be obtained by assigning values to the parameter t. For example:

- *If we let $t = 2$, we have the solution $x = 5, y = -5, z = 2$: $(5, -5, 2)$.*
- *If we let $t = 0$, we have the solution $x = 1, y = 1, z = 0$: $(1, 1, 0)$.*

Example 2.3.5. *Solve the following system of equations:*

$$\begin{aligned} x + 2y - 3z &= 5 \\ 2x + 4y - 6z &= 10 \\ 3x + 6y - 9z &= 15 \end{aligned}$$

Solution 2.3.5. *The reduced row-echelon form is given below:*

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This time the last two equations drop out. We are left with one equation and three variables. Again, there are an infinite number of solutions. But this time the answer must be expressed in terms of two arbitrary constants.

If we let $z = t$ and $y = s$, the first equation $x + 2y - 3z = 5$ results in $x = 5 - 2s + 3t$. We rewrite the parametric solution as:

$$\textit{Parametric Solution:} \quad x = 5 - 2s + 3t, \quad y = s, \quad z = t.$$

Summary 2.3.1: Systems of Equations - Special Cases

1. If any row of the reduced row-echelon form of the matrix gives a false statement such as $0 = 1$, the system is inconsistent and has no solution.
2. If the reduced row echelon form has fewer equations than the variables and the system is consistent, then the system has an infinite number of solutions. Remember the rows that contain all zeros are dropped.
   (a) If a system has an infinite number of solutions, the solution must be expressed in the parametric form.
   (b) The number of arbitrary parameters equals the number of variables minus the number of equations.

2.4 Inverse Matrices

In this section you will learn to:

1. Find the inverse of a matrix, if it exists.

2. Use inverses to solve linear systems.

In this section, we will learn to find the inverse of a matrix, if it exists. Later, we will use matrix inverses to solve linear systems.

Definition 2.4.1. *An $n \times n$ matrix has an **inverse** if there exists a matrix B such that $AB = BA = I_n$, where I_n is an $n \times n$ identity matrix. The **inverse** of a matrix A, if it exists, is denoted by the symbol A^{-1}.*

Example 2.4.1. *Given matrices A and B below, verify that they are inverses.*

$$A = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$$

Solution 2.4.1. *The matrices are inverses if the product AB and BA both equal the identity matrix of dimension 2×2, denoted as I_2:*

$$AB = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

and

$$BA = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Clearly, that is the case; therefore, the matrices A and B are inverses of each other.

Example 2.4.2. *Find the inverse of the matrix $A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$.*

Solution 2.4.2. *Suppose A has an inverse, and it is denoted as $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $AB = I_2$:*

$$\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

After multiplying the matrices on the left side, we get the system:

$$\begin{aligned} 3a + c &= 1 \\ 3b + d &= 0 \\ 5a + 2c &= 0 \\ 5b + 2d &= 1 \end{aligned}$$

Solving this system, we find $a = 2$, $b = -1$, $c = -5$, and $d = 3$. Therefore, the inverse of matrix A is $B = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$.

In this problem, finding the inverse of matrix A amounted to solving the system of equations:

$$\begin{aligned} 3a + c &= 1 \\ 3b + d &= 0 \\ 5a + 2c &= 0 \\ 5b + 2d &= 1 \end{aligned}$$

Actually, it can be written as two systems, one with variables a and c, and the other with b and d. The augmented matrices for both are given below.

$$\left[\begin{array}{cc|c} 3 & 1 & 1 \\ 5 & 2 & 0 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{cc|c} 3 & 1 & 0 \\ 5 & 2 & 1 \end{array}\right]$$

As we look at the two augmented matrices, we notice that the coefficient matrix for both the matrices is the same. This implies the row operations of the Gauss-Jordan method will also be the same. A great deal of work can be saved if the two right-hand columns are grouped together to form one augmented matrix as below.

$$\left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 5 & 2 & 0 & 1 \end{array}\right]$$

And solving this system, we get

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -5 & 3 \end{array}\right]$$

The matrix on the right side of the vertical line is the A^{-1} matrix. What you just witnessed is no coincidence. This is the method that is often employed in finding the inverse of a matrix.

Summary 2.4.1:

The Method for Finding the Inverse of a Matrix

1. Write the augmented matrix $[A|I_n]$.
2. Write the augmented matrix in step 1 in reduced row echelon form.
3. If the reduced row echelon form in 2 is $[I_n|B]$, then B is the inverse of A.
4. If the left side of the row reduced echelon is not an identity matrix, the inverse does not exist.

Example 2.4.3. *Given the matrix A below, find its inverse.*

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Solution 2.4.3. *We write the augmented matrix as follows.*

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

We will reduce this matrix using the Gauss-Jordan method. Multiplying the first row by -2 *and adding it to the second row, we get*

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 5 & -2 & -2 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

If we swap the second and third rows, we get

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \\ 0 & 5 & -2 & -2 & 1 & 0 \end{array}\right]$$

Divide the second row by -2*. The result is*

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1/2 & 0 & 0 & -1/2 \\ 0 & 5 & -2 & -2 & 1 & 0 \end{array}\right]$$

Let us do two operations here. 1) *Add the second row to the first.* 2) *Add* -5 *times the second row to the third. And we get*

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 1 & 0 & -1/2 \\ 0 & 1 & -1/2 & 0 & 0 & -1/2 \\ 0 & 0 & 1/2 & -2 & 1 & 5/2 \end{array}\right]$$

Multiplying the third row by 2 *results in*

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 1 & 0 & -1/2 \\ 0 & 1 & -1/2 & 0 & 0 & -1/2 \\ 0 & 0 & 1 & -4 & 2 & 5 \end{array}\right]$$

Multiply the third row by $1/2$ *and add it to the second. Also, multiply the third row by* $-1/2$ *and add it to the first.*

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & -3 \\ 0 & 1 & 0 & -2 & 1 & 2 \\ 0 & 0 & 1 & -4 & 2 & 5 \end{array}\right]$$

Therefore, the inverse of matrix A is $A^{-1} = \begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix}$.

One should verify the result by multiplying the two matrices to see if the product does, indeed, equal the identity matrix.

Now that we know how to find the inverse of a matrix, we will use inverses to solve systems of equations. The method is analogous to solving a simple equation like the one below.

$$\frac{2}{3}x = 4$$

Example 2.4.4. *Solve the following equation:*

$$x = 4$$

Solution 2.4.4. *To solve the above equation, we multiply both sides of the equation by the multiplicative inverse of $\frac{2}{3}$, which happens to be $\frac{3}{2}$. We get*

$$\frac{3}{2} \cdot \frac{2}{3}x = 4 \cdot \frac{3}{2}$$

Hence,

$$x = 6.$$

We use example 2.4.4 as an analogy to show how linear systems of the form $AX = B$ are solved. To solve a linear system, we first write the system in the matrix equation $AX = B$, where A is the coefficient matrix, X is the matrix of variables, and B is the matrix of constant terms. We then multiply both sides of this equation by the multiplicative inverse of the matrix A. Consider the following example.

Example 2.4.5. *Solve the following system*

$$\begin{aligned} 3x + y &= 3 \\ 5x + 2y &= 4 \end{aligned}$$

Solution 2.4.5. *To solve the above equation, first we express the system as*

$$AX = B$$

where A is the coefficient matrix, and B is the matrix of constant terms. We get

$$\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

To solve this system, we multiply both sides of the matrix equation $AX = B$ by A^{-1}. Matrix multiplication is not commutative, so we need to multiply by A^{-1} on the left on both sides of the equation.

Matrix A is the same matrix A whose inverse we found in Example 2.4.2, so $A^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$.

Multiplying both sides by A^{-1}, we get

$$\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Therefore, $x = 2$, and $y = -3$.

Example 2.4.6. *Solve the following system:*

$$\begin{aligned} x - y + z &= 6 \\ 2x + 3y &= 1 \\ -2y + z &= 5 \end{aligned}$$

Solution 2.4.6. *To solve the above equation, we write the system in matrix form $AX = B$ as follows:*

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 5 \end{bmatrix}$$

To solve this system, we need the inverse of A. From Example 2.4.3, A^{-1} is given by

$$A^{-1} = \begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix}$$

Multiplying both sides of the matrix equation $AX = B$ on the left by A^{-1}, we get

$$\begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

After multiplying the matrices, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

Therefore, $x = 2$, $y = -1$, and $z = 3$.

We remind the reader that not every system of equations can be solved by the matrix inverse method. Although the Gauss-Jordan method works for every situation, the matrix inverse method works only in cases where the inverse of the square matrix exists. In such cases the system has a unique solution.

Summary 2.4.2: Finding the Inverse of a Matrix

1. Write the augmented matrix $[A|I_n]$.
2. Write the augmented matrix in step 1 in reduced row echelon form.
3. If the reduced row echelon form in step 2 is $[I_n|B]$, then B is the inverse of A.
4. If the left side of the row reduced echelon is not an identity matrix, the inverse does not exist.

Summary 2.4.3: Solving a System of Equations When a Unique Solution Exists

1. Express the system in the matrix equation $AX = B$.
2. To solve the equation $AX = B$, multiply both sides by A^{-1}:

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B \quad \text{where } I \text{ is the identity matrix}$$

2.5 Application of Matrices in Cryptography

In this section, you will learn to:

1. Encode a message using matrix multiplication.
2. Decode a coded message using the matrix inverse and matrix multiplication.

Encryption dates back approximately 4000 years. Historical accounts indicate that the Chinese, Egyptians, Indians, and Greeks encrypted messages in some way for various purposes. One famous encryption scheme is called the Caesar cipher, also called a substitution cipher, used by Julius Caesar, involved shifting letters in the alphabet, such as replacing A by C, B by D, C by E, etc., to encode a message. Substitution ciphers are too simple in design to be considered secure today.

In the middle ages, European nations began to use encryption. A variety of encryption methods were used in the US from the Revolutionary War, through the Civil War, and on into modern times.

Applications of mathematical theory and methods to encryption became widespread in military usage in the 20th century. The military would encode messages before sending, and the recipient would decode the message, in order to send information about military operations in a manner that kept the information safe if the message was intercepted. In World War II, encryption played an important role, as both Allied and Axis powers sent encrypted messages and devoted significant resources to strengthening their own encryption while also trying to break the opposition's encryption.

In this section, we will examine a method of encryption that uses matrix multiplication and matrix inverses. This method, known as the Hill Algorithm, was created by Lester Hill, a mathematics professor who taught at several US colleges and also was involved with military encryption. The Hill algorithm marks the introduction of modern mathematical theory and methods to the field of cryptography.

These days, the Hill Algorithm is not considered a secure encryption method; it is relatively easy to break with modern technology. However, in 1929 when it was developed, modern computing technology did not exist. This method, which we can handle easily with today's technology, was too cumbersome to use with hand calculations. Hill devised a mechanical encryption machine to help with the mathematics; his machine relied on gears and levers but never gained widespread use. Hill's method was considered sophisticated and powerful in its time and is one of many methods influencing techniques in use today. Other encryption methods at that time also utilized special coding machines. Alan Turing, a computer scientist pioneer in the field of artificial intelligence, invented a machine that was able to decrypt messages encrypted by the German Enigma machine, helping to turn the tide of World War II.

With the advent of the computer age and internet communication, the use of encryption has become widespread in communication and in keeping private data secure; it is no longer limited to military uses. Modern encryption methods are more complicated, often combining several steps or methods to encrypt data to keep it more secure and harder to break. Some modern methods make use of matrices as part of the encryption and decryption process; other fields of mathematics such as number theory play a large role in modern cryptography.

2.5.1 Using Matrices for Encoding and Decoding

To use matrices in encoding and decoding secret messages, our procedure is as follows.

We first convert the secret message into a string of numbers by arbitrarily assigning a number to each letter of the message. Next, we convert this string of numbers into a new set of numbers by multiplying the string by a square matrix of our choice that has an inverse. This new set of numbers represents the coded message.

To decode the message, we take the string of coded numbers and multiply it by the inverse of the matrix to get the original string of numbers. Finally, by associating the numbers with their corresponding letters, we obtain the original message.

In this section, we will use the correspondence shown below where letters A to Z correspond to the numbers 1 to 26, a space is represented by the number 27, and punctuation is ignored.

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13

N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	26

Example 2.5.1. *Use matrix A to encode the message: ATTACK NOW!*

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

Solution 2.5.1. *We divide the letters of the message into groups of two.*

$$AT \quad TA \quad CK \quad \_N \quad OW$$

We assign the numbers to these letters from the above table, and convert each pair of numbers into 2×1 matrices. In the case where a single letter is left over on the end, a space is added to make it into a pair.

$$\begin{bmatrix} A \\ T \end{bmatrix} = \begin{bmatrix} 1 \\ 20 \end{bmatrix}, \quad \begin{bmatrix} T \\ A \end{bmatrix} = \begin{bmatrix} 20 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} C \\ K \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}, \quad \begin{bmatrix} - \\ N \end{bmatrix} = \begin{bmatrix} 27 \\ 14 \end{bmatrix}, \quad \begin{bmatrix} O \\ W \end{bmatrix} = \begin{bmatrix} 15 \\ 23 \end{bmatrix}$$

So at this stage, our message expressed as 2×1 matrices is as follows.

$$\begin{bmatrix} 1 \\ 20 \end{bmatrix}, \quad \begin{bmatrix} 20 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 11 \end{bmatrix}, \quad \begin{bmatrix} 27 \\ 14 \end{bmatrix}, \quad \begin{bmatrix} 15 \\ 23 \end{bmatrix}$$

Now to encode, we multiply, on the left, each matrix of our message by the matrix A.

For example, the product of A with our first matrix is:

$$A \cdot \begin{bmatrix} 1 \\ 20 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 20 \end{bmatrix} = \begin{bmatrix} 41 \\ 61 \end{bmatrix}$$

And the product of A with our second matrix is:

$$A \cdot \begin{bmatrix} 20 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 20 \\ 1 \end{bmatrix} = \begin{bmatrix} 22 \\ 23 \end{bmatrix}$$

Multiplying matrix A by each matrix in our list, in turn, gives the desired coded message:

$$\begin{bmatrix} 41 \\ 61 \end{bmatrix}, \quad \begin{bmatrix} 22 \\ 23 \end{bmatrix}, \quad \begin{bmatrix} 25 \\ 36 \end{bmatrix}, \quad \begin{bmatrix} 55 \\ 69 \end{bmatrix}, \quad \begin{bmatrix} 61 \\ 84 \end{bmatrix}$$

Example 2.5.2. *Decode the following message that was encoded using matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$:*

$$\begin{bmatrix} 21 \\ 26 \end{bmatrix}, \quad \begin{bmatrix} 37 \\ 53 \end{bmatrix}, \quad \begin{bmatrix} 45 \\ 54 \end{bmatrix}, \quad \begin{bmatrix} 74 \\ 101 \end{bmatrix}, \quad \begin{bmatrix} 53 \\ 69 \end{bmatrix}$$

Solution 2.5.2. *Since this message was encoded by multiplying by the matrix A in Example 2.4.2, we decode this message by first multiplying each matrix, on the left, by the inverse of matrix A given below.*

$$A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

For example:

$$\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 21 \\ 26 \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \end{bmatrix}$$

By multiplying each of the matrices in our list by the matrix A^{-1}, we get the following.

$$\begin{bmatrix}11\\5\end{bmatrix}, \quad \begin{bmatrix}5\\16\end{bmatrix}, \quad \begin{bmatrix}27\\9\end{bmatrix}, \quad \begin{bmatrix}20\\27\end{bmatrix}, \quad \begin{bmatrix}21\\16\end{bmatrix}$$

Finally, by associating the numbers with their corresponding letters, we obtain:

$$\begin{bmatrix}K\\E\end{bmatrix}, \quad \begin{bmatrix}E\\P\end{bmatrix}, \quad \begin{bmatrix}\_\\T\end{bmatrix}, \quad \begin{bmatrix}I\\U\end{bmatrix}, \quad \begin{bmatrix}P\\\_\end{bmatrix}$$

And the message reads: KEEP IT UP.

Now suppose we wanted to use a 3×3 matrix to encode a message, then instead of dividing the letters into groups of two, we would divide them into groups of three.

Example 2.5.3. *Using the matrix* $B = \begin{bmatrix}1 & 1 & -1\\1 & 0 & 1\\2 & 1 & 1\end{bmatrix}$*, encode the message: ATTACK NOW!*

Solution 2.5.3. *We divide the letters of the message into groups of three.*

$$ATT \quad ACK \quad \_NO \quad W\_\_$$

Note that since the single letter "W" was left over on the end, we added two spaces to make it into a triplet.

Now we assign the numbers their corresponding letters from the table, and convert each triplet of numbers into 3×1 matrices. We get

$$\begin{bmatrix}A\\T\\T\end{bmatrix} = \begin{bmatrix}1\\20\\20\end{bmatrix}, \quad \begin{bmatrix}A\\C\\K\end{bmatrix} = \begin{bmatrix}1\\3\\11\end{bmatrix}, \quad \begin{bmatrix}-\\N\\O\end{bmatrix} = \begin{bmatrix}27\\14\\15\end{bmatrix}, \quad \begin{bmatrix}W\\-\\-\end{bmatrix} = \begin{bmatrix}23\\27\\27\end{bmatrix}$$

So far we have,

$$\begin{bmatrix}1\\20\\20\end{bmatrix}, \quad \begin{bmatrix}1\\3\\11\end{bmatrix}, \quad \begin{bmatrix}27\\14\\15\end{bmatrix}, \quad \begin{bmatrix}23\\27\\27\end{bmatrix}$$

We multiply, on the left, each matrix of our message by the matrix B. For example,

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 20 \\ 20 \end{bmatrix} = \begin{bmatrix} 1 \\ 21 \\ 42 \end{bmatrix}$$

By multiplying each of the matrices in (III) by the matrix B, we get the desired coded message as follows:

$$\begin{bmatrix} 1 \\ 21 \\ 42 \end{bmatrix}, \quad \begin{bmatrix} -7 \\ 12 \\ 16 \end{bmatrix}, \quad \begin{bmatrix} 26 \\ 42 \\ 83 \end{bmatrix}, \quad \begin{bmatrix} 23 \\ 50 \\ 100 \end{bmatrix}$$

If we need to decode this message, we simply multiply the coded message by B^{-1}, and associate the numbers with the corresponding letters of the alphabet.

In Example 2.5.4 we will demonstrate how to use matrix B^{-1} to decode an encrypted message.

Example 2.5.4. *Decode the following message that was encoded using matrix* $B = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$*:*

$$\begin{bmatrix} 11 \\ 20 \\ 43 \end{bmatrix}, \quad \begin{bmatrix} 25 \\ 10 \\ 41 \end{bmatrix}, \quad \begin{bmatrix} 22 \\ 14 \\ 41 \end{bmatrix}$$

Solution 2.5.4. *Since this message was encoded by multiplying by the matrix B. We first determine the inverse of B.*

$$B^{-1} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & -3 & 2 \\ -1 & -1 & 1 \end{bmatrix}$$

To decode the message, we multiply each matrix, on the left, by B^{-1}. For example,

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & -3 & 2 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 11 \\ 20 \\ 43 \end{bmatrix} = \begin{bmatrix} 8 \\ 15 \\ 12 \end{bmatrix}$$

Multiplying each of the matrices in our list by the matrix B^{-1} *gives the following:*

$$\begin{bmatrix} 8 \\ 15 \\ 12 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 27 \\ 6 \end{bmatrix}, \quad \begin{bmatrix} 9 \\ 18 \\ 5 \end{bmatrix}$$

Finally, by associating the numbers with their corresponding letters, we obtain:

$$\begin{bmatrix} H \\ O \\ L \end{bmatrix}, \quad \begin{bmatrix} D \\ \_ \\ F \end{bmatrix}, \quad \begin{bmatrix} I \\ R \\ E \end{bmatrix}$$

The message reads: HOLD FIRE.

Summary 2.5.1: Encoding and Decoding

To Encode a Message

1. Divide the letters of the message into groups of two or three.
2. Convert each group into a string of numbers by assigning a number to each letter of the message. Remember to assign letters to blank spaces.
3. Convert each group of numbers into column matrices.
4. Convert these column matrices into a new set of column matrices by multiplying them with a compatible square matrix of your choice that has an inverse. This new set of numbers or matrices represents the coded message.

To Decode a Message

1. Take the string of coded numbers and multiply it by the inverse of the matrix that was used to encode the message.
2. Associate the numbers with their corresponding letters.

2.6 Applications – Leontief Models

In this section you will learn

1. Application of matrices to model closed economic systems

2. Application of matrices to model open economic systems

In the 1930s, Wassily Wassilyevich Leontief (holder of one of the greatest names ever) used matrices to model economic systems. His models, often referred to as the input-output models, divide the economy into sectors where each sector produces goods and services not only for itself but also for other sectors. These sectors are dependent on each other, and the total input always equals the total output. In 1973, he won the Nobel Prize in Economics for his work in this field. In this section, we look at both the closed and the open models that he developed.

2.6.1 The Closed Model

As an example of the closed model, we look at a very simple economy, where there are only three sectors: food, shelter, and clothing.

Example 2.6.1. *We assume that in a village there is a farmer, carpenter, and a tailor, who provide the three essential goods: food, shelter, and clothing. Suppose the farmer himself consumes* 40% *of the food he produces, and gives* 40% *to the carpenter, and* 20% *to the tailor. Thirty percent of the carpenter's production is consumed by himself,* 40% *by the farmer, and* 30% *by the carpenter. Fifty percent of the tailor's production is used by himself,* 30% *by the farmer, and* 20% *by the tailor. Write the matrix that describes this closed model.*

Solution 2.6.1. *The table below describes the above information.*

	Proportion produced by the farmer	*Proportion produced by the carpenter*	*Proportion produced by the tailor*
The proportion used by the farmer	*.40*	*.40*	*.30*
The proportion used by the carpenter	*.40*	*.30*	*.20*
The proportion used by the tailor	*.20*	*.30*	*.50*

In matrix form, it can be written as follows.

$$A = \begin{bmatrix} .40 & .40 & .30 \\ .40 & .30 & .20 \\ .20 & .30 & .50 \end{bmatrix}$$

This matrix is called the input-output matrix. It is important that we read the matrix correctly. For example, the entry A_{23}, the entry in row 2 and column 3, represents the following.

$$A_{23} = 20\% \textit{ of the tailor's production is used by the carpenter.}$$

$$A_{33} = 50\% \textit{ of the tailor's production is used by the tailor.}$$

Example 2.6.2. *In Example 2.6.1 above, how much should each person get for his efforts?*

Solution 2.6.2. *We choose the following variables.*

$$\begin{aligned} x &= \textit{Farmer's pay} \\ y &= \textit{Carpenter's pay} \\ z &= \textit{Tailor's pay} \end{aligned}$$

As we said earlier, in this model input must equal output. That is, the amount paid by each equals the amount received by each.

Let us say the farmer gets paid x dollars. Let us now look at the farmer's expenses. The farmer uses up 40% of his own production, that is, of the x dollars he gets paid, he pays himself $.40x$ dollars, he pays $.40y$ dollars to the *carpenter, and .30z to the tailor. Since the expenses equal the wages, we get the following equation.*

$$x = .40x + .40y + .30z$$

In the same manner, we get

$$y = .40x + .30y + .20z$$
$$z = .20x + .30y + .50z$$

The above system can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} .40 & .40 & .30 \\ .40 & .30 & .20 \\ .20 & .30 & .50 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This system is often referred to as $X = AX$.

Simplification results in the system of equations $(I - A)X = 0$

$$\begin{bmatrix} .60 & -.40 & -.30 \\ -.40 & .70 & -.20 \\ -.20 & -.30 & .50 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We put this into an augmented matrix

$$\left[\begin{array}{ccc|c} .60 & -.40 & -.30 & 0 \\ -.40 & .70 & -.20 & 0 \\ -.20 & -.30 & .50 & 0 \end{array}\right]$$

Solving for x, y, and z using the Gauss-Jordan method, we get

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{29}{26} & 0 \\ 0 & 1 & -\frac{12}{13} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This gives parametric equations:

$$x = \frac{29}{26}t, \quad y = \frac{12}{13}t, \quad z = t$$

Since we are only trying to determine the proportions of the pay, we can choose t to be any value. Suppose we let $t = \$2600$, then we get

$$x = \$2900, \quad y = \$2400, \quad z = \$2600$$

Note 2.6.1. *The use of a graphing calculator or computer application in solving the systems of linear matrix equations in these problems is strongly recommended.*

2.6.2 The Open Model

The open model is more realistic as it deals with the economy where sectors of the economy not only satisfy each other's needs but also satisfy some outside demands. In this case, the outside demands are put on by the consumer. But the basic assumption is still the same: whatever is produced is consumed.

Let us again look at a very simple scenario. Suppose the economy consists of three people: the farmer F, the carpenter C, and the tailor T. A part of the farmer's production is used by all three, and the rest is used by the consumer. In the same manner, a part of the carpenter's and the tailor's production is used by all three, and the rest is used by the consumer.

Let us assume that whatever the farmer produces, 20% is used by him, 15% by the carpenter, 10% by the tailor, and the consumer uses the other \$40 billion worth of food. Ten percent of the carpenter's production is used by him, 25% by the farmer, 5% by the tailor, and \$50 billion worth by the consumer. Fifteen percent of the clothing is used by the tailor, 10% by the farmer, 5% by the carpenter, and the remaining \$60 billion worth by the consumer. We write the internal consumption in the following table and express the demand as the matrix D.

	F produces	C produces	T produces
F uses	0.20	0.25	0.10
C uses	0.15	0.10	0.05
T uses	0.10	0.05	0.15

The consumer demand for each industry in billions of dollars is given by the matrix $D = \begin{bmatrix} 40 \\ 50 \\ 60 \end{bmatrix}$.

Example 2.6.3. *In the example above, what should be, in billions of dollars, the required output by each industry to meet the demand given by the matrix D?*

Solution 2.6.3. *We choose the following variables.*

$$\begin{aligned} x &= \textit{Farmer's output} \\ y &= \textit{Carpenter's output} \\ z &= \textit{Tailor's output} \end{aligned}$$

In the closed model, our equation was $X = AX$, that is, the total input equals the total output. This time our equation is similar with the exception of the demand by the consumer.

So our equation for the open model should be $X = AX + D$, where D represents the demand matrix.

We express it as follows:

$$X = AX + D$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} .20 & .25 & .10 \\ .15 & .10 & .05 \\ .10 & .05 & .15 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 40 \\ 50 \\ 60 \end{bmatrix}$$

To solve this system, we write it as

$$X = AX + D$$

$$(I - A)X = D$$

$$X = (I - A)^{-1}D$$

where I is a 3×3 identity matrix.

$$I - A = \begin{bmatrix} .80 & -.25 & -.10 \\ -.15 & .90 & -.05 \\ -.10 & -.05 & .85 \end{bmatrix}$$

$$(I - A)^{-1} = \begin{bmatrix} 1.3445 & .3835 & .1807 \\ .2336 & 1.1814 & .097 \\ .1719 & .1146 & 1.2034 \end{bmatrix}$$

$$X = \begin{bmatrix} 1.3445 & .3835 & .1807 \\ .2336 & 1.1814 & .097 \\ .1719 & .1146 & 1.2034 \end{bmatrix} \begin{bmatrix} 40 \\ 50 \\ 60 \end{bmatrix}$$

$$X = \begin{bmatrix} 83.7999 \\ 74.2341 \\ 84.8138 \end{bmatrix}$$

The three industries must produce the following amount of goods in billions of dollars.

$$\begin{aligned} Farmer &= 83.7999 \\ Carpenter &= 74.2341 \\ Tailor &= 84.8138 \end{aligned}$$

We will do one more problem like the one above, except this time we give the amount of internal and external consumption in dollars and ask for the proportion of the amounts consumed by each of the industries. In other words, we ask for the matrix A.

Example 2.6.4. *Suppose an economy consists of three industries F, C, and T. Each of the industries produces for internal consumption among themselves, as well as for external demand by the consumer. The table shows the use of each industry's production in dollars.*

	F	***C***	***T***	***Demand***	***Total***
F	40	50	60	100	250
C	30	40	40	110	220
T	20	30	30	120	200

The first row says that of the \$250 dollars worth of production by the industry F, \$40 is used by F, \$50 is used by C, \$60 is used by T, and the remainder of \$100 is used by the consumer. The other rows are described in a similar manner.

Once again, the total input equals the total output. Find the proportion of the amounts consumed by each of the industries. In other words, find the matrix A.

Solution 2.6.4. *We are being asked to determine the following:*

How much of the production of each of the three industries, F, C, and T is required to produce one unit of F? The same way, how much of the production *of each of the three industries, F, C, and T is required to produce one unit of C? And finally, how much of the production of each of the three industries, F, C, and T is required to produce one unit of T?*

Since we are looking for proportions, we need to divide the production of each industry by the total production for each industry.

We analyze as follows: To produce 250 units of F, 30 units of C, and 20 units of T, the required units are 40, 30, and 20 respectively. Therefore, to produce 1 unit of each, we divide by 250:

$$\text{For F: } \frac{40}{250}, \text{ For C: } \frac{30}{250}, \text{ For T: } \frac{20}{250}$$

Similarly, for 220 units of C, the required units are 50, 40, and 30 respectively. To produce 1 unit of C, we divide by 220:

$$\text{For F: } \frac{50}{220}, \text{ For C: } \frac{40}{220}, \text{ For T: } \frac{30}{220}$$

And for 200 units of T, the required units are 60, 40, and 30 respectively. To produce 1 unit of T, we divide by 200:

$$\text{For F: } \frac{60}{200}, \text{ For C: } \frac{40}{200}, \text{ For T: } \frac{30}{200}$$

These fractions represent the units of F, C, and T required to produce 1 unit of each.

We obtain the following matrix:

$$A = \begin{bmatrix} \frac{40}{250} & \frac{50}{220} & \frac{60}{200} \\ \frac{30}{250} & \frac{40}{220} & \frac{40}{200} \\ \frac{20}{250} & \frac{30}{220} & \frac{30}{200} \end{bmatrix} = \begin{bmatrix} .1600 & .2273 & .3000 \\ .1200 & .1818 & .2000 \\ .0800 & .1364 & .1500 \end{bmatrix}$$

Clearly $AX + D = X$

$$\begin{bmatrix} .1600 & .2273 & .3000 \\ .1200 & .1818 & .2000 \\ .0800 & .1364 & .1500 \end{bmatrix} \begin{bmatrix} 250 \\ 220 \\ 200 \end{bmatrix} + \begin{bmatrix} 100 \\ 110 \\ 120 \end{bmatrix} = \begin{bmatrix} 250 \\ 220 \\ 200 \end{bmatrix}$$

Summary 2.6.1: Leontief's Models

Leontief's Closed Model

1. All consumption is within the industries. There is no external demand.
2. Input equals output.
3. $X = AX$ or $(I - A)X = 0$

Leontief's Open Model

1. In addition to internal consumption, there is an outside demand by the consumer.
2. Input equals output.
3. $X = AX + D$ or $X = (I - A)^{-1}D$

Chapter 3

Linear Programming with Geometry

In this chapter, you will learn to:

1. Solve linear programming problems that maximize the objective function.
2. Solve linear programming problems that minimize the objective function.

3.1 Maximization Applications

In this section, you will learn to:

1. Recognize the typical form of a linear programming problem.
2. Formulate maximization linear programming problems.
3. Graph feasibility regions for maximization linear programming problems.
4. Determine optimal solutions for maximization linear programming problems.

Application problems in business, economics, and social and life sciences often ask us to make decisions on the basis of certain conditions. The conditions or constraints often take the form of inequalities. In this section, we will begin to formulate, analyze, and solve such problems, at a simple level, to understand the many components of such a problem.

A typical linear programming problem consists of finding an extreme value of a linear function subject to certain constraints. We are either trying to maximize or minimize the value of this linear function, such as to maximize profit or revenue, or to minimize cost. That is why these linear programming problems are classified as maximization or minimization problems, or just optimization problems. The function we are trying to optimize is called an objective function, and the conditions that must be satisfied are called constraints.

A typical example is to maximize profit from producing several products, subject to limitations on materials or resources needed for producing these items; the problem requires us to determine the amount of each item produced. Another type of problem involves scheduling; we need to determine how much time to devote to each of several activities in order to maximize income from (or minimize cost of) these activities, subject to limitations on time and other resources available for each activity.

In this chapter, we will work with problems that involve only two variables, and therefore, can be solved by graphing.

