Applied Finite Mathematics

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Chapter 1

Linear Equations

In this chapter, you will learn to:

1. Graph a linear equation.
2. Find the slope of a line.
3. Determine an equation of a line.
4. Solve linear systems.
5. Do application problems using linear equations.

1.1 Graphing a Linear Equation

In this section, you will learn to:

1. Graph a line when you know its equation.
2. Graph a line when you are given its equation in parametric form.
3. Graph and find equations of vertical and horizontal lines.

1.1.1 Graphing a Line from its Equation

Equations whose graphs are straight lines are called linear equations. The following are some examples of linear equations:

$$\begin{aligned} 2x - 3y &= 6, \\ 3x &= 4y - 7, \\ y &= 2x - 5, \\ 2y &= 3, \\ x - 2 &= 0. \end{aligned}$$

A line is completely determined by two points. Therefore, to graph a linear equation, we need to find the coordinates of two points. This can be accomplished by choosing an arbitrary value for x or y and then solving for the other variable.

Example 1.1.1. *Graph the line $y = 3x + 2$.*

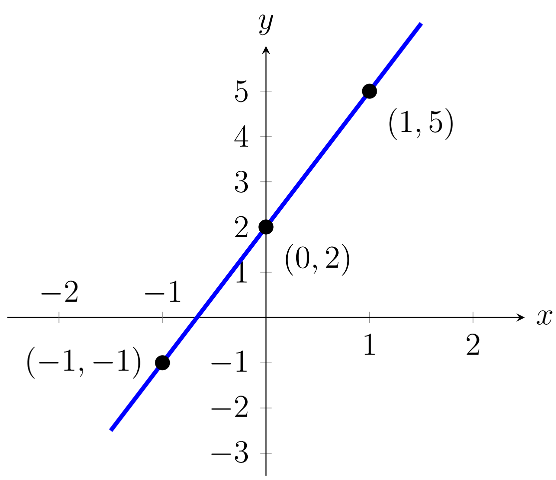
Solution 1.1.1. *We need to find the coordinates of at least two points. We arbitrarily choose $x = -1$, $x = 0$, and $x = 1$.*

If $x = -1$, then $y = 3(-1) + 2$ or $y = -1$. Therefore, $(-1, -1)$ is a point on this line.

If $x = 0$, then $y = 3(0) + 2$ or $y = 2$. Hence the point $(0, 2)$.

If $x = 1$, then $y = 5$, and we get the point $(1, 5)$. Below, the results are summarized, and the line is graphed.

x	y
-1	-1
0	2
1	5



Example 1.1.2. *Graph the line:* $2x + y = 4$

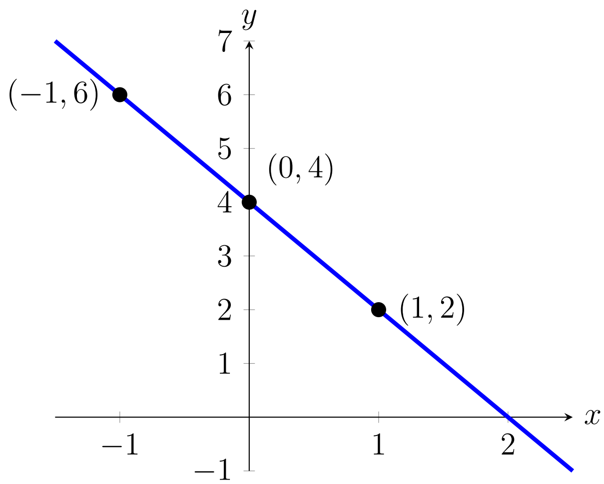
Solution 1.1.2. *Again, we need to find coordinates of at least two points. We arbitrarily choose* $x = -1$*,* $x = 0$*, and* $y = 2$*.*

If $x = -1$*, then* $2(-1) + y = 4$ *which results in* $y = 6$*. Therefore,* $(-1, 6)$ *is a point on this line.*

If $x = 0$*, then* $2(0) + y = 4$*, which results in* $y = 4$*. Hence the point* $(0, 4)$*.*

If $y = 2$*, then* $2x + 2 = 4$*, which yields* $x = 1$*, and gives the point* $(1, 2)$*. The table below shows the points, and the line is graphed.*

x	y
-1	6
0	4
1	2



1.1.2 Intercepts:

The points at which a line crosses the coordinate axes are called the intercepts. When graphing a line by plotting two points, using the intercepts is often preferred because they are easy to find.

- To find the value of the x-intercept, we let $y = 0$.
- To find the value of the y-intercept, we let $x = 0$.

Example 1.1.3. *Find the intercepts of the line:* $2x - 3y = 6$*, and graph.*

Solution 1.1.3. *To find the x-intercept, let* $y = 0$ *in the equation, and solve for* x.

$$\begin{aligned} 2x - 3(0) &= 6 \\ 2x &= 6 \\ x &= 3 \end{aligned}$$

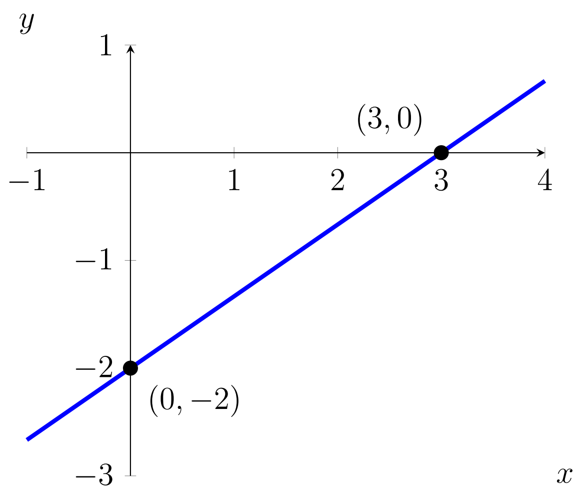
Therefore, the x-intercept is the point $(3, 0)$.

To find the y-intercept, let $x = 0$ *in the equation, and solve for* y.

$$\begin{aligned} 2(0) - 3y &= 6 \\ 0 - 3y &= 6 \\ -3y &= 6 \\ y &= -2 \end{aligned}$$

Therefore, the y-intercept is the point $(0, -2)$.

To graph the line, plot the points for the x-intercept $(3, 0)$ *and the y-intercept* $(0, -2)$, *and use them to draw the line.*



1.1.3 Graphing a Line from Its Equation in Parametric Form

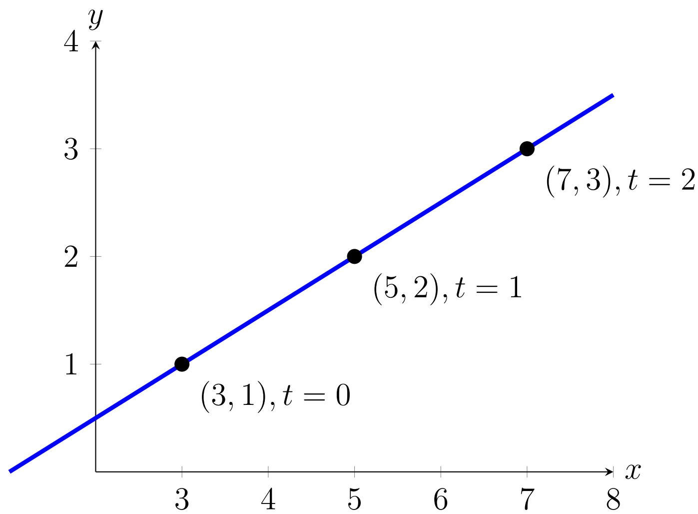
In higher math, equations of lines are sometimes written in parametric form. For example, $x = 3 + 2t, y = 1 + t$. The letter t is called the parameter or the dummy variable.

Parametric lines can be graphed by finding values for x and y by substituting numerical values for t. Plot the points using their (x, y) coordinates and use the points to draw the line.

Example 1.1.4. *Graph the line given by the parametric equations:* $x = 3 + 2t$, $y = 1 + t$

Solution 1.1.4. *Let* $t = 0, 1$ *and* 2*; for each value of* t*, find the corresponding values for* x *and* y*. The results are given in the table below.*

t	x	y
0	3	1
1	5	2
2	7	3



1.1.4 Horizontal and Vertical Lines

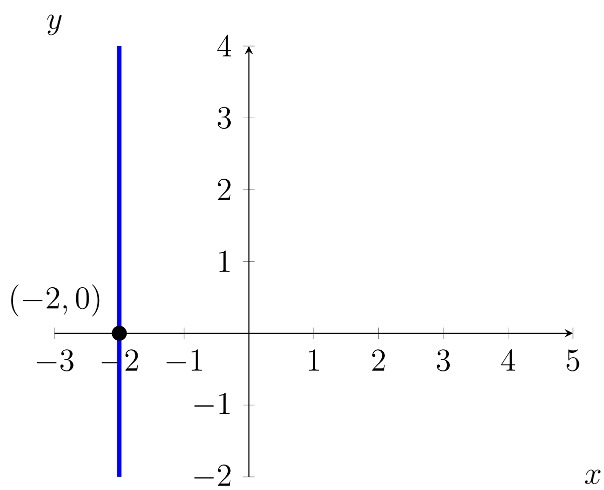
When an equation of a line has only one variable, the resulting graph is a horizontal or a vertical line.

The graph of the line $x = a$, where a is a constant, is a vertical line that passes through the point $(a, 0)$. Every point on this line has the x-coordinate equal to a, regardless of the y-coordinate.

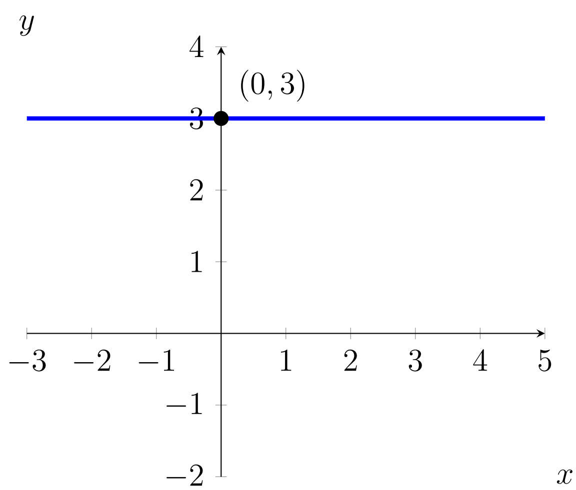
The graph of the line $y = b$, where b is a constant, is a horizontal line that passes through the point $(0, b)$. Every point on this line has the y-coordinate equal to b, regardless of the x-coordinate.

Example 1.1.5. *Graph the lines:* $x = -2$*, and* $y = 3$.

Solution 1.1.5. *The graph of the line* $x = -2$ *is a vertical line that has the* x*-coordinate* -2 *no matter what the* y*-coordinate is. The graph is a vertical line passing through point* $(-2, 0)$.



The graph of the line $y = 3$ is a horizontal line that has the y-coordinate 3 regardless of what the x-coordinate is. Therefore, the graph is a horizontal line that passes through point $(0, 3)$.



1.2 Slope of a Line

In this section, you will learn to:

1. Find the slope of a line.
2. Graph the line if a point and the slope are given.

In the last section, we learned to graph a line by choosing two points on the line. A graph of a line can also be determined if one point and the "steepness" of the line is known. The number that refers to the steepness or inclination of a line is called the slope of the line. From previous math courses, many of you remember slope as the "rise over run," or "the vertical change over the horizontal change" and have often seen it expressed as:

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

We give a precise definition.

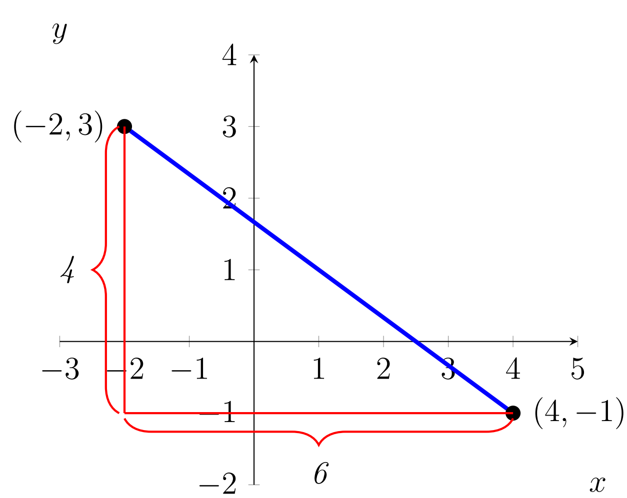
Definition 1.2.1. *If* (x_1, y_1) *and* (x_2, y_2) *are two different points on a line, the slope of the line is*

$$slope = m = \frac{y_2 - y_1}{x_2 - x_1}$$

Example 1.2.1. *Find the slope of the line passing through points* $(-2, 3)$ *and* $(4, -1)$*, and graph the line.*

Solution 1.2.1. *Let* $(x_1, y_1) = (-2, 3)$ *and* $(x_2, y_2) = (4, -1)$*, then the slope is*

$$slope = m = \frac{-1 - 3}{4 - (-2)} = \frac{-4}{6} = -\frac{2}{3}$$



To give the reader a better understanding, both the vertical change, -4, and the horizontal change, 6, are shown in the above figure.

When two points are given, it does not matter which point is denoted as (x_1, y_1) and which (x_2, y_2). The value for the slope will be the same.

In Example 1.2.1, if we instead choose $(x_1, y_1) = (4, -1)$ and $(x_2, y_2) = (-2, 3)$, then we will get the same value for the slope as we obtained earlier.

The steps involved are as follows:

$$\mathrm{m} = \frac{3 - (-1)}{-2 - 4} = \frac{4}{-6} = -\frac{2}{3}$$

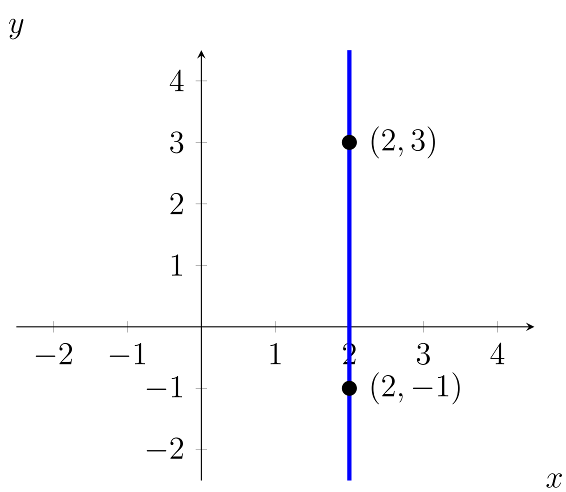
The student should further observe that

- If a line rises when going from left to right, then it has a positive slope. In this situation, as the value of x increases, the value of y also increases.
- If a line falls going from left to right, it has a negative slope; as the value of x increases, the value of y decreases.

Example 1.2.2. *Find the slope of the line that passes through the points $(2, 3)$ and $(2, -1)$, and graph.*

Solution 1.2.2. *Let $(x_1, y_1) = (2, 3)$ and $(x_2, y_2) = (2, -1)$, then the slope is*

$$m = \frac{-1 - 3}{2 - 2} = \frac{-4}{0} = \textit{undefined.}$$

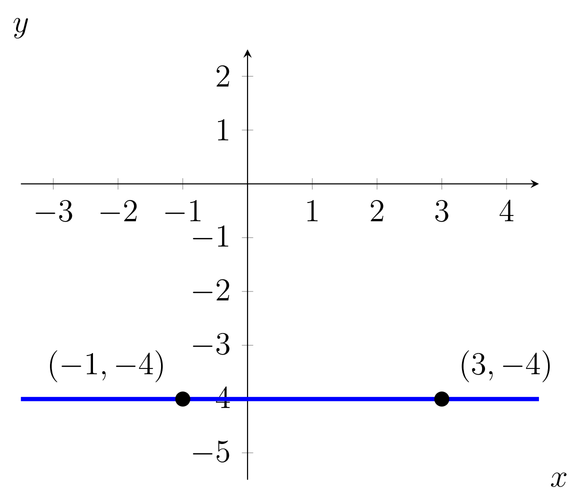


Note 1.2.1. *The slope of a vertical line is undefined.*

Example 1.2.3. *Find the slope of the line that passes through the points* $(-1, -4)$ *and* $(3, -4)$*.*

Solution 1.2.3. *Let* $(x_1, y_1) = (-1, -4)$ *and* $(x_2, y_2) = (3, -4)$*, then the slope is*

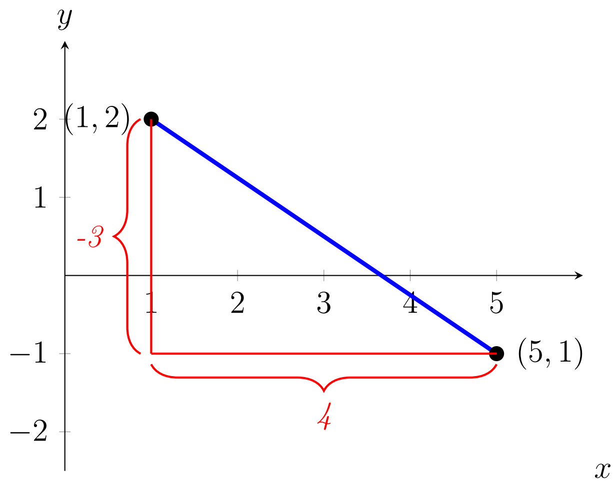
$$m = \frac{-4 - (-4)}{3 - (-1)} = \frac{0}{4} = 0.$$



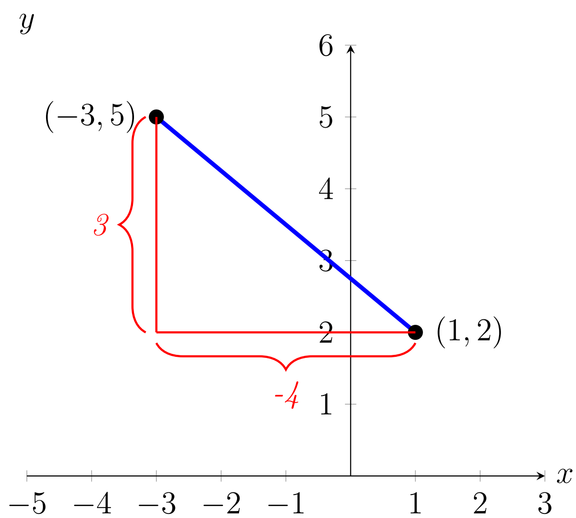
Note 1.2.2. *The slope of a horizontal line is* 0*.*

Example 1.2.4. *Graph the line that passes through the point* $(1, 2)$ *and has a slope of* $-\frac{3}{4}$*.*

Solution 1.2.4. *The slope equals* $\frac{rise}{run}$*. The fact that the slope is* $\frac{-3}{4}$ *means that for every rise of* -3 *units (fall of 3 units), there is a run of 4 units. So if from the given point* $(1, 2)$ *we go down 3 units and go right 4 units, we reach the point* $(5, -1)$*. The graph is obtained by connecting these two points.*



Alternatively, since $\frac{3}{-4}$ *represents the same number, the line can be drawn by starting at the point* $(1, 2)$ *and choosing a rise of 3 units followed by a run of* -4 *units. So from the point* $(1, 2)$*, we go up 3 units and to the left 4 units, thus reaching the point* $(-3, 5)$*, which is also on the same line. See figure below.*

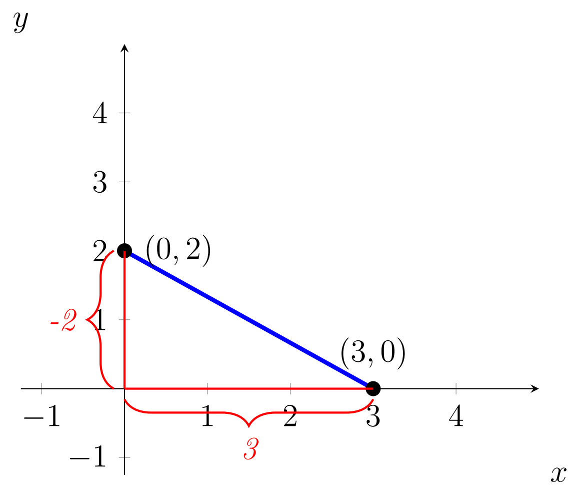


Example 1.2.5. *Find the slope of the line* $2x + 3y = 6$*.*

Solution 1.2.5. *In order to find the slope of this line, we will choose any two points on this line. Again, the selection of* x *and* y *intercepts seems to be a good choice. The* x*-intercept is* $(3, 0)$*, and the* y*-intercept is* $(0, 2)$*. Therefore,* *the slope is*

$$m = \frac{2-0}{3-0} = \frac{-2}{3}.$$

The graph below shows the line and the x-intercepts and y-intercepts:



Example 1.2.6. *Find the slope of the line $y = 3x + 2$.*

Solution 1.2.6. *We again find two points on the line, say $(0, 2)$ and $(1, 5)$. Therefore, the slope is*

$$m = \frac{5-2}{1-0} = \frac{3}{1} = 3.$$

Look at the slopes and the y-intercepts of the following lines.

Line	Slope	Y-Intercept
$y = 3x + 2$	3	2
$y = -2x + 5$	-2	5
$y = \frac{3}{2}x - 4$	$\frac{3}{2}$	-4

It is no coincidence that when an equation of the line is solved for y, the coefficient of the x term represents the slope, and the constant term represents the y-intercept. In other words, for the line $y = mx + b$, m is the slope, and b is the y-intercept.

Example 1.2.7. *Determine the slope and y-intercept of the line $2x+3y = 6$.*

Solution 1.2.7. *We solve for y:*

$$2x + 3y = 6$$

$$3y = -2x + 6$$
$$y = -\frac{2}{3}x + 2$$

The slope is equal to the coefficient of the x term, which is $-\frac{2}{3}$. The y-intercept is equal to the constant term, which is 2.

