Applied Finite Mathematics

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March 29, 2024

Contents

Chapter 1

Linear Equations

In this chapter, you will learn to:

1. Graph a linear equation.
2. Find the slope of a line.
3. Determine an equation of a line.
4. Solve linear systems.
5. Do application problems using linear equations.

1.1 Graphing a Linear Equation

In this section, you will learn to:

1. Graph a line when you know its equation.
2. Graph a line when you are given its equation in parametric form.
3. Graph and find equations of vertical and horizontal lines.

1.1.1 Graphing a Line from its Equation

Equations whose graphs are straight lines are called linear equations. The following are some examples of linear equations:

$$
\begin{aligned}
2x - 3y &= 6, \\
3x &= 4y - 7, \\
y &= 2x - 5, \\
2y &= 3, \\
x - 2 &= 0.
\end{aligned}
$$

A line is completely determined by two points. Therefore, to graph a linear equation, we need to find the coordinates of two points. This can be accomplished by choosing an arbitrary value for x or y and then solving for the other variable.

Example 1.1.1. *Graph the line $y = 3x + 2$.*

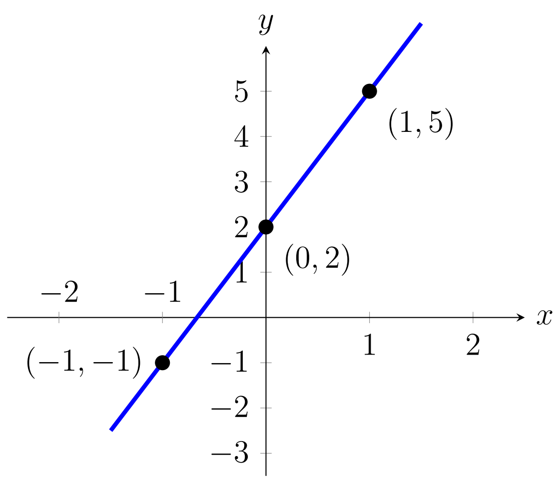
Solution 1.1.1. *We need to find the coordinates of at least two points. We arbitrarily choose $x = -1$, $x = 0$, and $x = 1$.*

If $x = -1$, then $y = 3(-1) + 2$ or $y = -1$. Therefore, $(-1, -1)$ is a point on this line.

If $x = 0$, then $y = 3(0) + 2$ or $y = 2$. Hence the point $(0, 2)$.

If $x = 1$, then $y = 5$, and we get the point $(1, 5)$. Below, the results are summarized, and the line is graphed.

x	y
-1	-1
0	2
1	5



Example 1.1.2. *Graph the line:* $2x + y = 4$

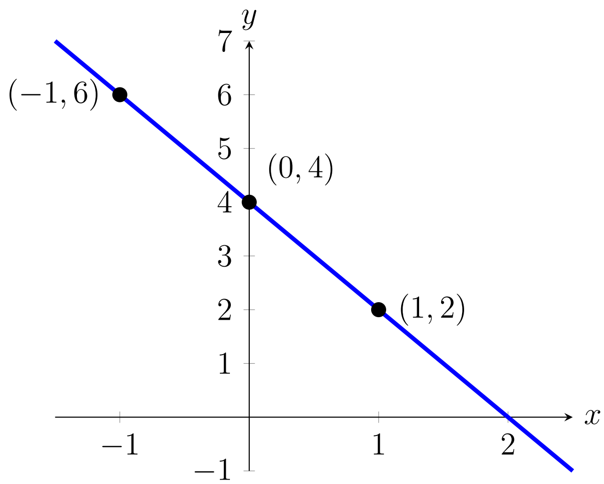
Solution 1.1.2. *Again, we need to find coordinates of at least two points. We arbitrarily choose* $x = -1$, $x = 0$, *and* $y = 2$.

If $x = -1$, *then* $2(-1) + y = 4$ *which results in* $y = 6$. *Therefore,* $(-1, 6)$ *is a point on this line.*

If $x = 0$, *then* $2(0) + y = 4$, *which results in* $y = 4$. *Hence the point* $(0, 4)$.

If $y = 2$, *then* $2x + 2 = 4$, *which yields* $x = 1$, *and gives the point* $(1, 2)$. *The table below shows the points, and the line is graphed.*

x	y
-1	6
0	4
1	2



1.1.2 Intercepts:

The points at which a line crosses the coordinate axes are called the intercepts. When graphing a line by plotting two points, using the intercepts is often preferred because they are easy to find.

- To find the value of the x-intercept, we let $y = 0$.
- To find the value of the y-intercept, we let $x = 0$.

Example 1.1.3. *Find the intercepts of the line: $2x - 3y = 6$, and graph.*

Solution 1.1.3. *To find the x-intercept, let $y = 0$ in the equation, and solve for x.*

$$\begin{aligned} 2x - 3(0) &= 6 \\ 2x &= 6 \\ x &= 3 \end{aligned}$$

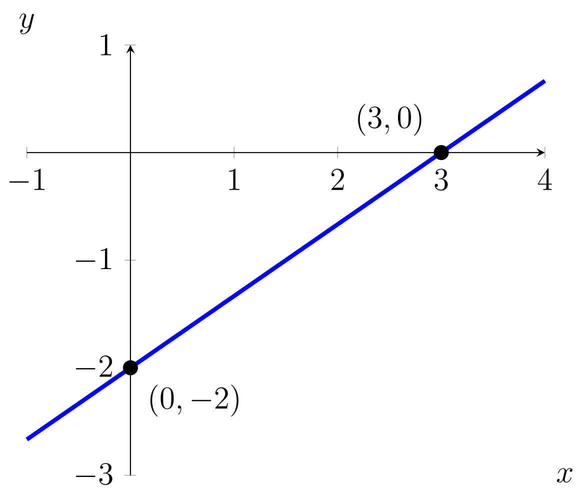
Therefore, the x-intercept is the point $(3, 0)$.

To find the y-intercept, let $x = 0$ in the equation, and solve for y.

$$\begin{aligned} 2(0) - 3y &= 6 \\ 0 - 3y &= 6 \\ -3y &= 6 \\ y &= -2 \end{aligned}$$

Therefore, the y-intercept is the point $(0, -2)$*.*

To graph the line, plot the points for the x-intercept $(3, 0)$ *and the y-intercept* $(0, -2)$*, and use them to draw the line.*



1.1.3 Graphing a Line from Its Equation in Parametric Form

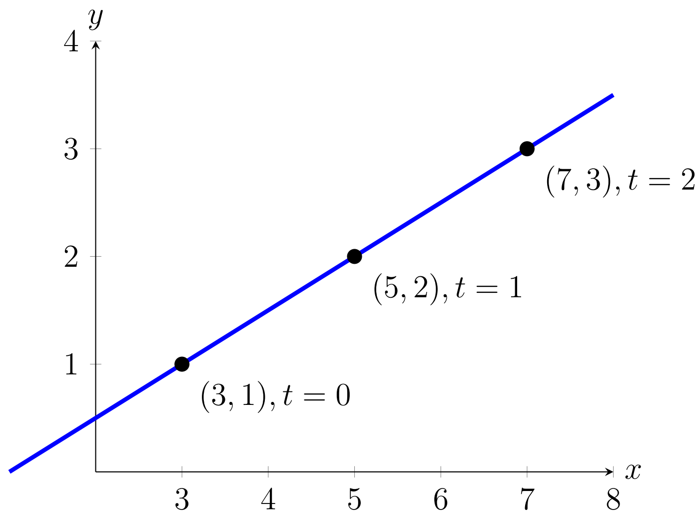
In higher math, equations of lines are sometimes written in parametric form. For example, $x = 3 + 2t, y = 1 + t$. The letter t is called the parameter or the dummy variable.

Parametric lines can be graphed by finding values for x and y by substituting numerical values for t. Plot the points using their (x, y) coordinates and use the points to draw the line.

Example 1.1.4. *Graph the line given by the parametric equations:* $x = 3 + 2t$, $y = 1 + t$

Solution 1.1.4. *Let* $t = 0, 1$ *and* 2*; for each value of* t*, find the corresponding values for* x *and* y*. The results are given in the table below.*

t	x	y
0	3	1
1	5	2
2	7	3



1.1.4 Horizontal and Vertical Lines

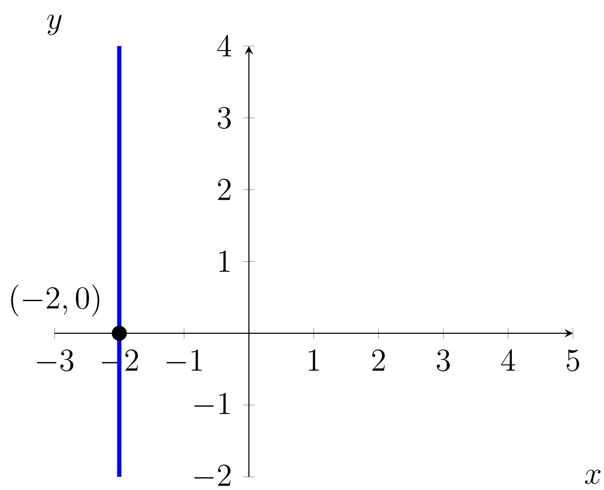
When an equation of a line has only one variable, the resulting graph is a horizontal or a vertical line.

The graph of the line $x = a$, where a is a constant, is a vertical line that passes through the point $(a, 0)$. Every point on this line has the x-coordinate equal to a, regardless of the y-coordinate.

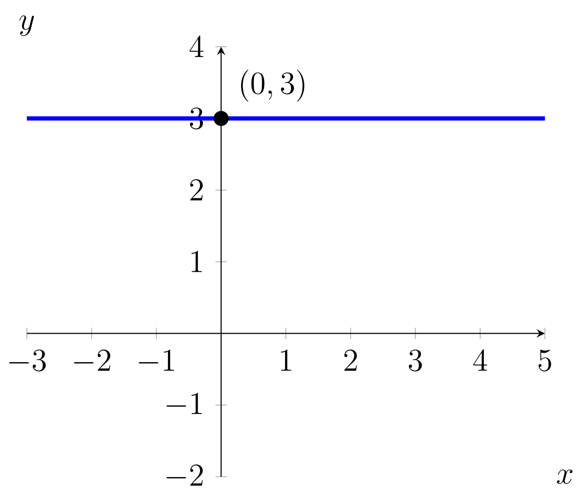
The graph of the line $y = b$, where b is a constant, is a horizontal line that passes through the point $(0, b)$. Every point on this line has the y-coordinate equal to b, regardless of the x-coordinate.

Example 1.1.5. *Graph the lines: $x = -2$, and $y = 3$.*

Solution 1.1.5. *The graph of the line $x = -2$ is a vertical line that has the x-coordinate -2 no matter what the y-coordinate is. The graph is a vertical line passing through point $(-2, 0)$.*



The graph of the line $y = 3$ is a horizontal line that has the y-coordinate 3 regardless of what the x-coordinate is. Therefore, the graph is a horizontal line that passes through point $(0, 3)$.



1.2 Slope of a Line

In this section, you will learn to:

1. Find the slope of a line.
2. Graph the line if a point and the slope are given.

In the last section, we learned to graph a line by choosing two points on the line. A graph of a line can also be determined if one point and the "steepness" of the line is known. The number that refers to the steepness or inclination of a line is called the slope of the line. From previous math courses, many of you remember slope as the "rise over run," or "the vertical change over the horizontal change" and have often seen it expressed as:

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

We give a precise definition.

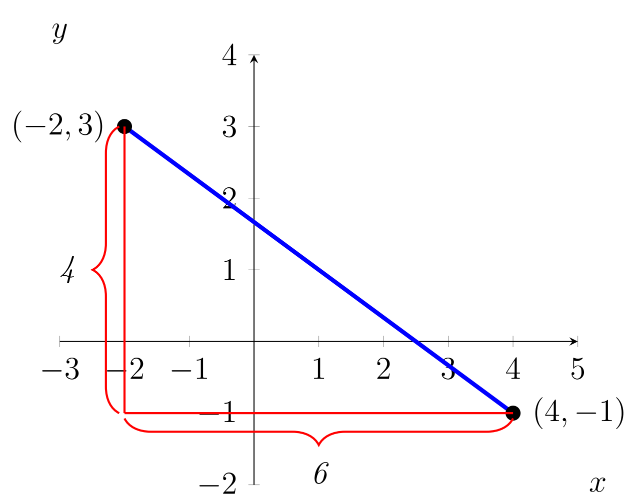
Definition 1.2.1. *If (x_1, y_1) and (x_2, y_2) are two different points on a line, the slope of the line is*

$$slope = m = \frac{y_2 - y_1}{x_2 - x_1}$$

Example 1.2.1. *Find the slope of the line passing through points $(-2, 3)$ and $(4, -1)$, and graph the line.*

Solution 1.2.1. *Let $(x_1, y_1) = (-2, 3)$ and $(x_2, y_2) = (4, -1)$, then the slope is*

$$slope = m = \frac{-1 - 3}{4 - (-2)} = \frac{-4}{6} = -\frac{2}{3}$$



To give the reader a better understanding, both the vertical change, -4, *and the horizontal change,* 6, *are shown in the above figure.*

When two points are given, it does not matter which point is denoted as (x_1, y_1) and which (x_2, y_2). The value for the slope will be the same.

In Example 1.2.1, if we instead choose $(x_1, y_1) = (4, -1)$ and $(x_2, y_2) = (-2, 3)$, then we will get the same value for the slope as we obtained earlier.

The steps involved are as follows:

$$\mathrm{m} = \frac{3 - (-1)}{-2 - 4} = \frac{4}{-6} = -\frac{2}{3}$$

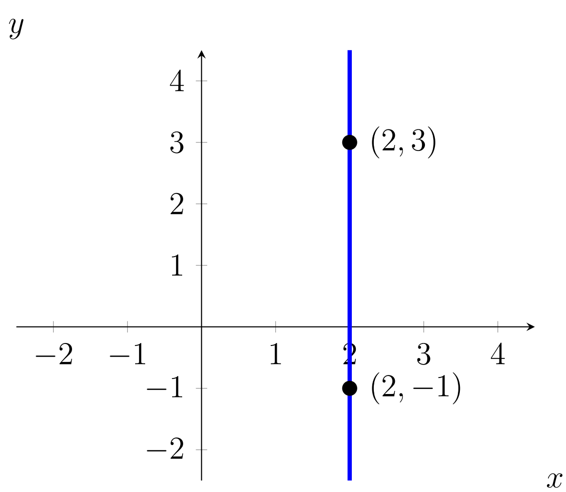
The student should further observe that

- If a line rises when going from left to right, then it has a positive slope. In this situation, as the value of x increases, the value of y also increases.
- If a line falls going from left to right, it has a negative slope; as the value of x increases, the value of y decreases.

Example 1.2.2. *Find the slope of the line that passes through the points* $(2, 3)$ *and* $(2, -1)$*, and graph.*

Solution 1.2.2. *Let* $(x_1, y_1) = (2, 3)$ *and* $(x_2, y_2) = (2, -1)$*, then the slope is*

$$m = \frac{-1 - 3}{2 - 2} = \frac{-4}{0} = \textit{undefined.}$$

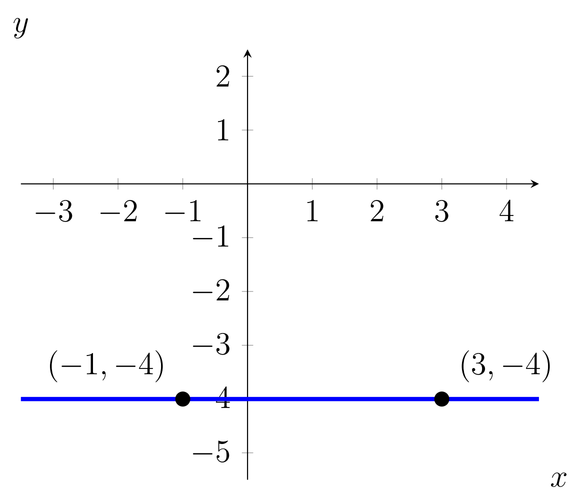


Note 1.2.1. *The slope of a vertical line is undefined.*

Example 1.2.3. *Find the slope of the line that passes through the points* $(-1,-4)$ *and* $(3,-4)$.

Solution 1.2.3. *Let* $(x_1, y_1) = (-1,-4)$ *and* $(x_2, y_2) = (3,-4)$, *then the slope is*

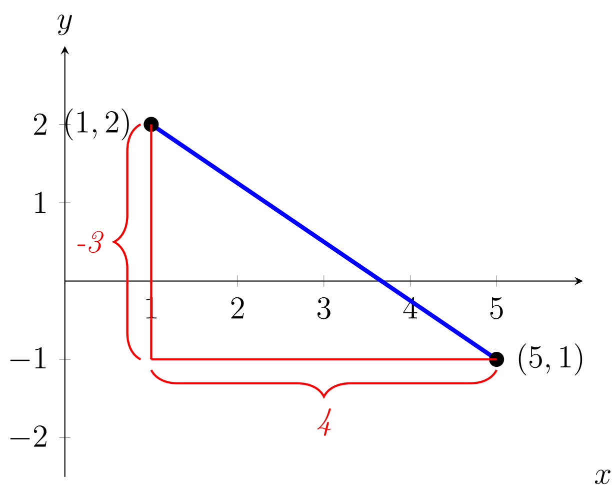
$$m = \frac{-4-(-4)}{3-(-1)} = \frac{0}{4} = 0.$$



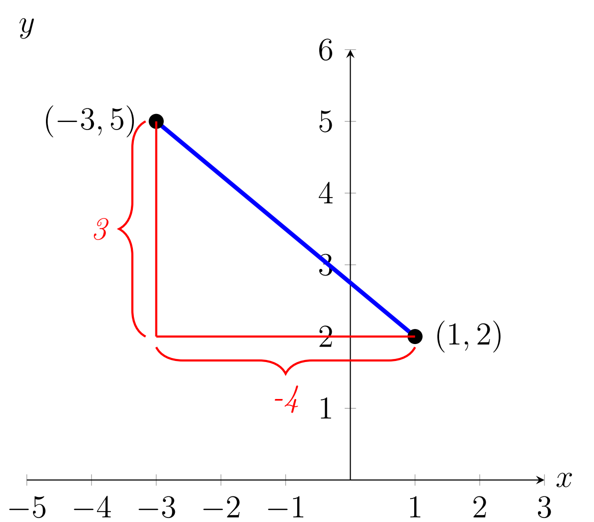
Note 1.2.2. *The slope of a horizontal line is* 0.

