# Math for Computer Sciences

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This book was written for Spokane Community College's Math 104 - Computer Math. Many of the exercises in this text are also available through the on-line homework system *MyOpenMath*. For access to these on-line exercises contact Zachery.Solheim@SCC.Spokane.edu.

# Contents

# Chapter 0

# Introduction: Reasoning and Problem Solving

*"Mathematics is not about numbers, equations, computations, or algorithms: it is about understanding."* - William P. Thurston*

*"You should carefully study the Art of Reasoning, as it is what most people are very deficient in, and I know few things more disagreeable than to argue, or even converse with a man who has no idea of inductive and deductive philosophy."* - William John Wills†

On form of reasoning, deductive reasoning, begins with accepted assumptions to arrive at specific conclusions. On the other hand, inductive reasoning uses observations to come to larger, more general statements. For example, observing the sun rise and set day after day one might come to the conclusion that the sun will rise and set *every* day - this type of reasoning is induction. To turn this example into one of deduction we can simply reverse the order of events. *Assuming* the sun will rise and set each day, one comes to the conclusion that the sun will rise and set tomorrow.

As you read the following chapters and complete the exercises you'll be practicing both inductive

*Mathematicians: An Outer View of the Inner World by Mariana Cook

†20th Natural Philosophy Alliance Proceedings, Volume 10, By David de Hilster

and deductive reasoning. For example, deductive reasoning will be used while simplifying algebraic expressions and inductive reasoning will help explain the tools within algebra and arithmetic.

While induction and deduction are necessary tools they also have their downfalls. If a new gambler follows inductive reasoning, for example, and wins his first four hands of blackjack then he may assume he'll win the fifth. This is possible, of course, but in a fair game it's very unlikely. The point here is that without enough data, induction can lead to incorrect conclusions. Meanwhile, deduction can cause problems if the premise is false; if you walk into a casino under the assumption you'll win, you'll probably be very disappointed.

As you work through this text you'll be applying your abilities to reason and problem solve. In fact, one of your goals in this coarse is to sharpen these skills. That being said, Polya's[‡] principles of problem solving make up an excellent guide to follow.

- First Principle: Understand the problem

  Understanding a problem in any context includes identifying the goal and any useful information but may also require rephrasing the problem, or performing a little extra research when a word or phrase within the problem isn't itself well understood. To help understand a problem you might draw a picture (especially for geometry problems) or organize information in a table.

  Keep in mind that, when you come to a new type of problem, 70-80% of your time will probably be spent at this stage.

- Second Principle: Come up with a plan

  Once you understand the problem you can identify useful tools and take out of consideration anything that won't be helpful. At this stage you might be looking for patterns, identifying useful formulas, or even coming up with something new - as necessary as logical reasoning is, your intuition can be incredibly useful here.

- Third Principle: Follow your plan

  While this is comparatively simpler than the previous two stages, make sure you don't rush through this stage. Follow your plan carefully and be persistent.

[‡]How to Solve It by G. Polya

- Fourth Principle: Review

  Always make sure to go back and review your understanding of the problem, your plan and its implementation. In the event that your plan failed, work to understand why and use this understanding to devise a new plan. Even when your plan succeeds, reviewing how and why it was successful will give you a better understanding of how to approach similar problems. This is especially true of plans coming out of intuition; you can't rely on a gut feeling giving good results every time, so going back to understand why it gave you success will help you in the future.

# Chapter 1

# Arithmetic and Algebra

## 1.1 Order of Operations

The order of operations details in which order operations should be performed. A common, helpful acronym for the order of operations is P.E.M.D.A.S., which stands for: Parenthesis, Exponents, Multiplication, Division, Addition, Subtraction.

While this acronym is helpful, it can be misleading, so we'll make some minor changes. First, along with parenthesis in the top tier of the order of operations are all types of grouping symbols. Grouping symbols include parenthesis, square brackets, and set brackets/curly brackets; these types of grouping symbols do not have any extra operation associated to them. Some other common grouping symbols, which do have an associated operation, including absolute value brackets, the radical symbol, and the division bar or fraction bar.

Parenthesis

Exponents

Multiplication

Division

Addition

Subtraction

**Example 1:** Evaluate each of the following expressions.

For each of these we need to keep in mind the order of operations. Having grouping symbols in the top tier of the order of operations means that simplification inside grouping symbols should occur first.

(a) $(3+2)^2$

Inside the grouping symbols we have "3+2". Since 3+2=5, our first step is:

$$(3+2)^2 = (5)^2$$

Since no more simplification can be performed inside the parenthesis, we can move on to the next tier in the order of operations: exponents.

$$(5)^2 = (5) * (5) = 25$$

With no more operations to be performed, our answer is 25.

(b) $-(3-5) * (4+7)$

In this expression we have two sets of grouping symbols; inside the first set $3-5$ simplifies to -2, and in the second set $4+7$ yields 11.

$$-(3-5) * (4+7) = -(-2) * (11)$$

With no exponents in the expression, we can move on to the next tier: multiplication and division.

$$-(-2) * (11) = -(-22)$$

Finally, with the multiplication completed, we can move on to addition and subtraction. It isn't obvious, but at the front of our expression we do have something qualifying as subtraction: the "$-$" at the front of the expression tells us to take the opposite of the value in parenthesis and is a form of subtraction. Since the opposite of $-22$ is 22, our result is $-(-22) = 22$:

$$-(3-5)*(4+7) = 22$$

(c) $(-75+60)^2$

Much like the first expression we evaluated, we'll simplify inside the parenthesis first: $-75+60 = -15$. With simplification inside the parenthesis completed, we can move on to the exponent:

$$(-75+60)^2 = (-15)^2 = (-15)*(-15) = 225$$

Remember here that the product of two negative numbers is positive.

(d) $2.5*\sqrt{25-16}$

Our final expression in this example involves a square root; remember that this is a type of grouping symbols with an associated operation. Must like the expressions with parenthesis we just saw, simplification inside the grouping symbol should be our first step:

$$2.5*\sqrt{25-16} = 2.5*\sqrt{9}$$

Our next step, and a part of grouping symbols being in the top tier of the order of operations, is to perform any operations associated to the grouping symbols. For this step, note that $\sqrt{9} = 3$ because $3^2$=9. This gives us

$$2.5*\sqrt{9} = 2.5*3$$

The evaluation can be completed by performing the indicated multiplication, $2.5*3 = 7.5$, so that

$$2.5*\sqrt{25-16} = 7.5$$

The second note to make on P.E.M.D.A.S. is that multiplication and division fall under the same tier in the order of operations, just as addition and subtraction are in the same tier in the order of operations. This means that when an expression involves both multiplication and division (or addition and subtraction), the operations should be performed as they appear, read from left to right; multiplication **does not** have precedence over division and addition **does not** have precedence over subtraction.

**Example 2:** Evaluate each of the following expressions.

Moving through these, remember to simplify inside grouping symbols first (if grouping symbols appear) then to perform multiplication and division as the operation appears as you read from left to right, followed by addition and subtraction.

(a) Evaluating $6 \div 3 * 5$

Since the only operations here are multiplication and division, the operations should be performed in the order they appear. The first operation that's read is division: $6 \div 3 = 2$.

The next operation to be performed is multiplication:

$$6 \div 3 * 5 = 2 * 5 = 10$$

(b) Evaluating $6 \div (3 * 2)$

Though this looks much like the last expression we evaluated, the inclusion of parenthesis will make a big difference. Because simplification inside grouping symbols should occur first, our first step will be the multiplication: $2 * 3 = 6$. This gives us

$$6 \div (3 * 2) = 6 \div (6)$$

Now, performing the division, we get $6 \div 6 = 1$, so

$$6 \div (3 * 2) = 1$$

(c) Evaluating $30 \div 5 * 4 \div 2$

Again we have only multiplication and division to perform, so each operation should be completed as they appear from left to right. Our first step is to divide: $30 \div 5 = 6$.

The second step to complete is the multiplication between 5 and 4. Combining this with our first step we get

$$30 \div 5 * 4 \div 2 = 6 * 4 \div 2 = 24 \div 2$$

The evaluation can be completed by performing the division:

$$24 \div 2 = 12$$

(d) Evaluating $6 - 1 + 2$

With only subtraction and addition in this expression, we'll first subtract, and then add:

$$\begin{aligned} 6 - 1 + 2 &= 5 + 2 \\ &= 7 \end{aligned}$$

(e) Evaluating $6 - (1 + 2)$

This time the inclusion of parenthesis gives priority to the addition, so we will first add and then subtract:

$$\begin{aligned} 6 - (1 + 2) &= 6 - (3) \\ &= 3 \end{aligned}$$

(f) Evaluating $9 - 5 + 14 - 2$

Out final expression has three operations; read from left to right, they are subtraction, addition, and subtraction, and this is the order in which they should be performed.

$$\begin{aligned} 9 - 5 + 14 - 2 &= 4 + 14 - 2 \\ &= 18 - 2 \\ &= 16 \end{aligned}$$

There are some properties and tools that allow us to work around a strict interpretation of the order of operations. These include the following:

- The distributive property of multiplication over addition: $a * (b + c) = a * b + a * c$
- The commutative property of multiplication: $a * b = b * a$
- The commutative property of addition: $a + b = b + a$

- The associative property of multiplication: $(a * b) * c = a * (b * c)$
- The associative property of addition: $(a + b) + c = a + (b + c)$
- The ability to write division as multiplication: $a \div b = a * \frac{1}{b}$
- The ability to write subtraction as addition: $a - b = a + (-b)$

**Example 3:** Suppose you want to find the balance of your checking account after paying several bills. Specifically, assume your checking account holds \$1,450 and you pay \$49 for internet, \$575 for rent, and \$87 for utilities. How much remains in your checking account?

We can approach this in one of two ways: deducting each bill one at a time, or adding up the expenditures and subtracting this total. These different approaches utilize different tools but give the same result.

<u>Subtracting each bill one at a time:</u>

$$\begin{aligned} & 1450 - 49 - 575 - 87 \\ = \;& 1401 - 575 - 87 \\ = \;& 826 - 87 \\ = \;& 739 \end{aligned}$$

<u>Subtracting the total expenditures:</u>

$$\begin{aligned} & 1450 - 49 - 575 - 87 \\ = \;& 1450 - (49 + 575 + 87) \\ = \;& 1450 - (711) \\ = \;& 739 \end{aligned}$$

Subtracting the total expenditures utilizes the distributive property and the ability to rewrite subtraction as addition. With each method, we see the remaining balance is \$739.

Other useful properties include the existence of identities and inverses.

- Adding a number and the *additive identity* returns the original number.
- Adding a number to its *additive inverse* returns the additive identity; additive inverses are also called opposites.
- Multiplying a number by the *multiplicative identity* returns the original number.

- Multiplying a number by its *multiplicative inverse* returns the multiplicative identity; multiplicative inverses are also called reciprocals.

**Example 4:** Identify each of the following:

(a) The additive identity.

The additive identity is zero because, for any number $n$, $n + 0 = n$. For example, $12 + 0 = 12$.

(b) The additive inverse of 5.

The additive inverse of 5 is $-5$ because $5 + (-5) = 0$. In general, the additive inverse of a number $n$ is its opposite, $-n$.

(c) The multiplicative identity.

The multiplicative identity is 1 because, for any number $n$, $n * 1 = n$.

(d) The multiplicative inverse of 5.

The multiplicative inverse of 5 is $\frac{1}{5}$ because $5 * \frac{1}{5} = 1$. In general, the multiplicative inverse of a (non-zero) number $n$ is $\frac{1}{n}$. (We can also specify for a fraction $\frac{m}{n}$ the multiplicative inverse is $\frac{n}{m}$, which will be further explained as we investigate the multiplication of fractions.)

## 1.2 Rational Numbers: A Review of Fractions and Decimals

The examples seen up to this point have mostly involved *integers*; numbers like $-5$, 0, 2, or 512. A ratio of two integers is called a *rational number*. For example, $-\frac{1}{2}$, $\frac{7}{3}$, and 5 are all rational numbers. A rational number like $\frac{7}{3}$ is called an *improper fraction* because the numerator (7) is greater than the denominator (3). $\frac{7}{3}$ could also be written as a *mixed number*: $2\frac{1}{3}$.

To convert from mixed number notation into improper fraction notation we use multiplication and addition of whole numbers; converting in the opposite direction, we use long division with remainder notation. For example, to write $3\frac{4}{5}$ as an improper fraction, multiply 3 by 5 and add 4 to the product: $3 * 5 + 4 = 15 + 4 = 19$. Then the improper fraction form of $3\frac{4}{5}$ is $\frac{19}{5}$.

On the other hand, to write $\frac{19}{5}$ as a mixed number first divide 19 by 5: $19 \div 5 = 3R4$. The quotient is the whole number while the remainder is the numerator, so $\frac{19}{5}$ is equivalent to $3\frac{4}{5}$.

Conversions from one format to the other will be seen periodically in the examples that follow. The focus of this section, however, is to review the multiplication, division, addition, and subtraction of rational numbers.

**Example 5:** Simplify each expression.

Multiplying fractions is as easy as multiplying integers; we multiply numerator by numerator and denominator by denominator. Before, or after, the multiplication is complete it may be possible to reduce; cancel factors common to both numerator and denominator.

(a) $\dfrac{3}{4} * \dfrac{7}{8}$:
For this expression we can simply multiply the two numerators by one another and multiply the denominators by one another.

$$\frac{3}{4} * \frac{7}{8} = \frac{3 * 7}{4 * 8} = \frac{21}{32}$$

(b) $\dfrac{5}{6} * \dfrac{9}{13}$:
We can take the exact same steps as shown in part 1:

$$\frac{5}{6} * \frac{9}{13} = \frac{5 * 9}{6 * 13} = \frac{45}{78}$$

However, this time more can be done; specifically, the fraction can be reduced because 45 and 78 are both divisible by 3:

$$\frac{45}{78} = \frac{15 * 3}{26 * 3} = \frac{15 * \cancel{3}}{26 * \cancel{3}} = \frac{15}{26}$$

(c) $\dfrac{8}{27} * \dfrac{15}{28}$:
Again, we could take the same initial steps as seen in the previous examples. On the other hand, we could reduce *before* multiplying. Notice that 8, in a numerator, and 28, in a

denominator, are both divisible by 4:

$$\frac{8}{27} * \frac{15}{28} = \frac{2*4}{27} * \frac{15}{7*4} = \frac{2*\cancel{4}}{27} * \frac{15}{7*\cancel{4}} = \frac{2}{27} * \frac{15}{7}$$

Furthermore, both 27 and 15 are divisible by 3, so we can reduce some more:

$$\frac{2}{27} * \frac{15}{7} = \frac{2}{9*3} * \frac{5*3}{7} = \frac{2}{9*\cancel{3}} * \frac{5*\cancel{3}}{7} = \frac{2}{9} * \frac{5}{7}$$

Finally, let's multiply the fractions:

$$\frac{2}{9} * \frac{5}{7} = \frac{2*5}{9*7} = \frac{10}{63}$$

(d) $4\frac{1}{5} * \frac{2}{9}$:
Our first step here is to convert the mixed number into an improper fraction:

$$4\frac{1}{5} = \frac{4*5+1}{5} = \frac{21}{5}$$

From here we can reduce between numerators and denominators, then multiply.

$$\begin{aligned}
\frac{21}{5} * \frac{2}{9} &= \frac{7*3}{5} * \frac{2}{3*3} && \text{(Factor common to numerator and denominator)} \\
&= \frac{7*\cancel{3}}{5} * \frac{2}{3*\cancel{3}} && \text{(Cancel common factor of 9)} \\
&= \frac{7}{5} * \frac{2}{3} \\
&= \frac{7*2}{5*3} && \text{(Multiply numerators, multiply denominators)} \\
&= \frac{14}{15}
\end{aligned}$$

**Example 6:** To divide a number by a fraction we rewrite the division as multiplication, which turns the divisor into its reciprocal. From here, the multiplication is completed as discussed in the previous example.

(a) Evaluate $\frac{2}{5} \div \frac{10}{3}$:

Our first step is to rewrite the division as multiplication:

$$\frac{2}{5} \div \frac{10}{3} = \frac{2}{5} * \frac{3}{10}$$

From here we can continue as in our previous examples:

$$\begin{aligned}
\frac{2}{5} * \frac{3}{10} &= \frac{1 * 2}{5} * \frac{3}{5 * 2} \\
&= \frac{1 * \cancel{2}}{5} * \frac{3}{5 * \cancel{2}} \\
&= \frac{1}{5} * \frac{3}{5} \\
&= \frac{1 * 3}{5 * 5} \\
&= \frac{3}{25}
\end{aligned}$$

(b) Evaluate $16 \div \frac{8}{9}$:

Again we should first write the division as multiplication:

$$16 \div \frac{8}{9} = 16 * \frac{9}{8}$$

Since we have a whole number factor, we'll write the whole number as a fraction then continue through the same steps as before:

$$\begin{aligned}
16 * \frac{9}{8} &= \frac{16}{1} * \frac{9}{8} \\
&= \frac{2*8}{1} * \frac{9}{1*8} \\
&= \frac{2*\cancel{8}}{1} * \frac{9}{1*\cancel{8}} \\
&= \frac{2}{1} * \frac{9}{1} \\
&= \frac{2*9}{1*1} \\
&= \frac{18}{1} \\
&= 18
\end{aligned}$$

Unlike multiplication and division, addition and subtraction of fractions require the denominators be the same; finding a common denominator for fractions is the same as finding a common multiple of the denominators. Once a common denominator is in place, the addition/subtraction is performed between the numerators.

**Example 7:** Evaluate each expression.

(a) $\frac{2}{9} + \frac{5}{6}$

To find the least common denominator, we search for the smallest number divisible by both 9 and 6, the denominators of our fractions. One approach is to list multiples of 6 and 9 until a common multiple is found. Other approaches use the prime factorization of the numbers. Below we see multiple ways of finding the least common denominator, or LCD.

Listing

| 6 | 9 |
|---|---|
| 6 | 9 |
| 12 | 18 |
| 18 | 27 |
| ⋮ | ⋮ |

"Cake Method"

| | 6 | 9 |
|---|---|---|
| 3 | 2 | 3 |
| 2 | 1 | 3 |
| 3 | 1 | 1 |

$3 \times 2 \times 3 = 18$

| Number | Prime Factors | |
|---|---|---|
| | 2 | 3 |
| 6 | 1 | 1 |
| 9 | 0 | 2 |
| Largest: | 1 | 2 |
| $2^1 \times 3^2 = 18$ | | |

Whichever method is used, our next step is to rewrite the fractions so that the denominator is the LCD, 18.

Since $9 * 2 = 18$ we'll multiply numerator and denominator of $\frac{2}{9}$ by 2:

$$\frac{2}{9} * \frac{2}{2} = \frac{4}{18}$$

Since $6 * 3 = 18$ we'll multiply numerator and denominator of $\frac{5}{6}$ by 3:

$$\frac{5}{6} * \frac{3}{3} = \frac{15}{18}$$

With the fractions rewritten, we can complete the addition:

$$\begin{aligned} \frac{2}{9} + \frac{5}{6} &= \frac{4}{18} + \frac{15}{18} \\ &= \frac{4+15}{18} \\ &= \frac{19}{18} \end{aligned}$$

Notice that the addition only affected the numerators; this will be true whenever we add or subtract fractions.

At this stage we should reduce if possible. Since reduction is not possible with $\frac{19}{18}$, we can leave the answer as is or rewrite our answer as a mixed number. Therefore, our answer is $\frac{19}{18}$ or $1\frac{1}{18}$.

As we progress through the remaining examples, this process will be streamlined with some comments or notes on select steps. The steps we've taken are as follows:

1. Identify the LCD
2. Rewrite the fractions using the LCD
3. Add numerators
4. Reduce if possible

(b) $\dfrac{5}{12} - \dfrac{11}{42}$

Below we have the three methods discussed for identifying the LCD.

Listing

| 12 | 42 |
|---|---|
| 12 | 42 |
| 24 | 84 |
| 36 | 126 |
| 48 | 168 |
| 60 | 210 |
| 72 | 252 |
| 84 | 294 |

"Cake Method"

| | 12 | 42 |
|---|---|---|
| 2 | 6 | 21 |
| 3 | 2 | 7 |
| 2 | 1 | 7 |
| 7 | 1 | 1 |

$2 \times 3 \times 2 \times 7 = 84$

| Number | Prime Factors | | |
|---|---|---|---|
| | 2 | 3 | 7 |
| 12 | 2 | 1 | 0 |
| 42 | 1 | 1 | 1 |
| Largest: | 2 | 1 | 1 |
| $2^2 \times 3^1 \times 7^1 = 84$ | | | |

Once the LCD is found, the fractions can be rewritten and the subtraction preformed:

$$
\begin{aligned}
\frac{7}{12} - \frac{11}{42} &= \frac{7}{12} * \frac{7}{7} - \frac{11}{42} * \frac{2}{2} \\
&= \frac{49}{84} - \frac{22}{84} && \text{(Rewrite fractions using LCD)} \\
&= \frac{49 - 22}{84} \\
&= \frac{27}{84} && \text{(Subtract numerators)} \\
&= \frac{9}{28} && \text{(Reduced: common factor of 3)}
\end{aligned}
$$

(c) $1\dfrac{7}{8} + \dfrac{5}{6}$

Notice a slight difference between this expression and the last two: there is a mixed number involved. Perhaps the simplest way of handling this is to rewrite the mixed number as an improper fraction.

$$1\frac{7}{8} = \frac{1 * 8 + 7}{8} = \frac{15}{8}$$

With the mixed number addressed we can continue as before, starting with identifying the LCD:

Listing

| 6 | 8 |
|---|---|
| 6 | 8 |
| 12 | 16 |
| 18 | 24 |
| 24 | 32 |
| ⋮ | ⋮ |

"Cake Method"

| | 6 | 8 |
|---|---|---|
| 2 | 3 | 4 |
| 3 | 1 | 4 |
| 2 | 1 | 2 |
| 2 | 1 | 1 |

$2 \times 3 \times 2 \times 2 = 24$

| Number | Prime Factors | |
|---|---|---|
| | 2 | 3 |
| 6 | 1 | 1 |
| 8 | 3 | 0 |
| Largest: | 3 | 1 |
| $2^3 \times 3^1 = 24$ | | |

$$
\begin{aligned}
1\frac{7}{8} + \frac{5}{6} &= \frac{15}{8} + \frac{5}{6} && \text{(Writing mixed number as improper fraction)} \\
&= \frac{15}{8} * \frac{3}{3} + \frac{5}{6} * \frac{4}{4} \\
&= \frac{45}{24} - \frac{20}{24} && \text{(Rewrite using LCD)} \\
&= \frac{45 + 20}{24} \\
&= \frac{65}{24} && \text{(Add numerators)} \\
&= 2\frac{17}{24} && \text{(Optional: Write as mixed number)}
\end{aligned}
$$

While all rational number can be written as fractions, they can also be written as decimal numbers; a decimal number is rational when its fraction part (the part following the decimal point) either terminates or repeats a sequence of digits infinitely.