In the next chapter, we'll learn an algorithm to find a solution numerically. That will provide us with a tool to solve problems with more than two variables. At that time, with a little more knowledge about linear programming, we'll also explore the many ways these techniques are used in business and wide variety of other fields.

We begin by solving a maximization problem.

Example 3.1.1. *Niki holds two part-time jobs, Job I and Job II. She never wants to work more than a total of 12 hours a week. She has determined that for every hour she works at Job I, she needs 2 hours of preparation time, and for every hour she works at Job II, she needs one hour of preparation time, and she cannot spend more than 16 hours for preparation. If Niki makes $40 an hour at Job I, and $30 an hour at Job II, how many hours should she work per week at each job to maximize her income?*

Solution 3.1.1. *We start by choosing our variables. Let x be the number of* *hours per week Niki will work at Job I, and y the number of hours per week she will work at Job II.*

Now we write the objective function. Since Niki gets paid \$40 an hour at Job I, and \$30 an hour at Job II, her total income I is given by the following equation.

$$I = 40x + 30y$$

Our next task is to find the constraints. The constraints based on the problem description are:

$$x + y \leq 12$$

$$2x + y \leq 16$$

$$x \geq 0, \quad y \geq 0$$

We have formulated the problem as follows: Maximize

$$I = 40x + 30y$$

Subject to:

$$x + y \leq 12$$

$$2x + y \leq 16$$

$$x \geq 0; \quad y \geq 0$$

To solve the problem, we graph the constraints and shade the region that satisfies all the inequality constraints. We graph the lines by plotting the x-intercept and y-intercept and use a test point to determine which portion of the plane to shade.

In this example, after graphing the lines representing the constraints and using the origin (0,0) as a test point, we find that the feasible region is the area below and to the left of both constraint lines, above the x-axis, and to the right of the y-axis.

The shaded region where all conditions are satisfied is called the feasibility region or the feasibility polygon. The Fundamental Theorem of Linear Programming states that the maximum (or minimum) value of the objective function always takes place at the vertices of the feasibility region. Therefore, we will identify all the vertices (corner points) of the feasibility region. We

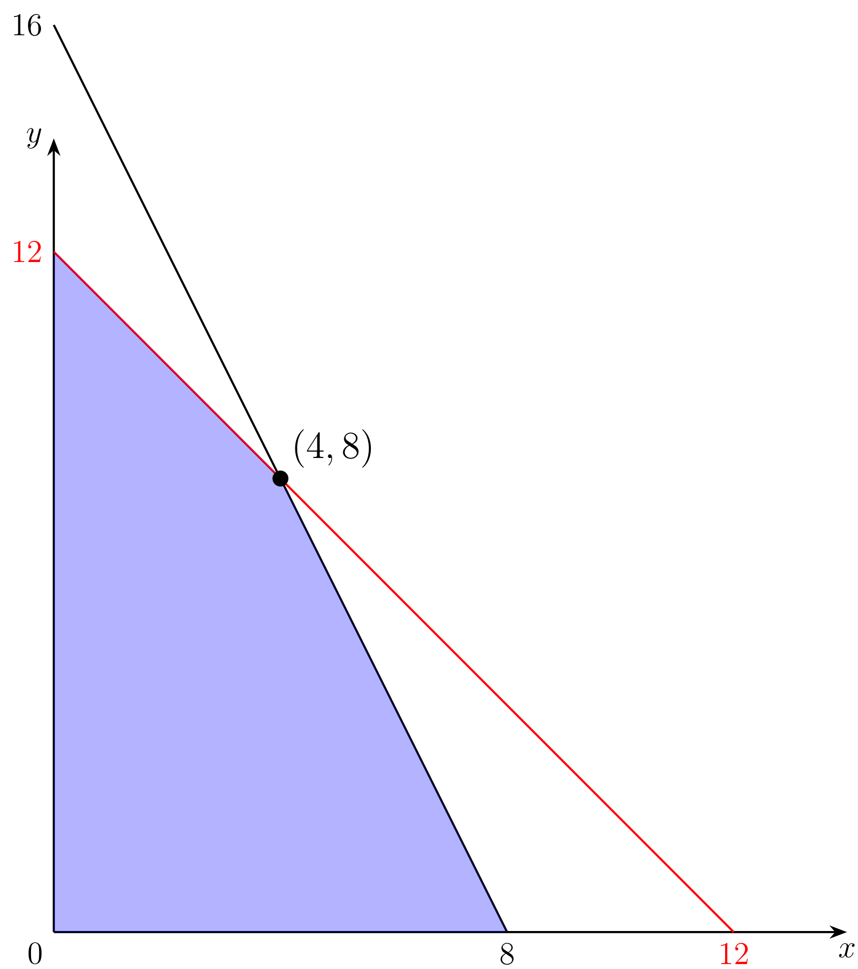


Figure 3.1: The red line is $x + y = 12$, the black line is $2x + y = 16$, and the blue region is the feasible region.

call these points critical points. They are listed as $(0, 0)$, $(0, 12)$, $(4, 8)$, and $(8, 0)$.

To maximize Niki's income, we will substitute these points in the objective function to see which point gives us the highest income per week. We list the results below:

Critical Points	**Income**
$(0, 0)$	$40(0) + 30(0) = \$0$
$(0, 12)$	$40(0) + 30(12) = \$360$
$(4, 8)$	$40(4) + 30(8) = \$400$
$(8, 0)$	$40(8) + 30(0) = \$320$

Clearly, the point $(4, 8)$ gives the most profit: \$400. Therefore, we conclude that Niki should work 4 hours at Job I and 8 hours at Job II.

Example 3.1.2. *A factory manufactures two types of gadgets, regular and premium. Each gadget requires the use of two operations, assembly and finishing, and there are at most 12 hours available for each operation. A regular gadget requires 1 hour of assembly and 2 hours of finishing, while a premium gadget needs 2 hours of assembly and 1 hour of finishing. Due to other restrictions, the company can make at most 7 gadgets a day. If a profit of \$20 is realized for each regular gadget and \$30 for a premium gadget, how many of each should be manufactured to maximize profit?*

Solution 3.1.2. *We choose our variables. Let x be the number of regular gadgets manufactured each day, and y be the number of premium gadgets manufactured each day.*

The objective function is

$$P = 20x + 30y$$

We now write the constraints. The company can make at most 7 gadgets a day, giving us:

$$x + y \leq 7$$

The regular gadget requires one hour of assembly and the premium gadget two hours, with at most 12 hours available for assembly:

$$x + 2y \leq 12$$

Similarly, for finishing, we have:

$$2x + y \leq 12$$

The non-negativity constraints are:

$$x \geq 0, \quad y \geq 0$$

We formulate the problem as follows: Maximize $P = 20x + 30y$ *Subject to:*

$$x + y \leq 7$$

$$x + 2y \leq 12$$

$$2x + y \leq 12$$

$$x \geq 0; \quad y \geq 0$$

We next graph the constraints and feasibility region.

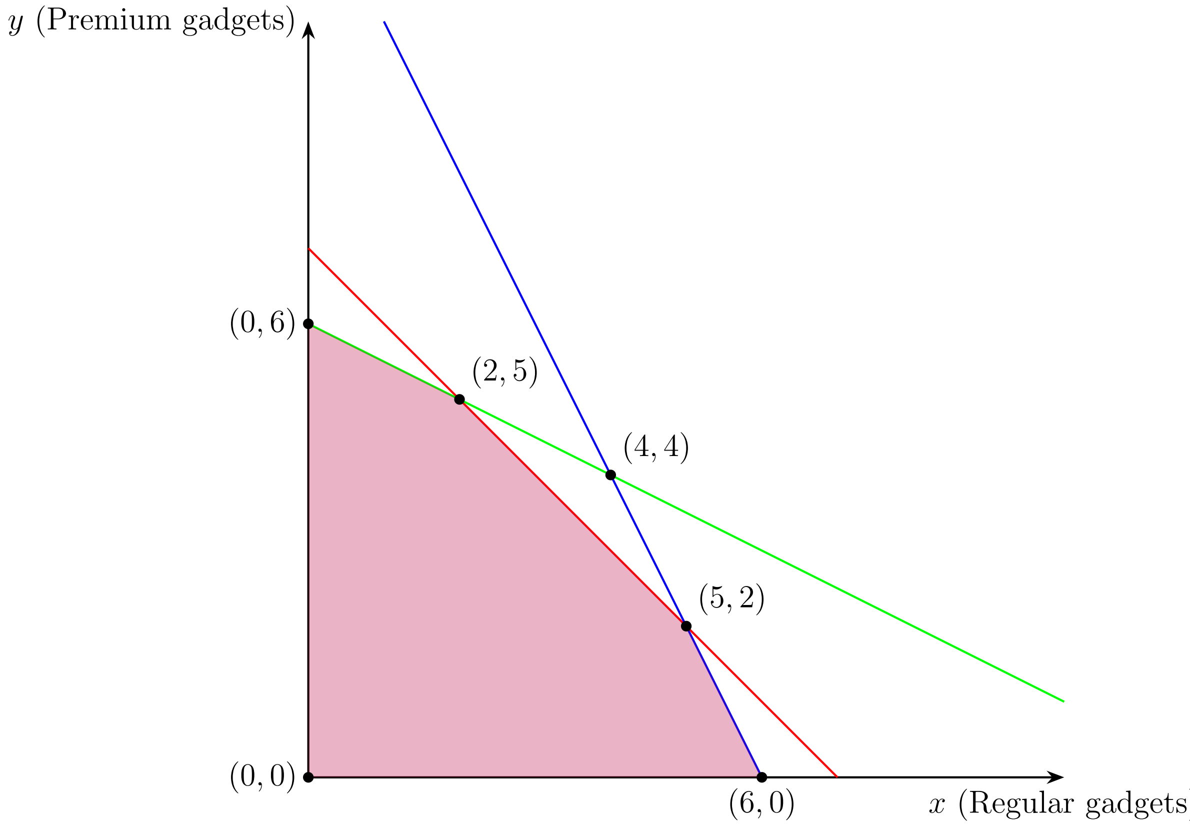


Figure 3.2: Feasibility region for the gadget factory optimization problem

Again, we have shaded the feasibility region, where all constraints are satisfied. Since the extreme value of the objective function always takes place at the vertices of the feasibility region, we identify all the critical points. They are listed as $(0,0)$, $(0,6)$, $(2,5)$, $(5,2)$, and $(6,0)$. Notice, $(4,4)$ is **not** a critical point because it is not on the edge of the critical region. To maximize profit, we will substitute these points in the objective function to see which point gives us the maximum profit each day. The results are listed below:

Critical Point	**Income**
$(0,0)$	$20(0)+30(0)=\$0$
$(0,6)$	$20(0)+30(6)=\$180$
$(2,5)$	$20(2)+30(5)=\$190$
$(5,2)$	$20(5)+30(2)=\$160$
$(6,0)$	$20(6)+30(0)=\$120$

The point $(2,5)$ gives the most profit, and that profit is \$190. Therefore, we conclude that we should manufacture 2 regular gadgets and 5 premium gadgets daily to obtain the maximum profit of \$190.

So far, we have focused on "standard maximization problems" in which:

1. The objective function is to be maximized.
2. All constraints are of the form $ax+by \leq c$.
3. All variables are constrained to be non-negative ($x \geq 0$, $y \geq 0$).

We will next consider an example where that is not the case. Our next problem is said to have "mixed constraints" since some of the inequality constraints are of the form $ax+by \leq c$ and some are of the form $ax+by \geq c$. The non-negativity constraints are still an important requirement in any linear program.

Example 3.1.3. *Solve the following maximization problem graphically.*

$$\begin{aligned} \textit{Maximize} \quad & P = 10x+15y \\ \textit{Subject to:} \quad & x+y \geq 1 \\ & x+2y \leq 6 \\ & 2x+y \leq 6 \\ & x \geq 0; \quad y \geq 0 \end{aligned}$$

Solution 3.1.3. *The graph is shown below.*

The five critical points are listed in the figure above. The reader should observe that the first constraint $x+y \geq 1$ requires that the feasibility region

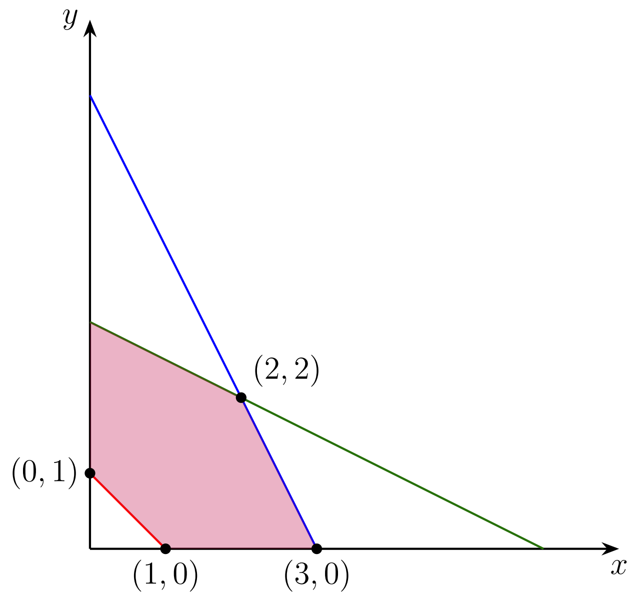


Figure 3.3: The red line is $x + y = 1$, the green line is $x + 2y = 6$ and the blue line is $2x + y = 6$.

must be bounded below by the line $x + y = 1$*; the test point* $(0, 0)$ *does not satisfy* $x + y \geq 1$*, so we shade the region on the opposite side of the line from the test point* $(0, 0)$.

Critical Point	*Income*
$(1, 0)$	$10(1) + 15(0) = \$10$
$(3, 0)$	$10(3) + 15(0) = \$30$
$(2, 2)$	$10(2) + 15(2) = \$50$
$(0, 3)$	$10(0) + 15(3) = \$45$
$(0, 1)$	$10(0) + 15(1) = \$15$

Clearly, the point $(2, 2)$ *maximizes the objective function to a maximum value of* 50*. It is important to observe that if the point* $(0, 0)$ *lies on the line for a constraint, then* $(0, 0)$ *could not be used as a test point. We would need to select any other point that does not lie on the line to use as a test point in that situation.*

Finally, we address an important question: Is it possible to determine the point that gives the maximum value without calculating the value at each critical point?

The answer is yes.

For example 3.1.2, we substituted the points $(0,0)$, $(0,6)$, $(2,5)$, $(5,2)$, and $(6,0)$ in the objective function $P = 20x+30y$, and we got the values \$0, \$180, \$190, \$160, \$120, respectively. Sometimes that is not the most efficient way of finding the optimum solution. Instead, we could find the optimal value by also graphing the objective function.

To determine the largest P, we graph $P = 20x + 30y$ for any value P of our choice. Let us say, we choose $P = 60$. We graph $20x + 30y = 60$.

Now we move the line parallel to itself, that is, keeping the same slope at all times. Since we are moving the line parallel to itself, the slope is kept the same, and the only thing that is changing is the P. As we move away from the origin, the value of P increases. The largest possible value of P is realized when the line touches the last corner point of the feasibility region.

The figure below shows the movements of the line, and the optimum solution is achieved at the point $(2,5)$. In maximization problems, as the line is being moved away from the origin, this optimum point is the farthest critical point.

Summary 3.1.1: Maximization Linear Programming Problems

1. Write the objective function.
2. Write the constraints.
   (a) For the standard maximization linear programming problems, constraints are of the form: $ax + by \leq c$.
   (b) Since the variables are non-negative, we include the constraints: $x \geq 0$, $y \geq 0$.
3. Graph the constraints.
4. Shade the feasibility region.
5. Find the corner points.
6. Determine the corner point that gives the maximum value.
   (a) This is done by finding the value of the objective function at each corner point.
   (b) This can also be done by moving the line associated with the objective function.

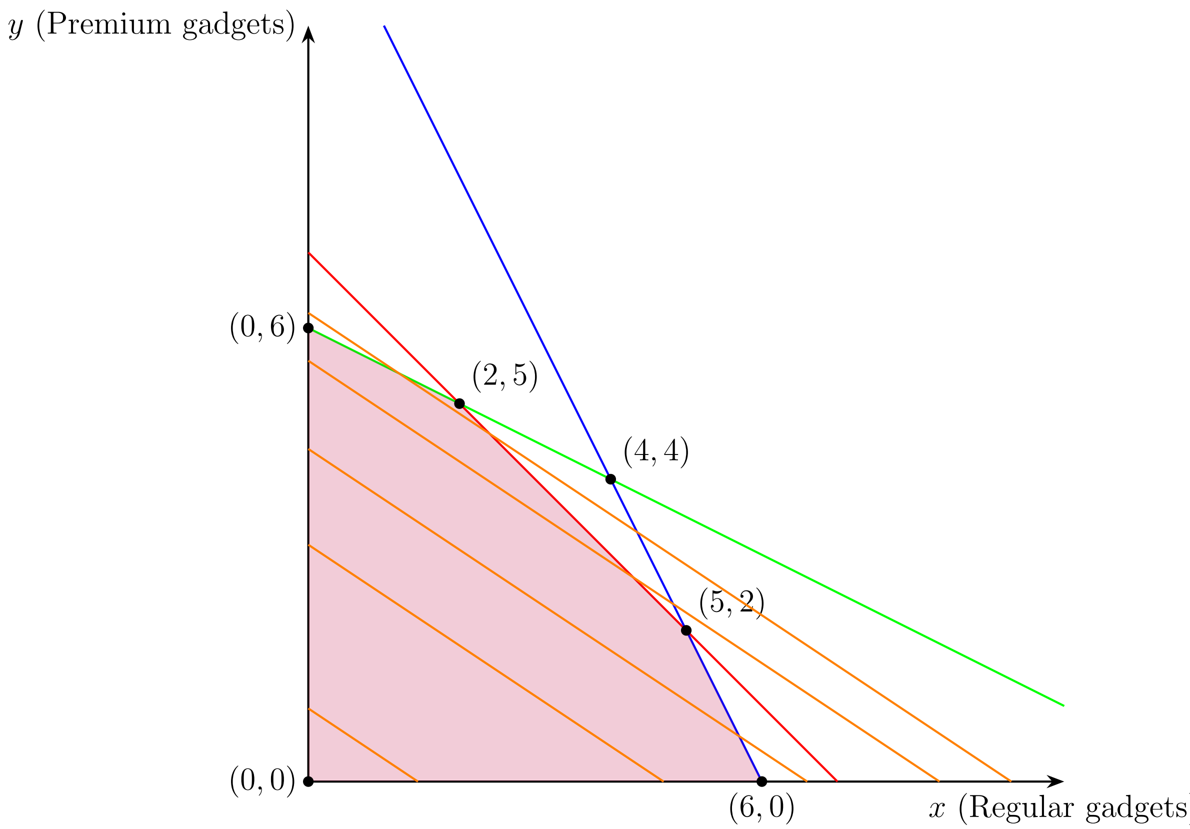


Figure 3.4: Feasibility region for the gadget factory optimization problem with profit lines.

3.2 Minimization Applications

In this section, you will learn to:

1. Formulate minimization linear programming problems.
2. Graph feasibility regions for minimization linear programming problems.
3. Determine optimal solutions for minimization linear programming problems.

Minimization linear programming problems are solved in much the same way as the maximization problems.

For the standard minimization linear program, the constraints are of the form $ax + by \geq c$, as opposed to the form $ax + by \leq c$ for the standard maximization problem. As a result, the feasible solution extends indefinitely to the upper right of the first quadrant, and is unbounded. But that is not a concern, since in order to minimize the objective function, the line associated with the objective function is moved towards the origin, and the critical point that minimizes the function is closest to the origin.

However, one should be aware that in the case of an unbounded feasibility region, the possibility of no optimal solution exists.

Example 3.2.1. *At a university, Professor Symons wishes to employ two people, John and Mary, to grade papers for his classes. John is a graduate student and can grade 20 papers per hour; John earns \$15 per hour for grading papers. Mary is a post-doctoral associate and can grade 30 papers per hour; Mary earns \$25 per hour for grading papers. Each must be employed at least one hour a week to justify their employment. If Professor Symons has at least 110 papers to be graded each week, how many hours per week should he employ each person to minimize the cost?*

Solution 3.2.1. *We choose the variables as follows: Let x be the number of hours per week John is employed, and y be the number of hours per week Mary is employed.*

The objective function is

$$C = 15x + 25y$$

The constraints are that each must work at least one hour each week:

$$x \geq 1$$

$$y \geq 1$$

John can grade 20 papers per hour and Mary 30 papers per hour, with at least 110 papers to be graded per week:

$$20x + 30y \geq 110$$

Additionally, x and y are non-negative:

$$x \geq 0$$

$$y \geq 0$$

The problem is thus formulated as: Minimize $C = 15x + 25y$ *Subject to:*

$$x \geq 1$$

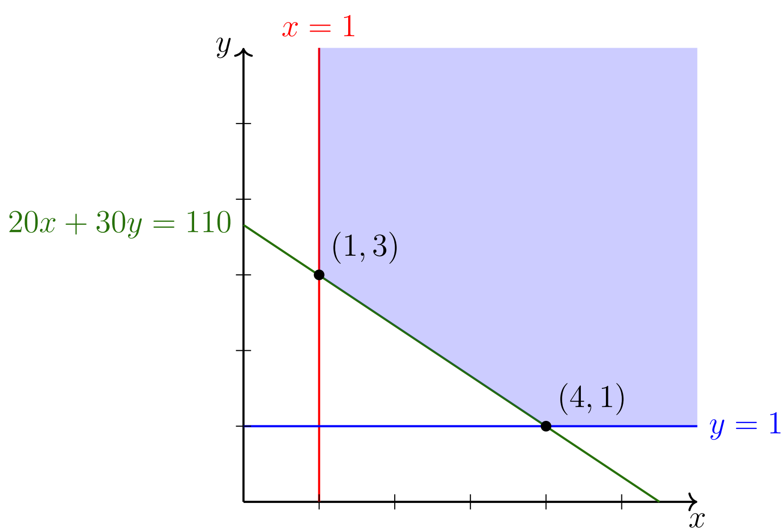
$$y \geq 1$$

$$20x + 30y \geq 110$$

$$x \geq 0$$

$$y \geq 0$$

To solve the problem, we graph the constraints as follows:



Again, we have shaded the feasibility region, where all constraints are satisfied. If we used test point $(0, 0)$ *that does not lie on any of the constraints, we observe that* $(0, 0)$ *does not satisfy any of the constraints* $x \geq 1$, $y \geq 1$, *and* $20x + 30y \geq 110$. *Thus, all the shading for the feasibility region lies on the opposite side of the constraint lines from the point* $(0, 0)$.

Alternatively, we could use test point $(4, 6)$, *which also does not lie on any of the constraint lines. We'd find that* $(4, 6)$ *does satisfy all of the inequality constraints. Consequently, all the shading for the feasibility region lies on the same side of the constraint lines as the point* $(4, 6)$.

Since the extreme value of the objective function always takes place at the vertices of the feasibility region, we identify the two critical points, $(1, 3)$ *and* $(4, 1)$*. To minimize cost, we will substitute these points in the objective function to see which point gives us the minimum cost each week. The results are listed below:*

Critical points	*Income*
$(1, 3)$	$15(1) + 25(3) = \$90$
$(4, 1)$	$15(4) + 25(1) = \$85$

The point $(4, 1)$ *gives the least cost, and that cost is $85. Therefore, we conclude that in order to minimize grading costs, Professor Symons should employ John for 4 hours a week and Mary for 1 hour a week at a cost of $85 per week.*

Example 3.2.2. *Professor Hamer is on a low cholesterol diet. During lunch at the college cafeteria, he always chooses between two meals, Pasta or Tofu. The table below lists the amount of protein, carbohydrates, and vitamins each meal provides along with the amount of cholesterol he is trying to minimize. Mr. Hamer needs at least 200 grams of protein, 960 grams of carbohydrates, and 40 grams of vitamins for lunch each month. Over this time period, how many days should he have the Pasta meal, and how many days the Tofu meal so that he gets the adequate amount of protein, carbohydrates, and vitamins and at the same time minimizes his cholesterol intake?*

	Pasta	***Tofu***
Protein (g)	*8*	*16*
Carbohydrates (g)	*60*	*40*
Vitamin C (g)	*2*	*2*
Cholesterol (mg)	*60*	*50*

Solution 3.2.2. *We choose the variables as follows: Let* x *be the number of days Mr. Hamer eats Pasta, and* y *the number of days he eats Tofu.*

The objective function for minimizing cholesterol intake is

$$C = 60x + 50y$$

The constraints for protein, carbohydrates, and vitamins are as follows:

$$\begin{aligned} 8x + 16y &\geq 200 \\ 60x + 40y &\geq 960 \\ 2x + 2y &\geq 40 \end{aligned}$$

Additionally, x and y are non-negative:

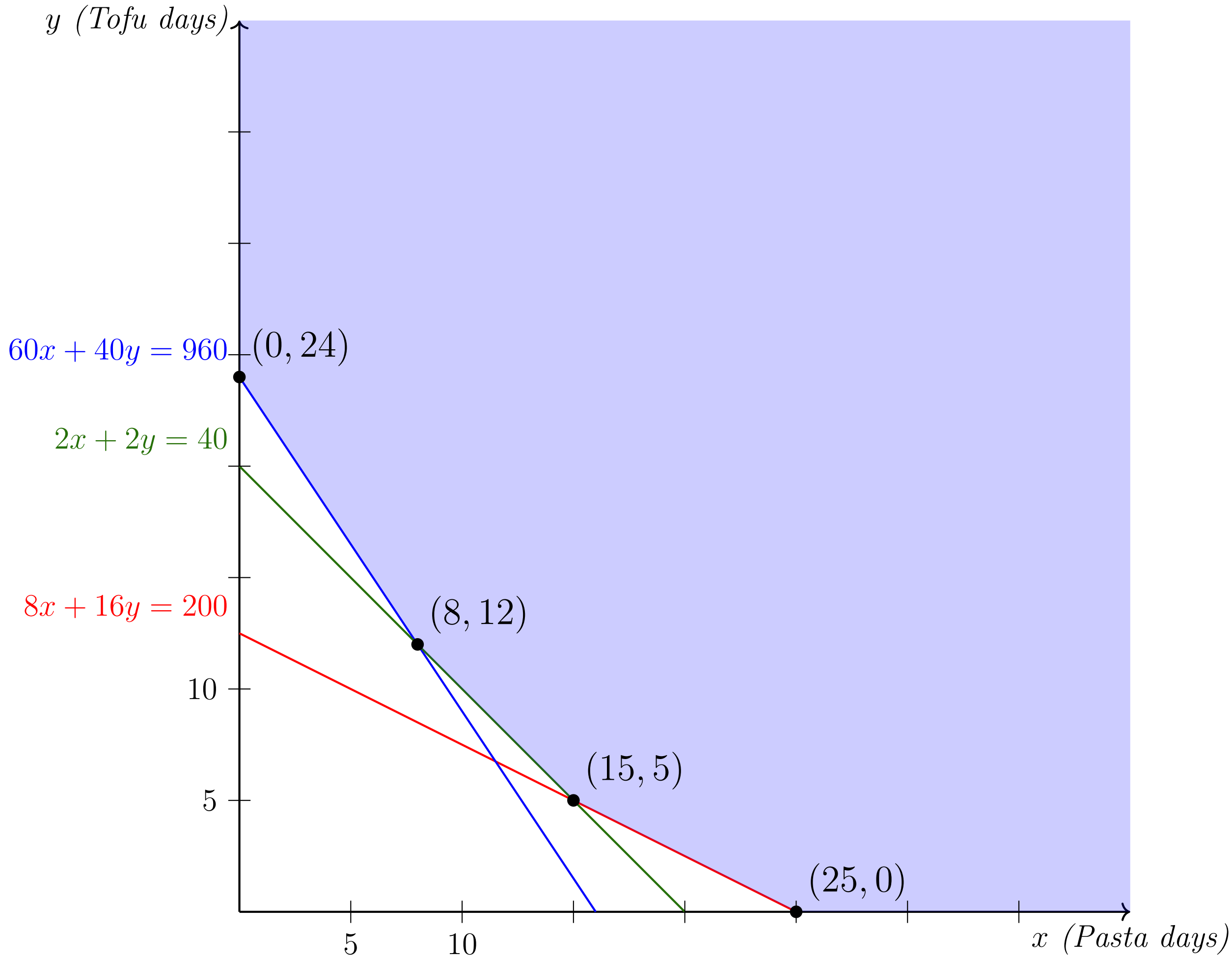
$$x \geq 0$$

$$y \geq 0$$

We summarize the problem as: Minimize $C = 60x + 50y$ Subject to:

$$\begin{aligned} 8x + 16y &\geq 200 \\ 60x + 40y &\geq 960 \\ 2x + 2y &\geq 40 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

To solve the problem, we graph the constraints and shade the feasibility region.



We have shaded the unbounded feasibility region, where all constraints are satisfied. To minimize the objective function, we find the vertices of the feasibility region. These vertices are $(0, 24)$, $(8, 12)$, $(15, 5)$, *and* $(25, 0)$. *To minimize cholesterol, we will substitute these points in the objective function to see which point gives us the smallest value. The results are listed below:*

Critical points	*Cholesterol*
$(0, 24)$	$60(0) + 50(24) = 1200\,mg$
$(8, 12)$	$60(8) + 50(12) = 1080\,mg$
$(15, 5)$	$60(15) + 50(5) = 1150\,mg$
$(25, 0)$	$60(25) + 50(0) = 1500\,mg$

The point $(8, 12)$ *gives the least cholesterol, which is 1080 mg. This states that for every 20 meals, Professor Hamer should eat Pasta for 8 days and Tofu for 12 days.*

We must be aware that in some cases, a linear program may not have an optimal solution.

- A linear program can fail to have an optimal solution if there is no feasibility region. If the inequality constraints are not compatible, there may not be a region in the graph that satisfies all the constraints. If the linear program does not have a feasible solution satisfying all constraints, then it cannot have an optimal solution.

- A linear program can fail to have an optimal solution if the feasibility region is unbounded. The two minimization linear programs we examined had unbounded feasibility regions. The feasibility region was bounded by constraints on some sides but was not entirely enclosed by the constraints. Both of the minimization problems had optimal solutions. However, if we were to consider a maximization problem with a similar unbounded feasibility region, the linear program would have no optimal solution. No matter what values of x and y were selected, we could always find other values of x and y that would produce a higher value for the objective function. In other words, if the value of the objective function can be increased without bound in a linear program with an unbounded feasible region, there is no optimal maximum solution.

Although the method of solving minimization problems is similar to that of maximization problems, we still feel that we should summarize the steps involved.

Summary 3.2.1: Minimization Linear Programming Problems

1. Write the objective function.
2. Write the constraints.
   (a) For standard minimization linear programming problems, constraints are of the form: $ax + by \geq c$.
   (b) Since the variables are non-negative, include the constraints: $x \geq 0$; $y \geq 0$.
3. Graph the constraints.
4. Shade the feasibility region.
5. Find the corner points.
6. Determine the corner point that gives the minimum value.
   (a) This can be done by finding the value of the objective function at each corner point.
   (b) This can also be done by moving the line associated with the objective function.
   (c) There is the possibility that the problem has no solution.

Chapter 4

Linear Programming, Simplex Method

In this chapter, you will learn to:

1. Investigate real world applications of linear programming and related methods.
2. Solve linear programming maximization problems using the simplex method.
3. Solve linear programming minimumization problems using the simplex method.

4.1 Linear Programming Applications in Business, Finance, Medicine, and Social Science

In this section, you will learn about:

1. real world applications of linear programming and related methods

The linear programs we solved in chapter 3 contain only two variables, x and y, so that we could solve them graphically. In practice, linear programs can contain thousands of variables and constraints.

Later in this chapter we'll learn to solve linear programs with more than two variables using the simplex algorithm, which is a numerical solution method that uses matrices and row operations. However, in order to make the problems practical for learning purposes, our problems will still have only several variables.

Now that we understand the main concepts behind linear programming, we can also consider how linear programming is currently used in large scale real-world applications.

Linear programming is used in business and industry in production planning, transportation and routing, and various types of scheduling. Airlines use linear programs to schedule their flights, taking into account both scheduling aircraft and scheduling staff. Delivery services use linear programs to schedule and route shipments to minimize shipment time or minimize cost. Retailers use linear programs to determine how to order products from manufacturers and organize deliveries with their stores. Manufacturing companies use linear programming to plan and schedule production. Financial institutions use linear programming to determine the mix of financial products they offer, or to schedule payments transferring funds between institutions. Health care institutions use linear programming to ensure the proper supplies are available when needed. And as we'll see below, linear programming has also been used to organize and coordinate life saving health care procedures.