Example 1.2.4. *Graph the line that passes through the point* $(1, 2)$ *and has a slope of* $-\frac{3}{4}$.

Solution 1.2.4. *The slope equals* $\frac{rise}{run}$. *The fact that the slope is* $\frac{-3}{4}$ *means that for every rise of* -3 *units (fall of 3 units), there is a run of 4 units. So if from the given point* $(1, 2)$ *we go down 3 units and go right 4 units, we reach the point* $(5,-1)$. *The graph is obtained by connecting these two points.*



Alternatively, since $\frac{3}{-4}$ *represents the same number, the line can be drawn by starting at the point* $(1, 2)$ *and choosing a rise of 3 units followed by a run of* -4 *units. So from the point* $(1, 2)$*, we go up 3 units and to the left 4 units, thus reaching the point* $(-3, 5)$*, which is also on the same line. See figure below.*

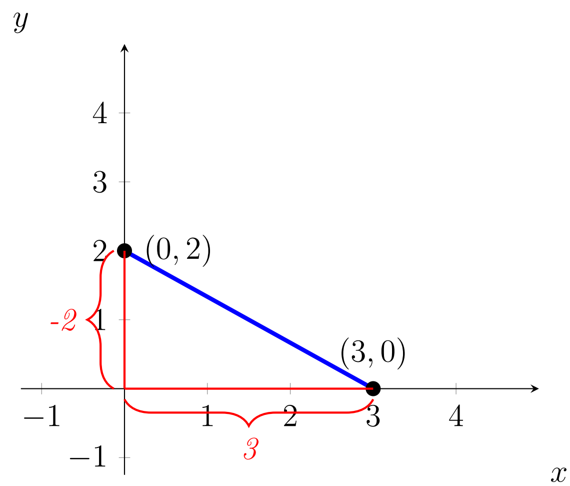


Example 1.2.5. *Find the slope of the line* $2x + 3y = 6$.

Solution 1.2.5. *In order to find the slope of this line, we will choose any two points on this line. Again, the selection of* x *and* y *intercepts seems to be a good choice. The* x*-intercept is* $(3, 0)$*, and the* y*-intercept is* $(0, 2)$*. Therefore,* *the slope is*

$$m = \frac{2-0}{3-0} = \frac{-2}{3}.$$

The graph below shows the line and the x-intercepts and y-intercepts:



Example 1.2.6. *Find the slope of the line $y = 3x + 2$.*

Solution 1.2.6. *We again find two points on the line, say $(0, 2)$ and $(1, 5)$. Therefore, the slope is*

$$m = \frac{5-2}{1-0} = \frac{3}{1} = 3.$$

Look at the slopes and the y-intercepts of the following lines.

Line	Slope	Y-Intercept
$y = 3x + 2$	3	2
$y = -2x + 5$	-2	5
$y = \frac{3}{2}x - 4$	$\frac{3}{2}$	-4

It is no coincidence that when an equation of the line is solved for y, the coefficient of the x term represents the slope, and the constant term represents the y-intercept. In other words, for the line $y = mx + b$, m is the slope, and b is the y-intercept.

Example 1.2.7. *Determine the slope and y-intercept of the line $2x+3y = 6$.*

Solution 1.2.7. *We solve for y:*

$$2x + 3y = 6$$

$$3y = -2x + 6$$

$$y = -\frac{2}{3}x + 2$$

The slope is equal to the coefficient of the x term, which is $-\frac{2}{3}$. The y-intercept is equal to the constant term, which is 2.

1.3 Determining the Equation of a Line

In this section, you will learn to:

1. Find an equation of a line if a point and the slope are given.

2. Find an equation of a line if two points are given.

So far, we were given an equation of a line and were asked to give information about it. For example, we were asked to find points on the line, find its slope, and even find intercepts. Now we are going to reverse the process. That is, we will be given either two points or a point and the slope of a line, and we will be asked to find its equation.

An equation of a line can be written in three forms: the slope-intercept form, the point-slope form, or the standard form. We will discuss each of them in this section.

A line is completely determined by two points or by a point and slope. The information we are given about a particular line will influence which form of the equation is most convenient to use. Once we know any form of the equation of a line, it is easy to re-express the equation in the other forms if needed.

The Slope-Intercept Form of a Line: $y = mx + b$

In the last section, we learned that the equation of a line whose slope $= m$ and y-intercept $= b$ is $y = mx + b$. This is called the slope-intercept form of the line and is the most commonly used form.

Example 1.3.1. *Find an equation of a line whose slope is 5, and y-intercept is 3.*

Solution 1.3.1. *Since the slope is $m = 5$, and the y-intercept is $b = 3$, the equation is $y = 5x + 3$.*

Example 1.3.2. *Find the equation of the line that passes through the point* $(2, 7)$ *and has slope* 3.

Solution 1.3.2. *Since* $m = 3$, *the partial equation is* $y = 3x + b$. *Now* b *can be determined by substituting the point* $(2, 7)$ *in the equation* $y = 3x + b$.

$7 = 3(2) + b$

$b = 1$

Therefore, the equation is $y = 3x + 1$.

Example 1.3.3. *Find an equation of the line that passes through the points* $(-1, 2)$ *and* $(1, 8)$.

Solution 1.3.3. $m = \frac{8-2}{1-(-1)} = \frac{6}{2} = 3$.

So the partial equation is $y = 3x + b$. *We can use either of the two points* $(-1, 2)$ *or* $(1, 8)$ *to find* b. *Substituting* $(-1, 2)$ *gives*

$2 = 3(-1) + b$

$5 = b$

So the equation is $y = 3x + 5$.

Example 1.3.4. *Find an equation of the line that has x-intercept* 3, *and y-intercept* 4.

Solution 1.3.4. *The x-intercept* $= 3$, *and y-intercept* $= 4$ *correspond to the points* $(3, 0)$ *and* $(0, 4)$, *respectively.*

$m = \frac{4-0}{0-3} = \frac{-4}{3}$

We are told the y-intercept is 4; *thus* $b = 4$.

Therefore, the equation is $y = -\frac{4}{3}x + 4$.

The Point-Slope Form of a Line: $y - y1 = m(x - x1)$

The point-slope form is useful when we know two points on the line and want to find the equation of the line.

Let L be a line with slope m, and known to contain a specific point $(x1, y1)$. If (x, y) is any other point on the line L, then the definition of a slope leads us to the point-slope form or point-slope formula.

The slope is $\frac{y-y1}{x-x1} = m$

Multiplying both sides by $(x - x1)$ gives the point-slope form:

$y - y1 = m(x - x1)$

Example 1.3.5. *Find the point-slope form of the equation of a line that has slope* 1.5 *and passes through the point* $(12, 4)$*.*

Solution 1.3.5. *Substituting the point* $(x1, y1) = (12, 4)$ *and* $m = 1.5$ *in the point-slope formula, we get*

$y - 4 = 1.5(x - 12)$

The student may be tempted to simplify this into the slope-intercept form $y = mx + b$*. But since the problem specifically requests point-slope form, we will not simplify it.*

The Standard Form of a Line: $Ax + By = C$

Another useful form of the equation of a line is the standard form.

If we know the equation of a line in point-slope form, $y - y1 = m(x - x1)$, or if we know the equation of the line in slope-intercept form $y = mx + b$, we can simplify the formula to have all terms for the x and y variables on one side of the equation, and the constant on the other side of the equation.

The result is referred to as the standard form of the line: $Ax + By = C$.

Example 1.3.6. *Using the point-slope formula, find the standard form of an equation of the line that passes through the point (2, 3) and has slope* $-\frac{3}{5}$*.*

Solution 1.3.6. *Substituting the point* $(2, 3)$ *and* $m = -\frac{3}{5}$ *in the point-slope formula, we get*

$$y - 3 = -\frac{3}{5}(x - 2).$$

Multiplying both sides by 5 gives us

$$5(y - 3) = -3(x - 2),$$

$$5y - 15 = -3x + 6,$$

$$3x + 5y = 21 \textit{ Standard Form.}$$

Example 1.3.7. *Find the standard form of the line that passes through the points (1, -2) and (4, 0).*

Solution 1.3.7. *First, we find the slope:* $m = \frac{0-(-2)}{4-1} = \frac{2}{3}$.

Then, the point-slope form is: $y - (-2) = \frac{2}{3}(x-1)$.

Multiplying both sides by 3 gives us

$$3(y+2) = 2(x-1),$$

$$3y + 6 = 2x - 2,$$

$$-2x + 3y = -8,$$

$$2x - 3y = 8 \textit{ Standard Form.}$$

Example 1.3.8. *Write the equation* $y = -\frac{2}{3}x + 3$ *in the standard form.*

Solution 1.3.8. *Multiplying both sides of the equation by 3, we get*

$$3y = -2x + 9,$$

$$2x + 3y = 9 \textit{ Standard Form.}$$

Example 1.3.9. *Write the equation* $3x - 4y = 10$ *in the slope-intercept form.*

Solution 1.3.9. *Solving for* y*, we get*

$$-4y = -3x + 10,$$

$$y = \frac{3}{4}x - \frac{5}{2} \textit{ Slope Intercept Form.}$$

Example 1.3.10. *Find the slope of the following lines, by inspection.*

1. $3x - 5y = 10$
2. $2x + 7y = 20$
3. $4x - 3y = 8$

Solution 1.3.10.

1. *For* $3x - 5y = 10$*, we have* $A = 3$ *and* $B = -5$*, therefore,* $m = -\frac{A}{B} = -\frac{3}{-5} = \frac{3}{5}$.
2. *For* $2x + 7y = 20$*, we have* $A = 2$ *and* $B = 7$*, therefore,* $m = -\frac{A}{B} = -\frac{2}{7}$.

3. *For* $4x - 3y = 8$, *we have* $A = 4$ *and* $B = -3$, *therefore,* $m = -\frac{A}{B} = -\frac{4}{-3} = \frac{4}{3}$.

Example 1.3.11. *Find an equation of the line that passes through* $(2, 3)$ *and has slope* $-\frac{4}{5}$.

Solution 1.3.11. *Since the slope of the line is* $-\frac{4}{5}$, *we know that the left side of the equation is* $4x + 5y$, *and the partial equation is going to be*

$$4x + 5y = c.$$

Of course, c *can easily be found by substituting for* x *and* y.

$$4(2) + 5(3) = c,$$

$$8 + 15 = c,$$

$$23 = c.$$

The desired equation is

$$4x + 5y = 23.$$

If you use this method often enough, you can do these problems very quickly. We summarize the forms for equations of a line below:

Summary 1.1. *Equations of Lines*

- *Slope-Intercept form:* $y = mx + b$, *where* m *is the slope and* b *is the* y*-intercept.*
- *Point-Slope form:* $y - y_1 = m(x - x_1)$, *where* m *is the slope and* (x_1, y_1) *is a point on the line.*
- *Standard form:* $Ax + By = C$.
- *Horizontal Line:* $y = b$, *where* b *is the* y*-intercept.*
- *Vertical Line:* $x = a$, *where* a *is the* x*-intercept.*

1.4 Applications

In this section, you will learn to use linear functions to model real-world applications.

Now that we have learned to determine equations of lines, we get to apply these ideas in a variety of real-life situations. Read the problem carefully. Highlight important information. Keep track of which values correspond to the independent variable (x) and which correspond to the dependent variable (y).

Example 1.4.1. *A taxi service charges \$0.50 per mile plus a \$5 flat fee. What will be the cost of traveling 20 miles? What will be cost of traveling x miles?*

Solution 1.4.1. *Let x be the distance traveled, in miles, and y be the cost in dollars.*

The cost of traveling 20 miles is $y = (0.50)(20) + 5 = 10 + 5 = 15$ *dollars.*

The cost of traveling x miles is $y = (0.50)(x) + 5 = 0.50x + 5$ *dollars.*

In this problem, \$0.50 per mile is referred to as the variable cost, and the flat charge \$5 as the fixed cost. Now if we look at our cost equation $y = 0.50x+5$, *we can see that the variable cost corresponds to the slope and the fixed cost to the y-intercept.*

Example 1.4.2. *The variable cost to manufacture a product is \$10 per item and the fixed cost \$2500. If x represents the number of items manufactured and y represents the total cost, write the cost function.*

Solution 1.4.2. *The variable cost of \$10 per item tells us that* $m = 10$. *The fixed cost represents the y-intercept, so* $b = 2500$. *Therefore, the cost equation is* $y = 10x + 2500$.

Example 1.4.3. *It costs \$750 to manufacture 25 items, and \$1000 to manufacture 50 items. Assuming a linear relationship holds, find the cost equation, and use this function to predict the cost of 100 items.*

Solution 1.4.3. *Let x be the number of items manufactured, and let y be the cost.*

Solving this problem is equivalent to finding an equation of a line that passes through the points $(25, 750)$ *and* $(50, 1000)$.

$m = \frac{1000-750}{50-25} = 10$

Therefore, the partial equation is $y = 10x + b$.

By substituting one of the points in the equation, we get $b = 500$.

Therefore, the cost equation is $y = 10x + 500$.

To find the cost of 100 items, substitute $x = 100$ *in the equation* $y = 10x+500$.

So the cost $= y = 10(100) + 500 = 1500$.

It costs $1500 to manufacture 100 items.

Example 1.4.4. *The freezing temperature of water in Celsius is 0 degrees, and in Fahrenheit, it's 32 degrees. The boiling temperatures of water in Celsius and Fahrenheit are 100 degrees and 212 degrees, respectively. Write a conversion equation from Celsius to Fahrenheit and use this equation to convert 30 degrees Celsius into Fahrenheit.*

Solution 1.4.4. *Let's look at what is given:*

Celsius	Fahrenheit
0	32
100	212

Solving this problem is equivalent to finding an equation of a line that passes through the points $(0, 32)$ *and* $(100, 212)$. *Since we are finding a linear relationship, we are looking for an equation* $y = mx + b$, *or in this case,* $F = mC + b$, *where* C *represents the temperature in Celsius, and* F *represents the temperature in Fahrenheit.*

The slope $m = \frac{212-32}{100-0} = 95$.

The equation is $F = 95C + b$.

Substituting the point $(0, 32)$, *we get* $F = 95C + 32$.

To convert 30 degrees Celsius into Fahrenheit, substitute $C = 30$ *in the equation:*

$$F = 95C + 32$$

$$F = 95(30) + 32 = 86$$

Example 1.4.5. *The population of Canada in the year 1980 was 24.5 million, and in the year 2010, it was 34 million. The population of Canada over that time period can be approximately modeled by a linear function. Let* x

represent time as the number of years after 1980, and let y represent the size of the population.

a. Write the linear function that gives a relationship between the time and the population.

b. Assuming the population continues to grow linearly in the future, use this equation to predict the population of Canada in the year 2025.

Solution 1.4.5. *The problem can be made easier by using 1980 as the base year, which means we choose the year 1980 as the year zero. This will make the year 2010 correspond to year 30. Now, let's look at the information we have:*

Year	Population
0 (1980)	24.5 million
30 (2010)	34 million

a. Solving this problem is equivalent to finding an equation of a line that passes through the points (0, 24.5) and (30, 34). We use these two points to find the slope:

$$m = \frac{34 - 24.5}{30 - 0} = \frac{9.5}{30} = 0.32$$

The y-intercept occurs when $x = 0$, *so* $b = 24.5$.

So, the equation relating time (x) and population (y) is:

$$y = 0.32x + 24.5$$

b. Now, to predict the population in the year 2025, we let $x = 2025 - 1980 = 45$:

$$y = 0.32x + 24.5$$

$$y = 0.32(45) + 24.5 = 38.9$$

In the year 2025, we predict that the population of Canada will be 38.9 million people.

Note that we assumed the population trend will continue to be linear. Therefore, if population trends change and this assumption does not continue to be true in the future, this prediction may not be accurate.

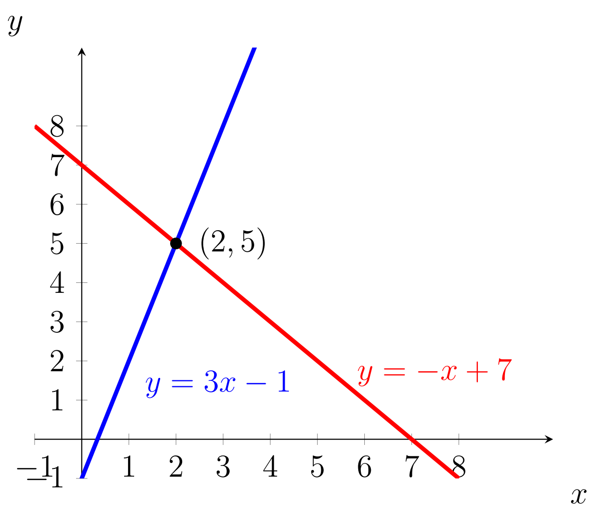
1.5 More Applications

1.5.1 Finding the Point of Intersection of Two Lines

In this section, we will do application problems that involve the intersection of lines. Therefore, before we proceed any further, we will first learn how to find the intersection of two lines.