While the decimal point is usually omitted when writing integers, it would fall immediately after the ones-digit. When we line up two integers to add or subtract, the decimal point in each number lines up by default because we're lining up the digits (ones-digit above ones-digit, tens above tens, etc.). Adding rational, decimal numbers requires the exact same organization.

**Example 8:** Evaluate each expression.

(a) 2.67+3.05

$$\begin{array}{rcccc} & & \overset{1}{} & & \\ & 2 & . & 6 & 7 \\ + & 3 & . & 0 & 5 \\ \hline & 5 & . & 7 & 2 \end{array}$$

(b) 6.25-0.905

$$\begin{array}{rccccc} & 5 & & 12 & 4 & 10 \\ & \cancel{6} & . & \cancel{2} & \cancel{5} & \cancel{0} \\ - & 0 & . & 9 & 0 & 5 \\ \hline & 5 & . & 3 & 4 & 5 \end{array}$$

(c) 0.287+0.91

$$\begin{array}{rccccc} & 1 & & & & \\ & 0 & . & 2 & 8 & 7 \\ + & 0 & . & 9 & 1 & 0 \\ \hline & 1 & . & 1 & 9 & 7 \end{array}$$

**Example 9:** Multiplication of decimal numbers is, again, performed just as the multiplication of integers.

(a) Evaluate $11.5 * 0.24$

$$\left.\begin{array}{r} 11.5 \\ \times 0.24 \end{array}\right\}\text{Stack the numbers, flush right}$$

$$\begin{array}{rl} \hline 460 & \text{Multiply 4 by 115} \\ +230\text{x} & \text{Multiply 2 by 115, indent from the right one digit} \\ \hline 2.760 & \text{Decimal point belongs three digits in from the right} \end{array}$$

To explain the placement of the decimal point, consider these numbers written as fractions:

$$11.5 = 11\frac{5}{10} \quad \textit{or} \quad \frac{115}{10} \qquad\qquad 0.24 = \frac{24}{100}$$

Because 11.5 involves one digit to the right of the decimal point, the tenths place, the denominator is 10. Because 0.24 uses two digits to the right of the decimal point, hundredths, the denominator is 100. When we multiply these fractions we'll have a new denominator of 1000:

$$\frac{115}{10} * \frac{24}{100} = \frac{2760}{1000} = 2.760 = 2.76$$

Rather than using fractions to determine where the decimal point will belong each time we multiply decimal numbers, count the number of digits to the right of the decimal point in each number - the sum of these values is the number of digits that should be to the right of the decimal point in the product, including zeros.

(b) Evaluate $0.56 * 0.3$

| | |
|---|---|
| 0.56 | 2 digits to the right of the decimal point |
| $\times$ 0.3 | 1 digit to the right of the decimal point |
| 168 | Multiply 56 by 3 |
| +000x | Multiply 56 by 0, indent from the right one digit |
| 0.168 | Decimal point belongs three digits in from the right |

(c) Evaluate $1.67 * 0.105$

| | |
|---|---|
| 1.67 | 2 digits to the right of the decimal point |
| $\times$0.105 | 3 digit to the right of the decimal point |
| 835 | Multiply 167 by 5 |
| 000x | Multiply 167 by 0, indent from the right one digit |
| +167xx | Multiply 167 by 1, indent from the right two digits |
| .17535 | Decimal point belongs 5 digits in from the right |

The set-up for dividing decimal numbers is identical to the division of integers. To explain the placement of the decimal point we'll write the division in fraction notation and use the multiplicative identity in a creative way.

**Example 10:** Evaluate each expression.

(a) $2.64 \div 1.1$

Rather than dividing by 1.1 explicitly, it will be easier to divide by a whole number. To do this, view the division as a fraction, multiplying numerator and denominator by a power of 10 to move the decimal point.

$$\frac{2.64}{1.1} * \frac{10}{10} = \frac{26.4}{11}$$

This means that dividing 2.64 by 1.1 is equivalent to dividing 26.4 by 11. Notice that the same effect can be obtained by simply moving the decimal point in the divisor right until a whole number is found, and moving the decimal point in the dividend the same number of digits.

$$\begin{array}{r}
2.4 \\
11\overline{)26.4} \\
\underline{-22\phantom{.4}} \\
44 \\
\underline{-44} \\
0
\end{array}$$

Therefore, $2.64 \div 1.1 = 2.4$. Notice that the decimal point in the quotient (2.4) is placed directly above the decimal point in the dividend (26.4).

(b) $6.81 \div 0.15$

Move the decimal point right 2 digits to make 0.15 a whole number, 15; this time our answer is 45.4.

$$\begin{array}{r}
45.4 \\
15\overline{)681.\phantom{0}} \\
\underline{-60\phantom{1.0}} \\
81\phantom{.0} \\
\underline{-75\phantom{.0}} \\
60 \\
\underline{-60} \\
0
\end{array}$$

(c) $2.5 \div 6$

No need to move the decimal point; 6 is already a whole number.

$$\begin{array}{r} 0.4166 \\ 6\overline{)2.5\phantom{000}} \\ \underline{-24}\phantom{000} \\ 10\phantom{00} \\ \underline{-6}\phantom{00} \\ 40\phantom{0} \\ \underline{-36}\phantom{0} \\ 40 \\ \underline{-36} \\ 4 \end{array}$$

Notice that the remainder has started repeating; in the last three steps we found a remainder of 4. This means that the 6 in the quotient will also continue repeating, infinitely. To denote this infinite repetition we can write the quotient using ellipsis or over-bar notation: 0.41666..., or $0.41\bar{6}$. The repeating sixes followed by ellipsis is meant to convey that the repetition continues, and the bar above the 6 denotes the same thing.

Percentages are one application of decimal numbers. The word "percent" means "per hundred"; for example, 42.5% means 42.5 hundredths, which can be written $\frac{42.5}{100}$ (which simplifies to $\frac{17}{40}$)or, more commonly, 0.425.

**Example 11:** Convert each percentage into decimal form and each decimal number into a percentage.

(a) 20%

20% means 20 per hundred, or 20 hundredths. This can be translated directly into decimal form as 0.20, or simply 0.2.

(b) 105%

105% means 105 per hundred. We can take the same approach as in the previous problem, or we could divide 105 by 100. This would yield 1.05.

**Note:** Notice in the previous two examples we could have simply moved the decimal point in the percentage number two places to the left. This is because division by 100 (implied by "per hundred") results in moving the decimal point two places to the left.

(c) 0.012

To convert 0.012 into a percentage we can utilize the previous note. If we can move the decimal point two places to the left when converting from percentage to decimal form, then converting from a decimal to a percentage can be accomplished by moving the decimal point two places to the right:

$$0.012 = 1.2\%$$

(d) 120.5

Following the same tool used in part 3, we can move the decimal point in 120.5 two places to the right to get

$$120.5 = 12050\%$$

## 1.3 Exponents and Radicals

An **exponent** counts the number of factors of its base; for example, in the expression $2^5$ the exponent is 5, the base is 2, and the expression means the same as $2*2*2*2*2$. This definition of an exponent can be given symbolically as

$$x^n = \underbrace{x * x * ... * x}_{n \text{ factors of } x}$$

Using this definition several properties of exponents can be derived.

1. Product Rule - Product of exponential expressions with equal bases: $b^n * b^m = b^{n+m}$
2. Quotient Rule - Quotient of exponential expressions with equal bases: $\frac{b^n}{b^m} = b^{n-m}$, $b \neq 0$
3. Power to a Power - An exponential expression raised to a power: $(b^n)^m = b^{n*m}$
4. Power of a Product - A product raised to a power: $(a*b)^n = a^n * b^n$
5. Power of a Quotient - A quotient or fraction raised to a power: $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$, $b \neq 0$

6. Zero Exponent Property: $b^0 = 1$, $b \neq 0$

7. Negative Exponent Property: $b^{-n} = \dfrac{1}{b^n}$ or $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$, $b \neq 0$

8. Fractional Exponent Property: $b^{\frac{1}{n}} = \sqrt[n]{b}$

**Example 12:** Using the properties of exponents, simplify the expressions below. In this example we'll start simplifying expressions with variables - we'll assume the variables represent unknown positive values.

(a) $3^5 * 3^2$

Here we have a product where the factors have the same base: 3. Since the base is the same, we can use the product rule which says to add the exponents:

$$3^5 * 3^2 = 3^{5+2} = 3^7 = 243$$

(b) $\dfrac{3}{2^{-5}}$

Since we have a negative exponent in the expression, we can use the negative exponent property. This turns $2^{-5}$ into $1/2^5$, and the original expression into

$$\frac{3}{1/2^5}$$

But now we have fractions within fractions. To completely simplify the expression, we'll rewrite the fraction as division:

$$\frac{3}{1/2^5} = 3 \div \frac{1}{2^5} = 3 * \frac{2^5}{1} = 3 * 2^5 = 3 * 32 = 96$$

To shorten this work, notice that we can change the sign of an exponent by moving the factor the exponent is attached to into the numerator of the overall expression if the factor originally appears in the denominator (or into the denominator if the factor originally appears in the numerator). We'll use this in our next example.

(c) $\dfrac{3 * 5^{-2}}{2^{-3}}$

$$\begin{aligned}\frac{3 * 5^{-2}}{2^{-3}} &= \frac{3 * 2^3}{5^2} && (5^{-2} \text{ becomes } 5^2 \text{ when moved into the denominator})\\ &= \frac{3 * 8}{25} \\ &= \frac{24}{25}\end{aligned}$$

(d) $\dfrac{3^7}{3^9}$

In this expression we have a quotient where both the divisor and dividend (denominator and numerator) are powers of 3; since the bases are the same we can use the quotient rule and subtract the exponent in the denominator from the exponent in the numerator.

$$\frac{3^7}{3^9} = 3^{7-9} = 3^{-2}$$

We don't want to end with a negative exponent, so we'll use the negative exponent property:

$$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$$

(e) $\left(\dfrac{12x}{5n^7}\right)^{-2}$

There are several properties we can use here: the negative exponent property, the power of a quotient property, the power of a product property, and the power to a power property. We can simplify the expression using these properties one at a time:

$$\begin{aligned}
\left(\frac{12x}{5n^7}\right)^{-2} &= \left(\frac{5n^7}{12x}\right)^2 && \text{(Negative Exponent Property)}\\
&= \frac{(5n^7)^2}{(12x)^2} && \text{(Power of a Quotient Property)}\\
&= \frac{5^2(n^7)^2}{12^2x^2} && \text{(Power of a Product Property)}\\
&= \frac{25n^{7*2}}{144x^2} && \text{(Power to a Power Property)}\\
&= \frac{25n^{14}}{144x^2}
\end{aligned}$$

(f) $\dfrac{a^3b^8c^4}{b^5a^3c^7}$

Again, we'll be able to use multiple properties here. We'll start with the Quotient Rule, combining $a$'s, $b$'s, and $c$'s:

$$\begin{aligned}
\frac{a^3b^8c^4}{b^5a^3c^7} &= a^{3-3}b^{8-5}c^{4-7} && \text{(Quotient Rule)}\\
&= a^0b^3c^{-3}\\
&= b^3c^{-3} && \text{(Zero Exponent Property)}\\
&= \frac{b^3}{c^3} && \text{(Negative Exponent Property)}
\end{aligned}$$

One property we didn't use in the last example is the Rational Exponent Property. Before using this property we need to discuss radicals.

Radicals are defined in terms of powers; for example, the *square root* is defined by the second power (squaring). More specifically, the square root of 64 is 8 because $8^2 = 64$. In general,

$$\sqrt{a} = b \quad \text{if} \quad b^2 = a \quad (\text{and } b \geq 0)$$

This can be extended to any natural power. The cube root of 64 is 4 because $4^3 = 64$, and the fourth root of 81 is 3 because $3^4 = 81$. In general, an $n^{th}$ root is defined as follows:

$$\sqrt[n]{a} = b \quad \text{if} \quad b^n = a \quad (\text{and } b \geq 0 \text{ if } n \text{ is even})$$

The type of root is indicated by the *index*, and the expression of which a root is being taken is called the *radicand*.

$$\sqrt[INDEX]{RADICAND}$$

**Example 13:** Use the definition of radicals to simplify each of the expressions below. Assume variables are nonnegative.

(a) $\sqrt{25}$

$\sqrt{25} = 5$ because $5^2 = 25$

(b) $\sqrt[3]{-125}$

$\sqrt[3]{-125} = -5$ because $(-5)^3 = -125$

(c) $3\sqrt{81}$

$\sqrt{81} = 9$ because $9^2 = 81$,
so $3\sqrt{81} = 3 * 9 = 27$

(d) $4\sqrt[3]{27}$

$\sqrt[3]{27} = 2$ because $3^3 = 27$,
so $4\sqrt[3]{27} = 4 * 3 = 12$

(e) $\sqrt[4]{x^{12}}$

$\sqrt[4]{x^{12}} = x^3$ because $(x^3)^4 = x^{12}$

(f) $\sqrt[5]{x^{20}}$

$\sqrt[5]{x^{20}} = x^4$ because $(x^4)^5 = x^{20}$

**Example 14:** Attempt to simplify $\sqrt{-9}$. What issues arise?

We may be tempted to answer -3, but $(-3)^2 = (-3) * (-3) = 9$. Since no real number, when squared, yields -9, the square root of -9 is undefined in the real numbers[a]. More generally, the expression $\sqrt[n]{x}$ is defined for any number $x$ when $n$ is odd, but when $n$ is even $\sqrt[n]{x}$ is defined only when $x$ is greater than or equal to zero.

[a] An expression such as $\sqrt{-9}$ is defined in the complex numbers; however, we will not be investigating the complex numbers in this text.

**Example 15:** Because radicals can be written using exponents, the properties of exponents can be extended to apply to radicals. Use the properties of exponents to simplify the expressions below and assume all variables are nonnegative.

(a) $\sqrt[4]{x^{12}}$

$$\begin{aligned} \sqrt[4]{x^{12}} &= (x^{12})^{\frac{1}{4}} && \text{(Rational Exponent)} \\ &= x^{\frac{12}{4}} && \text{(Power to a Power)} \\ &= x^3 \end{aligned}$$

(b) $\sqrt[5]{x^{20}}$

$$\begin{aligned} \sqrt[5]{x^{20}} &= (x^{20})^{\frac{1}{5}} && \text{(Rational Exponent)} \\ &= x^{\frac{20}{5}} && \text{(Power to a Power)} \\ &= x^4 \end{aligned}$$

(c) $\sqrt{9x^2}$

$$\begin{aligned} \sqrt{9x^2} &= (9x^2)^{\frac{1}{2}} && \text{(Rational Exponent)} \\ &= 9^{\frac{1}{2}}(x^2)^{\frac{1}{2}} && \text{(Power of a Product)} \\ &= \sqrt{9}(x^2)^{\frac{1}{2}} && \text{(Rational Exponent)} \\ &= 3x^{\frac{2}{2}} && \text{(Power to a Power)} \\ &= 3x \end{aligned}$$

(d) $\sqrt[4]{81x^{12}}$

$$\begin{aligned}
\sqrt[4]{81x^{12}} &= (81x^{12})^{\frac{1}{4}} && \text{(Rational Exponent)}\\
&= 81^{\frac{1}{4}}(x^{12})^{\frac{1}{4}} && \text{(Power of a Product)}\\
&= \sqrt[4]{81}(x^{12})^{\frac{1}{4}} && \text{(Rational Exponent)}\\
&= 3x^{\frac{12}{4}} && \text{(Power to a Power)}\\
&= 3x^3
\end{aligned}$$

**Example 16:** Simplify the expressions below assuming variables are nonnegative.

(a) $18^{\frac{1}{3}} * 12^{\frac{1}{3}}$

$$\begin{aligned}
&18^{\frac{1}{3}} * 12^{\frac{1}{3}}\\
&= (18 * 12)^{\frac{1}{3}} && \text{(Power of a Product)}\\
&= (216)^{\frac{1}{3}}\\
&= \sqrt[3]{216} && \text{(Rational Exponent)}\\
&= 6
\end{aligned}$$

(b) $\sqrt{x^7} * \sqrt{x^5}$

$$\begin{aligned}
&\sqrt{x^7} * \sqrt{x^5}\\
&= (x^7)^{\frac{1}{2}} * (x^5)^{\frac{1}{2}} && \text{(Rational Exponent)}\\
&= (x^7 * x^5)^{\frac{1}{2}} && \text{(Power of a Product)}\\
&= (x^{7+5})^{\frac{1}{2}} && \text{(Product Rule)}\\
&= (x^{12})^{\frac{1}{2}}\\
&= x^{\frac{12}{2}} && \text{(Power to a Power)}\\
&= x^6
\end{aligned}$$

## 1.4 Scientific Notation

One application of exponents is called *scientific notation*. A number is in scientific notation when it is in the form

$$a * 10^n$$

where $1 \leq |a| < 10$ and $n$ is an integer. Below we see numbers written in standard notation, in classic scientific notation, and in scientific notation as displayed by a computer.

| Standard Notation | Scientific Notation: Classic | Scientific Notation: Computer |
|---|---|---|
| -1,234,500,000 | $-1.2345 * 10^9$ | $-1.2345\ E + 9$ |
| 0.0000678 | $6.78 * 10^{-5}$ | $6.78\ E - 5$ |

To write a number in scientific notation, consider the standard notation form as being multiplied by $10^0$ (which is 1). Then move the decimal until there is exactly one nonzero digit to the left of the decimal point, adding 1 to the exponent for each place the decimal is moved to the left, or subtracting 1 from the exponent for each place the decimal is moved to the right.

**Example 17:** Write each number in scientific notation.

(a) 23580000000000

(b) 0.0000000124

$$\begin{aligned} 23580000000000 &= 2\underbrace{3580000000000}_{\text{13 digits}}*10^0 \\ &= 2.358 * 10^{0+13} \\ &= 2.358 * 10^{13} \end{aligned}$$

$$\begin{aligned} 0.0000000124 &= 0.\underbrace{00000001}_{\text{8 digits}}24 * 10^0 \\ &= 1.24 * 10^{0-8} \\ &= 1.24 * 10^{-8} \end{aligned}$$

**Example 18:** Operating on numbers in scientific notation is a lot like our work in Section 1.3 simplifying variable expressions. Useful properties here include the commutative and associative properties of multiplication, the product rule and the quotient rule. Multiply or divide in the following expressions as appropriate, writing all answers in scientific notation.

(a) $(2.44 * 10^{24})(8.6 * 10^{15})$

Recall that the commutative property of multiplication allows us to rearrange factors while the associative property let's us conveniently group factors.

$$\begin{aligned}
(2.44 * 10^{24})(8.6 * 10^{15}) &= 2.44 * 10^{24} * 8.6 * 10^{15} \\
&= 2.44 * 8.6 * 10^{24} * 10^{15} && \text{(Commutative Property)} \\
&= (2.44 * 8.6)(10^{24} * 10^{15}) && \text{(Associative Property)} \\
&= 20.984 * 10^{39} && \text{(Product Rule)} \\
&= 2.0984 * 10^{40} && \text{(Rewriting in scientific notation)}
\end{aligned}$$

(b) $\dfrac{3.405 * 10^9}{1.5 * 10^{12}}$

Note that our final answers in this section can have negative exponents, unlike our work in previous sections.

$$\begin{aligned}
\frac{3.405 * 10^9}{1.5 * 10^{12}} &= \frac{3.405}{1.5} * \frac{10^9}{10^{12}} \\
&= 2.27 * 10^{-3}
\end{aligned}$$

(c) $\dfrac{(1.44 * 10^{10})(9.252 * 10^5)}{(3.6 * 10^{-14})(4 * 10^{-20})}$

Our third expressions involves both multiplication and division of numbers in scientific notation. This can be approached in multiple ways, but we'll take the most obvious path.

$$\begin{aligned}
\frac{(1.44 * 10^{10})(9.252 * 10^5)}{(3.6 * 10^{-14})(4 * 10^{-20})} &= \frac{(1.44 * 10^{10} * 9.252 * 10^5)}{(3.6 * 10^{-14} * 4 * 10^{-20})} \\
&= \frac{(1.44 * 9.252 * 10^{10} * 10^5)}{(3.6 * 4 * 10^{-14} * 10^{-20})} \\
&= \frac{13.32288 * 10^{15}}{14.4 * 10^{-34}} \\
&= \frac{13.32288}{14.4} * \frac{10^{15}}{10^{-34}} \\
&= 0.9252 * 10^{49} \\
&= 9.252 * 10^{48}
\end{aligned}$$

**Example 19:** According to TOP500[a], a biannually published list of the five hundred fastest noncommercial computer systems, as of November 2015 the fastest computer in the world is China's Tianhe-2 with a processing speed of 33.86 petaflops ($10^{15}$ floating-point operations per second). How many floating-point operations can the Tianhe-2 perform per hour? (Answer in scientific notation.)