1.3 Determining the Equation of a Line

In this section, you will learn to:

1. Find an equation of a line if a point and the slope are given.
2. Find an equation of a line if two points are given.

So far, we were given an equation of a line and were asked to give information about it. For example, we were asked to find points on the line, find its slope, and even find intercepts. Now we are going to reverse the process. That is, we will be given either two points or a point and the slope of a line, and we will be asked to find its equation.

An equation of a line can be written in three forms: the slope-intercept form, the point-slope form, or the standard form. We will discuss each of them in this section.

A line is completely determined by two points or by a point and slope. The information we are given about a particular line will influence which form of the equation is most convenient to use. Once we know any form of the equation of a line, it is easy to re-express the equation in the other forms if needed.

The Slope-Intercept Form of a Line: $y = mx + b$

In the last section, we learned that the equation of a line whose slope $= m$ and y-intercept $= b$ is $y = mx + b$. This is called the slope-intercept form of the line and is the most commonly used form.

Example 1.3.1. *Find an equation of a line whose slope is 5, and y-intercept is 3.*

Solution 1.3.1. *Since the slope is $m = 5$, and the y-intercept is $b = 3$, the equation is $y = 5x + 3$.*

Example 1.3.2. *Find the equation of the line that passes through the point* $(2, 7)$ *and has slope* 3.

Solution 1.3.2. *Since* $m = 3$, *the partial equation is* $y = 3x + b$. *Now* b *can be determined by substituting the point* $(2, 7)$ *in the equation* $y = 3x + b$.

$7 = 3(2) + b$

$b = 1$

Therefore, the equation is $y = 3x + 1$.

Example 1.3.3. *Find an equation of the line that passes through the points* $(-1, 2)$ *and* $(1, 8)$.

Solution 1.3.3. $m = \frac{8-2}{1-(-1)} = \frac{6}{2} = 3$.

So the partial equation is $y = 3x + b$. *We can use either of the two points* $(-1, 2)$ *or* $(1, 8)$ *to find* b. *Substituting* $(-1, 2)$ *gives*

$2 = 3(-1) + b$

$5 = b$

So the equation is $y = 3x + 5$.

Example 1.3.4. *Find an equation of the line that has x-intercept* 3, *and y-intercept* 4.

Solution 1.3.4. *The x-intercept* $= 3$, *and y-intercept* $= 4$ *correspond to the points* $(3, 0)$ *and* $(0, 4)$, *respectively.*

$m = \frac{4-0}{0-3} = \frac{-4}{3}$

We are told the y-intercept is 4; *thus* $b = 4$.

Therefore, the equation is $y = -\frac{4}{3}x + 4$.

The Point-Slope Form of a Line: $y - y1 = m(x - x1)$

The point-slope form is useful when we know two points on the line and want to find the equation of the line.

Let L be a line with slope m, and known to contain a specific point $(x1, y1)$. If (x, y) is any other point on the line L, then the definition of a slope leads us to the point-slope form or point-slope formula.

The slope is $\frac{y-y1}{x-x1} = m$

Multiplying both sides by $(x - x1)$ gives the point-slope form:

$y - y1 = m(x - x1)$

Example 1.3.5. *Find the point-slope form of the equation of a line that has slope* 1.5 *and passes through the point* $(12, 4)$.

Solution 1.3.5. *Substituting the point* $(x1, y1) = (12, 4)$ *and* $m = 1.5$ *in the point-slope formula, we get*

$y - 4 = 1.5(x - 12)$

The student may be tempted to simplify this into the slope-intercept form $y = mx + b$. *But since the problem specifically requests point-slope form, we will not simplify it.*

The Standard Form of a Line: $Ax + By = C$

Another useful form of the equation of a line is the standard form.

If we know the equation of a line in point-slope form, $y - y1 = m(x - x1)$, or if we know the equation of the line in slope-intercept form $y = mx + b$, we can simplify the formula to have all terms for the x and y variables on one side of the equation, and the constant on the other side of the equation.

The result is referred to as the standard form of the line: $Ax + By = C$.

Example 1.3.6. *Using the point-slope formula, find the standard form of an equation of the line that passes through the point (2, 3) and has slope* $-\frac{3}{5}$.

Solution 1.3.6. *Substituting the point* $(2, 3)$ *and* $m = -\frac{3}{5}$ *in the point-slope formula, we get*

$$y - 3 = -\frac{3}{5}(x - 2).$$

Multiplying both sides by 5 gives us

$$5(y - 3) = -3(x - 2),$$

$$5y - 15 = -3x + 6,$$

$$3x + 5y = 21 \textit{ Standard Form.}$$

Example 1.3.7. *Find the standard form of the line that passes through the points (1, -2) and (4, 0).*

Solution 1.3.7. *First, we find the slope:* $m = \frac{0-(-2)}{4-1} = \frac{2}{3}$.

Then, the point-slope form is: $y - (-2) = \frac{2}{3}(x - 1)$.

Multiplying both sides by 3 gives us

$$3(y + 2) = 2(x - 1),$$

$$3y + 6 = 2x - 2,$$

$$-2x + 3y = -8,$$

$$2x - 3y = 8 \text{ Standard Form.}$$

Example 1.3.8. *Write the equation* $y = -\frac{2}{3}x + 3$ *in the standard form.*

Solution 1.3.8. *Multiplying both sides of the equation by 3, we get*

$$3y = -2x + 9,$$

$$2x + 3y = 9 \text{ Standard Form.}$$

Example 1.3.9. *Write the equation* $3x - 4y = 10$ *in the slope-intercept form.*

Solution 1.3.9. *Solving for* y*, we get*

$$-4y = -3x + 10,$$

$$y = \frac{3}{4}x - \frac{5}{2} \text{ Slope Intercept Form.}$$

Example 1.3.10. *Find the slope of the following lines, by inspection.*

1. $3x - 5y = 10$
2. $2x + 7y = 20$
3. $4x - 3y = 8$

Solution 1.3.10.
1. *For* $3x - 5y = 10$*, we have* $A = 3$ *and* $B = -5$*, therefore,* $m = -\frac{A}{B} = -\frac{3}{-5} = \frac{3}{5}$.
2. *For* $2x + 7y = 20$*, we have* $A = 2$ *and* $B = 7$*, therefore,* $m = -\frac{A}{B} = -\frac{2}{7}$.

3. *For* $4x - 3y = 8$, *we have* $A = 4$ *and* $B = -3$, *therefore,* $m = -\frac{A}{B} = -\frac{4}{-3} = \frac{4}{3}$.

Example 1.3.11. *Find an equation of the line that passes through* $(2, 3)$ *and has slope* $-\frac{4}{5}$.

Solution 1.3.11. *Since the slope of the line is* $-\frac{4}{5}$, *we know that the left side of the equation is* $4x + 5y$, *and the partial equation is going to be*

$$4x + 5y = c.$$

Of course, c *can easily be found by substituting for* x *and* y.

$$4(2) + 5(3) = c,$$

$$8 + 15 = c,$$

$$23 = c.$$

The desired equation is

$$4x + 5y = 23.$$

If you use this method often enough, you can do these problems very quickly. We summarize the forms for equations of a line below:

Summary 1.3.1. *Equations of Lines*

- *Slope-Intercept form:* $y = mx + b$, *where* m *is the slope and* b *is the* y*-intercept.*
- *Point-Slope form:* $y - y_1 = m(x - x_1)$, *where* m *is the slope and* (x_1, y_1) *is a point on the line.*
- *Standard form:* $Ax + By = C$.
- *Horizontal Line:* $y = b$, *where* b *is the* y*-intercept.*
- *Vertical Line:* $x = a$, *where* a *is the* x*-intercept.*

1.4 Applications

In this section, you will learn to use linear functions to model real-world applications.

Now that we have learned to determine equations of lines, we get to apply these ideas in a variety of real-life situations. Read the problem carefully. Highlight important information. Keep track of which values correspond to the independent variable (x) and which correspond to the dependent variable (y).

Example 1.4.1. *A taxi service charges \$0.50 per mile plus a \$5 flat fee. What will be the cost of traveling 20 miles? What will be cost of traveling x miles?*

Solution 1.4.1. *Let x be the distance traveled, in miles, and y be the cost in dollars.*

The cost of traveling 20 miles is $y = (0.50)(20) + 5 = 10 + 5 = 15$ *dollars.*

The cost of traveling x miles is $y = (0.50)(x) + 5 = 0.50x + 5$ *dollars.*

In this problem, \$0.50 per mile is referred to as the variable cost, and the flat charge \$5 as the fixed cost. Now if we look at our cost equation $y = 0.50x+5$*, we can see that the variable cost corresponds to the slope and the fixed cost to the y-intercept.*

Example 1.4.2. *The variable cost to manufacture a product is \$10 per item and the fixed cost \$2500. If x represents the number of items manufactured and y represents the total cost, write the cost function.*

Solution 1.4.2. *The variable cost of \$10 per item tells us that* $m = 10$*. The fixed cost represents the y-intercept, so* $b = 2500$*. Therefore, the cost equation is* $y = 10x + 2500$*.*

Example 1.4.3. *It costs \$750 to manufacture 25 items, and \$1000 to manufacture 50 items. Assuming a linear relationship holds, find the cost equation, and use this function to predict the cost of 100 items.*

Solution 1.4.3. *Let x be the number of items manufactured, and let y be the cost.*

Solving this problem is equivalent to finding an equation of a line that passes through the points $(25, 750)$ *and* $(50, 1000)$*.*

$m = \frac{1000-750}{50-25} = 10$

Therefore, the partial equation is $y = 10x + b$*.*

By substituting one of the points in the equation, we get $b = 500$.

Therefore, the cost equation is $y = 10x + 500$.

To find the cost of 100 items, substitute $x = 100$ *in the equation* $y = 10x+500$.

So the cost $= y = 10(100) + 500 = 1500$.

It costs $1500 to manufacture 100 items.

Example 1.4.4. *The freezing temperature of water in Celsius is 0 degrees, and in Fahrenheit, it's 32 degrees. The boiling temperatures of water in Celsius and Fahrenheit are 100 degrees and 212 degrees, respectively. Write a conversion equation from Celsius to Fahrenheit and use this equation to convert 30 degrees Celsius into Fahrenheit.*

Solution 1.4.4. *Let's look at what is given:*

Celsius	*Fahrenheit*
0	*32*
100	*212*

Solving this problem is equivalent to finding an equation of a line that passes through the points $(0, 32)$ *and* $(100, 212)$. *Since we are finding a linear relationship, we are looking for an equation* $y = mx + b$, *or in this case,* $F = mC + b$, *where* C *represents the temperature in Celsius, and* F *represents the temperature in Fahrenheit.*

The slope $m = \frac{212-32}{100-0} = 95$.

The equation is $F = 95C + b$.

Substituting the point $(0, 32)$, *we get* $F = 95C + 32$.

To convert 30 degrees Celsius into Fahrenheit, substitute $C = 30$ *in the equation:*

$$F = 95C + 32$$

$$F = 95(30) + 32 = 86$$

Example 1.4.5. *The population of Canada in the year 1980 was 24.5 million, and in the year 2010, it was 34 million. The population of Canada over that time period can be approximately modeled by a linear function. Let* x

represent time as the number of years after 1980, and let y represent the size of the population.

a. Write the linear function that gives a relationship between the time and the population.

b. Assuming the population continues to grow linearly in the future, use this equation to predict the population of Canada in the year 2025.

Solution 1.4.5. *The problem can be made easier by using 1980 as the base year, which means we choose the year 1980 as the year zero. This will make the year 2010 correspond to year 30. Now, let's look at the information we have:*

Year	Population
0 (1980)	24.5 million
30 (2010)	34 million

a. Solving this problem is equivalent to finding an equation of a line that passes through the points (0, 24.5) and (30, 34). We use these two points to find the slope:

$$m = \frac{34 - 24.5}{30 - 0} = \frac{9.5}{30} = 0.32$$

The y-intercept occurs when $x = 0$, *so* $b = 24.5$.

So, the equation relating time (x) and population (y) is:

$$y = 0.32x + 24.5$$

b. Now, to predict the population in the year 2025, we let $x = 2025 - 1980 = 45$*:*

$$y = 0.32x + 24.5$$

$$y = 0.32(45) + 24.5 = 38.9$$

In the year 2025, we predict that the population of Canada will be 38.9 million people.

Note that we assumed the population trend will continue to be linear. Therefore, if population trends change and this assumption does not continue to be true in the future, this prediction may not be accurate.

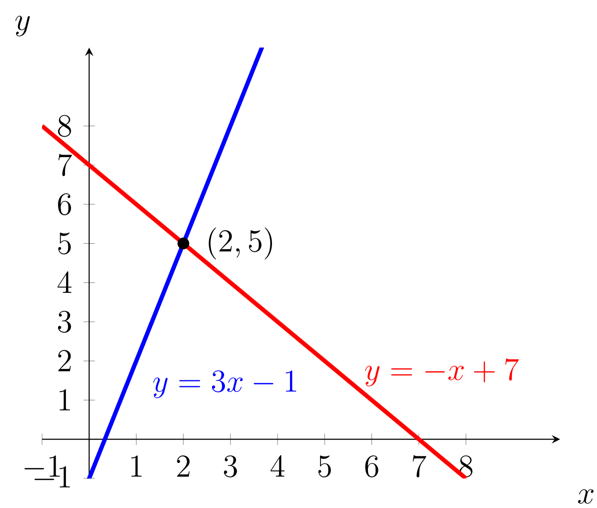
1.5 More Applications

1.5.1 Finding the Point of Intersection of Two Lines

In this section, we will do application problems that involve the intersection of lines. Therefore, before we proceed any further, we will first learn how to find the intersection of two lines.

Example 1.5.1. *Find the intersection of the line* $y = 3x - 1$ *and the line* $y = -x + 7$.

Solution 1.5.1. *We graph both lines on the same axes, as shown below, and read the solution* $(2, 5)$.



Finding an intersection of two lines graphically is not always easy or practical; therefore, we will now learn to solve these problems algebraically.

At the point where two lines intersect, the x *and* y *values for both lines are the same. So in order to find the intersection, we either let the* x*-values or the* y*-values equal.*

If we were to solve the above example algebraically, it will be easier to let the

y-values equal. Since $y = 3x - 1$ for the first line, and $y = -x + 7$ for the second line, by letting the y-values equal, we get:

$$\begin{aligned} 3x - 1 &= -x + 7 \\ 4x &= 8 \\ x &= 2 \end{aligned}$$

By substituting $x = 2$ in any of the two equations, we obtain $y = 5$. Hence, the solution is $(2, 5)$.

1.5.2 Solving Systems of Equations: The Elimination Method

A common algebraic method used to solve systems of equations is called the elimination method. The objective is to eliminate one of the two variables by adding the left and right sides of the equations together. Once one variable is eliminated, we have an equation with only one variable that can be solved. Finally, by substituting the value of the variable that has been found in one of the original equations, we can get the value of the other variable.

Example 1.5.2. *Find the intersection of the lines $2x + y = 7$ and $3x - y = 3$ by the elimination method.*

Solution 1.5.2. *We add the left and right sides of the two equations:*

$$\begin{aligned} 2x + y &= 7 \\ 3x - y &= 3 \\ 5x &= 10 \\ x &= 2 \end{aligned}$$

Now we substitute $x = 2$ into any of the two equations and solve for y:

$$\begin{aligned} 2(2) + y &= 7 \\ 4 + y &= 7 \\ y &= 3 \end{aligned}$$

Therefore, the solution is $(2, 3)$*.*

Example 1.5.3. *Solve the system of equations* $x + 2y = 3$ *and* $2x + 3y = 4$ *by the elimination method.*

Solution 1.5.3. *If we add the two equations directly, none of the variables are eliminated. However, the variable* x *can be eliminated by multiplying the first equation by* -2 *and leaving the second equation unchanged:*

$$\begin{aligned} -2x - 4y &= -6 \\ 2x + 3y &= 4 \\ -y &= -2 \\ y &= 2 \end{aligned}$$

Substituting $y = 2$ *into* $x + 2y = 3$*, we get:*

$$\begin{aligned} x + 2(2) &= 3 \\ x + 4 &= 3 \\ x &= -1 \end{aligned}$$

Therefore, the solution is $(-1, 2)$*.*

Example 1.5.4. *Solve the system of equations* $3x - 4y = 5$ *and* $4x - 5y = 6$*.*

Solution 1.5.4. *This time, we multiply the first equation by* -4 *and the second by* 3 *before adding (the choice of numbers is not unique):*

$$\begin{aligned} -12x + 16y &= -20 \\ 12x - 15y &= 18 \\ y &= -2 \end{aligned}$$

By substituting $y = -2$ into any one of the equations, we get:

$$\begin{aligned} 3x - 4(-2) &= 5 \\ 3x + 8 &= 5 \\ 3x &= -3 \\ x &= -1 \end{aligned}$$

Hence, the solution is $(-1, -2)$.

1.5.3 Supply, Demand, and the Equilibrium Market Price

In a free market economy, the supply curve for a commodity is the number of items of a product that can be made available at different prices, and the demand curve is the number of items the consumer will buy at different prices. As the price of a product increases, its demand decreases, and supply increases. On the other hand, as the price decreases, the demand increases, and supply decreases. The equilibrium price is reached when the demand equals the supply.

Example 1.5.5. *The supply curve for a product is given by $y = 3.5x - 14$, and the demand curve for the same product is given by $y = -2.5x + 34$, where x is the price and y is the number of items produced. Find the following:*

1. *How many items will be supplied at a price of \$10?*
2. *How many items will be demanded at a price of \$10?*
3. *Determine the equilibrium price.*
4. *How many items will be produced at the equilibrium price?*

Solution 1.5.5.

1. *To find the number of items supplied at a price of \$10, we substitute $x = 10$ into the supply equation $y = 3.5x - 14$. Therefore, $y = 3.5(10) - 14 = 21$ items will be supplied.*
2. *To find the number of items demanded at a price of \$10, we substitute $x = 10$ into the demand equation $y = -2.5x + 34$. Therefore, $y = -2.5(10) + 34 = 9$ items will be demanded.*

3. *To determine the equilibrium price, we set the supply equal to the demand:*

$$3.5x - 14 = -2.5x + 34$$

Solving for x:

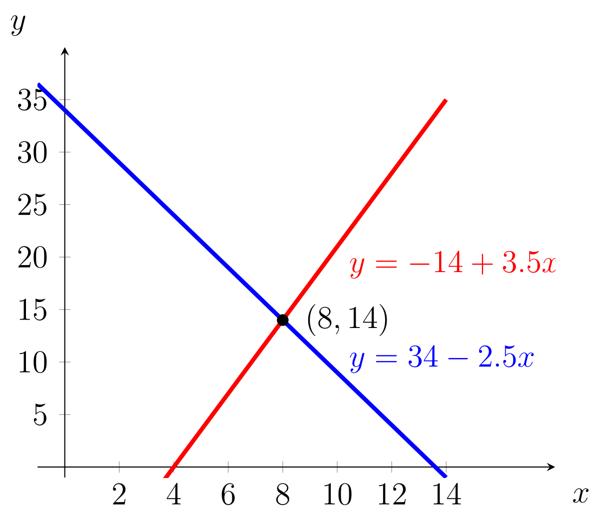
$$6x = 48$$

$$x = 8$$

So, the equilibrium price is $x = 8$.

4. *To find how many items will be produced at the equilibrium price, we substitute $x = 8$ into either the supply or the demand equation. Using the supply equation, we get $y = 3.5(8) - 14 = 14$ items will be produced.*

The graph shows the intersection of the supply and demand functions and their point of intersection, $(8, 14)$.



1.5.4 Break-Even Point

In a business, profit is generated by selling products. If a company sells x number of items at a price P, then the revenue R is the price multiplied by the number of items sold: $R = P \cdot x$. The production costs C are the sum of the variable costs and the fixed costs, often written as $C = mx + b$, where x is the number of items manufactured.

- The slope m is called the marginal cost and represents the cost to produce one additional item or unit.
- The variable cost, mx, depends on how much is being produced.
- The fixed cost b is constant and does not change regardless of production quantity.

Profit is equal to revenue minus cost: $Profit = R - C$. A company makes a profit if the revenue is greater than the cost, and there is a loss if the cost is greater than the revenue. The point on the graph where the revenue equals the cost is called the break-even point, and at this point, the profit is 0.

Example 1.5.6. *If the revenue function of a product is $R = 5x$ and the cost function is $C = 3x + 12$, find the following:*

1. *If 4 items are produced, what will the revenue be?*
2. *What is the cost of producing 4 items?*
3. *How many items should be produced to break even?*
4. *What will be the revenue and cost at the break-even point?*

Solution 1.5.6.