Example 1.5.1. *Find the intersection of the line $y = 3x - 1$ and the line $y = -x + 7$.*

Solution 1.5.1. *We graph both lines on the same axes, as shown below, and read the solution $(2, 5)$.*



Finding an intersection of two lines graphically is not always easy or practical; therefore, we will now learn to solve these problems algebraically.

At the point where two lines intersect, the x and y values for both lines are the same. So in order to find the intersection, we either let the x-values or the y-values equal.

If we were to solve the above example algebraically, it will be easier to let the

y-values equal. Since $y = 3x - 1$ for the first line, and $y = -x + 7$ for the second line, by letting the y-values equal, we get:

$$\begin{aligned} 3x - 1 &= -x + 7 \\ 4x &= 8 \\ x &= 2 \end{aligned}$$

By substituting $x = 2$ in any of the two equations, we obtain $y = 5$. Hence, the solution is $(2, 5)$.

1.5.2 Solving Systems of Equations: The Elimination Method

A common algebraic method used to solve systems of equations is called the elimination method. The objective is to eliminate one of the two variables by adding the left and right sides of the equations together. Once one variable is eliminated, we have an equation with only one variable that can be solved. Finally, by substituting the value of the variable that has been found in one of the original equations, we can get the value of the other variable.

Example 1.5.2. *Find the intersection of the lines $2x + y = 7$ and $3x - y = 3$ by the elimination method.*

Solution 1.5.2. *We add the left and right sides of the two equations:*

$$\begin{aligned} 2x + y &= 7 \\ 3x - y &= 3 \\ 5x &= 10 \\ x &= 2 \end{aligned}$$

Now we substitute $x = 2$ into any of the two equations and solve for y:

$$\begin{aligned} 2(2) + y &= 7 \\ 4 + y &= 7 \\ y &= 3 \end{aligned}$$

Therefore, the solution is $(2, 3)$*.*

Example 1.5.3. *Solve the system of equations* $x + 2y = 3$ *and* $2x + 3y = 4$ *by the elimination method.*

Solution 1.5.3. *If we add the two equations directly, none of the variables are eliminated. However, the variable* x *can be eliminated by multiplying the first equation by* -2 *and leaving the second equation unchanged:*

$$\begin{aligned} -2x - 4y &= -6 \\ 2x + 3y &= 4 \\ -y &= -2 \\ y &= 2 \end{aligned}$$

Substituting $y = 2$ *into* $x + 2y = 3$*, we get:*

$$\begin{aligned} x + 2(2) &= 3 \\ x + 4 &= 3 \\ x &= -1 \end{aligned}$$

Therefore, the solution is $(-1, 2)$*.*

Example 1.5.4. *Solve the system of equations* $3x - 4y = 5$ *and* $4x - 5y = 6$*.*

Solution 1.5.4. *This time, we multiply the first equation by* -4 *and the second by* 3 *before adding (the choice of numbers is not unique):*

$$\begin{aligned} -12x + 16y &= -20 \\ 12x - 15y &= 18 \\ y &= -2 \end{aligned}$$

By substituting $y = -2$ into any one of the equations, we get:

$$\begin{aligned} 3x - 4(-2) &= 5 \\ 3x + 8 &= 5 \\ 3x &= -3 \\ x &= -1 \end{aligned}$$

Hence, the solution is $(-1, -2)$.

1.5.3 Supply, Demand, and the Equilibrium Market Price

In a free market economy, the supply curve for a commodity is the number of items of a product that can be made available at different prices, and the demand curve is the number of items the consumer will buy at different prices. As the price of a product increases, its demand decreases, and supply increases. On the other hand, as the price decreases, the demand increases, and supply decreases. The equilibrium price is reached when the demand equals the supply.

Example 1.5.5. *The supply curve for a product is given by $y = 3.5x - 14$, and the demand curve for the same product is given by $y = -2.5x + 34$, where x is the price and y is the number of items produced. Find the following:*

1. *How many items will be supplied at a price of \$10?*
2. *How many items will be demanded at a price of \$10?*
3. *Determine the equilibrium price.*
4. *How many items will be produced at the equilibrium price?*

Solution 1.5.5.

1. *To find the number of items supplied at a price of \$10, we substitute $x = 10$ into the supply equation $y = 3.5x - 14$. Therefore, $y = 3.5(10) - 14 = 21$ items will be supplied.*
2. *To find the number of items demanded at a price of \$10, we substitute $x = 10$ into the demand equation $y = -2.5x + 34$. Therefore, $y = -2.5(10) + 34 = 9$ items will be demanded.*

3. *To determine the equilibrium price, we set the supply equal to the demand:*

$$3.5x - 14 = -2.5x + 34$$

Solving for x*:*

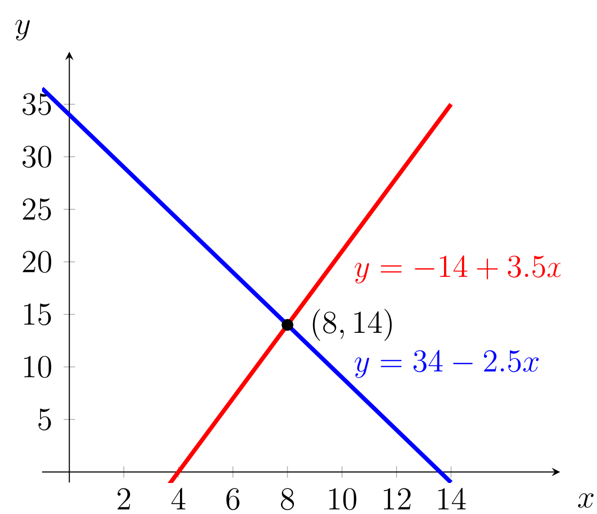
$$6x = 48$$

$$x = 8$$

So, the equilibrium price is $x = 8$*.*

4. *To find how many items will be produced at the equilibrium price, we substitute* $x = 8$ *into either the supply or the demand equation. Using the supply equation, we get* $y = 3.5(8) - 14 = 14$ *items will be produced.*

The graph shows the intersection of the supply and demand functions and their point of intersection, $(8, 14)$*.*



1.5.4 Break-Even Point

In a business, profit is generated by selling products. If a company sells x number of items at a price P, then the revenue R is the price multiplied by the number of items sold: $R = P \cdot x$. The production costs C are the sum of the variable costs and the fixed costs, often written as $C = mx + b$, where x is the number of items manufactured.

- The slope m is called the marginal cost and represents the cost to produce one additional item or unit.
- The variable cost, mx, depends on how much is being produced.
- The fixed cost b is constant and does not change regardless of production quantity.

Profit is equal to revenue minus cost: $Profit = R - C$. A company makes a profit if the revenue is greater than the cost, and there is a loss if the cost is greater than the revenue. The point on the graph where the revenue equals the cost is called the break-even point, and at this point, the profit is 0.

Example 1.5.6. *If the revenue function of a product is $R = 5x$ and the cost function is $C = 3x + 12$, find the following:*

1. *If 4 items are produced, what will the revenue be?*
2. *What is the cost of producing 4 items?*
3. *How many items should be produced to break even?*
4. *What will be the revenue and cost at the break-even point?*

Solution 1.5.6.

1. *To find the revenue when 4 items are produced, we substitute $x = 4$ in the revenue equation $R = 5x$, and the answer is $R = 20$.*
2. *To find the cost of producing 4 items, we substitute $x = 4$ in the cost equation $C = 3x + 12$, and the answer is $C = 24$.*
3. *To determine the number of items required to break even, we set the revenue equal to the cost:*

   $$5x = 3x + 12$$

   Solving for x:

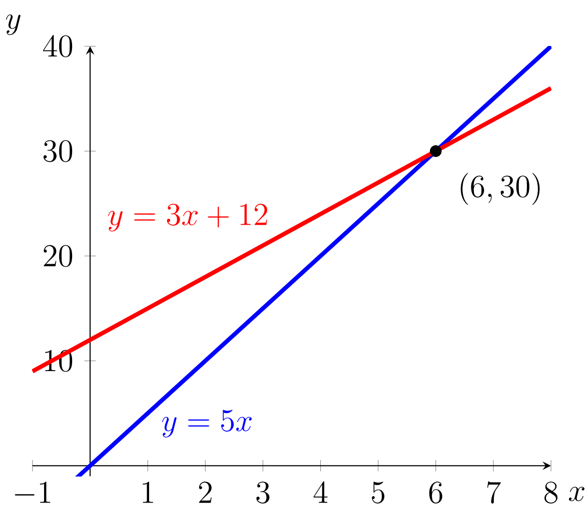
   $$2x = 12$$

   $$x = 6$$

   So, 6 items should be produced to break even.
4. *At the break-even point, when $x = 6$, we can substitute $x = 6$ in either the revenue or the cost equation to find that both revenue and cost are*

equal to 30*. Therefore, the revenue and cost at the break-even point are both* 30*.*

The graph below shows the intersection of the revenue and cost functions and their point of intersection, $(6, 30)$*.*



Chapter 2

Matrices

In this chapter, you will learn to:

1. Do matrix operations.
2. Solve linear systems using the Gauss-Jordan method.
3. Solve linear systems using the matrix inverse method.
4. Do application problems

2.1 Introduction to Matrices

In this section, you will learn to:

1. Add and subtract matrices.
2. Multiply a matrix by a scalar.
3. Multiply two matrices.

A matrix is a 2-dimensional array of numbers arranged in rows and columns. Matrices provide a method of organizing, storing, and working with mathematical information. They have numerous applications and uses in the real world.

(TODO: fix references here)Matrices are particularly useful in working with models based on systems of linear equations, which we'll explore in sections

2.2, 2.3, and 2.4 of this chapter. They are also used in encryption (section 2.5) and economic modeling (section 2.6).

Furthermore, matrices play a crucial role in optimization problems in (TODO: fix references here) Chapter 4, such as maximizing profit or revenue and minimizing costs. They are used in business for scheduling, routing transportation and shipments, and managing inventory. Matrices are applicable in various fields where data organization and problem-solving are essential.

The use of matrices has expanded with the increase in available data across different domains. They are fundamental tools for organizing data and solving problems in science fields like physics, chemistry, biology, genetics, meteorology, and economics. In computer science, matrix mathematics is foundational for animation in movies and video games.

Moreover, matrices are used in analyzing network diagrams, such as social media connections on platforms like Facebook, LinkedIn, etc. The mathematics of network diagrams falls under "graph theory" and relies on matrices to organize information in graphs that depict connections and associations in a network.

A matrix is a rectangular array of numbers. Matrices are useful in organizing and manipulating large amounts of data. In order to get some idea of what matrices are all about, we will look at the following example.

Example 2.1.1. *Fine Furniture Company makes chairs and tables at its San Jose, Hayward, and Oakland factories. The total production, in hundreds, from the three factories for the years 2014 and 2015 is listed in the table below.*

	2014	*2015*		
	CHAIRS	*TABLES*	*CHAIRS*	*TABLES*
SAN JOSE	30	18	36	20
HAYWARD	20	12	24	18
OAKLAND	16	10	20	12

1. *Represent the production for the years 2014 and 2015 as the matrices A and B.*

2. *Find the difference in sales between the years 2014 and 2015.*

3. *The company predicts that in the year 2020 the production at these factories will be double that of the year 2014. What will the production be for the year 2020?*

Solution 2.1.1. 1. *The matrices are as follows:*

$$A = \begin{bmatrix} 30 & 18 \\ 20 & 12 \\ 16 & 10 \end{bmatrix} \qquad B = \begin{bmatrix} 36 & 20 \\ 24 & 18 \\ 20 & 12 \end{bmatrix}$$

2. *We are looking for the matrix $B-A$. When two matrices have the same number of rows and columns, they can be added or subtracted entry by entry. Therefore, we get:*

$$B - A = \begin{bmatrix} 36-30 & 20-18 \\ 24-20 & 18-12 \\ 20-16 & 12-10 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 4 & 6 \\ 4 & 2 \end{bmatrix}$$

3. *We would like a matrix that is twice the matrix of 2014, i.e., $2A$. Whenever a matrix is multiplied by a number, each entry is multiplied by the number.*

$$2A = 2\begin{bmatrix} 30 & 18 \\ 20 & 12 \\ 16 & 10 \end{bmatrix} = \begin{bmatrix} 60 & 36 \\ 40 & 24 \\ 32 & 20 \end{bmatrix}$$

2.1.1 Vocabulary

Before we go any further, we need to familiarize ourselves with some terms that are associated with matrices.

The numbers in a matrix are called the entries or the elements of a matrix.

Whenever we talk about a matrix, we need to know its size or dimension. The dimension of a matrix is the number of rows and columns it has. When we say a matrix is a "3 by 4 matrix," we are saying that it has 3 rows and 4 columns. The rows are always mentioned first, and the columns second. This means that a 3×4 matrix does not have the same dimension as a 4×3 matrix.

$$A = \begin{bmatrix} 1 & 4 & -2 & 0 \\ 3 & -1 & 7 & 9 \\ 6 & 2 & 0 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 9 & 8 \\ -3 & 0 & 1 \\ 6 & 5 & -2 \\ -4 & 7 & 8 \end{bmatrix}$$

Matrix A has dimensions 3×4 — Matrix B has dimensions 4×3

A matrix that has the same number of rows as columns is called a square matrix. A matrix with all entries zero is called a zero matrix. A square matrix with 1's along the main diagonal and zeros everywhere else, is called an identity matrix. When a square matrix is multiplied by an identity matrix of same size, the matrix remains the same.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix I is a 3×3 identity matrix

A matrix with only one row is called a row matrix or a row vector, and a matrix with only one column is called a column matrix or a column vector. Two matrices are equal if they have the same size and the corresponding entries are equal. We can perform arithmetic operations with matrices. Next we will define and give examples illustrating the operations of matrix addition and subtraction, scalar multiplication, and matrix multiplication. Note that matrix multiplication is quite different from what you would intuitively expect, so pay careful attention to the explanation. Note also that the ability to perform matrix operations depends on the matrices involved being compatible in size, or dimensions, for that operation. The definition of compatible dimensions is different for different operations, so note the requirements carefully for each.

2.1.2 Matrix Addition and Subtraction

If two matrices have the same size, they can be added or subtracted. The operations are performed on corresponding entries.

Example 2.1.2. *Given the matrices A, B, C, and D below:*

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 1 \\ 5 & 0 & 3 \end{bmatrix}, \ B = \begin{bmatrix} 2 & -1 & 3 \\ 2 & 4 & 2 \\ 3 & 6 & 1 \end{bmatrix}, \ C = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}, \ D = \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix}$$

Find, if possible:

1. $A + B$
2. $C - D$
3. $A + D$

Solution 2.1.2.

- *We add each element of A to the corresponding entry of B:*

$$A + B = \begin{bmatrix} 3 & 1 & 7 \\ 4 & 7 & 3 \\ 8 & 6 & 4 \end{bmatrix}$$

- *We perform the subtraction entry by entry for $C - D$:*

$$C - D = \begin{bmatrix} 6 \\ 5 \\ -1 \end{bmatrix}$$

- *The sum $A+D$ cannot be found because the two matrices have different sizes. Two matrices can only be added or subtracted if they have the same dimension.*

2.1.3 Multiplying a Matrix by a Scalar

If a matrix is multiplied by a scalar, each entry is multiplied by that scalar.

Example 2.1.3. *Given the matrix A and C in the previous example, find $2A$ and $-3C$.*

Solution 2.1.3.

- *To find $2A$, we multiply each entry of matrix A by 2:*

$$2A = \begin{bmatrix} 2 & 4 & 8 \\ 4 & 6 & 2 \\ 10 & 0 & 6 \end{bmatrix}$$

- *To find $-3C$, we multiply each entry of C by -3:*

$$-3C = \begin{bmatrix} -12 \\ -6 \\ -9 \end{bmatrix}$$

2.1.4 Multiplication of Two Matrices

To multiply a matrix by another is not as easy as the addition, subtraction, or scalar multiplication of matrices. Because of its wide use in application problems, it is important that we learn it well. Therefore, we will try to learn the process in a step by step manner.

Example 2.1.4. *Given* $A = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$ *and* $B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, *find the product* AB.

Solution 2.1.4. *The product is a 1×1 matrix whose entry is obtained by multiplying the corresponding entries and then forming the sum:*

$$AB = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 2a + 3b + 4c$$

Note that AB is a 1×1 matrix, and its only entry is $2a + 3b + 4c$.

Example 2.1.5. *Given* $A = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$ *and* $B = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$, *find the product* AB.

Solution 2.1.5. *Again, we multiply the corresponding entries and add:*

$$AB = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} = (2 \cdot 5) + (3 \cdot 6) + (4 \cdot 7) = 10 + 18 + 28 = 56$$

Example 2.1.6. *Given* $A = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$ *and* $B = \begin{bmatrix} 5 & 3 \\ 6 & 4 \\ 7 & 5 \end{bmatrix}$, *find the product* AB.

Solution 2.1.6. *We know how to multiply a row matrix by a column matrix. To find the product AB, in this example, we will multiply the row matrix A to both the first and second columns of matrix B, resulting in a 1×2 matrix:*

$$AB = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 6 & 4 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} (2 \cdot 5) + (3 \cdot 6) + (4 \cdot 7) & (2 \cdot 3) + (3 \cdot 4) + (4 \cdot 5) \end{bmatrix} = \begin{bmatrix} 56 & 38 \end{bmatrix}$$

We multiplied a 1×3 matrix by a matrix whose size is 3×2. So unlike addition and subtraction, it is possible to multiply two matrices with different dimensions if the number of entries in the rows of the first matrix is the same as the number of entries in the columns of the second matrix.