Since we're given the rate in floating-point operations *per second*, we need to convert hours to minutes and minutes to seconds. In problems like this, using the units alongside the computations can help us be sure we're operating correctly.

$$\frac{33.86 * 10^{15} \text{floating point operations}}{1 \text{ second}} * \frac{60 \text{seconds}}{1 \text{minute}} * \frac{60 \text{minutes}}{1 \text{hour}}$$

$$= \frac{33.86 * 10^{15} \text{floating point operations}}{1 \cancel{\text{ second}}} * \frac{60 \cancel{\text{seconds}}}{1 \cancel{\text{minute}}} * \frac{60 \cancel{\text{minutes}}}{1 \text{hour}}$$

From this we can see that, after canceling, the only units remaining are floating-point operations in the numerator and hour in the denominator, so evaluating will yield floating-point operations per hour.

$$\begin{aligned}33.86 * 10^{15} * 60 * 60 &= 33.86 * 10^{15} * 3600 \\ &= 121896 * 10^{15} \\ &= 1.21896 * 10^{20}\end{aligned}$$

Therefore, the Tianhe-2 could perform $1.21896 * 10^{20}$ floating-point operations per hour.

[a] https://www.top500.org/lists/2015/11/, accessed May 2016

# 1.5 Exercises

## 1.5.1 Order of Operations

Evaluate each expression.

1. $13 - 29 + 4$
2. $16 \div 2 * 8$
3. $-4 * (-3)^2$
4. $12 + 6 * (9 - 4)$
5. $(18 - 5) - (9 - 24)$
6. $(7 + 4 * (-3))^2$
7. $\dfrac{5+3}{3-1}$
8. $12 * (6 - 6)$
9. $(-3)^2(-2)^3$
10. $24 - 6 * (7 - 3)$
11. $18 \div 3 * 2 + 1$
12. $16 + 12 - 9 + 3$

## 1.5.2 Fractions and Decimals

Evaluate each expression.

13. $\frac{36}{96}$
14. $\frac{3}{4} - \frac{1}{4}$
15. $\frac{5}{7} + \frac{9}{7}$
16. $\frac{2}{15} * \frac{10}{27}$
17. $\frac{4}{9} \div 6$
18. $0.25 + 1.286$
19. $25.68 - 40.3$
20. $24.6 \div 0.12$
21. $6.65 * 0.501$
22. $\dfrac{5}{6} - \dfrac{1}{4} + \dfrac{1}{3}$
23. $8.25{+}1.993$
24. $1.7 * \frac{2}{3}$

## 1.5.3 Exponents and Radicals

Simplify each expression. Assume variables are positive.

25. $3^4 * 3^{-6}$
26. $2.4^2$
27. $25^{-2} * 5$
28. $\dfrac{4^5}{4^7}$
29. $(-5)^0$
30. $-5^0$
31. $(3 * 5)^2$
32. $36 * \left(\dfrac{2}{3}\right)^{-3}$

33. $(6+7)^2$

34. $\frac{5}{9}\left(1\frac{1}{5}\right)^2$

35. $x^6x^9$

36. $\frac{w^3}{w^7}$

37. $\frac{v^7}{v^4}$

38. $\frac{n^{-3}n^{12}}{n^{-4}}$

39. $y^3y^{-8}$

40. $m^{-14}m^{14}$

41. $\sqrt{16}$

42. $121^{\frac{1}{2}}$

43. $\sqrt[4]{81}$

44. $1728^{\frac{1}{3}}$

45. $\sqrt[5]{-1}$

46. $\sqrt[3]{64}$

47. $\sqrt{x^6}$

48. $\sqrt[5]{y^{110}}$

49. $\sqrt{49s^8}$

50. $(x^{12})^{\frac{1}{3}}$

51. $(64c^9z^{-30})^{\frac{1}{3}}$

52. $\frac{x^5 * x^7}{x^{24}}$

### 1.5.4 Scientific Notation

Write the given number in scientific notation.

53. 0.2675

54. 1300000000

55. 25040000000000

56. 0.0000000023

57. 60000000

58. 10

59. 1

60. 0.00000000030005

Write the given number in standard notation.

61. $3.21 * 10^2$

62. $1.001 * 10^{-4}$

63. $2.51 * 10^{13}$

64. $6 * 10^{-15}$

65. $4.78 * 10^{-8}$

66. $9.99 * 10^{11}$

Perform the multiplication or division; write your answer in scientific notation.

67. $(2.4 * 10^5)(5 * 10^{-18})$

68. $\frac{9.6 * 10^{14}}{1.5 * 10^9}$

69. $(7.42 * 10^{-25})(1.6 * 10^{-11})$

70. $\frac{2.08 * 10^{60}}{1.3 * 10^{-20}}$

71. $\frac{(2.1 * 10^{-8})(3.4 * 10^7)}{1.4 * 10^{-15}}$

72. $\frac{(2.46 * 10^{12})(6.25 * 10^{24})}{(1.5 * 10^8)(4.1 * 10^{48})}$

### 1.5.5 Mixed Problems

For problems 73-78, evaluate the given expression.

73. $2 * (16+9)^{\frac{1}{2}}$

74. $\dfrac{9^{\frac{3}{2}} - 25^{\frac{1}{2}}}{2 * 3^2 - 6}$

75. $\sqrt{12^2 - (3^3 - 4)}$

76. $(16+5) * (5-8)$

77. $\dfrac{6 \div 12 * (5-3)}{(3-2)^2}$

78. $4 - 4^2 + (6-3)^2$

79. What is the additive identity and why?

80. The number system we work in is base-10 because each digit counts the number of a specific power of ten. For example, 7,205 is seven thousands, two hundreds, zero tens, and five ones, or

$$7 * 10^3 + 2 * 10^2 + 0 * 10^1 + 5 * 10^0$$

This is called the *decimal expansion* of 7,205. Write the decimal expansion of the given number.

(a) 57,109

(b) 42.506

81. Simplify each expression if possible; if an expression is undefined, explain why it is undefined.

(a) $5^0$

(b) $5 * 0$

(c) $\frac{5}{0}$

82. Simplify each expression; assume all variables are positive.

(a) $\dfrac{3 - 2(6-3)}{5^2 - 4^2}$

(b) $\dfrac{\sqrt{25x^3}}{\sqrt{81x^5}}$

83. The NASA data archive at the Goddard Space Flight Center contains 25 terabytes of data from over 1000 science missions and investigations. (1 terabyte = $10^{12}$ bytes). How many CD-roms does this equal if the capacity of a CD-rom is about $6 * 10^8$ bytes? Round your answer to two significant figures and leave your answer in scientific notation.*

*This problem and others like it can be found at nasa.gov (http://spacemath.gsfc.nasa.gov/Modules/2page9.pdf)

# Chapter 2

# Number Systems

## 2.1 Decimal

A *number system* is a method of expressing numbers using a predefined set of symbols, and following a set of rules in combining symbols to represent different numbers. In the table below we see the four number systems we'll investigate along with some terminology and information about each number system. The first number system we'll discuss is called the decimal system.

| | Number System | | | |
|---|---|---|---|---|
| Terminology | **Decimal** | Binary | Octal | Hexadecimal |
| Radix | **ten** | two | eight | sixteen |
| Symbols | **0,1,2,3,4,5,6,7,8,9** | 0,1 | 0,1,2,3,4,5,6,7 | 0,1,2,3,4,5,6,7,<br>8,9,A,B,C,D,E,F |
| Weight (Base) | **Powers of 10 (10)** | Powers of 2 (2) | Powers of 8 (8) | Powers of 16 (16) |
| Symbol Name | **digit** | bit | | nibble/hex digit |

Each of the systems described in the table above are *positional*, meaning that the value of a symbols depends on its position relative to the symbols around it. The *radix* of such a number system is the number of usable symbols. In the decimal system we have ten symbols: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. Based on the position of one of these symbols it is assigned a weight. The decimal system is a base-10 number system, so the weights are powers of ten; for example, a weight could be $10^0 = 1$, $10^2 = 100$, or $10^{-2} = 0.01$ depending on its position. In the decimal system, an individual symbol is called a digit.

**Example 1:** In the number 12,201, only three of the numerical symbols are used. As we see in the table below, the rightmost digit has a weight of $10^0$, or 1, and as we move left the weight increases by a factor of ten each time.

| Number | 1 | 2 | 2 | 0 | 1 |
|---|---|---|---|---|---|
| Weight | $10^4$ | $10^3$ | $10^2$ | $10^1$ | $10^0$ |
| Weight Name | ten-thousands | thousands | hundreds | tens | ones |

Note that we can also express fractional numbers in the decimal system, using the decimal point to separate the integer part of a number from its fractional part. For example, in the number 3.21, 3 has a weight of $10^0$, 2 has a weight of $10^{-1}$, and 1 has a weight of $10^{-2}$.

As we'll see in later chapters, it is sometimes useful to write out a number with each symbol's weight explicitly stated. This is called expanded form or expanded notation.

**Example 2:** Give the *decimal expansion* of the decimal number 1,035,084.

$$1 \times 10^6 + 0 \times 10^5 + 3 \times 10^4 + 5 \times 10^3 + 0 \times 10^2 + 8 \times 10^1 + 4 \times 10^0$$

The last two pieces of terminology in this section are least significant digit and most significant digit. The *least significant digit*, or LSD, is the rightmost digit of a number (non-zero if the digit falls to the right of the decimal point), while the *most significant digit*, or MSD, is the leftmost non-zero digit of a number.

**Example 3:** Identify the MSD and LSD in the number 1,035,084.

The MSD is 1 while the LSD is 4.

## 2.2 Binary

| | Number System | | | |
|---|---|---|---|---|
| Terminology | Decimal | **Binary** | Octal | Hexadecimal |
| Radix | ten | **two** | eight | sixteen |
| Symbols | 0,1,2,3,4,5,6,7,8,9 | **0,1** | 0,1,2,3,4,5,6,7 | 0,1,2,3,4,5,6,7,<br>8,9,A,B,C,D,E,F |
| Weight (Base) | Powers of 10 | **Powers of 2** | Powers of 8 | Powers of 16 |
| Symbol Name | digit | **bit** | | nibble/hex digit |

As the name indicates, the binary system is base-2, meaning that placement weight is given by powers of two.

Since we'll be working in multiple number systems, determining which system we're working in at any given time will be necessary. It is common to use subscripts to denote the base of a number. For example $101_{10}$ is in decimal* while $101_2$ is in binary. We also must be able to convert between different systems; how can a binary number be converted into decimal and visa-versa?

Converting from binary to decimal is relatively simple: write the *binary expansion* of the number, then simplify.

**Example 4:** Convert $101_2$ into a decimal number.

$$\begin{aligned} 101_2 &= 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 && \text{(Binary Expansion)} \\ &= 4 + 0 + 1 \\ &= 5 \end{aligned}$$

*Due to the fact that the decimal system is the most commonly used number system, base-10 numbers will generally not include a subscript while numbers in other bases will include the appropriate subscript.

**Example 5:** Converting from decimal to binary can be a little trickier, and there are two common approaches we could take. For example, consider the decimal number 101. One method of converting this into binary involves finding the largest power of two, no greater than 101, which in this case is $2^6$, and then investigating the lesser powers. These lesser powers would be $2^5 = 32$, $2^4 = 16$, $2^3 = 8$, $2^2 = 4$, $2^1 = 2$, and $2^0 = 1$. This approach would continue with the following steps:

- Since 64 is the largest power of two which is also less than or equal to 101, consider 101-64=37.
- Since $2^5 = 32$ is the largest power of two which is also less than or equal to 37, consider 37-32=5.
- Since $2^2 = 4$ is the largest power of two which is also less than or equal to 5, consider 5-4=1.
- Since $2^0 = 1$ is the largest power of two with is also less than or equal to 1, consider 1-1=0.
- Since a difference of zero has been reached, we need go no further.

From the steps above, a value of one-hundred one is attained by adding the sixth, fifth, second, and zeroth powers of two. Therefore the expansion of the binary equivalent would look like

$$1 * 2^6 + 1 * 2^5 + 0 * 2^4 + 0 * 2^3 + 1 * 2^2 + 0 * 2^1 + 1 * 2^0$$

giving us the binary equivalent of 101:

$$1100101_2$$

This approach may seem cumbersome, especially when converting larger values. One reason is that this method requires either repeated computation or memorization of powers. The first ten powers of two are shown here to aid those using the method just described.

| | | | | | |
|---|---|---|---|---|---|
| $2^0$ | $=$ | 1 | $2^5$ | $=$ | 32 |
| $2^1$ | $=$ | 2 | $2^6$ | $=$ | 64 |
| $2^2$ | $=$ | 4 | $2^7$ | $=$ | 128 |
| $2^3$ | $=$ | 8 | $2^8$ | $=$ | 256 |
| $2^4$ | $=$ | 16 | $2^9$ | $=$ | 512 |

**Example 6:** A second approach relies on division and can be easily altered to work for the other number systems we'll discuss. The idea is to divide the decimal number by the base of the number system into which we're converting; a remainder indicates the value of the symbol to be used, moving from the LSD to the MSD. For example, the steps of converting 101 into binary are as follows:

- $101 \div 2 = 50R1$. The remainder is 1, so the LSD, in binary, is 1.
- Considering the quotient from the previous step, $50 \div 2 = 25R0$. The remainder is 0, so the next bit will be zero.
- $25 \div 2 = 12R1$
- $12 \div 2 = 6R0$
- $6 \div 2 = 3R0$
- $3 \div 2 = 1R1$
- $1 \div 2 = 0R1$, so the MSD is 1.

Moving through these steps, the remainders give the bits of the binary equivalent to 101 which, again, is

$$1100101_2$$

Another advantage this second approach has over the first is that *every* bit of the binary number appears in the process, not just those bits using the symbol "1".

This process works because each subsequent step of division acts like division of the original number, 101, by greater and greater powers of two. This means that, at each step, the remainder will indicate whether or not a lesser power of two is used in the binary representation.

- In the first step, because 2 does not go into 101 without remainder, a one is necessary. With a one accounted for, we have 101-1=100 left.
- In the second step, because 2 does go into 50 without remainder (equivalently, 4 does go into 100 without remainder), a two is not necessary.
- In the third step, because 2 does not go into 25 without remainder (equivalently, 8 doesn't go into 100 without remainder), a four is necessary. With a four accounted for, we have 100-4=96 left.
- In the fourth step, because 2 does go into 12 without remainder (equivalently, 16 does go into 96 without remainder), an eight is not necessary. Etc.

Similar processes can be used to convert values with fractional parts between decimal and binary. The conversion from binary to decimal can, again, be completed by simplifying the binary expansion. The conversion from decimal to binary requires a slight modification, however.

**Example 7:** To convert $1001.1101_2$ into its decimal equivalent we can operate on the binary expansion:

$$\begin{aligned} 1001.1101_2 &= 1*2^3+0*2^2+0*2^1+1*2^0+1*2^{-1}+1*2^{-2}+0*2^{-3}+1*2^{-4} \\ &= 8+0+0+1+0.5+0.25+0.0625 \\ &= 9.8125 \end{aligned}$$

**Example 8:** Consider the decimal number 5.375; we've seen how to convert the integer part of this number into binary, and the result is $5 = 101_2$.

The fractional part will be converted into binary through multiplication rather than division.

$$\begin{array}{r} 0.375 \\ \times \quad 2 \\ \hline 0.750 \end{array} \qquad \begin{array}{r} 0.75 \\ \times \quad 2 \\ \hline 1.50 \end{array} \qquad \begin{array}{r} 0.5 \\ \times \quad 2 \\ \hline 1.0 \end{array}$$

Integer part: 0
This will be the bit in
the halves ($2^{-1}$) place.

Integer part: 1
This will be the bit in
the quarters ($2^{-2}$) place.

Integer part: 1
This will be the bit in
the eighths ($2^{-3}$) place.

Once no fractional part remains, the process is complete. Therefore, the binary equivalent of 5.375 is

$$101.011_2$$

## 2.3 Octal and Hexadecimal

| | Number System | | | |
|---|---|---|---|---|
| Terminology | Decimal | Binary | **Octal** | Hexadecimal |
| Radix | ten | two | **eight** | sixteen |
| Symbols | 0,1,2,3,4,5,6,7,8,9 | 0,1 | **0,1,2,3,4,5,6,7** | 0,1,2,3,4,5,6,7,<br>8,9,A,B,C,D,E,F |
| Weight (Base) | Powers of 10 | Powers of 2 | **Powers of 8** | Powers of 16 |
| Symbol Name | digit | bit | | nibble/hex digit |

In computing, a *word*† is the unit of data used by a processor. Most computers today use 32- or 64-bit words; however, in the 60's and 70's many computers utilized 12-, 24-, or 36-bit words. As a result, octal (base-8 number system) was a convenient abbreviation for the binary in computing because 12, 24, and 36 are divisible by three and every octal digit is equivalent to a 3-bit word.

In our first example we'll explore how to convert between decimal and octal, before moving on to hexadecimal, saving conversion between binary, octal, and hexadecimal for the end of this section.

†Some texts defines a word as "a group of one or more bytes", where a byte is a group of eight bits. We will be more lax with our definition and will refer to a group of $n$ bits as an "$n$-bit word."

**Example 9:** Convert 123 to its octal equivalent, and convert $123_8$ to its decimal equivalent.

For each conversion we can borrow from our work with binary. To convert $123_8$ into decimal we can operate on the *octal expansion*:

$$\begin{aligned} 123_8 &= 1*8^2 + 2*8^1 + 3*8^0 && \text{(Octal Expansion)} \\ &= 64 + 16 + 3 \\ &= 83 \end{aligned}$$

The process we used to convert from decimal to binary can be modified to convert into octal; rather than dividing by 2 we'll divide by 8.

- $123 \div 8 = 15R3$ - the LSD is 3
- $15 \div 8 = 1R7$
- $1 \div 8 = 0R1$ - the MSD is 1

Again, focusing on the remainders, the octal equivalent to 123 is

$$173_8$$

| | Number System | | | |
|---|---|---|---|---|
| Terminology | Decimal | Binary | Octal | **Hexadecimal** |
| Radix | ten | two | eight | **sixteen** |
| Symbols | 0,1,2,3,4,5,6,7,8,9 | 0,1 | 0,1,2,3,4,5,6,7 | **0,1,2,3,4,5,6,7,** 8,9,A,B,C,D,E,F |
| Weight (Base) | Powers of 10 | Powers of 2 | Powers of 8 | **Powers of 16** |
| Symbol Name | digit | bit | | **nibble/hex digit** |

A symbol in hexadecimal (base-16 number system) may be referred to as a hex digit or as a nibble (a 4-bit word) because each 4-bit word is equivalent to a hex digit and visa-versa. The hexadecimal system has become a more convenient abbreviation for the 32- and 64-bit words of today's computers, since 32 and 64 are multiples of 4.

Because the hexadecimal system requires 16 symbols we include the first six letters of the alphabet, with $A = 10$, $B = 11$, and so on. Regarding the notation for labeling a number as hexadecimal, we'll use subscripts as previously discussed; however, another notation concatenates a hex number with the letter H; for example, 12D0H is the hexadecimal number $12D0_{16}$.

**Example 10:** What is the hexadecimal equivalent of 1000?

Converting a number from decimal to hexadecimal is much like a similar conversion into binary or octal. Repeatedly dividing by the base, in this case 16, and recording the remainder until there is nothing left to operate on.

- $1000 \div 16 = 62R8$ - the LSD is 8
- $62 \div 16 = 3R14$ - remember, $14 = E_{16}$
- $3 \div 16 = 0R3$ - the MSD is 3

Therefore, we have $1000 = 3E8_{16}$.

**Example 11:** What is the decimal equivalent of $123_{16}$? $12D0_{16}$?

Again we can return to a familiar process: operating on the expanded form. The hexadecimal expansion of $123_{16}$ is

$$1 * 16^2 + 2 * 16^1 + 3 * 16^0$$

Simplifying this expression yields 291, so $123_{16} = 291$.

We can approach the second hexadecimal number in the same way, keeping in mind that

$D_{16} = 13$:

$$12D0_{16} = 1 * 16^3 + 2 * 16^2 + 13 * 16^1 + 0 * 16^0$$

Simplifying the right hand side of the equation above gives us 4816.

**Example 12:** What are the binary, octal, and hexadecimal equivalents of 150?

For binary, we have

- $150 \div 2 = 75R0$ - the LSD is 0
- $75 \div 2 = 37R1$
- $37 \div 2 = 18R1$
- $18 \div 2 = 9R0$
- $9 \div 2 = 4R1$
- $4 \div 2 = 2R0$
- $2 \div 2 = 1R0$
- $1 \div 2 = 0R1$ - the MSD is 1

So our binary equivalent is $10010110_2$. Finding the octal equivalent, while shorter, will look similar:

- $150 \div 8 = 18R6$ - the LSD is 6
- $18 \div 8 = 2R2$
- $2 \div 8 = 0R2$ - the MSD is 2

$226_8$ is our octal equivalent. Finally, for hexadecimal, we have $96_{16}$:

- $150 \div 16 = 9R6$ - the LSD is 6

- $9 \div 16 = 0R9$ - the MSD is 9

At this point you may be wondering if there was another way to get our octal and hexadecimal values, using our work for the binary equivalent. In fact, we could have split the binary number into pieces with each piece translatable into octal or hexadecimal.

To convert from binary into octal, note that a three-bit word is equivalent to an octal digit. This is because, using three bits, we can express the whole values from 0 to 7:

$$\begin{array}{cccc} 000_2 = 0 & 001_2 = 1 & 010_1 = 2 & 011_2 = 3 \\ 100_2 = 4 & 101_2 = 5 & 110_2 = 6 & 111_2 = 7 \end{array}$$

A single octal digit has the same capability. We can expand on this, considering the possible values of a six-bit word:

$$\begin{array}{cccc} 000000_2 = 0 & 000001_2 = 1 & 000010_2 = 2 & 000011_2 = 3 \\ 000100_2 = 4 & 000101_2 = 5 & 000110_2 = 6 & 000111_2 = 7 \\ 001000_2 = 8 & 001001_2 = 9 & 001010_2 = 10 & 001011_2 = 11 \\ 001100_2 = 12 & 001101_2 = 13 & 001110_2 = 14 & 001111_2 = 15 \\ 010000_2 = 16 & 010001_2 = 17 & 010010_2 = 18 & 010011_2 = 19 \\ \vdots & \vdots & \vdots & \vdots \\ 111100_2 = 60 & 111101_2 = 61 & 111110_2 = 62 & 111111_2 = 63 \end{array}$$

Notice that the maximum value of a six-bit word is 63, the same as a two octal-digit number: $77_8 = 7 * 8^1 + 7 * 8^0 = 56 + 7 = 63$. This leads to the idea that a single octal digit is equivalent to a three-bit word. Using this idea, we can convert from binary to octal by splitting a binary number into words of length three, converting each word into an octal digit.