In some of the applications, the techniques used are related to linear programming but are more sophisticated than the methods we study in this class. One such technique is called integer programming. In these situations, answers must be integers to make sense, and can not be fractions. Problems where solutions must be integers are more difficult to solve than the linear programs we've worked with. In fact, many of our problems have been very carefully constructed for learning purposes so that the answers just happen to turn out to be integers, but in the real world unless we specify that as a restriction, there is no guarantee that a linear program will produce integer solutions. There are also related techniques that are called non-linear programs, where the functions defining the objective function and/or some or all of the constraints may be non-linear rather than straight lines.

Many large businesses that use linear programming and related methods have analysts on their staff who can perform the analyses needed, including linear programming and other mathematical techniques. Consulting firms specializing in use of such techniques also aid businesses who need to apply these methods to their planning and scheduling processes.

When used in business, many different terms may be used to describe the use of techniques such as linear programming as part of mathematical business models. Optimization, operations research, business analytics, data science, industrial engineering hand management science are among the terms used to describe mathematical modelling techniques that may include linear programming and related met

In the rest of this section we'll explore six real world applications, and investigate what they are trying to accomplish using optimization, as well as what their constraints might represent.

4.1.1 Airline Scheduling

Airlines use techniques that include and are related to linear programming to schedule their aircraft to flights on various routes and to schedule crews to the flights. In addition, airlines also use linear programming to determine ticket pricing for various types of seats and levels of service or amenities, as well as the timing at which ticket prices change.

The process of scheduling aircraft and departure times on flight routes can be expressed as a model that minimizes cost, of which the largest component is generally fuel costs. Constraints involve considerations such as:

- Each aircraft needs to complete a daily or weekly tour to return back to its point of origin.
- Scheduling sufficient flights to meet demand on each route.
- Scheduling the right type and size of aircraft on each route to be appropriate for the route and for the demand for the number of passengers.
- Aircraft must be compatible with the airports it departs from and arrives at - not all airports can handle all types of planes.

A model to accomplish this could contain thousands of variables and constraints. Highly trained analysts determine ways to translate all the constraints into mathematical inequalities or equations to put into the model.

After aircraft are scheduled, crews need to be assigned to flights. Each flight needs a pilot, a co-pilot, and flight attendants. Each crew member needs to complete a daily or weekly tour to return back to his or her home base. Additional constraints on flight crew assignments take into account factors such as:

- Pilot and co-pilot qualifications to fly the particular type of aircraft they are assigned to.
- Flight crew have restrictions on the maximum amount of flying time per day and the length of mandatory rest periods between flights or per day that must meet certain minimum rest time regulations.
- Numbers of crew members required for a particular type or size of aircraft.

When scheduling crews to flights, the objective function would seek to minimize total flight crew costs, determined by the number of people on the crew and pay rates of the crew members. However, the cost for any particular route might not end up being the lowest possible for that route, depending on tradeoffs to the total cost of shifting different crews to different routes.

An airline can also use linear programming to revise schedules on short notice on an emergency basis when there is a schedule disruption, such as due to weather. In this case, the considerations to be managed involve:

- Getting aircraft and crews back on schedule as quickly as possible.
- Moving aircraft from storm areas to areas with calm weather to keep the aircraft safe from damage and ready to come back into service as quickly and conveniently as possible.
- Ensuring crews are available to operate the aircraft and that crews continue to meet mandatory rest period requirements and regulations.

4.1.2 Kidney Donation Chain

For patients who have kidney disease, a transplant of a healthy kidney from a living donor can often be a lifesaving procedure. Criteria for a kidney donation procedure include the availability of a donor who is healthy enough to donate a kidney, as well as a compatible match between the patient and donor for blood type and several other characteristics. Ideally, if a patient needs a kidney donation, a close relative may be a match and can be the kidney donor. However, often there is not a relative who is a close enough match to be the donor. Considering donations from unrelated donors allows for a larger pool of potential donors. Kidney donations involving unrelated donors can sometimes be arranged through a chain of donations that pair patients with donors. For example, a kidney donation chain with three donors might operate as follows:

- Donor A donates a kidney to Patient B.
- Donor B, who is related to Patient B, donates a kidney to Patient C.
- Donor C, who is related to Patient C, donates a kidney to Patient A, who is related to Donor A.

Linear programming is one of several mathematical tools that have been used to help efficiently identify a kidney donation chain. In this type of model, patient/donor pairs are assigned compatibility scores based on characteristics of patients and potential donors. The objective is to maximize the total compatibility scores. Constraints ensure that donors and patients are paired only if compatibility scores are sufficiently high to indicate an acceptable match.

4.1.3 Advertisements in Online Marketing

Did you ever make a purchase online and then notice that as you browse websites, search, or use social media, you now see more ads related the item you purchased? Marketing organizations use a variety of mathematical techniques, including linear programming, to determine individualized advertising placement purchases.

Instead of advertising randomly, online advertisers want to sell bundles of advertisements related to a particular product to batches of users who are more likely to purchase that product. Based on an individual's previous browsing and purchase selections, he or she is assigned a "propensity score" for making a purchase if shown an ad for a certain product. The company placing the ad generally does not know individual personal information based on the history of items viewed and purchased, but instead has aggregated information for groups of individuals based on what they view or purchase. However, the company may know more about an individual's history if he or she logged into a website making that information identifiable, within the privacy provisions and terms of use of the site.

The company's goal is to buy ads to present to specified size batches of people who are browsing. The linear program would assign ads and batches of people to view the ads using an objective function that seeks to maximize advertising response modeled using the propensity scores. The constraints are to stay within the restrictions of the advertising budget.

4.1.4 Loans

A car manufacturer sells its cars though dealers. Dealers can offer loan financing to customers who need to take out loans to purchase a car. Here we will consider how car manufacturers can use linear programming to determine the specific characteristics of the loan they offer to a customer who purchases a car. In a future chapter, we will learn how to do the financial calculations related to loans.

A customer who applies for a car loan fills out an application. This provides the car dealer with information about that customer. In addition, the car dealer can access a credit bureau to obtain information about a customer's credit score.

Based on this information obtained about the customer, the car dealer offers a loan with certain characteristics, such as interest rate, loan amount, and length of loan repayment period.

Linear programming can be used as part of the process to determine the characteristics of the loan offer. The linear program seeks to maximize the profitability of its portfolio of loans. The constraints limit the risk that the customer will default and will not repay the loan. The constraints also seek to minimize the risk of losing the loan customer if the conditions of the loan are not favorable enough; otherwise, the customer may find another lender, such as a bank, which can offer a more favorable loan.

4.1.5 Production Planning and Scheduling in Manufacturing

Consider the example of a company that produces yogurt. There are different varieties of yogurt products in a variety of flavors. Yogurt products have a short shelf life; it must be produced on a timely basis to meet demand, rather than drawing upon a stockpile of inventory as can be done with a product that is not perishable. Most ingredients in yogurt also have a short shelf life, so can not be ordered and stored for long periods of time before use; ingredients must be obtained in a timely manner to be available when needed but still be fresh. Linear programming can be used in both production planning and scheduling.

To start the process, sales forecasts are developed to determine demand to know how much of each type of product to make. There are often various manufacturing plants at which the products may be produced. The appropriate ingredients need to be at the production facility to produce the products assigned to that facility. Transportation costs must be considered, both for obtaining and delivering ingredients to the correct facilities, and for transport of finished product to the sellers. The linear program that monitors production planning and scheduling must be updated frequently - daily or even twice each day - to take into account variations from a master plan.

4.1.6 Bike Share Programs

Over 600 cities worldwide have bikeshare programs. Although bikeshare programs have been around for a long time, they have proliferated in the past decade as technology has developed new methods for tracking the bicycles.

Bikeshare programs vary in the details of how they work, but most typically people pay a fee to join and then can borrow a bicycle from a bike share station and return the bike to the same or a different bike share station. Over time the bikes tend to migrate; there may be more people who want to pick up a bike at station A and return it at station B than there are people who want to do the opposite. In chapter 8, we'll investigate a technique that can be used to predict the distribution of bikes among the stations.

Once other methods are used to predict the actual and desired distributions of bikes among the stations, bikes may need to be transported between stations to even out the distribution. Bikeshare programs in large cities have used methods related to linear programming to help determine the best routes and methods for redistributing bicycles to the desired stations once the desire distributions have been determined. The optimization model would seek to minimize transport costs and/or time subject to constraints of having sufficient bicycles at the various stations to meet demand.

4.2 Maximization by the Simplex Method

In this section, you will learn to:

1. Solve linear programming maximization problems using the Simplex Method by
   (a) Identifing and set up a linear program in standard maximization form
   (b) Converting inequality constraints to equations using slack variables
   (c) Setting up the initial simplex tableau using the objective function and slack equations
   (d) Finding the optimal simplex tableau by performing pivoting operations
   (e) Identifing the optimal solution from the optimal simplex tableau

In the last chapter, we used the geometrical method to solve linear programming problems, but the geometrical approach will not work for problems that have more than two variables. In real-life situations, linear programming problems consist of literally thousands of variables and are solved by computers. We can solve these problems algebraically, but that will not be very efficient. Suppose we were given a problem with, say, 5 variables and 10 constraints. By choosing all combinations of five equations with five unknowns, we could find all the corner points, test them for feasibility, and come up with the solution, if it exists. But the trouble is that even for a problem with so few variables, we will get more than 250 corner points, and testing each point will be very tedious. So we need a method that has a systematic algorithm and can be programmed for a computer. The method has to be efficient enough so we wouldn't have to evaluate the objective function at each corner point. We have just such a method, and it is called the simplex method.

The simplex method was developed during the Second World War by Dr. George Dantzig. His linear programming models helped the Allied forces with transportation and scheduling problems. In 1979, a Soviet scientist named Leonid Khachian developed a method called the ellipsoid algorithm, which was supposed to be revolutionary, but as it turned out, it is not any better than the simplex method. In 1984, Narendra Karmarkar, a research scientist at AT&T Bell Laboratories developed Karmarkar's algorithm, which has been proven to be four times faster than the simplex method for certain problems. But the simplex method still works the best for most problems.

The simplex method uses an approach that is very efficient. It does not compute the value of the objective function at every point; instead, it begins with a corner point of the feasibility region where all the main variables are zero and then systematically moves from corner point to corner point while improving the value of the objective function at each stage. The process continues until the optimal solution is found.

To learn the simplex method, we try a rather unconventional approach. We first list the algorithm, and then work a problem. We justify the reasoning behind each step during the process. A thorough justification is beyond the scope of this course.

We start out with an example we solved in the last chapter by the graphical method. This will provide us with some insight into the simplex method and at the same time give us the chance to compare a few of the feasible solutions we obtained previously by the graphical method.

But first, we list the algorithm for the simplex method.

Summary 4.2.1: The Simplex Method

Here are the steps to solve a linear programming problem using the Simplex Method:

1. **Set up the problem.**
   - Write the objective function and the inequality constraints.
2. **Convert the inequalities into equations.**
   - Add one slack variable for each inequality.
3. **Construct the initial simplex tableau.**
   - Write the objective function as the bottom row.
4. **Identify the pivot column.**
   - The most negative entry in the bottom row identifies the pivot column.
5. **Calculate the quotients.**
   - Divide the far-right column by the identified pivot column to find quotients.
   - The smallest positive quotient identifies a row, and its corresponding element is the pivot element.
6. **Perform pivoting.**
   - Make all other entries in the pivot column zero by using the Gauss-Jordan method.
7. **Repeat if necessary.**
   - If there are still negative entries in the bottom row, go back to step 4.
8. **Read off your answers.**
   - Get the variables using the columns with 1 and 0s. All other variables are zero.
   - The maximum value you are looking for appears in the bottom right-hand corner.

Now, we use the simplex method to solve example 4.2.1 solved geometrically in example 3.1.1.

Example 4.2.1. *Niki holds two part-time jobs, Job I and Job II. She never wants to work more than a total of 12 hours a week. For every hour she works at Job I, she needs 2 hours of preparation time, and for every hour at Job II, she needs one hour of preparation time. She cannot spend more than 16 hours on preparation. If Niki makes $40 an hour at Job I and $30 an hour at Job II, how many hours should she work at each job to maximize* *her income?*

Solution 4.2.1.

1. ***Set up the problem.*** *Write the objective function and the constraints. Since the simplex method is used for problems that consist of many variables, it is not practical to use the variables x, y, z etc. We use symbols x_1, x_2, x_3, and so on.*

   Let x_1 = The number of hours per week Niki will work at Job I.
   and x_2 = The number of hours per week Niki will work at Job II.

   It is customary to choose the variable that is to be maximized as Z. The problem is formulated the same way as we did in the last chapter.

$$\text{Maximize} \quad Z = 40x_1 + 30x_2$$

   Subject to:

$$\begin{aligned} x_1 + x_2 &\leq 12 \\ 2x_1 + x_2 &\leq 16 \\ x_1, x_2 &\geq 0 \end{aligned}$$

2. ***Convert the inequalities into equations.*** *This is done by adding one slack variable for each inequality.*

   For example, to convert the inequality $x_1 + x_2 \leq 12$ into an equation, we add a non-negative variable y_1, and we get

$$x_1 + x_2 + y_1 = 12$$

   Here the variable y_1 picks up the slack, and it represents the amount by which $x_1 + x_2$ falls short of 12. In this problem, if Niki works fewer than 12 hours, say 10, then y_1 is 2. Later when we read off the final solution from the simplex table, the values of the slack variables will identify the unused amounts.

   We rewrite the objective function $Z = 40x_1 + 30x_2$ as $-40x_1 - 30x_2 + Z = 0$.

After adding the slack variables, our problem reads

Objective function:

$$-40x_1 - 30x_2 + Z = 0$$

Subject to constraints:

$$\begin{aligned} x_1 + x_2 + y_1 &= 12 \\ 2x_1 + x_2 + y_2 &= 16 \\ x_1, x_2 &\geq 0 \end{aligned}$$

3. ***Construct the initial simplex tableau.*** *Each inequality constraint appears in its own row. (The non-negativity constraints do not appear as rows in the simplex tableau.) Write the objective function as the bottom row.*

   Now that the inequalities are converted into equations, we can represent the problem into an augmented matrix called the initial simplex tableau as follows.

$$\begin{array}{ccccc|c} x_1 & x_2 & y_1 & y_2 & Z & C \\ 1 & 1 & 1 & 0 & 0 & 12 \\ 2 & 1 & 0 & 1 & 0 & 16 \\ \hline -40 & -30 & 0 & 0 & 1 & 0 \end{array}$$

   Here the vertical line separates the left hand side of the equations from the right side. The horizontal line separates the constraints from the objective function. The right side of the equation is represented by the column C.

   The reader may observe that the last four columns of this matrix look like the final matrix for the solution of a system of equations. If we arbitrarily choose $x_1 = 0$ and $x_2 = 0$, we get

$$\begin{array}{c} \begin{array}{cccc} y_1 & y_2 & Z & C \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

which reads $y_1 = 12,\ y_2 = 16,\ Z = 0.$

The solution obtained by arbitrarily assigning values to some variables and then solving for the remaining variables is called the basic solution associated with the tableau. So the above solution is the basic solution associated with the initial simplex tableau. We can label the basic solution variable in the right of the last column as shown in the table below.

$$\begin{array}{ccccc|cc} x_1 & x_2 & y_1 & y_2 & Z & C & \\ 1 & 1 & 1 & 0 & 0 & 12 & y_1 \\ 2 & 1 & 0 & 1 & 0 & 16 & y_2 \\ \hline -40 & -30 & 0 & 0 & 1 & 0 & Z \end{array}$$

4. ***The most negative entry in the bottom row identifies the pivot column.*** *The most negative entry in the bottom row is -40; therefore, the column 1 is identified.*

$$\begin{array}{ccccc|c} x_1 & x_2 & y_1 & y_2 & Z & C \\ 1 & 1 & 1 & 0 & 0 & 12 \\ 2 & 1 & 0 & 1 & 0 & 16 \\ \hline -40 & -30 & 0 & 0 & 1 & 0 \end{array}$$

Why do we choose the most negative entry in the bottom row?

The most negative entry in the bottom row represents the largest coefficient in the objective function; the coefficient whose entry will increase the value of the objective function the quickest.

The simplex method begins at a corner point where all the main variables, the variables that have symbols such as x_1, x_2, x_3, *etc., are zero. It then moves from a corner point to the adjacent corner point always increasing the value of the objective function. In the case of the objective function* $Z = 40x_1 + 30x_2$, *it will make more sense to increase the value of* x_1 *rather than* x_2. *The variable* x_1 *represents the number of hours per week Niki works at Job I. Since Job I pays \$40 per hour as opposed to Job II which pays only \$30, the variable* x_1 *will increase the objective function by \$40 for a unit of increase in the variable* x_1.

5. ***Calculate the quotients. The smallest quotient identifies a row.*** *The element in the intersection of the column identified in step 4 (marked with* $\uparrow$*) and the row identified in this step is identified as the pivot element.*

   Following the algorithm, in order to calculate the quotient, we divide the entries in the far right column by the entries in column 1, excluding the entry in the bottom row.

x_1	x_2	y_1	y_2	Z	C		
1	1	1	0	0	12		$12/1 = 12$
2	1	0	1	0	16	$\leftarrow$	$16/2 = 8$
-40	-30	0	0	1	0		
$\uparrow$							

   The smallest of the two quotients, 12 and 8, is 8. Therefore row 2 is identified. The intersection of column 1 and row 2 is the entry 2, which has been highlighted. This is our pivot element.

   Why do we find quotients, and why does the smallest quotient identify a row?

   When we choose the most negative entry in the bottom row, we are trying to increase the value of the objective function by bringing in the variable x_1*. But we cannot choose any value for* x_1*. For instance, letting* $x_1 = 100$ *is not possible because Niki never wants to work more than 12 hours at both jobs combined:* $x_1 + x_2 \leq 12$*. Therefore, the maximum she can work is 12 hours for* x_1*, meaning the preparation time for Job I is two times the time spent on the job. Since she never wants to spend more than 16 hours for preparation, the maximum time she can work is* $\frac{16}{2} = 8$ *hours. Using the pivot element guarantees that we do not violate the constraints.*

   Why do we identify the pivot element?

   The simplex method begins at a corner point where all the main variables, the variables that have symbols such as x_1, x_2, x_3*, etc., are zero. It then moves from a corner point to the adjacent corner point always improving the value of the objective function. The value of the objective function is improved by changing the number of units of the variables.*

We may add the number of units of one variable, while throwing away the units of another. Pivoting allows us to do just that.

The variable whose units are being added is called the entering variable, and the variable whose units are being replaced is called the departing variable. The entering variable in the above table is x_1, and it was identified by the most negative entry in the bottom row. The departing variable y_2 was identified by the lowest of all quotients.

6. ***Perform pivoting to make all other entries in this column zero.*** *In Chapter 2, we used pivoting to obtain the row echelon form of an augmented matrix. Pivoting is a process of obtaining a 1 in the location of the pivot element (marked by a box below), and then making all other entries zeros in that column. We've highlighted the pivot row to make it easier to track. So now our job is to make our pivot element a 1 by dividing the entire second row by 2. The result follows.*

x_1	x_2	y_1	y_2	Z	C
1	1	1	0	0	12
$\boxed{2}$	1	0	1	0	16
−40	−30	0	0	1	0

To obtain a zero in the entry first above the pivot element, we multiply the second row by −1 and add it to row 1. We get

x_1	x_2	y_1	y_2	Z	C
0	1/2	1	−1/2	0	4
$\boxed{1}$	1/2	0	1/2	0	8
−40	−30	0	0	1	0

To obtain a zero in the element below the pivot, we multiply the second row by 40 and add it to the last row.

x_1	x_2	y_1	y_2	Z	C
0	1/2	1	−1/2	0	4
$\boxed{1}$	1/2	0	1/2	0	8
0	−10	0	20	1	320

We now determine the basic solution associated with this tableau. By arbitrarily choosing $x_2 = 0$ *and* $y_2 = 0$*, we obtain* $x_1 = 8$*,* $y_1 = 4$*, and* $Z = 320$*. If we write the augmented matrix, whose left side is a matrix with columns that have one 1 and all other entries zeros, we get the following matrix stating the same thing.*

$$\begin{array}{cccc} x_1 & y_1 & Z & C \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 320 \end{array}\right]$$

We can restate the solution associated with this matrix as $x_1 = 8$*,* $x_2 = 0$*,* $y_1 = 4$*,* $y_2 = 0$*, and* $z = 320$*. At this stage, it reads that if Niki works 8 hours at Job I and no hours at Job II, her profit* z *will be \$320. Recall from Example 1 in Section 3.1 that* $(8, 0)$ *was one of our corner points. Here* $y_1 = 4$ *and* $y_2 = 0$ *mean that she will be left with 4 hours of working time and no preparation time.*

7. ***When there are no more negative entries in the bottom row, we are finished; otherwise, we start again from step 4.*** *Since there is still a negative entry,* -10*, in the bottom row, we need to begin, again, from step 4. This time we will not repeat the details of every step; instead, we will identify the column and row that give us the pivot element, and highlight the pivot element. The result is as follows.*

$$\begin{array}{ccccc|cl} x_1 & x_2 & y_1 & y_2 & Z & C & \\ 0 & 1/2 & 1 & -1/2 & 0 & 4 & \leftarrow \quad 4/(1/2) = 8 \\ 1 & 1/2 & 0 & 1/2 & 0 & 8 & \qquad 8/(1/2) = 16 \\ \hline 0 & -10 & 0 & 20 & 1 & 320 & \\ & \uparrow & & & & & \end{array}$$

We make the pivot element 1 by multiplying row 1 by 2, and we get

$$\begin{array}{ccccc|c} x_1 & x_2 & y_1 & y_2 & Z & C \\ 0 & \boxed{1} & 2 & -1 & 0 & 8 \\ 1 & 1/2 & 0 & 1/2 & 0 & 8 \\ \hline 0 & -10 & 0 & 20 & 1 & 320 \end{array}$$

Now to make all other entries as zeros in this column, we first multiply row 1 by $-\frac{1}{2}$ *and add it to row 2, and then multiply row 1 by 10 and add it to the bottom row.*

$$\begin{array}{ccccc|c} x_1 & x_2 & y_1 & y_2 & Z & C \\ 0 & 1 & 2 & -1 & 0 & 8 \\ 1 & 0 & -1 & 1 & 0 & 4 \\ \hline 0 & 0 & 20 & 10 & 1 & 400 \end{array}$$

We no longer have negative entries in the bottom row, therefore we are finished.

Why are we finished when there are no negative entries in the bottom row?

The answer lies in the bottom row. The bottom row corresponds to the equation:

$$0x_1 + 0x_2 + 20y_1 + 10y_2 + Z = 400 \quad \textit{or} \quad Z = 400 - 20y_1 - 10y_2$$

Since all variables are non-negative, the highest value Z *can ever achieve is 400, and that will happen only when* y_1 *and* y_2 *are zero.*

8. ***Read off your answers.*** *We now read off our answers, that is, we determine the basic solution associated with the final simplex tableau. Again, we look at the columns that have a* 1 *and all other entries zeros. Since the columns labeled* y_1 *and* y_2 *are not such columns, we arbitrarily choose* $y_1 = 0$*, and* $y_2 = 0$*, and we get*

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & Z & C \end{array} \\ \left[\begin{array}{ccc|c} 0 & 1 & 0 & 8 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 400 \end{array}\right] \end{array}$$

The matrix reads $x_2 = 8$*,* $x_1 = 4$*, and* $Z = 400$*.*

The final solution says that if Niki works 4 hours at Job I and 8 hours at Job II, she will maximize her income to \$400. Since both slack variables are zero, it means that she would have used up all the working time, as well as the preparation time, and none will be left.

4.3 Minimization by the Simplex Method

In this section, you will learn to solve linear programming minimization problems using the simplex method.

1. Identify and set up a linear program in standard minimization form.
2. Formulate a dual problem in standard maximization form.
3. Use the simplex method to solve the dual maximization problem.
4. Identify the optimal solution to the original minimization problem from the optimal simplex tableau.

In this section, we will solve the standard linear programming minimization problems using the simplex method. Once again, we remind the reader that in the standard minimization problems all constraints are of the form $ax + by \geq c$.

The procedure to solve these problems was developed by Dr. John Von Neuman. It involves solving an associated problem called the dual problem. To every minimization problem there corresponds a dual problem. The solution of the dual problem is used to find the solution of the original problem. The dual problem is a maximization problem, which we learned to solve in the last section. We first solve the dual problem by the simplex method.

From the final simplex tableau, we then extract the solution to the original minimization problem. Before we go any further, however, we first learn to convert a minimization problem into its corresponding maximization problem called its dual.

Example 4.3.1. *Convert the following minimization problem into its dual.*

Minimize

$$Z = 12x_1 + 16x_2$$

Subject to:

$$\begin{aligned} x_1 + 2x_2 &\geq 40 \\ x_1 + x_2 &\geq 30 \\ x_1 \geq 0; \quad x_2 &\geq 0 \end{aligned}$$

Solution 4.3.1. *To achieve our goal, we first express our problem as the following matrix.*

$$\begin{array}{cc|c} 1 & 2 & 40 \\ 1 & 1 & 30 \\ \hline 12 & 16 & 0 \end{array}$$

Observe that this table looks like an initial simplex tableau without the slack variables. Next, we write a matrix whose columns are the rows of this matrix, and the rows are the columns. Such a matrix is called a transpose of the original matrix. We get:

$$\begin{array}{cc|c} 1 & 1 & 12 \\ 2 & 1 & 16 \\ \hline 40 & 30 & 0 \end{array}$$

The following maximization problem associated with the above matrix is called its dual.

Maximize

$$Z = 40y_1 + 30y_2$$

Subject to:

$$\begin{aligned} y_1 + y_2 &\leq 12 \\ 2y_1 + y_2 &\leq 16 \\ y_1 \geq 0; \quad y_2 &\geq 0 \end{aligned}$$

Note that we have chosen the variables as y's, instead of x's, to distinguish the two problems.

Example 4.3.2. *Solve both the minimization problem and its dual maximization problem graphically.*

Solution 4.3.2. *Our minimization problem is as follows.*

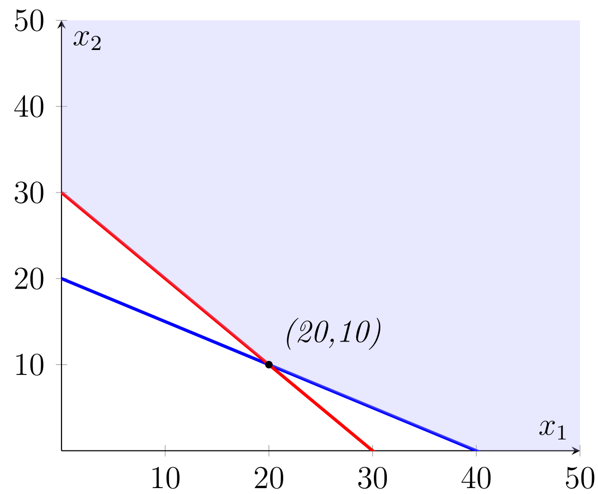
Minimize

$$Z = 12x_1 + 16x_2$$

Subject to:

$$\begin{aligned} x_1 + 2x_2 &\geq 40 \\ x_1 + x_2 &\geq 30 \\ x_1 \geq 0; \quad x_2 &\geq 0 \end{aligned}$$

We now graph the inequalities:



We have plotted the graph, shaded the feasibility region, and labeled the corner points. The corner point (20, 10) gives the lowest value for the objective function and that value is 400.

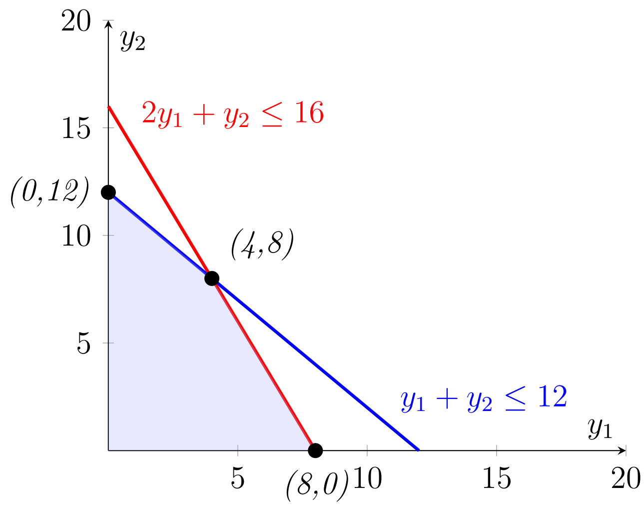
Now its dual is: Maximize

$$Z = 40y_1 + 30y_2$$

Subject to:

$$\begin{aligned} y_1 + y_2 &\leq 12 \\ 2y_1 + y_2 &\leq 16 \\ y_1 \geq 0; \quad y_2 &\geq 0 \end{aligned}$$

We graph the inequalities:



Again, we have plotted the graph, shaded the feasibility region, and labeled the corner points. The corner point (4, 8) gives the highest value for the objective function, with a value of 400.

The reader may recognize that Example 4.3.2 above is the same as Example 3.1.1 in section 3.1. It is also the same problem as Example 4.2.1 in section 4.2, where we solved it by the simplex method.

We observe that the minimum value of the minimization problem is the same as the maximum value of the maximization problem; in Example 4.3.2 the minimum and maximum are both 400. This is not a coincidence. We state the duality principle.

Definition 4.3.1. ***The Duality Principle*** *The objective function of the minimization problem reaches its minimum if and only if the objective function of its dual reaches its maximum. And when they do, they are equal.*

Our next goal is to extract the solution for our minimization problem in Example 4.3.1 from the corresponding dual. To do this, we solve the dual by the simplex method.

Example 4.3.3. *Find the solution to the minimization problem in Example 4.3.1 by solving its dual using the simplex method. We rewrite our problem: Minimize*

$$Z = 12x_1 + 16x_2$$

Subject to:

$$x_1 + 2x_2 \geq 40$$
$$x_1 + x_2 \geq 30$$
$$x_1 \geq 0; \quad x_2 \geq 0$$

Solution 4.3.3. *The dual is: Maximize*

$$Z = 40y_1 + 30y_2$$

Subject to:

$$y_1 + y_2 \leq 12$$
$$2y_1 + y_2 \leq 16$$
$$y_1 \geq 0; \quad y_2 \geq 0$$

Recall that we solved the above problem by the simplex method in Example 4.2.1, section 4.2. Therefore, we only show the initial and final simplex tableau.

The initial simplex tableau is:

$$\begin{array}{ccccc|cc}
y_1 & y_2 & x_1 & x_2 & Z & C & \\
1 & 1 & 1 & 0 & 0 & 12 & y_1 \\
2 & 1 & 0 & 1 & 0 & 16 & y_2 \\
\hline
-40 & -30 & 0 & 0 & 1 & 0 & Z
\end{array}$$

Observe an important change. Here our main variables are y_1 and y_2 and the slack variables are x_1 and x_2.

The final simplex tableau reads as follows:

$$\begin{array}{ccccc|c}
y_1 & y_2 & x_1 & x_2 & Z & C \\
0 & 1 & 2 & -1 & 0 & 8 \\
1 & 0 & -1 & 1 & 0 & 4 \\
\hline
0 & 0 & 20 & 10 & 1 & 400
\end{array}$$

A closer look at this table reveals that the x_1 and x_2 values along with the minimum value for the minimization problem can be obtained from the last row of the final tableau. We have highlighted these values by the arrows.

$$\begin{array}{ccccc|c}
y_1 & y_2 & x_1 & x_2 & Z & C \\
0 & 1 & 2 & -1 & 0 & 8 \\
1 & 0 & -1 & 1 & 0 & 4 \\
\hline
0 & 0 & 20 & 10 & 1 & 400 \\
 & & \uparrow & \uparrow & & \uparrow
\end{array}$$

We restate the solution as follows: The minimization problem has a minimum value of 400 at the corner point $(20, 10)$.

Summary 4.3.1: Minimization by the Simplex Method

1. Set up the problem.
2. Write a matrix whose rows represent each constraint with the objective function as its bottom row.
3. Write the transpose of this matrix by interchanging the rows and columns.
4. Now write the dual problem associated with the transpose.
5. Solve the dual problem by the simplex method learned in section 4.2.
6. The optimal solution is found in the bottom row of the final matrix in the columns corresponding to the slack variables, and the minimum value of the objective function is the same as the maximum value of the dual.