Example 2.1.7. *Given* $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}$ *and* $B = \begin{bmatrix} 5 & 3 \\ 6 & 4 \\ 7 & 5 \end{bmatrix}$, *find the product AB.*

Solution 2.1.7. *This time we are multiplying two rows of matrix A with two columns of matrix B. Since the number of entries in each row of A is the same as the number of entries in each column of B, the product is possible. We do exactly what we did in the last example. The only difference is that matrix A has one more row.*

We multiply the first row of matrix A with the two columns of B, one at a time, and then repeat the process with the second row of A. We get:

$$AB = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 6 & 4 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} (2 \cdot 5 + 3 \cdot 6 + 4 \cdot 7) & (2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5) \\ (1 \cdot 5 + 2 \cdot 6 + 3 \cdot 7) & (1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5) \end{bmatrix}$$

$$AB = \begin{bmatrix} 56 & 38 \\ 38 & 26 \end{bmatrix}$$

Example 2.1.8. *Given matrices* $E = \begin{bmatrix} 1 & 2 \\ 4 & 2 \\ 3 & 1 \end{bmatrix}$, $F = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$, $G = \begin{bmatrix} 4 & 1 \end{bmatrix}$, *and* $H = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$, *find the following products if possible:*

1. EF

2. *FE*
3. *FH*
4. *GH*
5. *HG*

Solution 2.1.8. 1. *To find EF, we multiply the rows of E with the columns of F. The result is:*

$$EF = \begin{bmatrix} 1 & 2 \\ 4 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} (1 \cdot 2 + 2 \cdot 3) & (1 \cdot -1 + 2 \cdot 2) \\ (4 \cdot 2 + 2 \cdot 3) & (4 \cdot -1 + 2 \cdot 2) \\ (3 \cdot 2 + 1 \cdot 3) & (3 \cdot -1 + 1 \cdot 2) \end{bmatrix} = \begin{bmatrix} 8 & 3 \\ 14 & 0 \\ 9 & -1 \end{bmatrix}$$

2. *Product FE is not possible because F has two entries in each row, while E has three entries in each column.*

3. $FH = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} (2 \cdot -3 + -1 \cdot -1) \\ (3 \cdot -3 + 2 \cdot -1) \end{bmatrix} = \begin{bmatrix} -5 \\ -11 \end{bmatrix}$

4. $GH = \begin{bmatrix} 4 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \end{bmatrix} = (4 \cdot -3 + 1 \cdot -1) = -13$

5. $HG = \begin{bmatrix} -3 \\ -1 \end{bmatrix} \begin{bmatrix} 4 & 1 \end{bmatrix} = \begin{bmatrix} (-3 \cdot 4 & -3 \cdot 1) \\ (-1 \cdot 4 & -1 \cdot 1) \end{bmatrix} = \begin{bmatrix} -12 & -3 \\ -4 & -1 \end{bmatrix}$

We summarize some important properties of matrix multiplication that we observed in the previous examples.

- For the product AB to exist, the number of columns of matrix A must equal the number of rows of matrix B.
- If matrix A has dimensions $m \times n$ and matrix B has dimensions $n \times p$, then the product AB will have dimensions $m \times p$.
- Matrix multiplication is not commutative; that is, in general, AB does not equal BA.

Example 2.1.9. *Given matrices* $R = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 5 \\ 2 & 3 & 1 \end{bmatrix}$, $S = \begin{bmatrix} 0 & -1 & 2 \\ 3 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$, *and* $T = \begin{bmatrix} -2 & 3 & 0 \\ -3 & 2 & 2 \\ -1 & 1 & 0 \end{bmatrix}$, *find* $2RS - 3ST$.

Solution 2.1.9. *Solution: To find $2RS-3ST$, we first compute the products RS and ST:*

$$RS = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 5 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 2 \\ 3 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (1\cdot 0+0\cdot 3+2\cdot 4) & (1\cdot -1+0\cdot 1+2\cdot 2) & (1\cdot 2+0\cdot 0+2\cdot 1) \\ (2\cdot 0+1\cdot 3+5\cdot 4) & (2\cdot -1+1\cdot 1+5\cdot 2) & (2\cdot 2+1\cdot 0+5\cdot 1) \\ (2\cdot 0+3\cdot 3+1\cdot 4) & (2\cdot -1+3\cdot 1+1\cdot 2) & (2\cdot 2+3\cdot 0+1\cdot 1) \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 3 & 4 \\ 23 & 9 & 9 \\ 13 & 3 & 5 \end{bmatrix}$$

Next, we compute ST:

$$ST = \begin{bmatrix} 0 & -1 & 2 \\ 3 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 3 & 0 \\ -3 & 2 & 2 \\ -1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (0\cdot -2+-1\cdot -3+2\cdot -1) & (0\cdot 3+-1\cdot 2+2\cdot 1) & (0\cdot 0+-1\cdot 2+2\cdot 0) \\ (3\cdot -2+1\cdot -3+0\cdot -1) & (3\cdot 3+1\cdot 2+0\cdot 1) & (3\cdot 0+1\cdot 1+0\cdot 0) \\ (4\cdot -2+2\cdot -3+1\cdot -1) & (4\cdot 3+2\cdot 2+1\cdot 1) & (4\cdot 0+2\cdot 1+1\cdot 0) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -2 \\ -9 & 11 & 2 \\ -15 & 17 & 4 \end{bmatrix}$$

Now we can find $2RS-3ST$:

$$2RS - 3ST = 2\cdot \begin{bmatrix} 8 & 3 & 4 \\ 23 & 9 & 9 \\ 13 & 3 & 5 \end{bmatrix} - 3\cdot \begin{bmatrix} 1 & 0 & -2 \\ -9 & 11 & 2 \\ -15 & 17 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 6 & 8 \\ 46 & 18 & 18 \\ 26 & 6 & 10 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 6 \\ -27 & 33 & 6 \\ -45 & 51 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 6 & 14 \\ 73 & -15 & 12 \\ 71 & -45 & -2 \end{bmatrix}$$

The result of $2RS - 3ST$ is a matrix with dimensions 3×3.

Example 2.1.10. *Given matrix $F = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$, find F^2.*

Solution 2.1.10. *F^2 is found by multiplying matrix F by itself, using matrix multiplication.*

$$F^2 = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 2 + (-1) \cdot 3 & 2 \cdot (-1) + (-1) \cdot 2 \\ 3 \cdot 2 + 2 \cdot 3 & 3 \cdot (-1) + 2 \cdot 2 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 12 & 1 \end{bmatrix}$$

Note that F^2 is not found by squaring each entry of matrix F. The process of raising a matrix to a power, such as finding F^2, is only possible if the matrix is a square matrix.

2.1.5 Systems of Linear Equations

Using matrices to represent a system of linear equations is a powerful technique that allows for efficient solving of such systems. In this method, we define matrices as follows:

- Matrix A represents the coefficients of the variables in the system and is called the coefficient matrix.
- Matrix X is a column matrix that contains the variables of the system.
- Matrix B is a column matrix that contains the constants of the system.

By defining these matrices, we can represent a system of linear equations as the matrix equation $AX = B$, where A, X, and B are matrices. This representation simplifies the process of solving linear systems and allows us to apply matrix operations to find the solution.

In the next sections, we will delve deeper into how to use matrices to solve linear systems and explore various methods and techniques for efficient computation and analysis. Matrix representation is widely used in mathematical modeling, engineering, economics, and various other fields where systems of linear equations arise.

Example 2.1.11. *Verify that the system of two linear equations with two unknowns:*

$$\begin{aligned} ax + by &= h \\ cx + dy &= k \end{aligned}$$

can be written as $AX = B$, where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad B = \begin{bmatrix} h \\ k \end{bmatrix}.$$

Solution 2.1.11. *If we multiply the matrices A and X, we get*

$$AX = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}.$$

If $AX = B$, then

$$\begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} = \begin{bmatrix} h \\ k \end{bmatrix}.$$

If two matrices are equal, then their corresponding entries are equal. It follows that

$$\begin{aligned} ax + by &= h \\ cx + dy &= k \end{aligned}$$

Example 2.1.12. *Express the following system as a matrix equation in the form $AX = B$.*

$$\begin{aligned} 2x + 3y - 4z &= 5 \\ 3x + 4y - 5z &= 6 \\ 5x - 6z &= 7 \end{aligned}$$

Solution 2.1.12. *This system of equations can be expressed in the form $AX = B$ as shown below.*

$$\begin{bmatrix} 2 & 3 & -4 \\ 3 & 4 & -5 \\ 5 & 0 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$$

2.2 Systems of Linear Equations; Gauss-Jordan Method

In this section you will learn to

1. Represent a system of linear equations as an augmented matrix
2. Solve the system using elementary row operations.

In this section, we learn to solve systems of linear equations using a process called the Gauss-Jordan method. The process begins by first expressing the system as a matrix, and then reducing it to an equivalent system by simple row operations. The process is continued until the solution is obvious from the matrix. The matrix that represents the system is called the augmented matrix, and the arithmetic manipulation that is used to move from a system to a reduced equivalent system is called a row operation.

Example 2.2.1. *Write the following system as an augmented matrix.*

$$\begin{aligned} 2x + 3y - 4z &= 5 \\ 3x + 4y - 5z &= -6 \\ 4x + 5y - 6z &= 7 \end{aligned}$$

Solution 2.2.1. *We express the above information in matrix form. Since a system is entirely determined by its coefficient matrix and by its matrix of constant terms, the augmented matrix will include only the coefficient matrix and the constant matrix. So the augmented matrix we get is as follows:*

$$\left[\begin{array}{ccc|c} 2 & 3 & -4 & 5 \\ 3 & 4 & -5 & -6 \\ 4 & 5 & -6 & 7 \end{array}\right]$$

In the last section, we expressed the system of equations as $AX = B$, where A represented the coefficient matrix, and B the matrix of constant terms. As an augmented matrix, we write the matrix as $[A|B]$. It is clear that all of the information is maintained in this matrix form, and only the letters x, y, and z are missing. A student may choose to write x, y, and z on top of the first three columns to help ease the transition.

Example 2.2.2. *For the following augmented matrix, write the system of equations it represents.*

$$\left[\begin{array}{ccc|c} 1 & 3 & -5 & 2 \\ 2 & 0 & -3 & -5 \\ 3 & 2 & -3 & -1 \end{array}\right]$$

Solution 2.2.2. *The system is readily obtained as below.*

$$\begin{aligned} x + 3y - 5z &= 2 \\ 2x - 3z &= -5 \\ 3x + 2y - 3z &= -1 \end{aligned}$$

Once a system is expressed as an augmented matrix, the Gauss-Jordan method reduces the system into a series of equivalent systems by using the row operations. This row reduction continues until the system is expressed in what is called the reduced row echelon form. The reduced row echelon form of the coefficient matrix has 1's along the main diagonal and zeros elsewhere. The solution is readily obtained from this form.

The method is not much different form the algebraic operations we employed in the elimination method in the first chapter. The basic difference is that it is algorithmic in nature, and, therefore, can easily be programmed on a computer.

We will next solve a system of two equations with two unknowns, using the elimination method, and then show that the method is analogous to the Gauss-Jordan method.

Example 2.2.3. *Solve the following system by the elimination method.*

$$\begin{aligned} x + 3y &= 7 \\ 3x + 4y &= 11 \end{aligned}$$

Solution 2.2.3. *We multiply the first equation by -3 and add it to the second equation.*

$$\begin{aligned} -3x - 9y &= -21 \\ 3x + 4y &= 11 \end{aligned}$$

This transforms our original system into an equivalent system:

$$\begin{aligned} x + 3y &= 7 \\ -5y &= -10 \end{aligned}$$

Dividing the second equation by -5, we get the next equivalent system.

$$\begin{aligned} x + 3y &= 7 \\ y &= 2 \end{aligned}$$

Multiplying the second equation by -3 and adding it to the first, we get

$$\begin{aligned} x &= 1 \\ y &= 2 \end{aligned}$$

Example 2.2.4. *Solve the following system from Example 3 by the Gauss-Jordan method, and show the similarities in both methods by writing the equations next to the matrices.*

$$\begin{aligned} x + 3y &= 7 \\ 3x + 4y &= 11 \end{aligned}$$

Solution 2.2.4. *The augmented matrix for the system is as follows.*

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 3 & 4 & 11 \end{array}\right] \quad \Leftrightarrow \quad \begin{aligned} x + 3y &= 7 \\ 3x + 4y &= 11 \end{aligned}$$

We multiply the first row by -3 and add it to the second row.

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & -5 & -10 \end{array}\right] \quad \Leftrightarrow \quad \begin{aligned} x + 3y &= 7 \\ -5y &= -10 \end{aligned}$$

Dividing the second row by -5, we get,

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & 1 & 2 \end{array}\right] \quad \Leftrightarrow \quad \begin{aligned} x + 3y &= 7 \\ y &= 2 \end{aligned}$$

Finally, we multiply the second row by -3 and add to the first row, and we get,

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right] \quad \Leftrightarrow \quad \begin{aligned} x &= 1 \\ y &= 2 \end{aligned}$$

2.2.1 Row Operations in Gauss-Jordan Method

The Gauss-Jordan method employs three fundamental row operations:

1. Any two rows in the augmented matrix may be interchanged.
2. Any row may be multiplied by a non-zero constant.
3. A constant multiple of a row may be added to another row.

One can easily see that these three row operations may make the system look different, but they do not change the solution of the system.

Example of Row Interchange

Consider the system of equations with two unknowns:

$$\begin{aligned} x + 3y &= 7 \\ 3x + 4y &= 11 \end{aligned}$$

If we interchange the rows, we get:

$$\begin{aligned} 3x + 4y &= 11 \\ x + 3y &= 7 \end{aligned}$$

Clearly, this system has the same solution as the original.

Example of Multiplying a Row by a Constant

Consider the system again:

$$\begin{aligned} x + 3y &= 7 \\ 3x + 4y &= 11 \end{aligned}$$

Multiplying the first row by -3, we get:

$$\begin{aligned} -3x - 9y &= -21 \\ 3x + 4y &= 11 \end{aligned}$$

Once again, this new system has the same solution as the original.

Example of Adding a Constant Multiple of One Row to Another

For the system:

$$\begin{aligned} x + 3y &= 7 \\ 3x + 4y &= 11 \end{aligned}$$

If we multiply the first row by -3 and add it to the second row, we get:

$$\begin{aligned} x + 3y &= 7 \\ -5y &= -10 \end{aligned}$$

The solution remains unchanged.

Now that we understand how the three row operations work, it is time to introduce the Gauss-Jordan method to solve systems of linear equations. As mentioned earlier, the Gauss-Jordan method starts out with an augmented matrix, and by a series of row operations ends up with a matrix that is in the reduced row echelon form. A matrix is in the reduced row echelon form if the first nonzero entry in each row is a 1, and the columns containing these 1's have all other entries as zeros. The reduced row echelon form also requires that the leading entry in each row be to the right of the leading entry in the row above it, and the rows containing all zeros be moved down to the bottom. We state the Gauss-Jordan method as follows.

Gauss-Jordan Method Steps

Here are the steps of the Gauss-Jordan method for solving linear systems:

1. Write the augmented matrix.
2. Interchange rows if necessary to obtain a non-zero number in the first row, first column.
3. Use a row operation to get a 1 as the entry in the first row and first column.
4. Use row operations to make all other entries as zeros in column one.
5. Interchange rows if necessary to obtain a nonzero number in the second row, second column. Use a row operation to make this entry 1. Use row operations to make all other entries as zeros in column two.

6. Repeat step 5 for row 3, column 3. Continue moving along the main diagonal until you reach the last row, or until the number is zero.

7. The final matrix is called the reduced row-echelon form.

Example 2.2.5. *Solve the following system by the Gauss-Jordan method:*

$$\begin{array}{rcrcrcr} 2x & + & y & + & 2z & = & 10 \\ x & + & 2y & + & z & = & 8 \\ 3x & + & y & - & z & = & 2 \end{array}$$

Solution 2.2.5. *We write the augmented matrix.*

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 10 \\ 1 & 2 & 1 & 8 \\ 3 & 1 & -1 & 2 \end{array}\right]$$

We want a 1 in row one, column one. This can be obtained by dividing the first row by 2, or interchanging the second row with the first. Interchanging the rows is a better choice because that way we avoid fractions.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 2 & 1 & 2 & 10 \\ 3 & 1 & -1 & 2 \end{array}\right] \quad \textit{we interchanged row 1(R1) and row 2(R2)}$$

We need to make all other entries zeros in column 1. To make the entry (2) a zero in row 2, column 1, we multiply row 1 by -2 and add it to the second row. We get,

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -3 & 0 & -6 \\ 3 & 1 & -1 & 2 \end{array}\right] \quad -2R1 + R2$$

To make the entry (3) a zero in row 3, column 1, we multiply row 1 by -3 and add it to the third row. We get,

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -3 & 0 & -6 \\ 0 & -5 & -4 & -22 \end{array}\right] \quad -3R1 + R3$$

So far we have made a 1 in the left corner and all other entries zeros in that column. Now we move to the next diagonal entry, row 2, column 2. We need *to make this entry(-3) a 1 and make all other entries in this column zeros. To make row 2, column 2 entry a 1, we divide the entire second row by -3.*

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 1 & 0 & 2 \\ 0 & -5 & -4 & -22 \end{array}\right] \; R2 \cdot \frac{1}{(-3)}$$

Next, we make all other entries zeros in the second column.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -4 & -12 \end{array}\right] \; -2R2 + R1 \textit{ and } 5R2 + R3$$

We make the last diagonal entry a 1, by dividing row 3 by -4.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \; R3 \cdot \frac{1}{(-4)}$$

Finally, we make all other entries zeros in column 3.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \; -R3 + R1$$

Clearly, the solution reads $x = 1$, $y = 2$, *and* $z = 3$.