**Example 13:** Convert $110101010_2$ into octal.

To complete this conversion, we can convert three-bit pieces individually, starting with the least significant: 010, 101, and 110.

- $010_2 = 2$ - the octal LSD is 2
- $101_2 = 5$
- $110_2 = 6$ - the octal MSD is 6

Therefore $652_8 = 110101010_2$.

Considering our previous example, where we found $10010110_2 = 150$, the fact that eight bits are used is not an issue. Even though eight is not divisible by three, we can extend the binary number to include more bits: $10010110_2 = 010010110_2$. Now we can split this into three-bit pieces as before.

- $110_2 = 6$ - the octal LSD is 6
- $010_2 = 2$
- $010_2 = 2$ - the octal MSD is 2

Again, we come up with the octal equivalent of $226_8$.

The process for converting binary into hexadecimal is very similar, as is the logic. Arguing that every four-bit word is equivalent to a hex-digit, we can split any binary number into four-bit pieces and convert each piece individually into hexadecimal.

Returning to Example 12, rather than converting directly from decimal into hexadecimal we could have used our work in binary. Since $150 = 10010110_2$, split the binary number into four-bit pieces:

- $0110_2 = 6$ - the hexadecimal LSD is 6
- $1001_2 = 9$ - the hexadecimal MSD is 9

Again, we get the same result found in the previous example, namely $150 = 96_{16}$.

## 2.4 Arithmetic with Binary Numbers

We've seen how to represent positive rational numbers in binary, but we can also write negative integers in binary. We do this by reserving a "sign bit". The leftmost bit (or MSD) of a unit of storage is reserved for the sign of the number; this is called the *sign bit*.

- Positive numbers have the sign bit set to zero, and
- negative numbers have the sign bit set to one.

Working with 4-bit words, the table to the write lists the positive and negative numbers that are possible. Comparing these positive and negative binary numbers in the table you may notice a pattern; we'll be discussing this very soon as **2's Complement Notation** is introduced.

| Positive | Negative |
|---|---|
| $0000_2 = 0$ | $1111_2 = -1$ |
| $0001_2 = 1$ | $1110_2 = -2$ |
| $0010_2 = 2$ | $1101_2 = -3$ |
| $0011_2 = 3$ | $1100_2 = -4$ |
| $0100_2 = 4$ | $1011_2 = -5$ |
| $0101_2 = 5$ | $1010_2 = -6$ |
| $0110_2 = 6$ | $1001_2 = -7$ |
| $0111_2 = 7$ | $1000_2 = -8$ |

Obviously 4-bit words are not long enough for us to store or generate very much information, but they will be adequate examples to help us understand binary arithmetic.

**Example 14:** Addition of binary numbers is very much like addition of decimal numbers. Complete the decimal addition below, investigate the process, and discover the binary addition algorithm to complete the binary addition.

Decimal addition: 483+276

- The first step is to add 6 and 3 to get 9:

$$\begin{array}{r} 483 \\ +276 \\ \hline 9 \end{array}$$

- Next, we'll add 8 and 7 to get 15. The 5 will drop into the sum and the 1 will be carried:

$$\begin{array}{r} 1\phantom{00} \\ 483 \\ +276 \\ \hline 59 \end{array}$$

- Our last step is to add 1, 4, and 2 to get 7:

$$\begin{array}{r} 1\phantom{00} \\ 483 \\ +276 \\ \hline 759 \end{array}$$

Binary addition: $0010_2 + 0011_2$

- The first step is to add 0 and 1 to get 1; since $1 = 1_2$, we can drop the 1 into the sum:

$$\begin{array}{r} 0010_2 \\ +0011_2 \\ \hline 1_2 \end{array}$$

- Next, we'll add 1 and 1 to get 2; since $2 = 10_2$, the 0 will drop into the sum and the 1 will be carried:

$$\begin{array}{r} 1\phantom{00}\phantom{_2} \\ 0010_2 \\ +0011_2 \\ \hline 01_2 \end{array}$$

- Then adding 1, 0, and 0 we get 1, or $1_2$. Finally, adding 0 and 0, we get $0_2$:

$$\begin{array}{r} 1\phantom{00}\phantom{_2} \\ 0010_2 \\ +0011_2 \\ \hline 0101_2 \end{array}$$

Notice that the binary addition follows the same pattern as decimal addition. This is true for octal addition and hexadecimal addition as well, with each using the algorithm below:

- Stack numbers of the same base with equally weighted symbols one over the other.
- Add equally weighted symbols, starting with the LSD of the numbers to be added.
- If the sum of two symbols is less than the base, the sum of the two symbols is dropped into the overall sum.

- If the sum of two symbols is greater than or equal to the base, drop the LSD of the sum into the overall sum and carry the MSD into the next column of the addition.

**Example 15:** Convert 3 and 4 to 4-bit words and add the two numbers to get the binary equivalent of positive 7.

First note that $3 = 0011_2$ and$4 = 0100_2$. Then we can follow the algorithm just described and, in this case, no carrying is necessary.

$$\begin{array}{r} 0011_2 \\ +0100_2 \\ \hline 0111_2 \end{array}$$

**Example 16:** Convert 7 and 10 to 6-bit words and add the two numbers to get the binary equivalent of positive 17.

First note that $7 = 000111_2$ and$10 = 001010_2$. Then we can follow the addition algorithm. This time carrying will result from addition in the twos, fours, and eights places:

$$\begin{array}{r} 111\phantom{0_2} \\ 000111_2 \\ +001010_2 \\ \hline 010001_2 \end{array}$$

**Example 17:** Since $0001_2 = +1$ one might imagine that simply changing the sign bit would give the opposite, that $1001_2 = -1$. Based on what we know about addition of binary numbers, why can't this be the case?

The sum of 1 and $-1$ should be zero. However, if $1001_2 = -1$, then according to our addition algorithm we have $0 = 1 + (-1) = 1010_2$. This doesn't make any sense; we already know that the (4-bit) binary equivalent of 0 is simply $0000_2$.

$$\begin{array}{r} 0001_2 \\ +1001_2 \\ \hline 1010_2 \end{array}$$

As we saw in the last example, simply changing the MSD from 0 to 1 does not give us the opposite of a binary number. Instead, we follow the process below to find the opposite of a number, be it positive or negative:

- Start with the binary representation of the number.

- Flip the bits (change all ones to zeros and all zeros to ones).
- Add 1 to the result.

This process is known as taking *2's complement* of a number.

**Example 18:** Find the binary representation of -6, then use addition to verify the result.

- Find the binary equivalent of 6:
  $6 = 0110_2$
- Flip the bits: $1001_2$
- Add 1: $1010_2$

Verify:

$$\begin{array}{r} 0110_2 \\ +1010_2 \\ \hline 10000_2 \end{array} \quad =0000_2$$

Notice the leading bit, colored red, in the original sum; this bit will often arise when adding signed binary numbers and is disregarded. Since we added two signed, four-bit words the sum should also be a signed, four-bit word and so the extra fifth bit that can result from such addition is ignored.

Note that taking 2's complement of a binary number changes the sign of the number, so just as taking 2's complement of a positive number gives the negative number with the same absolute value, taking 2's complement of a negative number gives the positive number with the same absolute value. Also, zero is grouped with the positive binary numbers; there is no negative zero (2's complement of $0000_2$ is again $0000_2$).

**Example 19:** Add $1101_2$ $(-3)$ and $1110_2$ $(-2)$.

$$\begin{array}{r} 1101_2 \\ +1110_2 \\ \hline 11011_2 \end{array} \quad =1011_2$$

**Example 20:** Subtraction can always be expressed as addition $(a - b = a + (-b))$, and this is the approach we'll take with subtraction of binary numbers. Subtract 3 from 4 by adding $1101_2$ $(-3)$ and $0100_2$ $(4)$.

$$\begin{array}{r} 0100_2 \\ +1101_2 \\ \hline 10001_2 \end{array} \quad =0001_2$$

**Example 21:** What is the value of $0010_2$, $1011_2$, and the sum of these two binary numbers?

Finding the value of $0010_2$ is straightforward; using our work in the previous sections, the binary expansion leads us to two as this number's value. However, the second binary number is negative since the MSD is 1. To find the values of this number we'll first take 2's complement, which leads us to $0101_2$. Since this has a value of five, the original binary number is negative five.

The only things left to do is check the sum. Now, we know what the sum of 2 and $-5$ should be $-3$, but let's actually find the binary sum:

$$\begin{array}{r} 0010_2 \\ +1011_2 \\ \hline 1101_2 \end{array}$$

- Flip the bits on our sum ($1101_2$): $0010_2$
- Add 1: $0011_2$
- Since $0011_2$ has a value of three, $1101_2$ has a value of negative three.

**Example 22:** When performing binary computations, an error known as *overflow* occurs when there are not enough bits to represent the result. Working with 4-bit words, we cannot represent an integer greater than 7 or less than -8. Attempt to complete the addition below and determine whether or not overflow occurs.

(a) $1011_2 + 0110_2$

$$\begin{array}{r} 1011_2 \\ +0110_2 \\ \hline 10001_2 \end{array}$$

Here we *do not* have overflow: the sum of negative five and positive six is positive one. The additional (red) digit can be dropped without issue.

(b) $1000_2 + 1011_2$

$$\begin{array}{r} 1000_2 \\ +1011_2 \\ \hline 10011_2 \end{array}$$

Here we *do* have overflow: the sum of negative eight and negative five is not positive three.

(c) $0101_2 + 0110_2$

$$\begin{array}{r} 0101_2 \\ +0110_2 \\ \hline 1011_2 \end{array}$$

Here we *do* have overflow: the sum of positive five and positive six is not negative five. Notice that an extra fifth bit does not have to occur for overflow to occur.

## 2.5 Exercises

### 2.5.1 Decimal

For problems 1-15, write the decimal expansion of the given number.

1. 12
2. 30
3. 42
4. 6
5. 15
6. 85
7. 100
8. 256
9. 203
10. 205
11. 753
12. 489
13. 1,480
14. 3,062
15. 12,528

For problems 16-21, identify the MSD and the LSD and the weight associated to each.

16. 50
17. 1,246
18. 401
19. 59
20. 107
21. 352

### 2.5.2 Binary

For problems 22-30, write the binary expansion of the given number.

22. $101_2$
23. $11_2$
24. $1011_2$
25. $1111_2$
26. $1101_2$
27. $0101_2$
28. $11100100_2$
29. $01011100_2$
30. $10011011_2$

For problems 31-45, find the decimal equivalent of the given binary number.

31. $1_2$
32. $0_2$
33. $10_2$
34. $11_2$
35. $00_2$
36. $01_2$
37. $0100_2$
38. $1010_2$
39. $1011_2$
40. $110100_2$
41. $010110_2$
42. $100011_2$

43. $01100111_2$ 44. $01001001_2$ 45. $11111111_2$

For problems 46-60, find the binary equivalent of the given decimal number.

46. 3 47. 0 48. 5 49. 8 50. 1

51. 12 52. 16 53. 15 54. 24 55. 17

56. 32 57. 50 58. 75 59. 97 60. 124

Find the binary equivalent of the given decimal number, or the decimal equivalent of the given binary number.

61. 3.5 62. 0.75 63. $1.1_2$ 64. 18.625 65. $10.11_2$

66. $0.101_2$ 67. 2.25 68. $101.011_2$ 69. $0.111_2$ 70. 10.0625

Challenge: Find the binary equivalent of the following decimal numbers. Hint - use the bar notation used to express repeating decimal numbers. For example, 0.3333... (3 repeating infinitely) can be written $0.\bar{3}$ and 0.7121212... can be expressed as $0.7\overline{12}$.

71. 0.75 72. 5.1875 73. 1.2 74. 0.66

### 2.5.3 Octal and Hexadecimal

Find both the octal and the hexadecimal equivalent of the following decimal numbers.

75. 5 76. 8 77. 13 78. 16 79. 24

80. 35 81. 36 82. 94 83. 128 84. 150

Find both the octal the hexadecimal equivalent of the following binary numbers.

85. $11_2$ 86. $101_2$ 87. $1011_2$ 88. $10110_2$ 89. $01001_2$

90. $101010_2$ 91. $011000_2$ 92. $11000111_2$ 93. $01111010_2$ 94. $10011101_2$

Find the decimal and (8-bit) binary equivalents of each number.

95. $11_8$ 96. $27_8$ 97. $30_{16}$ 98. $52_8$ 99. $64_{16}$

100. $100_8$ 101. $C5_{16}$ 102. $AB_{16}$ 103. $64_8$ 104. $E0_{16}$

Writing questions: Other number systems in history have included the use of other bases such as 60, 20, and 5. Some of the following problems involve numbers in bases other than 2, 8, 10, or 16.

105. Describe how you could find the decimal equivalent of a number from a base-7 number system which uses the symbols 0, 1, 2, 3, 4, 5, and 6. For example, how could you convert from $130_7$ to 70.

106. Suppose a base-20 number system used the symbols 0-9 and A-J, with values associated to A-J much like values are associated to A-F in the hexadecimal system. Describe how to find the base-20 equivalent of a decimal number. For example, how could you convert from 350 to $HA_{20}$?

107. Direct conversions from binary to octal, or from binary to hexadecimal, are relatively simple. On the other hand, converting from octal to hexadecimal can be done directly but is simpler when using an intermediary number system like binary. Why are direct conversions from binary to octal (or binary to hexadecimal) simpler than a direct conversion between hexadecimal and octal? How could the hexadecimal equivalent of an octal number be found without using an intermediary number system like binary or decimal?

### 2.5.4 Arithmetic with Binary Numbers

Give a signed binary equivalent of the given decimal number using 2's compliment notation.

108. $3$ 109. $-4$ 110. $12$ 111. $-10$ 112. $-13$

113. $15$ 114. $-15$ 115. $-24$ 116. $26$ 117. $32$

Each of the following is a signed binary number; find the decimal equivalent of each number.

118. $0110_2$ 119. $1010_2$ 120. $1001_2$ 121. $0011_2$ 122. $1111_2$ 123. $1100\ 0100_2$

124. $0110\ 1111_2$ 125. $0101\ 1000_2$ 126. $1101\ 1000_2$ 127. $1110\ 0101_2$

From problems 128-139 do the following:

(a) Rewrite any subtraction as addition.

(b) Find the signed, binary representation of each decimal number. (Be sure that both binary numbers use the same number of bits; remember that the MSD for each is the sign-bit.)

(c) Perform the binary addition.

(d) Convert the sum back into decimal.

128. $8+2$ 129. $12+7$ 130. $5-2$ 131. $5+2$ 132. $1-7$ 133. $1+7$

134. $12-16$ 135. $14+12$ 136. $-6-4$ 137. $-3+14$ 138. $11+20$ 139. $-7-15$

# Chapter 3

# Logic and Set Theory

## 3.1 Sets and Variables

Set theory is the branch of mathematics focusing on the study of sets, which are simply collections of objects. The inclusion of set theory here is meant to help understand the logical connectives we'll see through the rest of this chapter and for the same reason further discussion of variables is included.

Again, a set is just a collection of objects and, in set theory, these are usually mathematical objects: numbers, operations, relations, et cetera. An object in a set is called an element, or member, of the set and the symbol "$\in$" is used to connect an element to its set. For example, $a \in A$ is read "$a$ is an element of the set $A$". Similarly, $A \ni a$ is read "the set $A$ contains $a$". To show that an element is not in a given set the symbol "$\notin$" is used in the same way, with $b \notin A$ reading "$b$ is not an element of the set $A$". Note that elements need not be included in (or excluded from) a set one at a time. For example, $a, z \in A$ means that both $a$ and $z$ are elements of the set $A$. Some common sets, and examples of this notation, are given below.

- $\varnothing$ is the empty set, or the set containing no elements, also written $\{\}$.
- $\mathbb{N}$ is the set of natural numbers: $1 \in \mathbb{N}$, but $0 \notin \mathbb{N}$.

- $\mathbb{Z}$ is the set of integers: $-1, 1 \in \mathbb{Z}$, but $1.25 \notin \mathbb{Z}$.
- $\mathbb{Q}$ is the set of rational numbers: $1\frac{2}{5}, 0 \in \mathbb{Q}$, but $\pi, \sqrt{2} \notin \mathbb{Q}$.
- $\mathbb{R}$ is the set of real numbers: $1, 2.6, \sqrt{3} \in \mathbb{R}$ but for the imaginary unit $i$, $i \notin \mathbb{R}$.

There are a few common methods for describing a set: by description, by listing, and by set-builder notation. For example, suppose $A$ is the set of natural numbers from one to ten. Then $A$ could be defined in the following ways.

- $A$ is the set of natural numbers from 1 to 10. Defined by description.
- $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ Defined by listing elements.
- $A = \{x | x \in \mathbb{N}, 1 \leq x \leq 10\}$ Defined by set-builder notation.
- $A = \{x : x \in \mathbb{N}, 1 \leq x \leq 10\}$ Defined by set-builder notation.

Looking through these different definitions we make note of a couple of things. First, when defining by listing or by set-builder notation, the symbols { and } are used. These are left and right set-brackets, respectively. While these can be used as grouping symbols in mathematical expressions, we'll use them only in our set definitions. Second, there are two versions of the set-builder notation given above with only a slight difference between the two: the use of ":" versus the use of "|". Each symbol is read as "such that" in set-builder notation and both are included here because both are widely used. The version you see will depend on the text book you read.

While a variable was used to represent numerical values in our examples of set-builder notation, that is not all a variable can be used for. For example, a variable can take on the meaning of some non-numerical object as seen in the set below.

$$F = \{x : x \text{ is a four-legged animal}\}$$

Variables can also represent statements which may be true or false; this usage of variables will be seen as we move on to symbolic logic. But first, to help us transition into logic, we'll look at some operations that can be performed on or between sets. These are finding the union or intersection of two sets, or finding the complement of a set.

Before defining the complement of a set we need to understand the universe of discourse, also called the universal set. The universal set is the set of all objects under consideration within a situation or discussion. Generally denoted $U$, the universal set is usually defined prior to operating on the primary sets under consideration. All other sets we use within this discussion must contain only elements from $U$ or no element at all. If, for example, $U$ is the set of integers from 0 to 10 then we could discuss the set $A = \{1, 2, 3, 4, 5\}$.

The complement of a set, $A$, is all elements in $U$ that are not in $A$. Continuing with our definitions of $A$ and $U$ from the previous paragraph, the complement of $A$ would be $\{0, 6, 7, 8, 9, 10\}$. Note that the notation used for the complement of a set differs from text to text, with the most common being $A'$, $A^c$, and $\bar{A}$; we'll use the first of these,

$$A' = \{0, 6, 7, 8, 9, 10\}$$

Another set operation, the intersection of two sets, is very self-descriptive; taking the intersection of two sets, say $A$ and $B$, results in a set containing any element that is common to $A$ and $B$. For example, if $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 2, 4, 6, 8, 10\}$ then the intersection of $A$ and $B$, denoted $A \cap B$, is $\{2, 4\}$:

$$A \cap B = \{2, 4\}$$

The union of two sets, on the other hand, joins all elements of both sets. Continuing to let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 2, 4, 6, 8, 10\}$, the union of $A$ and $B$, denoted $A \cup B$, is $\{0, 1, 2, 3, 4, 5, 6, 8, 10\}$:

$$A \cup B = \{0, 1, 2, 3, 4, 5, 6, 8, 10\}$$

**Example 1:** Let $U = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$. Given the definitions of sets $A$, $B$, $C$, $D$, and $E$ below, find the indicated unions, intersections, and complements.

| | | |
|---|---|---|
| $A = \{1, 2, 3, 4\}$ | $C = \{-4, -3, -2, -1, 0\}$ | $E = \{-3, -1, 1, 3\}$ |
| $B = \{0, 1, 2, 3, 4\}$ | $D = \{-4, -2, 0, 2, 4\}$ | |

(a) $A'$

Remember that $A'$ is the set of elements not in $A$. The elements in the universal set not included in $A$ are $-4$, $-3$, $-2$, $-1$, and 0, so $A' = \{-4, -3, -2, -1, 0\}$ or simply $A' = C$.

(b) $D'$

Since $D$ contains only the even integers from $U$, the complement of $D$ will contain only the odds: $D' = \{-3, -1, 1, 3\}$ or $D' = E$.

(c) $A \cup C$

The union of $A$ and $C$ will contain all elements from $A$ and all elements from $C$:

$$A \cup C = \{1, 2, 3, 4\} \cup \{-4, -3, -2, -1, 0\} = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\} = U$$

(d) $D \cup C$

Again we're joining two sets, but this time the result won't be an already defined set:

$$D \cup C = \{-4, -2, 0, 2, 4\} \cup \{-4, -3, -2, -1, 0\} = \{-4, -3, -2, -1, 0, 2, 4\}$$

(e) $B \cap C$

To find the intersection of $B$ and $C$ we need only find the elements common to $B$ and $C$. Since $B = \{0, 1, 2, 3, 4\}$ and $C = \{-4, -3, -2, -1, 0\}$, the only element contained by both $B$ and $C$ is 0. So $B \cap C = \{0\}$.

(f) $A \cap E$

With $A$ being the positive elements of $U$ and $E$ containing the odds, the elements appearing in both are the positive odds: $A \cap E = \{1, 3\}$.

(g) $(A \cup D)'$

With this example we have two operations to complete: finding a union and taking a complement. But which operation should be performed first?

Just as we have an order of operations for arithmetic and algebra, we can implement a order of operations, or order of precedence, in set theory. As usual, grouping symbols (parenthesis) are first in line. These are followed by complementing a set, followed by intersection, followed by union. Usually parenthesis are included to indicate the order in which unions and intersections should be performed, but when no parenthesis are given an intersection takes precedence over a union.