Chapter 5

Exponential and Logarithmic Functions

In this chapter, you will:

1. Examine exponential and logarithmic functions and their properties.
2. Identify exponential growth and decay functions and use them to model applications.
3. Use the natural base e to represent exponential functions.
4. Use logarithmic functions to solve equations involving exponential functions.

5.1 Exponential Growth and Decay Models

In this section, you will learn to:

1. Recognize and model exponential growth and decay.
2. Compare linear and exponential growth.
3. Distinguish between exponential and power functions.

5.1.1 Comparing Exponential and Linear Growth

Consider two social media sites that are expanding the number of users they have:

- Site A has 10,000 users and expands by adding 1,500 new users each month.

- Site B has 10,000 users and expands by increasing the number of users by 10% each month.

The number of users for Site A can be modeled as linear growth. The number of users increases by a constant number, 1500, each month. If x represents the number of months that have passed and y is the number of users, the number of users after x months is given by $y = 10000 + 1500x$.

For Site B, the user base expands by a constant percent each month, rather than by a constant number. Growth that occurs at a constant percent each unit of time is called exponential growth.

We can compare the growth for each site by examining the number of users for the first 12 months. The table shows the calculations for the first 4 months only, but the same calculation process is used to complete the remaining months.

Month	Users at Site A	Users at Site B
0	10000	10000
1	10000 + 1500 = 11500	$10000 + 10\%$ of 10000 $= 10000 + 0.10(10000) = 10000(1.10) = 11000$
2	11500 + 1500 = 13000	$11000 + 10\%$ of 11000 $= 11000 + 0.10(11000) = 11000(1.10) = 12100$
3	13000 + 1500 = 14500	$12100 + 10\%$ of 12100 $= 12100 + 0.10(12100) = 12100(1.10) = 13310$
4	14500 + 1500 = 16000	$13310 + 10\%$ of 13310 $= 13310 + 0.10(13310) = 13310(1.10) = 14641$
5	17500	16105
6	19000	17716
7	20500	19487
8	22000	21436
9	23500	23579
10	25000	25937
11	26500	28531
12	28000	31384

For Site B, we can re-express the calculations to observe the patterns and develop a formula for the number of users after x months:

Month 1: $y = 10000(1.1) = 11000$
Month 2: $y = 11000(1.1) = 10000(1.1)(1.1) = 10000(1.1)^2 = 12100$
Month 3: $y = 12100(1.1) = 10000(1.1)^2(1.1) = 10000(1.1)^3 = 13310$
Month 4: $y = 13310(1.1) = 10000(1.1)^3(1.1) = 10000(1.1)^4 = 14641$

By observing the patterns in the calculations for months 2, 3, and 4, we can generalize the formula. After x months, the number of users y is given by the function $y = 10000(1.1)^x$.

5.1.2 Using Exponential Functions to Model Growth and Decay

In exponential growth, the value of the dependent variable y increases at a constant percentage rate as the value of the independent variable x (or t) increases. Examples of exponential growth functions include:

- The number of residents of a city or nation that grows at a constant percent rate.

- The amount of money in a bank account that earns interest if money is deposited at a single point in time and left in the bank to compound without any withdrawals.

In exponential decay, the value of the dependent variable y decreases at a constant percentage rate as the value of the independent variable x (or t) increases. Examples of exponential decay functions include:

- Value of a car or equipment that depreciates at a constant percent rate over time.

- The amount a drug that still remains in the body as time passes after it is ingested.

- The amount of radioactive material remaining over time as a radioactive substance decays.

Exponential functions often model quantities as a function of time; thus, we often use the letter t as the independent variable instead of x.

Summary 5.1.1: Exponential Growth

1. Quantity grows by a constant percent per unit of time.
2. $y = ab^x$
3. a is a positive number representing the initial value of the function when $x = 0$.
4. b is a real number that is greater than 1: $b > 1$.
5. The growth rate r is a positive number, $r > 0$ where $b = 1 + r$ (so that $r = b - 1$).

Summary 5.1.2: Exponential Decay

1. Quantity decreases by a constant percent per unit of time.
2. $y = ab^x$
3. a is a positive number representing the initial value of the function when $x = 0$.
4. b is a real number that is between 0 and 1: $0 < b < 1$.
5. The decay rate r is a negative number, $r < 0$ where $b = 1 + r$ (so that $r = b - 1$).

In general, the domain of exponential functions is the set of all real numbers. The range of an exponential growth or decay function is the set of all positive real numbers.

In most applications, the independent variable x or t represents time. When the independent variable represents time, we may choose to restrict the domain so that the independent variable can have only non-negative values for the application to make sense. If we restrict the domain, then the range is also restricted.

- For an exponential growth function $y = ab^x$ with $b > 1$ and $a > 0$, if we restrict the domain so that $x \geq 0$, then the range is $y \geq a$.
- For an exponential decay function $y = ab^x$ with $0 < b < 1$ and $a > 0$, if we restrict the domain so that $x \geq 0$, then the range is $0 < y \leq a$.

Example 5.1.1. *Consider the growth models for social media sites A and B, where x = number of months since the site was started and y = number of users. The number of users for Site A follows the linear growth model:*

$$y = 10000 + 1500x.$$

The number of users for Site B follows the exponential growth model:

$$y = 10000 \cdot (1.1)^x$$

For each site, use the function to calculate the number of users at the end of the first year, to verify the values in the table. Then use the functions to predict the number of users after 30 months.

Solution 5.1.1. *Since x is measured in months, then $x = 12$ at the end of one year.*

Linear Growth Model: *When* $x = 12$ *months, then* $y = 10000 + 1500(12) = 28000$ *users. When* $x = 30$ *months, then* $y = 10000 + 1500(30) = 55000$ *users.*

Exponential Growth Model: *When* $x = 12$ *months, then* $y = 10000 \cdot (1.1)^{12} =$ *31384 users. When* $x = 30$ *months, then* $y = 10000 \cdot (1.1)^{30} =$ *174494 users.*

We see that as x, the number of months, gets larger, the exponential growth function grows larger faster than the linear function (even though in the initial stages the linear function grew faster). This is an important characteristic of exponential growth: exponential growth functions always grow faster and larger in the long run than linear growth functions.

It is helpful to use function notation, writing $y = f(x) = ab^x$, to specify the value of x at which the function is evaluated.

Example 5.1.2. *A forest has a population of 2000 squirrels that is increasing at the rate of 3% per year. Let* t *be the number of years and* $y = f(t)$ *the number of squirrels at time* t*.*

1. *Find the exponential growth function that models the number of squirrels in the forest at the end of* t *years.*

2. *Use the function to find the number of squirrels after 5 years and after 10 years.*

   Solution 5.1.2. *The exponential decay function is* $y = g(t) = ab^t$*, , where the initial population* $a = 2000$ *squirrels, and the growth rate* $r = 3\% = 0.03$ *per year, hence* $b = 1 + r = 1.03$*.*

1. *The exponential growth function is* $y = f(t) = 2000(1.03^t)$

2. *After 5 years, the population is* $y = f(5) = 2000(1.03^5) \approx 2315.25$ *squirrels. After 10 years, the population is* $y = f(10) = 2000(1.03^{10}) \approx 2691.70$ *squirrels.*

Example 5.1.3. *A large lake has a population of 1000 frogs. Unfortunately, the frog population is decreasing at the rate of 5% per year. Let* t *represent the number of years and* $y = g(t)$ *the number of frogs in the lake at time* t*.*

1. *Find the exponential decay function that models the frog population.*

2. *Calculate the population size after 10 years.*

Solution 5.1.3. *The exponential decay function is* $y = g(t) = ab^t$*, where* $a = 1000$ *is the initial population of frogs, and the decay rate is 5% per year, which translates to* $r = -0.05$*. Therefore,* $b = 1 + r = 0.95$*.*

1. *The function modeling the frog population is* $y = g(t) = 1000(0.95)^t$*.*
2. *After 10 years, the population is* $y = g(10) = 1000(0.95)^{10} \approx 599$ *frogs, showing a significant decrease due to the yearly decline.*

Example 5.1.4. *A bacteria population is described by the function* $y = f(t) = 100(2^t)$*, where* t *is the time in hours, and* y *is the number of bacteria.*

1. *What is the initial population?*
2. *What is change after the first hour?*
3. *How long does it take for the population to reach 800 bacteria?*

Solution 5.1.4.

1. *The initial population is* $y = f(0) = 100(2^0) = 100$ *bacteria, as* $a = 100$ *in the function* $f(t)$*.*
2. *After one hour, the population doubles to* $y = f(1) = 100(2^1) = 200$ *bacteria.*
3. *To find when the population reaches 800, solve* $800 = 100(2^t)$*:*

$$\begin{aligned} 800 &= 100(2^t) \\ 8 &= 2^t \\ 2^3 &= 2^t \\ \Rightarrow t &= 3. \end{aligned}$$

It takes 3 hours for the population to grow to 800 bacteria.

Important notes about Example 5.1.4:

1. In solving $8 = 2t$, we "knew" that t is 3. But we usually cannot know the value of the variable just by looking at the equation. Later, we will use logarithms to solve equations that have the variable in the exponent.

2. To solve $800 = 100(2^t)$, we divided both sides by 100 to isolate the exponential expression 2^t. We cannot multiply 100 by 2. Even if we write it as $800 = 100(2)t$, which is equivalent, we still cannot multiply 100 by 2. The exponent applies only to the quantity immediately before it, so the exponent t applies only to the base of 2.

5.1.3 Comparing Linear, Exponential, and Power Functions

To identify the type of function from its formula, we need to carefully note the position that the variable occupies in the formula.

Summary 5.1.3: Comparing Functions

- A **linear function** can be written in the form $y = ax + b$. As we studied in Chapter 1, there are other forms in which linear equations can be written, but linear functions can all be rearranged to have the form $y = mx + b$.
- An **exponential function** has the form $y = ab^x$, where the variable x is in the exponent. The base b is a positive number:
  - If $b > 1$, the function represents exponential growth.
  - If $0 < b < 1$, the function represents exponential decay.
- A **power function** has the form $y = cx^p$, where the variable x is in the base. The exponent p is a non-zero number.

We compare three functions:

- linear function $y = f(x) = 2x$
- exponential function $y = g(x) = 2^x$
- power function $y = h(x) = x^2$

x	$f(x) = 2x$	$g(x) = 2^x$	$h(x) = x^2$
0	0	1	0
1	2	2	1
2	4	4	4
3	6	8	9
4	8	16	16
5	10	32	25
6	12	64	36
10	20	1024	100

For the functions in the previous table: the linear function $y = f(x) = 2x$, the exponential function $y = g(x) = 2^x$, and the power function $y = h(x) = x^2$, if we restrict the domain to $x \geq 0$ only, then all these functions are growth functions. When $x \geq 0$, the value of y increases as the value of x increases.

The exponential growth function grows larger faster than the linear and power functions as x gets large. This is always true of exponential growth functions as x gets large enough.

Notice that for equal intervals of change in x

- with a linear function y increases by addition of a constant amount.
- with an exponential function, y is mulitplied by a constant amount.
- with a power function neither of these is true.

Example 5.1.5. *Classify the functions below as exponential, linear, or power functions.*

a. $y = 10x^3$

b. $y = 1000 - 30x$

c. $y = 1000(1.05^x)$

d. $y = 500(0.75^x)$

e. $y = 10\sqrt[3]{x} = x^{1/3}$

f. $y = 5x - 1$

g. $y = \frac{6}{x^2} = 6x^{-2}$

Solution 5.1.5. *The exponential functions are*

c. $y = 1000(1.05^x)$ *The variable is in the exponent; the base is the number* $b = 1.05$.

d. $y = 500(0.75^x)$ *The variable is in the exponent; the base is the number* $b = 0.75$.

The linear functions are

b. $y = 1000 - 30x$

f. $y = 5x - 1$

The power functions are

a. $y = 10x^3$ *The variable is the base; the exponent is a fixed number,* $p = 3$.

e. $y = 10\sqrt[3]{x} = 10x^{1/3}$ *The variable is the base; the exponent is a number,* $p = 1/3$.

g. $y = \frac{6}{x^2} = 6x^{-2}$ *The variable is the base; the exponent is a number,* $p = -2$.

5.1.4 The Natural Base

The number e is often used as the base of an exponential function and is called the natural base. It is approximately 2.71828 and is an irrational number with an infinite never repeating decimal expansion. The reader may be familiar with another famous irrational number, π. Section 6.2.4 shows how the value of e enters the world of finance and why this number is mathematically important.

When e is the base in an exponential growth or decay function, it is referred to as continuous growth or continuous decay. We will use e in Chapter 6 in financial calculations when we examine interest that compounds continuously.

Any exponential function can be written in the form $y = ae^{kx}$, where k is called the continuous growth or decay rate:

- If $k > 0$, the function represents exponential growth.
- If $k < 0$, the function represents exponential decay.

The initial value a is the starting amount of whatever is growing or decaying.

We can rewrite the function in the form $y = ab^x$, where $b = e^k$.

In general, if we know one form of the equation, we can find the other forms. For now, we have not yet covered the skills to find k when we know b. After we learn about logarithms later in this chapter, we will find k using the natural logarithm: $k = \ln(b)$.

The table below summarizes the forms of exponential growth and decay functions.

	$\boldsymbol{y = ab^x}$	$\boldsymbol{y = a(1+r)^x}$	$\boldsymbol{y = ae^{kx}, k \neq 0}$
Initial value	$a > 0$	$a > 0$	$a > 0$
Relationship between b, r, k	$b > 0$	$b = 1 + r$	$b = e^k$ and $k = \ln b$
Growth	$b > 1$	$r > 0$	$k > 0$
Decay	$0 < b < 1$	$r < 0$	$k < 0$

Example 5.1.6. *The value of houses in a city are increasing at a continuous growth rate of* 6% *per year. For a house that currently costs* \$400,000*:*

1. *Write the exponential growth function in the form $y = ae^{kx}$.*
2. *What would be the value of this house 4 years from now?*
3. *Rewrite the exponential growth function in the form $y = ab^x$.*
4. *Find and interpret r.*

Solution 5.1.6.

1. *The initial value of the house is $a = \$400,000$. The problem states that the continuous growth rate is 6% per year, so $k = 0.06$. The growth function is: $y = 400000e^{0.06x}$.*
2. *After 4 years, the value of the house is $y = 400000e^{0.06(4)} \approx \$508,500$.*
3. *To rewrite $y = 400000e^{0.06x}$ in the form $y = ab^x$, we use the fact that $b = e^k$. Therefore, $b = e^{0.06} \approx 1.061836547 \approx 1.0618$ and the function becomes $y = 400000(1.0618)^x$.*
4. *To find r, we use the fact that $b = 1 + r$. Given $b = 1.0618$, we solve $1 + r = 1.0618$ which gives $r = 0.0618$. The value of the house is*

increasing at an annual rate of 6.18%.

Example 5.1.7. *Suppose that the value of a certain model of new car decreases at a continuous decay rate of* 8% *per year. For a car that costs* \$20,000 *when new:*

1. *Write the exponential decay function in the form* $y = ae^{kx}$.
2. *What would be the value of this car 5 years from now?*
3. *Rewrite the exponential decay function in the form* $y = ab^x$.
4. *Find and interpret* r.

Solution 5.1.7.

1. *The initial value of the car is* $a = \$20,000$. *The problem states that the continuous decay rate is* 8% *per year, so* $k = -0.08$. *The decay function is:* $y = 20000e^{-0.08x}$.
2. *After 5 years, the value of the car is* $y = 20000e^{-0.08(5)} \approx \$13,406.40$.
3. *To rewrite* $y = 20000e^{-0.08x}$ *in the form* $y = ab^x$, *we use the fact that* $b = e^k$. *Therefore,* $b = e^{-0.08} \approx 0.9231163464 \approx 0.9231$ *and the function becomes* $y = 20000(0.9231)^x$.
4. *To find* r, *we use the fact that* $b = 1 + r$. *Given* $b = 0.9231$, *we solve* $1 + r = 0.9231$ *which gives* $r = -0.0769$. *The value of the car is decreasing at an annual rate of* 7.69%.

5.2 Graphing Exponential Functions

In this section, you will:

1. examine properties of exponential functions
2. examine graphs of exponential functions

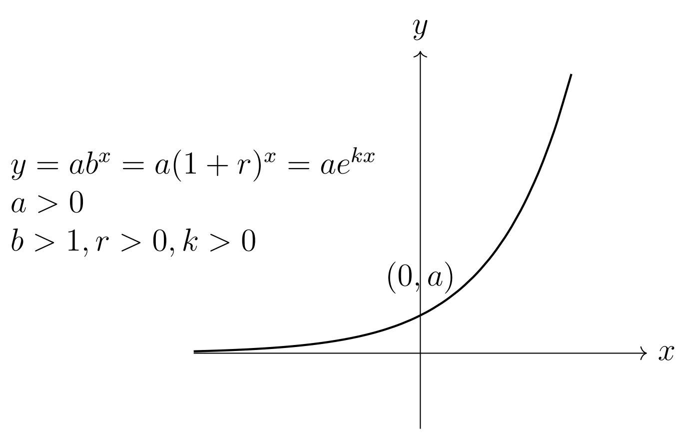
An exponential function can be written in forms $f(x) = ab^x = a(1 + r)^x = ae^{kx}$:

- a is the initial value because $f(0) = a$.
- In the growth and decay models that we examine in this finite math textbook, $a > 0$.

- b is often called the growth factor. We restrict b to be positive ($b > 0$) because even roots of negative numbers are undefined. We want the function to be defined for all values of x, but b^x would be undefined for some values of x if $b < 0$.
- r is called the growth or decay rate. In the formula for the functions, we use r in decimal form, but in the context of a problem we usually state r as a percent.
- k is called the continuous growth rate or continuous decay rate.

5.2.1 Properties of Exponential Growth Functions

- The function $y = f(x) = ab^x$ represents growth if $b > 1$ and $a > 0$.
- The growth rate r is positive when $b > 1$. Because $b = 1 + r > 1$, then $r = b - 1 > 0$.
- The function $y = f(x) = ae^{kx}$ represents growth if $k > 0$ and $a > 0$.
- The function is an increasing function; y increases as x increases.



There are some properties the reader should notice:

- **Domain:** All real numbers can be input to an exponential function. Mathematicians would write $\{x \in \mathbb{R}\}$ or simply $\mathbb{R}$.
- **Range:** If $a > 0$, the range is the set of all positive real numbers. Mathematicians would write $\{x \in \mathbb{R} : x > 0\}$.
- The graph is always above the x axis.

- **Horizontal Asymptote:** when $b > 1$, the horizontal asymptote is the negative x axis, as x becomes large negative. Using mathematical notation: as $x \to -\infty$, then $y \to 0$.
- The vertical intercept is the point $(0, a)$ on the y-axis.
- There is no horizontal intercept because the function does not cross the x-axis.

5.2.2 Properties of Exponential Decay Functions

- The function $y = f(x) = ab^x$ represents decay if $0 < b < 1$ and $a > 0$.
- The growth rate r is negative when $0 < b < 1$. Because $b = 1 + r < 1$, then $r = b - 1 < 0$.
- The function $y = f(x) = ae^{kx}$ represents decay if $k < 0$ and $a > 0$.
- The function is a decreasing function; y decreases as x increases.
- **Domain:** All real numbers can be input to an exponential function. Mathematicians would write $\{x \in \mathbb{R}\}$ or simply $\mathbb{R}$.
- **Range:** If $a > 0$, the range is the set of all positive real numbers. Mathematicians would write $\{x \in \mathbb{R} : x > 0\}$.
- **Horizontal Asymptote:** when $b < 1$, the horizontal asymptote is the positive x axis as x becomes large positive. Using mathematical notation: as $x \to \infty$, then $y \to 0$.
- The vertical intercept is the point $(0, a)$ on the y-axis. There is no horizontal intercept because the function does not cross the x-axis.

The graphs for exponential growth and decay functions are displayed in the figure 5.2.2 below for comparison.

5.2.3 An Exponential Function Is a One-to-One Function

Observe that in the graph of an exponential function, each y value on the graph occurs only once. Therefore, every y value in the range corresponds to only one x value. So, for any particular value of y, you can use the graph

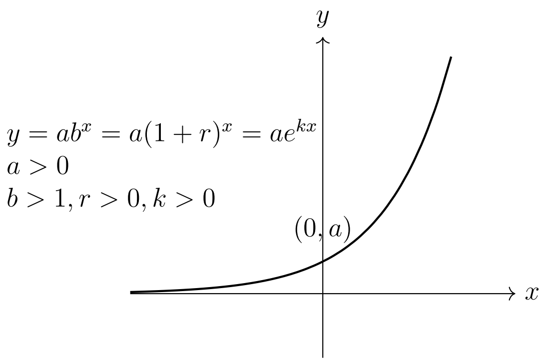


Figure 5.1: Exponential Growth

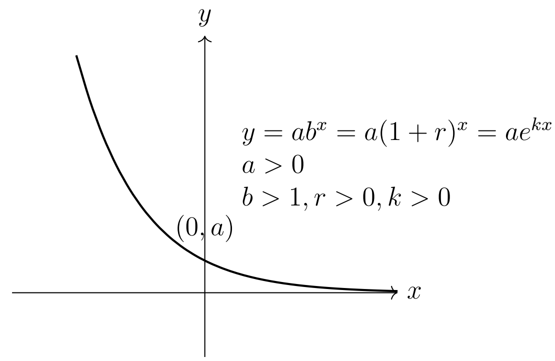


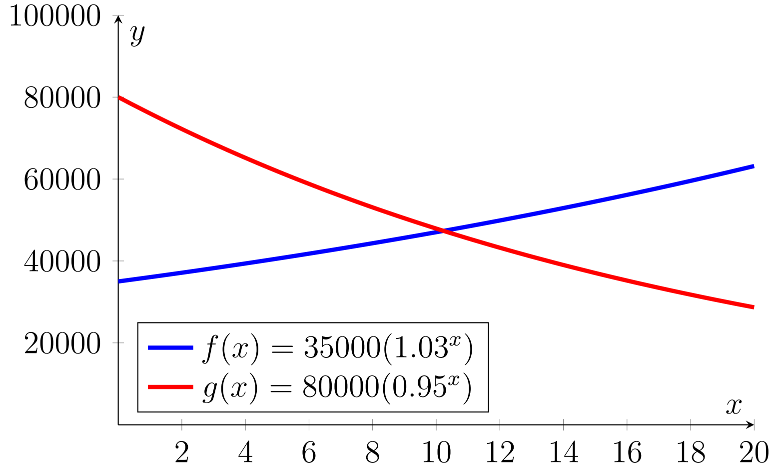
Figure 5.2: Exponential Decay

to see which value of x is the input to produce that y value as output. This property is called ”one-to-one”.

Because for each value of the output y, you can uniquely determine the value of the corresponding input x, thus every exponential function has an inverse function. The inverse function of an exponential function is a logarithmic function, which we will investigate in the next section.

Example 5.2.1. *x years after the year 2025, the population of the city of Fulton is given by the function $y = f(x) = 35000(1.03^x)$, and x years after the year 2025, the population of the city of Greenville is given by the function $y = g(x) = 80000(0.95^x)$. Compare the graphs of these functions.*

Solution 5.2.1. *Here are graphs:*



Some observations we may make:

- *Fulton's population is undergoing Exponential Growth.*
  - $b = 1.03 > 1$ *and* $r = 0.03 > 0$
  - *The population is increasing.*
- *Greenville's population is undergoing Exponential Decay.*
  - $b = 0.95 < 1$ *and* $r = -0.05 < 0$
  - *The population is decreasing.*
- *The initial population of Fulton in* 2025 *is 35000.*
- *The initial population of Greenville in* 2025 *is 80000.*
- *In general, the domains of both functions* $f(x)$ *and* $g(x)$ *are all real numbers, but realistically the populations of Fulton and Greenville cannot follow this model for all such values of* x*. Depending on how they were created, the models may be good enough for use for* x *representing several years in the past to several years in the future.*
- *If the model holds, the populations of the two cities will be approximately equal sometime around* 2035.

5.3 Logarithms and Logarithmic Functions

In this section, you will learn:

- The definition of a logarithmic function as the inverse of the exponential function.
- How to write equivalent logarithmic and exponential expressions.
- The definition of common logarithms and natural logarithms.
- Properties of logarithms and the Log Rules.

5.3.1 Define the Logarithm

Suppose that a population of 50 flies is expected to double every week, leading to a function of the form $f(x) = 50 \cdot (2)^x$, where x represents the number of weeks that have passed. When will this population reach 500? Trying to solve this problem leads to $500 = 50 \cdot (2)^x$. Dividing both sides by 50 to isolate the exponential leads to $10 = 2^x$.

While we have set up exponential models and used them to make predictions, you may have noticed that solving exponential equations has not yet been mentioned. The reason is simple: none of the algebraic tools discussed so far are sufficient to solve exponential equations. Consider the equation $2^x = 10$ above. We know that $2^3 = 8$ and $2^4 = 16$, so it is clear that x must be some value between 3 and 4 since $g(x) = 2^x$ is increasing. We could use technology to create a table of values or graph to better estimate the solution, but we would like to find an algebraic way to solve the equation.

We need an inverse operation to exponentiation in order to solve for the variable if the variable is in the exponent. As we learned in algebra class, the inverse function for an exponential function is a logarithmic function. We also learned that an exponential function has an inverse function, because each output (y) value corresponds to only one input (x) value. The name given to this property was "one-to-one".

Definition 5.3.1. *The* ***logarithm*** *(base b), written* $\log_b(x)$*, is the inverse of the exponential function (base b),* b^x*.*

$$y = \log_b(x) \quad \Leftrightarrow \quad b^y = x$$

Some Notes

- In general, $b^a = c$ is called an **exponential equation** and is equivalent to the **logarithmic equation** $\log_b(c) = a$.
- The base b must be positive: $b > 0$
- Since the logarithm and exponential are inverses, it follows that:

$$\log_b(b^x) = x \quad \text{and} \quad b^{\log_b(x)} = x$$

- Since log is a function, it is most correctly written as $\log_b(c)$, using parentheses to denote function evaluation, just as we would with $f(c)$. However, when the input is a single variable or number, it is common to see the parentheses dropped and the expression written as $\log_b c$.

Example 5.3.1. *Write these exponential equations as logarithmic equations:*

1. $2^3 = 8$
2. $5^2 = 25$
3. $10^{-3} = \frac{1}{1000}$

Solution 5.3.1.

1. $2^3 = 8$ *can be written as a logarithmic equation as* $\log_2(8) = 3$
2. $5^2 = 25$ *can be written as a logarithmic equation as* $\log_5(25) = 2$
3. $10^{-3} = \frac{1}{1000}$ *can be written as a logarithmic equation as* $\log_{10}\left(\frac{1}{1000}\right) = -3$

Example 5.3.2. *Write these logarithmic equations as exponential equations:*

1. $\log_6(\sqrt{6}) = \frac{1}{2}$
2. $\log_3(9) = 2$

Solution 5.3.2.

1. $\log_6(\sqrt{6}) = \frac{1}{2}$ *can be written as an exponential equation as* $6^{\frac{1}{2}} = \sqrt{6}$
2. $\log_3(9) = 2$ *can be written as an exponential equation as* $3^2 = 9$

By establishing the relationship between exponential and logarithmic functions, we can now solve basic logarithmic and exponential equations by rewriting.

Example 5.3.3. *Solve* $\log_4(x) = 2$ *for x.*

Solution 5.3.3. *By rewriting this expression as an exponential,* $4^2 = x$*, so* $x = 16$*.*

Example 5.3.4. *Solve* $2^x = 10$ *for x.*

Solution 5.3.4. *By rewriting this expression as a logarithm, we get* $x = \log_2(10)$*. Using a computer utility we find* $x \approx 3.32192809489$*.*

While this does define a solution, you may find it somewhat unsatisfying since it is difficult to compare this expression to the decimal estimate we made earlier. Also, giving an exact expression for a solution is not always useful—often we really need a decimal approximation to the solution. Luckily, this is a task that calculators and computers are quite adept at. Unfortunately for us, most calculators will only evaluate logarithms of two bases: base 10 and base e. Computer utilities such as Desmos and Wolfram Alpha compute such things nicely. Please note, even with many decimal places computers and calculators are providing estimates.

5.3.2 Natural and Base 10 Logarithms

Definition 5.3.2. *The **natural** logarithm is the logarithm with e as the base (*$\log_e(x)$*). It is often written as* $\ln(x)$*.*

Definition 5.3.3. *The **base 10** (or common) logarithm is the logarithm with 10 as the base (*$\log_1 0(x)$*). It is often written as* $\log(x)$*.*

Example 5.3.5. *Evaluate* $\log(1000)$ *using the definition of the common log.*

Solution 5.3.5. *The table shows values of the common log:*

number	***number as exponential***	log(***number***)
1000	10^3	*3*
100	10^2	*2*
10	10^1	*1*
1	10^0	*0*
0.1	10^{-1}	*-1*
0.01	10^{-2}	*-2*
0.001	10^{-3}	*-3*

To evaluate $\log(1000)$*, we can say*

$$x = \log(1000)$$

Then rewrite the equation in exponential form using the common log base of 10

$$10^x = 1000$$

From this, we might recognize that 1000 is the cube of 10, so

$$x = 3.$$

Alternatively, we can use the inverse property of logs to write

$$\log_{10}(10^3) = 3$$

Example 5.3.6. *Evaluate* $\log(1/1,000,000)$

Solution 5.3.6. *To evaluate* $\log(1/1,000,000)$*, we can say*

$$x = \log(1/1,000,000) = \log(1/10^6) = \log(10^{-6})$$

Then rewrite the equation in exponential form:

$$10^x = 10^{-6}$$

Therefore $x = -6$.

Alternatively, we can use the inverse property of logs to find the answer:

$$\log_{10}(10^{-6}) = -6$$

Example 5.3.7. *Evaluate*

1. $\ln e^5$
2. $\ln \sqrt{e}$.

Solution 5.3.7.

1. *To evaluate* $\ln e^5$, *we can say*

   $$x = \ln e^5$$

   Then rewrite into exponential form using the natural log base of e

   $$e^x = e^5$$

   Therefore $x = 5$

   Alternatively, we can use the inverse property of logs to write $\ln(e^5) = 5$

2. *To evaluate* $\ln \sqrt{e}$, *we recall that roots are represented by fractional exponents*

   $$x = \ln \sqrt{e} = \ln(e^{1/2})$$

   Then rewrite into exponential form using the natural log base of e

   $$e^x = e^{1/2}$$

   Therefore $x = 1/2$

   Alternatively, we can use the inverse property of logs to write $\ln(e^{1/2}) = 1/2$

Example 5.3.8. *Evaluate the following using your calculator or computer:*

1. $\log 500$
2. $\ln 500$

Solution 5.3.8.

1. $\log 500 \approx 2.69897$
2. $\ln 500 \approx 6.214608$

5.3.3 Properties of Logarithms

Summary 5.3.1:

1. **Exponent Property**:
$$\log_b(A^p) = p\log_b(A)$$
2. **Product Property**:
$$\log_b(AC) = \log_b(A) + \log_b(C)$$
3. **Quotient Property**:
$$\log_b\left(\frac{A}{C}\right) = \log_b(A) - \log_b(C)$$

5.4 Graphs and Properties of Logarithmic Functions

In this section, you will:

1. examine properties of logarithmic functions
2. examine graphs of logarithmic functions
3. examine the relationship between graphs of exponential and logarithmic functions

Recall that the exponential function $f(x) = 2^x$ produces this table of values

x	-3	-2	-1	0	1	2	3
$f(x)$	1/8	1/4	1/2	1	2	4	8

Since the logarithmic function is an inverse of the exponential, $g(x) = \log_2(x)$

produces the table of values

x	1/8	1/4	1/2	1	2	4	8
$g(x)$	-3	-2	-1	0	1	2	3

In this second table, notice that:

- As the input increases, the output increases.
- As input increases, the output increases more slowly.
- Since the exponential function only outputs positive values, the logarithm can only accept positive values as inputs, so the domain of the log function is $(0, \infty)$.
- Since the exponential function can accept all real numbers as inputs, the logarithm can have any real number as output, so the range is all real numbers or $(-\infty, \infty)$.