1. *To find the revenue when 4 items are produced, we substitute $x = 4$ in the revenue equation $R = 5x$, and the answer is $R = 20$.*
2. *To find the cost of producing 4 items, we substitute $x = 4$ in the cost equation $C = 3x + 12$, and the answer is $C = 24$.*
3. *To determine the number of items required to break even, we set the revenue equal to the cost:*

   $$5x = 3x + 12$$

   Solving for x:

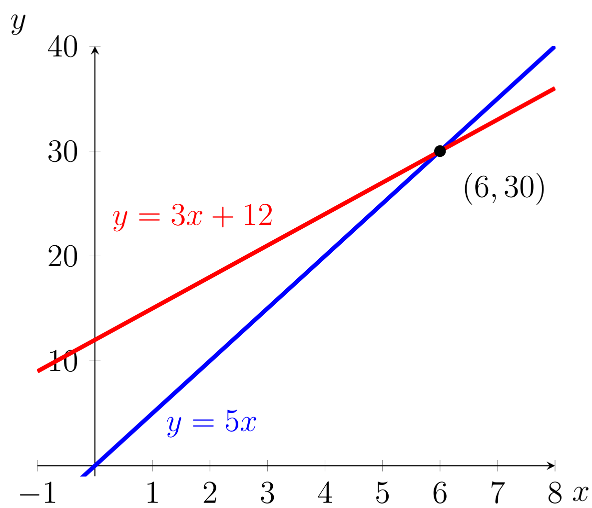
   $$2x = 12$$

   $$x = 6$$

   So, 6 items should be produced to break even.
4. *At the break-even point, when $x = 6$, we can substitute $x = 6$ in either the revenue or the cost equation to find that both revenue and cost are*

> *equal to* 30*. Therefore, the revenue and cost at the break-even point are both* 30*.*

The graph below shows the intersection of the revenue and cost functions and their point of intersection, $(6, 30)$*.*



Chapter 2

Matrices

In this chapter, you will learn to:

1. Do matrix operations.
2. Solve linear systems using the Gauss-Jordan method.
3. Solve linear systems using the matrix inverse method.
4. Do application problems

2.1 Introduction to Matrices

In this section, you will learn to:

1. Add and subtract matrices.
2. Multiply a matrix by a scalar.
3. Multiply two matrices.

A matrix is a 2-dimensional array of numbers arranged in rows and columns. Matrices provide a method of organizing, storing, and working with mathematical information. They have numerous applications and uses in the real world.

(TODO: fix references here)Matrices are particularly useful in working with models based on systems of linear equations, which we'll explore in sections

2.2, 2.3, and 2.4 of this chapter. They are also used in encryption (section 2.5) and economic modeling (section 2.6).

Furthermore, matrices play a crucial role in optimization problems in (TODO: fix references here) Chapter 4, such as maximizing profit or revenue and minimizing costs. They are used in business for scheduling, routing transportation and shipments, and managing inventory. Matrices are applicable in various fields where data organization and problem-solving are essential.

The use of matrices has expanded with the increase in available data across different domains. They are fundamental tools for organizing data and solving problems in science fields like physics, chemistry, biology, genetics, meteorology, and economics. In computer science, matrix mathematics is foundational for animation in movies and video games.

Moreover, matrices are used in analyzing network diagrams, such as social media connections on platforms like Facebook, LinkedIn, etc. The mathematics of network diagrams falls under "graph theory" and relies on matrices to organize information in graphs that depict connections and associations in a network.

A matrix is a rectangular array of numbers. Matrices are useful in organizing and manipulating large amounts of data. In order to get some idea of what matrices are all about, we will look at the following example.

Example 2.1.1. *Fine Furniture Company makes chairs and tables at its San Jose, Hayward, and Oakland factories. The total production, in hundreds, from the three factories for the years 2014 and 2015 is listed in the table below.*

	2014	*2015*		
	CHAIRS	*TABLES*	*CHAIRS*	*TABLES*
SAN JOSE	30	18	36	20
HAYWARD	20	12	24	18
OAKLAND	16	10	20	12

1. *Represent the production for the years 2014 and 2015 as the matrices A and B.*

2. *Find the difference in sales between the years 2014 and 2015.*

3. *The company predicts that in the year 2020 the production at these factories will be double that of the year 2014. What will the production be for the year 2020?*

Solution 2.1.1. 1. *The matrices are as follows:*

$$A = \begin{bmatrix} 30 & 18 \\ 20 & 12 \\ 16 & 10 \end{bmatrix} \qquad B = \begin{bmatrix} 36 & 20 \\ 24 & 18 \\ 20 & 12 \end{bmatrix}$$

2. *We are looking for the matrix $B-A$. When two matrices have the same number of rows and columns, they can be added or subtracted entry by entry. Therefore, we get:*

$$B - A = \begin{bmatrix} 36-30 & 20-18 \\ 24-20 & 18-12 \\ 20-16 & 12-10 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 4 & 6 \\ 4 & 2 \end{bmatrix}$$

3. *We would like a matrix that is twice the matrix of 2014, i.e., $2A$. Whenever a matrix is multiplied by a number, each entry is multiplied by the number.*

$$2A = 2\begin{bmatrix} 30 & 18 \\ 20 & 12 \\ 16 & 10 \end{bmatrix} = \begin{bmatrix} 60 & 36 \\ 40 & 24 \\ 32 & 20 \end{bmatrix}$$

2.1.1 Vocabulary

Before we go any further, we need to familiarize ourselves with some terms that are associated with matrices.

The numbers in a matrix are called the entries or the elements of a matrix.

Whenever we talk about a matrix, we need to know its size or dimension. The dimension of a matrix is the number of rows and columns it has. When we say a matrix is a "3 by 4 matrix," we are saying that it has 3 rows and 4 columns. The rows are always mentioned first, and the columns second. This means that a 3×4 matrix does not have the same dimension as a 4×3 matrix.

$$A = \begin{bmatrix} 1 & 4 & -2 & 0 \\ 3 & -1 & 7 & 9 \\ 6 & 2 & 0 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 9 & 8 \\ -3 & 0 & 1 \\ 6 & 5 & -2 \\ -4 & 7 & 8 \end{bmatrix}$$

Matrix A has dimensions 3×4 — Matrix B has dimensions 4×3

A matrix that has the same number of rows as columns is called a square matrix. A matrix with all entries zero is called a zero matrix. A square matrix with 1's along the main diagonal and zeros everywhere else, is called an identity matrix. When a square matrix is multiplied by an identity matrix of same size, the matrix remains the same.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix I is a 3×3 identity matrix

A matrix with only one row is called a row matrix or a row vector, and a matrix with only one column is called a column matrix or a column vector. Two matrices are equal if they have the same size and the corresponding entries are equal. We can perform arithmetic operations with matrices. Next we will define and give examples illustrating the operations of matrix addition and subtraction, scalar multiplication, and matrix multiplication. Note that matrix multiplication is quite different from what you would intuitively expect, so pay careful attention to the explanation. Note also that the ability to perform matrix operations depends on the matrices involved being compatible in size, or dimensions, for that operation. The definition of compatible dimensions is different for different operations, so note the requirements carefully for each.

2.1.2 Matrix Addition and Subtraction

If two matrices have the same size, they can be added or subtracted. The operations are performed on corresponding entries.

Example 2.1.2. *Given the matrices A, B, C, and D below:*

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 1 \\ 5 & 0 & 3 \end{bmatrix}, \ B = \begin{bmatrix} 2 & -1 & 3 \\ 2 & 4 & 2 \\ 3 & 6 & 1 \end{bmatrix}, \ C = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}, \ D = \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix}$$

Find, if possible:

1. $A + B$
2. $C - D$
3. $A + D$

Solution 2.1.2.

- *We add each element of A to the corresponding entry of B:*

$$A + B = \begin{bmatrix} 3 & 1 & 7 \\ 4 & 7 & 3 \\ 8 & 6 & 4 \end{bmatrix}$$

- *We perform the subtraction entry by entry for $C - D$:*

$$C - D = \begin{bmatrix} 6 \\ 5 \\ -1 \end{bmatrix}$$

- *The sum $A+D$ cannot be found because the two matrices have different sizes. Two matrices can only be added or subtracted if they have the same dimension.*

2.1.3 Multiplying a Matrix by a Scalar

If a matrix is multiplied by a scalar, each entry is multiplied by that scalar.

Example 2.1.3. *Given the matrix A and C in the previous example, find $2A$ and $-3C$.*

Solution 2.1.3.

- *To find $2A$, we multiply each entry of matrix A by 2:*

$$2A = \begin{bmatrix} 2 & 4 & 8 \\ 4 & 6 & 2 \\ 10 & 0 & 6 \end{bmatrix}$$

- *To find $-3C$, we multiply each entry of C by -3:*

$$-3C = \begin{bmatrix} -12 \\ -6 \\ -9 \end{bmatrix}$$

2.1.4 Multiplication of Two Matrices

To multiply a matrix by another is not as easy as the addition, subtraction, or scalar multiplication of matrices. Because of its wide use in application problems, it is important that we learn it well. Therefore, we will try to learn the process in a step by step manner.

Example 2.1.4. *Given $A = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, find the product AB.*

Solution 2.1.4. *The product is a 1×1 matrix whose entry is obtained by multiplying the corresponding entries and then forming the sum:*

$$AB = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 2a + 3b + 4c$$

Note that AB is a 1×1 matrix, and its only entry is $2a + 3b + 4c$.

Example 2.1.5. *Given $A = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$, find the product AB.*

Solution 2.1.5. *Again, we multiply the corresponding entries and add:*

$$AB = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} = (2 \cdot 5) + (3 \cdot 6) + (4 \cdot 7) = 10 + 18 + 28 = 56$$

Example 2.1.6. *Given $A = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 3 \\ 6 & 4 \\ 7 & 5 \end{bmatrix}$, find the product AB.*

Solution 2.1.6. *We know how to multiply a row matrix by a column matrix. To find the product AB, in this example, we will multiply the row matrix A to both the first and second columns of matrix B, resulting in a 1×2 matrix:*

$$AB = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 6 & 4 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} (2 \cdot 5) + (3 \cdot 6) + (4 \cdot 7) & (2 \cdot 3) + (3 \cdot 4) + (4 \cdot 5) \end{bmatrix} = \begin{bmatrix} 56 & 38 \end{bmatrix}$$

We multiplied a 1×3 matrix by a matrix whose size is 3×2. So unlike addition and subtraction, it is possible to multiply two matrices with different dimensions if the number of entries in the rows of the first matrix is the same as the number of entries in the columns of the second matrix.

Example 2.1.7. *Given $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 3 \\ 6 & 4 \\ 7 & 5 \end{bmatrix}$, find the product AB.*

Solution 2.1.7. *This time we are multiplying two rows of matrix A with two columns of matrix B. Since the number of entries in each row of A is the same as the number of entries in each column of B, the product is possible. We do exactly what we did in the last example. The only difference is that matrix A has one more row.*

We multiply the first row of matrix A with the two columns of B, one at a time, and then repeat the process with the second row of A. We get:

$$AB = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 6 & 4 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} (2 \cdot 5 + 3 \cdot 6 + 4 \cdot 7) & (2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5) \\ (1 \cdot 5 + 2 \cdot 6 + 3 \cdot 7) & (1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5) \end{bmatrix}$$

$$AB = \begin{bmatrix} 56 & 38 \\ 38 & 26 \end{bmatrix}$$

Example 2.1.8. *Given matrices $E = \begin{bmatrix} 1 & 2 \\ 4 & 2 \\ 3 & 1 \end{bmatrix}$, $F = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$, $G = \begin{bmatrix} 4 & 1 \end{bmatrix}$, and $H = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$, find the following products if possible:*

1. *EF*

2. *FE*

3. *FH*

4. *GH*

5. *HG*

Solution 2.1.8. *1. To find EF, we multiply the rows of E with the columns of F. The result is:*

$$EF = \begin{bmatrix} 1 & 2 \\ 4 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} (1 \cdot 2 + 2 \cdot 3) & (1 \cdot -1 + 2 \cdot 2) \\ (4 \cdot 2 + 2 \cdot 3) & (4 \cdot -1 + 2 \cdot 2) \\ (3 \cdot 2 + 1 \cdot 3) & (3 \cdot -1 + 1 \cdot 2) \end{bmatrix} = \begin{bmatrix} 8 & 3 \\ 14 & 0 \\ 9 & -1 \end{bmatrix}$$

2. *Product FE is not possible because F has two entries in each row, while E has three entries in each column.*

3. $FH = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} (2 \cdot -3 + -1 \cdot -1) \\ (3 \cdot -3 + 2 \cdot -1) \end{bmatrix} = \begin{bmatrix} -5 \\ -11 \end{bmatrix}$

4. $GH = \begin{bmatrix} 4 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \end{bmatrix} = (4 \cdot -3 + 1 \cdot -1) = -13$

5. $HG = \begin{bmatrix} -3 \\ -1 \end{bmatrix} \begin{bmatrix} 4 & 1 \end{bmatrix} = \begin{bmatrix} (-3 \cdot 4 & -3 \cdot 1) \\ (-1 \cdot 4 & -1 \cdot 1) \end{bmatrix} = \begin{bmatrix} -12 & -3 \\ -4 & -1 \end{bmatrix}$

We summarize some important properties of matrix multiplication that we observed in the previous examples.

- For the product AB to exist, the number of columns of matrix A must equal the number of rows of matrix B.
- If matrix A has dimensions $m \times n$ and matrix B has dimensions $n \times p$, then the product AB will have dimensions $m \times p$.
- Matrix multiplication is not commutative; that is, in general, AB does not equal BA.

Example 2.1.9. *Given matrices* $R = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 5 \\ 2 & 3 & 1 \end{bmatrix}$, $S = \begin{bmatrix} 0 & -1 & 2 \\ 3 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$, *and* $T = \begin{bmatrix} -2 & 3 & 0 \\ -3 & 2 & 2 \\ -1 & 1 & 0 \end{bmatrix}$, *find $2RS - 3ST$.*

Solution 2.1.9. *Solution: To find $2RS-3ST$, we first compute the products RS and ST:*

$$RS = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 5 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 2 \\ 3 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (1\cdot 0 + 0\cdot 3 + 2\cdot 4) & (1\cdot -1 + 0\cdot 1 + 2\cdot 2) & (1\cdot 2 + 0\cdot 0 + 2\cdot 1) \\ (2\cdot 0 + 1\cdot 3 + 5\cdot 4) & (2\cdot -1 + 1\cdot 1 + 5\cdot 2) & (2\cdot 2 + 1\cdot 0 + 5\cdot 1) \\ (2\cdot 0 + 3\cdot 3 + 1\cdot 4) & (2\cdot -1 + 3\cdot 1 + 1\cdot 2) & (2\cdot 2 + 3\cdot 0 + 1\cdot 1) \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 3 & 4 \\ 23 & 9 & 9 \\ 13 & 3 & 5 \end{bmatrix}$$

Next, we compute ST:

$$ST = \begin{bmatrix} 0 & -1 & 2 \\ 3 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 3 & 0 \\ -3 & 2 & 2 \\ -1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (0\cdot -2 + -1\cdot -3 + 2\cdot -1) & (0\cdot 3 + -1\cdot 2 + 2\cdot 1) & (0\cdot 0 + -1\cdot 2 + 2\cdot 0) \\ (3\cdot -2 + 1\cdot -3 + 0\cdot -1) & (3\cdot 3 + 1\cdot 2 + 0\cdot 1) & (3\cdot 0 + 1\cdot 1 + 0\cdot 0) \\ (4\cdot -2 + 2\cdot -3 + 1\cdot -1) & (4\cdot 3 + 2\cdot 2 + 1\cdot 1) & (4\cdot 0 + 2\cdot 1 + 1\cdot 0) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -2 \\ -9 & 11 & 2 \\ -15 & 17 & 4 \end{bmatrix}$$

Now we can find $2RS - 3ST$:

$$2RS - 3ST = 2\cdot \begin{bmatrix} 8 & 3 & 4 \\ 23 & 9 & 9 \\ 13 & 3 & 5 \end{bmatrix} - 3\cdot \begin{bmatrix} 1 & 0 & -2 \\ -9 & 11 & 2 \\ -15 & 17 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 6 & 8 \\ 46 & 18 & 18 \\ 26 & 6 & 10 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 6 \\ -27 & 33 & 6 \\ -45 & 51 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 6 & 14 \\ 73 & -15 & 12 \\ 71 & -45 & -2 \end{bmatrix}$$

The result of $2RS - 3ST$ *is a matrix with dimensions* 3×3.

Example 2.1.10. *Given matrix* $F = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$, *find* F^2.

Solution 2.1.10. F^2 *is found by multiplying matrix* F *by itself, using matrix multiplication.*

$$F^2 = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 2 + (-1) \cdot 3 & 2 \cdot (-1) + (-1) \cdot 2 \\ 3 \cdot 2 + 2 \cdot 3 & 3 \cdot (-1) + 2 \cdot 2 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 12 & 1 \end{bmatrix}$$

Note that F^2 *is not found by squaring each entry of matrix* F. *The process of raising a matrix to a power, such as finding* F^2, *is only possible if the matrix is a square matrix.*

2.1.5 Systems of Linear Equations

Using matrices to represent a system of linear equations is a powerful technique that allows for efficient solving of such systems. In this method, we define matrices as follows:

- Matrix A represents the coefficients of the variables in the system and is called the coefficient matrix.
- Matrix X is a column matrix that contains the variables of the system.
- Matrix B is a column matrix that contains the constants of the system.

By defining these matrices, we can represent a system of linear equations as the matrix equation $AX = B$, where A, X, and B are matrices. This representation simplifies the process of solving linear systems and allows us to apply matrix operations to find the solution.

In the next sections, we will delve deeper into how to use matrices to solve linear systems and explore various methods and techniques for efficient computation and analysis. Matrix representation is widely used in mathematical modeling, engineering, economics, and various other fields where systems of linear equations arise.

Example 2.1.11. *Verify that the system of two linear equations with two unknowns:*

$$\begin{aligned} ax + by &= h \\ cx + dy &= k \end{aligned}$$

can be written as $AX = B$, where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad B = \begin{bmatrix} h \\ k \end{bmatrix}.$$

Solution 2.1.11. *If we multiply the matrices A and X, we get*

$$AX = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}.$$

If $AX = B$, then

$$\begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} = \begin{bmatrix} h \\ k \end{bmatrix}.$$

If two matrices are equal, then their corresponding entries are equal. It follows that

$$\begin{aligned} ax + by &= h \\ cx + dy &= k \end{aligned}$$

Example 2.1.12. *Express the following system as a matrix equation in the form $AX = B$.*

$$\begin{aligned} 2x + 3y - 4z &= 5 \\ 3x + 4y - 5z &= 6 \\ 5x - 6z &= 7 \end{aligned}$$

Solution 2.1.12. *This system of equations can be expressed in the form $AX = B$ as shown below.*

$$\begin{bmatrix} 2 & 3 & -4 \\ 3 & 4 & -5 \\ 5 & 0 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$$

2.2 Systems of Linear Equations; Gauss-Jordan Method

In this section you will learn to

1. Represent a system of linear equations as an augmented matrix
2. Solve the system using elementary row operations.

In this section, we learn to solve systems of linear equations using a process called the Gauss-Jordan method. The process begins by first expressing the system as a matrix, and then reducing it to an equivalent system by simple row operations. The process is continued until the solution is obvious from the matrix. The matrix that represents the system is called the augmented matrix, and the arithmetic manipulation that is used to move from a system to a reduced equivalent system is called a row operation.

Example 2.2.1. *Write the following system as an augmented matrix.*

$$\begin{aligned} 2x + 3y - 4z &= 5 \\ 3x + 4y - 5z &= -6 \\ 4x + 5y - 6z &= 7 \end{aligned}$$

Solution 2.2.1. *We express the above information in matrix form. Since a system is entirely determined by its coefficient matrix and by its matrix of constant terms, the augmented matrix will include only the coefficient matrix and the constant matrix. So the augmented matrix we get is as follows:*

$$\left[\begin{array}{ccc|c} 2 & 3 & -4 & 5 \\ 3 & 4 & -5 & -6 \\ 4 & 5 & -6 & 7 \end{array}\right]$$

In the last section, we expressed the system of equations as $AX = B$, where A represented the coefficient matrix, and B the matrix of constant terms. As an augmented matrix, we write the matrix as $[A|B]$. It is clear that all of the information is maintained in this matrix form, and only the letters x, y, and z are missing. A student may choose to write x, y, and z on top of the first three columns to help ease the transition.

Example 2.2.2. *For the following augmented matrix, write the system of equations it represents.*

$$\left[\begin{array}{ccc|c} 1 & 3 & -5 & 2 \\ 2 & 0 & -3 & -5 \\ 3 & 2 & -3 & -1 \end{array}\right]$$

Solution 2.2.2. *The system is readily obtained as below.*

$$\begin{aligned} x + 3y - 5z &= 2 \\ 2x - 3z &= -5 \\ 3x + 2y - 3z &= -1 \end{aligned}$$

Once a system is expressed as an augmented matrix, the Gauss-Jordan method reduces the system into a series of equivalent systems by using the row operations. This row reduction continues until the system is expressed in what is called the reduced row echelon form. The reduced row echelon form of the coefficient matrix has 1's along the main diagonal and zeros elsewhere. The solution is readily obtained from this form.

The method is not much different form the algebraic operations we employed in the elimination method in the first chapter. The basic difference is that it is algorithmic in nature, and, therefore, can easily be programmed on a computer.

We will next solve a system of two equations with two unknowns, using the elimination method, and then show that the method is analogous to the Gauss-Jordan method.

Example 2.2.3. *Solve the following system by the elimination method.*

$$\begin{aligned} x + 3y &= 7 \\ 3x + 4y &= 11 \end{aligned}$$

Solution 2.2.3. *We multiply the first equation by -3 and add it to the second equation.*

$$\begin{aligned} -3x - 9y &= -21 \\ 3x + 4y &= 11 \end{aligned}$$

This transforms our original system into an equivalent system:

$$\begin{aligned} x + 3y &= 7 \\ -5y &= -10 \end{aligned}$$

Dividing the second equation by -5, we get the next equivalent system.

$$\begin{aligned} x + 3y &= 7 \\ y &= 2 \end{aligned}$$

Multiplying the second equation by -3 and adding it to the first, we get

$$\begin{aligned} x &= 1 \\ y &= 2 \end{aligned}$$

Example 2.2.4. *Solve the following system from Example 3 by the Gauss-Jordan method, and show the similarities in both methods by writing the equations next to the matrices.*

$$\begin{aligned} x + 3y &= 7 \\ 3x + 4y &= 11 \end{aligned}$$

Solution 2.2.4. *The augmented matrix for the system is as follows.*

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 3 & 4 & 11 \end{array}\right] \quad \Leftrightarrow \quad \begin{aligned} x + 3y &= 7 \\ 3x + 4y &= 11 \end{aligned}$$

We multiply the first row by -3 and add it to the second row.