For this example, we'll first find the union of $A$ and $D$, then take the complement of the result.

$$\begin{aligned}(A \cup D)' &= (\{1,2,3,4\} \cup \{-4,-2,0,2,4\})' \\ &= \{-4,-2,0,1,2,3,4\}' \\ &= \{-3,-1\}\end{aligned}$$

(h) $(B \cap E)'$

Following the order of precedence, we'll be finding the intersection of $B$ and $E$ then taking the complement of the result.

$$\begin{aligned}(B \cap E)' &= (\{0,1,2,3,4\} \cap \{-3,-1,1,3\})' \\ &= \{1,3\}' \\ &= \{-4,-3,-2,-1,0,2,4\}\end{aligned}$$

(i) $C \cup B \cap E$

Since intersection takes precedence over union, the first operation we'll complete is the intersection. The intersection of $E$ and $B$ gives us $\{1,3\}$. Then, finding the union of this set and $C$, we have

$$\{-4,-3,-2,-1,0\} \cup \{1,3\} = \{-4,-3,-2,-1,0,1,3\}$$

(j) $(C \cup B) \cap E$

This difference between this and the previous example is the parenthesis around $C \cup B$. These parenthesis indicate the union should be performed first:

$$
\begin{aligned}
(C \cup B) \cap E &= (\{-4,-3,-2,-1,0\} \cup \{0,1,2,3,4\}) \cap E \\
&= \{-4,-3,-2,-1,0,1,2,3,4\} \cap E \\
&= U \cap E \\
&= E
\end{aligned}
$$

With the union of $B$ and $C$ being the universal set $U$, finding the intersection of $E$ and $U$ takes no work at all; since all of $E$ is within $U$, the elements common to both $E$ and $U$ will be every element of $E$.

(k) $(A' \cap B) \cap E$

First note that $A' \cap B = \{0\}$. Then we have

$$
\begin{aligned}
(A' \cap B) \cap E &= \{0\} \cap E \\
&= \{0\} \cap \{-3,-1,1,3\} \\
&= \varnothing
\end{aligned}
$$

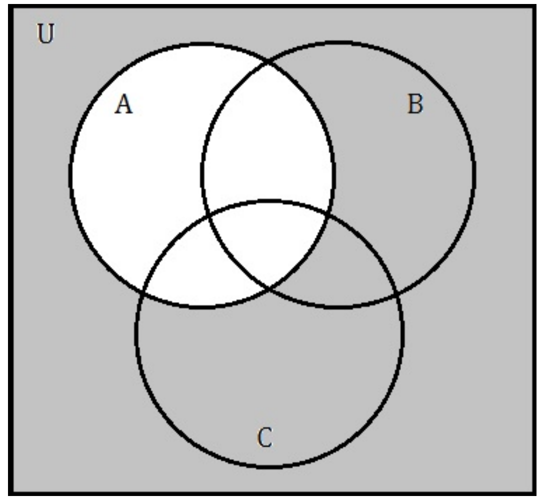
Our next example looks at sets without focusing on individual elements. To help us understand unions, intersections, and complements in our next example Venn diagrams are used; these depict the relationships between sets without necessarily focusing on specific elements. In Venn diagrams, the universal set is generally a rectangular region containing the other sets with which we're concerned. These other sets are usually depicted by circular regions but can (and sometimes must) be modeled with other shapes.

**Example 2:** Let $U$ be the set of students at South State College. Below are definitions for several sets; use these to help describe the unions, intersections, and complements of sets.

- $A$: The set of students majoring in arts and sciences.
- $B$: The set of students majoring in business administration.
- $C$: The set of students majoring in education.

(a) $A'$ (Right)

Since $A$ is the set of students majoring in the arts and sciences, $A'$ will be the set of students *not* majoring in arts and sciences. These include students with other majors, whether or not the major is mentioned in the definitions above, as well as students with no major at all. This also includes students who have two or more majors so long as none of those majors is in the arts and sciences.

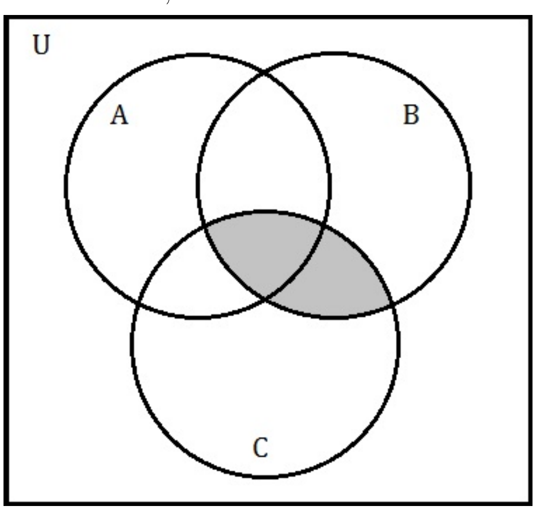


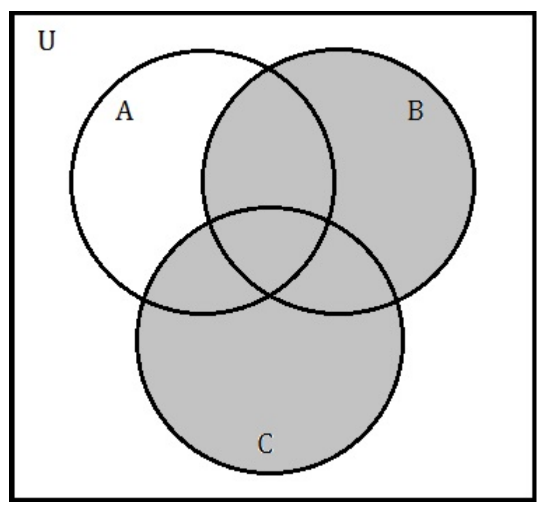

(b) $B \cap C$ (Below)

This intersection of sets holds those students majoring in both education and business administration; these students may have a third (or fourth, or fifth, etc.) major but their majors must include both education, and business administration.

(c) $B \cup C$ (Below)

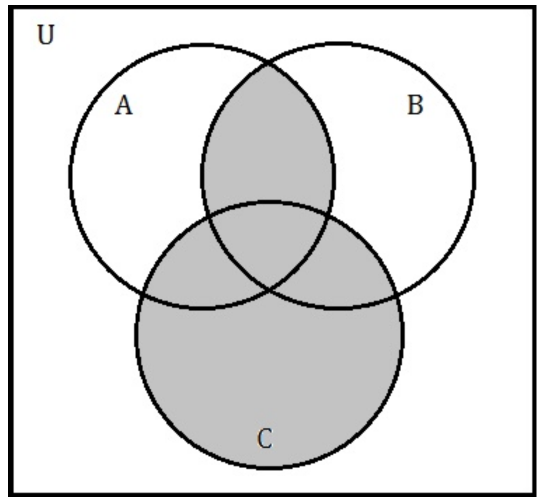
Because the union of sets includes all elements from both sets, $B \cup C$ includes any student majoring in business administration, education, or both.



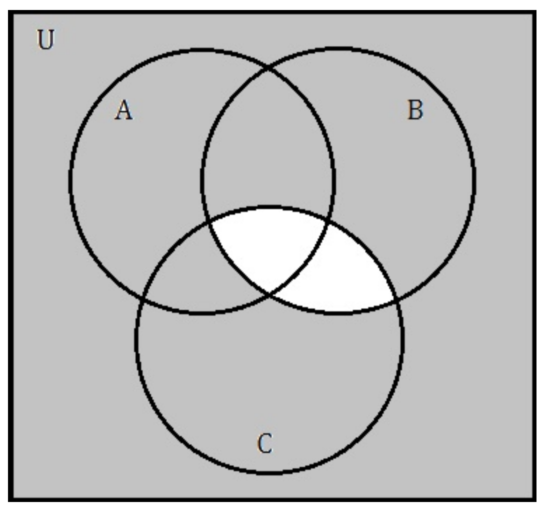
(d) $A \cap B \cup C$ (Right)

Remember that we're following an order of precedence in set theory; in this case, the intersection occurs before the union. With $A \cap B$ we have students majoring in *both* arts and sciences and business administration. This intersection is followed by a union. The final set, $A \cap B \cup C$, will include students majoring in *both* arts and sciences and business administration, students majoring in education, and student majoring in all three disciplines.




(e) $(B \cap C)'$ (Right)

Again we must follow the order of precedence. First, consider the intersection: $B \cap C$ includes those students majoring in both business administration and education. With the complement, $(B \cap C)'$, we get those students *not* majoring in both business administration and education. Again, these students may have any other major, or no major at all. This also includes business administration majors that aren't also majoring in education, as well as students majoring in education that aren't also business administration majors.



## 3.2 AND, OR, and NOT

The logical operations we'll see here act very much like the set theory operations of the last section. These operations are AND, OR, and NOT; symbolically, these are denoted $\wedge$, $\vee$, and $\neg$. In this section, the variables will represent statements rather than sets or elements. These statements must be able to take on a value of true or false. For example, the statement "it is raining" may be true or false and so may be represented by a variable. On the other hand, "the length of a rope" is not something that can be true or false and so cannot be represented by a variable in logic*. The truth of a statement can be represented by T or 1 (for true) and F or 0 (for false). The use of 1 and 0 for true and false (respectively) will help us later connect our work in logic and number systems.

While most of our work with logic will be more general, our first few examples will use specific variable definitions to help make sense of the logical operations. These introductory examples will use the following variable definitions:

- $p$: The sky is cloudy.
- $q$: It is warm outside.

The first operation to discuss here is NOT. The symbolic statement $\neg p$, read "not $p$", means "the sky is not cloudy". The NOT operation negates the statement it precedes. If a statement starts out as true then its negation is false; if a statement is originally false then its negation is true. Like the complement of set theory, NOT is in the second tier of the order of (logical) operations after grouping symbols.

To the right we see our first example of a truth table in which logical statements are at the top of a column representing the possible truth values of that statement; this truth table describes the NOT operation. From the first row, if "it is raining" is true then "it is not raining" is false. In the second row, if "it is raining" is false then "it is not raining" is true.

| $p$ | $\neg p$ |
|---|---|
| 1 | 0 |
| 0 | 1 |

Our second operation is AND. Much like an element must be in both sets for it to be in the intersection of the sets, an AND between two statements requires both statements to be true for

*This type of variable definition is, however, appropriate in algebra where a variable may model physical measurements rather than truth value.

the result to be true. For example, with $p \wedge q$, if it is in fact raining and it is warm outside then then $p \wedge q$ is also true; but, if either $p$ or $q$ is false (if it is not raining or it is not warm outside) then $p \wedge q$ is false. The truth table for this can be seen below-left.

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 0 | 0 |
| 0 | 1 | 0 |
| 0 | 0 | 0 |

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 0 | 1 |
| 0 | 1 | 1 |
| 0 | 0 | 0 |

The last operation we have is OR (above-right). The statement $p \vee q$, "it is raining or it is warm outside", is true if either or *both* of $p$ or $q$ is true. This may feel counter-intuitive; the everyday usage of "or" is actually the logical exclusive-or (XOR, denoted $\oplus$). Our OR operation does, however, resemble the union of set theory. Recall that if an element was in one or both of two given sets then that same element would be in the union of the two sets. For example, an element $x$ is not in the union of sets $A$ and $B$, $A \cup B$, only when $x$ is in neither $A$ nor $B$, much like the statement $p \vee q$ is false only when both $p$ and $q$ are false.

Moving away from these specific variable definitions for the moment, our next few examples look at how these logical operations interact and how truth tables for more complex statements can be constructed.

**Example 3:** Complete a truth table for the following logical statement: $p \vee (q \wedge r)$.

You may have noticed in the truth tables we've seen so far that every possible situation is considered. When dealing with only two variables, $p$ and $q$, the truth table included the case where both statements are true, both are false, and one is true while the other is false. Since we have three variables in this example's statement there will be even more situations to consider. To the right, we have all cases listed out: in the first row $p$, $q$, and $r$ are all true, in the second $p$ and $q$ are true while $r$ is false, et cetera.

| $p$ | $q$ | $r$ | $p \vee (q \wedge r)$ |
|---|---|---|---|
| 1 | 1 | 1 | |
| 1 | 1 | 0 | |
| 1 | 0 | 1 | |
| 1 | 0 | 0 | |
| 0 | 1 | 1 | |
| 0 | 1 | 0 | |
| 0 | 0 | 1 | |
| 0 | 0 | 0 | |

Note that there are 8 cases to consider; in general, if a statement uses $n$ variables then there will be $2^n$ rows in the statement's truth table.

The truth table above is not complete and, rather than attempting to find the truth value for each case all at once, it will be simpler if we break the larger statement into smaller pieces.

Since there are two operations being performed, AND and OR, we'll look at each case in two steps. Since there are parenthesis around $q \wedge r$ this is the first part of the statement we'll evaluate.

Remember that the AND logical operations only gives a result of "true", or 1, when both connected variables are also true. So $q \wedge r$ is true only when both $q$ and $r$ are true, but is otherwise false.

| $p$ | $q$ | $r$ | $q \wedge r$ | $p \vee (q \wedge r)$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 0 | 1 |
| 1 | 0 | 0 | 0 | 1 |
| 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 |

Once the AND operation has been performed we can consider the overall statement; in doing this we'll focus on the first and fourth columns of the truth table.

Since only one of the statement needs to be true for the OR operation to have a result of "true", the first five rows/cases make the overall statement true while the last three make the overall statement false.

**Example 4:** Complete a truth table for the following logical statement: $\neg(p \wedge q) \wedge (\neg p \vee \neg q)$.

While this statement only involves two variables there are more operations being performed. First, identifying the columns our truth table will include, we have

$$p, q, p \wedge q, \neg(p \wedge q), \neg p, \neg q, \neg p \vee \neg q, \text{ and } \neg(p \wedge q) \wedge (\neg p \vee \neg q)$$

Since there are only two variables in use we'll need only four rows. Furthermore, the first three columns will look like a truth table we've already seen, with the fourth column being the negation of the third:

| $p$ | $q$ | $p \wedge q$ | $\neg(p \wedge q)$ | $\neg p$ | $\neg q$ | $\neg p \vee \neg q$ | $\neg(p \wedge q) \wedge (\neg p \vee \neg q)$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 0 | | | | |
| 1 | 0 | 0 | 1 | | | | |
| 0 | 1 | 0 | 1 | | | | |
| 0 | 0 | 0 | 1 | | | | |

Filling in the remaining columns, we have the negation of $p$ and of $q$, an OR connecting these negations, and the overall statement. For the last column, we can compare the fourth and seventh columns.

| $p$ | $q$ | $p \wedge q$ | $\neg(p \wedge q)$ | $\neg p$ | $\neg q$ | $\neg p \vee \neg q$ | $\neg(p \wedge q) \wedge (\neg p \vee \neg q)$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 1 | 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 |
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |

## 3.3 Conditionals and Biconditionals

Two more important logical connectives are relations: the conditional, also known as an "if-then", and the biconditional, or "if and only if". The notation for these are shown below.

| Name | Notation | Example | |
|---|---|---|---|
| | | Symbolic Statement | English Translation |
| Conditional | $\Rightarrow$ or $\Leftarrow$ | $p \Rightarrow q$ or $q \Leftarrow p$ | "If $p$ then $q$", "$p$ implies $q$", or "$q$ is implied by $p$". |
| Biconditional | $\Leftrightarrow$ | $p \Leftrightarrow q$ | "$p$ if and only if $q$" or "$p$ is equivalent to $q$" |

For the most part these connectives, the conditional and the biconditional, will act exactly as you'd expect them to, given the translations above. Since the biconditional can be read "is equivalent to" it makes sense for the result of a biconditional to be true when the statements it's connecting are the same: both true or both false. This is in fact what happens. The statement $p \Leftrightarrow q$ is true when $p$ and $q$ are both true or both false; otherwise, $p \Leftrightarrow q$ is false.

The result of a conditional, on the other hand, can depend on the position of the variable. In the statement $p \Rightarrow q$, $p$ is called the premise and $q$ is the conclusion. A conditional statement is only false when the premise is true but the conclusion is false; if the premise is false then the conditional statement is vacuously true. To help explain why this is the case we'll consider $p$ and $q$ with definitions.

- $p$: John misses his final exam.
- $q$: John fails his class.

Given the variable definitions above, the statement $p \Rightarrow q$ means "if John misses his final exam then John fails his class". Below-right we see the truth table for this conditional statement, so let's go through and explain each row.

- First row: John misses his final and fails his class - the statement is true.
- Second row: John misses his final but does not fail the class - the statement is false.
- Third and fourth rows: John did not miss his final - since the premise is false, whether he passes the class or not is irrelevant to the truth of the overall statement; whether he passes the class or not may then be determined based on past performance or performance on the final. Since the stated relationship between missing a final and failing a class hasn't been contradicted, the statement is vacuously true.

| $p$ | $q$ | $p \Rightarrow q$ |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 0 | 0 |
| 0 | 1 | 1 |
| 0 | 0 | 1 |

Considering the same variable definitions, let's investigate the biconditional statement $p \Leftrightarrow q$. This statement could be read "John missing his final exam is equivalent to John failing his class".

- First and fourth rows: John misses his final and fails his class (first row) or does not miss the final and doesn't fail (fourth row) - the statement is true.
- Second and third rows: John misses his final but does not fail the class (second) or doesn't miss his final but *does* fail the class - in either case the equivalence between missing the final and failing the class has been contradicted, so the statement is false.

| $p$ | $q$ | $p \Leftrightarrow q$ |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 0 | 0 |
| 0 | 1 | 0 |
| 0 | 0 | 1 |

With these relationships, some interesting results can be obtained, which we'll see in the examples and exercises that follow. Before moving on to our examples, however, note that there is an order of precedence in logic as there was in set theory. As usual, grouping symbols are often provided to indicate precedence and avoid ambiguity, but when parenthesis are not included the order of precedence among logical operations is as follows: first, NOT ($\neg$); second, AND ($\wedge$); third, OR ($\vee$); fourth, IF-THEN ($\Rightarrow$); last, IFF ($\Leftrightarrow$).

**Example 5:** Construct a truth table for the following statement:$(p \Rightarrow q) \Rightarrow p$.

| $p$ | $q$ | $p \Rightarrow q$ | $(p \Rightarrow q) \Rightarrow p$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 |
| 0 | 1 | 1 | 0 |
| 0 | 0 | 1 | 0 |

**Example 6:** Construct a truth table for the following statement: $(\neg(p \Rightarrow q)) \Leftrightarrow (p \wedge \neg q)$.

| $p$ | $q$ | $p \Rightarrow q$ | $\neg(p \Rightarrow q)$ | $\neg q$ | $p \Rightarrow \neg q$ | $(\neg(p \Rightarrow q)) \Leftrightarrow (p \wedge \neg q)$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 0 | 0 | 1 |
| 1 | 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 | 1 | 0 | 1 |

This last example is our first *tautology*: a statement that is true under all circumstances. Since this biconditional statement is a tautology we have an alternative way to express the negation of a conditional statement using only the simpler operations AND and NOT. Our next example is one of DeMorgan's Laws and has implications in set theory. The second of DeMorgan's Laws appears in the exercise set of this chapter.

**Example 7:** Construct a truth table for the following statement: $\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$.

| $p$ | $q$ | $p \vee q$ | $\neg(p \vee q)$ | $\neg p$ | $\neg q$ | $\neg p \wedge \neg q$ | $\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 |
| 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 |
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |

Relating this back to set theory, suppose $p$ and $q$ have definitions as follows, where $A$ and $B$ are sets:

- $p$: $x \in A$
- $q$: $x \in B$

Then the statement $\neg(p \vee q)$ could be read "it is not true that $x$ is an element of $A$ or $x$ is an element of $B$". The other half of this statement, $\neg p \wedge \neg q$, could be read "$x$ is not an element of $A$ and $x$ is not an element of $B$". Because the overall biconditional statement is a tautology, the two statements, "it is not true that $x$ is an element of $A$ or $x$ is an element of $B$" and "$x$ is not an element of $A$ and $x$ is not an element of $B$", are in fact equivalent. Written in the notation of set theory, our logical statement with the given definitions gives us the following equality: $(A \cup B)' = A' \cap B'$.

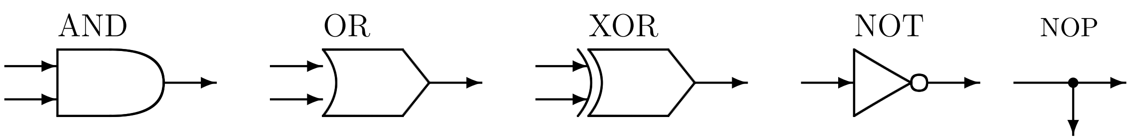
Our final example is yet another tautology. In this case we see the relationship between conditionals and a biconditional.

**Example 8:** Construct a truth table for the following statement:
$((p \Rightarrow q) \wedge (q \Rightarrow p)) \Leftrightarrow (p \Leftrightarrow q)$.

| $p$ | $q$ | $p \Rightarrow q$ | $q \Rightarrow p$ | $(p \Rightarrow q) \wedge (q \Rightarrow p)$ | $p \Leftrightarrow q$ | $((p \Rightarrow q) \wedge (q \Rightarrow p)) \Leftrightarrow (p \Leftrightarrow q)$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 | 0 | 1 |
| 0 | 1 | 1 | 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 1 | 1 | 1 | 1 |

## 3.4 Logic Circuits

Programming languages use the same kind of logical operations we've discussed here like NOT, AND, OR, and IFTHEN, as well as some we haven't seen including XOR and IFTHENELSE. These logical operations can be represented by logic gates (or a combination of logic gates) which together form logic circuits. Next are the logic gates we'll cover in this section.



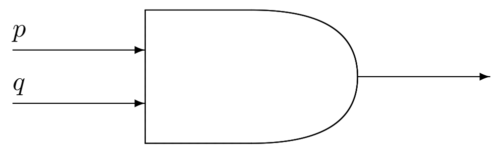

Notice that the gates AND, OR, and XOR have two arrows leading into them from the left and one exiting them from the right. The two arrows entering the gate indicate inputs or values on which the logical operation is being performed. Since AND, OR, and XOR are operations connecting two variables we have two arrows leading into the gate, while NOT and NOP operate on only one variable at a time and so need only one arrow entering the gate.