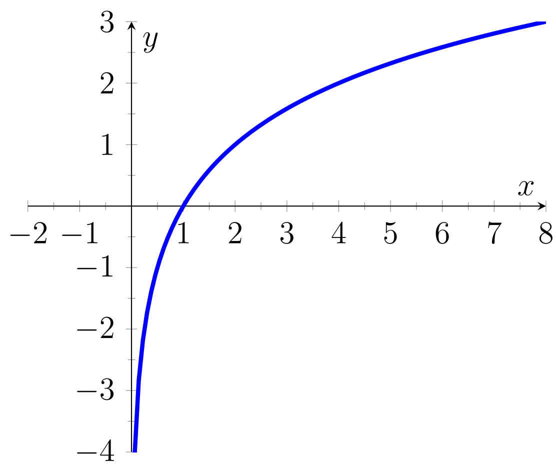
Plotting the graph of $g(x) = \log_2(x)$ from the points in the table, notice that as the input values for x approach zero, the output of the function grows very large in the negative direction, indicating a vertical asymptote at $x = 0$.

In symbolic notation we write

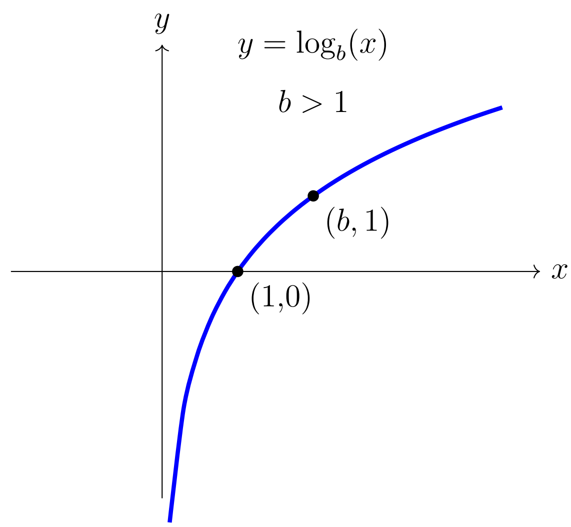
$$\text{as } x \to 0^+, f(x) \to -\infty$$

and

$$\text{as } x \to \infty, f(x) \to \infty$$



Graphically, in the function $g(x) = \log_b(x)$, $b > 1$, we observe the following properties:



- The graph has a horizontal intercept at (1, 0).

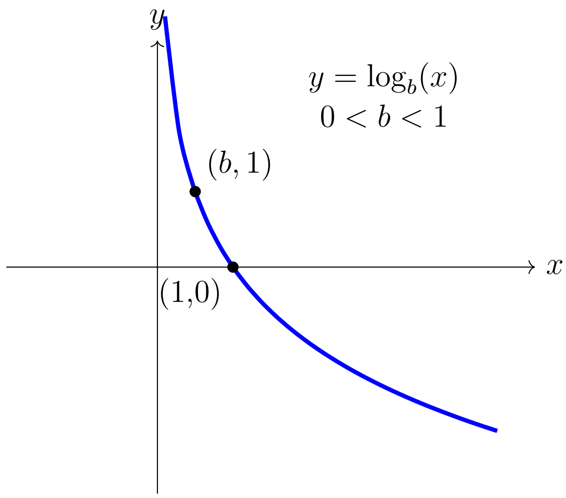
- The line $x = 0$ (the y-axis) is a vertical asymptote; as $x \to 0^+$, $y \to -\infty$.

- The graph is increasing if $b > 1$.

- The domain of the function is $x > 0$, or $(0, \infty)$.

- The range of the function is all real numbers, or $(-\infty, \infty)$.

However if the base b is less than 1, $0 < b < 1$, then the graph appears as below.



- The graph has a horizontal intercept at (1, 0).
- The line $x = 0$ (the y-axis) is a vertical asymptote; as $x \to 0^+$, $y \to -\infty$.
- The graph is decreasing if $0 < b < 1$.
- The domain of the function is $x > 0$, or $(0, \infty)$.
- The range of the function is all real numbers, or $(-\infty, \infty)$.

When graphing a logarithmic function, it can be helpful to remember that the graph will pass through the points $(1, 0)$ and $(b, 1)$.

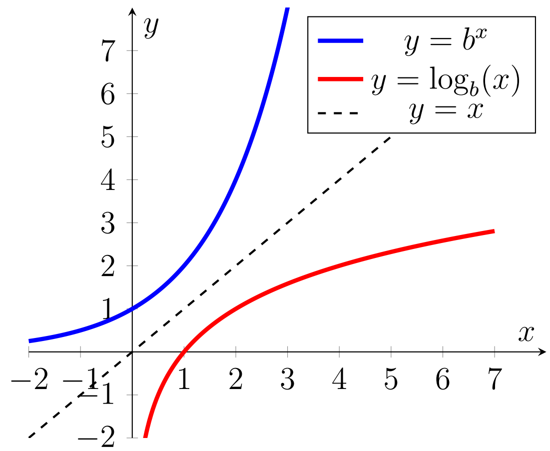
Finally, we compare the graphs of $y = b^x$ and $y = \log_b(x)$, shown below on the same axes.

Because the functions are inverse functions of each other, for every specific ordered pair (h, k) on the graph of $y = bx$, we find the point (k, h) with the coordinates reversed on the graph of $y = \log_b(x)$.

In other words, if the point with $x = h$ and $y = k$ is on the graph of $y = bx$, then the point with $x = k$ and $y = h$ lies on the graph of $y = \log_b(x)$.

The domain of $y = bx$ is the range of $y = \log_b(x)$. The range of $y = bx$ is the domain of $y = \log_b(x)$.

For this reason, the graphs appear as reflections, or mirror images, of each other across the diagonal line $y = x$. This is because the inputs and outputs are swapped for a function and its inverse.



5.5 Application Problems with Exponential and Logarithmic Functions

In this section, you will:

1. Review strategies for solving equations arising from exponential formulas.
2. Solve application problems involving exponential functions and logarithmic functions.

5.5.1 Strategies for Solving Equations That Contain Exponents

When solving application problems that involve exponential and logarithmic functions, we need to pay close attention to the position of the variable in the equation to determine the proper way solve the equation we investigate solving equations that contain exponents.

Suppose we have an equation in the form:

$$\text{value} = \text{coefficient}(\text{base})^{\text{exponent}}$$

We consider four strategies for solving the equation:

- If the coefficient, base, and exponent are all known, we only need to evaluate the expression for coefficient$(\text{base})^{\text{exponent}}$ to evaluate its value.
- If the variable is the coefficient, evaluate the expression for $(\text{base})^{\text{exponent}}$. Then it becomes a linear equation which we solve by dividing to isolate the variable.
- If the variable is in the exponent, use logarithms to solve the equation.
- If the variable is not in the exponent, but is in the base, use roots to solve the equation.

Example 5.5.1. *Suppose that a stock's price is rising at the rate of 7% per year, and that it continues to increase at this rate. If the value of one share of this stock is $43 now, find the value of one share of this stock three years from now.*

Solution 5.5.1. *Let y be the value of the stock after t years: $y = ab^t$.*

The problem tells us that $a = 43$ and $r = 0.07$, so $b = 1+r = 1+0.07 = 1.07$.

Therefore, function y is $43(1.07)^t$.

In this case we know that $t = 3$ years, and we need to evaluate y when $t = 3$.

At the end of 3 years, the value of this one share of this stock will be

$$y = 43(1.07)^3 = \$52.68$$

Example 5.5.2. *The value of a new car depreciates (decreases) after it is purchased. Suppose that the value of the car depreciates according to an exponential decay model. Suppose that the value of the car is $12000 at the end of 5 years and that its value has been decreasing at the rate of 9% per year. Find the value of the car when it was new.*

Solution 5.5.2. *Let y be the value of the car after t years: $y = ab^t$.*

Given $r = -0.09$ and $b = 1 + r = 1 + (-0.09) = 0.91$.

The function is $y = a(0.91)^t$.

For $t = 5$, and $y = 12000$; substituting these values gives $12000 = a(0.91)^5$.

Solve for a:

First, evaluate $(0.91)^5$ *and then solve the resulting linear equation to find* a*.*

$$12000 = a(.624)$$

$$a = \frac{12000}{0.624} = \$19,230.77$$

The car's value was $19,230.77 when it was new.

Example 5.5.3. *A national park has a population of 5000 deer in the year 2024. Conservationists are concerned because the deer population is decreasing at the rate of 7% per year. If the population continues to decrease at this rate, how long will it take until the population is only 3000 deer?*

Solution 5.5.3. *Let* y *be the number of deer in the national park* t *years after the year 2024:* $y = ab^t$*.*

Given $r = -0.07$ *and* $b = 1 + r = 1 + (-0.07) = 0.93$*, the initial population is* $a = 5000$*.*

The exponential decay function is $y = 5000(0.93)^t$*.*

To find when the population will be 3000, substitute $y = 3000$*:*

$$3000 = 5000(0.93)^t$$

Next, divide both sides by 5000 to isolate the exponential expression:

$$\frac{3000}{5000} = (0.93)^t$$

$$0.6 = 0.93^t$$

Rewrite the equation in logarithmic form; then use the change of base formula to evaluate.

$$t = \log_{0.93}(0.6)$$

$$t = \frac{\ln(0.6)}{\ln(0.93)} \approx 7.039 \textit{ years}$$

After 7.039 *years, there are 3000 deer.*

In Example 5.5.3, we needed to state the answer to several decimal places of precision to remain accurate. Evaluating the original function using a rounded value of $t = 7$ years gives a value that is close to 3000, but not exactly 3000.

$$y = 5000(0.93)^7 = 3008.5 \text{ deer}$$

However using $t = 7.039$ years produces a value of 3000 for the population of deer

$$y = 5000(0.93)^{7.039} = 3000.0016 \approx 3000 \text{ deer}$$

Example 5.5.4. *A video posted on YouTube initially had 80 views as soon as it was posted. The total number of views to date has been increasing exponentially according to the exponential growth function* $y = 80e^{0.2t}$*, where* t *represents time measured in days since the video was posted. How many days does it take until 2500 people have viewed this video?*

Solution 5.5.4. *Let* y *be the total number of views* t *days after the video is initially posted. We are given that the exponential growth function is* $y = 80e^{0.2t}$ *and we want to find the value of* t *for which* $y = 2500$*. Substitute* $y = 2500$ *into the equation and use natural log to solve for* t*.*

First, divide both sides of the equation by 80 to isolate the exponential term:

$$\frac{2500}{80} = e^{0.2t}$$

This simplifies to:

$$31.25 = e^{0.2t}$$

Taking the natural logarithm of both sides gives us:

$$0.2t = \ln(31.25)$$

Dividing both sides by 0.2 to solve for t yields:

$$t = \frac{\ln(31.25)}{0.2}$$

Using a calculator, we find that:

$$t \approx \frac{3.442}{0.2} \approx 17.2 \text{ days}$$

Thus, the video will have 2500 total views approximately 17 days after it was posted.

Example 5.5.5. *A statistician creates a website to analyze sports statistics. His business plan states that his goal is to accumulate 50,000 followers by the end of 2 years (24 months from now). He hopes that if he achieves this goal his site will be purchased by a sports news outlet. The initial user base of people signed up as a result of pre-launch advertising is 400 people. Find the monthly growth rate needed if the user base is to accumulate to 50,000 users at the end of 24 months.*

Solution 5.5.5. *Let y be the total user base t months after the site is launched.*

The growth function for this site is $y = 400(1+r)^t$.

We don't know the growth rate r. We do know that when $t = 24$ months, then $y = 50000$.

Substitute the values of y and t; then we need to solve for r.

$$\frac{50000}{400} = (1+r)^{24}$$

Divide both sides by 400 to isolate $(1+r)^{24}$ on one side of the equation

$$125 = (1+r)^{24}$$

Because the variable in this equation is in the base, we use roots:

$$\sqrt[24]{125} = 1 + r$$

$$125^{1/24} = 1 + r$$

$$1.2228 \approx 1 + r$$

$$r \approx 0.2228$$

The website's user base needs to increase at the rate of 22.28% per month in order to accumulate 50,000 users by the end of 24 months.

Example 5.5.6. *A fact sheet on caffeine dependence from Johns Hopkins Medical Center states that the half-life of caffeine in the body is between 4 and 6 hours. Assuming that the typical half-life of caffeine in the body is 5 hours for the average person and that a typical cup of coffee has 120 mg of caffeine.*

1. *Write the decay function.*
2. *Find the hourly rate at which caffeine leaves the body.*
3. *How long does it take until only 20 mg of caffeine is still in the body?*

Solution 5.5.6.

1. *Let y be the total amount of caffeine in the body t hours after drinking the coffee. Exponential decay function $y = ab^t$ models this situation. The initial amount of caffeine is $a = 120$.*

   We don't know b or t, but we know that the half-life of caffeine in the body is 5 hours. This tells us that when $t = 5$, then there is half the initial amount of caffeine remaining in the body.

$$y = 120b^t$$
$$\frac{1}{2}(120) = 120b^5$$
$$60 = 120b^5$$

   Divide both sides by 120 to isolate the expression b^5 that contains the variable.

$$\frac{60}{120} = b^5$$

$$0.5 = b^5$$

The variable is in the base and the exponent is a number. Use roots to solve for b:

$$b = \sqrt[5]{0.5}$$
$$b \approx 0.87$$

We can now write the decay function for the amount of caffeine (in mg.) remaining in the body t hours after drinking a cup of coffee with 120 mg of caffeine

$$y = f(t) = 120(0.87)^t$$

2. *Use $b = 1 + r$ to find the decay rate r. Because $b = 0.87$ and the amount of caffeine in the body is decreasing over time, the value of r will be negative.*

$$0.87 = 1 + r$$
$$r = -0.13$$

The decay rate is 13%; the amount of caffeine in the body decreases by 13% per hour.

3. *To find the time at which only 20 mg of caffeine remains in the body, substitute $y = 20$ and solve for the corresponding value of t.*

$$20 = 120(0.87)^t$$

Divide both sides by 120 to isolate the exponential expression.

$$\frac{20}{120} = (0.87)^t$$
$$0.1667 = (0.87)^t$$

Rewrite the expression in logarithmic form and then use the change of base formula to evaluate.

$$t = \log_{0.87}(0.1667)$$
$$t = \frac{\ln(0.1667)}{\ln(0.87)}$$
$$t \approx 12.9 \text{ hours}$$

After 12.9 hours, 20 mg of caffeine remains in the body.

5.5.2 Expressing Exponential Functions in Different Forms

Now that we've developed our equation solving skills, we revisit the question of expressing exponential functions equivalently in the forms $y = ab^t$ and $y = ae^{kt}$.

We've already determined that if given the form $y = ae^{kt}$, it is straightforward to find b.

Example 5.5.7. *For the following examples, assume t is measured in years.*

1. *Express $y = 3500e^{0.25t}$ in form $y = ab^t$ and find the annual percentage growth rate.*
2. *Express $y = 28000e^{-0.32t}$ in form $y = ab^t$ and find the annual percentage decay rate.*

Solution 5.5.7.

1. *To express the function in the form $y = ab^t$: We start with the equation*

$$y = ab^t.$$

This can also be written as

$$y = a(e^{kt}) \quad \textit{where} \quad e^{kt} = b.$$

Thus,

$$e^k = b.$$

For this example,

$$b = e^{0.25} \approx 1.284.$$

We rewrite the growth function as

$$y = 3500(1.284)^t.$$

To find r, recall that

$$b = 1 + r.$$

So,

$$1.284 = 1 + r$$

which gives us

$$r = 0.284.$$

The continuous growth rate is $k = 0.25$ *and the annual percentage growth rate is 28.4%.*

2. *To express* $y = 28000e^{-0.32t}$ *in the form* $y = ab^t$*: We start with the equation*

$$y = ab^t.$$

This can also be written as

$$y = a(e^{kt}) \quad \textit{where} \quad e^{kt} = b.$$

Thus,

$$e^k = b.$$

For this example,

$$b = e^{-0.32} \approx 0.7261.$$

We rewrite the decay function as

$$y = 28000(0.7261)^t.$$

To find r*, recall that*

$$b = 1 + r.$$

So,

$$0.7261 = 1 + r$$

which gives us

$$r = -0.2739.$$

The continuous decay rate is $k = -0.32$ *and the annual percentage decay rate is 27.39%.*

In the sentence, we omit the negative sign when stating the annual percentage decay rate because we have used the word "decay" to indicate that r *is negative.*

Example 5.5.8. *Express the following:*

1. $y = 4200(1.078)^t$ *in the form* $y = ae^{kt}$
2. $y = 150(0.73)^t$ *in the form* $y = ae^{kt}$

Solution 5.5.8. 1. *To express* $y = 4200(1.078)^t$ *in the form* $y = ae^{kt}$*: Start with the equation*

$$y = ae^{kt}$$

which can also be written as

$$y = ab^t.$$

We need to find the constant k *such that*

$$e^k = b.$$

Given $b = 1.078,$ *we have*

$$e^k = 1.078.$$

Therefore,

$$k = \ln(1.078) \approx 0.0751.$$

We rewrite the growth function as

$$y = 4200e^{0.0751t}.$$

2. *To express* $y = 150(0.73)^t$ *in the form* $y = ae^{kt}$*: Start with the equation*

$$y = ae^{kt}$$

which can also be written as

$$y = ab^t.$$

Here, we need to find the constant k *such that*

$$e^k = b.$$

Given $b = 0.73,$ *we have*

$$e^k = 0.73.$$

Therefore,

$$k = \ln(0.73) \approx -0.3147.$$

We rewrite the decay function as

$$y = 150e^{-0.3147t}.$$

Example 5.5.9. *Suppose that Vinh invests \$10000 in an investment earning 5% per year. He wants to know how long it will take his investment to accumulate to \$12000, and how long it would take to accumulate to \$15000.*

Solution 5.5.9. *We start by writing the exponential growth function that models the value of this investment as a function of the time since the \$10000 is initially invested*

$$y = 10000(1.05)^t$$

We divide both sides by 10000 to isolate the exponential expression on one side.

$$\frac{y}{10000} = 1.05^t$$

Next we take the natural logarithm of both sides of the equation.

$$\ln(1.05^t) = \left(\frac{y}{10000}\right)$$

Use the exponent property of logarithms to get the power out of the logarithm

$$t\ln(1.05) = \ln\left(\frac{y}{10000}\right)$$

Dividing both sides by $\ln(1.05)$

$$t = \frac{\ln\left(\frac{y}{10000}\right)}{\ln(1.05)}$$

To find the number of years until the value of this investment is \$12000, we substitute $y = 12000$ *into* y *and evaluate to find* t*:*

$$t = \frac{\ln\left(\frac{12000}{10000}\right)}{\ln(1.05)} = \frac{\ln(1.2)}{\ln(1.05)} \approx 3.74 \text{ years}$$

To find the number of years until the value of this investment is \$15000, we substitute $y = 15000$ *into* y *and evaluate to find* t*:*

$$t = \frac{\ln\left(\frac{15000}{10000}\right)}{\ln(1.05)} = \frac{\ln(1.5)}{\ln(1.05)} \approx 8.31 \text{ years}$$

Chapter 6

Mathematics of Finance

In this chapter, you will learn to:

1. Solve financial problems that involve simple interest.
2. Solve problems involving compound interest.
3. Find the future value of an annuity, and the amount of payments to a sinking fund.
4. Find the future value of an annuity, and an installment payment on a loan.

6.1 Simple Interest and Discount

In this section, you will learn to:

1. Find simple interest.
2. Find present value.
3. Find discounts and proceeds.

6.1.1 Simple Interest

It costs to borrow money. The rent one pays for the use of money is called the interest. The amount of money that is being borrowed or loaned is called the principal or present value. Simple interest is paid only on the original amount borrowed. When the money is loaned out, the person who borrows the money generally pays a fixed rate of interest on the principal for the time period he keeps the money. Although the interest rate is often specified for a year, it may be specified for a week, a month, or a quarter, etc. The credit card companies often list their charges as monthly rates, sometimes it is as high as 1.5% a month.

Summary 6.1.1: Simple Interest

If an amount P is borrowed for a time t at an interest rate of r per time period, then the simple interest is given by

$$I = P \cdot r \cdot t$$

The total amount A, also called the accumulated value or the future value, is given by

$$A = P + I = P + Prt$$

or

$$A = P(1 + rt)$$

where interest rate r is expressed in decimals.

Example 6.1.1. *Ursula borrows \$600 for 5 months at a simple interest rate of 15% per year. Find the interest, and the total amount she is obligated to pay?*

Solution 6.1.1. *The interest is computed by multiplying the principal with the interest rate and the time.*

$$I = Prt$$

$$I = \$600(0.15)\frac{5}{12} = \$37.50$$

The total amount is

$$A = P + I = \$600 + \$37.50 = \$637.50$$

Incidentally, the total amount can be computed directly as

$$A = P(1 + rt) = \$600[1 + (0.15)(5/12)] = \$600(1 + 0.0625) = \$637.50$$

Example 6.1.2. *Jose deposited \$2500 in an account that pays 6% simple interest. How much money will he have at the end of 3 years?*

Solution 6.1.2. *The total amount or the future value is given by* $A = P(1+rt)$.

$$A = \$2500[1 + (0.06)(3)]$$

$$A = \$2950$$

Example 6.1.3. *Darnel owes a total of \$3060 which includes 12% interest for the three years he borrowed the money. How much did he originally borrow?*

Solution 6.1.3. *This time we are asked to compute the principal* P.

$$\$3060 = P[1 + (.12)(3)]$$

$$\$3060 = P(1.36)$$

$$P = \frac{\$3060}{1.36}$$

$$P = \$2250$$

Darnel originally borrowed \$2250.

Example 6.1.4. *A Visa credit card company charges a 1.5% finance charge each month on the unpaid balance. If Martha owed \$2350 and has not paid her bill for three months, how much does she owe now?*

Solution 6.1.4. *Before we attempt the problem, the reader should note that in this problem the rate of finance charge is given per month and not per year.*

The total amount Martha owes is the previous unpaid balance plus the finance charge.

$$A = \$2350 + \$2350(0.015)(3) = \$2350 + \$105.75 = \$2455.75$$

Alternatively, again, we can compute the amount directly by using formula $A = P(1 + rt)$

$$A = \$2350[1 + (.015)(3)] = \$2350(1.045) = \$2455.75$$

6.1.2 Discounts and Proceeds

Banks often deduct the simple interest from the loan amount at the time that the loan is made. When this happens, we say the loan has been discounted. The interest that is deducted is called the discount, and the actual amount that is given to the borrower is called the proceeds. The amount the borrower is obligated to repay is called the maturity value.

Summary 6.1.2: Discounts and Proceeds

If an amount M is borrowed for a time t at a discount rate of r per year, then the discount D is

$$D = M \cdot r \cdot t$$

The proceeds P, the actual amount the borrower gets, is given by

$$P = M - D$$

$$P = M - Mrt$$

or

$$P = M(1 - rt)$$

where interest rate r is expressed in decimals.

Example 6.1.5. *Francisco borrows \$1200 for 10 months at a simple interest rate of 15% per year. Determine the discount and the proceeds.*

Solution 6.1.5. *The discount D is the interest on the loan that the bank deducts from the loan amount.*

$$D = Mrt$$

$$D = \$1200\,(0.15)\left(\frac{10}{12}\right) = \$150$$

Therefore, the bank deducts \$150 from the maturity value of \$1200, and gives Francisco \$1050. Francisco is obligated to repay the bank \$1200. In this case, the discount D = \$150, and the proceeds P = \$1200 - \$150 = \$1050.

Example 6.1.6. *If Francisco wants to receive \$1200 for 10 months at a simple interest rate of 15% per year, what amount of loan should he apply for?*

Solution 6.1.6. *In this problem, we are given the proceeds P and are being asked to find the maturity value M.*

$$P = \$1200, \quad r = 0.15, \quad t = \frac{10}{12}$$

. *We need to find M. We know $P = M - D$ but also $D = Mrt$ therefore $P = M - Mrt = M(1 - rt)$*

$$\$1200 = M\left[1 - (0.15)\left(\frac{10}{12}\right)\right]$$

We need to solve for M.

$$\$1200 = M(1 - 0.125)$$

$$\$1200 = M(0.875)$$
$$\frac{\$1200}{0.875} = M$$
$$\$1371.43 = M$$

Therefore, Francisco should ask for a loan for \$1371.43. The bank will discount \$171.43 and Francisco will receive \$1200.

6.2 Compound Interest

In this section you will learn to:

1. Find the future value of a lump-sum.
2. Find the present value of a lump-sum.
3. Find the effective interest rate.

6.2.1 Compound Interest

In the last section, we examined problems involving simple interest. Simple interest is generally charged when the lending period is short and often less than a year. When the money is loaned or borrowed for a longer time period, if the interest is paid (or charged) not only on the principal, but also on the past interest, then we say the interest is compounded.

Suppose we deposit \$200 in an account that pays 8% interest. At the end of one year, we will have \$200 + \$200(0.08) = \$200(1 + .08) = \$216.

Now suppose we put this amount, \$216, in the same account. After another year, we will have \$216 + \$216(0.08) = \$216(1 + .08) = \$233.28.

An initial deposit of \$200 has accumulated to \$233.28 in two years. Further note that had it been simple interest, this amount would have accumulated to only \$232. The reason the amount is slightly higher is because the interest (\$16) we earned the first year, was put back into the account. And this \$16 amount itself earned interest of \$16(0.08) = \$1.28, thus resulting in the increase. So we have earned interest on the principal as well as on the past interest, and that is why we call it compound interest.

Now suppose we leave this amount, \$233.28, in the bank for another year, the final amount will be \$233.28 + \$233.28(0.08) = \$233.28(1 + .08) = \$251.94.

Now let us look at the mathematical part of this problem so that we can devise an easier way to solve these problems. After one year, we had \$200(1 + .08) = \$216. After two years, \$216 = $\$200(1 + .08)^2$. But \$216 = \$200(1 + .08), therefore, the above expression becomes

$$\$200(1 + .08)(1 + .08) = \$200(1 + .08)^2 = \$233.28$$

After three years, we get

$$\$233.28(1 + .08) = \$200(1 + .08)(1 + .08)(1 + .08)$$

which can be written as

$$\$200(1 + .08)^3 = \$251.94$$

Suppose we are asked to find the total amount at the end of 5 years, we will get

$$200(1 + .08)^5 = \$293.87$$

We summarize the compound interest calculations as follows:

The original amount	\$200	= \$200
The amount after one year	$\$200(1 + .08)$	= \$216
The amount after two years	$\$200(1 + .08)^2$	= \$233.28
The amount after three years	$\$200(1 + .08)^3$	= \$251.94
The amount after five years	$\$200(1 + .08)^5$	= \$293.87
The amount after t years	$\$200(1 + .08)^t$	

6.2.2 Compounding Periods

Banks often compound interest more than one time a year. Consider a bank that pays 8% interest but compounds it four times a year, or quarterly. This means that every quarter the bank will pay an interest equal to one-fourth of 8%, or 2%.

Now if we deposit \$200 in the bank, after one quarter we will have $\$200(1 + .08/4)$ or \$204. After two quarters, we will have $\$200(1 + .08/4)^2$ or \$208.08. After one year, we will have $\$200(1 + .08/4)^4$ or \$216.49. After three years, we will have $\$200(1 + .08/4)^12$ or \$253.65, etc.

The original amount	\$200	= \$200
The amount after one quarter	$\$200\left(1 + \frac{.08}{4}\right)$	= \$204
The amount after two quarters	$\$200\left(1 + \frac{.08}{4}\right)^2$	= \$208.08
The amount after one year	$\$200\left(1 + \frac{.08}{4}\right)^4$	= \$216.49
The amount after two years	$\$200\left(1 + \frac{.08}{4}\right)^8$	= \$234.31
The amount after three years	$\$200\left(1 + \frac{.08}{4}\right)^{12}$	= \$253.65
The amount after five years	$\$200\left(1 + \frac{.08}{4}\right)^{20}$	= \$297.19
The amount after t years	$\$200\left(1 + \frac{.08}{4}\right)^{4t}$	

The general formula for compound interest is given by the

Summary 6.2.1: Compound Interest Formula

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where P is the principal amount, r is the annual interest rate, n is the number of times interest is compounded per year, and t is the time in years.

Example 6.2.1. *If \$3500 is invested at 9% compounded monthly, what will* *the future value be in four years?*

Solution 6.2.1. *Clearly an interest of* $\frac{0.09}{12}$ *is paid every month for four years. The interest is compounded* $4 \cdot 12 = 48$ *times over the four-year period. We get*

$$A = \$3500\left(1 + \frac{.09}{12}\right)^{48} = \$3500(1.0075)^{48} = \$5009.92$$

\$3500 invested at 9% compounded monthly will accumulate to \$5009.92 in four years.

Example 6.2.2. *How much should be invested in an account paying 9% compounded daily for it to accumulate to \$5,000 in five years?*

Solution 6.2.2. *We know the future value, but need to find the principal.*

$$\$5000 = P\left(1 + \frac{.09}{365}\right)^{365(5)}$$

$$\$5000 = P(1.568225)$$

$$P = \$3188.32$$

\$3188.32 invested in an account paying 9% compounded daily will accumulate to \$5,000 in five years.

Example 6.2.3. *If \$4,000 is invested at 4% compounded annually, how long will it take to accumulate to \$6,000?*

Solution 6.2.3. $n = 1$ *because annual compounding means compounding only once per year. The formula simplifies to* $A = (1 + r)^t$ *when* $n = 1$.

$$\$6000 = \$4000(1.04)^t$$

Dividing by 4000 yields

$$\frac{6000}{4000} = (1.04)^t$$

$$1.5 = 1.04^t$$

We use logarithms to solve for the value of t *because the variable* t *is in the exponent.*

$$\log(1.5) = \log(1.04^t)$$

Using the Exponent Rule for logarithms we can take the power down

$$\log(1.5) = t\log(1.04)$$

then solve for t by dividing:

$$t = \frac{\ln(1.5)}{\ln(1.04)} \approx 10.33 \text{ years}$$

It takes about 10 years and 4 months for \$4000 to accumulate to \$6000 if invested at 4% interest, compounded annually.

Example 6.2.4. *If \$5,000 is invested now for 6 years what interest rate compounded quarterly is needed to obtain an accumulated value of \$8,000.*

Solution 6.2.4. *We have $n = 4$ for quarterly compounding.*

$$\begin{aligned}
\$8,000 &= \$5,000\left(1+\frac{r}{4}\right)^{4\cdot 6}\\
\$8,000 &= \$5,000\left(1+\frac{r}{4}\right)^{24}\\
1.6 &= \left(1+\frac{r}{4}\right)^{24}
\end{aligned}$$

We use roots to solve for r because the variable r is in the base, whereas the exponent is a known number.

$$\begin{aligned}
\sqrt[24]{1.6} &= 1+\frac{r}{4}\\
1.6^{\frac{1}{24}} &= 1+\frac{r}{4}
\end{aligned}$$

Evaluating the left side of the equation gives

$$\begin{aligned}
1.0197765 &= 1+\frac{r}{4}\\
0.0197765 &= \frac{r}{4}\\
r &= 4(0.0197765) = 0.0791
\end{aligned}$$

An interest rate of 7.91% is needed in order for \$5,000 invested now to accumulate to \$8,000 at the end of 6 years, with interest compounded quarterly.

6.2.3 Effective Interest Rate

Banks are required to state their interest rate in terms of an "effective yield" or "effective interest rate", for comparison purposes. The effective rate is also called the Annual Percentage Yield (APY) or Annual Percentage Rate (APR).

The effective rate is the interest rate compounded annually would be equivalent to the stated rate and compounding periods. The next example shows how to calculate the effective rate. To examine several investments to see which has the best rate, we find and compare the effective rate for each investment.

Example 6.2.5 illustrates how to calculate the effective rate.

Example 6.2.5. *If Bank A pays 7.2% interest compounded monthly, what is the effective interest rate? If Bank B pays 7.25% interest compounded semiannually, what is the effective interest rate? Which bank pays more interest?*

Solution 6.2.5. *Bank A: Suppose we deposit \$1 in this bank and leave it for a year, we will get*

$$r_{effective} = \left(1 + \frac{0.072}{12}\right)^{12} = 1.0744$$

$$r_{effective} = 1.0744 - 1 = 0.0744$$

We earned interest of \$1.0744 - \$1.00 = \$0.0744 on an investment of \$1.

The effective interest rate is 7.44%, often referred to as the APY or APR.

Bank B: The effective rate is calculated as

$$r_{effective} = \left(1 + \frac{0.0725}{2}\right)^{2} - 1 = .0738$$

The effective interest rate is 7.38%.

Bank A pays slightly higher interest, with an effective rate of 7.44%, compared to Bank B with effective rate 7.38%.