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & -5 & -10 \end{array}\right] \quad \Leftrightarrow \quad \begin{aligned} x + 3y &= 7 \\ -5y &= -10 \end{aligned}$$

Dividing the second row by -5, we get,

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & 1 & 2 \end{array}\right] \quad \Leftrightarrow \quad \begin{aligned} x + 3y &= 7 \\ y &= 2 \end{aligned}$$

Finally, we multiply the second row by -3 and add to the first row, and we get,

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right] \quad \Leftrightarrow \quad \begin{aligned} x &= 1 \\ y &= 2 \end{aligned}$$

2.2.1 Row Operations in Gauss-Jordan Method

The Gauss-Jordan method employs three fundamental row operations:

1. Any two rows in the augmented matrix may be interchanged.
2. Any row may be multiplied by a non-zero constant.
3. A constant multiple of a row may be added to another row.

One can easily see that these three row operations may make the system look different, but they do not change the solution of the system.

Example of Row Interchange

Consider the system of equations with two unknowns:

$$\begin{aligned} x + 3y &= 7 \\ 3x + 4y &= 11 \end{aligned}$$

If we interchange the rows, we get:

$$\begin{aligned} 3x + 4y &= 11 \\ x + 3y &= 7 \end{aligned}$$

Clearly, this system has the same solution as the original.

Example of Multiplying a Row by a Constant

Consider the system again:

$$\begin{aligned} x + 3y &= 7 \\ 3x + 4y &= 11 \end{aligned}$$

Multiplying the first row by -3, we get:

$$\begin{aligned} -3x - 9y &= -21 \\ 3x + 4y &= 11 \end{aligned}$$

Once again, this new system has the same solution as the original.

Example of Adding a Constant Multiple of One Row to Another

For the system:

$$\begin{aligned} x + 3y &= 7 \\ 3x + 4y &= 11 \end{aligned}$$

If we multiply the first row by -3 and add it to the second row, we get:

$$\begin{aligned} x + 3y &= 7 \\ -5y &= -10 \end{aligned}$$

The solution remains unchanged.

Now that we understand how the three row operations work, it is time to introduce the Gauss-Jordan method to solve systems of linear equations. As mentioned earlier, the Gauss-Jordan method starts out with an augmented matrix, and by a series of row operations ends up with a matrix that is in the reduced row echelon form. A matrix is in the reduced row echelon form if the first nonzero entry in each row is a 1, and the columns containing these 1's have all other entries as zeros. The reduced row echelon form also requires that the leading entry in each row be to the right of the leading entry in the row above it, and the rows containing all zeros be moved down to the bottom. We state the Gauss-Jordan method as follows.

Gauss-Jordan Method Steps

Here are the steps of the Gauss-Jordan method for solving linear systems:

1. Write the augmented matrix.
2. Interchange rows if necessary to obtain a non-zero number in the first row, first column.
3. Use a row operation to get a 1 as the entry in the first row and first column.
4. Use row operations to make all other entries as zeros in column one.
5. Interchange rows if necessary to obtain a nonzero number in the second row, second column. Use a row operation to make this entry 1. Use row operations to make all other entries as zeros in column two.

6. Repeat step 5 for row 3, column 3. Continue moving along the main diagonal until you reach the last row, or until the number is zero.
7. The final matrix is called the reduced row-echelon form.

Example 2.2.5. *Solve the following system by the Gauss-Jordan method:*

$$\begin{array}{rcrcrcr} 2x & + & y & + & 2z & = & 10 \\ x & + & 2y & + & z & = & 8 \\ 3x & + & y & - & z & = & 2 \end{array}$$

Solution 2.2.5. *We write the augmented matrix.*

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 10 \\ 1 & 2 & 1 & 8 \\ 3 & 1 & -1 & 2 \end{array}\right]$$

We want a 1 in row one, column one. This can be obtained by dividing the first row by 2, or interchanging the second row with the first. Interchanging the rows is a better choice because that way we avoid fractions.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 2 & 1 & 2 & 10 \\ 3 & 1 & -1 & 2 \end{array}\right] \quad \textit{we interchanged row 1(R1) and row 2(R2)}$$

We need to make all other entries zeros in column 1. To make the entry (2) a zero in row 2, column 1, we multiply row 1 by -2 and add it to the second row. We get,

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -3 & 0 & -6 \\ 3 & 1 & -1 & 2 \end{array}\right] \quad -2R1 + R2$$

To make the entry (3) a zero in row 3, column 1, we multiply row 1 by -3 and add it to the third row. We get,

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -3 & 0 & -6 \\ 0 & -5 & -4 & -22 \end{array}\right] \quad -3R1 + R3$$

So far we have made a 1 in the left corner and all other entries zeros in that column. Now we move to the next diagonal entry, row 2, column 2. We need *to make this entry(-3) a 1 and make all other entries in this column zeros. To make row 2, column 2 entry a 1, we divide the entire second row by -3.*

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 1 & 0 & 2 \\ 0 & -5 & -4 & -22 \end{array}\right] \quad R2 \cdot \frac{1}{(-3)}$$

Next, we make all other entries zeros in the second column.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -4 & -12 \end{array}\right] \quad -2R2 + R1 \textit{ and } 5R2 + R3$$

We make the last diagonal entry a 1, by dividing row 3 by -4.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad R3 \cdot \frac{1}{(-4)}$$

Finally, we make all other entries zeros in column 3.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad -R3 + R1$$

Clearly, the solution reads $x = 1$, $y = 2$, *and* $z = 3$.

Before we leave this section, we mention some terms we may need in the fourth chapter.

The process of obtaining a 1 in a location, and then making all other entries zeros in that column, is called pivoting.

The number that is made a 1 is called the pivot element, and the row that contains the pivot element is called the pivot row.

We often multiply the pivot row by a number and add it to another row to obtain a zero in the latter. The row to which a multiple of pivot row is added is called the target row.

2.3 Systems of Linear Equations – Special Cases

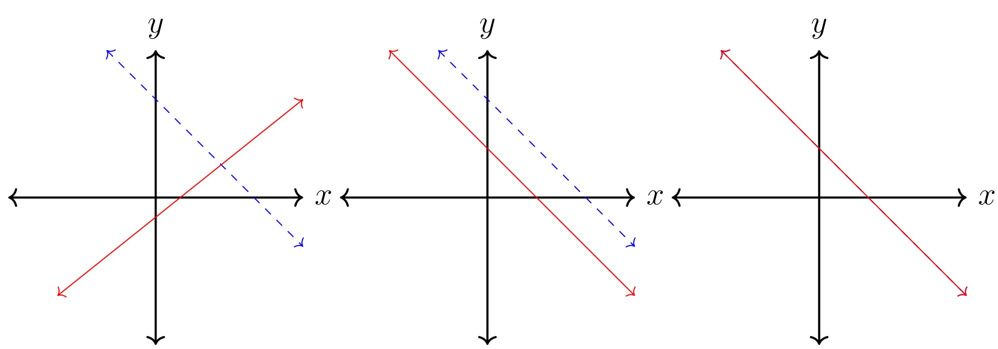
In this section you will learn to:

1. Determine the linear systems that have no solution.
2. Solve the linear systems that have infinitely many solutions.

If we consider the intersection of two lines in a plane, three things can happen.

1. The lines intersect in exactly one point. This is called an independent system.
2. The lines are parallel, so they do not intersect. This is called an inconsistent system.
3. The lines coincide; they intersect at infinitely many points. This is a dependent system.

The figures below show all three cases:



Every system of equations has either one solution, no solution, or infinitely many solutions.

In the last section, we used the Gauss-Jordan method to solve systems that had exactly one solution. In this section, we will determine the systems that have no solution, and solve the systems that have infinitely many solutions.

Example 2.3.1. *Solve the following system of equations using the Gauss-Jordan method:*

$$x + y = 7$$
$$x + y = 9$$

Solution 2.3.1. *Let us use the Gauss-Jordan method to solve this system. The augmented matrix is*

$$\left[\begin{array}{cc|c} 1 & 1 & 7 \\ 1 & 1 & 9 \end{array}\right]$$

If we multiply the first row by -1 *and add it to the second row, we get*

$$\left[\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 0 & 2 \end{array}\right]$$

Since 0 *cannot equal* 2*, the last equation cannot be true for any choices of* x *and* y*. Alternatively, it is clear that the two lines are parallel; therefore, they do not intersect.*

In the examples that follow, we are going to start using a calculator to row reduce the augmented matrix, in order to focus on understanding the answer rather than focusing on the process of carrying out the row operations.

Example 2.3.2. *Solve the following system of equations:*

$$\begin{aligned} 2x + 3y - 4z &= 7 \\ 3x + 4y - 2z &= 9 \\ 5x + 7y - 6z &= 20 \end{aligned}$$

Solution 2.3.2. *We represent the system as an augmented matrix:*

$$\left[\begin{array}{ccc|c} 2 & 3 & -4 & 7 \\ 3 & 4 & -2 & 9 \\ 5 & 7 & -6 & 20 \end{array}\right]$$

By obtaining the reduced row-echelon form from a matrix calculator, we get:

$$\left[\begin{array}{ccc|c} 1 & 0 & 10 & 0 \\ 0 & 1 & -8 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The bottom row implies $0x + 0y + 0z = 1$*, which is a contradiction. Thus, the system is inconsistent and has no solution.*

Example 2.3.3. *Solve the following system of equations:*

$$\begin{aligned} x + y &= 7 \\ x + y &= 7 \end{aligned}$$

Solution 2.3.3. *The problem asks for the intersection of two identical lines, meaning the lines coincide and intersect at an infinite number of points.*

A few intersection points are listed as follows: (3, 4), (5, 2), (-1, 8), (-6, 13), etc. However, when a system has an infinite number of solutions, the solution is often expressed in parametric form. This can be done by assigning an arbitrary constant, t, to one of the variables and solving for the remaining variables. If we let $y = t$, then $x = 7 - t$. In other words, all ordered pairs of the form $(7 - t, t)$ satisfy the given system of equations.

Alternatively, solving with the Gauss-Jordan method, we obtain the reduced row-echelon form below, which includes a row of all zeros that can be ignored since it provides no additional information about the values of x and y that solve the system.

$$\left[\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 0 & 0 \end{array}\right]$$

This leaves us with only one equation but two variables. Whenever there are more variables than equations, the solution must be expressed as a parametric solution in terms of an arbitrary constant, as shown above.

Parametric Solution: $x = 7 - t$, $y = t$.

Example 2.3.4. *Solve the following system of equations:*

$$\begin{aligned} x + y + z &= 2 \\ 2x + y - z &= 3 \\ 3x + 2y &= 5 \end{aligned}$$

Solution 2.3.4. *The augmented matrix and the reduced row-echelon form are given below:*

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & -1 & 3 \\ 3 & 2 & 0 & 5 \end{array}\right] \quad \to \quad \left[\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since the last equation dropped out, we are left with two equations and three variables. This means the system has an infinite number of solutions. We express those solutions in the parametric form by letting the last variable z equal the parameter t.

The first equation reads $x - 2z = 1$, therefore, $x = 1 + 2z$. The second equation reads $y + 3z = 1$, therefore, $y = 1 - 3z$. And now if we let $z = t$, the parametric solution is expressed as follows:

$$\textit{Parametric Solution:} \quad x = 1 + 2t, \quad y = 1 - 3t, \quad z = t.$$

The reader should note that particular solutions, or specific solutions, to the system can be obtained by assigning values to the parameter t. For example:

- *If we let $t = 2$, we have the solution $x = 5, y = -5, z = 2$: $(5, -5, 2)$.*
- *If we let $t = 0$, we have the solution $x = 1, y = 1, z = 0$: $(1, 1, 0)$.*

Example 2.3.5. *Solve the following system of equations:*

$$\begin{aligned} x + 2y - 3z &= 5 \\ 2x + 4y - 6z &= 10 \\ 3x + 6y - 9z &= 15 \end{aligned}$$

Solution 2.3.5. *The reduced row-echelon form is given below:*

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This time the last two equations drop out. We are left with one equation and three variables. Again, there are an infinite number of solutions. But this time the answer must be expressed in terms of two arbitrary constants.

If we let $z = t$ and $y = s$, the first equation $x + 2y - 3z = 5$ results in $x = 5 - 2s + 3t$. We rewrite the parametric solution as:

$$\textit{Parametric Solution:} \quad x = 5 - 2s + 3t, \quad y = s, \quad z = t.$$

Summary 2.3.1.

1. *If any row of the reduced row-echelon form of the matrix gives a false statement such as* $0 = 1$*, the system is inconsistent and has no solution.*
2. *If the reduced row echelon form has fewer equations than the variables and the system is consistent, then the system has an infinite number of solutions. Remember the rows that contain all zeros are dropped.*
   (a) *If a system has an infinite number of solutions, the solution must be expressed in the parametric form.*
   (b) *The number of arbitrary parameters equals the number of variables minus the number of equations.*

2.4 Inverse Matrices

In this section you will learn to:

1. Find the inverse of a matrix, if it exists.
2. Use inverses to solve linear systems.

In this section, we will learn to find the inverse of a matrix, if it exists. Later, we will use matrix inverses to solve linear systems.

Definition 2.4.1. *An* $n \times n$ *matrix has an* ***inverse*** *if there exists a matrix* B *such that* $AB = BA = I_n$*, where* I_n *is an* $n \times n$ *identity matrix. The* ***inverse*** *of a matrix* A*, if it exists, is denoted by the symbol* A^{-1}*.*

Example 2.4.1. *Given matrices* A *and* B *below, verify that they are inverses.*

$$A = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$$

Solution 2.4.1. *The matrices are inverses if the product* AB *and* BA *both equal the identity matrix of dimension* 2×2*, denoted as* I_2*:*

$$AB = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

and

$$BA = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Clearly, that is the case; therefore, the matrices A and B are inverses of each other.

Example 2.4.2. *Find the inverse of the matrix $A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$.*

Solution 2.4.2. *Suppose A has an inverse, and it is denoted as $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $AB = I_2$:*

$$\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

After multiplying the matrices on the left side, we get the system:

$$\begin{aligned} 3a + c &= 1 \\ 3b + d &= 0 \\ 5a + 2c &= 0 \\ 5b + 2d &= 1 \end{aligned}$$

Solving this system, we find $a = 2$, $b = -1$, $c = -5$, and $d = 3$. Therefore, the inverse of matrix A is $B = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$.

In this problem, finding the inverse of matrix A amounted to solving the system of equations:

$$\begin{aligned} 3a + c &= 1 \\ 3b + d &= 0 \\ 5a + 2c &= 0 \\ 5b + 2d &= 1 \end{aligned}$$

Actually, it can be written as two systems, one with variables a and c, and the other with b and d. The augmented matrices for both are given below.

$$\left[\begin{array}{cc|c} 3 & 1 & 1 \\ 5 & 2 & 0 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{cc|c} 3 & 1 & 0 \\ 5 & 2 & 1 \end{array}\right]$$

As we look at the two augmented matrices, we notice that the coefficient matrix for both the matrices is the same. This implies the row operations of the Gauss-Jordan method will also be the same. A great deal of work can be saved if the two right-hand columns are grouped together to form one augmented matrix as below.

$$\left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 5 & 2 & 0 & 1 \end{array}\right]$$

And solving this system, we get

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -5 & 3 \end{array}\right]$$

The matrix on the right side of the vertical line is the A^{-1} matrix. What you just witnessed is no coincidence. This is the method that is often employed in finding the inverse of a matrix.

Summary 2.4.1. ***The Method for Finding the Inverse of a Matrix***

1. *Write the augmented matrix $[A|I_n]$.*
2. *Write the augmented matrix in step 1 in reduced row echelon form.*
3. *If the reduced row echelon form in 2 is $[I_n|B]$, then B is the inverse of A.*
4. *If the left side of the row reduced echelon is not an identity matrix, the inverse does not exist.*

Example 2.4.3. *Given the matrix A below, find its inverse.*

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Solution 2.4.3. *We write the augmented matrix as follows.*

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

We will reduce this matrix using the Gauss-Jordan method. Multiplying the *first row by -2 and adding it to the second row, we get*

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 5 & -2 & -2 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

If we swap the second and third rows, we get

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \\ 0 & 5 & -2 & -2 & 1 & 0 \end{array}\right]$$

Divide the second row by -2. The result is

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1/2 & 0 & 0 & -1/2 \\ 0 & 5 & -2 & -2 & 1 & 0 \end{array}\right]$$

Let us do two operations here. 1) *Add the second row to the first.* 2) *Add -5 times the second row to the third. And we get*

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 1 & 0 & -1/2 \\ 0 & 1 & -1/2 & 0 & 0 & -1/2 \\ 0 & 0 & 1/2 & -2 & 1 & 5/2 \end{array}\right]$$

Multiplying the third row by 2 results in

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 1 & 0 & -1/2 \\ 0 & 1 & -1/2 & 0 & 0 & -1/2 \\ 0 & 0 & 1 & -4 & 2 & 5 \end{array}\right]$$

Multiply the third row by $1/2$ and add it to the second. Also, multiply the third row by $-1/2$ and add it to the first.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & -3 \\ 0 & 1 & 0 & -2 & 1 & 2 \\ 0 & 0 & 1 & -4 & 2 & 5 \end{array}\right]$$

Therefore, the inverse of matrix A is $A^{-1} = \begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix}$.

One should verify the result by multiplying the two matrices to see if the product does, indeed, equal the identity matrix.

Now that we know how to find the inverse of a matrix, we will use inverses to solve systems of equations. The method is analogous to solving a simple equation like the one below.

$$\frac{2}{3}x = 4$$

Example 2.4.4. *Solve the following equation:*

$$x = 4$$

Solution 2.4.4. *To solve the above equation, we multiply both sides of the equation by the multiplicative inverse of* $\frac{2}{3}$*, which happens to be* $\frac{3}{2}$*. We get*

$$\frac{3}{2} \cdot \frac{2}{3}x = 4 \cdot \frac{3}{2}$$

Hence,

$$x = 6.$$

We use example 2.4.4 as an analogy to show how linear systems of the form $AX = B$ are solved. To solve a linear system, we first write the system in the matrix equation $AX = B$, where A is the coefficient matrix, X is the matrix of variables, and B is the matrix of constant terms. We then multiply both sides of this equation by the multiplicative inverse of the matrix A. Consider the following example.

Example 2.4.5. *Solve the following system*

$$\begin{aligned} 3x + y &= 3 \\ 5x + 2y &= 4 \end{aligned}$$

Solution 2.4.5. *To solve the above equation, first we express the system as*

$$AX = B$$

where A is the coefficient matrix, and B is the matrix of constant terms. We get

$$\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

To solve this system, we multiply both sides of the matrix equation $AX = B$ by A^{-1}. Matrix multiplication is not commutative, so we need to multiply by A^{-1} on the left on both sides of the equation.

Matrix A is the same matrix A whose inverse we found in Example 2.4.2, so $A^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$.

Multiplying both sides by A^{-1}, we get

$$\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Therefore, $x = 2$, and $y = -3$.

Example 2.4.6. *Solve the following system:*

$$\begin{aligned} x - y + z &= 6 \\ 2x + 3y &= 1 \\ -2y + z &= 5 \end{aligned}$$

Solution 2.4.6. *To solve the above equation, we write the system in matrix form $AX = B$ as follows:*

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 5 \end{bmatrix}$$

To solve this system, we need the inverse of A. From Example 2.4.3, A^{-1} is given by

$$A^{-1} = \begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix}$$

Multiplying both sides of the matrix equation $AX = B$ on the left by A^{-1}, we get

$$\begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

After multiplying the matrices, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

Therefore, $x = 2$, $y = -1$, and $z = 3$.

We remind the reader that not every system of equations can be solved by the matrix inverse method. Although the Gauss-Jordan method works for every situation, the matrix inverse method works only in cases where the inverse of the square matrix exists. In such cases the system has a unique solution.

Summary 2.4.2.
The Method for Finding the Inverse of a Matrix

1. *Write the augmented matrix* $[A|I_n]$.
2. *Write the augmented matrix in step 1 in reduced row echelon form.*
3. *If the reduced row echelon form in step 2 is* $[I_n|B]$*, then* B *is the inverse of* A.
4. *If the left side of the row reduced echelon is not an identity matrix, the inverse does not exist.*

The Method for Solving a System of Equations When a Unique Solution Exists

1. *Express the system in the matrix equation* $AX = B$.
2. *To solve the equation* $AX = B$*, multiply both sides by* A^{-1}:

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B \quad \text{where } I \text{ is the identity matrix}$$

2.5 Application of Matrices in Cryptography

In this section, you will learn to:

1. Encode a message using matrix multiplication.
2. Decode a coded message using the matrix inverse and matrix multiplication.

Encryption dates back approximately 4000 years. Historical accounts indicate that the Chinese, Egyptians, Indians, and Greeks encrypted messages in some way for various purposes. One famous encryption scheme is called the Caesar cipher, also called a substitution cipher, used by Julius Caesar, involved shifting letters in the alphabet, such as replacing A by C, B by D, C by E, etc., to encode a message. Substitution ciphers are too simple in design to be considered secure today.

In the middle ages, European nations began to use encryption. A variety of encryption methods were used in the US from the Revolutionary War, through the Civil War, and on into modern times.

Applications of mathematical theory and methods to encryption became widespread in military usage in the 20th century. The military would encode messages before sending, and the recipient would decode the message, in order to send information about military operations in a manner that kept the information safe if the message was intercepted. In World War II, encryption played an important role, as both Allied and Axis powers sent encrypted messages and devoted significant resources to strengthening their own encryption while also trying to break the opposition's encryption.

In this section, we will examine a method of encryption that uses matrix multiplication and matrix inverses. This method, known as the Hill Algorithm, was created by Lester Hill, a mathematics professor who taught at several US colleges and also was involved with military encryption. The Hill algorithm marks the introduction of modern mathematical theory and methods to the field of cryptography.