Of these gates, the operation of two have not yet been covered: the "exclusive or", denoted XOR or $\oplus$, and "no operation" or NOP. The "exclusive or" functions more in line with how the word "or" is used in conversation. For example, the logical statement "$p \oplus q$" is true when exactly one of $p$ or $q$ is true, but not when both are true. To the right we have a full truth table for this operation. While the XOR operation can be expressed using the basic operations AND, OR, and NOT, using the XOR gate will save us both time and space as we look at and construct logic circuits. The NOP gate performs no operation but is used as a splitter, taking an input and carrying it in multiple directions.

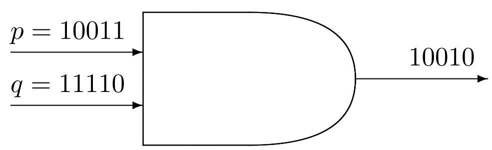
| $p$ | $q$ | $p \oplus q$ |
|---|---|---|
| 1 | 1 | 0 |
| 1 | 0 | 1 |
| 0 | 1 | 1 |
| 0 | 0 | 0 |

As we move on to investigate logic circuits keep in mind that these circuits are less restricted than the truth tables we've seen. In the truth tables, all combinations of true and false, or one and zero, are considered so that all possible outcomes are observed. With a circuit, on the other hand, an infinite number of inputs are possible and the length of the input is not fixed. For example, in a truth table for a logical statement involving two variables there are four true/false combinations to consider. In a circuit modeling the same statement the variables can both take on a single value, two values, or a dozen values; any sequence of ones and zeros can be fed into the circuit as long as all variables have the same length. For a better understanding of this, consider the logical statement $p \wedge q$.

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 0 | 0 |
| 0 | 1 | 0 |
| 0 | 0 | 0 |




Above we have the truth table for $p \wedge q$ as well as the circuit for $p \wedge q$. Unlike the truth table, which has fixed inputs (the columns for $p$ and $q$, so $p = 1100$ and $q = 1010$), the variables of the circuit are meant to take on *user defined* binary values. For example, we could let $p = 10011$ and let $q = 11110$ and the circuit would operate as follows:



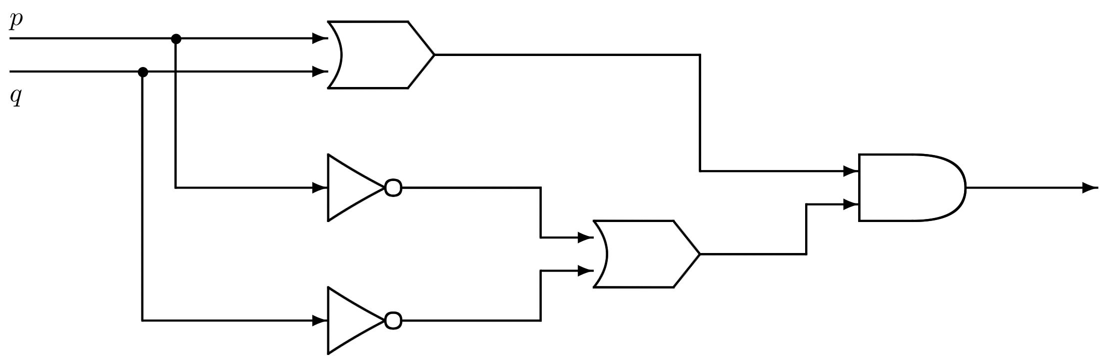

This circuit is taking in the binary values assigned to $p$ and $q$ and operating on bits with the same weight. The one's bits are 1 (for $p$) and 0 (for $q$) and so the resulting one's bit is 0; similarly, the two's bits are 1 and 1, so the resulting two's bit is 1. Because the gate operates on bits with equal weight as it operates on $p$ and $q$, the user defined values must be the same length. For example, $p = 101$ and $q = 11$ have different lengths, but $q$ can be expanded to $q = 011$; $q$ has the same value but its written length is now the same as that of $p$.

Note: In the context of circuits the reader may find it useful to think of bits not as a statement of "true" or "false", but as switches; with this mindset, a 1 represents a switch set to "on" and a 0 represents a switch set to "off".

**Example 9:** Construct a circuit for the statement $(p \vee q) \wedge (\neg p \vee \neg q)$. Find the output of the circuit if $p = 1001$ and $q = 1101$.

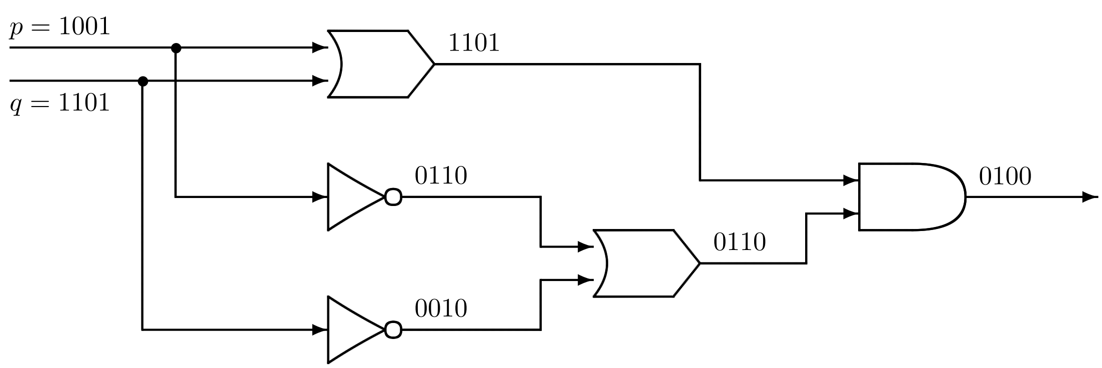
To construct a circuit modeling a logical statement, close attention to implied order is necessary. Following parenthesis and the order of precedence:

- $p$ and $q$ will be run through an OR gate,
- $p$ and $q$ will be negated and these negations run through an OR gate,
- and finally the results of the previous two steps will be run through an AND gate.




Before we feed the values of $p$ and $q$ through this circuit, take a moment to review this circuit and see how it works; check that it does in fact model the symbolic statement we started with. Can you think of a way to be sure that this is the correct circuit?

Once the values for $p$ and $q$ are chosen and fed into the circuit we can see how each gate operates on its input.




You might notice that the last circuit left a bit as true, a switch set to "on", when exactly one of $p$ or $q$ started with that same bit as true but otherwise bits were left as false, a switch set to "off". This circuit is a lengthier version of the simpler XOR gate. Symbolically:

$$((p \vee q) \wedge (\neg p \vee \neg q)) \Leftrightarrow (p \oplus q)$$

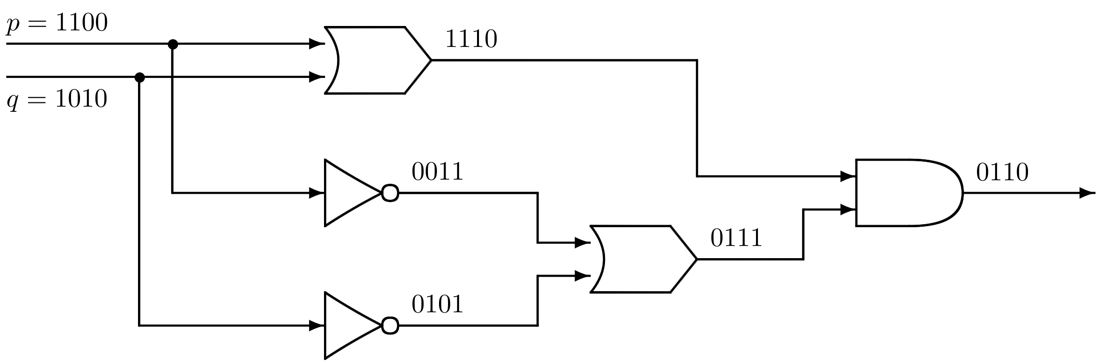
To check this equivalence, we'll answer an earlier question: how can we be sure that this is the correct circuit? Assigning the values that $p$ and $q$ would take on in a truth table and running them through the circuit, then comparing this result to the result of a truth table. If the result found in the circuit is the same as the final column of our truth table then the circuit effectively models the logical statement.

Since we're also interested in the equivalence between the given logical statement and the shortened $p \oplus q$, the truth table for both statements are provided below. Remember that in a truth table the given values for $p$ and $q$ would be 1100 and 1010 respectively.

| $p$ | $q$ | $p \vee q$ | $\neg p$ | $\neg q$ | $\neg p \vee \neg q$ | $(p \vee q) \wedge (\neg p \vee \neg q)$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 | 0 | 1 | 1 |
| 0 | 0 | 0 | 1 | 1 | 1 | 0 |

| $p$ | $q$ | $p \oplus q$ |
|---|---|---|
| 1 | 1 | 0 |
| 1 | 0 | 1 |
| 0 | 1 | 1 |
| 0 | 0 | 0 |

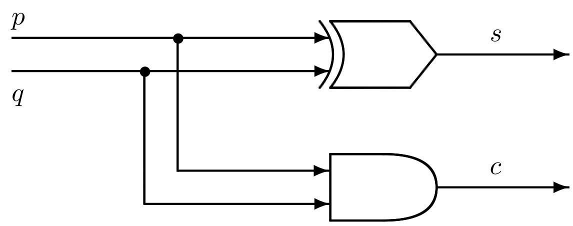
Above we can see that the final columns of the truth tables are identical; therefore, the logical statements are equivalent. Now let's compare these results to those of the circuit:




The results of the circuit are identical to those of the truth table for the statement the circuit was meant to model; this is the correct circuit.

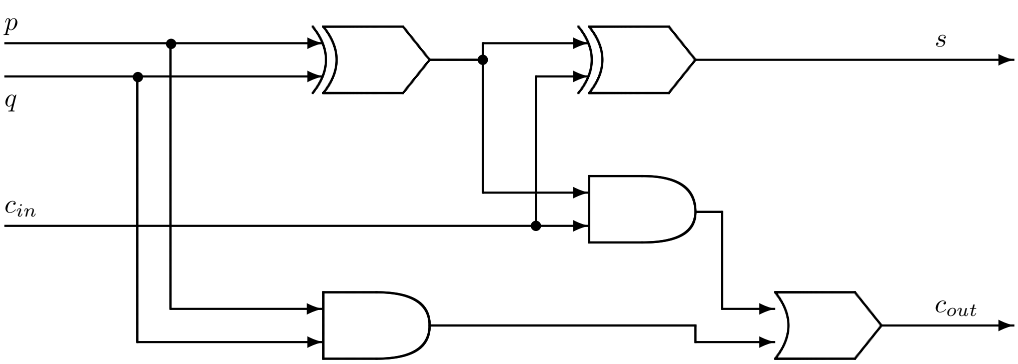
**Example 10:** Describe the operation of the circuit below assuming single-bit values are assigned to $p$ and $q$.

Restricting this circuit to take in only single-bit values makes it incredibly useful, operating much like something you've likely used many times: a calculator. At least, a very, very simple calculator.

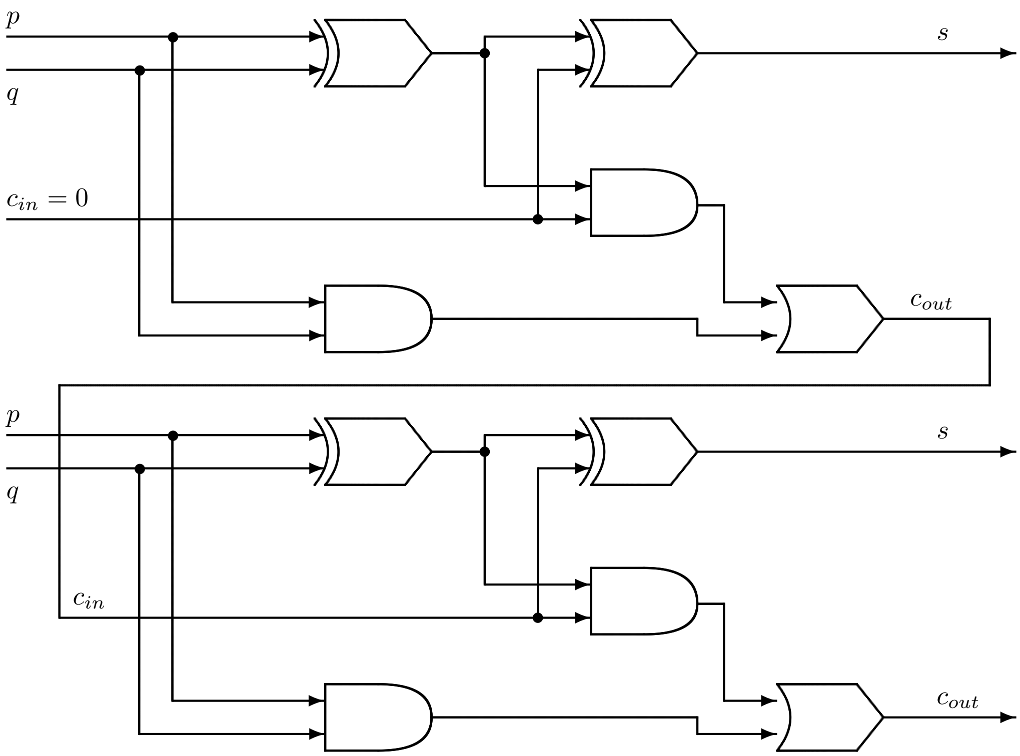



The last circuit is called a "half-adder", performing binary addition on single bit numbers, even carrying when necessary. Two single-bit numbers, $p$ and $q$, go in and a two-bit number comes out; concatenating $c$ and $s$, $p + q = cs$ where $s$ is the sum without carry and $c$ is the carried bit. For example, if $p = 1$ and $q = 1$ then $s = 0$ and $c = 1$, so that $1 + 1 = 10$. This simple half-adder can be expanded on to create a full-adder, and stringing together several full-adders facilitates addition of multiple-bit binary numbers.

Below we have a full-adder. Notice that there are two-versions of the carry variable here: $c_{in}$ and $c_{out}$. The first of these, $c_{in}$, is the value carried from a previous step of addition (from the preceding full-adder), while $c_{out}$ is the carrying occurring within a full-adder. Since full-adders are meant to be strung together to perform addition on multiple-bit binary numbers, $c_{out}$ from one full-adder leads to $c_{in}$ in a succeeding full-adder.




To bring full-adders together and perform addition on multiple-bit binary numbers, two full-adders are drawn with $c_{out}$ from one adder connecting to $c_{in}$ in the succeeding adder as shown below. Since no addition is performed before $p$ and $q$ are fed into the circuit, the initial $c_{in}$ starts as zero.




Notice that the top half of this diagram must be operating on the one's bits of $p$ and $q$ and the resulting sum ($s$) is the one's bit of the sum of $p$ and $q$. The bottom half is operating on the two's bits of $p$ and $q$ and the resulting $s$ is the two's bit of the sum of $p$ and $q$, with the final value of $c_{out}$ being the four's bit.

These full-adders can continue to be strung together to add binary numbers of as great a length as desired. To add $n$-bit binary numbers, $n$ full-adders must be strung together.

## 3.5 Exercises

### 3.5.1 Set Theory

Given the definitions of sets $A$, $B$, $C$, $D$, $E$, and the universal set $U$ (provided below), define the indicated set.

- $A = \{0, 1, 2, 3, 5, 8, 13\}$
- $B = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- $C = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18\}$
- $D = \{2, 3, 5, 7, 11, 13, 17, 19\}$
- $E = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$
- $U = \{x : x \in \mathbb{Z}, 0 \leq x \leq 19\}$

1. $B'$
2. $A \cup C$
3. $B \cap D$
4. $(C')'$
5. $(A \cup B)'$
6. $A' \cap B'$
7. $E' \cup D'$
8. $(E \cap D)'$
9. $E \cap C$
10. $(B' \cap D)'$
11. $B \cup D'$
12. $A \cap (B \cup C)$
13. $(A \cap B) \cup (A \cap C)$
14. $B \cup (C \cap D)$
15. $(B \cup C) \cap (B \cup D)$

Given the definitions for sets $A$, $B$, $C$, and the universal set $U$, define the indicated set.

- $U$: U.S. based companies
- $A$: Companies with more than 20 employees
- $B$: Companies based in Washington state.
- $C$: Companies whose annual revenue is at least $1,000,000.

16. $A'$
17. $B'$
18. $C'$
19. $A \cap B$
20. $B \cup C$
21. $A \cap B \cap C$
22. $(A \cup B)'$
23. $(A \cap C) \cup B$
24. $A \cap (C \cup B)$
25. $B \cup (A \cap C)'$

### 3.5.2 AND, OR, and NOT

Construct a truth table for the given logical statement.

26. $p \vee q$
27. $p \wedge q$
28. $\neg p$
29. $\neg(\neg p)$
30. $\neg(\neg(\neg p))$
31. $(p \vee q) \vee r$
32. $p \vee (q \vee r)$
33. $(p \wedge q) \wedge r$
34. $p \wedge (q \wedge r)$
35. $p \vee (q \wedge r)$
36. $(p \vee q) \wedge r$
37. $\neg(p \wedge q)$
38. $\neg(p \vee q)$
39. $\neg p \vee \neg q$
40. $\neg p \wedge \neg q$
41. $\neg(p \wedge \neg p)$
42. $(p \wedge q) \vee (p \wedge r)$
43. $(\neg p \wedge q) \wedge (p \wedge \neg q)$
44. $\neg(p \vee \neg(q \wedge r))$

For problems 45-49, assume the following variable definitions: $p$ - the house includes at least three bedrooms, $q$ - the house includes at least two full bathrooms, and $r$ - the house costs at most \$180,000.

Write the given symbolic statement in English.

45. $\neg r$
46. $p \vee q$
47. $\neg p \wedge r$
48. $p \wedge q \wedge r$
49. $\neg((p \wedge q) \vee r)$

For problems 50-54 use the set definitions and logical variable definitions provided below.

- $U = \{x : x \in \mathbb{N}, x < 10\}$
- $A = \{2, 4, 6, 8\}$
- $B$ is the set of prime numbers less than 10.
- $p$: $x \in A$
- $q$: $x \in U$
- $r$: $x \in B$

Find all values of $x$ in $U$, if any exist, which make the given statement true.

50. $q$
51. $\neg r$
52. $p \wedge r$
53. $\neg q \wedge p$
54. $\neg p \vee r$

### 3.5.3 Conditionals and Biconditionals

Construct a truth table for the given logical statement; identify any tautology as such.

55. $p \Rightarrow p$ 56. $p \Leftrightarrow p$ 57. $p \Rightarrow q$ 58. $p \Leftarrow q$ 59. $p \Leftrightarrow q$

60. $(p \Rightarrow q) \wedge (p \Leftarrow q)$ 61. $(p \wedge \neg q) \Rightarrow \neg q$ 62. $p \Rightarrow (q \Rightarrow p)$ 63. $(p \Rightarrow q) \Rightarrow p$

64. $\neg(q \Rightarrow p) \Rightarrow \neg p$ 65. $(p \Rightarrow \neg p) \Rightarrow p$ 66. $((p \Rightarrow q) \Rightarrow p) \Rightarrow p$

67. $(p \Rightarrow r) \Rightarrow ((p \Rightarrow q) \Rightarrow r)$ 68. $(q \Rightarrow r) \Rightarrow ((p \Rightarrow q) \Rightarrow r)$ 69. $(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$

70. $((p \vee q) \vee r) \Leftrightarrow (p \vee (q \vee r))$ 71. $((p \wedge q) \wedge r) \Leftrightarrow (p \wedge (q \wedge r))$

72. $((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$ 73. $((p \Rightarrow q) \wedge (p \Rightarrow r)) \Leftrightarrow (p \Rightarrow (q \wedge r))$

74. $(p \wedge (q \vee r)) \Leftrightarrow ((p \vee q) \wedge (p \vee r))$ 75. $(p \vee (q \wedge r)) \Leftrightarrow ((p \wedge q) \vee (p \wedge r))$

76. $\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$

### 3.5.4 Logic Circuits

For problems 77-88, draw the circuit modeling the given logical statement.

77. $p \wedge q$ 78. $p \vee q$ 79. $\neg p$ 80. $p \oplus q$ 81. $\neg(p \oplus q)$

82. $\neg(p \wedge q)$ 83. $\neg p \vee \neg q$ 84. $p \vee (q \wedge r)$ 85. $(p \wedge q) \vee (p \wedge r)$

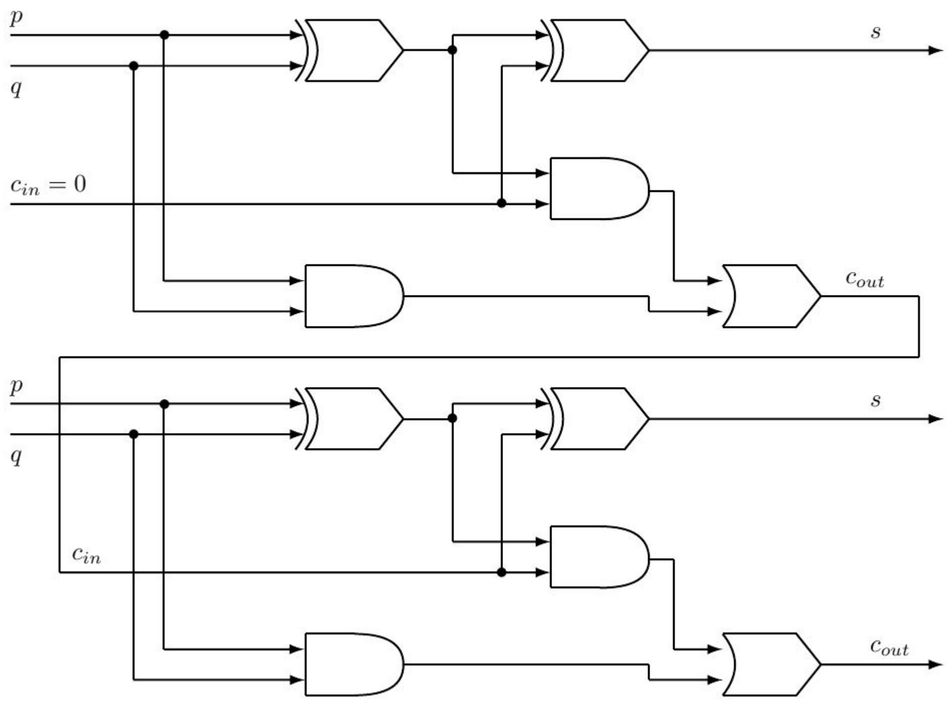
86. $(p \oplus q) \wedge r$ 87. $(\neg(p \wedge q)) \wedge (p \vee q)$ 88. $(\neg(p \wedge q)) \wedge (p \vee q) \wedge r$

89. Construct a circuit that would give the opposite (in two's complement notation) of a signed, two-bit number ($00_2 = 0$, $01_2 = 1$, $10_2 = -2$, or $11_2 = -1$). (Hint: Modify the two-bit adder discussed at the end of this section. Let $p$ be the entered two-bit number and fix $q$.)