6.2.4 Continuous Compounding

Interest can be compounded yearly, semiannually, quarterly, monthly, and daily. Using the same calculation methods, we could compound every hour, every minute, and even every second. As the compounding period gets shorter and shorter, we move toward the concept of continuous compounding.

But what do we mean when we say the interest is compounded continuously, and how do we compute such amounts? When interest is compounded "infinitely many times", we say that the interest is compounded continuously. Our next objective is to derive a formula to model continuous compounding.

Suppose we put \$1 in an account that pays 100% interest. If the interest is compounded once a year, the total amount after one year will be \$1(1 + 1) = \$2. If the interest is compounded semiannually, in one year we will have $\$1(1+\frac{1}{2})^2 = \$2.25$. If the interest is compounded quarterly, in one year we will have $\$1(1+\frac{1}{4})^4 = \$2.44$. If the interest is compounded monthly, in one year we will have $\$1(1+\frac{1}{12})^{12} = \$2.61$. If the interest is compounded daily, in one year we will have $\$1(1+\frac{1}{365})^{365} = \$2.71$.

We show the results as follows:

Frequency of compounding	Formula	Total amount
Annually	$\$1(1+1)$	\$2
Semiannually	$\$1(1+\frac{1}{2})^2$	\$2.25
Quarterly	$\$1(1+\frac{1}{4})^4$	\$2.44140625
Monthly	$\$1(1+\frac{1}{12})^{12}$	\$2.61303529
Daily	$\$1(1+\frac{1}{365})^{365}$	\$2.71456748
Hourly	$\$1(1+\frac{1}{8760})^{8760}$	\$2.71812699
Every minute	$\$1(1+\frac{1}{525600})^{525600}$	\$2.71827922
Every Second	$\$1(1+\frac{1}{31536000})^{31536000}$	\$2.71828247
Continuously	\$1(2.718281828...)	\$2.718281828...

We have noticed that the \$1 we invested does not grow without bound. It starts to stabilize to an irrational number 2.718281828... given the name "e" after the great mathematician Euler. In mathematics, we say that as n becomes infinitely large, the expression equals e. Therefore, it is natural that the number e plays a part in continuous compounding.

It can be shown that as n becomes infinitely large, the expression

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

Therefore, it follows that if we invest \$P at an interest rate r per year, compounded continuously, after t years the final amount will be given by

Summary 6.2.2: Continous Compounding

$$A = P \cdot e^{rt}$$

with P the principal, r the interest rate, t the time in years, and A the value after compounding for those t years

Example 6.2.6. *\$3500 is invested at 9% compounded continuously. Find the future value in 4 years.*

Solution 6.2.6. *Using the formula for the continuous compounding, we get* $A = Pe^{rt}$.

$$A = \$3500e^{0.09(4)}$$
$$A = \$3500e^{0.36}$$
$$A = \$5016.65$$

Example 6.2.7. *If an amount is invested at 7% compounded continuously, what is the effective interest rate?*

Solution 6.2.7. *If we deposit \$1 in the bank at 7% compounded continuously for one year, and subtract that \$1 from the final amount, we get the effective interest rate in decimals.*

$$r_{effective} = 1e^{0.07} - 1$$
$$r_{effective} = 1.0725 - 1$$
$$r_{effective} = 0.0725 \quad or \quad 7.25\%$$

Example 6.2.8. *If an amount is invested at 7% compounded continuously, how long will it take to double?*

Solution 6.2.8. *We use the model:* $Pe^{0.07t} = A$.

We don't know the initial value of the principal but we do know that the accumulated value is double (twice) the principal.

$$\begin{aligned}
Pe^{0.07t} &= 2P \\
\frac{Pe^{0.07t}}{P} &= \frac{2P}{P} \\
e^{0.07t} &= 2 \\
0.07t &= \ln(2) \\
t &= \frac{\ln(2)}{0.07} \\
t &= 9.9 \textit{ years}
\end{aligned}$$

It takes 9.9 years for money to double if invested at 7% continuous interest.

6.3 Annuities and Sinking Funds

In this section, you will learn to:

1. Find the future value of an annuity.
2. Find the amount of payments to a sinking fund.

6.3.1 Ordinary Annuity

In the first two sections of this chapter, we examined problems where an amount of money was deposited lump sum in an account and was left there for the entire time period. Now we will do problems where timely payments are made in an account. When a sequence of payments of some fixed amount are made in an account at equal intervals of time, we call that an annuity. And this is the subject of this section.

To develop a formula to find the value of an annuity, we will need to recall the formula for the sum of a geometric series.

A geometric series is of the form: $a + ax + ax^2 + ax^3 + \ldots + ax^n$.

In a geometric series, each subsequent term is obtained by multiplying the preceding term by a number, called the common ratio. A geometric series is completely determined by knowing its first term, the common ratio, and the number of terms.

The following are some examples of geometric series:

- $3 + 6 + 12 + 24 + 48$ has first term $a = 3$ and common ratio $x = 2$.
- $2 + 6 + 18 + 54 + 162$ has first term $a = 2$ and common ratio $x = 3$.
- $37 + 3.7 + .37 + .037 + .0037$ has first term $a = 35$ and common ratio $x = 0.1$.

In your algebra class, you developed a formula for finding the sum of a geometric series. You probably used r as the symbol for the ratio, but we are using x because r is the symbol we have been using for the interest rate.

The formula for the sum of a geometric series with first term a and common ratio x is:

$$S = a\left(\frac{1 - x^n}{1 - x}\right)$$

We will use this formula to find the value of an annuity.

Consider the following example.

Example 6.3.1. *If at the end of each month a deposit of \$500 is made in an account that pays 8% compounded monthly, what will the final amount be after five years?*

Solution 6.3.1. *There are 60 deposits made in this account. The first payment stays in the account for 59 months, the second payment for 58 months, the third for 57 months, and so on.*

The first payment of \$500 will accumulate to an amount of

$$\$500\left(1 + \frac{0.08}{12}\right)^{59}.$$

The second payment of \$500 will accumulate to an amount of

$$\$500\left(1 + \frac{0.08}{12}\right)^{58}.$$

The third payment will accumulate to

$$\$500\left(1+\frac{0.08}{12}\right)^{57}.$$

And so on...

Finally, the next to last (59th) payment will accumulate to

$$\$500(1+\frac{0.08}{12})^{1}$$

. The last payment is taken out the same time it is made, and will not earn any interest.

To find the total amount in five years, we need to add the accumulated value of these sixty payments.

In other words, we need to find the sum of the following series:

$$\$500\left(1+\frac{0.08}{12}\right)^{59}+\$500\left(1+\frac{0.08}{12}\right)^{58}+\$500\left(1+\frac{0.08}{12}\right)^{57}+\ldots+\$500$$

Written backwards, we have

$$\$500+\$500\left(1+\frac{0.08}{12}\right)+\$500\left(1+\frac{0.08}{12}\right)^{2}+\ldots+\$500\left(1+\frac{0.08}{12}\right)^{59}$$

This is a geometric series with $a = \$500$, $r = (1+\frac{0.08}{12})$, and $n = 59$. The sum is

$$\$500\left(\frac{(1+\frac{0.08}{12})^{60}-1}{\frac{0.08}{12}}\right)$$

$$= \$500(73.47686)$$

$$= \$36738.43$$

When the payments are made at the end of each period rather than at the beginning, we call it an ordinary annuity.

Summary 6.3.1: Future Value of an Ordinary Annuity

If a payment of m dollars is made in an account n times a year at an interest rate r, then the final amount A after t years is given by:

$$A = \frac{m\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]}{\frac{r}{n}}$$

The future value is also called the accumulated value. Note that the formula assumes that the payment period is the same as the compounding period. If these are not the same, then this formula does not apply

Example 6.3.2. *Tanya deposits \$300 at the end of each quarter in her savings account. If the account earns 5.75% compounded quarterly, how much money will she have in 4 years?*

Solution 6.3.2. *The future value of this annuity can be found using the above formula.*

$$\begin{aligned} A &= \$300\left(\frac{(1+\frac{0.0575}{4})^{4\cdot 4}-1}{\frac{0.0575}{4}}\right) \\ A &= \$300(17.8463) \\ A &= \$5353.89 \end{aligned}$$

If Tanya deposits \$300 into a savings account earning 5.75% compounded quarterly for 4 years, then at the end of 4 years she will have \$5,353.89.

Example 6.3.3. *Robert needs \$5,000 in three years. How much should he deposit each month in an account that pays 8% compounded monthly in order to achieve his goal?*

Solution 6.3.3. *If Robert saves m dollars per month, after three years he will have*

$$m\left(\frac{(1+\frac{0.08}{12})^{36}-1}{\frac{0.08}{12}}\right)$$

But we'd like this amount to be \$5,000. Therefore,

$$m\left(\frac{(1+\frac{0.08}{12})^{36}-1}{\frac{0.08}{12}}\right) = \$5000$$

$$m(40.5356) = \$5000$$
$$m = \frac{\$5000}{40.5356} = \$123.35$$

Robert needs to deposit \$123.35 at the end of each month for 3 years into an account paying 8% compounded monthly in order to have \$5,000 at the end of 5 years.

6.3.2 Sinking Funds

When a business deposits money at regular intervals into an account in order to save for a future purchase of equipment, the savings fund is referred to as a "sinking fund". Calculating the sinking fund deposit uses the same method as the previous problem.

Example 6.3.4. *A business needs \$450,000 in five years. How much should be deposited each quarter in a sinking fund that earns 9% compounded quarterly to have this amount in five years?*

Solution 6.3.4. *Again, suppose that m dollars are deposited each quarter in the sinking fund. After five years, the future value of the fund should be \$450,000. This suggests the following relationship:*

$$m\left[\frac{(1+\frac{0.09}{4})^{20}-1}{\frac{0.09}{4}}\right] = \$450,000$$

$$m(24.9115) = 450,000$$
$$m = \frac{450000}{24.9115} = \$18,063.93$$

The business needs to deposit \$18,063.93 at the end of each quarter for 5 years into a sinking fund earning interest of 9% compounded quarterly in order to have \$450,000 at the end of 5 years.

6.3.3 Annuities Due

If the payment is made at the beginning of each period, rather than at the end, we call it an annuity due. The formula for the annuity due can be derived in a similar manner. Reconsider Example 6.3.1, with the change that the deposits are made at the beginning of each month.

Example 6.3.5. *If at the beginning of each month a deposit of \$500 is made in an account that pays 8% compounded monthly, what will the final amount be after five years?*

Solution 6.3.5. *There are 60 deposits made in this account. The first payment stays in the account for 60 months, the second payment for 59 months, the third for 58 months, and so on.*

The first payment of \$500 will accumulate to an amount of

$$\$500\left(1+\frac{0.08}{12}\right)^{59}.$$

The second payment of \$500 will accumulate to an amount of

$$\$500\left(1+\frac{0.08}{12}\right)^{58}.$$

The third payment will accumulate to

$$\$500\left(1+\frac{0.08}{12}\right)^{57}.$$

And so on...

Finally, the last (60th) payment will accumulate a month's interest to

$$\$500\left(1+\frac{0.08}{12}\right)^{1}$$

.

To find the total amount in five years, we need to find the sum of the series: In other words, we need to find the sum of the following series:

$$\$500\left(1+\frac{0.08}{12}\right)^{60}+\$500\left(1+\frac{0.08}{12}\right)^{58}+\$500\left(1+\frac{0.08}{12}\right)^{57}+\ldots+\$500\left(1+\frac{0.08}{12}\right)^{1}$$

Written backwards, we have

$$\$500\left(1+\frac{0.08}{12}\right)+\$500\left(1+\frac{0.08}{12}\right)^{2}+\ldots+\$500\left(1+\frac{0.08}{12}\right)^{60}$$

This isn't a geometric series, but if we add \$500 to the front of the series and subtract it from the back we haven't changed the value, but we'll have a geometric series.

$$\$500+\$500\left(1+\frac{0.08}{12}\right)+\$500\left(1+\frac{0.08}{12}\right)^2+\ldots+\$500\left(1+\frac{0.08}{12}\right)^{60}-\$500$$

Except for the last term, we have a geometric series with $a = \$500$*,* $r = (1 + .08/12)$*, and* $n = 60$*. Therefore the sum is*

$$A = \frac{\$500[(1 + .08/12)^{61} - 1]}{.08/12} - \$500$$

$$A = \$500(74.9667) - \$500$$

$$A = \$37483.35 - \$500$$

$$A = \$36983.35$$

So, in the case of an annuity due, to find the future value, we increase the number of periods n by 1, and subtract one payment.

Summary 6.3.2: Future Value of an Annuity Due

If a payment of m dollars is made in an account n times a year (at the beginning of each month) at an interest rate r, then the final amount A after t years is given by:

$$A = \frac{m\left[\left(1+\frac{r}{n}\right)^{nt+1} - 1\right]}{\frac{r}{n}} - m$$

Most of the problems we are going to do in this chapter involve ordinary annuities, therefore, we will down play the significance of the last formula for the annuity due. We mentioned the formula for the annuity due only for completeness.

6.4 Present Value of an Annuity and Installment Payment

In this section, you will learn to:

1. Find the present value of an annuity.
2. Find the amount of installment payment on a loan.

In Section 6.2, we learned to find the future value of a lump sum, and in Section 6.3, we learned to find the future value of an annuity. With these two concepts in hand, we will now learn to amortize a loan, and to find the present value of an annuity.

The present value of an annuity is the amount of money we would need now in order to be able to make the payments in the annuity in the future. In other word, the present value is the value now of a future stream of payments.

We start by breaking this down step by step to understand the concept of the present value of an annuity. After that, the examples provide a more efficient way to do the calculations by working with concepts and calculations we have already explored in Sections 6.2 and 6.3.

Suppose Carlos owns a small business and employs an assistant manager to help him run the business. Assume it is January 1 now. Carlos plans to pay his assistant manager a \$1000 bonus at the end of this year and another \$1000 bonus at the end of the following year. Carlos' business had good profits this year so he wants to put the money for his assistant's future bonuses into a savings account now. The money he puts in now will earn interest at the rate of 4% per year compounded annually while in the savings account.

How much money should Carlos put into the savings account now so that he will be able to withdraw \$1000 one year from now and another \$1000 two years from now?

At first, this sounds like a sinking fund. But it is different. In a sinking fund, we put money into the fund with periodic payments to save to accumulate to a specified lump sum that is the future value at the end of a specified time period.

In this case we want to put a lump sum into the savings account now, so that lump sum is our principal, P. Then we want to withdraw that amount as a series of period payments; in this case the withdrawals are an annuity with \$1000 payments at the end of each of two years.

We need to determine the amount we need in the account now, the present value, to be able to make withdraw the periodic payments later.

We use the compound interest formula from Section 6.2 with $r = 0.04$ and $n = 1$ for annual compounding to determine the present value of each payment of \$1000.

Consider the first payment of \$1000 at the end of year 1. Let P_1 be its present value

$$\$1000 = P(1.04)^1 \quad \text{so} \quad P_1 = \$961.54$$

Now consider the second payment of \$1000 at the end of year 2. Let P_2 be its present value

$$\$1000 = P(1.04)^2 \quad \text{so} \quad P_2 = \$924.56$$

To make the \$1000 payments at the specified times in the future, the amount that Carlos needs to deposit now is the present value.

The calculation above was useful to illustrate the meaning of the present value of an annuity.

But it is not an efficient way to calculate the present value. If we were to have a large number of annuity payments, the step-by-step calculation would be long and tedious.

Example 6.4.1 investigates and develops an efficient way to calculate the present value of an annuity, by relating the future (accumulated) value of an annuity and its present value.

Example 6.4.1. *Suppose you have won a lottery that pays \$1,000 per month for the next 20 years. But, you prefer to have the entire amount now. If the interest rate is 8%, how much will you accept?*

Solution 6.4.1. *This classic present value problem needs our complete attention because the rationalization we use to solve this problem will be used again in the problems to follow.*

Consider, for argument purposes, that two people Mr. Cash, and Mr. Credit have won the same lottery of \$1,000 per month for the next 20 years. Mr. Credit is happy with his \$1,000 monthly payment, but Mr. Cash wants to have the entire amount now.

Our job is to determine how much Mr. Cash should get. We reason as follows:

If Mr. Cash accepts \$P dollars, then the \$P dollars deposited at 8% for 20 years should yield the same amount as the \$1,000 monthly payments for 20 years. In other words, we are comparing the future values for both Mr. Cash and Mr. Credit, and we would like the future values to equal.

Since Mr. Cash is receiving a lump sum of x dollars, its future value is given by the lump sum formula we studied in Section 6.2, and it is

$$A = P(1 + 0.08/12)^{240}$$

Since Mr. Credit is receiving a sequence of payments, or an annuity, of \$1,000 per month, its future value is given by the annuity formula we learned in Section 6.3. This value is

$$A = \frac{\$1,000[(1 + 0.08/12)^{240} - 1]}{0.08/12}$$

The only way Mr. Cash will agree to the payment he receives is if these two future values are equal. So we set them equal and solve for the unknown.

$$P(1 + 0.08/12)^{240} = \frac{\$1,000[(1 + 0.08/12)^{240} - 1]}{0.08/12}$$

$$P(4.9268) = \$1,000(589.02041)$$

$$P = \frac{\$589,020.41}{4.9268}$$

$$P = \$119,554.36$$

The present value of an ordinary annuity of \$1,000 each month for 20 years at 8% is \$119,554.36.

The reader should also note that if Mr. Cash takes his lump sum of P = \$119,554.36 and invests it at 8% compounded monthly, he will have an accumulated value of A=\$589,020.41 in 20 years.

6.4.1 Installment Payment on a Loan

If a person or business needs to buy or pay for something now (a car, a home, college tuition, equipment for a business) but does not have the money, they can borrow the money as a loan. They receive the loan amount called the principal (or present value) now and are obligated to pay back the principal in the future over a stated amount of time (term of the loan), as regular periodic payments with interest.

Example 6.4.2 examines how to calculate the loan payment, using reasoning similar to Example 6.4.1.

Example 6.4.2. *Find the monthly payment for a car costing \$15,000 if the loan is amortized over five years at an interest rate of 9%.*

Solution 6.4.2. *Consider the scenario where Mr. Cash pays cash and Mr. Credit wishes to make monthly payments. We are to determine the monthly payment amount for Mr. Credit.*

Since Mr. Cash is paying a lump sum of \$15,000, his future value is given by the lump sum formula:

$$A = P(1 + r/n)^{nt} = \$15,000(1 + .09/12)^{60} = \$15,000(1.5657)$$

For Mr. Credit to make a sequence of payments, or an annuity, of m dollars per month, and its future value is given by the annuity formula:

$$A = m\left[\frac{(1 + r/n)^{nt} - 1}{r/n}\right] = m\left[\frac{(1 + .09/12)^{12(5)} - 1}{.09/12}\right] = m(75.4241)$$

Setting the two future values equal and solving for m, we find the monthly payment m:

$$\$15,000(1.5657) = m(75.4241)$$

$$m = \frac{\$15,000 \cdot 1.5657}{75.4241} = \$311.38$$

Therefore, the monthly payment needed to repay the loan is \$311.38 for five years.

Summary 6.4.1: Present Value of an Annuity

If a payment of m dollars is made in an account n times a year at an interest r, then the present value P of the annuity after t years is

$$P(1+r/n)^{nt} = \frac{m[(1+r/n)^{nt}-1]}{r/n}$$

Summary 6.4.2: Present Value of an Annuity

When used for a loan, the amount P is the loan amount, and m is the periodic payment needed to repay the loan over a term of t years with n payments per year at an interest r.

$$P(1+r/n)^{nt} = \frac{m[(1+r/n)^{nt}-1]}{r/n}$$

Note that the formula assumes that the payment period is the same as the compounding period. If these are not the same, then this formula does not apply.

6.4.2 Alternate Formula to find Present Value of an Annuity

Finally, we note that many finite mathematics and finance books develop the formula for the present value of an annuity differently.

Instead of using the formula:

$$P\left(1+\frac{r}{n}\right)^{nt} = \frac{m[(1+\frac{r}{n})^{nt}-1]}{\frac{r}{n}}$$

and solving for the present value P after substituting the numerical values for the other items in the formula, many textbooks first solve the formula for P in order to develop a new formula for the present value. Then the numerical information can be substituted into the present value formula and evaluated, without needing to solve algebraically for P.

Starting with Summary 6.4.1:

$$P\left(1+\frac{r}{n}\right)^{nt} = \frac{m[(1+\frac{r}{n})^{nt}-1]}{\frac{r}{n}}$$

Divide both sides by $(1+\frac{r}{n})^{nt}$ to isolate P,

$$P = m\frac{\left[(1+\frac{r}{n})^{nt}-1\right]}{\frac{r}{n}} \cdot \frac{1}{(1+\frac{r}{n})^{nt}}$$

simplify to arrive at

Summary 6.4.3: Present Value of an Annuity (Alternate Formula)

$$P = m\frac{\left[1-(1+\frac{r}{n})^{-nt}\right]}{\frac{r}{n}}$$

The authors of this book believe that it is easier to use the formula in Sumamry 6.4.1 and solve for P or m as needed. In this approach there are fewer formulas to understand, and many students find it easier to learn. In the problems the rest of this chapter, when a problem requires the calculation of the present value of an annuity, formula in Sumamry 6.4.1 will be used.

However, some people prefer formula in Summary 6.4.3, and it is mathematically correct to use that method. Note that if you choose to use formula in Summary 6.4.3, you need to be careful with the negative exponents in the formula. And if you needed to find the periodic payment, you would still need to do the algebra to solve for the value of m.

It would be a good idea to check with your instructor to see if he or she has a preference. In fact, you can usually tell your instructor's preference by noting how he or she explains and demonstrates these types of problems in class.

6.5 Miscellaneous Application Problems

In this section, you will learn to apply concepts of compound interest for savings and annuities to:

1. Find the outstanding balance, partway through the term of a loan, of the future payments still remaining on the loan.

2. Perform financial calculations in situations involving several stages of savings and/or annuities.

3. Find the fair market value of a bond.

4. Construct an amortization schedule for a loan.

We have already developed the tools to solve most finance problems. Now we use these tools to solve some application problems.

6.5.1 Outstanding Balance on a Loan

One of the most common problems deals with finding the balance owed at a given time during the life of a loan. Suppose a person buys a house and amortizes the loan over 30 years, but decides to sell the house a few years later. At the time of the sale, he is obligated to pay off his lender; therefore, he needs to know the balance he owes. Since most long-term loans are paid off prematurely, we are often confronted with this problem.

To find the outstanding balance of a loan at a specified time, we need to find the present value P of all future payments that have not yet been paid. In this case, t does not represent the entire term of the loan. Instead:

- t represents the time that still remains on the loan
- nt represents the total number of future payments.

Example 6.5.1. *Mr. Jackson bought his house in 2004, and financed the loan for 30 years at an interest rate of 7.8%. His monthly payment was \$1260. In 2024, Mr. Jackson decides to pay off the loan. Find the balance of the loan he still owes.*

Solution 6.5.1. *The reader should note that the original amount of the loan is not mentioned in the problem. That is because we don't need to know that to find the balance.*

The original loan was for 30 years. 20 years have past so there are $10 = 30 - 20$ *years still remaining.* $12(10) = 120$ *payments still remain to be paid on this loan.*

As for the bank or lender is concerned, Mr. Jackson is obligated to pay \$1260 each month for 10 more years; he still owes a total of 120 payments. But since Mr. Jackson wants to pay it all off now, we need to find the present value P at the time of repayment of the remaining 10 years of payments of \$1260 each month.

Using the formula we get for the present value of an annuity, we get

$$P\left(1+\frac{0.078}{12}\right)^{120} = \$1260\left[\frac{(1+\frac{0.078}{12})^{120}-1}{\frac{0.078}{12}}\right]$$

$$P(2.17597) = \$227957.85$$

$$P = \$104761.48$$

Summary 6.5.1: Finding the Outstanding Balance of a Loan

If a loan has a payment of m dollars made n times a year at an interest r, then the outstanding value of the loan when there are t years still remaining on the loan is given by P:

$$P\left(1+\frac{r}{n}\right)^{nt} = \frac{m\left[(1+\frac{r}{n})^{nt}-1\right]}{\frac{r}{n}}$$

Note that t is not the original term of the loan but instead t is the amount of time still remaining in the future nt is the number of payments still remaining in the future.

Note that there are other methods to find the outstanding balance on a loan, but the method illustrated above is the easiest.

One alternate method would be to use an amortization schedule, as illustrated toward the end of this section. An amortization schedule shows the payments, interest, and outstanding balance step by step after each loan payment. An amortization schedule is tedious to calculate by hand but can be easily constructed using spreadsheet software.

Another way to find the outstanding balance, which we will not illustrate here, is to find the difference $A - B$, where:

- A is the original loan amount (principal) accumulated to the date on which we want to find the outstanding balance (using compound interest formula).

- B is the accumulated value of all payments that have been made as of the date on which we want to find the outstanding balance (using formula for the accumulated value of an annuity).

In this case, we would need to do a compound interest calculation and an annuity calculation; then, we need to find the difference between them. Three calculations are needed instead of one.

It is a mathematically acceptable way to calculate the outstanding balance. However, it is strongly recommended that students use the method explained in the box above and illustrated in Example 6.5.1, as it is much simpler.

6.5.2 Problems Involving Multiple Stages of Savings and/or Annuities

1. Suppose a baby, Aisha, is born and her grandparents invest \$5000 in a college fund. The money remains invested for 18 years until Aisha enters college, and then is withdrawn in equal semiannual payments over the 4 years that Aisha expects to need to finish college. The college investment fund earns 5% interest compounded semiannually. How much money can Aisha withdraw from the account every six months while she is in college?

2. Aisha graduates college and starts a job. She saves \$1000 each quarter, depositing it into a retirement savings account. Suppose that Aisha saves for 30 years and then retires. At retirement she wants to withdraw money as an annuity that pays a constant amount every month for 25 years. During the savings phase, the retirement account earns 6% interest compounded quarterly. During the annuity payout phase, the retirement account earns 4.8% interest compounded monthly. Calculate Aisha's monthly retirement annuity payout.

These problems appear complicated. But each can be broken down into two smaller problems involving compound interest on savings or involving annuities. Often the problem involves a savings period followed by an annuity period; the accumulated value from the first part of the problem may become a present value in the second part. Read each problem carefully to determine what is needed.

Example 6.5.2. *Suppose a baby, Aisha, is born and her grandparents invest*

\$8000 in a college fund. The money remains invested for 18 years until Aisha enters college, and then is withdrawn in equal semiannual payments over the 4 years that Aisha expects to attend college. The college investment fund earns 5% interest compounded semiannually. How much money can Aisha withdraw from the account every six months while she is in college?

Solution 6.5.2.

Part 1: Accumulation of College Savings: *Find the accumulated value at the end of 18 years of a sum of \$8000 invested at 5% compounded semiannually.*

$$A = \$8000\left(1+\frac{0.05}{2}\right)^{2(18)} = \$8000(1.025)^{36} = \$8000(2.432535) = \$19460.28$$

Part 2: Semiannual annuity payout from savings to put toward college expenses. *Find the amount of the semiannual payout for four years using the accumulated savings from part 1 of the problem with an interest rate of 5% compounded semiannually.*

$$A = \$19460.28$$

in Part 1 is the accumulated value at the end of the savings period. This becomes the present value $P = \$19460.28$ *when calculating the semiannual payments in Part 2.*

$$\$19460.28\left(1+\frac{0.05}{2}\right)^{2(4)} = \frac{m\left[\left(1+\frac{0.05}{2}\right)^{2(4)}-1\right]}{\left(\frac{0.05}{2}\right)}$$

$$\$23710.46 = m(8.73612)$$

$$m = \$2714.07$$

Aisha will be able to withdraw \$2714.07 semiannually for her college expenses.

Example 6.5.3. *Aisha graduates college and starts a job. She saves \$1000 each quarter, depositing it into a retirement savings account. Suppose that Aisha saves for 30 years and then retires. At retirement she wants to withdraw money as an annuity that pays a constant amount every month for 25 years. During the savings phase, the retirement account earns 6% interest* *compounded quarterly. During the annuity payout phase, the retirement account earns 4.8% interest compounded monthly. Calculate Aisha's monthly retirement annuity payout.*

Solution 6.5.3.

Part 1: Accumulation of Retirement Savings: *Find the accumulated value at the end of 30 years of \$1000 deposited at the end of each quarter into a retirement savings account earning 6% interest compounded quarterly.*

$$A = \frac{\$1000\left[\left(1+\frac{0.06}{4}\right)^{4(30)}-1\right]}{\left(\frac{0.06}{4}\right)}$$

$$A = \$331288.19$$

Part 2: Monthly retirement annuity payout: *Find the amount of the monthly annuity payments for 25 years using the accumulated savings from part 1 of the problem with an interest rate of 4.8% compounded monthly.*

$$A = \$331288.19$$

in Part 1 is the accumulated value at the end of the savings period. This amount will become the present value $P = \$331288.19$ *when calculating the monthly retirement annuity payments in Part 2.*

$$\$331288.19\left(1+\frac{0.048}{12}\right)^{12 \cdot 25} = \frac{m\left[\left(1+\frac{0.048}{12}\right)^{12 \cdot 25}-1\right]}{\frac{0.048}{12}}$$

$$\$1097285.90 = m(578.04483)$$

$$m = \$1898.27$$

Aisha will have a monthly retirement annuity income of \$1898.27 when she retires.

6.5.3 Fair Market Value of a Bond

Whenever a business, and for that matter the U. S. government, needs to raise money it does it by selling bonds. A bond is a certificate of promise that states the terms of the agreement. Usually the business sells bonds for the face amount of \$1,000 each for a stated term, a period of time ending at a specified maturity date.

The person who buys the bond, the bondholder, pays \$1,000 to buy the bond.

The bondholder is promised two things: First that he will get his \$1,000 back at the maturity date, and second that he will receive a fixed amount of interest every six months.

As the market interest rates change, the price of the bond starts to fluctuate. The bonds are bought and sold in the market at their fair market value.

The interest rate a bond pays is fixed, but if the market interest rate goes up, the value of the bond drops since the money invested in the bond could earn more if invested elsewhere. When the value of the bond drops, we say it is trading at a discount.

On the other hand, if the market interest rate drops, the value of the bond goes up since the bond now yields a higher return than the market interest rate, and we say it is trading at a premium.

Example 6.5.4. *The Orange Computer Company needs to raise money to expand. It issues a 10-year \$1,000 bond that pays \$30 every six months. If the current market interest rate is 7%, what is the fair market value of the bond?*

Solution 6.5.4. *The bond certificate promises us two things – An amount of \$1,000 to be paid in 10 years, and a semi-annual payment of \$30 for ten years. Therefore, to find the fair market value of the bond, we need to find the present value of the lump sum of \$1,000 we are to receive in 10 years, as well as, the present value of the \$30 semi-annual payments for the 10 years.*

We will let P_1 be the present value of the (face amount of \$1,000

$$P_1\left(1+\frac{0.07}{2}\right)^{20} = \$1,000$$

Since the interest is paid twice a year, the interest is compounded twice a year and $n = 2(10) = 20$

$$P_1(1.9898) = \$1,000$$

$$P_1 = \$502.56$$

We will let P_2 be the present value of the \$30 semi-annual payments is

$$P_2\left(1+\frac{0.07}{2}\right)^{20} = \$30\left[\frac{\left(1+\frac{0.07}{2}\right)^{20}-1}{\left(\frac{0.07}{2}\right)}\right]$$

$$P_2(1.9898) = 848.39$$

$$P_2 = \$426.37$$

The present value of the lump-sum \$1,000 = \$502.56 The present value of the \$30 semi-annual payments = \$426.37 The fair market value of the bond is $P = P_1 + P_2 = \$502.56 + \$426.37 = \$928.93$

Note that because the market interest rate of 7% is higher than the bond's implied interest rate of 6% implied by the semiannual payments, the bond is selling at a discount; its fair market value of \$928.93 is less than its face value of \$1000.

Example 6.5.5. *A state issues a 15 year \$1000 bond that pays \$25 every six months. If the current market interest rate is 4%, what is the fair market value of the bond?*

Solution 6.5.5. *The bond certificate promises two things – an amount of \$1,000 to be paid in 15 years, and semi-annual payments of \$25 for 15 years. To find the fair market value of the bond, we find the present value of the \$1,000 face value we are to receive in 15 years and add it to the present value of the \$25 semi-annual payments for the 15 years. In this example,* $n = 2(15) = 30$.