These days, the Hill Algorithm is not considered a secure encryption method; it is relatively easy to break with modern technology. However, in 1929 when it was developed, modern computing technology did not exist. This method, which we can handle easily with today's technology, was too cumbersome to use with hand calculations. Hill devised a mechanical encryption machine to help with the mathematics; his machine relied on gears and levers but never gained widespread use. Hill's method was considered sophisticated and powerful in its time and is one of many methods influencing techniques in use today. Other encryption methods at that time also utilized special coding machines. Alan Turing, a computer scientist pioneer in the field of artificial intelligence, invented a machine that was able to decrypt messages encrypted by the German Enigma machine, helping to turn the tide of World War II.

With the advent of the computer age and internet communication, the use of encryption has become widespread in communication and in keeping private data secure; it is no longer limited to military uses. Modern encryption methods are more complicated, often combining several steps or methods to encrypt data to keep it more secure and harder to break. Some modern methods make use of matrices as part of the encryption and decryption process; other fields of mathematics such as number theory play a large role in modern cryptography.

2.5.1 Using Matrices for Encoding and Decoding

To use matrices in encoding and decoding secret messages, our procedure is as follows.

We first convert the secret message into a string of numbers by arbitrarily assigning a number to each letter of the message. Next, we convert this string of numbers into a new set of numbers by multiplying the string by a square matrix of our choice that has an inverse. This new set of numbers represents the coded message.

To decode the message, we take the string of coded numbers and multiply it by the inverse of the matrix to get the original string of numbers. Finally, by associating the numbers with their corresponding letters, we obtain the original message.

In this section, we will use the correspondence shown below where letters A to Z correspond to the numbers 1 to 26, a space is represented by the number 27, and punctuation is ignored.

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13

N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	26

Example 2.5.1. *Use matrix A to encode the message: ATTACK NOW!*

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

Solution 2.5.1. *We divide the letters of the message into groups of two.*

$$AT \quad TA \quad CK \quad \_N \quad OW$$

We assign the numbers to these letters from the above table, and convert each pair of numbers into 2×1 matrices. In the case where a single letter is left over on the end, a space is added to make it into a pair.

$$\begin{bmatrix} A \\ T \end{bmatrix} = \begin{bmatrix} 1 \\ 20 \end{bmatrix}, \quad \begin{bmatrix} T \\ A \end{bmatrix} = \begin{bmatrix} 20 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} C \\ K \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}, \quad \begin{bmatrix} - \\ N \end{bmatrix} = \begin{bmatrix} 27 \\ 14 \end{bmatrix}, \quad \begin{bmatrix} O \\ W \end{bmatrix} = \begin{bmatrix} 15 \\ 23 \end{bmatrix}$$

So at this stage, our message expressed as 2×1 *matrices is as follows.*

$$\begin{bmatrix} 1 \\ 20 \end{bmatrix}, \quad \begin{bmatrix} 20 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 11 \end{bmatrix}, \quad \begin{bmatrix} 27 \\ 14 \end{bmatrix}, \quad \begin{bmatrix} 15 \\ 23 \end{bmatrix}$$

Now to encode, we multiply, on the left, each matrix of our message by the matrix A*.*

For example, the product of A *with our first matrix is:*

$$A \cdot \begin{bmatrix} 1 \\ 20 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 20 \end{bmatrix} = \begin{bmatrix} 41 \\ 61 \end{bmatrix}$$

And the product of A *with our second matrix is:*

$$A \cdot \begin{bmatrix} 20 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 20 \\ 1 \end{bmatrix} = \begin{bmatrix} 22 \\ 23 \end{bmatrix}$$

Multiplying matrix A *by each matrix in our list, in turn, gives the desired coded message:*

$$\begin{bmatrix} 41 \\ 61 \end{bmatrix}, \quad \begin{bmatrix} 22 \\ 23 \end{bmatrix}, \quad \begin{bmatrix} 25 \\ 36 \end{bmatrix}, \quad \begin{bmatrix} 55 \\ 69 \end{bmatrix}, \quad \begin{bmatrix} 61 \\ 84 \end{bmatrix}$$

Example 2.5.2. *Decode the following message that was encoded using matrix* $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$*:*

$$\begin{bmatrix} 21 \\ 26 \end{bmatrix}, \quad \begin{bmatrix} 37 \\ 53 \end{bmatrix}, \quad \begin{bmatrix} 45 \\ 54 \end{bmatrix}, \quad \begin{bmatrix} 74 \\ 101 \end{bmatrix}, \quad \begin{bmatrix} 53 \\ 69 \end{bmatrix}$$

Solution 2.5.2. *Since this message was encoded by multiplying by the matrix* A *in Example 2.4.2, we decode this message by first multiplying each matrix, on the left, by the inverse of matrix* A *given below.*

$$A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

For example:

$$\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 21 \\ 26 \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \end{bmatrix}$$

By multiplying each of the matrices in our list by the matrix A^{-1}, we get the following.

$$\begin{bmatrix} 11 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} 5 \\ 16 \end{bmatrix}, \quad \begin{bmatrix} 27 \\ 9 \end{bmatrix}, \quad \begin{bmatrix} 20 \\ 27 \end{bmatrix}, \quad \begin{bmatrix} 21 \\ 16 \end{bmatrix}$$

Finally, by associating the numbers with their corresponding letters, we obtain:

$$\begin{bmatrix} K \\ E \end{bmatrix}, \quad \begin{bmatrix} E \\ P \end{bmatrix}, \quad \begin{bmatrix} \_ \\ T \end{bmatrix}, \quad \begin{bmatrix} I \\ U \end{bmatrix}, \quad \begin{bmatrix} P \\ \_ \end{bmatrix}$$

And the message reads: KEEP IT UP.

Now suppose we wanted to use a 3×3 matrix to encode a message, then instead of dividing the letters into groups of two, we would divide them into groups of three.

Example 2.5.3. *Using the matrix* $B = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$, *encode the message: ATTACK NOW!*

Solution 2.5.3. *We divide the letters of the message into groups of three.*

$$ATT \quad ACK \quad \_NO \quad W\_\_$$

Note that since the single letter "W" was left over on the end, we added two spaces to make it into a triplet.

Now we assign the numbers their corresponding letters from the table, and convert each triplet of numbers into 3×1 matrices. We get

$$\begin{bmatrix} A \\ T \\ T \end{bmatrix} = \begin{bmatrix} 1 \\ 20 \\ 20 \end{bmatrix}, \quad \begin{bmatrix} A \\ C \\ K \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 11 \end{bmatrix}, \quad \begin{bmatrix} - \\ N \\ O \end{bmatrix} = \begin{bmatrix} 27 \\ 14 \\ 15 \end{bmatrix}, \quad \begin{bmatrix} W \\ - \\ - \end{bmatrix} = \begin{bmatrix} 23 \\ 27 \\ 27 \end{bmatrix}$$

So far we have,

$$\begin{bmatrix} 1 \\ 20 \\ 20 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \\ 11 \end{bmatrix}, \quad \begin{bmatrix} 27 \\ 14 \\ 15 \end{bmatrix}, \quad \begin{bmatrix} 23 \\ 27 \\ 27 \end{bmatrix}$$

We multiply, on the left, each matrix of our message by the matrix B. For example,

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 20 \\ 20 \end{bmatrix} = \begin{bmatrix} 1 \\ 21 \\ 42 \end{bmatrix}$$

By multiplying each of the matrices in (III) by the matrix B, we get the desired coded message as follows:

$$\begin{bmatrix} 1 \\ 21 \\ 42 \end{bmatrix}, \quad \begin{bmatrix} -7 \\ 12 \\ 16 \end{bmatrix}, \quad \begin{bmatrix} 26 \\ 42 \\ 83 \end{bmatrix}, \quad \begin{bmatrix} 23 \\ 50 \\ 100 \end{bmatrix}$$

If we need to decode this message, we simply multiply the coded message by B^{-1}, and associate the numbers with the corresponding letters of the alphabet.

In Example 2.5.4 we will demonstrate how to use matrix B^{-1} to decode an encrypted message.

Example 2.5.4. *Decode the following message that was encoded using matrix $B = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$:*

$$\begin{bmatrix} 11 \\ 20 \\ 43 \end{bmatrix}, \quad \begin{bmatrix} 25 \\ 10 \\ 41 \end{bmatrix}, \quad \begin{bmatrix} 22 \\ 14 \\ 41 \end{bmatrix}$$

Solution 2.5.4. *Since this message was encoded by multiplying by the matrix B. We first determine the inverse of B.*

$$B^{-1} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & -3 & 2 \\ -1 & -1 & 1 \end{bmatrix}$$

To decode the message, we multiply each matrix, on the left, by B^{-1}. For example,

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & -3 & 2 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 11 \\ 20 \\ 43 \end{bmatrix} = \begin{bmatrix} 8 \\ 15 \\ 12 \end{bmatrix}$$

Multiplying each of the matrices in our list by the matrix B^{-1} gives the following:

$$\begin{bmatrix} 8 \\ 15 \\ 12 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 27 \\ 6 \end{bmatrix}, \quad \begin{bmatrix} 9 \\ 18 \\ 5 \end{bmatrix}$$

Finally, by associating the numbers with their corresponding letters, we obtain:

$$\begin{bmatrix} H \\ O \\ L \end{bmatrix}, \quad \begin{bmatrix} D \\ \_ \\ F \end{bmatrix}, \quad \begin{bmatrix} I \\ R \\ E \end{bmatrix}$$

The message reads: HOLD FIRE.

Summary 2.5.1.
To Encode a Message

1. *Divide the letters of the message into groups of two or three.*
2. *Convert each group into a string of numbers by assigning a number to each letter of the message. Remember to assign letters to blank spaces.*
3. *Convert each group of numbers into column matrices.*
4. *Convert these column matrices into a new set of column matrices by multiplying them with a compatible square matrix of your choice that has an inverse. This new set of numbers or matrices represents the coded message.*

To Decode a Message

1. *Take the string of coded numbers and multiply it by the inverse of the matrix that was used to encode the message.*
2. *Associate the numbers with their corresponding letters.*

2.6 Applications – Leontief Models

In this section you will learn

1. Application of matrices to model closed economic systems
2. Application of matrices to model open economic systems

In the 1930s, Wassily Wassilyevich Leontief (holder of one of the greatest names ever) used matrices to model economic systems. His models, often referred to as the input-output models, divide the economy into sectors where each sector produces goods and services not only for itself but also for other sectors. These sectors are dependent on each other, and the total input always equals the total output. In 1973, he won the Nobel Prize in Economics for his work in this field. In this section, we look at both the closed and the open models that he developed.

2.6.1 The Closed Model

As an example of the closed model, we look at a very simple economy, where there are only three sectors: food, shelter, and clothing.

Example 2.6.1. *We assume that in a village there is a farmer, carpenter,* *and a tailor, who provide the three essential goods: food, shelter, and clothing. Suppose the farmer himself consumes* 40% *of the food he produces, and gives* 40% *to the carpenter, and* 20% *to the tailor. Thirty percent of the carpenter's production is consumed by himself,* 40% *by the farmer, and* 30% *by the carpenter. Fifty percent of the tailor's production is used by himself,* 30% *by the farmer, and* 20% *by the tailor. Write the matrix that describes this closed model.*

Solution 2.6.1. *The table below describes the above information.*

	Proportion produced by the farmer	*Proportion produced by the carpenter*	*Proportion produced by the tailor*
The proportion used by the farmer	*.40*	*.40*	*.30*
The proportion used by the carpenter	*.40*	*.30*	*.20*
The proportion used by the tailor	*.20*	*.30*	*.50*

In matrix form, it can be written as follows.

$$A = \begin{bmatrix} .40 & .40 & .30 \\ .40 & .30 & .20 \\ .20 & .30 & .50 \end{bmatrix}$$

This matrix is called the input-output matrix. It is important that we read the matrix correctly. For example, the entry A_{23}, the entry in row 2 and column 3, represents the following.

$$A_{23} = 20\% \text{ of the tailor's production is used by the carpenter.}$$
$$A_{33} = 50\% \text{ of the tailor's production is used by the tailor.}$$

Example 2.6.2. *In Example 2.6.1 above, how much should each person get for his efforts?*

Solution 2.6.2. *We choose the following variables.*

$$x = \text{Farmer's pay}$$
$$y = \text{Carpenter's pay}$$
$$z = \text{Tailor's pay}$$

As we said earlier, in this model input must equal output. That is, the amount paid by each equals the amount received by each.

Let us say the farmer gets paid x dollars. Let us now look at the farmer's expenses. The farmer uses up 40% of his own production, that is, of the x dollars he gets paid, he pays himself $.40x$ dollars, he pays $.40y$ dollars to the carpenter, and $.30z$ to the tailor. Since the expenses equal the wages, we get the following equation.

$$x = .40x + .40y + .30z$$

In the same manner, we get

$$y = .40x + .30y + .20z$$

$$z = .20x + .30y + .50z$$

The above system can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} .40 & .40 & .30 \\ .40 & .30 & .20 \\ .20 & .30 & .50 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This system is often referred to as $X = AX$.

Simplification results in the system of equations $(I - A)X = 0$

$$\begin{bmatrix} .60 & -.40 & -.30 \\ -.40 & .70 & -.20 \\ -.20 & -.30 & .50 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We put this into an augmented matrix

$$\left[\begin{array}{ccc|c} .60 & -.40 & -.30 & 0 \\ -.40 & .70 & -.20 & 0 \\ -.20 & -.30 & .50 & 0 \end{array}\right]$$

Solving for x, y, and z using the Gauss-Jordan method, we get

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{29}{26} & 0 \\ 0 & 1 & -\frac{12}{13} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This gives parametric equations:

$$x = \frac{29}{26}t, \quad y = \frac{12}{13}t, \quad z = t$$

Since we are only trying to determine the proportions of the pay, we can choose t to be any value. Suppose we let $t = \$2600$*, then we get*

$$x = \$2900, \quad y = \$2400, \quad z = \$2600$$

Note 2.6.1. *The use of a graphing calculator or computer application in solving the systems of linear matrix equations in these problems is strongly recommended.*

2.6.2 The Open Model

The open model is more realistic as it deals with the economy where sectors of the economy not only satisfy each other's needs but also satisfy some outside demands. In this case, the outside demands are put on by the consumer. But the basic assumption is still the same: whatever is produced is consumed.

Let us again look at a very simple scenario. Suppose the economy consists of three people: the farmer F, the carpenter C, and the tailor T. A part of the farmer's production is used by all three, and the rest is used by the consumer. In the same manner, a part of the carpenter's and the tailor's production is used by all three, and the rest is used by the consumer.

Let us assume that whatever the farmer produces, 20% is used by him, 15% by the carpenter, 10% by the tailor, and the consumer uses the other \$40 billion worth of food. Ten percent of the carpenter's production is used by him, 25% by the farmer, 5% by the tailor, and \$50 billion worth by the consumer. Fifteen percent of the clothing is used by the tailor, 10% by the farmer, 5% by the carpenter, and the remaining \$60 billion worth by the consumer. We write the internal consumption in the following table and express the demand as the matrix D.

	F produces	C produces	T produces
F uses	0.20	0.25	0.10
C uses	0.15	0.10	0.05
T uses	0.10	0.05	0.15

The consumer demand for each industry in billions of dollars is given by the matrix $D = \begin{bmatrix} 40 \\ 50 \\ 60 \end{bmatrix}$.

Example 2.6.3. *In the example above, what should be, in billions of dollars, the required output by each industry to meet the demand given by the matrix D?*

Solution 2.6.3. *We choose the following variables.*

$$\begin{aligned} x &= \textit{Farmer's output} \\ y &= \textit{Carpenter's output} \\ z &= \textit{Tailor's output} \end{aligned}$$

In the closed model, our equation was $X = AX$, that is, the total input equals the total output. This time our equation is similar with the exception of the demand by the consumer.

So our equation for the open model should be $X = AX + D$, where D represents the demand matrix.

We express it as follows:

$$X = AX + D$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} .20 & .25 & .10 \\ .15 & .10 & .05 \\ .10 & .05 & .15 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 40 \\ 50 \\ 60 \end{bmatrix}$$

To solve this system, we write it as

$$X = AX + D$$

$$(I - A)X = D$$

$$X = (I - A)^{-1}D$$

where I is a 3×3 identity matrix.

$$I - A = \begin{bmatrix} .80 & -.25 & -.10 \\ -.15 & .90 & -.05 \\ -.10 & -.05 & .85 \end{bmatrix}$$

$$(I-A)^{-1} = \begin{bmatrix} 1.3445 & .3835 & .1807 \\ .2336 & 1.1814 & .097 \\ .1719 & .1146 & 1.2034 \end{bmatrix}$$

$$X = \begin{bmatrix} 1.3445 & .3835 & .1807 \\ .2336 & 1.1814 & .097 \\ .1719 & .1146 & 1.2034 \end{bmatrix} \begin{bmatrix} 40 \\ 50 \\ 60 \end{bmatrix}$$

$$X = \begin{bmatrix} 83.7999 \\ 74.2341 \\ 84.8138 \end{bmatrix}$$

The three industries must produce the following amount of goods in billions of dollars.

$$\begin{aligned} \mathit{Farmer} &= 83.7999 \\ \mathit{Carpenter} &= 74.2341 \\ \mathit{Tailor} &= 84.8138 \end{aligned}$$

We will do one more problem like the one above, except this time we give the amount of internal and external consumption in dollars and ask for the proportion of the amounts consumed by each of the industries. In other words, we ask for the matrix A.

Example 2.6.4. *Suppose an economy consists of three industries F, C, and T. Each of the industries produces for internal consumption among themselves, as well as for external demand by the consumer. The table shows the use of each industry's production in dollars.*

	F	***C***	***T***	***Demand***	***Total***
F	40	50	60	100	250
C	30	40	40	110	220
T	20	30	30	120	200

The first row says that of the \$250 dollars worth of production by the industry F, \$40 is used by F, \$50 is used by C, \$60 is used by T, and the remainder *of \$100 is used by the consumer. The other rows are described in a similar manner.*

Once again, the total input equals the total output. Find the proportion of the amounts consumed by each of the industries. In other words, find the matrix A.

Solution 2.6.4. *We are being asked to determine the following:*

How much of the production of each of the three industries, F, C, and T is required to produce one unit of F? The same way, how much of the production of each of the three industries, F, C, and T is required to produce one unit of C? And finally, how much of the production of each of the three industries, F, C, and T is required to produce one unit of T?

Since we are looking for proportions, we need to divide the production of each industry by the total production for each industry.

We analyze as follows: To produce 250 units of F, 30 units of C, and 20 units of T, the required units are 40, 30, and 20 respectively. Therefore, to produce 1 unit of each, we divide by 250:

$$\textit{For F: } \frac{40}{250}, \ \textit{For C: } \frac{30}{250}, \ \textit{For T: } \frac{20}{250}$$

Similarly, for 220 units of C, the required units are 50, 40, and 30 respectively. To produce 1 unit of C, we divide by 220:

$$\textit{For F: } \frac{50}{220}, \ \textit{For C: } \frac{40}{220}, \ \textit{For T: } \frac{30}{220}$$

And for 200 units of T, the required units are 60, 40, and 30 respectively. To produce 1 unit of T, we divide by 200:

$$\textit{For F: } \frac{60}{200}, \ \textit{For C: } \frac{40}{200}, \ \textit{For T: } \frac{30}{200}$$

These fractions represent the units of F, C, and T required to produce 1 unit of each.

We obtain the following matrix:

$$A = \begin{bmatrix} \frac{40}{250} & \frac{50}{220} & \frac{60}{200} \\ \frac{30}{250} & \frac{40}{220} & \frac{40}{200} \\ \frac{20}{250} & \frac{30}{220} & \frac{30}{200} \end{bmatrix} = \begin{bmatrix} .1600 & .2273 & .3000 \\ .1200 & .1818 & .2000 \\ .0800 & .1364 & .1500 \end{bmatrix}$$

Clearly $AX + D = X$

$$\begin{bmatrix} .1600 & .2273 & .3000 \\ .1200 & .1818 & .2000 \\ .0800 & .1364 & .1500 \end{bmatrix} \begin{bmatrix} 250 \\ 220 \\ 200 \end{bmatrix} + \begin{bmatrix} 100 \\ 110 \\ 120 \end{bmatrix} = \begin{bmatrix} 250 \\ 220 \\ 200 \end{bmatrix}$$

Summary 2.6.1.
Leontief's Closed Model

1. *All consumption is within the industries. There is no external demand.*
2. *Input equals output.*
3. $X = AX$ *or* $(I - A)X = 0$

Leontief's Open Model

1. *In addition to internal consumption, there is an outside demand by the consumer.*
2. *Input equals output.*
3. $X = AX + D$ *or* $X = (I - A)^{-1}D$

Chapter 3

Linear Programming with Geometry

In this chapter, you will learn to:

1. Solve linear programming problems that maximize the objective function.

2. Solve linear programming problems that minimize the objective function.

3.1 Maximization Applications

In this section, you will learn to:

1. Recognize the typical form of a linear programming problem.

2. Formulate maximization linear programming problems.

3. Graph feasibility regions for maximization linear programming problems.

4. Determine optimal solutions for maximization linear programming problems.

Application problems in business, economics, and social and life sciences often ask us to make decisions on the basis of certain conditions. The conditions or constraints often take the form of inequalities. In this section, we will begin to formulate, analyze, and solve such problems, at a simple level, to understand the many components of such a problem.

A typical linear programming problem consists of finding an extreme value of a linear function subject to certain constraints. We are either trying to maximize or minimize the value of this linear function, such as to maximize profit or revenue, or to minimize cost. That is why these linear programming problems are classified as maximization or minimization problems, or just optimization problems. The function we are trying to optimize is called an objective function, and the conditions that must be satisfied are called constraints.