Before we leave this section, we mention some terms we may need in the fourth chapter.

The process of obtaining a 1 in a location, and then making all other entries zeros in that column, is called pivoting.

The number that is made a 1 is called the pivot element, and the row that contains the pivot element is called the pivot row.

We often multiply the pivot row by a number and add it to another row to obtain a zero in the latter. The row to which a multiple of pivot row is added is called the target row.

2.3 Systems of Linear Equations – Special Cases

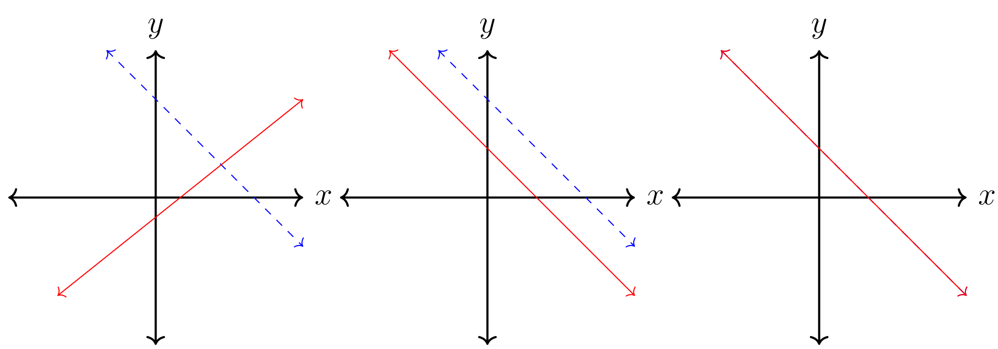
In this section you will learn to:

1. Determine the linear systems that have no solution.
2. Solve the linear systems that have infinitely many solutions.

If we consider the intersection of two lines in a plane, three things can happen.

1. The lines intersect in exactly one point. This is called an independent system.
2. The lines are parallel, so they do not intersect. This is called an inconsistent system.
3. The lines coincide; they intersect at infinitely many points. This is a dependent system.

The figures below show all three cases:



Every system of equations has either one solution, no solution, or infinitely many solutions.

In the last section, we used the Gauss-Jordan method to solve systems that had exactly one solution. In this section, we will determine the systems that have no solution, and solve the systems that have infinitely many solutions.

Example 2.3.1. *Solve the following system of equations using the Gauss-Jordan method:*

$$x + y = 7$$
$$x + y = 9$$

Solution 2.3.1. *Let us use the Gauss-Jordan method to solve this system. The augmented matrix is*

$$\left[\begin{array}{cc|c} 1 & 1 & 7 \\ 1 & 1 & 9 \end{array}\right]$$

If we multiply the first row by -1 and add it to the second row, we get

$$\left[\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 0 & 2 \end{array}\right]$$

Since 0 cannot equal 2, the last equation cannot be true for any choices of x and y. Alternatively, it is clear that the two lines are parallel; therefore, they do not intersect.

In the examples that follow, we are going to start using a calculator to row reduce the augmented matrix, in order to focus on understanding the answer rather than focusing on the process of carrying out the row operations.

Example 2.3.2. *Solve the following system of equations:*

$$\begin{aligned} 2x + 3y - 4z &= 7 \\ 3x + 4y - 2z &= 9 \\ 5x + 7y - 6z &= 20 \end{aligned}$$

Solution 2.3.2. *We represent the system as an augmented matrix:*

$$\left[\begin{array}{ccc|c} 2 & 3 & -4 & 7 \\ 3 & 4 & -2 & 9 \\ 5 & 7 & -6 & 20 \end{array}\right]$$

By obtaining the reduced row-echelon form from a matrix calculator, we get:

$$\left[\begin{array}{ccc|c} 1 & 0 & 10 & 0 \\ 0 & 1 & -8 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The bottom row implies $0x + 0y + 0z = 1$, which is a contradiction. Thus, the system is inconsistent and has no solution.

Example 2.3.3. *Solve the following system of equations:*

$$x + y = 7$$
$$x + y = 7$$

Solution 2.3.3. *The problem asks for the intersection of two identical lines, meaning the lines coincide and intersect at an infinite number of points.*

A few intersection points are listed as follows: (3, 4), (5, 2), (-1, 8), (-6, 13), etc. However, when a system has an infinite number of solutions, the solution is often expressed in parametric form. This can be done by assigning an arbitrary constant, t*, to one of the variables and solving for the remaining variables. If we let* $y = t$*, then* $x = 7 - t$*. In other words, all ordered pairs of the form* $(7 - t, t)$ *satisfy the given system of equations.*

Alternatively, solving with the Gauss-Jordan method, we obtain the reduced row-echelon form below, which includes a row of all zeros that can be ignored since it provides no additional information about the values of x *and* y *that solve the system.*

$$\left[\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 0 & 0 \end{array}\right]$$

This leaves us with only one equation but two variables. Whenever there are more variables than equations, the solution must be expressed as a parametric solution in terms of an arbitrary constant, as shown above.

Parametric Solution: $x = 7 - t$, $y = t$.

Example 2.3.4. *Solve the following system of equations:*

$$x + y + z = 2$$
$$2x + y - z = 3$$
$$3x + 2y = 5$$

Solution 2.3.4. *The augmented matrix and the reduced row-echelon form are given below:*

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & -1 & 3 \\ 3 & 2 & 0 & 5 \end{array}\right] \quad \rightarrow \quad \left[\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since the last equation dropped out, we are left with two equations and three variables. This means the system has an infinite number of solutions. We express those solutions in the parametric form by letting the last variable z equal the parameter t.

The first equation reads $x - 2z = 1$, therefore, $x = 1 + 2z$. The second equation reads $y + 3z = 1$, therefore, $y = 1 - 3z$. And now if we let $z = t$, the parametric solution is expressed as follows:

$$\textit{Parametric Solution:} \quad x = 1 + 2t, \quad y = 1 - 3t, \quad z = t.$$

The reader should note that particular solutions, or specific solutions, to the system can be obtained by assigning values to the parameter t. For example:

- *If we let $t = 2$, we have the solution $x = 5, y = -5, z = 2$: $(5, -5, 2)$.*
- *If we let $t = 0$, we have the solution $x = 1, y = 1, z = 0$: $(1, 1, 0)$.*

Example 2.3.5. *Solve the following system of equations:*

$$\begin{aligned}
x + 2y - 3z &= 5 \\
2x + 4y - 6z &= 10 \\
3x + 6y - 9z &= 15
\end{aligned}$$

Solution 2.3.5. *The reduced row-echelon form is given below:*

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This time the last two equations drop out. We are left with one equation and three variables. Again, there are an infinite number of solutions. But this time the answer must be expressed in terms of two arbitrary constants.

If we let $z = t$ and $y = s$, the first equation $x + 2y - 3z = 5$ results in $x = 5 - 2s + 3t$. We rewrite the parametric solution as:

$$\textit{Parametric Solution:} \quad x = 5 - 2s + 3t, \quad y = s, \quad z = t.$$

Summary 2.1.

1. *If any row of the reduced row-echelon form of the matrix gives a false statement such as* $0 = 1$*, the system is inconsistent and has no solution.*
2. *If the reduced row echelon form has fewer equations than the variables and the system is consistent, then the system has an infinite number of solutions. Remember the rows that contain all zeros are dropped.*
   (a) *If a system has an infinite number of solutions, the solution must be expressed in the parametric form.*
   (b) *The number of arbitrary parameters equals the number of variables minus the number of equations.*

2.4 Inverse Matrices

In this section you will learn to:

1. Find the inverse of a matrix, if it exists.
2. Use inverses to solve linear systems.

In this section, we will learn to find the inverse of a matrix, if it exists. Later, we will use matrix inverses to solve linear systems.

Definition 2.4.1. *An* $n \times n$ *matrix has an* ***inverse*** *if there exists a matrix* B *such that* $AB = BA = I_n$*, where* I_n *is an* $n \times n$ *identity matrix. The* ***inverse*** *of a matrix* A*, if it exists, is denoted by the symbol* A^{-1}*.*

Example 2.4.1. *Given matrices* A *and* B *below, verify that they are inverses.*

$$A = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$$

Solution 2.4.1. *The matrices are inverses if the product* AB *and* BA *both equal the identity matrix of dimension* 2×2*, denoted as* I_2*:*

$$AB = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

and

$$BA = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Clearly, that is the case; therefore, the matrices A and B are inverses of each other.

Example 2.4.2. *Find the inverse of the matrix $A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$.*

Solution 2.4.2. *Suppose A has an inverse, and it is denoted as $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $AB = I_2$:*

$$\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

After multiplying the matrices on the left side, we get the system:

$$\begin{aligned} 3a + c &= 1 \\ 3b + d &= 0 \\ 5a + 2c &= 0 \\ 5b + 2d &= 1 \end{aligned}$$

Solving this system, we find $a = 2$, $b = -1$, $c = -5$, and $d = 3$. Therefore, the inverse of matrix A is $B = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$.

In this problem, finding the inverse of matrix A amounted to solving the system of equations:

$$\begin{aligned} 3a + c &= 1 \\ 3b + d &= 0 \\ 5a + 2c &= 0 \\ 5b + 2d &= 1 \end{aligned}$$

Actually, it can be written as two systems, one with variables a and c, and the other with b and d. The augmented matrices for both are given below.

$$\left[\begin{array}{cc|c} 3 & 1 & 1 \\ 5 & 2 & 0 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{cc|c} 3 & 1 & 0 \\ 5 & 2 & 1 \end{array}\right]$$

As we look at the two augmented matrices, we notice that the coefficient matrix for both the matrices is the same. This implies the row operations of the Gauss-Jordan method will also be the same. A great deal of work can be saved if the two right-hand columns are grouped together to form one augmented matrix as below.

$$\left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 5 & 2 & 0 & 1 \end{array}\right]$$

And solving this system, we get

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -5 & 3 \end{array}\right]$$

The matrix on the right side of the vertical line is the A^{-1} matrix. What you just witnessed is no coincidence. This is the method that is often employed in finding the inverse of a matrix.

Summary 2.2. ***The Method for Finding the Inverse of a Matrix***

1. *Write the augmented matrix $[A|I_n]$.*
2. *Write the augmented matrix in step 1 in reduced row echelon form.*
3. *If the reduced row echelon form in 2 is $[I_n|B]$, then B is the inverse of A.*
4. *If the left side of the row reduced echelon is not an identity matrix, the inverse does not exist.*

Example 2.4.3. *Given the matrix A below, find its inverse.*

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Solution 2.4.3. *We write the augmented matrix as follows.*

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

We will reduce this matrix using the Gauss-Jordan method. Multiplying the first row by -2 and adding it to the second row, we get

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 5 & -2 & -2 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

If we swap the second and third rows, we get

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \\ 0 & 5 & -2 & -2 & 1 & 0 \end{array}\right]$$

Divide the second row by -2. *The result is*

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1/2 & 0 & 0 & -1/2 \\ 0 & 5 & -2 & -2 & 1 & 0 \end{array}\right]$$

Let us do two operations here. 1) *Add the second row to the first.* 2) *Add* -5 *times the second row to the third. And we get*

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 1 & 0 & -1/2 \\ 0 & 1 & -1/2 & 0 & 0 & -1/2 \\ 0 & 0 & 1/2 & -2 & 1 & 5/2 \end{array}\right]$$

Multiplying the third row by 2 *results in*

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 1 & 0 & -1/2 \\ 0 & 1 & -1/2 & 0 & 0 & -1/2 \\ 0 & 0 & 1 & -4 & 2 & 5 \end{array}\right]$$

Multiply the third row by $1/2$ *and add it to the second. Also, multiply the third row by* $-1/2$ *and add it to the first.*

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & -3 \\ 0 & 1 & 0 & -2 & 1 & 2 \\ 0 & 0 & 1 & -4 & 2 & 5 \end{array}\right]$$

Therefore, the inverse of matrix A *is* $A^{-1} = \begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix}$.

One should verify the result by multiplying the two matrices to see if the product does, indeed, equal the identity matrix.

Now that we know how to find the inverse of a matrix, we will use inverses to solve systems of equations. The method is analogous to solving a simple equation like the one below.

$$\frac{2}{3}x = 4$$

Example 2.4.4. *Solve the following equation:*

$$x = 4$$

Solution 2.4.4. *To solve the above equation, we multiply both sides of the equation by the multiplicative inverse of $\frac{2}{3}$, which happens to be $\frac{3}{2}$. We get*

$$\frac{3}{2} \cdot \frac{2}{3}x = 4 \cdot \frac{3}{2}$$

Hence,

$$x = 6.$$

We use example 2.4.4 as an analogy to show how linear systems of the form $AX = B$ are solved. To solve a linear system, we first write the system in the matrix equation $AX = B$, where A is the coefficient matrix, X is the matrix of variables, and B is the matrix of constant terms. We then multiply both sides of this equation by the multiplicative inverse of the matrix A. Consider the following example.

Example 2.4.5. *Solve the following system*

$$\begin{aligned} 3x + y &= 3 \\ 5x + 2y &= 4 \end{aligned}$$

Solution 2.4.5. *To solve the above equation, first we express the system as*

$$AX = B$$

where A is the coefficient matrix, and B is the matrix of constant terms. We get

$$\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

To solve this system, we multiply both sides of the matrix equation $AX = B$ by A^{-1}. Matrix multiplication is not commutative, so we need to multiply by A^{-1} on the left on both sides of the equation.

Matrix A is the same matrix A whose inverse we found in Example 2.4.2, so $A^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$.

Multiplying both sides by A^{-1}, we get

$$\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Therefore, $x = 2$, and $y = -3$.

Example 2.4.6. *Solve the following system:*

$$\begin{aligned} x - y + z &= 6 \\ 2x + 3y &= 1 \\ -2y + z &= 5 \end{aligned}$$

Solution 2.4.6. *To solve the above equation, we write the system in matrix form $AX = B$ as follows:*

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 5 \end{bmatrix}$$

To solve this system, we need the inverse of A. From Example 2.4.3, A^{-1} is given by

$$A^{-1} = \begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix}$$

Multiplying both sides of the matrix equation $AX = B$ on the left by A^{-1}, we get

$$\begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

After multiplying the matrices, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

Therefore, $x = 2$, $y = -1$, and $z = 3$.

We remind the reader that not every system of equations can be solved by the matrix inverse method. Although the Gauss-Jordan method works for every situation, the matrix inverse method works only in cases where the inverse of the square matrix exists. In such cases the system has a unique solution.

Summary 2.3.

The Method for Finding the Inverse of a Matrix

1. *Write the augmented matrix* $[A|I_n]$.
2. *Write the augmented matrix in step 1 in reduced row echelon form.*
3. *If the reduced row echelon form in step 2 is* $[I_n|B]$, *then* B *is the inverse of* A.
4. *If the left side of the row reduced echelon is not an identity matrix, the inverse does not exist.*

The Method for Solving a System of Equations When a Unique Solution Exists

1. *Express the system in the matrix equation* $AX = B$.
2. *To solve the equation* $AX = B$, *multiply both sides by* A^{-1}:

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B \quad \text{where } I \text{ is the identity matrix}$$

2.5 Application of Matrices in Cryptography

In this section, you will learn to:

1. Encode a message using matrix multiplication.
2. Decode a coded message using the matrix inverse and matrix multiplication.

Encryption dates back approximately 4000 years. Historical accounts indicate that the Chinese, Egyptians, Indians, and Greeks encrypted messages in some way for various purposes. One famous encryption scheme is called the Caesar cipher, also called a substitution cipher, used by Julius Caesar, involved shifting letters in the alphabet, such as replacing A by C, B by D, C by E, etc., to encode a message. Substitution ciphers are too simple in design to be considered secure today.

In the middle ages, European nations began to use encryption. A variety of encryption methods were used in the US from the Revolutionary War, through the Civil War, and on into modern times.

Applications of mathematical theory and methods to encryption became widespread in military usage in the 20th century. The military would encode messages before sending, and the recipient would decode the message, in order to send information about military operations in a manner that kept the information safe if the message was intercepted. In World War II, encryption played an important role, as both Allied and Axis powers sent encrypted messages and devoted significant resources to strengthening their own encryption while also trying to break the opposition's encryption.

In this section, we will examine a method of encryption that uses matrix multiplication and matrix inverses. This method, known as the Hill Algorithm, was created by Lester Hill, a mathematics professor who taught at several US colleges and also was involved with military encryption. The Hill algorithm marks the introduction of modern mathematical theory and methods to the field of cryptography.

These days, the Hill Algorithm is not considered a secure encryption method; it is relatively easy to break with modern technology. However, in 1929 when it was developed, modern computing technology did not exist. This method, which we can handle easily with today's technology, was too cumbersome to use with hand calculations. Hill devised a mechanical encryption machine to help with the mathematics; his machine relied on gears and levers but never gained widespread use. Hill's method was considered sophisticated and powerful in its time and is one of many methods influencing techniques in use today. Other encryption methods at that time also utilized special coding machines. Alan Turing, a computer scientist pioneer in the field of artificial intelligence, invented a machine that was able to decrypt messages encrypted by the German Enigma machine, helping to turn the tide of World War II.