We will let P_1 be the present value of the lump-sum \$1,000

$$P_1(1+0.04/2)^{30} = \$1,000$$

$$P_1 = \$552.07$$

We will let P_2 be the present value of the \$25 semi-annual payments is

$$P_2(1+0.04/2)^{30} = \$25\left[\frac{(1+0.04/2)^{30}-1}{(0.04/2)}\right]$$

$$P_2(1.18114) = \$1014.20$$

$$P_2 = \$559.90$$

The present value of the lump-sum \$1,000 = \$552.07 The present value of the \$25 semi-annual payments = \$559.90 Therefore, the fair market value of the bond is

$$P = P_1 + P_2 = \$552.07 + \$559.90 = \$1111.97$$

Because the market interest rate of 4% is lower than the interest rate of 5% implied by the semiannual payments, the bond is selling at a premium: the fair market value of \$1,111.97 is more than the face value of \$1,000.

Summary 6.5.2: Fair Market Value of a Bond

To Find the Fair Market Value of a Bond, first find the present value of the face amount A that is payable at the maturity date:

$$A = P_1 \left(1 + \frac{r}{n}\right)^{nt}$$

Solve to find P_1.
Then find the present value of the semiannually payments of Sm over the term of the bond:

$$P_2 \left(1 + \frac{r}{n}\right)^{nt} = \frac{m\left[(1+\frac{r}{n})^{nt} - 1\right]}{r/n}$$

Solve to find P_2. The fair market value (or present value or price or current value) of the bond is the sum of the present values calculated above:

$$P = P_1 + P_2$$

6.5.4 Amortization Schedule for a Loan

An amortization schedule is a table that lists all payments on a loan, splits them into the portion devoted to interest and the portion that is applied to repay principal, and calculates the outstanding balance on the loan after each payment is made.

Example 6.5.6. *An amount of \$500 is borrowed for 6 months at a rate of 12%. Make an amortization schedule showing the monthly payment, the* *monthly interest on the outstanding balance, the portion of the payment contributing toward reducing the debt, and the outstanding balance.*

Solution 6.5.6. *The reader can verify that the monthly payment is $86.27.*

The first month, the outstanding balance is $500, and therefore, the monthly interest on the outstanding balance is

$$(\textit{outstanding balance})(\textit{monthly interest rate}) = (\$500)(0.12/12) = \$5$$

This means, the first month, out of the $86.27 payment, $5 goes toward the interest and the remaining $81.27 toward the balance leaving a new balance of $500 - $81.27 = $418.73.

Similarly, the second month, the outstanding balance is $418.73, and the monthly interest on the outstanding balance is $4.19. Again, out of the $86.27 payment, $4.19 goes toward the interest and the remaining $82.08 toward the balance leaving a new balance of $418.73 - $82.08 = $336.65. The process continues in the table below.

Payment #	*Payment*	*Interest*	*Debt Payment*	*Balance*
1	*$86.27*	*$5*	*$81.27*	*$418.73*
2	*$86.27*	*$4.19*	*$82.08*	*$336.65*
3	*$86.27*	*$3.37*	*$82.90*	*$253.75*
4	*$86.27*	*$2.54*	*$83.73*	*$170.02*
5	*$86.27*	*$1.70*	*$84.57*	*$85.45*
6	*$86.27*	*$0.85*	*$85.42*	*$0.03*

Note that the last balance of 3 cents is due to error in rounding off.

An amortization schedule is usually lengthy and tedious to calculate by hand. For example, an amortization schedule for a 30-year mortgage loan with monthly payments would have $(12)(30) = 360$ rows of calculations in the amortization schedule table. A car loan with 5 years of monthly payments would have $12(5) = 60$ rows of calculations in the amortization schedule table. However, it would be straightforward to use a spreadsheet application on a computer or write a little code to do these repetitive calculations.

Most of the other applications in this section's problem set are reasonably straightforward and can be solved by taking a little extra care in interpreting them. And remember, there is often more than one way to solve a problem.

6.6 Classification of Finance Problems

In this section, you will review the concepts of Chapter 6 to:

1. re-examine the types of financial problems and classify them.
2. re-examine the vocabulary words used in describing financial calculations

We'd like to remind the reader that the hardest part of solving a finance problem is determining the category it falls into. So in this section, we will emphasize the classification of problems rather than finding the actual solution. We suggest that the student read each problem carefully and look for the word or words that may give clues to the kind of problem that is presented. For instance, students often fail to distinguish a lump-sum problem from an annuity. Since the payments are made each period, an annuity problem contains words such as each, every, per, etc. One should also be aware that in the case of a lump-sum, only a single deposit is made, while in an annuity numerous deposits are made at equal spaced time intervals. To help interpret the vocabulary used in the problems, we include a glossary at the end of this section.

Students often confuse the present value with the future value. For example, if a car costs \$15,000, then this is its present value. Surely, you cannot convince the dealer to accept \$15,000 in some future time, say, in five years. Recall how we found the installment payment for that car. We assumed that two people, Mr. Cash and Mr. Credit, were buying two identical cars both costing \$15,000 each. To settle the argument that both people should pay exactly the same amount, we put Mr. Cash's cash of \$15,000 in the bank as a lump-sum and Mr. Credit's monthly payments of x dollars each as an annuity. Then we make sure that the future values of these two accounts are equal. As you remember, at an interest rate of 9%

The future value of Mr. Cash's lump-sum was

$$\$15,000\left(1+\frac{0.09}{12}\right)^{60}.$$

The future value of Mr. Credit's annuity was

$$\frac{m\left[\left(1+\frac{0.09}{12}\right)^{60}-1\right]}{\frac{0.09}{12}}.$$

To solve the problem, we set the two expressions equal and solve for m. The present value of an annuity is found in exactly the same way. For example, suppose Mr. Credit is told that he can buy a particular car for \$311.38 a month for five years, and Mr. Cash wants to know how much he needs to pay. We are finding the present value of the annuity of \$311.38 per month, which is the same as finding the price of the car. This time our unknown quantity is the price of the car. Now suppose the price of the car is P, then

the future value of Mr. Cash's lump-sum is

$$P\left(1+\frac{0.09}{12}\right)^{60},$$

and the future value of Mr. Credit's annuity is

$$\frac{\$311.38\left[\left(1+\frac{0.09}{12}\right)^{60}-1\right]}{\frac{0.09}{12}}.$$

Setting them equal we get,

$$P\left(1+\frac{0.09}{12}\right)^{60}=\frac{\$311.38\left[\left(1+\frac{0.09}{12}\right)^{60}-1\right]}{\frac{0.09}{12}}$$

$$P(1.5657)=(\$311.38)(75.4241)$$

$$P(1.5657)=\$23,485.57$$

$$P=\$15,000.04$$

6.6.1 Classification of Problems and Equations for Solutions

We now list six problems that form a basis for all finance problems. Further, we classify these problems and give an equation for the solution.

Example 6.6.1. *If \$2,000 is invested at 7% compounded quarterly, what will the final amount be in 5 years?*

Solution 6.6.1. *This is about the future (accumulated) value of a lump-sum. The future value can be calculated using the formula for compound interest:*

$$FV = A = \$2000\left(1 + \frac{0.07}{4}\right)^{4 \cdot 5}$$

Example 6.6.2. *How much should be invested at 8% compounded yearly, for the final amount to be \$5,000 in five years?*

Solution 6.6.2. *This is about the present value of a lump-sum. The present value required for the future value of \$5,000 can be calculated as:*

$$PV(1 + 0.08)^5 = \$5,000$$

Example 6.6.3. *If \$200 is invested each month at 8.5% compounded monthly, what will the final amount be in 4 years?*

Solution 6.6.3. *The future value of an annuity formula is used in this case:*

$$FV = A = \frac{\$200\left[\left(1 + \frac{0.085}{12}\right)^{12 \times 4} - 1\right]}{\frac{0.085}{12}}$$

Example 6.6.4. *How much should be invested each month at 9% for it to accumulate to \$8,000 in three years?*

Solution 6.6.4. *This is a sinking fund payment problem where the formula is:*

$$m\frac{\left[\left(1 + \frac{0.09}{12}\right)^{12 \cdot 3} - 1\right]}{\frac{0.09}{12}} = \$8,000$$

Example 6.6.5. *Keith has won a lottery paying him \$2,000 per month for the next 10 years. He'd rather have the entire sum now. If the interest rate is 7.6%, how much should he receive?*

Solution 6.6.5. *This is about the present value of an annuity. The present value of an annuity is calculated with:*

$$PV\left(1+\frac{0.076}{12}\right)^{12\cdot 10} = \frac{\$2,000\left[\left(1+\frac{0.076}{12}\right)^{12\cdot 10}-1\right]}{\frac{0.076}{12}}$$

Example 6.6.6. *Mr. A has just donated \$25,000 to his alma mater. Mr. B would like to donate an equivalent amount, but would like to pay by monthly payments over a five year period. If the interest rate is 8.2%, determine the size of the monthly payment?*

Solution 6.6.6. *The monthly payment can be found using the formula for the present value of an annuity due to the installment payment plan:*

$$m\frac{\left[(1+\frac{0.082}{12})^{60}-1\right]}{\frac{0.082}{12}} = \$25,000\left(1+\frac{0.082}{12}\right)^{60}$$

6.7 Financial Calculations Vocabulary

As we've seen in these examples, it's important to read the problems carefully to correctly identify the situation. It is essential to understand to vocabulary for financial problems. Many of the vocabulary words used are listed in the glossary below for easy reference.

Definition 6.7.1. *The **Term**, denoted by t, is the time period for a loan or investment. In this book t is represented in years and should be converted into years when it is stated in months or other units.*

Definition 6.7.2. *The **Principal**, denoted by P, refers to the amount of money borrowed in a loan. If a sum of money is invested for a period of time, the sum invested at the start is the Principal.*

Definition 6.7.3. *The **Present Value**, denoted by P, is the value of money at the beginning of the time period.*

Definition 6.7.4. *The **Accumulated Value** or **Future Value** refers to the value of money at the end of the time period.*

Definition 6.7.5. *The **Discount** occurs in loans involving simple interest if the interest is deducted from the loan amount at the beginning of the loan period, rather than being repaid at the end of the loan period.*

Definition 6.7.6. *The **Periodic Payment**, denoted by m, is the amount of a constant periodic payment that occurs at regular intervals during the time period under consideration, such as periodic payments made to repay a loan, regular periodic payments into a bank account as savings, or regular periodic payments to a retired person as an annuity.*

Definition 6.7.7. *The **Number of payment periods and compounding periods per year**, denoted by n, is considered to be the same in this book when dealing with periodic payments. While in general the compounding and payment periods do not have to be the same and calculations can become more complicated, formulas for different periods can be found in finance textbooks or various online resources. Technology such as online financial calculators, spreadsheet financial functions, or financial pocket calculators can be utilized for these calculations.*

Definition 6.7.8. *The **Number of periods**, denoted by nt, is calculated as $nt = (\text{number of periods per year})(\text{number of years})$. It gives the total number of payment and compounding periods.*

Definition 6.7.9. *The **Annual interest rate** or **Nominal rate** is the stated annual interest rate. This is expressed as a percent but converted to decimal form when used in financial calculation formulas. For example, if a bank account pays 3% interest compounded quarterly, then 3% is the nominal rate, and it is included in the financial formulas as $r = 0.03$.*

Definition 6.7.10. *The **Interest rate per compounding period**, $\frac{r}{n}$, is the interest rate for each compounding period. If a bank account pays 3% interest compounded quarterly, then $\frac{r}{n} = \frac{0.03}{4} = 0.0075$, corresponding to a rate of 0.75% per quarter. Some Finite Math books use the symbol i to represent $\frac{r}{n}$.*

Definition 6.7.11. *The **Effective Rate**, denoted by $r_{effective}$, or **Effective Annual Interest Rate**, also known as **Annual Percentage Yield (APY)** or **Annual Percentage Rate (APR)**, is the interest rate compounded annually that would yield the same interest as the stated compounded rate for the investment. The effective rate provides a uniform way for investors or borrowers to compare different interest rates with different compounding periods.*

Definition 6.7.12. ***Interest**, denoted by I, is money paid by a borrower for the use of money borrowed as a loan. It is also money earned over time when* *depositing money into a savings account, certificate of deposit, or money market account. When a person deposits money in a bank account, the depositor is essentially lending money to the bank temporarily, and the bank pays interest to the depositor.*

Definition 6.7.13. *A **Sinking Fund** is a fund established by making periodic payments into a savings or investment account over a period of time. The purpose of a sinking fund is to save for a future purchase, such as a business setting aside money to buy equipment at the end of the savings period.*

Definition 6.7.14. *An **Annuity** is a series of periodic payments. In this book, it refers to a stream of constant periodic payments made at the end of each compounding period for a certain amount of time. Commonly, the term annuity is used to describe a steady stream of payments received by an individual as retirement income, like from a pension. Annuity payments may be made at the end of each payment period (ordinary annuity) or at the beginning (annuity due). While compounding and payment periods can differ, this textbook only addresses cases where these periods are the same.*

Definition 6.7.15. *A **Lump Sum** refers to a single sum of money paid or deposited all at once, rather than distributed over time. An example includes lottery winnings when the recipient opts for a one-time "lump sum" payment instead of periodic payments over time. The term "lump sum" implies that the transaction is a one-off and not a sequence of periodic payments.*

Definition 6.7.16. *A **Loan** is an amount of money borrowed with an agreement that the borrower will repay the lender in the future, within a specified period known as the term of the loan. Repayment typically occurs through periodic payments until the loan is fully paid off by the end of the term. Some loans may be repaid in a single sum at the loan's end, with interest paid either periodically during the term or as a lump sum at the end, or through a discount at the start of the loan.*

Chapter 7

Sets and Counting

In this chapter, you will learn to:

1. Use set theory and Venn diagrams to solve counting problems.
2. Use the Multiplication Axiom to solve counting problems.
3. Use Permutations to solve counting problems.
4. Use Combinations to solve counting problems.
5. Use the Binomial Theorem to expand $(x + y)^n$.

7.1 Sets

In this section, you will learn to:

1. Use set notation to represent unions, intersections, and complements of sets.
2. Use Venn diagrams to solve counting problems.

7.1.1 Introduction to Sets

In this section, we will familiarize ourselves with set operations and notations, so that we can apply these concepts to both counting and probability problems. We begin by defining some terms.

A set is a collection of objects, and its members are called the elements of the set. We name the set by using capital letters and enclose its members in braces. Suppose we need to list the members of the chess club. We use the following set notation:

$$C = \{\text{Ken, Bob, Tran, Shanti, Eric}\}$$

7.1.2 Empty Set, Set Equality, Subsets

A set that has no members is called an empty set. The empty set is denoted by the symbol $\emptyset$.

Two sets are equal if they have the same elements.

A set A is a subset of a set B if every member of A is also a member of B. Suppose $C = \{\text{Al, Bob, Chris, David, Ed}\}$ and $A = \{\text{Bob, David}\}$. Then A is a subset of C, written as $A \subseteq C$.

Every set is a subset of itself, and the empty set is a subset of every set.

7.1.3 Union of Two Sets

Let A and B be two sets, then the union of A and B, written as $A \cup B$, is the set of all elements that are either in A or in B, or in both A and B.

7.1.4 Intersection of Two Sets

Let A and B be two sets, then the intersection of A and B, written as $A \cap B$, is the set of all elements that are common to both sets A and B.

A universal set U is the set consisting of all elements under consideration.

7.1.5 Complement of a Set, Disjoint Sets

Let A be any set, then the complement of set A, written as $\overline{A}$, is the set consisting of elements in the universal set U that are not in A. This can also be written as $U - A$, which is said "U less A".

Two sets A and B are called disjoint sets if their intersection is an empty set. Clearly, a set and its complement are disjoint; however, two sets can be disjoint and not be complements.

Example 7.1.1. *List all the subsets of the set of primary colors* $\{red, yellow, blue\}$.

Solution 7.1.1. *The subsets are* $\emptyset$, $\{red\}$, $\{yellow\}$, $\{blue\}$, $\{red, yellow\}$, $\{red, blue\}$, $\{yellow, blue\}$, $\{red, yellow, blue\}$.

Note that the empty set is a subset of every set, and a set is a subset of itself.

Example 7.1.2. *Let* $F = \{Aikman, Jackson, Rice, Sanders, Young\}$, *and let* $B = \{Griffey, Jackson, Sanders, Thomas\}$. *Find the intersection of the sets* F *and* B.

Solution 7.1.2. *The intersection of the two sets is the set whose elements belong to both sets. Therefore,* $F \cap B = \{Jackson, Sanders\}$

Example 7.1.3. *Find the union of the sets* $F and B$ *given as follows.*

$$F = \{Aikman, Jackson, Rice, Sanders, Young\}$$

$$B = \{Griffey, Jackson, Sanders, Thomas\}$$

Solution 7.1.3. *The union of two sets is the set whose elements are either in* A *or in* B *or in both* A *and* B. *Observe that when writing the union of two sets, the repetitions are avoided.*

$$F \cup B = \{Aikman, Griffey, Jackson, Rice, Sanders, Thomas, Young\}$$

Example 7.1.4. *Let the universal set* $U = \{red, orange, yellow, green, blue, indigo, violet\}$ *and* $P = \{red, yellow, blue\}$. *Find the complement of* P.

Solution 7.1.4. *The complement of a set* P *is the set consisting of elements in the universal set* U *that are not in* P. *Therefore,*

$$\overline{P} = \{orange, green, indigo, violet\}$$

To achieve a better understanding, suppose that the universal set U *represents the colors of the spectrum, and* P *represents those colors of the spectrum that are not primary colors.*

Example 7.1.5. *Let the universal set* $U = \{red, orange, yellow, green, blue, indigo, violet\}$ *and* $P = \{red,\ yellow,\ blue\}$. *Find a set* R *so that* R *is not the complement of* P *but* R *and* P *are disjoint.*

Solution 7.1.5. $R = \{orange,\ green\}$ *and* $P = \{red,\ yellow,\ blue\}$ *are disjoint because the intersection of the two sets is the empty set. The sets have no elements in common. However, they are not complements because their union* $P \cup R = \{red,\ yellow,\ blue,\ orange,\ green\}$ *is not equal to the universal set* U.

Example 7.1.6. *Let* $U = \{red,\ orange,\ yellow,\ green,\ blue,\ indigo,\ violet\}$, $P = \{red,\ yellow,\ blue\}$, $Q = \{red,\ green\}$, *and* $R = \{orange,\ green,\ indigo\}$. *Find* $\overline{P \cup Q} \cap \overline{R}$.

Solution 7.1.6. *We do the problems in steps:*

$$\begin{aligned}
P \cup Q &= \{red,\ yellow,\ blue,\ green\} \\
\overline{P \cup Q} &= \{orange,\ indigo,\ violet\} \\
\overline{R} &= \{red,\ yellow,\ blue,\ violet\} \\
\overline{P \cup Q} \cap \overline{R} &= \{violet\}
\end{aligned}$$

7.1.6 Venn Diagrams

We now use Venn diagrams to illustrate the relationships between sets. In the late 1800s, an English logician named John Venn developed a method to represent relationships between sets. He represented these relationships using diagrams, which are now known as Venn diagrams.

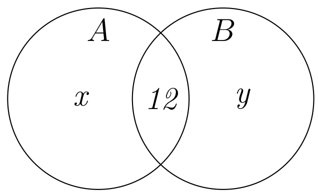
A Venn diagram represents a set as the interior of a circle. Often two or more circles are enclosed in a rectangle where the rectangle represents the universal set. To visualize an intersection or union of a set is easy. In this section, we will mainly use Venn diagrams to sort various populations and count objects.

Example 7.1.7. *Suppose a survey of car enthusiasts showed that over a certain time period, 30 drove cars with automatic transmissions, 20 drove cars with standard transmissions, and 12 drove cars of both types. If everyone in the survey drove cars with one of these transmissions, how many people participated in the survey?*

Solution 7.1.7. *We will use Venn diagrams to solve this problem.*

Let the set A represent those car enthusiasts who drove cars with automatic transmissions, and set S represent the car enthusiasts who drove the cars with standard transmissions. Now we use Venn diagrams to sort out the information given in this problem.

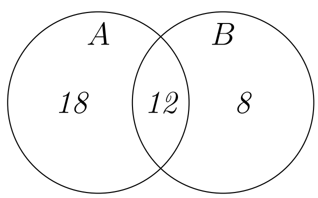
Since 12 people drove both cars, we place the number 12 in the region common to both sets. x represents the number of people who only drove cars with automatic transmissions. Similarly, y represents the number of drivers who only drove cars with standard transmissions.



Because 30 people drove cars with automatic transmissions, the circle A must contain 30 elements. This means that $x + 12 = 30$, or $x = 18$.

Similarly, since 20 people drove cars with standard transmissions, the circle S must contain 20 elements.

Thus, $y + 12 = 20$ which in turn makes $y = 8$.

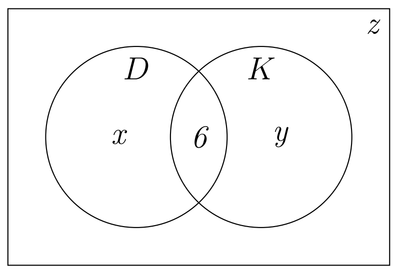


Now that all the information is sorted out, it is easy to read from the diagram that 18 people drove cars with automatic transmissions only, 12 people drove both types of cars, and 8 drove cars with standard transmissions only.

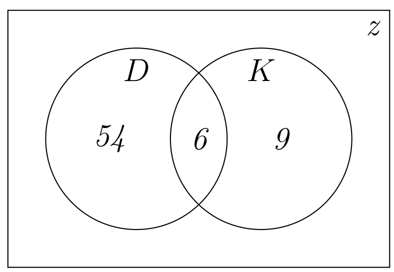
Therefore, $18 + 12 + 8 = 38$ people took part in the survey.

Example 7.1.8. *A survey of 100 people in California indicates that 60 people have visited Disneyland, 15 have visited Knott's Berry Farm, and 6 have visited both. How many people have visited neither place?*

Solution 7.1.8. *The problem is similar to the one in Example 7.1.7. Let the set D represent the people who have visited Disneyland, and K the set of people who have visited Knott's Berry Farm.*

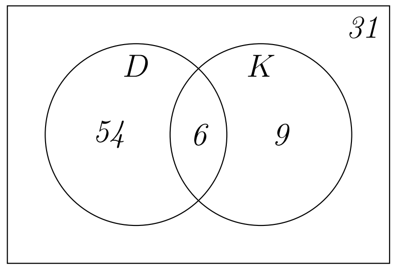


We fill the three regions associated with the sets D and K in the same manner as before. The number x represents the number of people that have been to Disneyland, but not Knotts. We know the total number of people who have been to Disneyland is 60, so $x + 6 = 60$. This means x must be 54. Similar reasoning will help us find that $y = 9$.



Since 100 people participated in the survey, the rectangle representing the universal set U must contain 100 objects. Let z represent those people in the universal set that are neither in the set D nor in K. This means $54 + 6 + 9 + z = 100$, or $z = 31$.

Therefore, there are 31 people in the survey who have visited neither place.



Example 7.1.9. *A survey of 100 exercise-conscious people resulted in the following information:*

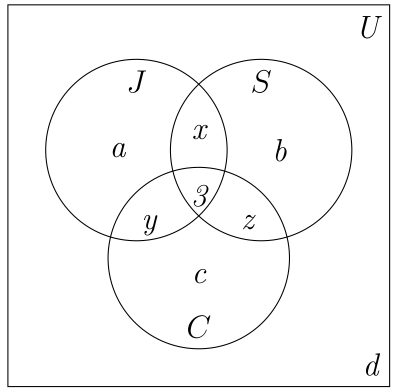
- *50 jog, 30 swim, and 35 cycle*
- *14 jog and swim*
- *7 swim and cycle*
- *9 jog and cycle*
- *3 people take part in all three activities*

1. *How many jog but do not swim or cycle?*
2. *How many take part in only one of the activities?*
3. *How many do not take part in any of these activities?*

Solution 7.1.9. *Let J represent the set of people who jog, S the set of people who swim, and C who cycle.*

In using Venn diagrams, our ultimate aim is to assign a number to each region. We always begin by first assigning the number to the innermost region and then working our way out.

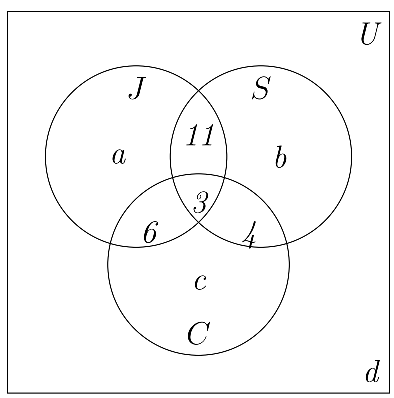
We'll show the solution step by step. As you practice working out such problems, you will find that with practice you will not need to draw multiple copies of the diagram.



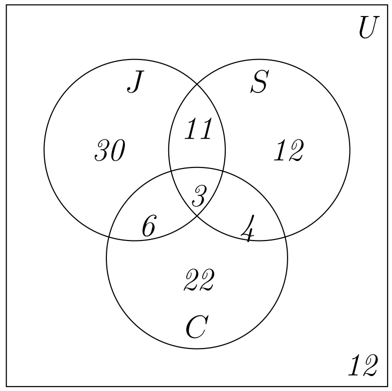
We place a 3 in the innermost region because it represents the number of people who participate in all three activities. Next, we label the regions we don't know about. Now, we compute x*,* y*, and* z*.*

Since 14 people jog and swim, $x + 3 = 14$*, or* $x = 11$*. The fact that 9 people jog and cycle results in* $y + 3 = 9$*, or* $y = 6$*. Since 7 people swim and cycle,* $z + 3 = 7$*, or* $z = 4$*.*

Updating our Venn diagram



Now we proceed to find the unknowns a, b, and c, as shown in Figure IV. Since 50 people jog, $a+11+6+3 = 50$*, or* $a = 30$*. 30 people swim, therefore,* $b + 11 + 4 + 3 = 30$*, or* $b = 12$*. 35 people cycle, therefore,* $c+6+4+3 = 35$*, or* $c = 22$*. By adding all the entries in all three sets, we get a sum of 88. Since 100 people were surveyed, the number, d, inside the universal set but outside of all three sets is given by* $100 - d = 88$*, or* $d = 12$*. Finally we put all the information in our diagram, all the information is sorted out, and the questions can readily be answered.*



1. *The number of people who jog but do not swim or cycle is 30.*

2. *The number who take part in only one of these activities is 30 + 12 + 22 = 64.*

3. *The number of people who do not take part in any of these activities is 12.*

7.2 Tree Diagrams and the Multiplication Axiom

In this section you will learn to

1. Use trees to count possible outcomes in a multi-step process

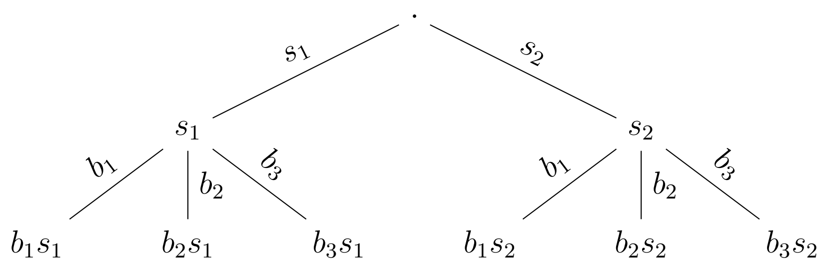
2. Use the multiplication axiom to count possible outcomes in a multi-stop process.

In this chapter, we are trying to develop counting techniques that will be used in the next chapter to study probability. One of the most fundamental of such techniques is called the Multiplication Axiom. Before we introduce the multiplication axiom, we first look at some examples.

Example 7.2.1. *If a woman has two blouses and three skirts, how many different outfits consisting of a blouse and a skirt can she wear?*

Solution 7.2.1. *Suppose we call the blouses b_1 and b_2, and skirts s_1, s_2, and s_3. We can have the following six outfits: b_1s_1, b_1s_2, b_1s_3, b_2s_1, b_2s_2, b_2s_3*

Alternatively, we can draw a tree diagram:



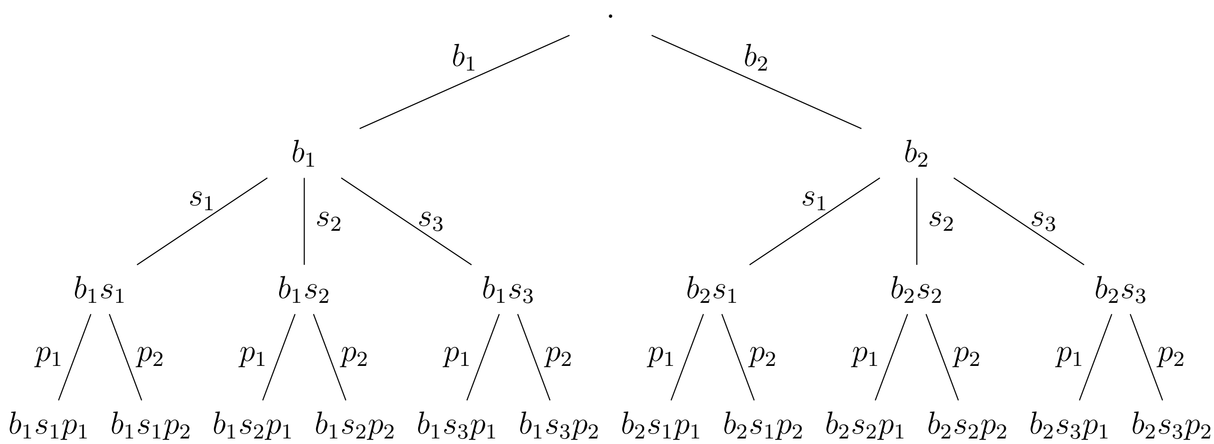
The tree diagram gives us all six possibilities. The method involves two steps. First the woman chooses a blouse. She has two choices: blouse one or blouse two. If she chooses blouse one, she has three skirts to match it with; skirt one, skirt two, or skirt three. Similarly if she chooses blouse two, she can match it with each of the three skirts, again. The tree diagram helps us visualize these possibilities.

The reader should note that the process involves two steps. For the first step of choosing a blouse, there are two choices, and for each choice of a blouse, there are three choices of choosing a skirt. So altogether there are $2 \cdot 3 = 6$ *possibilities.*

If, in the previous example, we add the shoes to the outfit, we have the following problem.

Example 7.2.2. *If a woman has two blouses, three skirts, and two pumps, how many different outfits consisting of a blouse, a skirt, and a pair of pumps can she wear?*

Solution 7.2.2. *Suppose we call the blouses* b_1 *and* b_2*, the skirts* s_1*,* s_2*, and* s_3*, and the pumps* p_1*, and* p_2*. The following tree diagram will illustrate all the possible outfits.*



We count the number of branches in the tree, and see that there are 12 different possibilities.

This time the method involves three steps. First, the woman chooses a blouse. She has two choices: blouse one or blouse two. Now suppose she chooses blouse one. This takes us to step two of the process which consists of choosing a skirt. She has three choices for a skirt, and let us suppose she chooses skirt two. Now that she has chosen a blouse and a skirt, we have moved to the third step of choosing a pair of pumps. Since she has two pairs of pumps, she has two choices for the last step. Let us suppose she chooses pumps two. She has chosen the outfit consisting of blouse one, skirt two, and pumps two, or $b_1s_2p_2$. *By looking at the different branches on the tree, one can easily see the other possibilities.*

The important thing to observe here, again, is that this is a three step process. There are two choices for the first step of choosing a blouse. For each choice of a blouse, there are three choices of choosing a skirt, and for each combination of a blouse and a skirt, there are two choices of selecting a pair of pumps.

All in all, we have $2 \cdot 3 \cdot 2 = 12$ *different possibilities.*

Tree diagrams help us visualize the different possibilities, but they are not practical when the possibilities are numerous. Besides, we are mostly interested in finding the number of elements in the set and not the actual list of all possibilities; once the problem is envisioned, we can solve it without a tree diagram. The two examples we just solved may have given us a clue to do just that.

Let us now try to solve Example 2 without a tree diagram. The problem involves three steps: choosing a blouse, choosing a skirt, and choosing a pair of pumps. The number of ways of choosing each are listed below. By multiplying these three numbers we get 12, which is what we got when we did the problem using a tree diagram.

The number of ways to choose a blouse	The number of ways to choose a skirt	The number of ways to choose pumps
2	3	2

The procedure we just employed is called the multiplication axiom.

Summary 7.2.1: THE MULTIPLICATION AXIOM

If a task can be done in m ways, and a second task can be done in n ways, then the operation involving the first task followed by the second can be performed in $m \cdot n$ ways.