A typical example is to maximize profit from producing several products, subject to limitations on materials or resources needed for producing these items; the problem requires us to determine the amount of each item produced. Another type of problem involves scheduling; we need to determine how much time to devote to each of several activities in order to maximize income from (or minimize cost of) these activities, subject to limitations on time and other resources available for each activity.

In this chapter, we will work with problems that involve only two variables, and therefore, can be solved by graphing.

In the next chapter, we'll learn an algorithm to find a solution numerically. That will provide us with a tool to solve problems with more than two variables. At that time, with a little more knowledge about linear programming, we'll also explore the many ways these techniques are used in business and wide variety of other fields.

We begin by solving a maximization problem.

Example 3.1.1. *Niki holds two part-time jobs, Job I and Job II. She never wants to work more than a total of 12 hours a week. She has determined that for every hour she works at Job I, she needs 2 hours of preparation time, and for every hour she works at Job II, she needs one hour of preparation time, and she cannot spend more than 16 hours for preparation. If Niki makes $40 an hour at Job I, and $30 an hour at Job II, how many hours should she work per week at each job to maximize her income?*

Solution 3.1.1. *We start by choosing our variables. Let x be the number of* *hours per week Niki will work at Job I, and y the number of hours per week she will work at Job II.*

Now we write the objective function. Since Niki gets paid \$40 an hour at Job I, and \$30 an hour at Job II, her total income I is given by the following equation.

$$I = 40x + 30y$$

Our next task is to find the constraints. The constraints based on the problem description are:

$$x + y \leq 12$$

$$2x + y \leq 16$$

$$x \geq 0, \quad y \geq 0$$

We have formulated the problem as follows: Maximize

$$I = 40x + 30y$$

Subject to:

$$x + y \leq 12$$

$$2x + y \leq 16$$

$$x \geq 0; \quad y \geq 0$$

To solve the problem, we graph the constraints and shade the region that satisfies all the inequality constraints. We graph the lines by plotting the x-intercept and y-intercept and use a test point to determine which portion of the plane to shade.

In this example, after graphing the lines representing the constraints and using the origin (0,0) as a test point, we find that the feasible region is the area below and to the left of both constraint lines, above the x-axis, and to the right of the y-axis.

The shaded region where all conditions are satisfied is called the feasibility region or the feasibility polygon. The Fundamental Theorem of Linear Programming states that the maximum (or minimum) value of the objective function always takes place at the vertices of the feasibility region. Therefore, we will identify all the vertices (corner points) of the feasibility region. We

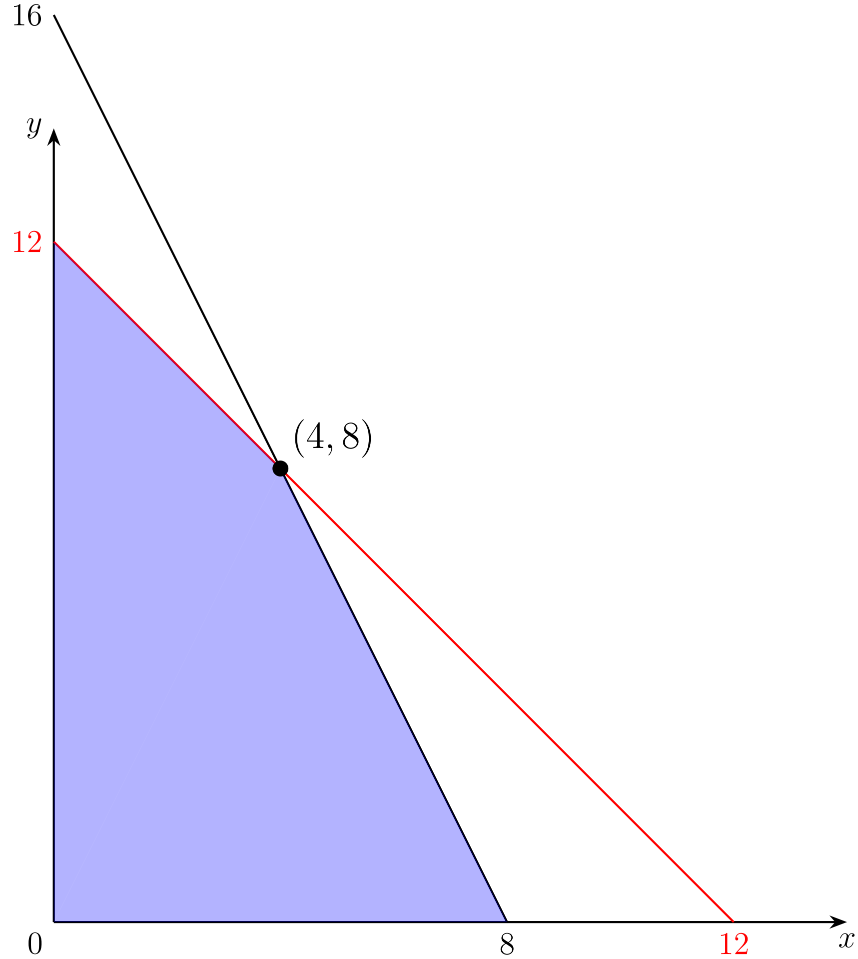


Figure 3.1: The red line is $x + y = 12$, the black line is $2x + y = 16$, and the blue region is the feasible region.

call these points critical points. They are listed as $(0, 0)$, $(0, 12)$, $(4, 8)$, and $(8, 0)$.

To maximize Niki's income, we will substitute these points in the objective function to see which point gives us the highest income per week. We list the results below:

Critical Points	**Income**
$(0,0)$	$40(0) + 30(0) = \$0$
$(0,12)$	$40(0) + 30(12) = \$360$
$(4,8)$	$40(4) + 30(8) = \$400$
$(8,0)$	$40(8) + 30(0) = \$320$

Clearly, the point $(4,8)$ gives the most profit: \$400. Therefore, we conclude that Niki should work 4 hours at Job I and 8 hours at Job II.

Example 3.1.2. *A factory manufactures two types of gadgets, regular and premium. Each gadget requires the use of two operations, assembly and finishing, and there are at most 12 hours available for each operation. A regular gadget requires 1 hour of assembly and 2 hours of finishing, while a premium gadget needs 2 hours of assembly and 1 hour of finishing. Due to other restrictions, the company can make at most 7 gadgets a day. If a profit of \$20 is realized for each regular gadget and \$30 for a premium gadget, how many of each should be manufactured to maximize profit?*

Solution 3.1.2. *We choose our variables. Let x be the number of regular gadgets manufactured each day, and y be the number of premium gadgets manufactured each day.*

The objective function is

$$P = 20x + 30y$$

We now write the constraints. The company can make at most 7 gadgets a day, giving us:

$$x + y \leq 7$$

The regular gadget requires one hour of assembly and the premium gadget two hours, with at most 12 hours available for assembly:

$$x + 2y \leq 12$$

Similarly, for finishing, we have:

$$2x + y \leq 12$$

The non-negativity constraints are:

$$x \geq 0, \quad y \geq 0$$

We formulate the problem as follows: Maximize $P = 20x + 30y$ *Subject to:*

$$x + y \leq 7$$

$$x + 2y \leq 12$$

$$2x + y \leq 12$$

$$x \geq 0; \quad y \geq 0$$

We next graph the constraints and feasibility region.

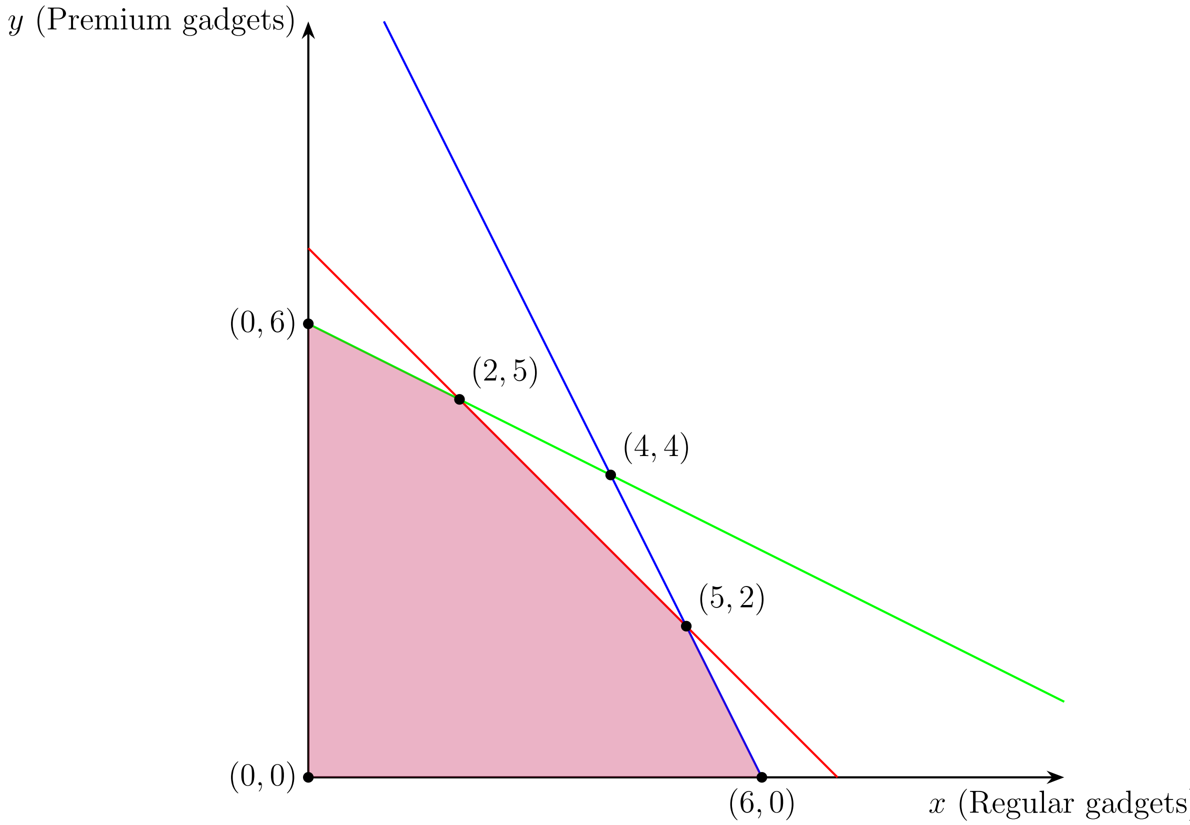


Figure 3.2: Feasibility region for the gadget factory optimization problem

Again, we have shaded the feasibility region, where all constraints are satisfied. Since the extreme value of the objective function always takes place at the vertices of the feasibility region, we identify all the critical points. They are listed as $(0,0)$, $(0,6)$, $(2,5)$, $(5,2)$, and $(6,0)$. Notice, $(4,4)$ is **not** a critical point because it is not on the edge of the critical region. To maximize profit, we will substitute these points in the objective function to see which point gives us the maximum profit each day. The results are listed below:

Critical Point	**Income**
$(0,0)$	$20(0)+30(0)=\$0$
$(0,6)$	$20(0)+30(6)=\$180$
$(2,5)$	$20(2)+30(5)=\$190$
$(5,2)$	$20(5)+30(2)=\$160$
$(6,0)$	$20(6)+30(0)=\$120$

The point $(2,5)$ gives the most profit, and that profit is \$190. Therefore, we conclude that we should manufacture 2 regular gadgets and 5 premium gadgets daily to obtain the maximum profit of \$190.

So far, we have focused on "standard maximization problems" in which:

1. The objective function is to be maximized.
2. All constraints are of the form $ax+by \leq c$.
3. All variables are constrained to be non-negative ($x \geq 0$, $y \geq 0$).

We will next consider an example where that is not the case. Our next problem is said to have "mixed constraints" since some of the inequality constraints are of the form $ax+by \leq c$ and some are of the form $ax+by \geq c$. The non-negativity constraints are still an important requirement in any linear program.

Example 3.1.3. *Solve the following maximization problem graphically.*

$$\begin{aligned} \textit{Maximize} \quad & P = 10x + 15y \\ \textit{Subject to:} \quad & x + y \geq 1 \\ & x + 2y \leq 6 \\ & 2x + y \leq 6 \\ & x \geq 0; \quad y \geq 0 \end{aligned}$$

Solution 3.1.3. *The graph is shown below.*

The five critical points are listed in the figure above. The reader should observe that the first constraint $x+y \geq 1$ requires that the feasibility region

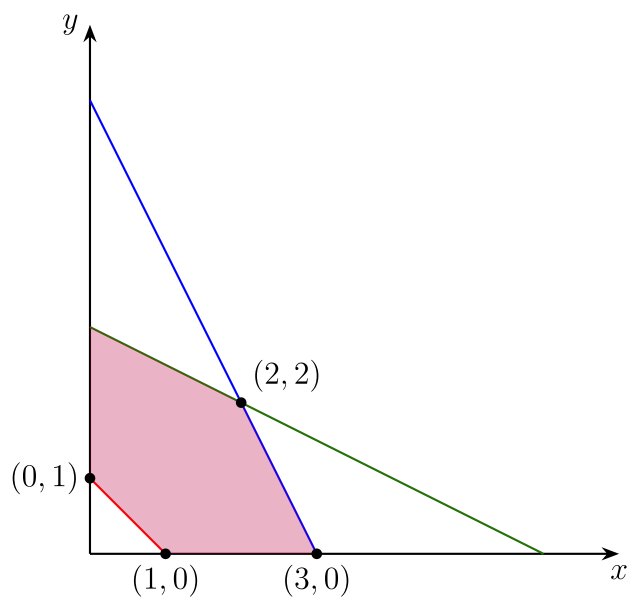


Figure 3.3: The red line is $x + y = 1$, the green line is $x + 2y = 6$ and the blue line is $2x + y = 6$.

must be bounded below by the line $x + y = 1$*; the test point* $(0, 0)$ *does not satisfy* $x + y \geq 1$*, so we shade the region on the opposite side of the line from the test point* $(0, 0)$*.*

Critical Point	*Income*
$(1, 0)$	$10(1) + 15(0) = \$10$
$(3, 0)$	$10(3) + 15(0) = \$30$
$(2, 2)$	$10(2) + 15(2) = \$50$
$(0, 3)$	$10(0) + 15(3) = \$45$
$(0, 1)$	$10(0) + 15(1) = \$15$

Clearly, the point $(2, 2)$ *maximizes the objective function to a maximum value of* 50*. It is important to observe that if the point* $(0, 0)$ *lies on the line for a constraint, then* $(0, 0)$ *could not be used as a test point. We would need to select any other point that does not lie on the line to use as a test point in that situation.*

Finally, we address an important question: Is it possible to determine the point that gives the maximum value without calculating the value at each critical point?

The answer is yes.

For example 3.1.2, we substituted the points $(0, 0)$, $(0, 6)$, $(2, 5)$, $(5, 2)$, and $(6, 0)$ in the objective function $P = 20x+30y$, and we got the values \$0, \$180, \$190, \$160, \$120, respectively. Sometimes that is not the most efficient way of finding the optimum solution. Instead, we could find the optimal value by also graphing the objective function.

To determine the largest P, we graph $P = 20x + 30y$ for any value P of our choice. Let us say, we choose $P = 60$. We graph $20x + 30y = 60$.

Now we move the line parallel to itself, that is, keeping the same slope at all times. Since we are moving the line parallel to itself, the slope is kept the same, and the only thing that is changing is the P. As we move away from the origin, the value of P increases. The largest possible value of P is realized when the line touches the last corner point of the feasibility region.

The figure below shows the movements of the line, and the optimum solution is achieved at the point $(2, 5)$. In maximization problems, as the line is being moved away from the origin, this optimum point is the farthest critical point.

Summary 3.1.1.
The Maximization Linear Programming Problems

1. *Write the objective function.*
2. *Write the constraints.*
   (a) *For the standard maximization linear programming problems, constraints are of the form:* $ax + by \leq c$.
   (b) *Since the variables are non-negative, we include the constraints:* $x \geq 0$, $y \geq 0$.
3. *Graph the constraints.*
4. *Shade the feasibility region.*
5. *Find the corner points.*
6. *Determine the corner point that gives the maximum value.*
   (a) *This is done by finding the value of the objective function at each corner point.*
   (b) *This can also be done by moving the line associated with the objective function.*

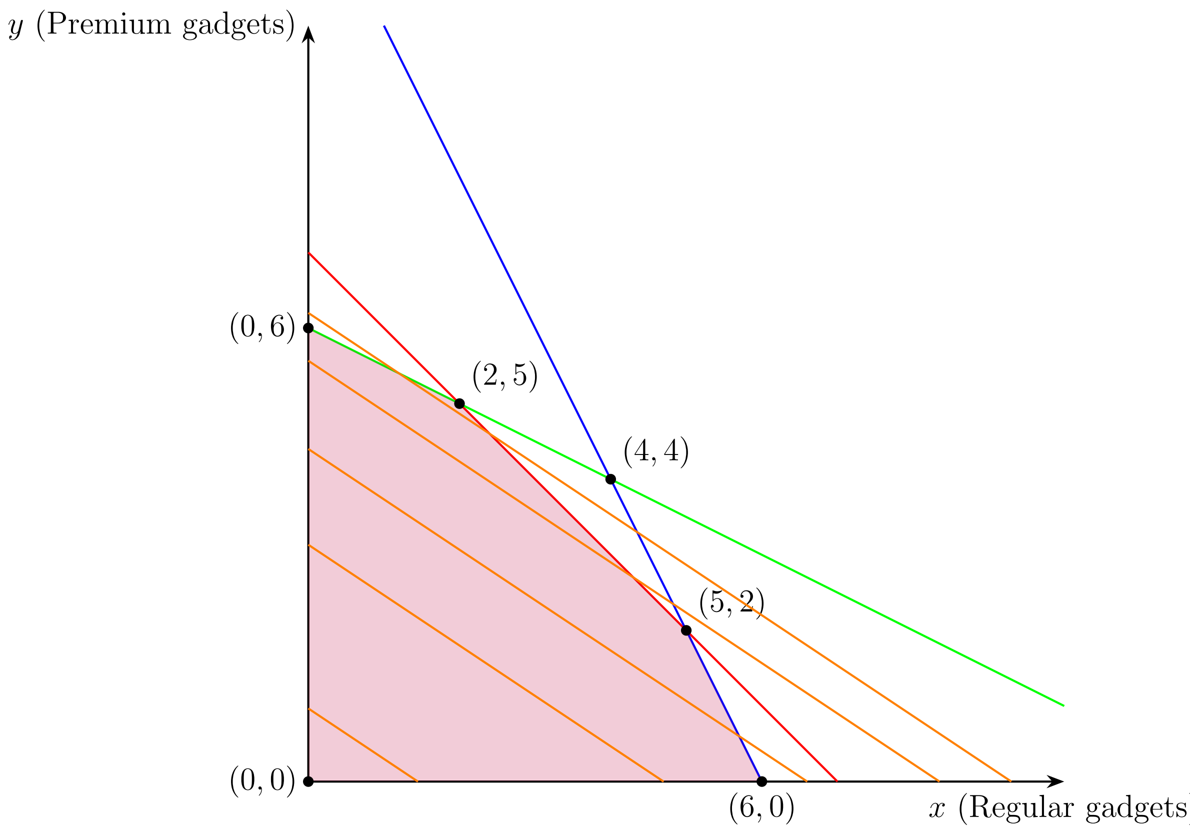


Figure 3.4: Feasibility region for the gadget factory optimization problem with profit lines.

3.2 Minimization Applications

In this section, you will learn to:

1. Formulate minimization linear programming problems.
2. Graph feasibility regions for minimization linear programming problems.
3. Determine optimal solutions for minimization linear programming problems.

Minimization linear programming problems are solved in much the same way as the maximization problems.

For the standard minimization linear program, the constraints are of the form $ax + by \geq c$, as opposed to the form $ax + by \leq c$ for the standard maximization problem. As a result, the feasible solution extends indefinitely to the upper right of the first quadrant, and is unbounded. But that is not a concern, since in order to minimize the objective function, the line associated with the objective function is moved towards the origin, and the critical point that minimizes the function is closest to the origin.

However, one should be aware that in the case of an unbounded feasibility region, the possibility of no optimal solution exists.

Example 3.2.1. *At a university, Professor Symons wishes to employ two people, John and Mary, to grade papers for his classes. John is a graduate student and can grade 20 papers per hour; John earns \$15 per hour for grading papers. Mary is a post-doctoral associate and can grade 30 papers per hour; Mary earns \$25 per hour for grading papers. Each must be employed at least one hour a week to justify their employment. If Professor Symons has at least 110 papers to be graded each week, how many hours per week should he employ each person to minimize the cost?*

Solution 3.2.1. *We choose the variables as follows: Let x be the number of hours per week John is employed, and y be the number of hours per week Mary is employed.*

The objective function is

$$C = 15x + 25y$$

The constraints are that each must work at least one hour each week:

$$x \geq 1$$

$$y \geq 1$$

John can grade 20 papers per hour and Mary 30 papers per hour, with at least 110 papers to be graded per week:

$$20x + 30y \geq 110$$

Additionally, x and y are non-negative:

$$x \geq 0$$

$$y \geq 0$$

The problem is thus formulated as: Minimize $C = 15x + 25y$ *Subject to:*

$$x \geq 1$$

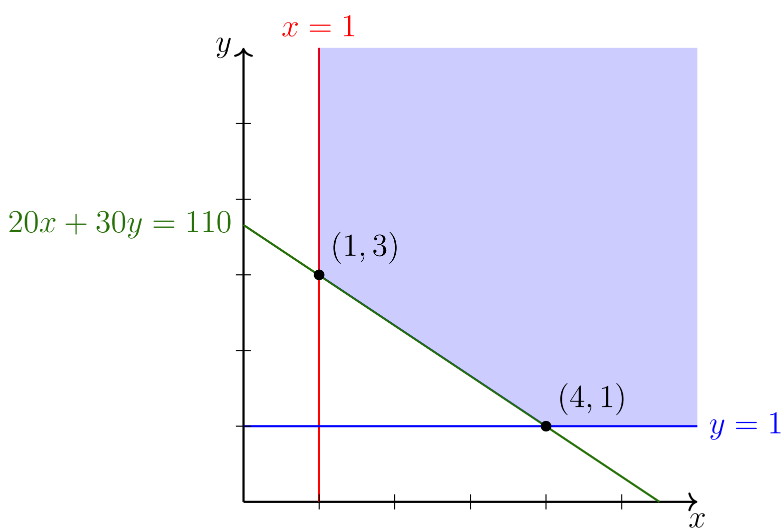
$$y \geq 1$$

$$20x + 30y \geq 110$$

$$x \geq 0$$

$$y \geq 0$$

To solve the problem, we graph the constraints as follows:



Again, we have shaded the feasibility region, where all constraints are satisfied. If we used test point $(0, 0)$ *that does not lie on any of the constraints, we observe that* $(0, 0)$ *does not satisfy any of the constraints* $x \geq 1$, $y \geq 1$, *and* $20x + 30y \geq 110$. *Thus, all the shading for the feasibility region lies on the opposite side of the constraint lines from the point* $(0, 0)$.