With the advent of the computer age and internet communication, the use of encryption has become widespread in communication and in keeping private data secure; it is no longer limited to military uses. Modern encryption methods are more complicated, often combining several steps or methods to encrypt data to keep it more secure and harder to break. Some modern methods make use of matrices as part of the encryption and decryption process; other fields of mathematics such as number theory play a large role in modern cryptography.

2.5.1 Using Matrices for Encoding and Decoding

To use matrices in encoding and decoding secret messages, our procedure is as follows.

We first convert the secret message into a string of numbers by arbitrarily assigning a number to each letter of the message. Next, we convert this string of numbers into a new set of numbers by multiplying the string by a square matrix of our choice that has an inverse. This new set of numbers represents the coded message.

To decode the message, we take the string of coded numbers and multiply it by the inverse of the matrix to get the original string of numbers. Finally, by associating the numbers with their corresponding letters, we obtain the original message.

In this section, we will use the correspondence shown below where letters A to Z correspond to the numbers 1 to 26, a space is represented by the number 27, and punctuation is ignored.

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13

N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	26

Example 2.5.1. *Use matrix A to encode the message: ATTACK NOW!*

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

Solution 2.5.1. *We divide the letters of the message into groups of two.*

$$AT \quad TA \quad CK \quad \_N \quad OW$$

We assign the numbers to these letters from the above table, and convert each pair of numbers into 2×1 matrices. In the case where a single letter is left over on the end, a space is added to make it into a pair.

$$\begin{bmatrix} A \\ T \end{bmatrix} = \begin{bmatrix} 1 \\ 20 \end{bmatrix}, \quad \begin{bmatrix} T \\ A \end{bmatrix} = \begin{bmatrix} 20 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} C \\ K \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}, \quad \begin{bmatrix} - \\ N \end{bmatrix} = \begin{bmatrix} 27 \\ 14 \end{bmatrix}, \quad \begin{bmatrix} O \\ W \end{bmatrix} = \begin{bmatrix} 15 \\ 23 \end{bmatrix}$$

So at this stage, our message expressed as 2×1 *matrices is as follows.*

$$\begin{bmatrix} 1 \\ 20 \end{bmatrix}, \quad \begin{bmatrix} 20 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 11 \end{bmatrix}, \quad \begin{bmatrix} 27 \\ 14 \end{bmatrix}, \quad \begin{bmatrix} 15 \\ 23 \end{bmatrix}$$

Now to encode, we multiply, on the left, each matrix of our message by the matrix A.

For example, the product of A *with our first matrix is:*

$$A \cdot \begin{bmatrix} 1 \\ 20 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 20 \end{bmatrix} = \begin{bmatrix} 41 \\ 61 \end{bmatrix}$$

And the product of A *with our second matrix is:*

$$A \cdot \begin{bmatrix} 20 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 20 \\ 1 \end{bmatrix} = \begin{bmatrix} 22 \\ 23 \end{bmatrix}$$

Multiplying matrix A *by each matrix in our list, in turn, gives the desired coded message:*

$$\begin{bmatrix} 41 \\ 61 \end{bmatrix}, \quad \begin{bmatrix} 22 \\ 23 \end{bmatrix}, \quad \begin{bmatrix} 25 \\ 36 \end{bmatrix}, \quad \begin{bmatrix} 55 \\ 69 \end{bmatrix}, \quad \begin{bmatrix} 61 \\ 84 \end{bmatrix}$$

Example 2.5.2. *Decode the following message that was encoded using matrix* $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$*:*

$$\begin{bmatrix} 21 \\ 26 \end{bmatrix}, \quad \begin{bmatrix} 37 \\ 53 \end{bmatrix}, \quad \begin{bmatrix} 45 \\ 54 \end{bmatrix}, \quad \begin{bmatrix} 74 \\ 101 \end{bmatrix}, \quad \begin{bmatrix} 53 \\ 69 \end{bmatrix}$$

Solution 2.5.2. *Since this message was encoded by multiplying by the matrix* A *in Example 2.4.2, we decode this message by first multiplying each matrix, on the left, by the inverse of matrix* A *given below.*

$$A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

For example:

$$\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 21 \\ 26 \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \end{bmatrix}$$

By multiplying each of the matrices in our list by the matrix A^{-1}*, we get the following.*

$$\begin{bmatrix} 11 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} 5 \\ 16 \end{bmatrix}, \quad \begin{bmatrix} 27 \\ 9 \end{bmatrix}, \quad \begin{bmatrix} 20 \\ 27 \end{bmatrix}, \quad \begin{bmatrix} 21 \\ 16 \end{bmatrix}$$

Finally, by associating the numbers with their corresponding letters, we obtain:

$$\begin{bmatrix} K \\ E \end{bmatrix}, \quad \begin{bmatrix} E \\ P \end{bmatrix}, \quad \begin{bmatrix} \_ \\ T \end{bmatrix}, \quad \begin{bmatrix} I \\ U \end{bmatrix}, \quad \begin{bmatrix} P \\ \_ \end{bmatrix}$$

And the message reads: KEEP IT UP.

Now suppose we wanted to use a 3×3 matrix to encode a message, then instead of dividing the letters into groups of two, we would divide them into groups of three.

Example 2.5.3. *Using the matrix* $B = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$*, encode the message: ATTACK NOW!*

Solution 2.5.3. *We divide the letters of the message into groups of three.*

$$ATT \quad ACK \quad \_NO \quad W\_\_$$

Note that since the single letter "W" was left over on the end, we added two spaces to make it into a triplet.

Now we assign the numbers their corresponding letters from the table, and convert each triplet of numbers into 3×1 *matrices. We get*

$$\begin{bmatrix} A \\ T \\ T \end{bmatrix} = \begin{bmatrix} 1 \\ 20 \\ 20 \end{bmatrix}, \quad \begin{bmatrix} A \\ C \\ K \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 11 \end{bmatrix}, \quad \begin{bmatrix} - \\ N \\ O \end{bmatrix} = \begin{bmatrix} 27 \\ 14 \\ 15 \end{bmatrix}, \quad \begin{bmatrix} W \\ - \\ - \end{bmatrix} = \begin{bmatrix} 23 \\ 27 \\ 27 \end{bmatrix}$$

So far we have,

$$\begin{bmatrix} 1 \\ 20 \\ 20 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \\ 11 \end{bmatrix}, \quad \begin{bmatrix} 27 \\ 14 \\ 15 \end{bmatrix}, \quad \begin{bmatrix} 23 \\ 27 \\ 27 \end{bmatrix}$$

We multiply, on the left, each matrix of our message by the matrix B. For example,

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 20 \\ 20 \end{bmatrix} = \begin{bmatrix} 1 \\ 21 \\ 42 \end{bmatrix}$$

By multiplying each of the matrices in (III) by the matrix B, we get the desired coded message as follows:

$$\begin{bmatrix} 1 \\ 21 \\ 42 \end{bmatrix}, \quad \begin{bmatrix} -7 \\ 12 \\ 16 \end{bmatrix}, \quad \begin{bmatrix} 26 \\ 42 \\ 83 \end{bmatrix}, \quad \begin{bmatrix} 23 \\ 50 \\ 100 \end{bmatrix}$$

If we need to decode this message, we simply multiply the coded message by B^{-1}, and associate the numbers with the corresponding letters of the alphabet.

In Example 2.5.4 we will demonstrate how to use matrix B^{-1} to decode an encrypted message.

Example 2.5.4. *Decode the following message that was encoded using matrix* $B = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$*:*

$$\begin{bmatrix} 11 \\ 20 \\ 43 \end{bmatrix}, \quad \begin{bmatrix} 25 \\ 10 \\ 41 \end{bmatrix}, \quad \begin{bmatrix} 22 \\ 14 \\ 41 \end{bmatrix}$$

Solution 2.5.4. *Since this message was encoded by multiplying by the matrix B. We first determine the inverse of B.*

$$B^{-1} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & -3 & 2 \\ -1 & -1 & 1 \end{bmatrix}$$

To decode the message, we multiply each matrix, on the left, by B^{-1}. For example,

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & -3 & 2 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 11 \\ 20 \\ 43 \end{bmatrix} = \begin{bmatrix} 8 \\ 15 \\ 12 \end{bmatrix}$$

Multiplying each of the matrices in our list by the matrix B^{-1} gives the following:

$$\begin{bmatrix} 8 \\ 15 \\ 12 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 27 \\ 6 \end{bmatrix}, \quad \begin{bmatrix} 9 \\ 18 \\ 5 \end{bmatrix}$$

Finally, by associating the numbers with their corresponding letters, we obtain:

$$\begin{bmatrix} H \\ O \\ L \end{bmatrix}, \quad \begin{bmatrix} D \\ \_ \\ F \end{bmatrix}, \quad \begin{bmatrix} I \\ R \\ E \end{bmatrix}$$

The message reads: HOLD FIRE.

Summary 2.4.
To Encode a Message

1. *Divide the letters of the message into groups of two or three.*
2. *Convert each group into a string of numbers by assigning a number to each letter of the message. Remember to assign letters to blank spaces.*
3. *Convert each group of numbers into column matrices.*
4. *Convert these column matrices into a new set of column matrices by multiplying them with a compatible square matrix of your choice that has an inverse. This new set of numbers or matrices represents the coded message.*

To Decode a Message

1. *Take the string of coded numbers and multiply it by the inverse of the matrix that was used to encode the message.*
2. *Associate the numbers with their corresponding letters.*

2.6 Applications – Leontief Models

In this section you will learn

1. Application of matrices to model closed economic systems
2. Application of matrices to model open economic systems

In the 1930s, Wassily Wassilyevich Leontief (holder of one of the greatest names ever) used matrices to model economic systems. His models, often referred to as the input-output models, divide the economy into sectors where each sector produces goods and services not only for itself but also for other sectors. These sectors are dependent on each other, and the total input always equals the total output. In 1973, he won the Nobel Prize in Economics for his work in this field. In this section, we look at both the closed and the open models that he developed.

2.6.1 The Closed Model

As an example of the closed model, we look at a very simple economy, where there are only three sectors: food, shelter, and clothing.

Example 2.6.1. *We assume that in a village there is a farmer, carpenter,* *and a tailor, who provide the three essential goods: food, shelter, and clothing. Suppose the farmer himself consumes 40% of the food he produces, and gives 40% to the carpenter, and 20% to the tailor. Thirty percent of the carpenter's production is consumed by himself, 40% by the farmer, and 30% by the carpenter. Fifty percent of the tailor's production is used by himself, 30% by the farmer, and 20% by the tailor. Write the matrix that describes this closed model.*

Solution 2.6.1. *The table below describes the above information.*

	Proportion produced by the farmer	*Proportion produced by the carpenter*	*Proportion produced by the tailor*
The proportion used by the farmer	*.40*	*.40*	*.30*
The proportion used by the carpenter	*.40*	*.30*	*.20*
The proportion used by the tailor	*.20*	*.30*	*.50*

In matrix form, it can be written as follows.

$$A = \begin{bmatrix} .40 & .40 & .30 \\ .40 & .30 & .20 \\ .20 & .30 & .50 \end{bmatrix}$$

This matrix is called the input-output matrix. It is important that we read the matrix correctly. For example, the entry A_{23}, the entry in row 2 and column 3, represents the following.

$$A_{23} = 20\% \text{ of the tailor's production is used by the carpenter.}$$
$$A_{33} = 50\% \text{ of the tailor's production is used by the tailor.}$$

Example 2.6.2. *In Example 2.6.1 above, how much should each person get for his efforts?*

Solution 2.6.2. *We choose the following variables.*

$$x = \text{Farmer's pay}$$
$$y = \text{Carpenter's pay}$$
$$z = \text{Tailor's pay}$$

As we said earlier, in this model input must equal output. That is, the amount paid by each equals the amount received by each.

Let us say the farmer gets paid x dollars. Let us now look at the farmer's expenses. The farmer uses up 40% of his own production, that is, of the x dollars he gets paid, he pays himself $.40x$ dollars, he pays $.40y$ dollars to the carpenter, and $.30z$ to the tailor. Since the expenses equal the wages, we get the following equation.

$$x = .40x + .40y + .30z$$

In the same manner, we get

$$y = .40x + .30y + .20z$$

$$z = .20x + .30y + .50z$$

The above system can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} .40 & .40 & .30 \\ .40 & .30 & .20 \\ .20 & .30 & .50 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This system is often referred to as $X = AX$.

Simplification results in the system of equations $(I - A)X = 0$

$$\begin{bmatrix} .60 & -.40 & -.30 \\ -.40 & .70 & -.20 \\ -.20 & -.30 & .50 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We put this into an augmented matrix

$$\left[\begin{array}{ccc|c} .60 & -.40 & -.30 & 0 \\ -.40 & .70 & -.20 & 0 \\ -.20 & -.30 & .50 & 0 \end{array}\right]$$

Solving for x, y, and z using the Gauss-Jordan method, we get

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{29}{26} & 0 \\ 0 & 1 & -\frac{12}{13} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This gives parametric equations:

$$x = \frac{29}{26}t, \quad y = \frac{12}{13}t, \quad z = t$$

Since we are only trying to determine the proportions of the pay, we can choose t to be any value. Suppose we let $t = \$2600$, then we get

$$x = \$2900, \quad y = \$2400, \quad z = \$2600$$

Note 2.6.1. *The use of a graphing calculator or computer application in solving the systems of linear matrix equations in these problems is strongly recommended.*

2.6.2 The Open Model

The open model is more realistic as it deals with the economy where sectors of the economy not only satisfy each other's needs but also satisfy some outside demands. In this case, the outside demands are put on by the consumer. But the basic assumption is still the same: whatever is produced is consumed.

Let us again look at a very simple scenario. Suppose the economy consists of three people: the farmer F, the carpenter C, and the tailor T. A part of the farmer's production is used by all three, and the rest is used by the consumer. In the same manner, a part of the carpenter's and the tailor's production is used by all three, and the rest is used by the consumer.

Let us assume that whatever the farmer produces, 20% is used by him, 15% by the carpenter, 10% by the tailor, and the consumer uses the other \$40 billion worth of food. Ten percent of the carpenter's production is used by him, 25% by the farmer, 5% by the tailor, and \$50 billion worth by the consumer. Fifteen percent of the clothing is used by the tailor, 10% by the farmer, 5% by the carpenter, and the remaining \$60 billion worth by the consumer. We write the internal consumption in the following table and express the demand as the matrix D.

	F produces	C produces	T produces
F uses	0.20	0.25	0.10
C uses	0.15	0.10	0.05
T uses	0.10	0.05	0.15

The consumer demand for each industry in billions of dollars is given by the matrix $D = \begin{bmatrix} 40 \\ 50 \\ 60 \end{bmatrix}$.

Example 2.6.3. *In the example above, what should be, in billions of dollars, the required output by each industry to meet the demand given by the matrix D?*

Solution 2.6.3. *We choose the following variables.*

$$\begin{aligned} x &= \textit{Farmer's output} \\ y &= \textit{Carpenter's output} \\ z &= \textit{Tailor's output} \end{aligned}$$

In the closed model, our equation was $X = AX$, that is, the total input equals the total output. This time our equation is similar with the exception of the demand by the consumer.

So our equation for the open model should be $X = AX + D$, where D represents the demand matrix.

We express it as follows:

$$X = AX + D$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} .20 & .25 & .10 \\ .15 & .10 & .05 \\ .10 & .05 & .15 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 40 \\ 50 \\ 60 \end{bmatrix}$$

To solve this system, we write it as

$$X = AX + D$$

$$(I - A)X = D$$

$$X = (I - A)^{-1}D$$

where I is a 3×3 identity matrix.

$$I - A = \begin{bmatrix} .80 & -.25 & -.10 \\ -.15 & .90 & -.05 \\ -.10 & -.05 & .85 \end{bmatrix}$$

$$(I-A)^{-1} = \begin{bmatrix} 1.3445 & .3835 & .1807 \\ .2336 & 1.1814 & .097 \\ .1719 & .1146 & 1.2034 \end{bmatrix}$$

$$X = \begin{bmatrix} 1.3445 & .3835 & .1807 \\ .2336 & 1.1814 & .097 \\ .1719 & .1146 & 1.2034 \end{bmatrix} \begin{bmatrix} 40 \\ 50 \\ 60 \end{bmatrix}$$

$$X = \begin{bmatrix} 83.7999 \\ 74.2341 \\ 84.8138 \end{bmatrix}$$

The three industries must produce the following amount of goods in billions of dollars.

$$\begin{aligned} Farmer &= 83.7999 \\ Carpenter &= 74.2341 \\ Tailor &= 84.8138 \end{aligned}$$

We will do one more problem like the one above, except this time we give the amount of internal and external consumption in dollars and ask for the proportion of the amounts consumed by each of the industries. In other words, we ask for the matrix A.

Example 2.6.4. *Suppose an economy consists of three industries F, C, and T. Each of the industries produces for internal consumption among themselves, as well as for external demand by the consumer. The table shows the use of each industry's production in dollars.*

	F	***C***	***T***	***Demand***	***Total***
F	40	50	60	100	250
C	30	40	40	110	220
T	20	30	30	120	200

The first row says that of the \$250 dollars worth of production by the industry F, \$40 is used by F, \$50 is used by C, \$60 is used by T, and the remainder *of $100 is used by the consumer. The other rows are described in a similar manner.*

Once again, the total input equals the total output. Find the proportion of the amounts consumed by each of the industries. In other words, find the matrix A.

Solution 2.6.4. *We are being asked to determine the following:*

How much of the production of each of the three industries, F, C, *and* T *is required to produce one unit of* F? *The same way, how much of the production of each of the three industries,* F, C, *and* T *is required to produce one unit of* C? *And finally, how much of the production of each of the three industries,* F, C, *and* T *is required to produce one unit of* T?