90. Construct a circuit that would give the opposite (in two's complement notation) of a signed, three-bit number. (Hint: Attach a third full-adder to the circuit below. See hint for previous problem.)

91. The last circuit illustrated in this chapter (shown below) could add two-bit binary numbers ($p$ and $q$). However, the different bits need to be stripped from $p$ and $q$ and entered into the

appropriate places, as explained following the initial appearance of this circuit. Furthermore, the sums of these bits are concatenated to get the actual sum of $p$ and $q$. How could this circuit be modified so that a user could enter in only $p$ and $q$ and have the circuit use the appropriate bits in the correct places? How could this same circuit be modified to give the full sum of $p$ and $q$ rather than the bits of the sum? (Hint: Modify the front end of the provided circuit with AND's and fixed values, and back end of the circuit with the appropriate logical operation.)



# Chapter 4

# Algebra

## 4.1 Writing Equations

The operations and relations expressed in equations and inequalities all have their own terminology. Below you can find some of the language associated to certain operations and relationships.

- Addition: more than, greater than, added to, sum of, increased by, etc.
- Subtraction: fewer than, less than, subtracted from, difference of, etc.
- Multiplication: (a fraction/percentage) of, (a number) times greater, multiplied by, product of, etc.
- Division: (a fraction) of, divided by/into, quotient of, per, etc.
- Equality: is, is the same as, is equal to, has the same value as, etc.
- Inequalities - the key word here (and for any relation) to distinguish an inequality from an operation is the word "is".
  (a) $>$: is greater than, is more than, etc.
  (b) $\geq$: is greater than or equal to, is at least, is no less than, is not less than, etc.

(c) $<$: is less than, is under, etc.

(d) $\leq$: is less than or equal to, is at most, is no larger than, is not larger than, etc.

**Example 1:** Express each sentence symbolically, as an equation.

(a) The sum of eight and fourteen is twenty-two.

"The sum of" means we will be adding. "The sum of eight and fourteen" tells us the numbers to be added are 8 and 14. The word "is" translates to equality, so "is twenty-two" translates to "=22":

$$8 + 14 = 22$$

(b) Twelve is equal to three more than nine.

This time we start with "Twelve is", which translates to "12=". Then we have "three more than nine"; since "more than" translates to addition, the phrase "three more than nine" means we'll add three to nine.

$$12 = 9 + 3$$

Since addition is commutative, we could also write $12 = 3 + 9$. However, understanding the implied order is important, especially with regards to subtraction and division.

(c) Eighteen is the same as six less than twenty-four.

Similar to the last example, "Eighteen is" translates directly into symbolic form: "18=". The phrase "six less than twenty-four" is also similar to what we saw in the previous problem, but here order will be more important. With "less than" implying subtraction, "six less than twenty-four" translates to "24-6".

$$18 = 24 - 6$$

(d) The difference of nine and thirteen is negative four.

We'll also be writing out subtraction in this equation, implied by "difference". Unlike the previous problem, however, the order of subtraction implied by the word "difference" is left to right; "the difference of nine and thirteen" translates to "9-13". As before, "is" means equality, so our equation is

$$9 - 13 = -4$$

(e) The product of six and nine and a half is fifty-seven.

Since "product" implies multiplication, "product of nine and thirteen" gives us "9*13". As usual, "is" implies equality, so this time we get the equation

$$6 * 9\frac{1}{2} = 57$$

(f) A third of twenty-four is eight.

This we could read in a couple of ways. The most obvious is to translate "of" as multiplication so that "a third of twenty-four" becomes "$\frac{1}{3} * 24$". However, this could also be understood as dividing twenty-four by three: "$24 \div 3$". Then, of course, "is eight" becomes "=8".

$$\frac{1}{3} * 24 = 8 \qquad \text{or} \qquad 24 \div 3 = 8$$

(g) The quotient of six and nine is two thirds.

This is another situation where order is important. Since "quotient" implies division, "the quotient of six and nine" will translate to division of the stated numbers, read left to right: "$6 \div 9$". Then our equation is

$$6 \div 9 = \frac{2}{3}$$

(h) Eight times the sum of five and four is seventy-two.

This sentence implies multiple operations: "eight times" means we'll be multiplying something by eight, while "the sum of five and four" means we'll be adding five and four. Since the sentence starts with "eight times the sum", it's the sum (or result of addition) that will be multiplied by eight, so we have to be sure that the addition is performed *before* the multiplication. This can be accomplished with parenthesis.

$$8 * (5 + 4) = 72$$

Another key to writing equations and inequalities is identifying and differentiating between constants and variables; whenever a numerical quantity is described but not explicitly given, that unknown numerical quantity can be represented by a variable. In applications or story problems, where there is a contextual mean to a variable, that variable should be defined.

**Example 2:** Identify the quantity we should represent symbolically as a variable.

(a) The product of a number and twelve is eight times the sum of the number and one.

In this case the nonspecific "a number" is our unknown and would be represented by a variable.

(b) The quotient of 8 and a number is equivalent to 0.125.

Again, "a number" is what would be represented by a variable in the symbolic equivalent of this sentence.

(c) John is 6 years older than Tim.

Here we see unknown quantities with contextual meaning. There are two unknowns in this situation: John's age and Tim's age. This is where a clear definition is necessary; letting a variable represent "age" is not enough, because there are two unknown ages. Instead we'll need two variables, one representing Tim's age and the other representing John's age.

(d) Jenny bought a dozen donuts; some were plain and the rest had sprinkles.

Again we have two unknowns: the number of plain donuts and the number of donuts with sprinkles. As in the last example, we'll need two variables, one for each unknown quantity.

(e) Joseph spent several hours writing code and didn't compile until he was finished; writing the code took one-hundred times the time it took for the code to compile.

In this situation, one variable would be used to represent the time it took for the code to compile and another would be used to represent the time it took Joseph to write the code.

**Example 3:** For each situation, write an equation and define your variable(s).

(a) The product of a number and twelve is eight times the sum of the number and one.

This sentence is much like those we saw in Example 1. The parts of this sentence can be translated directly into symbols, where "a number" is expressed as a variable.

$$n\text{: a number}$$
$$n * 12 = 8 * (n + 1)$$

(b) The quotient of 8 and a number is equivalent to 0.125.

As in the previous problem, this sentence can be translated directly into symbols with "a number" being replaced by our variable.

$$n\text{: a number}$$
$$8 \div n = 0.125$$

(c) A young river birch is twelve feet tall, seventeen feet shorter than a white birch.

In this situation we can use a single variable; while two heights are mentioned, one is known and the other is not. Specifically, the height of the white birch is not given, so our variable will represent the height of the white birch (in feet). Since the river birch is seventeen feet shorter than the white birch, the height of the river birch is the same as seventeen subtracted from the height of the white birch.

$$x\text{: height of white birch (ft)}$$
$$12 = x - 17$$

(d) John is 6 years older than Tim.

This time we have context to deal with. We already know how the variables should be defined, but we need to understand what operation(s) to use. The words "six years older than Tim" can most easily be translated adding six to Tim's age.

$$x\text{: John's age}$$
$$y\text{: Tim's age}$$
$$x = y + 6$$

(e) Jenny bought a dozen donuts; some were plain and the rest had sprinkles.

This time we have context but no words associated to operations or relations. However, since Jenny bought a dozen donuts, either plain or with sprinkles, the total of these two types will be twelve. In other words, if we add the number of plain donuts to the number of donuts with sprinkles we would get twelve.

$x$: number of plain donuts

$y$: number of donuts with sprinkles

$$x + y = 12$$

This can be understood in another way. If there are $x$ plain donuts and 12 donuts total, then there are $12 - x$ donuts left with sprinkles.

$$y = 12 - x$$

(f) Joseph spent several hours writing code and didn't compile until he was finished; writing the code took one-hundred times the time it took for the code to compile.

Again we already have the variable definitions, but this time there are some words implying specific operations. The phrase "one-hundred times" translates to multiplication by 100. Then this sentence could be rewritten as "the number of hours for writing the code is one-hundred times the number of hours for compiling".

$x$: hours for compiling

$y$: hours for writing code

$$y = 100 * x$$

## 4.2 Solving Equations

In this section we'll be solving linear equations and proportions. To solve an equation is to find its solution(s), and a *solution* of a single variable equation is any value that, when replacing the variable, makes the equation a true statement. Before we actually attempt to solve an equation, let's get a better understanding of what a solution is.

**Example 4:** Is the given value a solution of the given equation?

(a) 5; $2x - 7 = 3$

To see if 5 is a solution of this equation, we will replace $x$ with 5. If the resulting equation is true then 5 is a solution, other wise 5 is not a solution.

$$\begin{aligned} 2x - 7 &= 3 && \text{(The original equation.)} \\ 2(5) - 7 &= 3 && \text{(Substituting 5 in for } x.\text{)} \\ 10 - 7 &= 3 && \text{(Simplifying both sides of the equation.)} \\ 3 &= 3 && \text{(Complete simplification.)} \end{aligned}$$

The final line in the sequence of equations above is true, therefore 5 is a solution to the original equation.

(b) 21; $\dfrac{3}{x} = \dfrac{7}{1}$

Taking the same approach, we'll replace $x$ with 21 and see if the resulting equation is true.

$$\begin{aligned} \frac{3}{x} &= \frac{7}{1} && \text{(The original equation.)} \\ \frac{3}{21} &= \frac{7}{1} && \text{(Substituting 21 for } x.\text{)} \\ \frac{1}{7} &= 7 && \text{(Simplifying both sides of the equation.)} \end{aligned}$$

Since $\frac{1}{7}$ and 7 are not equal, the last line in the sequence of equations above is false. Therefore 21 is not a solution to the original equation.

(c) 1.09; $2(x-3) + x = 1 - (7 - 3x)$

$$\begin{aligned} 2(x-3) + x &= 1 - (7 - 3x) && \text{(The original equation.)} \\ 2(1.09 - 3) + 1.09 &= 1 - (7 - 3(1.09)) && \text{(Substituting 1.09 for } x.\text{)} \\ 2(-1.91) + 1.09 &= 1 - (7 - 3.27) && \text{(Simplifying both sides of the equation.)} \\ -3.82 + 1.09 &= 1 - (3.73) \\ -2.73 &= -2.73 \end{aligned}$$

Since the above equation is true, 1.09 is a solution of the original equation.

With a better understanding of what it means for a number to be a solution to an equation, let's begin solving some equations. This will require a couple of properties of equality:

- Addition Property of Equality: Given an equation, any single value can be added to (or subtracted from) each side of the equation and the result is an equivalent equation. Symbolically, $(a = b) \Rightarrow (a + c = b + c)$.
- Multiplication Property of Equality: Given an equation, each side of the equation can be multiplied by (or divided by) a single, non-zero value and the result is an equivalent equation. Symbolically, $(a = b) \Rightarrow (a * c = b * c)$

**Example 5:** Solve the given equation. (Find all solutions of the equation.)

(a) $12x = 96$

To solve this equation we will use the properties of equality to isolate $x$ on one side of the equation, with all other terms on the other side of the equality sign.

Right now, $x$ is being multiplied by 12; since we want to get $x$ by itself we need to remove the multiplication by 12. Since the opposite of multiplication by 12 is division by 12, let's divide *both sides of the equation* by 12:

$$
\begin{aligned}
12x &= 96 && \text{(Original equation.)} \\
\frac{12x}{12} &= \frac{96}{12} && \text{(Divide both sides by 12.)} \\
x &= 8 && \text{(Simplify both sides of the equation.)}
\end{aligned}
$$

This means that 8 is the solution of the original equation! To check our work, we could repeat the process we saw in Example 4.

While the details of our work in solving equations will vary from equation to equation, the basic idea used here will be used throughout this section. To solve an equation, we'll gather our variable to one side of the equality sign and, through opposite operations, undo the operations surrounding the variable.

(b) $x + 74 = 52$

In this case 74 is being added to $x$; to undo this, we'll subtract 74 from both sides of the equation.

$$\begin{aligned} x + 74 &= 52 && \text{(Original equation.)} \\ x + 74 - 74 &= 52 - 74 && \text{(Subtract 74 from both sides.)} \\ x &= -22 && \text{(Simplify both sides.)} \end{aligned}$$

Therefore, $-22$ is the solution.

(c) $\dfrac{x}{3} - 4 = 11$

This time we have two operations affecting $x$: division by 3 and subtraction by 4. Since there are multiple operations, this could be approached in multiple ways; below we see two methods of solving the equation, with each method finding the correct solution of 45.

$$\begin{aligned} \frac{x}{3} - 4 &= 11 && \text{(Original equation.)} \\ \frac{x}{3} - 4 + 4 &= 11 + 4 && \text{(Add 4.)} \\ \frac{x}{3} &= 15 && \text{(Simplify.)} \\ \frac{x}{3} * 3 &= 15 * 3 && \text{(Multiply by 3.)} \\ x &= 45 && \text{(Simplify.)} \end{aligned}$$

$$\begin{aligned} \frac{x}{3} - 4 &= 11 && \text{(Original equation.)} \\ \left(\frac{x}{3} - 4\right) * 3 &= 11 * 3 && \text{(Multiply by 3.)} \\ x - 12 &= 33 && \text{(Simplify.)} \\ x - 12 + 12 &= 33 + 12 && \text{(Add 12.)} \\ x &= 45 && \text{(Simplify.)} \end{aligned}$$

In each equation of the last example, the variable appeared only once. Since not all equations start off this way, a common step in solving equations will be to combine our variable terms into one.

**Example 6:** Solve each equation.

(a) $2x - 5 = 5(x - 1)$

Since $x$ appears twice in this equation, we'll need to combine them into a single term. However, $x$ cannot be subtracted from inside the parentheses. First, we'll simplify both sides of the equation.

$$\begin{aligned}
2x - 5 &= 5(x-1) && \text{(Original equation.)}\\
2x - 5 &= 5x - 5 && \text{(Simplify.)}\\
2x - 5 - 5x &= 5x - 5 - 5x && \text{(Subtract } 5x\text{.)}\\
-3x - 5 &= -5 && \text{(Simplify (combine like terms))}\\
-3x - 5 + 5 &= -5 + 5 && \text{(Add 5.)}\\
-3x &= 0 && \text{(Simplify.)}\\
(-3x) \div (-3) &= 0 \div (-3) && \text{(Divide by -3.)}\\
x &= 0 && \text{(Simplify.)}
\end{aligned}$$

(b) $\dfrac{2-x}{5} = \dfrac{x+1}{3}$

This is an example of a proportion: a statement of equality between two ratios. A common first step in solving proportions is to *cross-multiply*, which is a special case of the multiplication property of equality. Given a proportion $\frac{a}{b} = \frac{c}{d}$, cross-multiplication is the rewriting of the proportion as $d * a = b * c$.

$$\begin{aligned}
\frac{2-x}{5} &= \frac{x+1}{3} && \\
3(2-x) &= 5(x+1) && \text{(Cross-multiply.)} \\
6-3x &= 5x+5 && \text{(Simplify.)} \\
6-3x+3x &= 5x+5+3x && \text{(Add } 3x.\text{)} \\
6 &= 8x+5 && \text{(Simplify.)} \\
6-5 &= 8x+5-5 && \text{(Subtract 5.)} \\
1 &= 8x && \text{(Simplify.)} \\
\frac{1}{8} &= \frac{8x}{8} && \text{(Divide by 8.)} \\
\frac{1}{8} &= x && \text{(Simplify.)}
\end{aligned}$$

We've already seen a few examples of equations of two variables; these cannot be solved in that we cannot use the two properties of equations we've seen alone to find solutions. However, we can solve such *literal equations* for a particular variables, meaning we can isolate one of the variables on one side of the equation.

**Example 7:** Solve the following equation for y: $2x - 6y = 27$.

$$\begin{aligned}
2x-6y &= 27 && \text{(original equation.)} \\
2x-6y-2x &= 27-2x && \text{(Subtract 2x.)} \\
-6y &= 27-2x && \text{(Simplify.)} \\
\frac{-6y}{-6} &= \frac{27-2x}{-6} && \text{(Divide by -6.)} \\
y &= -\frac{9}{2}+\frac{x}{3} && \text{(Simplify.)}
\end{aligned}$$

Again, solving an equation like this for a given variable will not result in a numerical solution the way solving a single variable equations does. In fact, the equations we've seen with two variables have an infinite number of solutions taking the form of ordered pairs, as we'll discuss in the sections ahead.

**Example 8:** For each situation write an equation and, if possible, solve the equation. If it is not possible to solve, explain why.

(a) The quotient of 8 and a number is equivalent to 0.125.

In this case we have a single unknown: "a number". Using a variable to represent this unknown, the sentence translates to

$$\frac{8}{n} = 0.125$$

Since we have a single variable, we could solve this equation. With the variable in the denominator, it may be useful to view the equation as a proportion:

$$\frac{8}{n} = \frac{0.125}{1}$$

Now we can solve through cross-multiplication and the other tools we've discussed.

$$\begin{aligned}
\frac{8}{n} &= \frac{0.125}{1} \\
8 * 1 &= 0.125 * n && \text{(Cross-multiply)} \\
8 &= 0.125n && \text{(Simplify)} \\
\frac{8}{0.125} &= \frac{0.125n}{0.125} && \text{(Divide by 0.125)} \\
64 &= n
\end{aligned}$$

(b) The product of a number and twelve is eight times the sum of the number and one.

Again, our variable is represent the single unknown: "a number". Then the sentence above translates to

$$n * 12 = 8 * (n + 1)$$

This can be solved using our properties of equations:

$$\begin{aligned} n * 12 =& 8 * (n+1) \\ 12n =& 8n + 8 \\ 12n - 8n =& 8n + 8 - 8n \\ 4n =& 8 \\ \frac{4n}{4} =& \frac{8}{4} \\ n =& 2 \end{aligned}$$

(c) A young river birch is twelve feet tall, seventeen feet shorter than a white birch.

A statement we saw earlier, we already have an equation for this problem. With $x$ representing the height of the white birch, our equation is

$$12 = x - 17$$

This can be solved simply by adding 17 to both sides of the equation.

$$\begin{aligned} 12 =& x - 17 \\ 12 + 17 =& x - 17 + 17 \\ 29 =& x \end{aligned}$$

Since $x$ represents the height of the white birch, this solution means that the white birch is 29 feet tall.

(d) John is 6 years older than Tim.

Another problem we saw earlier, we arrived at the equation $x = y + 6$, where $x$ is John's age and $y$ is Tim's age. However, because there are two variables we cannot solve this equation and come up with a single value for $x$ or for $y$. In fact, there are an infinite number of numerical combinations for $x$ and $y$ that would make the equation true, as we'll see in the next section.

(e) Jenny bought a dozen donuts; some were plain and the rest had sprinkles; there were twice as many donuts with sprinkles as there were plain.

A slight variation on a problem we saw earlier, we won't need two variables for this situation. Suppose we stick with our first variable definition, letting $x$ represent the number of plain donuts. Then the number of sprinkled donuts can be represented using $12 - x$.

The problem states that "there were twice as many donuts with sprinkles as there were plain." Then two times the number of plain donuts is the same as the number of sprinkled donuts, or

$$2x = 12 - x$$

We can solve this in just a couple of steps:

$$\begin{aligned} 2x &= 12 - x \\ 2x + x &= 12 - x + x \\ 3x &= 12 \\ \frac{3x}{3} &= \frac{12}{3} \\ x &= 4 \end{aligned}$$

This solution means that there were 4 plain donuts and, since Jenny bought $12 - x$ donuts with sprinkles, there were 8 sprinkled donuts.

(f) Joseph spent several hours writing code and didn't compile until he was finished; writing the code took one-hundred times the time it took for the code to compile. It took half an hour for the code to finish compiling.

The equation we came up with for this problem in the previous section was

$$y = 100x$$

However, this time we know that it took half an hour for the code to finish compiling. Since $x$ represented the hours it took to compile the code and $y$ represented the number of hours Joseph spent writing the code, we can replace $x$ with 0.5 (don't use 30; $x$ is measuring time in hours, not minutes).

$$\begin{aligned} y &= 100x \\ &= 100 * 0.5 \\ &= 50 \end{aligned}$$

Then $y = 50$ means that it took Joseph 50 hours to write his code.

## 4.3 Two Variable Linear Equations

By now you've seen a few equations that involved more than one variable; these have been linear equations of two variables. Formally, a linear equation of two variables is any equation of two variables that can be written in *standard linear form*, $Ax + By = C$. However, most of the linear equations we'll deal with in this section will be in *slope-intercept form*, $y = mx + b$.

Linear equations of two variables have an infinite number of solutions, with each solution taking the form of an *ordered pair*, written $(x, y)$. The definition of solution does not change much as we move from single-variable to two-variable equations; an ordered pair is a solution to a given equation when replacing the variables with the values makes the equation a true statement.

**Example 9:** Of the ordered pairs $(1, 2)$, $(0, 0)$, $(-2, 5)$, and $(-6, -8)$, determine which are solutions to the equation $y = -x + 3$.