Alternatively, we could use test point $(4, 6)$, *which also does not lie on any of the constraint lines. We'd find that* $(4, 6)$ *does satisfy all of the inequality constraints. Consequently, all the shading for the feasibility region lies on the same side of the constraint lines as the point* $(4, 6)$.

Since the extreme value of the objective function always takes place at the vertices of the feasibility region, we identify the two critical points, $(1,3)$ *and* $(4,1)$*. To minimize cost, we will substitute these points in the objective function to see which point gives us the minimum cost each week. The results are listed below:*

Critical points	*Income*
$(1,3)$	$15(1)+25(3)=\$90$
$(4,1)$	$15(4)+25(1)=\$85$

The point $(4,1)$ *gives the least cost, and that cost is \$85. Therefore, we conclude that in order to minimize grading costs, Professor Symons should employ John for 4 hours a week and Mary for 1 hour a week at a cost of \$85 per week.*

Example 3.2.2. *Professor Hamer is on a low cholesterol diet. During lunch at the college cafeteria, he always chooses between two meals, Pasta or Tofu. The table below lists the amount of protein, carbohydrates, and vitamins each meal provides along with the amount of cholesterol he is trying to minimize. Mr. Hamer needs at least 200 grams of protein, 960 grams of carbohydrates, and 40 grams of vitamins for lunch each month. Over this time period, how many days should he have the Pasta meal, and how many days the Tofu meal so that he gets the adequate amount of protein, carbohydrates, and vitamins and at the same time minimizes his cholesterol intake?*

	Pasta	***Tofu***
Protein (g)	*8*	*16*
Carbohydrates (g)	*60*	*40*
Vitamin C (g)	*2*	*2*
Cholesterol (mg)	*60*	*50*

Solution 3.2.2. *We choose the variables as follows: Let* x *be the number of days Mr. Hamer eats Pasta, and* y *the number of days he eats Tofu.*

The objective function for minimizing cholesterol intake is

$$C = 60x + 50y$$

The constraints for protein, carbohydrates, and vitamins are as follows:

$$\begin{aligned} 8x + 16y &\geq 200 \\ 60x + 40y &\geq 960 \\ 2x + 2y &\geq 40 \end{aligned}$$

Additionally, x and y are non-negative:

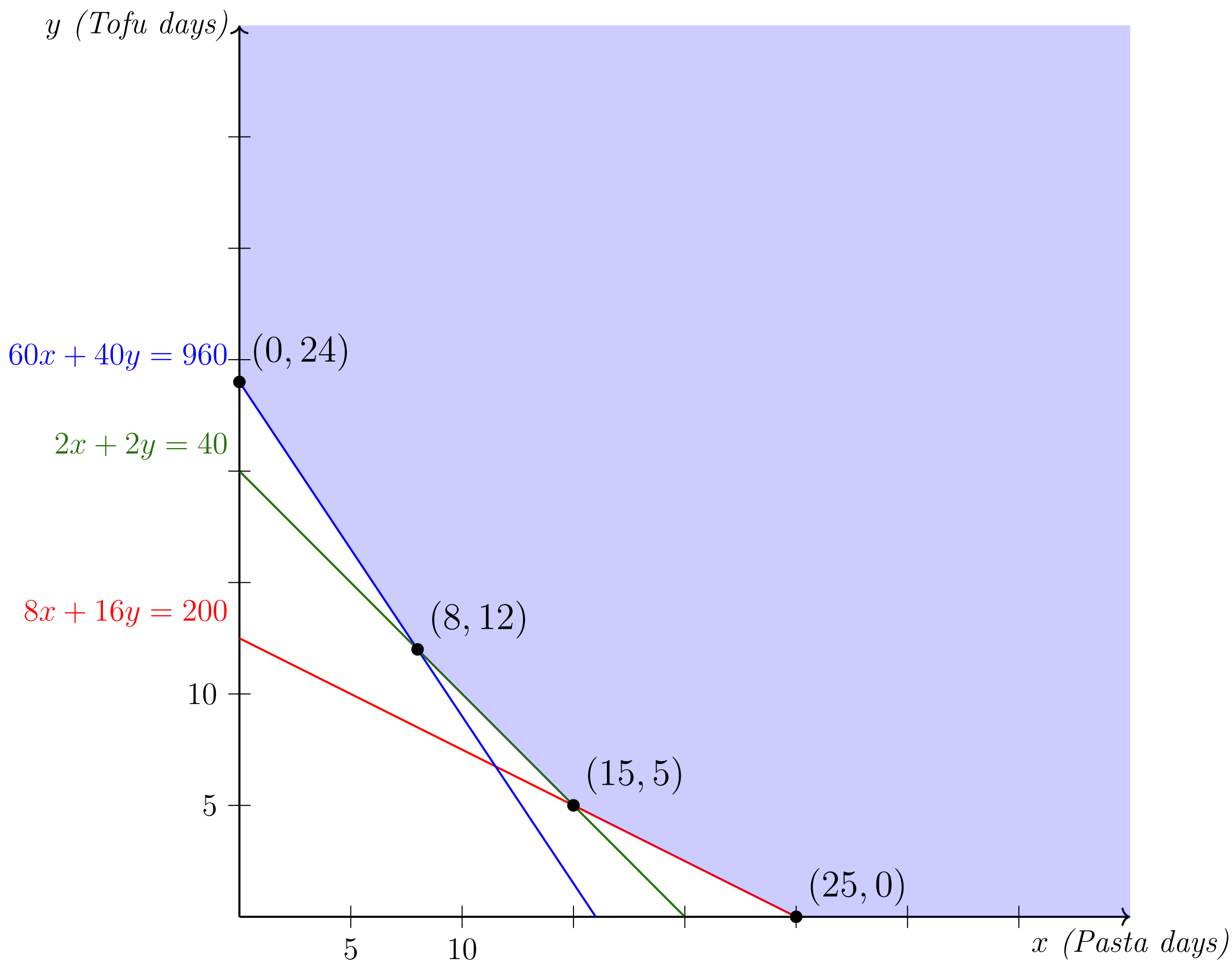
$$x \geq 0$$

$$y \geq 0$$

We summarize the problem as: Minimize $C = 60x + 50y$ Subject to:

$$\begin{aligned} 8x + 16y &\geq 200 \\ 60x + 40y &\geq 960 \\ 2x + 2y &\geq 40 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

To solve the problem, we graph the constraints and shade the feasibility region.



We have shaded the unbounded feasibility region, where all constraints are satisfied. To minimize the objective function, we find the vertices of the feasibility region. These vertices are $(0, 24)$, $(8, 12)$, $(15, 5)$, *and* $(25, 0)$. *To minimize cholesterol, we will substitute these points in the objective function to see which point gives us the smallest value. The results are listed below:*

Critical points	*Cholesterol*
$(0, 24)$	$60(0) + 50(24) = 1200\, mg$
$(8, 12)$	$60(8) + 50(12) = 1080\, mg$
$(15, 5)$	$60(15) + 50(5) = 1150\, mg$
$(25, 0)$	$60(25) + 50(0) = 1500\, mg$

The point $(8, 12)$ *gives the least cholesterol, which is 1080 mg. This states that for every 20 meals, Professor Hamer should eat Pasta for 8 days and Tofu for 12 days.*

We must be aware that in some cases, a linear program may not have an optimal solution.

- A linear program can fail to have an optimal solution if there is no feasibility region. If the inequality constraints are not compatible, there may not be a region in the graph that satisfies all the constraints. If the linear program does not have a feasible solution satisfying all constraints, then it cannot have an optimal solution.

- A linear program can fail to have an optimal solution if the feasibility region is unbounded. The two minimization linear programs we examined had unbounded feasibility regions. The feasibility region was bounded by constraints on some sides but was not entirely enclosed by the constraints. Both of the minimization problems had optimal solutions. However, if we were to consider a maximization problem with a similar unbounded feasibility region, the linear program would have no optimal solution. No matter what values of x and y were selected, we could always find other values of x and y that would produce a higher value for the objective function. In other words, if the value of the objective function can be increased without bound in a linear program with an unbounded feasible region, there is no optimal maximum solution.

Although the method of solving minimization problems is similar to that of maximization problems, we still feel that we should summarize the steps involved.

Summary 3.2.1.
Minimization Linear Programming Problems

1. *Write the objective function.*
2. *Write the constraints.*
   (a) *For standard minimization linear programming problems, constraints are of the form:* $ax + by \geq c$.
   (b) *Since the variables are non-negative, include the constraints:* $x \geq 0$; $y \geq 0$.
3. *Graph the constraints.*
4. *Shade the feasibility region.*
5. *Find the corner points.*
6. *Determine the corner point that gives the minimum value.*
   (a) *This can be done by finding the value of the objective function at each corner point.*
   (b) *This can also be done by moving the line associated with the objective function.*
   (c) *There is the possibility that the problem has no solution.*

Chapter 4

Linear Programming, Simplex Method

In this chapter, you will learn to:

1. Investigate real world applications of linear programming and related methods.
2. Solve linear programming maximization problems using the simplex method.
3. Solve linear programming minimumization problems using the simplex method.

4.1 Linear Programming Applications in Business, Finance, Medicine, and Social Science

In this section, you will learn about:

1. real world applications of linear programming and related methods

The linear programs we solved in chapter 3 contain only two variables, x and y, so that we could solve them graphically. In practice, linear programs can contain thousands of variables and constraints.

Later in this chapter we'll learn to solve linear programs with more than two variables using the simplex algorithm, which is a numerical solution method that uses matrices and row operations. However, in order to make the problems practical for learning purposes, our problems will still have only several variables.

Now that we understand the main concepts behind linear programming, we can also consider how linear programming is currently used in large scale real-world applications.

Linear programming is used in business and industry in production planning, transportation and routing, and various types of scheduling. Airlines use linear programs to schedule their flights, taking into account both scheduling aircraft and scheduling staff. Delivery services use linear programs to schedule and route shipments to minimize shipment time or minimize cost. Retailers use linear programs to determine how to order products from manufacturers and organize deliveries with their stores. Manufacturing companies use linear programming to plan and schedule production. Financial institutions use linear programming to determine the mix of financial products they offer, or to schedule payments transferring funds between institutions. Health care institutions use linear programming to ensure the proper supplies are available when needed. And as we'll see below, linear programming has also been used to organize and coordinate life saving health care procedures.

In some of the applications, the techniques used are related to linear programming but are more sophisticated than the methods we study in this class. One such technique is called integer programming. In these situations, answers must be integers to make sense, and can not be fractions. Problems where solutions must be integers are more difficult to solve than the linear programs we've worked with. In fact, many of our problems have been very carefully constructed for learning purposes so that the answers just happen to turn out to be integers, but in the real world unless we specify that as a restriction, there is no guarantee that a linear program will produce integer solutions. There are also related techniques that are called non-linear programs, where the functions defining the objective function and/or some or all of the constraints may be non-linear rather than straight lines.

Many large businesses that use linear programming and related methods have analysts on their staff who can perform the analyses needed, including linear programming and other mathematical techniques. Consulting firms specializing in use of such techniques also aid businesses who need to apply these methods to their planning and scheduling processes.

When used in business, many different terms may be used to describe the use of techniques such as linear programming as part of mathematical business models. Optimization, operations research, business analytics, data science, industrial engineering hand management science are among the terms used to describe mathematical modelling techniques that may include linear programming and related met

In the rest of this section we'll explore six real world applications, and investigate what they are trying to accomplish using optimization, as well as what their constraints might represent.

4.1.1 Airline Scheduling

Airlines use techniques that include and are related to linear programming to schedule their aircraft to flights on various routes and to schedule crews to the flights. In addition, airlines also use linear programming to determine ticket pricing for various types of seats and levels of service or amenities, as well as the timing at which ticket prices change.

The process of scheduling aircraft and departure times on flight routes can be expressed as a model that minimizes cost, of which the largest component is generally fuel costs. Constraints involve considerations such as:

- Each aircraft needs to complete a daily or weekly tour to return back to its point of origin.
- Scheduling sufficient flights to meet demand on each route.
- Scheduling the right type and size of aircraft on each route to be appropriate for the route and for the demand for the number of passengers.
- Aircraft must be compatible with the airports it departs from and arrives at - not all airports can handle all types of planes.

A model to accomplish this could contain thousands of variables and constraints. Highly trained analysts determine ways to translate all the constraints into mathematical inequalities or equations to put into the model.

After aircraft are scheduled, crews need to be assigned to flights. Each flight needs a pilot, a co-pilot, and flight attendants. Each crew member needs to complete a daily or weekly tour to return back to his or her home base. Additional constraints on flight crew assignments take into account factors such as:

- Pilot and co-pilot qualifications to fly the particular type of aircraft they are assigned to.
- Flight crew have restrictions on the maximum amount of flying time per day and the length of mandatory rest periods between flights or per day that must meet certain minimum rest time regulations.
- Numbers of crew members required for a particular type or size of aircraft.

When scheduling crews to flights, the objective function would seek to minimize total flight crew costs, determined by the number of people on the crew and pay rates of the crew members. However, the cost for any particular route might not end up being the lowest possible for that route, depending on tradeoffs to the total cost of shifting different crews to different routes.

An airline can also use linear programming to revise schedules on short notice on an emergency basis when there is a schedule disruption, such as due to weather. In this case, the considerations to be managed involve:

- Getting aircraft and crews back on schedule as quickly as possible.
- Moving aircraft from storm areas to areas with calm weather to keep the aircraft safe from damage and ready to come back into service as quickly and conveniently as possible.
- Ensuring crews are available to operate the aircraft and that crews continue to meet mandatory rest period requirements and regulations.

4.1.2 Kidney Donation Chain

For patients who have kidney disease, a transplant of a healthy kidney from a living donor can often be a lifesaving procedure. Criteria for a kidney donation procedure include the availability of a donor who is healthy enough to donate a kidney, as well as a compatible match between the patient and donor for blood type and several other characteristics. Ideally, if a patient needs a kidney donation, a close relative may be a match and can be the kidney donor. However, often there is not a relative who is a close enough match to be the donor. Considering donations from unrelated donors allows for a larger pool of potential donors. Kidney donations involving unrelated donors can sometimes be arranged through a chain of donations that pair patients with donors. For example, a kidney donation chain with three donors might operate as follows:

- Donor A donates a kidney to Patient B.
- Donor B, who is related to Patient B, donates a kidney to Patient C.
- Donor C, who is related to Patient C, donates a kidney to Patient A, who is related to Donor A.

Linear programming is one of several mathematical tools that have been used to help efficiently identify a kidney donation chain. In this type of model, patient/donor pairs are assigned compatibility scores based on characteristics of patients and potential donors. The objective is to maximize the total compatibility scores. Constraints ensure that donors and patients are paired only if compatibility scores are sufficiently high to indicate an acceptable match.

4.1.3 Advertisements in Online Marketing

Did you ever make a purchase online and then notice that as you browse websites, search, or use social media, you now see more ads related the item you purchased? Marketing organizations use a variety of mathematical techniques, including linear programming, to determine individualized advertising placement purchases.

Instead of advertising randomly, online advertisers want to sell bundles of advertisements related to a particular product to batches of users who are more likely to purchase that product. Based on an individual's previous browsing and purchase selections, he or she is assigned a "propensity score" for making a purchase if shown an ad for a certain product. The company placing the ad generally does not know individual personal information based on the history of items viewed and purchased, but instead has aggregated information for groups of individuals based on what they view or purchase. However, the company may know more about an individual's history if he or she logged into a website making that information identifiable, within the privacy provisions and terms of use of the site.

The company's goal is to buy ads to present to specified size batches of people who are browsing. The linear program would assign ads and batches of people to view the ads using an objective function that seeks to maximize advertising response modeled using the propensity scores. The constraints are to stay within the restrictions of the advertising budget.

4.1.4 Loans

A car manufacturer sells its cars though dealers. Dealers can offer loan financing to customers who need to take out loans to purchase a car. Here we will consider how car manufacturers can use linear programming to determine the specific characteristics of the loan they offer to a customer who purchases a car. In a future chapter, we will learn how to do the financial calculations related to loans.

A customer who applies for a car loan fills out an application. This provides the car dealer with information about that customer. In addition, the car dealer can access a credit bureau to obtain information about a customer's credit score.

Based on this information obtained about the customer, the car dealer offers a loan with certain characteristics, such as interest rate, loan amount, and length of loan repayment period.

Linear programming can be used as part of the process to determine the characteristics of the loan offer. The linear program seeks to maximize the profitability of its portfolio of loans. The constraints limit the risk that the customer will default and will not repay the loan. The constraints also seek to minimize the risk of losing the loan customer if the conditions of the loan are not favorable enough; otherwise, the customer may find another lender, such as a bank, which can offer a more favorable loan.

4.1.5 Production Planning and Scheduling in Manufacturing

Consider the example of a company that produces yogurt. There are different varieties of yogurt products in a variety of flavors. Yogurt products have a short shelf life; it must be produced on a timely basis to meet demand, rather than drawing upon a stockpile of inventory as can be done with a product that is not perishable. Most ingredients in yogurt also have a short shelf life, so can not be ordered and stored for long periods of time before use; ingredients must be obtained in a timely manner to be available when needed but still be fresh. Linear programming can be used in both production planning and scheduling.

To start the process, sales forecasts are developed to determine demand to know how much of each type of product to make. There are often various manufacturing plants at which the products may be produced. The appropriate ingredients need to be at the production facility to produce the products assigned to that facility. Transportation costs must be considered, both for obtaining and delivering ingredients to the correct facilities, and for transport of finished product to the sellers. The linear program that monitors production planning and scheduling must be updated frequently - daily or even twice each day - to take into account variations from a master plan.

4.1.6 Bike Share Programs

Over 600 cities worldwide have bikeshare programs. Although bikeshare programs have been around for a long time, they have proliferated in the past decade as technology has developed new methods for tracking the bicycles.

Bikeshare programs vary in the details of how they work, but most typically people pay a fee to join and then can borrow a bicycle from a bike share station and return the bike to the same or a different bike share station. Over time the bikes tend to migrate; there may be more people who want to pick up a bike at station A and return it at station B than there are people who want to do the opposite. In chapter 5, we'll investigate a technique that can be used to predict the distribution of bikes among the stations.

Once other methods are used to predict the actual and desired distributions of bikes among the stations, bikes may need to be transported between stations to even out the distribution. Bikeshare programs in large cities have used methods related to linear programming to help determine the best routes and methods for redistributing bicycles to the desired stations once the desire distributions have been determined. The optimization model would seek to minimize transport costs and/or time subject to constraints of having sufficient bicycles at the various stations to meet demand.

4.2 Maximization by the Simplex Method

In this section, you will learn to:

1. Solve linear programming maximization problems using the Simplex Method by
   (a) Identifing and set up a linear program in standard maximization form
   (b) Converting inequality constraints to equations using slack variables
   (c) Setting up the initial simplex tableau using the objective function and slack equations
   (d) Finding the optimal simplex tableau by performing pivoting operations
   (e) Identifing the optimal solution from the optimal simplex tableau

In the last chapter, we used the geometrical method to solve linear programming problems, but the geometrical approach will not work for problems that have more than two variables. In real-life situations, linear programming problems consist of literally thousands of variables and are solved by computers. We can solve these problems algebraically, but that will not be very efficient. Suppose we were given a problem with, say, 5 variables and 10 constraints. By choosing all combinations of five equations with five unknowns, we could find all the corner points, test them for feasibility, and come up with the solution, if it exists. But the trouble is that even for a problem with so few variables, we will get more than 250 corner points, and testing each point will be very tedious. So we need a method that has a systematic algorithm and can be programmed for a computer. The method has to be efficient enough so we wouldn't have to evaluate the objective function at each corner point. We have just such a method, and it is called the simplex method.

The simplex method was developed during the Second World War by Dr. George Dantzig. His linear programming models helped the Allied forces with transportation and scheduling problems. In 1979, a Soviet scientist named Leonid Khachian developed a method called the ellipsoid algorithm, which was supposed to be revolutionary, but as it turned out, it is not any better than the simplex method. In 1984, Narendra Karmarkar, a research scientist at AT&T Bell Laboratories developed Karmarkar's algorithm, which has been proven to be four times faster than the simplex method for certain problems. But the simplex method still works the best for most problems.

The simplex method uses an approach that is very efficient. It does not compute the value of the objective function at every point; instead, it begins with a corner point of the feasibility region where all the main variables are zero and then systematically moves from corner point to corner point while improving the value of the objective function at each stage. The process continues until the optimal solution is found.

To learn the simplex method, we try a rather unconventional approach. We first list the algorithm, and then work a problem. We justify the reasoning behind each step during the process. A thorough justification is beyond the scope of this course.