Since we are looking for proportions, we need to divide the production of each industry by the total production for each industry.

We analyze as follows: To produce 250 units of F, 30 units of C, and 20 units of T, the required units are 40, 30, and 20 respectively. Therefore, to produce 1 unit of each, we divide by 250:

$$\text{For F: } \frac{40}{250}, \text{ For C: } \frac{30}{250}, \text{ For T: } \frac{20}{250}$$

Similarly, for 220 units of C, the required units are 50, 40, and 30 respectively. To produce 1 unit of C, we divide by 220:

$$\text{For F: } \frac{50}{220}, \text{ For C: } \frac{40}{220}, \text{ For T: } \frac{30}{220}$$

And for 200 units of T, the required units are 60, 40, and 30 respectively. To produce 1 unit of T, we divide by 200:

$$\text{For F: } \frac{60}{200}, \text{ For C: } \frac{40}{200}, \text{ For T: } \frac{30}{200}$$

These fractions represent the units of F, C, and T required to produce 1 unit of each.

We obtain the following matrix:

$$A = \begin{bmatrix} \frac{40}{250} & \frac{50}{220} & \frac{60}{200} \\ \frac{30}{250} & \frac{40}{220} & \frac{40}{200} \\ \frac{20}{250} & \frac{30}{220} & \frac{30}{200} \end{bmatrix} = \begin{bmatrix} .1600 & .2273 & .3000 \\ .1200 & .1818 & .2000 \\ .0800 & .1364 & .1500 \end{bmatrix}$$

Clearly $AX + D = X$

$$\begin{bmatrix} .1600 & .2273 & .3000 \\ .1200 & .1818 & .2000 \\ .0800 & .1364 & .1500 \end{bmatrix} \begin{bmatrix} 250 \\ 220 \\ 200 \end{bmatrix} + \begin{bmatrix} 100 \\ 110 \\ 120 \end{bmatrix} = \begin{bmatrix} 250 \\ 220 \\ 200 \end{bmatrix}$$

Summary 2.5.

Leontief's Closed Model

1. *All consumption is within the industries. There is no external demand.*
2. *Input equals output.*
3. $X = AX$ *or* $(I - A)X = 0$

Leontief's Open Model

1. *In addition to internal consumption, there is an outside demand by the consumer.*
2. *Input equals output.*
3. $X = AX + D$ *or* $X = (I - A)^{-1}D$

Chapter 3

Linear Programming with Geometry

In this chapter, you will learn to:

1. Solve linear programming problems that maximize the objective function.
2. Solve linear programming problems that minimize the objective function.

3.1 Maximization Applications

In this section, you will learn to:

1. Recognize the typical form of a linear programming problem.
2. Formulate maximization linear programming problems.
3. Graph feasibility regions for maximization linear programming problems.
4. Determine optimal solutions for maximization linear programming problems.

Application problems in business, economics, and social and life sciences often ask us to make decisions on the basis of certain conditions. The conditions or constraints often take the form of inequalities. In this section, we will begin to formulate, analyze, and solve such problems, at a simple level, to understand the many components of such a problem.

A typical linear programming problem consists of finding an extreme value of a linear function subject to certain constraints. We are either trying to maximize or minimize the value of this linear function, such as to maximize profit or revenue, or to minimize cost. That is why these linear programming problems are classified as maximization or minimization problems, or just optimization problems. The function we are trying to optimize is called an objective function, and the conditions that must be satisfied are called constraints.

A typical example is to maximize profit from producing several products, subject to limitations on materials or resources needed for producing these items; the problem requires us to determine the amount of each item produced. Another type of problem involves scheduling; we need to determine how much time to devote to each of several activities in order to maximize income from (or minimize cost of) these activities, subject to limitations on time and other resources available for each activity.

In this chapter, we will work with problems that involve only two variables, and therefore, can be solved by graphing.

In the next chapter, we'll learn an algorithm to find a solution numerically. That will provide us with a tool to solve problems with more than two variables. At that time, with a little more knowledge about linear programming, we'll also explore the many ways these techniques are used in business and wide variety of other fields.

We begin by solving a maximization problem.

Example 3.1.1. *Niki holds two part-time jobs, Job I and Job II. She never wants to work more than a total of 12 hours a week. She has determined that for every hour she works at Job I, she needs 2 hours of preparation time, and for every hour she works at Job II, she needs one hour of preparation time, and she cannot spend more than 16 hours for preparation. If Niki makes \$40 an hour at Job I, and \$30 an hour at Job II, how many hours should she work per week at each job to maximize her income?*

Solution 3.1.1. *We start by choosing our variables. Let x be the number of* *hours per week Niki will work at Job I, and y the number of hours per week she will work at Job II.*

Now we write the objective function. Since Niki gets paid \$40 an hour at Job I, and \$30 an hour at Job II, her total income I is given by the following equation.

$$I = 40x + 30y$$

Our next task is to find the constraints. The constraints based on the problem description are:

$$x + y \leq 12$$

$$2x + y \leq 16$$

$$x \geq 0, \quad y \geq 0$$

We have formulated the problem as follows: Maximize

$$I = 40x + 30y$$

Subject to:

$$x + y \leq 12$$

$$2x + y \leq 16$$

$$x \geq 0; \quad y \geq 0$$

To solve the problem, we graph the constraints and shade the region that satisfies all the inequality constraints. We graph the lines by plotting the x-intercept and y-intercept and use a test point to determine which portion of the plane to shade.

In this example, after graphing the lines representing the constraints and using the origin (0,0) as a test point, we find that the feasible region is the area below and to the left of both constraint lines, above the x-axis, and to the right of the y-axis.

The shaded region where all conditions are satisfied is called the feasibility region or the feasibility polygon. The Fundamental Theorem of Linear Programming states that the maximum (or minimum) value of the objective function always takes place at the vertices of the feasibility region. Therefore, we will identify all the vertices (corner points) of the feasibility region. We

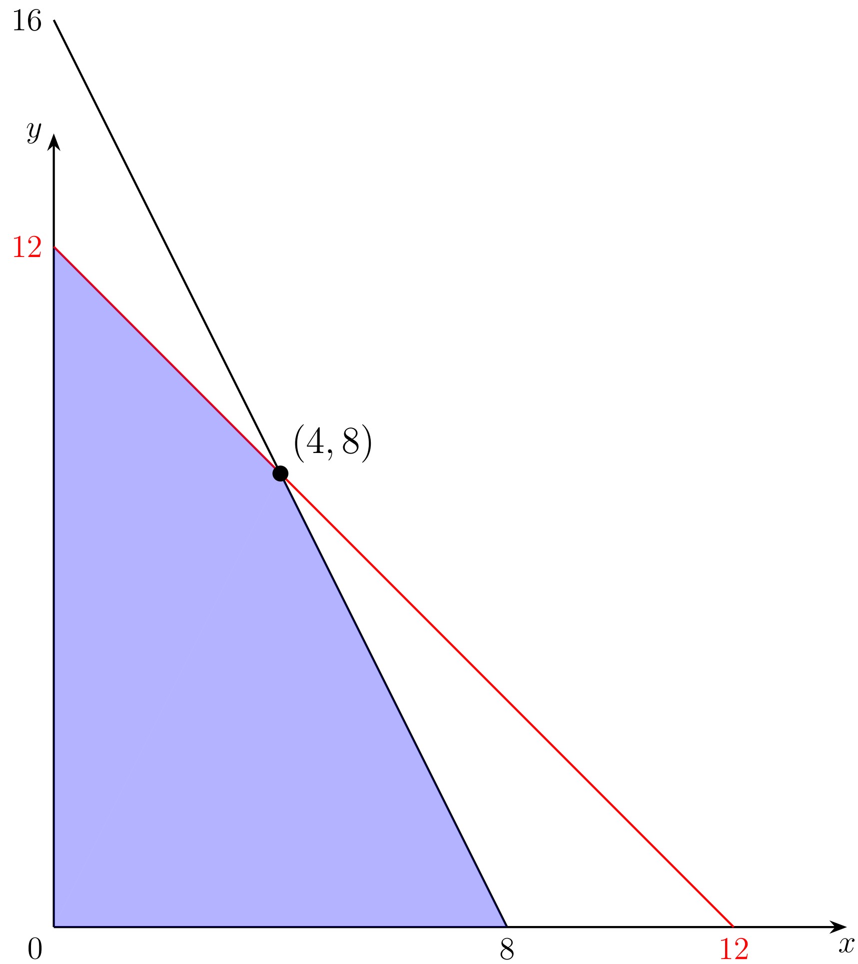


Figure 3.1: The red line is $x + y = 12$, the black line is $2x + y = 16$, and the blue region is the feasible region.

call these points critical points. They are listed as $(0, 0)$, $(0, 12)$, $(4, 8)$, and $(8, 0)$.

To maximize Niki's income, we will substitute these points in the objective function to see which point gives us the highest income per week. We list the results below:

Critical Points	**Income**
$(0, 0)$	$40(0) + 30(0) = \$0$
$(0, 12)$	$40(0) + 30(12) = \$360$
$(4, 8)$	$40(4) + 30(8) = \$400$
$(8, 0)$	$40(8) + 30(0) = \$320$

Clearly, the point $(4, 8)$ gives the most profit: \$400. Therefore, we conclude that Niki should work 4 hours at Job I and 8 hours at Job II.

Example 3.1.2. *A factory manufactures two types of gadgets, regular and premium. Each gadget requires the use of two operations, assembly and finishing, and there are at most 12 hours available for each operation. A regular gadget requires 1 hour of assembly and 2 hours of finishing, while a premium gadget needs 2 hours of assembly and 1 hour of finishing. Due to other restrictions, the company can make at most 7 gadgets a day. If a profit of \$20 is realized for each regular gadget and \$30 for a premium gadget, how many of each should be manufactured to maximize profit?*

Solution 3.1.2. *We choose our variables. Let x be the number of regular gadgets manufactured each day, and y be the number of premium gadgets manufactured each day.*

The objective function is

$$P = 20x + 30y$$

We now write the constraints. The company can make at most 7 gadgets a day, giving us:

$$x + y \leq 7$$

The regular gadget requires one hour of assembly and the premium gadget two hours, with at most 12 hours available for assembly:

$$x + 2y \leq 12$$

Similarly, for finishing, we have:

$$2x + y \leq 12$$

The non-negativity constraints are:

$$x \geq 0, \quad y \geq 0$$

We formulate the problem as follows: Maximize $P = 20x + 30y$ *Subject to:*

$$x + y \leq 7$$

$$x + 2y \leq 12$$

$$2x + y \leq 12$$

$$x \geq 0; \quad y \geq 0$$

We next graph the constraints and feasibility region.

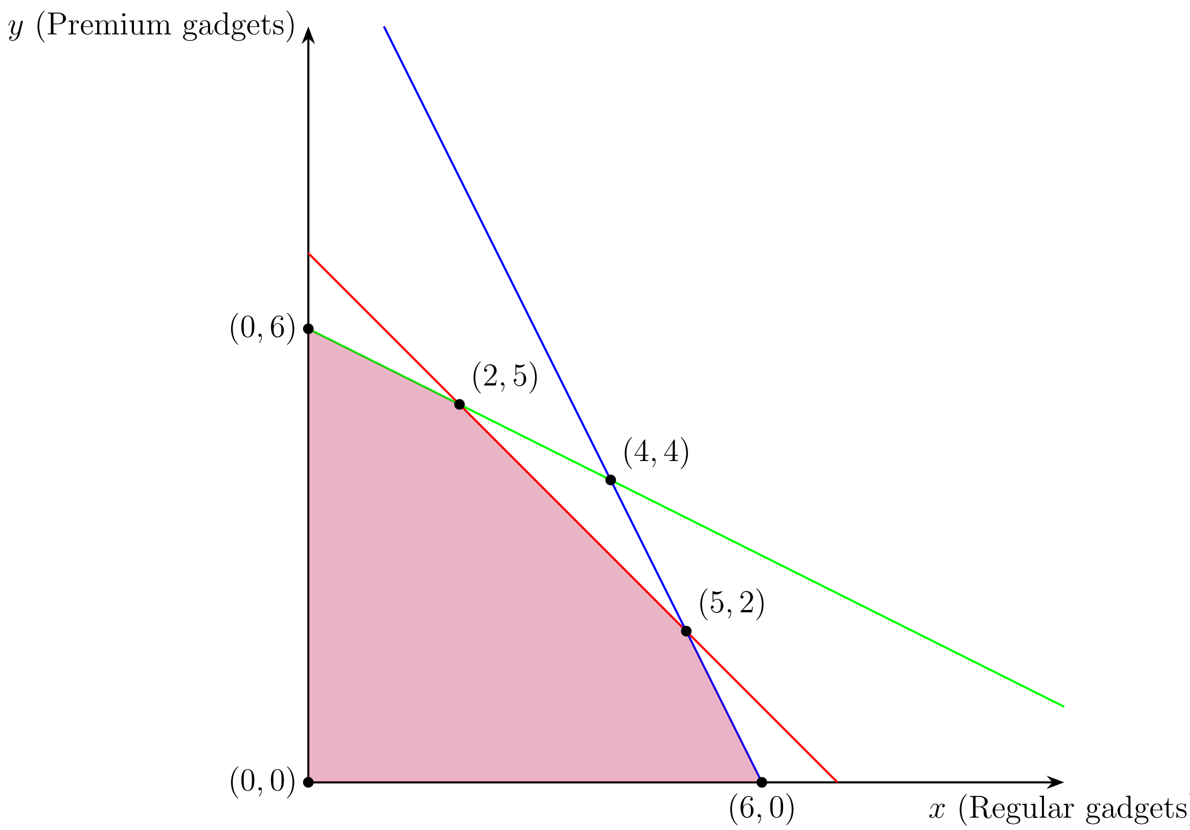


Figure 3.2: Feasibility region for the gadget factory optimization problem

Again, we have shaded the feasibility region, where all constraints are satisfied. Since the extreme value of the objective function always takes place at the vertices of the feasibility region, we identify all the critical points. They are listed as $(0,0)$, $(0,6)$, $(2,5)$, $(5,2)$, and $(6,0)$. Notice, $(4,4)$ is **not** a critical point because it is not on the edge of the critical region. To maximize profit, we will substitute these points in the objective function to see which point gives us the maximum profit each day. The results are listed below:

Critical Point	**Income**
$(0,0)$	$20(0) + 30(0) = \$0$
$(0,6)$	$20(0) + 30(6) = \$180$
$(2,5)$	$20(2) + 30(5) = \$190$
$(5,2)$	$20(5) + 30(2) = \$160$
$(6,0)$	$20(6) + 30(0) = \$120$

The point $(2,5)$ gives the most profit, and that profit is \$190. Therefore, we conclude that we should manufacture 2 regular gadgets and 5 premium gadgets daily to obtain the maximum profit of \$190.

So far, we have focused on "standard maximization problems" in which:

1. The objective function is to be maximized.
2. All constraints are of the form $ax + by \leq c$.
3. All variables are constrained to be non-negative ($x \geq 0$, $y \geq 0$).

We will next consider an example where that is not the case. Our next problem is said to have "mixed constraints" since some of the inequality constraints are of the form $ax + by \leq c$ and some are of the form $ax + by \geq c$. The non-negativity constraints are still an important requirement in any linear program.

Example 3.1.3. *Solve the following maximization problem graphically.*

$$\begin{aligned} \textit{Maximize} \quad & P = 10x + 15y \\ \textit{Subject to:} \quad & x + y \geq 1 \\ & x + 2y \leq 6 \\ & 2x + y \leq 6 \\ & x \geq 0; \quad y \geq 0 \end{aligned}$$

Solution 3.1.3. *The graph is shown below.*

The five critical points are listed in the figure above. The reader should observe that the first constraint $x + y \geq 1$ requires that the feasibility region

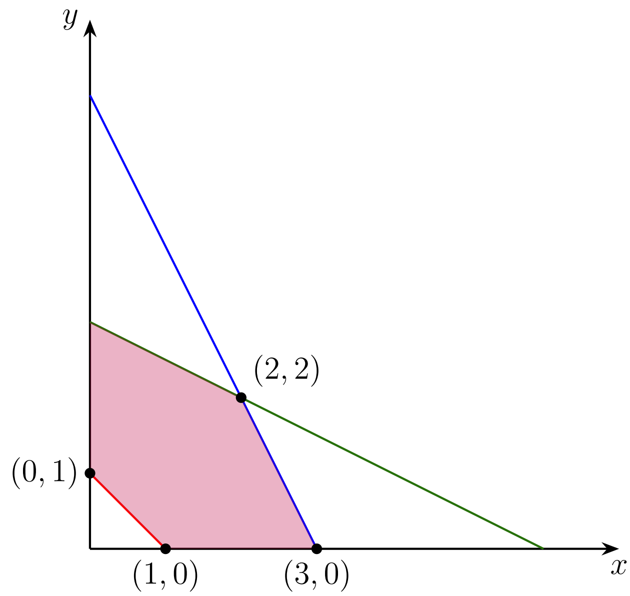


Figure 3.3: The red line is $x + y = 1$, the green line is $x + 2y = 6$ and the blue line is $2x + y = 6$.

must be bounded below by the line $x + y = 1$; the test point $(0, 0)$ does not satisfy $x + y \geq 1$, so we shade the region on the opposite side of the line from the test point $(0, 0)$.

Critical Point	*Income*
$(1, 0)$	$10(1) + 15(0) = \$10$
$(3, 0)$	$10(3) + 15(0) = \$30$
$(2, 2)$	$10(2) + 15(2) = \$50$
$(0, 3)$	$10(0) + 15(3) = \$45$
$(0, 1)$	$10(0) + 15(1) = \$15$

Clearly, the point $(2, 2)$ maximizes the objective function to a maximum value of 50. It is important to observe that if the point $(0, 0)$ lies on the line for a constraint, then $(0, 0)$ could not be used as a test point. We would need to select any other point that does not lie on the line to use as a test point in that situation.