The general multiplication axiom is not limited to just two tasks and can be used for any number of tasks.

Example 7.2.3. *A truck license plate consists of a letter followed by four digits. How many such license plates are possible?*

Solution 7.2.3. *Since there are 26 letters and 10 digits, we have the following choices for each.*

Letter	*Digit*	*Digit*	*Digit*	*Digit*
26	10	10	10	10

Therefore, the number of possible license plates is $26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 260000$.

Example 7.2.4. *In how many different ways can a 3-question true-false test be answered?*

Solution 7.2.4. *Since there are two choices for each question, we have*

Question 1	*Question 2*	*Question 3*
2	2	2

Applying the multiplication axiom, we get $2 \cdot 2 \cdot 2 = 8$ *different ways.*

We list all eight possibilities: TTT, TTF, TFT, TFF, FTT, FTF, FFT, FFF.

The reader should note that the first letter in each possibility is the answer corresponding to the first question, the second letter corresponds to the answer to the second question, and so on. For example, TFF, says that the answer to the first question is given as true, and the answers to the second and third questions false.

Example 7.2.5. *In how many different ways can four people be seated in a row?*

Solution 7.2.5. *Suppose we put four chairs in a row, and proceed to put four people in these seats. There are four choices for the first chair we choose. Once a person sits down in that chair, there are only three choices for the second chair, and so on. We list as shown below.*

4	3	2	1

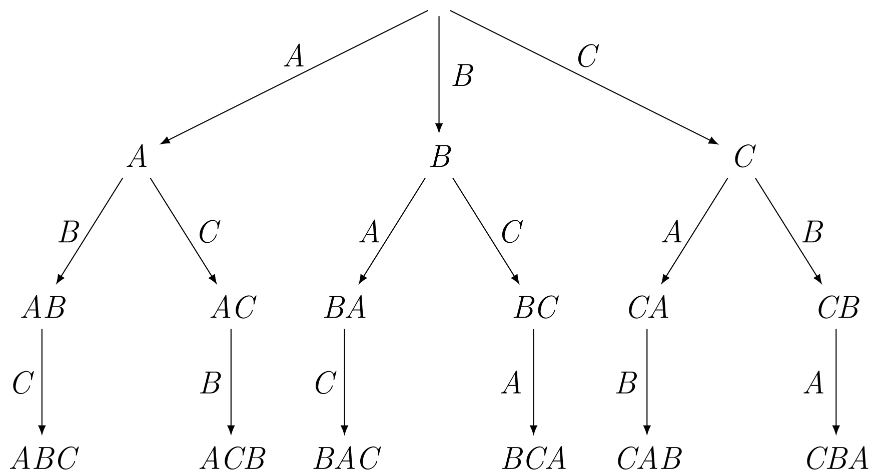
So there are altogether $4 \cdot 3 \cdot 2 \cdot 1 = 24$ *different ways.*

Example 7.2.6. *How many three-letter word sequences can be formed using the letters* $\{A,\ B,\ C\}$ *if no letter is to be repeated?*

Solution 7.2.6. *The problem is very similar to the previous example. Imagine a child having three building blocks labeled A, B, and C. Suppose he puts these blocks on top of each other to make word sequences. For the first letter he has three choices, namely A, B, or C. Let us suppose he chooses the first letter to be a B, then for the second block which must go on top of the first, he has only two choices: A or C. And for the last letter he has only one choice. We list the choices below.*

3	2	1

Therefore, $3 \cdot 2 \cdot 1 = 6$ *different word sequences can be formed.*



7.2.1 Permutations

In this section, you will learn to:

1. Count the number of possible permutations (ordered arrangement) of n items taken r at a time.
2. Count the number of possible permutations when there are conditions imposed on the arrangements.
3. Perform calculations using factorials.

In Example 7.2.6, we were asked to find the word sequences formed by using the letters $\{A, B, C\}$ if no letter is to be repeated. The tree diagram gave us the following six arrangements: ABC, ACB, BAC, BCA, CAB, and CBA.

Arrangements like these, where order is important and no element is repeated, are called permutations.

Definition 7.2.1. *A **permutation** of a set of elements is an ordered arrangement where each element is used once.*

Example 7.2.7. *How many three-letter word sequences can be formed using the letters $\{A,\ B,\ C,\ D\}$?*

Solution 7.2.7. *There are four choices for the first letter of our word, three choices for the second letter, and two choices for the third.*

4	3	2

Applying the multiplication axiom, we get $4 \cdot 3 \cdot 2 = 24$ *different arrangements.*

Example 7.2.8. *How many permutations of the letters of the word ARTICLE have consonants in the first and last positions?*

Solution 7.2.8. *In the word ARTICLE, there are 4 consonants.*

Since the first letter must be a consonant, we have four choices for the first position, and once we use up a consonant, there are only three consonants left for the last spot. We show as follows:

4	?	?	?	?	?	3

Since there are no more restrictions, we can go ahead and make the choices for the rest of the positions.

So far we have used up 2 letters, therefore, five remain. So for the next position there are five choices, for the position after that there are four choices, and so on. We get

4	5	4	3	2	1	3

So the total permutations are $4 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 3 = 1440$.

Example 7.2.9. *Given five letters* $\{A, B, C, D, E\}$. *Find the following:*

- *The number of four-letter word sequences.*
- *The number of three-letter word sequences.*
- *The number of two-letter word sequences.*

Solution 7.2.9. *The problem is easily solved by the multiplication axiom, and answers are as follows:*

- *The number of four-letter word sequences is* $5 \cdot 4 \cdot 3 \cdot 2 = 120$.
- *The number of three-letter word sequences is* $5 \cdot 4 \cdot 3 = 60$.
- *The number of two-letter word sequences is* $5 \cdot 4 = 20$.

We often encounter situations where we have a set of n objects and we are selecting r objects to form permutations. We refer to this as permutations of n objects taken r at a time, and we write it as nPr.

Therefore, the above example can also be answered as listed below:

1. The number of four-letter word sequences is $5P4 = 120$.
2. The number of three-letter word sequences is $5P3 = 60$.
3. The number of two-letter word sequences is $5P2 = 20$.

Before we give a formula for nPr, we'd like to introduce a symbol that we will use a great deal in this as well as in the next chapter.

Definition 7.2.2. *The **factorial** of a natural number n is given by*

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$$

Also, we define $0! = 1$.

Definition 7.2.3. *The number of permutations of n objects taken r at a time (with n and r natural numbers) denoted nPr is given by either*

$$nPr = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1)$$

or

$$nPr = \frac{n!}{(n-r)!}$$

Example 7.2.10. *Compute the following using both formulas.*

1. $6P3$
2. $7P2$

Solution 7.2.10. *We will identify n and r in each case and solve using the formulas provided.*

1. $6P3 = 6 \cdot 5 \cdot 4 = 120$, *or*

$$6P3 = \frac{6!}{(6-3)!} = \frac{6!}{3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 120$$

2. $7P2 = 7 \cdot 6 = 42$, *or*

$$7P2 = \frac{7!}{5!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 42$$

Next we consider some more permutation problems to get further insight into these concepts.

Example 7.2.11. *In how many different ways can 4 people be seated in a straight line if two of them insist on sitting next to each other?*

Solution 7.2.11. *Let us suppose we have four people A, B, C, and D. Further suppose that A and B want to sit together. For the sake of argument, we glue A and B together and treat them as one person* $\boxed{AB}$.

The four people are $\boxed{AB}\ CD$. *Since* $\boxed{AB}$ *is treated as one person, we have the following possible arrangements.*

$$\boxed{AB}\ CD,\ \boxed{AB}\ DC,\ C\ \boxed{AB}\ D,\ D\ \boxed{AB}\ C,\ CD\ \boxed{AB},\ DC\ \boxed{AB}$$

Note that there are six more such permutations because A and B could also be glued together in the order $\boxed{BA}$. *And they are*

$$\boxed{BA}\ CD,\ \boxed{BA}\ DC,\ C\ \boxed{BA}\ D,\ D\ \boxed{BA}\ C,\ CD\ \boxed{BA},\ DC\ \boxed{BA}$$

So altogether there are 12 different permutations.

Let us now do the problem using the multiplication axiom.

After we glue two of the people together and treat them as one person, we can say we have only three people. The multiplication axiom tells us that three people can be seated in $3!$ ways. Since two people can be glued together $2!$ ways, there are $3! \times 2! = 12$ different arrangements

Example 7.2.12. *You have 4 math books and 5 history books to put on a shelf that has 5 slots. In how many ways can the books be shelved if the first three slots are filled with math books and the next two slots are filled with history books?*

Solution 7.2.12. *We first do the problem using the multiplication axiom.*

Since the math books go in the first three slots, there are 4 choices for the first slot, 3 choices for the second and 2 choices for the third.

The fourth slot requires a history book, and has five choices. Once that choice is made, there are 4 history books left, and therefore, 4 choices for the last slot. The choices are shown below.

4	3	2	5	4

Therefore, the number of permutations are $4 \cdot 3 \cdot 2 \cdot 5 \cdot 4 = 480$*.*

Alternately, we can see that $4 \cdot 3 \cdot 2$ *is really same as* $4P3$*, and* $5 \cdot 4$ *is* $5P2$*.*

So the answer can be written as $(4P3)(5P2) = 480$*.*

Clearly, this makes sense. For every permutation of three math books placed in the first three slots, there are $5P2$ *permutations of history books that can be placed in the last two slots. Hence the multiplication axiom applies, and we have the answer* $(4P3)(5P2)$*.*

7.3 Circular Permutations and Permutations with Similar Elements

In this section you will learn to:

1. Count the number of possible permutations of items arranged in a circle.
2. Count the number of possible permutations when there are repeated items.

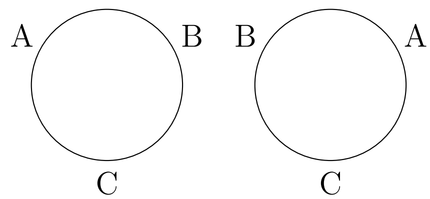
In this section, we will address the following two problems:

1. In how many different ways can five people be seated in a circle?
2. In how many different ways can the letters of the word MISSISSIPPI be arranged?

The first problem comes under the category of Circular Permutations, and the second under Permutations with Similar Elements.

7.3.1 Circular Permutations

Suppose we have three people named A, B, and C. We have already determined that they can be seated in a straight line in 3! or 6 ways. Our next problem is to see how many ways these people can be seated in a circle. We draw a diagram.



It happens that there are only two ways we can seat three people in a circle, relative to each other's positions. This kind of permutation is called a circular permutation. In such cases, no matter where the first person sits, the permutation is not affected. Each person can shift as many places as they like, and the permutation will not be changed. We are interested in the position of each person in relation to the others. Imagine the people on a merry-go-round; the rotation of the permutation does not generate a new permutation. So in circular permutations, the first person is considered a place holder, and where they sit does not matter.

Summary 7.3.1: Circular Permutations

The number of permutations of n elements in a circle is $(n-1)!$.

Another way to look at this is to say that there are $n!$ ways to place the people around the circle, but we have to divide by n because there are n different people to consider the start.

Example 7.3.1. *In how many different ways can five people be seated at a circular table?*

Solution 7.3.1. *We have already determined that the first person is just a place holder. Therefore, there is only one choice for the first spot. We have*

1	4	3	2	1

So the answer is 24.

Example 7.3.2. *At a scholarship dinner, there are four professors and four students. How many ways are there to sit if they want to sit alternately?*

Solution 7.3.2. *We again emphasize that the first person can sit anywhere without affecting the permutation.*

So there is only one choice for the first spot. Suppose a student sat down *first (since they're generally hungrier). The chair to the right must belong to a professor, and there are 4 choices. The next chair belongs to a student, so there are three choices and so on. We list the choices below.*

1	4	3	3	2	2	1	1

So the answer is 144.

7.3.2 Permutations with Similar Elements

Let us determine the number of distinguishable permutations of the letters ELEMENT. Suppose we make all the letters different by labeling the letters as follows.

$E_1LE_2ME_3NT$.

Since all the letters are now different, there are 7! different permutations.

Let us now look at one such permutation, say

$LE_1ME_2NE_3T$

Suppose we form new permutations from this arrangement by only moving the E's. Clearly, there are 3! or 6 such arrangements. We list them below.

$$\begin{gathered} LE_1ME_2NE_3T \\ LE_1ME_3NE_2T \\ LE_2ME_1NE_3T \\ LE_2ME_3NE_1T \\ LE_3ME_1NE_2T \\ LE_3ME_2NE_1T \end{gathered}$$

Because the E's are not different, there is only one arrangement LEMENET and not six. This is true for every permutation.

Let us suppose there are n different permutations of the letters ELEMENT.

Then there are $n \cdot 3!$ permutations of the letters $E_1LE_2ME_3NT$.

But we know there are 7! permutations of the letters $E_1LE_2ME_3NT$.

Therefore, $n \cdot 3! = 7!$

Or $n = \frac{7!}{3!}$.

Summary 7.3.2: Permutations with Similar Elements

The number of permutations of n elements taken n at a time, with r_1 elements of one kind, r_2 elements of another kind, and so on, is given by

$$\frac{n!}{r_1!\, r_2! \, \ldots \, r_k!}$$

Example 7.3.3. *Find the number of different permutations of the letters of the word MISSISSIPPI.*

Solution 7.3.3. *The word MISSISSIPPI has 11 letters. If the letters were all different there would have been* 11! *different permutations. But MISSISSIPPI has 4 S's, 4 I's, and 2 P's that are alike.*

So the answer is

$$\frac{11!}{4!4!2!} = 34,650.$$

Example 7.3.4. *If a coin is tossed six times, how many different outcomes consisting of 4 heads and 2 tails are there?*

Solution 7.3.4. *Again, we have permutations with similar elements.*

We are looking for permutations for the letters HHHHTT.

The answer is

$$\frac{6!}{4!2!} = 15.$$

Example 7.3.5. *In how many different ways can 4 nickels, 3 dimes, and 2 quarters be arranged in a row?*

Solution 7.3.5. *Assuming that all nickels are similar, all dimes are similar, and all quarters are similar, we have permutations with similar elements. Therefore, the answer is*

$$\frac{9!}{4!3!2!} = 1260.$$

Example 7.3.6. *A stock broker wants to assign 20 new clients equally to 4 of its salespeople. In how many different ways can this be done?*

Solution 7.3.6. *This means that each salesperson gets 5 clients. The problem can be thought of as an ordered partitions problem. In that case, using the formula we get*

$$\frac{20!}{5!5!5!5!} = 11,732,745,024.$$

Example 7.3.7. *A college has a straight row of 5 flagpoles at its main entrance. It has 3 identical green flags and 2 identical yellow flags. How many distinct arrangements of flags on the flagpoles are possible?*

Solution 7.3.7. *The problem can be thought of as distinct permutations of the letters GGGYY; that is arrangements of 5 letters, where 3 letters are similar, and the remaining 2 letters are similar:*

$$\frac{5!}{3!2!} = 10$$

Just to provide a little more insight into the solution, we list all 10 distinct permutations: GGGYY, GGYGY, GGYYG, GYGGY, GYGYG, GYYGG, YGGGY, YGGYG, YGYGG, YYGGG

7.4 Combinations

In this section, you will learn to:

1. Count the number of combinations of r out of n items (selections without regard to arrangement).
2. Use factorials to perform calculations involving combinations.

Suppose we have a set of three letters {A, B, C}, and we are asked to make two-letter word sequences. We have the following six permutations:

$$AB \quad BA \quad BC \quad CB \quad AC \quad CA$$

Now suppose we have a group of three people {A, B, C} as Alex, Blake, and Cam, respectively, and we are asked to form committees of two people each.

This time we have only three committees, namely:

$$AB \quad BC \quad AC$$

When forming committees, the order is not important because the committee that has Alex and Blake is no different than the committee that has Blake and Alex. As a result, we have only three committees and not six.

Forming word sequences is an example of permutations, while forming committees is an example of combinations - the topic of this section.

Permutations are those arrangements where order is important, while combinations are those arrangements where order is not significant. From now on, this is how we will tell permutations and combinations apart.

In the above example, there were six permutations, but only three combinations.

Just as the symbol nPr represents the number of permutations of n objects taken r at a time, nCr represents the number of combinations of n objects taken r at a time.

So in the above example, $3P2 = 6$, and $3C2 = 3$.

Our next goal is to determine the relationship between the number of combinations and the number of permutations in a given situation.

In the above example, if we knew that there were three combinations, we could have found the number of permutations by multiplying this number by 2!. That is because each combination consists of two letters, and that makes 2! permutations.

Example 7.4.1. *Given the set of letters $\{A, B, C, D\}$. Write the number of combinations of three letters, and then from these combinations determine the number of permutations.*

Solution 7.4.1. *We have the following four combinations.*

ABC, BCD, CDA, BDA

Since every combination has three letters, there are 3! *permutations for every combination. We list them below.*

$$\begin{array}{cccccc} ABC & ACB & BAC & BCA & CAB & CBA \\ BCD & BDC & CBD & CDB & DBC & DCB \\ CDA & CAD & DCA & DAC & ACD & ADC \\ BDA & BAD & DBA & DAB & ABD & ADB \end{array}$$

The number of permutations are 3! times the number of combinations; that is

$$4P3 = 3! \cdot 4C3$$

or

$$4C3 = \frac{4P3}{3!}$$

In general,

$$nCr = \frac{nPr}{r!}$$

Since

$$nPr = \frac{n!}{(n-r)!}$$

We have,

$$nCr = \frac{n!}{(n-r)! \cdot r!}$$

Summary 7.4.1: Combinations

A combination of a set of elements is an arrangement where each element is used once, and order is not important. The number of combinations of n objects taken r at a time is given by

$$nCr = \frac{n!}{(n-r)! \cdot r!}$$

where n and r are natural numbers.

Example 7.4.2. *Compute:*

1. $5C3$
2. $7C3$

Solution 7.4.2. *We use the above formula.*

1. $5C3 = \frac{5!}{(5-3)!3!} = \frac{5!}{2!3!} = 10$

2. $7C3 = \frac{7!}{(7-3)!3!} = \frac{7!}{4!3!} = 35$

Example 7.4.3. *In how many different ways can a student select to answer five questions from a test that has seven questions, if the order of the selection is not important?*

Solution 7.4.3. *Since the order is not important, it is a combination problem, and the answer is*

$$7C5 = 21.$$

Example 7.4.4. *How many line segments can be drawn by connecting any two of the six points that lie on the circumference of a circle?*

Solution 7.4.4. *Since the line that goes from point A to point B is same as the one that goes from B to A, this is a combination problem. It is a combination of 6 objects taken 2 at a time. Therefore, the answer is*

$$6C2 = \frac{6!}{4!2!} = 15.$$

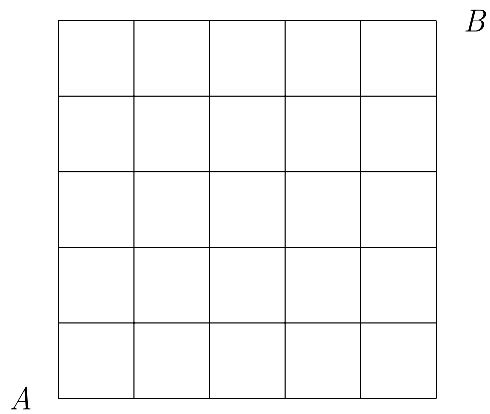
Example 7.4.5. *There are ten people at a party. If they all shake hands, how many hand-shakes are possible?*

Solution 7.4.5. *Note that between any two people there is only one hand shake. Therefore, we have*

$$10C2 = 45 \text{ hand-shakes.}$$

Example 7.4.6. *The shopping area of a town is in the shape of square that is 5 blocks by 5 blocks. How many different routes can a taxi driver take to go from one corner of the shopping area to the opposite cater-corner?*

Solution 7.4.6. *Let us suppose the taxi driver drives from the point A, the lower left hand corner, to the point B, the upper right hand corner as shown in the figure below.*



To reach his destination, he has to travel ten blocks; five horizontal, and five vertical. So if out of the ten blocks he chooses any five horizontal, the other five will have to be the vertical blocks, and vice versa.

Therefore, all he has to do is to choose 5 out of ten to be the horizontal blocks

The answer is $10C5$, *or 252.*

Alternately, the problem can be solved by permutations with similar elements. The taxi driver's route consists of five horizontal and five vertical blocks. If we call a horizontal block H, *and a vertical block* V, *then one possible route may be as follows.*

$$HHHHHVVVVV$$

Clearly there are

$$\frac{10!}{5! \cdot 5!} = 252$$

permutations.

Further note that by definition

$$10C5 = \frac{10!}{5! \cdot 5!}.$$

Example 7.4.7. *If a coin is tossed six times, in how many ways can it fall four heads and two tails?*

Solution 7.4.7. *First, we solve this problem using techniques from 7.3.2 - permutations with similar elements.*

We need 4 heads and 2 tails, that is we need to find all the rearrangements of HHHHTT. Since there are 4 Hs and 2 Ts. The number of permutations is given by

$$\frac{6!}{4!2!} = 15.$$

Now we solve this problem using combinations.

Suppose we have six spots to put the coins on. If we choose any four spots for heads, the other two will automatically be tails. So the problem is simply

$$6C4 = 15.$$

Incidentally, we could have easily chosen the two tails, instead. In that case, we would have gotten

$$6C2 = 15.$$

Further observe that by definition

$$6C4 = \frac{6!}{2!4!} \text{ and } 6C2 = \frac{6!}{4!2!}$$

Note 7.4.1. *It is true that,*

$$nCr = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!(r)!} = nC(n-r)$$

7.5 Combinations Involving Several Sets

In this section, you will learn to:

1. Count the number of items selected from more than one set.
2. Count the number of items selected when there are restrictions on the selections.

So far, we have solved the basic combination problem of r objects chosen from n different objects. Now we will consider certain variations of this problem.

Example 7.5.1. *How many five-person committees consisting of 2 faculty members and 3 students can be chosen from a total of 4 faculty members and 4 students?*

Solution 7.5.1. *We list 4 faculty members and 4 students as follows:*

$$F_1F_2F_3F_4S_1S_2S_3S_4$$

Since we want 5-person committees consisting of 2 faculty members and 3 students, we'll first form all possible two-faculty committees and all possible three-student committees. Clearly there are $4C2 = 6$ two-faculty committees, and $4C3 = 4$ three-student committees, we list them as follows:

2-Faculty Committees

- F_1F_2
- F_1F_3
- F_1F_4
- F_2F_3
- F_2F_4
- F_3F_4

3-Student Committees

- $S_1S_2S_3$
- $S_1S_2S_4$
- $S_1S_3S_4$
- $S_2S_3S_4$

For every 2-faculty committee there are four 3-student committees that can be chosen to make a 5-person committee. If we choose F_1F_2 as our 2-faculty committee, then we can choose any of $S_1S_2S_3$, $S_1S_2S_4$, $S_1S_3S_4$, or $S_2S_3S_4$ as our 3-student committees. As a result, we get

$$\boxed{F_1F_2}S_1S_2S_3,\ \boxed{F_1F_2}S_1S_2S_4,\ \boxed{F_1F_2}S_1S_3S_4,\ \boxed{F_1F_2}S_2S_3S_4$$

Similarly, if we choose F_1F_3 as our 2-faculty committee, then, again, we can choose any of $S_1S_2S_3$, $S_1S_2S_4$, $S_1S_3S_4$, or $S_2S_3S_4$ as our 3-student committees.

$$\boxed{F_1F_3}S_1S_2S_3,\ \boxed{F_1F_3}S_1S_2S_4,\ \boxed{F_1F_3}S_1S_3S_4,\ \boxed{F_1F_3}S_2S_3S_4$$

And so on.

Since there are six 2-faculty committees, and for every 2-faculty committee there are four 3-student committees, there are altogether $6 \cdot 4 = 24$ five-person committees.

In essence, we are applying the multiplication axiom to the different combinations.

Example 7.5.2. *A club consists of 4 freshmen, 5 sophomores, 5 juniors, and 6 seniors. How many ways can a committee of 4 people be chosen that includes:*

1. *One student from each class?*
2. *All juniors?*
3. *Two freshmen and 2 seniors?*
4. *No freshmen?*
5. *At least three seniors?*

Solution 7.5.2.

1. *Applying the multiplication axiom to the combinations involved, we get*

$$(4C1)(5C1)(5C1)(6C1) = 600$$

2. *We are choosing all 4 members from the 5 juniors, and none from the others.*

$$5C4 = 5$$

3. $4C2 \cdot 6C2 = 90$

4. *Since we don't want any freshmen on the committee, we need to choose all members from the remaining 16. That is*

$$16C4 = 1820$$

5. *Of the 4 people on the committee, we want at least three seniors. This can be done in two ways. We could have three seniors, and one non-senior, or all four seniors.*

$$(6C3)(14C1) + 6C4 = 295$$

Example 7.5.3. *How many five-letter word sequences consisting of 2 vowels and 3 consonants can be formed from the letters of the word INTRODUCE?*

Solution 7.5.3. *First we select a group of five letters consisting of 2 vowels and 3 consonants. Since there are 4 vowels and 5 consonants, we have*

$$(4C2)(5C3)$$

Since our next task is to make word sequences out of these letters, we multiply these by $5!$*:*

$$(4C2)(5C3)(5!) = 7200.$$

7.5.1 A Standard Deck of 52 Playing Cards

As in the previous example, many examples and homework problems in this book refer to a standard deck of 52 playing cards. Before we end this section, we take a minute to describe a standard deck of playing cards, as some readers may not be familiar with this.

A standard deck of 52 playing cards has 4 suits with 13 cards in each suit. The suits are diamonds (♢), hearts (♡), spades (♠), and clubs (♣). Each suit is associated with a color, either black (♠, ♣) or red (♢, ♡).

Each suit contains 13 denominations (or values) for cards: the nine numbers 2, 3, 4, ..., 10 and Jack(J), Queen (Q), King (K), Ace (A).

The Jack, Queen and King are called "face cards" because they have pictures on them. Therefore a standard deck has 12 face cards: (3 values JQK) x (4 suits ♡ ♢ ♠ ♣). There are two Jacks in profile (♡, ♠) and two Jacks in full face (♢, ♣), the Jacks in profile are sometimes refered to as the one-eyed Jacks.

We can visualize the 52 cards by the following display

Suit	**Color**	**Values (Denominations)**
♢ Diamonds	Red	2 3 4 5 6 7 8 9 10 J Q K A
♡ Hearts	Red	2 3 4 5 6 7 8 9 10 J Q K A
♠ Spades	Black	2 3 4 5 6 7 8 9 10 J Q K A
♣ Clubs	Black	2 3 4 5 6 7 8 9 10 J Q K A

Example 7.5.4. *A standard deck of playing cards has 52 cards consisting of 4 suits with 13 cards in each. In how many different ways can a 5-card hand consisting of four cards of one suit and one of another be drawn?*

Solution 7.5.4. *We will do the problem using the following steps.*

1. *Select a suit.*
2. *Select four cards from this suit.*
3. *Select another suit.*
4. *Select a card from that suit.*

Applying the multiplication axiom, we have

Ways of selecting the first suit	$4C1$
Ways of selecting 4 cards from this suit	$13C4$
Ways of selecting the next suit	$3C1$
Ways of selecting a card from that suit	$13C1$

$$(4C1) \cdot (13C4) \cdot (3C1) \cdot (13C1) = 11,540.$$

7.6 Binomial Theorem

In this section, you will learn to:

1. find the coefficients of a binomial expansion such as $(x+y)^n$ quickly and accurately

We end this chapter with one more application of combinations. Combinations are used in determining the coefficients of a binomial expansion such as $(x+y)^n$. Expanding a binomial expression by multiplying it out is a very tedious task, and is not practiced. Instead, a formula known as the Binomial Theorem is utilized to determine such an expansion. Before we introduce the Binomial Theorem, however, consider the following expansions.

$$
\begin{aligned}
(x+y)^2 &= x^2 + 2xy + y^2 \\
(x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\
(x+y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \\
(x+y)^5 &= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5 \\
(x+y)^6 &= x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6
\end{aligned}
$$

We make the following observations.

1. There are $n+1$ terms in the expansion $(x+y)^n$.
2. The sum of the powers of x and y is n.
3. The powers of x begin with n and decrease by one with each successive term. The powers of y begin with 0 and increase by one with each successive term.

Suppose we want to expand $(x+y)^3$. We first write the expansion without the coefficients. We temporarily substitute a blank in place of the coefficients.

$$(x+y)^3 = \underline{\quad}x^3 + \underline{\quad}x^2y + \underline{\quad}xy^2 + \underline{\quad}y^3 \tag{I}$$

Our next job is to replace each of the blanks in equation (I) with the corresponding coefficients that belong to this expansion. Clearly,

$$(x+y)^3 = (x+y)(x+y)(x+y)$$

If we multiply the right side and do not collect terms, we get the following.

$$xxx + xxy + xyx + xyy + yxx + yyx + yxy + yyy$$

Each product in the above expansion is the result of multiplying three variables by picking one from each of the factors $(x+y)(x+y)(x+y)$. For example, the product xxy is gotten by choosing x from the first factor, x from the second factor, and y from the third factor. There are three such products that simplify to x^2y, namely xxy, xyx, and yxx. These products take place when we choose an x from two of the factors and choose a y from the other factor. Clearly this can be done in $3C2$, or 3 ways. Therefore, the coefficient of the term x^2y is 3. The coefficients of the other terms are obtained in a similar manner.

We now replace the blanks with the coefficients in equation (I), and we get

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

Example 7.6.1. *Find the coefficient of the term x^2y^5 in the expansion $(x + y)^7$.*

Solution 7.6.1. *The expansion $(x + y)^7$ is $(x + y)(x + y)(x + y)(x + y)(x + y)(x + y)(x + y)$. In multiplying the right side, each product is gotten by picking an x or y from each of the seven factors $(x + y)$. The term x^2y^5 is obtained by choosing an x from two of the factors and a y from the other five factors. This can be done in $7C2$, or 21 ways. Therefore, the coefficient of the term x^2y^5 is 21.*

Example 7.6.2. *Expand $(x + y)^7$.*

Solution 7.6.2. *We first write the expansion without the coefficients.*

$$(x+y)^7 = \_\_\_x^7+\_\_\_x^6y+\_\_\_x^5y^2+\_\_\_x^4y^3+\_\_\_x^3y^4+\_\_\_x^2y^5+\_\_\_xy^6+\_\_\_y^7$$

Now we determine the coefficient of each term as we did in Example 1.

- *The coefficient of the term x^7 is $7C7$ or $7C0$ which equals 1.*
- *The coefficient of the term x^6y is $7C6$ or $7C1$ which equals 7.*
- *The coefficient of the term x^5y^2 is $7C5$ or $7C2$ which equals 21.*
- *The coefficient of the term x^4y^3 is $7C4$ or $7C3$ which equals 35,*
- *and so on.*

Substituting, we get:

$$(x + y)^7 = x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$$

Summary 7.6.1: Binomial Theorem

$$(x+y)^n = \binom{n}{0}x^n+\binom{n}{1}x^{n-1}y+\binom{n}{2}x^{n-2}y^2+\cdots+\binom{n}{n-1}xy^{n-1}+\binom{n}{n}y^n$$

Example 7.6.3. *Expand* $(3a-2b)^4$.

Solution 7.6.3. *If we let* $x = 3a$ *and* $y = -2b$*, and apply the Binomial Theorem, we get*

$$(3a-2b)^4$$

$$= 4C0(3a)^4+4C1(3a)^3(-2b)+4C2(3a)^2(-2b)^2+4C3(3a)(-2b)^3+4C4(-2b)^4$$

$$= 1(81a^4) + 4(27a^3)(-2b) + 6(9a^2)(4b^2) + 4(3a)(-8b^3) + 1(16b^4)$$

$$= 81a^4 - 216a^3b + 216a^2b^2 - 96a^3b + 16b^4$$

Example 7.6.4. *Find the fifth term of the expansion* $(3a-2b)^7$.

Solution 7.6.4. *The Binomial theorem tells us that in the* r*-th term of an expansion, the exponent of the* y *term is always one less than* r*, and the coefficient of the term is* nC_{r-1}*. Thus, for* $n = 7$ *and* $r - 1 = 5 - 1 = 4$*, so the coefficient is* $7C4 = 35$*. Therefore, the fifth term is*

$$(7C4)(3a)^{7-4}(-2b)^4 = 35(27a^3)(16b^4) = 15120a^3b^4$$

Chapter 8

More Probability