We start out with an example we solved in the last chapter by the graphical method. This will provide us with some insight into the simplex method and at the same time give us the chance to compare a few of the feasible solutions we obtained previously by the graphical method.

But first, we list the algorithm for the simplex method.

Summary 4.2.1.
The Simplex Method *Here are the steps to solve a linear programming problem using the Simplex Method:*

1. ***Set up the problem.***
   - *Write the objective function and the inequality constraints.*
2. ***Convert the inequalities into equations.***
   - *Add one slack variable for each inequality.*
3. ***Construct the initial simplex tableau.***
   - *Write the objective function as the bottom row.*
4. ***Identify the pivot column.***
   - *The most negative entry in the bottom row identifies the pivot column.*
5. ***Calculate the quotients.***
   - *Divide the far-right column by the identified pivot column to find quotients.*
   - *The smallest positive quotient identifies a row, and its corresponding element is the pivot element.*
6. ***Perform pivoting.***
   - *Make all other entries in the pivot column zero by using the Gauss-Jordan method.*
7. ***Repeat if necessary.***
   - *If there are still negative entries in the bottom row, go back to step 4.*
8. ***Read off your answers.***
   - *Get the variables using the columns with 1 and 0s. All other variables are zero.*
   - *The maximum value you are looking for appears in the bottom right-hand corner.*

Now, we use the simplex method to solve example 4.2.1 solved geometrically in example 3.1.1.

Example 4.2.1. *Niki holds two part-time jobs, Job I and Job II. She never wants to work more than a total of 12 hours a week. For every hour she works at Job I, she needs 2 hours of preparation time, and for every hour at Job II, she needs one hour of preparation time. She cannot spend more than 16 hours on preparation. If Niki makes \$40 an hour at Job I and \$30 an hour at Job II, how many hours should she work at each job to maximize* *her income?*

Solution 4.2.1.

1. ***Set up the problem.*** *Write the objective function and the constraints. Since the simplex method is used for problems that consist of many variables, it is not practical to use the variables x, y, z etc. We use symbols x_1, x_2, x_3, and so on.*

   Let x_1 = The number of hours per week Niki will work at Job I.
   and x_2 = The number of hours per week Niki will work at Job II.

   It is customary to choose the variable that is to be maximized as Z. The problem is formulated the same way as we did in the last chapter.

   $$\text{Maximize} \quad Z = 40x_1 + 30x_2$$

   Subject to:

   $$\begin{aligned} x_1 + x_2 &\leq 12 \\ 2x_1 + x_2 &\leq 16 \\ x_1, x_2 &\geq 0 \end{aligned}$$

2. ***Convert the inequalities into equations.*** *This is done by adding one slack variable for each inequality.*

   For example, to convert the inequality $x_1 + x_2 \leq 12$ into an equation, we add a non-negative variable y_1, and we get

   $$x_1 + x_2 + y_1 = 12$$

   Here the variable y_1 picks up the slack, and it represents the amount by which $x_1 + x_2$ falls short of 12. In this problem, if Niki works fewer than 12 hours, say 10, then y_1 is 2. Later when we read off the final solution from the simplex table, the values of the slack variables will identify the unused amounts.

   We rewrite the objective function $Z = 40x_1 + 30x_2$ as $-40x_1 - 30x_2 + Z = 0$.

After adding the slack variables, our problem reads

Objective function:

$$-40x_1 - 30x_2 + Z = 0$$

Subject to constraints:

$$\begin{aligned} x_1 + x_2 + y_1 &= 12 \\ 2x_1 + x_2 + y_2 &= 16 \\ x_1, x_2 &\geq 0 \end{aligned}$$

3. ***Construct the initial simplex tableau.*** *Each inequality constraint appears in its own row. (The non-negativity constraints do not appear as rows in the simplex tableau.) Write the objective function as the bottom row.*

   Now that the inequalities are converted into equations, we can represent the problem into an augmented matrix called the initial simplex tableau as follows.

$$\begin{array}{ccccc|c} x_1 & x_2 & y_1 & y_2 & Z & C \\ 1 & 1 & 1 & 0 & 0 & 12 \\ 2 & 1 & 0 & 1 & 0 & 16 \\ \hline -40 & -30 & 0 & 0 & 1 & 0 \end{array}$$

   Here the vertical line separates the left hand side of the equations from the right side. The horizontal line separates the constraints from the objective function. The right side of the equation is represented by the column C.

   The reader may observe that the last four columns of this matrix look like the final matrix for the solution of a system of equations. If we arbitrarily choose $x_1 = 0$ and $x_2 = 0$, we get

$$\begin{array}{cccc} y_1 & y_2 & Z & C \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

which reads $y_1 = 12,\ y_2 = 16,\ Z = 0.$

The solution obtained by arbitrarily assigning values to some variables and then solving for the remaining variables is called the basic solution associated with the tableau. So the above solution is the basic solution associated with the initial simplex tableau. We can label the basic solution variable in the right of the last column as shown in the table below.

x_1	x_2	y_1	y_2	Z	C	
1	1	1	0	0	12	y_1
2	1	0	1	0	16	y_2
-40	-30	0	0	1	0	Z

4. ***The most negative entry in the bottom row identifies the pivot column.*** *The most negative entry in the bottom row is -40; therefore, the column 1 is identified.*

x_1	x_2	y_1	y_2	Z	C
1	1	1	0	0	12
2	1	0	1	0	16
-40	-30	0	0	1	0

Why do we choose the most negative entry in the bottom row?

The most negative entry in the bottom row represents the largest coefficient in the objective function; the coefficient whose entry will increase the value of the objective function the quickest.

The simplex method begins at a corner point where all the main variables, the variables that have symbols such as x_1, x_2, x_3, *etc., are zero. It then moves from a corner point to the adjacent corner point always increasing the value of the objective function. In the case of the objective function* $Z = 40x_1 + 30x_2$, *it will make more sense to increase the value of* x_1 *rather than* x_2. *The variable* x_1 *represents the number of hours per week Niki works at Job I. Since Job I pays \$40 per hour as opposed to Job II which pays only \$30, the variable* x_1 *will increase the objective function by \$40 for a unit of increase in the variable* x_1.

5. ***Calculate the quotients. The smallest quotient identifies a row.*** *The element in the intersection of the column identified in step 4 (marked with ↑) and the row identified in this step is identified as the pivot element.*

   Following the algorithm, in order to calculate the quotient, we divide the entries in the far right column by the entries in column 1, excluding the entry in the bottom row.

$$
\begin{array}{ccccc|ccc}
x_1 & x_2 & y_1 & y_2 & Z & C & & \\
1 & 1 & 1 & 0 & 0 & 12 & & 12/1 = 12 \\
2 & 1 & 0 & 1 & 0 & 16 & \leftarrow & 16/2 = 8 \\
\hline
-40 & -30 & 0 & 0 & 1 & 0 & & \\
\uparrow & & & & & & &
\end{array}
$$

   The smallest of the two quotients, 12 and 8, is 8. Therefore row 2 is identified. The intersection of column 1 and row 2 is the entry 2, which has been highlighted. This is our pivot element.

   Why do we find quotients, and why does the smallest quotient identify a row?

   When we choose the most negative entry in the bottom row, we are trying to increase the value of the objective function by bringing in the variable x_1. But we cannot choose any value for x_1. For instance, letting $x_1 = 100$ is not possible because Niki never wants to work more than 12 hours at both jobs combined: $x_1 + x_2 \leq 12$. Therefore, the maximum she can work is 12 hours for x_1, meaning the preparation time for Job I is two times the time spent on the job. Since she never wants to spend more than 16 hours for preparation, the maximum time she can work is $\frac{16}{2} = 8$ hours. Using the pivot element guarantees that we do not violate the constraints.

   Why do we identify the pivot element?

   The simplex method begins at a corner point where all the main variables, the variables that have symbols such as x_1, x_2, x_3, etc., are zero. It then moves from a corner point to the adjacent corner point always improving the value of the objective function. The value of the objective function is improved by changing the number of units of the variables.

We may add the number of units of one variable, while throwing away the units of another. Pivoting allows us to do just that.

The variable whose units are being added is called the entering variable, and the variable whose units are being replaced is called the departing variable. The entering variable in the above table is x_1*, and it was identified by the most negative entry in the bottom row. The departing variable* y_2 *was identified by the lowest of all quotients.*

6. ***Perform pivoting to make all other entries in this column zero.*** *In Chapter 2, we used pivoting to obtain the row echelon form of an augmented matrix. Pivoting is a process of obtaining a* 1 *in the location of the pivot element (marked by a box below), and then making all other entries zeros in that column. We've highlighted the pivot row to make it easier to track. So now our job is to make our pivot element a* 1 *by dividing the entire second row by* 2*. The result follows.*

x_1	x_2	y_1	y_2	Z	C
1	1	1	0	0	12
$\boxed{2}$	1	0	1	0	16
−40	−30	0	0	1	0

To obtain a zero in the entry first above the pivot element, we multiply the second row by −1 *and add it to row* 1*. We get*

x_1	x_2	y_1	y_2	Z	C
0	1/2	1	−1/2	0	4
$\boxed{1}$	1/2	0	1/2	0	8
−40	−30	0	0	1	0

To obtain a zero in the element below the pivot, we multiply the second row by 40 *and add it to the last row.*

x_1	x_2	y_1	y_2	Z	C
0	1/2	1	−1/2	0	4
$\boxed{1}$	1/2	0	1/2	0	8
0	−10	0	20	1	320

We now determine the basic solution associated with this tableau. By arbitrarily choosing $x_2 = 0$ and $y_2 = 0$, we obtain $x_1 = 8$, $y_1 = 4$, and $Z = 320$. If we write the augmented matrix, whose left side is a matrix with columns that have one 1 and all other entries zeros, we get the following matrix stating the same thing.

$$\begin{array}{cccc} x_1 & y_1 & Z & C \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 320 \end{array}\right]$$

We can restate the solution associated with this matrix as $x_1 = 8$, $x_2 = 0$, $y_1 = 4$, $y_2 = 0$, and $z = 320$. At this stage, it reads that if Niki works 8 hours at Job I and no hours at Job II, her profit z will be \$320. Recall from Example 1 in Section 3.1 that $(8, 0)$ was one of our corner points. Here $y_1 = 4$ and $y_2 = 0$ mean that she will be left with 4 hours of working time and no preparation time.

7. ***When there are no more negative entries in the bottom row, we are finished; otherwise, we start again from step 4.*** *Since there is still a negative entry, -10, in the bottom row, we need to begin, again, from step 4. This time we will not repeat the details of every step; instead, we will identify the column and row that give us the pivot element, and highlight the pivot element. The result is as follows.*

$$\begin{array}{ccccc|ccc} x_1 & x_2 & y_1 & y_2 & Z & C & & \\ 0 & 1/2 & 1 & -1/2 & 0 & 4 & \leftarrow & 4/(1/2) = 8 \\ 1 & 1/2 & 0 & 1/2 & 0 & 8 & & 8/(1/2) = 16 \\ \hline 0 & -10 & 0 & 20 & 1 & 320 & & \\ & \uparrow & & & & & & \end{array}$$

We make the pivot element 1 by multiplying row 1 by 2, and we get

$$\begin{array}{ccccc|c} x_1 & x_2 & y_1 & y_2 & Z & C \\ 0 & \boxed{1} & 2 & -1 & 0 & 8 \\ 1 & 1/2 & 0 & 1/2 & 0 & 8 \\ \hline 0 & -10 & 0 & 20 & 1 & 320 \end{array}$$

Now to make all other entries as zeros in this column, we first multiply row 1 by $-\frac{1}{2}$ *and add it to row 2, and then multiply row 1 by 10 and add it to the bottom row.*

$$
\begin{array}{ccccc|c}
x_1 & x_2 & y_1 & y_2 & Z & C \\
0 & 1 & 2 & -1 & 0 & 8 \\
1 & 0 & -1 & 1 & 0 & 4 \\
\hline
0 & 0 & 20 & 10 & 1 & 400
\end{array}
$$

We no longer have negative entries in the bottom row, therefore we are finished.

Why are we finished when there are no negative entries in the bottom row?

The answer lies in the bottom row. The bottom row corresponds to the equation:

$$0x_1 + 0x_2 + 20y_1 + 10y_2 + Z = 400 \quad \text{or} \quad Z = 400 - 20y_1 - 10y_2$$

Since all variables are non-negative, the highest value Z *can ever achieve is 400, and that will happen only when* y_1 *and* y_2 *are zero.*

8. ***Read off your answers.*** *We now read off our answers, that is, we determine the basic solution associated with the final simplex tableau. Again, we look at the columns that have a* 1 *and all other entries zeros. Since the columns labeled* y_1 *and* y_2 *are not such columns, we arbitrarily choose* $y_1 = 0$, *and* $y_2 = 0$, *and we get*

$$
\begin{array}{c}
\begin{array}{cccc} x_1 & x_2 & Z & C \end{array} \\
\left[\begin{array}{ccc|c}
0 & 1 & 0 & 8 \\
1 & 0 & 0 & 4 \\
0 & 0 & 1 & 400
\end{array}\right]
\end{array}
$$

The matrix reads $x_2 = 8$, $x_1 = 4$, *and* $Z = 400$.

The final solution says that if Niki works 4 hours at Job I and 8 hours at Job II, she will maximize her income to \$400. Since both slack variables are zero, it means that she would have used up all the working time, as well as the preparation time, and none will be left.

4.3 Minimization by the Simplex Method

In this section, you will learn to solve linear programming minimization problems using the simplex method.

1. Identify and set up a linear program in standard minimization form.
2. Formulate a dual problem in standard maximization form.
3. Use the simplex method to solve the dual maximization problem.
4. Identify the optimal solution to the original minimization problem from the optimal simplex tableau.

In this section, we will solve the standard linear programming minimization problems using the simplex method. Once again, we remind the reader that in the standard minimization problems all constraints are of the form $ax + by \geq c$.

The procedure to solve these problems was developed by Dr. John Von Neuman. It involves solving an associated problem called the dual problem. To every minimization problem there corresponds a dual problem. The solution of the dual problem is used to find the solution of the original problem. The dual problem is a maximization problem, which we learned to solve in the last section. We first solve the dual problem by the simplex method.

From the final simplex tableau, we then extract the solution to the original minimization problem. Before we go any further, however, we first learn to convert a minimization problem into its corresponding maximization problem called its dual.

Example 4.3.1. *Convert the following minimization problem into its dual.*

Minimize

$$Z = 12x_1 + 16x_2$$

Subject to:

$$\begin{aligned} x_1 + 2x_2 &\geq 40 \\ x_1 + x_2 &\geq 30 \\ x_1 \geq 0; \quad x_2 &\geq 0 \end{aligned}$$

Solution 4.3.1. *To achieve our goal, we first express our problem as the following matrix.*

$$\begin{array}{cc|c} 1 & 2 & 40 \\ 1 & 1 & 30 \\ \hline 12 & 16 & 0 \end{array}$$

Observe that this table looks like an initial simplex tableau without the slack variables. Next, we write a matrix whose columns are the rows of this matrix, and the rows are the columns. Such a matrix is called a transpose of the original matrix. We get:

$$\begin{array}{cc|c} 1 & 1 & 12 \\ 2 & 1 & 16 \\ \hline 40 & 30 & 0 \end{array}$$

The following maximization problem associated with the above matrix is called its dual.

Maximize

$$Z = 40y_1 + 30y_2$$

Subject to:

$$\begin{aligned} y_1 + y_2 &\leq 12 \\ 2y_1 + y_2 &\leq 16 \\ y_1 \geq 0; \quad y_2 &\geq 0 \end{aligned}$$

Note that we have chosen the variables as y's, instead of x's, to distinguish the two problems.

Example 4.3.2. *Solve both the minimization problem and its dual maximization problem graphically.*

Solution 4.3.2. *Our minimization problem is as follows.*

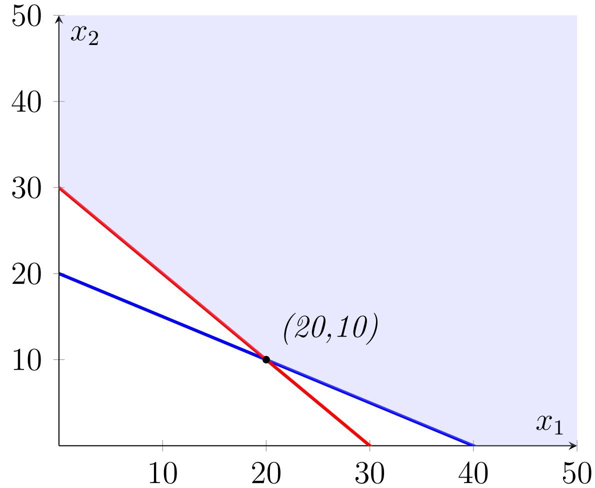
Minimize

$$Z = 12x_1 + 16x_2$$

Subject to:

$$\begin{aligned} x_1 + 2x_2 &\geq 40 \\ x_1 + x_2 &\geq 30 \\ x_1 \geq 0; \quad x_2 &\geq 0 \end{aligned}$$

We now graph the inequalities:



We have plotted the graph, shaded the feasibility region, and labeled the corner points. The corner point (20, 10) gives the lowest value for the objective function and that value is 400.

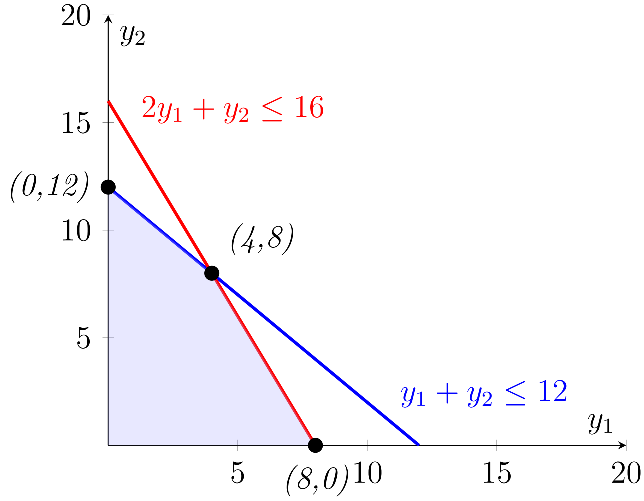
Now its dual is: Maximize

$$Z = 40y_1 + 30y_2$$

Subject to:

$$\begin{aligned} y_1 + y_2 &\leq 12 \\ 2y_1 + y_2 &\leq 16 \\ y_1 \geq 0; \quad y_2 &\geq 0 \end{aligned}$$

We graph the inequalities:



Again, we have plotted the graph, shaded the feasibility region, and labeled the corner points. The corner point (4, 8) gives the highest value for the objective function, with a value of 400.

The reader may recognize that Example 4.3.2 above is the same as Example 3.1.1 in section 3.1. It is also the same problem as Example 4.2.1 in section 4.2, where we solved it by the simplex method.

We observe that the minimum value of the minimization problem is the same as the maximum value of the maximization problem; in Example 4.3.2 the minimum and maximum are both 400. This is not a coincidence. We state the duality principle.

Definition 4.3.1. ***The Duality Principle*** *The objective function of the minimization problem reaches its minimum if and only if the objective function of its dual reaches its maximum. And when they do, they are equal.*

Our next goal is to extract the solution for our minimization problem in Example 4.3.1 from the corresponding dual. To do this, we solve the dual by the simplex method.

Example 4.3.3. *Find the solution to the minimization problem in Example 4.3.1 by solving its dual using the simplex method. We rewrite our problem: Minimize*

$$Z = 12x_1 + 16x_2$$

Subject to:

$$x_1 + 2x_2 \geq 40$$
$$x_1 + x_2 \geq 30$$
$$x_1 \geq 0; \quad x_2 \geq 0$$

Solution 4.3.3. *The dual is: Maximize*

$$Z = 40y_1 + 30y_2$$

Subject to:

$$y_1 + y_2 \leq 12$$
$$2y_1 + y_2 \leq 16$$
$$y_1 \geq 0; \quad y_2 \geq 0$$

Recall that we solved the above problem by the simplex method in Example 4.2.1, section 4.2. Therefore, we only show the initial and final simplex tableau.

The initial simplex tableau is:

y_1	y_2	x_1	x_2	Z	C	
1	1	1	0	0	12	y_1
2	1	0	1	0	16	y_2
-40	-30	0	0	1	0	Z

Observe an important change. Here our main variables are y_1 and y_2 and the slack variables are x_1 and x_2.

The final simplex tableau reads as follows:

y_1	y_2	x_1	x_2	Z	C
0	1	2	-1	0	8
1	0	-1	1	0	4
0	0	20	10	1	400

A closer look at this table reveals that the x_1 and x_2 values along with the minimum value for the minimization problem can be obtained from the last row of the final tableau. We have highlighted these values by the arrows.

$$\begin{array}{ccccc|c}
y_1 & y_2 & x_1 & x_2 & Z & C \\
0 & 1 & 2 & -1 & 0 & 8 \\
1 & 0 & -1 & 1 & 0 & 4 \\
\hline
0 & 0 & 20 & 10 & 1 & 400 \\
 & & \uparrow & \uparrow & & \uparrow
\end{array}$$

We restate the solution as follows: The minimization problem has a minimum value of 400 at the corner point $(20, 10)$.

Summary 4.3.1.
MINIMIZATION BY THE SIMPLEX METHOD

1. *Set up the problem.*
2. *Write a matrix whose rows represent each constraint with the objective function as its bottom row.*
3. *Write the transpose of this matrix by interchanging the rows and columns.*
4. *Now write the dual problem associated with the transpose.*
5. *Solve the dual problem by the simplex method learned in section 4.2.*
6. *The optimal solution is found in the bottom row of the final matrix in the columns corresponding to the slack variables, and the minimum value of the objective function is the same as the maximum value of the dual.*

Chapter 5

More Probability