Finally, we address an important question: Is it possible to determine the point that gives the maximum value without calculating the value at each critical point?

The answer is yes.

For example 3.1.2, we substituted the points $(0, 0)$, $(0, 6)$, $(2, 5)$, $(5, 2)$, and $(6, 0)$ in the objective function $P = 20x+30y$, and we got the values \$0, \$180, \$190, \$160, \$120, respectively. Sometimes that is not the most efficient way of finding the optimum solution. Instead, we could find the optimal value by also graphing the objective function.

To determine the largest P, we graph $P = 20x + 30y$ for any value P of our choice. Let us say, we choose $P = 60$. We graph $20x + 30y = 60$.

Now we move the line parallel to itself, that is, keeping the same slope at all times. Since we are moving the line parallel to itself, the slope is kept the same, and the only thing that is changing is the P. As we move away from the origin, the value of P increases. The largest possible value of P is realized when the line touches the last corner point of the feasibility region.

The figure below shows the movements of the line, and the optimum solution is achieved at the point $(2, 5)$. In maximization problems, as the line is being moved away from the origin, this optimum point is the farthest critical point.

Summary 3.1.
The Maximization Linear Programming Problems

1. *Write the objective function.*
2. *Write the constraints.*
   (a) *For the standard maximization linear programming problems, constraints are of the form:* $ax + by \leq c$.
   (b) *Since the variables are non-negative, we include the constraints:* $x \geq 0$, $y \geq 0$.
3. *Graph the constraints.*
4. *Shade the feasibility region.*
5. *Find the corner points.*
6. *Determine the corner point that gives the maximum value.*
   (a) *This is done by finding the value of the objective function at each corner point.*
   (b) *This can also be done by moving the line associated with the objective function.*

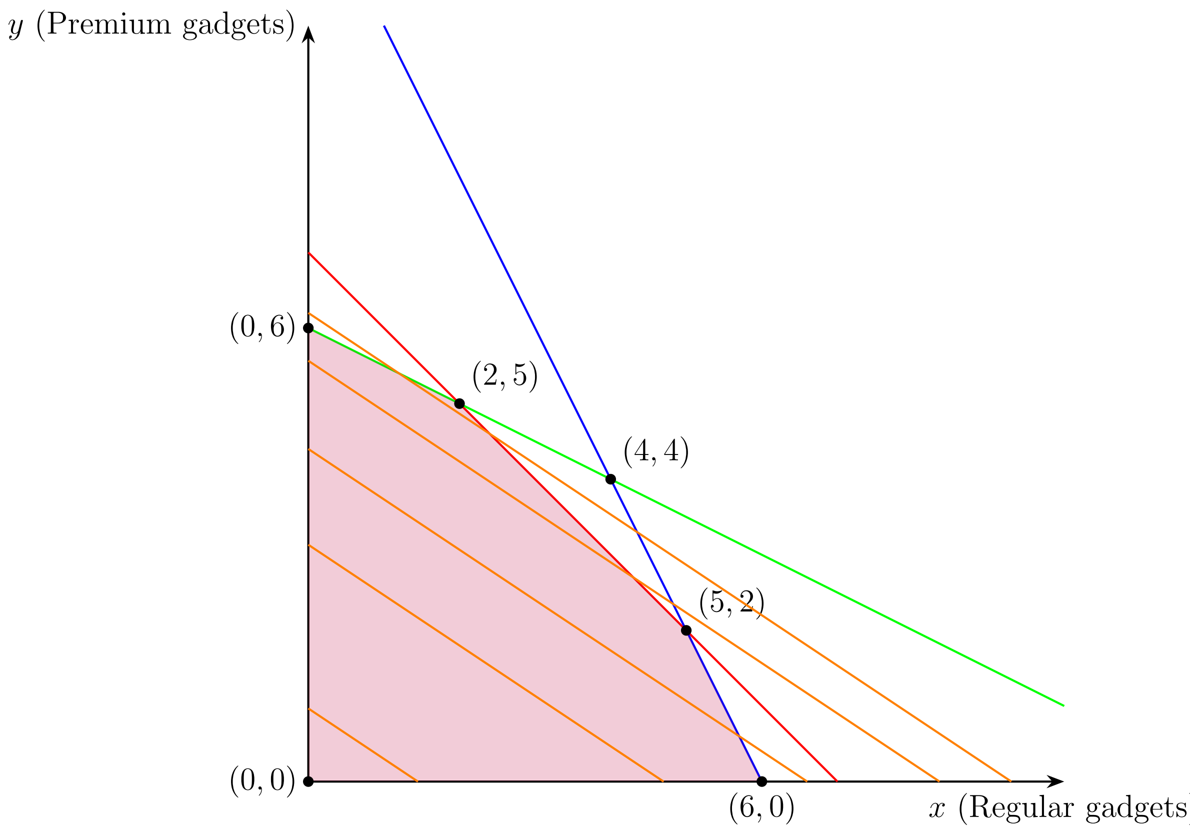


Figure 3.4: Feasibility region for the gadget factory optimization problem with profit lines.

3.2 Minimization Applications

In this section, you will learn to:

1. Formulate minimization linear programming problems.
2. Graph feasibility regions for minimization linear programming problems.
3. Determine optimal solutions for minimization linear programming problems.

Minimization linear programming problems are solved in much the same way as the maximization problems.

For the standard minimization linear program, the constraints are of the form $ax + by \geq c$, as opposed to the form $ax + by \leq c$ for the standard maximization problem. As a result, the feasible solution extends indefinitely to the upper right of the first quadrant, and is unbounded. But that is not a concern, since in order to minimize the objective function, the line associated with the objective function is moved towards the origin, and the critical point that minimizes the function is closest to the origin.

However, one should be aware that in the case of an unbounded feasibility region, the possibility of no optimal solution exists.

Example 3.2.1. *At a university, Professor Symons wishes to employ two people, John and Mary, to grade papers for his classes. John is a graduate student and can grade 20 papers per hour; John earns \$15 per hour for grading papers. Mary is a post-doctoral associate and can grade 30 papers per hour; Mary earns \$25 per hour for grading papers. Each must be employed at least one hour a week to justify their employment. If Professor Symons has at least 110 papers to be graded each week, how many hours per week should he employ each person to minimize the cost?*

Solution 3.2.1. *We choose the variables as follows: Let x be the number of hours per week John is employed, and y be the number of hours per week Mary is employed.*

The objective function is

$$C = 15x + 25y$$

The constraints are that each must work at least one hour each week:

$$x \geq 1$$

$$y \geq 1$$

John can grade 20 papers per hour and Mary 30 papers per hour, with at least 110 papers to be graded per week:

$$20x + 30y \geq 110$$

Additionally, x and y are non-negative:

$$x \geq 0$$

$$y \geq 0$$

The problem is thus formulated as: Minimize $C = 15x + 25y$ *Subject to:*

$$x \geq 1$$

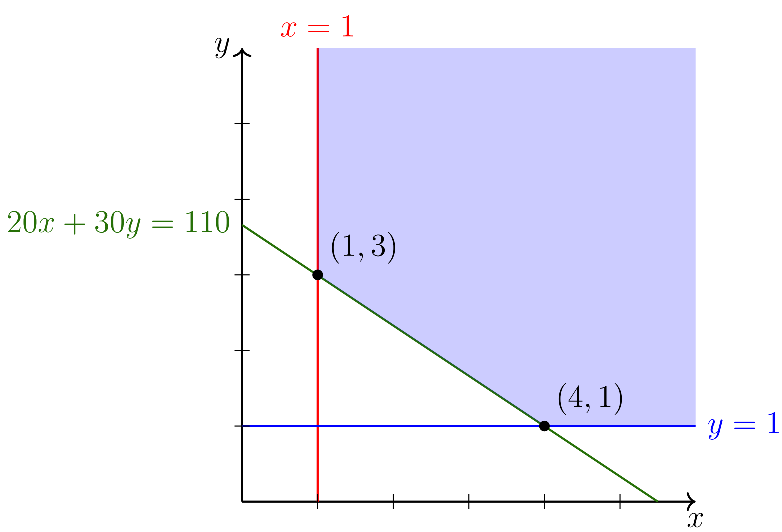
$$y \geq 1$$

$$20x + 30y \geq 110$$

$$x \geq 0$$

$$y \geq 0$$

To solve the problem, we graph the constraints as follows:



Again, we have shaded the feasibility region, where all constraints are satisfied. If we used test point $(0, 0)$ *that does not lie on any of the constraints, we observe that* $(0, 0)$ *does not satisfy any of the constraints* $x \geq 1$, $y \geq 1$, *and* $20x + 30y \geq 110$. *Thus, all the shading for the feasibility region lies on the opposite side of the constraint lines from the point* $(0, 0)$.

Alternatively, we could use test point $(4, 6)$, *which also does not lie on any of the constraint lines. We'd find that* $(4, 6)$ *does satisfy all of the inequality constraints. Consequently, all the shading for the feasibility region lies on the same side of the constraint lines as the point* $(4, 6)$.

Since the extreme value of the objective function always takes place at the vertices of the feasibility region, we identify the two critical points, $(1, 3)$ *and* $(4, 1)$*. To minimize cost, we will substitute these points in the objective function to see which point gives us the minimum cost each week. The results are listed below:*

Critical points	*Income*
$(1, 3)$	$15(1) + 25(3) = \$90$
$(4, 1)$	$15(4) + 25(1) = \$85$

The point $(4, 1)$ *gives the least cost, and that cost is $85. Therefore, we conclude that in order to minimize grading costs, Professor Symons should employ John for 4 hours a week and Mary for 1 hour a week at a cost of $85 per week.*

Example 3.2.2. *Professor Hamer is on a low cholesterol diet. During lunch at the college cafeteria, he always chooses between two meals, Pasta or Tofu. The table below lists the amount of protein, carbohydrates, and vitamins each meal provides along with the amount of cholesterol he is trying to minimize. Mr. Hamer needs at least 200 grams of protein, 960 grams of carbohydrates, and 40 grams of vitamins for lunch each month. Over this time period, how many days should he have the Pasta meal, and how many days the Tofu meal so that he gets the adequate amount of protein, carbohydrates, and vitamins and at the same time minimizes his cholesterol intake?*

	Pasta	***Tofu***
Protein (g)	*8*	*16*
Carbohydrates (g)	*60*	*40*
Vitamin C (g)	*2*	*2*
Cholesterol (mg)	*60*	*50*

Solution 3.2.2. *We choose the variables as follows: Let* x *be the number of days Mr. Hamer eats Pasta, and* y *the number of days he eats Tofu.*

The objective function for minimizing cholesterol intake is

$$C = 60x + 50y$$

The constraints for protein, carbohydrates, and vitamins are as follows:

$$\begin{aligned} 8x + 16y &\geq 200 \\ 60x + 40y &\geq 960 \\ 2x + 2y &\geq 40 \end{aligned}$$

Additionally, x *and* y *are non-negative:*

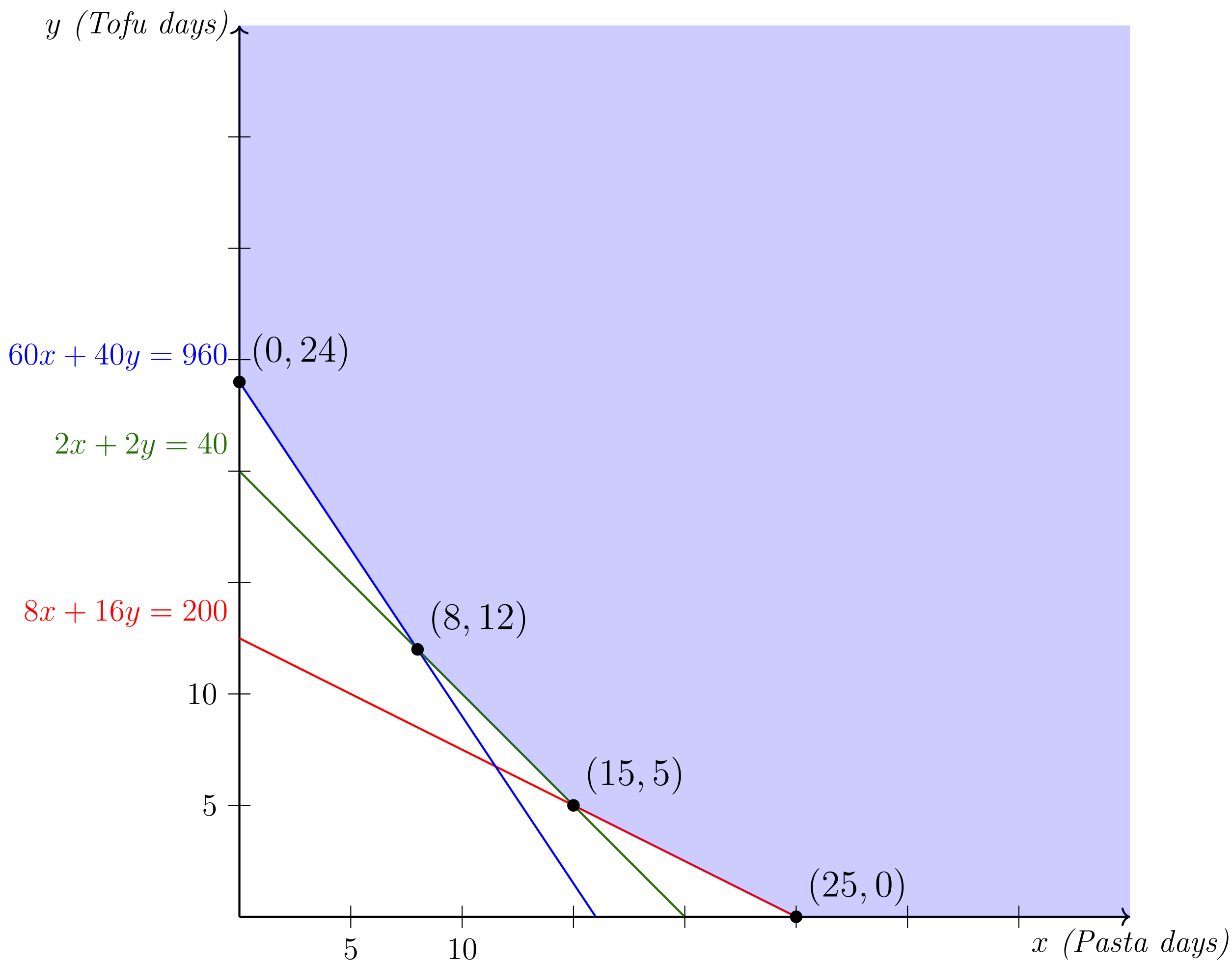
$$x \geq 0$$

$$y \geq 0$$

We summarize the problem as: Minimize $C = 60x + 50y$ *Subject to:*

$$\begin{aligned} 8x + 16y &\geq 200 \\ 60x + 40y &\geq 960 \\ 2x + 2y &\geq 40 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

To solve the problem, we graph the constraints and shade the feasibility region.



We have shaded the unbounded feasibility region, where all constraints are satisfied. To minimize the objective function, we find the vertices of the feasibility region. These vertices are $(0, 24)$, $(8, 12)$, $(15, 5)$, *and* $(25, 0)$. *To minimize cholesterol, we will substitute these points in the objective function to see which point gives us the smallest value. The results are listed below:*

Critical points	*Cholesterol*
$(0, 24)$	$60(0) + 50(24) = 1200\,mg$
$(8, 12)$	$60(8) + 50(12) = 1080\,mg$
$(15, 5)$	$60(15) + 50(5) = 1150\,mg$
$(25, 0)$	$60(25) + 50(0) = 1500\,mg$

The point $(8,12)$ *gives the least cholesterol, which is 1080 mg. This states that for every 20 meals, Professor Hamer should eat Pasta for 8 days and Tofu for 12 days.*

We must be aware that in some cases, a linear program may not have an optimal solution.

- A linear program can fail to have an optimal solution if there is no feasibility region. If the inequality constraints are not compatible, there may not be a region in the graph that satisfies all the constraints. If the linear program does not have a feasible solution satisfying all constraints, then it cannot have an optimal solution.

- A linear program can fail to have an optimal solution if the feasibility region is unbounded. The two minimization linear programs we examined had unbounded feasibility regions. The feasibility region was bounded by constraints on some sides but was not entirely enclosed by the constraints. Both of the minimization problems had optimal solutions. However, if we were to consider a maximization problem with a similar unbounded feasibility region, the linear program would have no optimal solution. No matter what values of x and y were selected, we could always find other values of x and y that would produce a higher value for the objective function. In other words, if the value of the objective function can be increased without bound in a linear program with an unbounded feasible region, there is no optimal maximum solution.

Although the method of solving minimization problems is similar to that of maximization problems, we still feel that we should summarize the steps involved.

Summary 3.2.
Minimization Linear Programming Problems

1. *Write the objective function.*
2. *Write the constraints.*
   (a) *For standard minimization linear programming problems, constraints are of the form:* $ax + by \geq c$.
   (b) *Since the variables are non-negative, include the constraints:* $x \geq 0$; $y \geq 0$.
3. *Graph the constraints.*
4. *Shade the feasibility region.*
5. *Find the corner points.*
6. *Determine the corner point that gives the minimum value.*
   (a) *This can be done by finding the value of the objective function at each corner point.*
   (b) *This can also be done by moving the line associated with the objective function.*
   (c) *There is the possibility that the problem has no solution.*