- $(1, 2)$ is a solution because substituting $x$ with 1 and $y$ with 2 turns the equation into a true statement:

$$\begin{aligned} &y = -x + 3 \\ &\hookrightarrow (2) = -(1) + 3 \\ &\hookrightarrow 2 = -1 + 3 \\ &\hookrightarrow 2 = 2 \end{aligned}$$

- $(0, 0)$ is *not* a solution because substituting $x$ with 0 and $y$ with 0 turns the equation into a false statement:

$$y = -x + 3$$
$$\hookrightarrow (0) = -(0) + 3$$
$$\hookrightarrow 0 = -0 + 3$$
$$\hookrightarrow 0 = 3$$

- $(-2, 5)$ is a solution because substituting $x$ with $-2$ and $y$ with 5 turns the equation into a true statement:

$$y = -x + 3$$
$$\hookrightarrow (5) = -(-2) + 3$$
$$\hookrightarrow 5 = 2 + 3$$
$$\hookrightarrow 5 = 5$$

- $(-6, -8)$ is *not* a solution because substituting $x$ with $-6$ and $y$ with $-8$ turns the equation into a false statement:

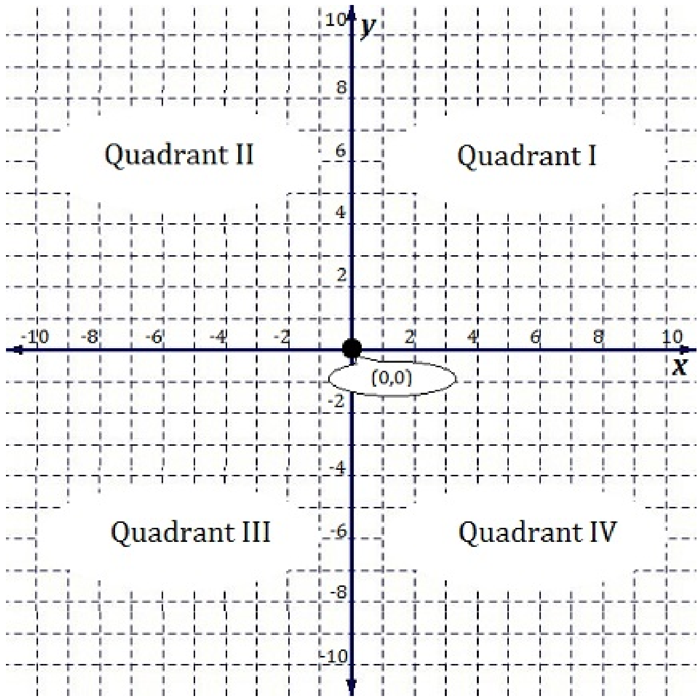
$$y = -x + 3$$
$$\hookrightarrow (-8) = -(-6) + 3$$
$$\hookrightarrow -8 = 6 + 3$$
$$\hookrightarrow -8 = 9$$

Unlike the single variable equations we saw in the last section, linear equations of two variables will have an infinite number of solutions. Since a linear equation of two variables has infinitely many solutions, it is convenient to give a graphical representation of the solution set of an equation. To do this, we utilize the Cartesian Coordinate System. Some important parts of system are described below.

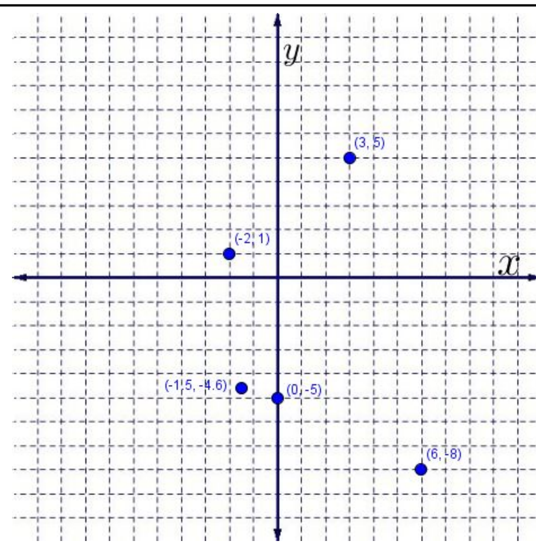
- Axes - An $x$-axis acts as a number line for $x$-values and a $y$-axis acts as a number line for $y$-values. Most commonly, the $x$-axis is horizontal while the $y$-axis is vertical, with the two axes intersecting when both $x$ and $y$ are zero.

- Coordinates - Any point in this plane can be described by an ordered pair with an $x$- and $y$-coordinate. For example, where the two axes cross, called the origin, is described by $(0, 0)$; in this case both the $x$-coordinate and the $y$-coordinate are 0. In general, in an ordered pair $(x, y)$ the first value is the $x$-coordinate and the second is the $y$-coordinate.

- Quadrants - The axes split the plane into four parts, called quadrants, as depicted in the graphic below.




**Example 10:** Plot and label the following points: $(-2, 1)$, $(3, 5)$, $(6, -8)$, $(0, -5)$, and $(-1.5, -4.6)$. (In the graph to the right, both $x$- and $y$-axes are scaled by 1's.)




Now, one of the defining characteristics of a linear equation is that the graphical representation of its solution set is a line; since two distinct points define a unique line, if we know two solutions of an equation we can give an accurate depiction of its solution set. While two is the minimal number of points necessary, finding three points is better because it gives you a way to check your work; if the three points do not fall on a single line then a mistake was made in finding the points.

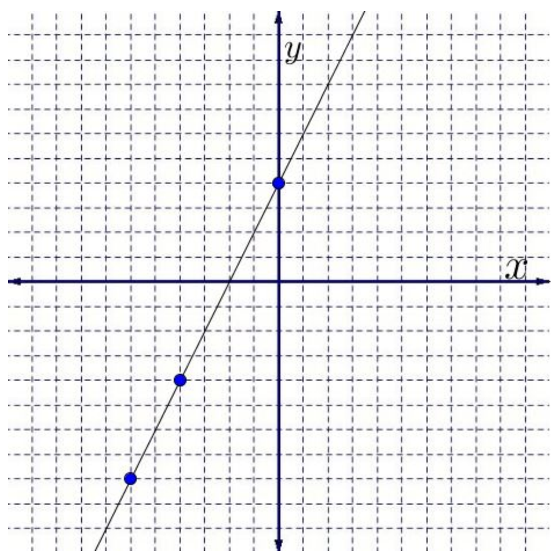
**Example 11:** Graph the solution set of each equation below.

(a) $y = 2x + 4$

We'll need to start by finding *at least* two solutions to the equation. We can do this by assuming a value for $x$ (or $y$) then solving the equation for $y$ (or $x$).

- If $x = 0$ then $y = 2(0) * 4$, so $y = 4$. This means that $(0, 4)$ is one solution to the equation.
- If $x = -4$ then $y = 2(-4) + 4$. This leads us to $y = -4$, so $(-4, -4)$ is a second solution.
- If $y = -8$ then $-8 = 2x + 4$. Solving this equation for $x$ gives $x = -6$, so $(-6, -8)$ is a third solution.

Plotting these three points and drawing the line that passes through them gives us the graph below, where each axis is scaled by 1's.

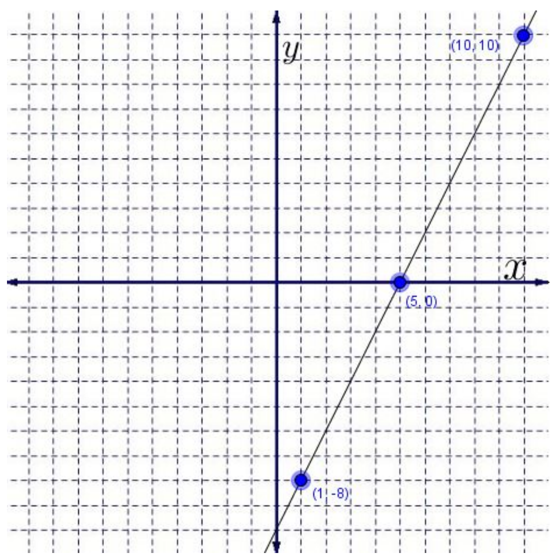



(b) $2x - y = 10$

The numbers you choose to use for $x$ or $y$ does not matter; any number can be used for either variable. This time around let's see what $x$ will be if $y$ is $-8$, 0, or 10:

- If $y = -8$ then $2x - (-8) = 10$, so $x = 1$ and $(1, -8)$ is one solution.
- If $y = 0$ then $2x - (0) = 10$. This leads us to $x = 5$, so $(5, 0)$ is another solution.
- If $y = 10$ then $2x - (10) = 10$. Solving this equation for $x$ gives $x = 10$, so $(10, 10)$ is yet another solution.

Again letting the axes be scaled by 1's, we get the graph below.




Notice that the equation in Example 11(b) was given in standard form rather than slope-intercept form; still we were able to find solutions to the equation and graph the solution set. However, slope-intercept form does have its advantages. Specifically, the *slope* and the *y-intercept* of a line appear explicitly in the slope-intercept form.

The $y$-intercept of a line is the point at which the line crosses the $y$-axis. The $y$-intercept of a line has coordinates $(0, b)$ where $b$ is the constant appearing in slope-intercept form, $y = mx + b$.

The slope of a line is the rate at which y-coordinates change with respect to a change in x-coordinates, and another defining feature of a linear equation is that, if it has a defined slope, the line's slope is constant (*any* two points on a given line will yield the same slope as any other two points on the same line). The slope of a line is represented by the constant $m$ in slope-intercept form, and it can be computed using the formula below:

$$m = \frac{y_1 - y_2}{x_1 - x_2}$$

In this formula, $(x_1, y_1)$ and $(x_2, y_2)$ are two given points of a line.

**Example 12:** Identify the slope and $y$-intercept of the lines described by the equations.

(a) $y = 2x - \frac{1}{5}$

Comparing this equation to the slope-intercept form, $y = mx + b$, we see that $m = 2$ while $b = -\frac{1}{5}$. Therefore, the slope of the line is 2 and the $y$-intercept is $(0, -\frac{1}{5})$.

(b) $y = -\frac{2x}{3} + 8$

Comparing this equation to the slope-intercept form, $y = mx + b$, we see that $m = -\frac{2}{3}$ and $b = 8$. Then the slope of the line is $-\frac{2}{3}$ and the $y$-intercept is $(0, 8)$.

While slope-intercept form is very useful, not all linear equations are given in slope-intercept form. In the next example we'll see how an equation in standard form can be rewritten in slope-intercept form.

**Example 13:** Consider the equation $3x - 9y = 108$.

(a) Rewrite this equation in slope-intercept form by solving for $y$:

- Subtract $3x$ from both sides of the equation.
- Divide both sides of the equation by $-9$.

- $-9y = -3x + 108$
- $y = \frac{1}{3}x - 12$

(b) Identify the slope and the $y$-intercept using your answer to part (a).

Since the equation in now in slope-intercept form, we can compare $y = mx + b$ to $y = \frac{1}{3}x - 12$ and see that the slope is $\frac{1}{3}$ and the $y$-intercept is $(0, -12)$.

(c) Find two solutions of the equation.

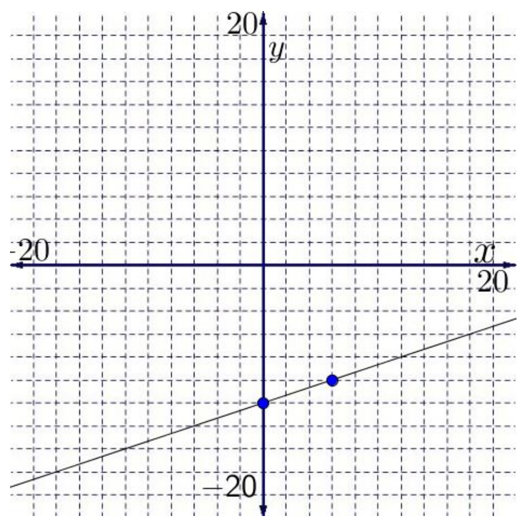
Since we know the $y$-intercept, we already know one solution: $(0, -12)$. For the second solution, let $x = 6$:

$$\begin{aligned} y &= \frac{1}{3}x - 12 \\ &= \frac{1}{3}(6) - 12 \\ &= 2 - 12 \\ &= -10 \end{aligned}$$

So a second solution is $(6, -10)$.

(d) Graph the solution set of the equation using the solutions found in part (c).

Since we know two solutions of the equation, we can plot the two points and draw the line connecting them as shown to the right. (Notice that the axes are scaled by 2's.)




(e) How does the slope, found in part (b), relate to your graph?

In our graph, notice that if we were to move from $(0, -12)$ to $(6, -10)$ we would be moving vertically 2 units and horizontally 6 units. The ratio of vertical movement to horizontal movement is often called "rise-over-run" and is equivalent to the slope.

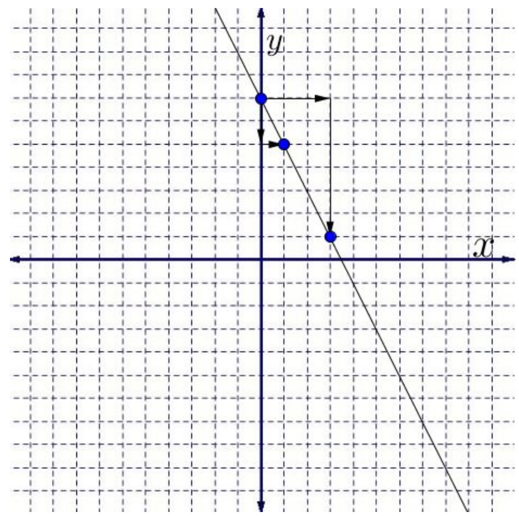
$$\frac{\text{rise}}{\text{run}} = \frac{\text{vertical movement}}{\text{horizontal movement}} = \frac{2}{6} = \frac{1}{3} = \text{slope}$$

This is an idea we can use to find solutions to an equation if one solution is already known.

**Example 14:** From each equation, identify the $y$-intercept and slope, and use these to graph the solution set of the equation. For each graph, scale both axes by 1's.

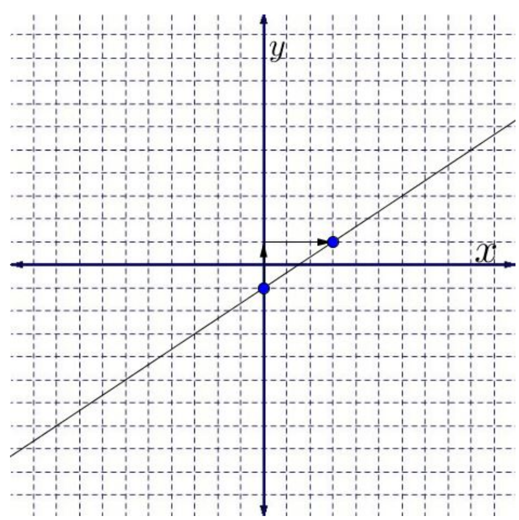
(a) $y = -2x + 7$

This equation is already in slope intercept form, so the slope and $y$-intercept can be quickly identified as $-2$ and $(0, 7)$ respectively. With a slope of $-2$, or $\frac{-2}{1}$, we can view this as a rise of $-2$ and a run of 1 to help us find a second point: $(0+1, 7-2) = (1, 5)$. In fact, we can use any fraction equivalent to $-2$ to help us find a second point. For example, $\frac{-6}{3} = -2$, so we could use a rise of $-6$ and a run of 3: $(0+3, 7-6) = (3, 1)$.



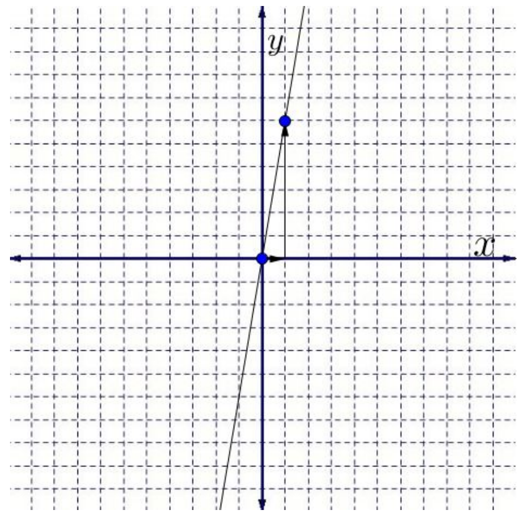

(b) $y = \frac{2}{3}x - 1$

This time around the slope of the line is $\frac{2}{3}$ and the $y$-intercept is $(0, -1)$. Using the slope to find a second point, we get $(0+3, -1+2) = (3, 1)$.



(c) $y = 6x$

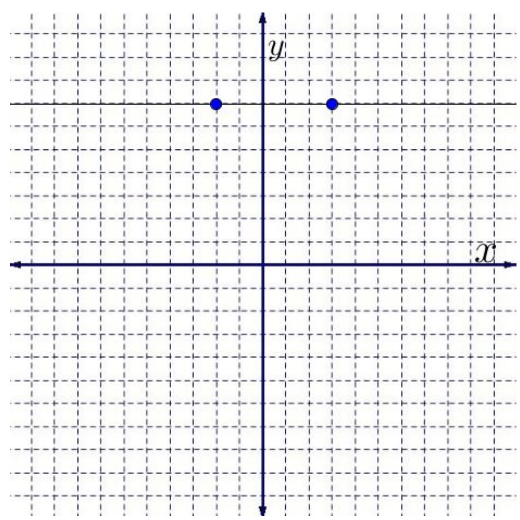
While it may not be obvious, this equation is also written in slope intercept form. In this case the slope is 6 and the $y$-intercept is $(0, 0)$. Since the slope is 6, from the origin we can rise 6 units and run 1 to get our second point, $(1, 6)$.




(d) $y = 7$

Our last linear equation could still be graphed using the slope-intercept approach we've used in the last three examples, with a slope of 0 and a $y$-intercept of $(0, 7)$.

On the other hand, we could make note of the fact that $x$ does not appear in this equation at all. This means there is no restriction on $x$, only on $y$; any point with a $y$-coordinate of 7 will be a point on our line. For example, $(-2, 7)$ and $(3, 7)$ will both be points on our line.




Notice that our graph depicts a horizontal line; any equation of the form $y = k$, where $k$ is any real number, describes a horizontal line. What should a line described by an equation like $x = k$ look like?

## 4.4 Exercises

### 4.4.1 Writing Equations

For problems 1-17 write each statement symbolically.

1. The sum of twenty-four and eighteen is forty-two.
2. The difference of eight and negative five is thirteen.
3. The quotient of eighty-four and two and one-tenth is forty.
4. The product of sixteen and negative three and five-tenths is negative fifty-six.
5. The sum of five and six-tenths and four and four-tenths is ten.
6. The difference of six and three-tenths and four and nine hundredths is two and twenty-one-hundredths.
7. Twelve more than the difference of negative twenty and eighteen is negative twenty-six.
8. Nine less than the product of six and five is twenty-one.
9. One more than the quotient of zero and eighteen is one.
10. The quotient of three and six, increased by one and six-tenths, is two and one-tenth.
11. The opposite of the sum of eight and negative twelve is four.
12. The product of the sum of eight and two and the difference of eight and two is sixty.
13. Six more than a number is twenty-eight.
14. The difference of a number and thirty-two and one-third is eighteen.
15. The product of sixteen and twelve is a number.
16. Three less than a number is one more than twice the number.
17. The product of one-third and a number is nine less than the number.

For problems 18-30, write an equation to model the situation. Define any variable used in the equation.

18. John had a twelve foot length of plywood and cut the board into two pieces. If one piece was

four and a half feet long, how long was the other piece?

19. In a jar containing thirty-two marbles, some are red and the rest are blue.

20. In a jar containing thirty-two marbles, some are red and the rest are blue. If seventeen are red, how many are blue?

21. If five paperback books cost \$32.20, what is the average cost of a paperback book?

22. Kelli has 15% of her paychecks withheld for income tax. If she earns \$2,592 in a month, before deductions, how much is withheld?

23. Amir spent \$5.56 on produce which cost \$0.98 per pound. How many pounds of produce did Amir purchase?

24. Paperback books cost \$7.10 apiece and hardback books cost \$15.70 apiece. Paula spent \$99.80 on books.

25. Paperback books cost \$7.10 apiece and hardback books cost \$15.70 apiece. Paula spent \$99.80 on books, only three of which were paperbacks. How many hardback books did she buy?

26. A textbook is advertised at \$32.70 but costs \$35.48 after sales tax. What is the sales tax rate? Round to the tenth of a percent if necessary.

27. A set of four tires is advertised at \$450.00 but costs \$497.25 after tax. What is the tax rate on the tires? Round to the tenth of a percent if necessary.

28. Amy is twice as old as her sister, Jenn. If the sum of their ages is 24, how old are the two sisters?

29. When two stackable bookshelves are stacked they have a combined height of six feet.

30. When two stackable bookshelves are stacked they have a combined height of six feet. If one bookshelf is 1.5 feet shorter than the other, how tall are the two bookshelves?

### 4.4.2 Solving Equations

Solve each equation below.

31. $x + 2 = -5$
32. $2.5n = 6$
33. $15 + t = 74$
34. $\frac{2}{3}y = 42$
35. $m - 2.6 = 1.29$
36. $-64 + w = -21$
37. $-4x = 0.28$
38. $n + \frac{5}{7} = \frac{2}{3}$
39. $11w = -550$
40. $12x = 0$
41. $m - 5.2 = \frac{7}{4}$
42. $-0.78z = -3.198$
43. $m + (-6) = -24$
44. $v + 46 = -95$
45. $12.06 + x = 9.2$
46. $c + \frac{8}{9} = \frac{5}{6}$
47. $\frac{4}{9}n = 2\frac{1}{8}$
48. $-\frac{7}{4}m = \frac{2}{3}$
49. $12 - x = 17$
50. $2n + 5 = 3 + n$
51. $5m + 2 = 3m - 8$
52. $-4n + 7 = 14$
53. $5.5(2m + 9) = 0$
54. $16 = 24 - 6y$
55. $-3(5x - 4) = 2(7 - x)$
56. $6(3 + 4m) = 14m$
57. $\frac{3}{2}(6 + 12x) = -5x$
58. $\frac{1}{3}y - 14 = \frac{1}{4}(n + 18)$
59. $6 - 0.3(9y - 11) = 0.3y$
60. $2w - 3.5 = 11.25 - 1.2w$
61. $-v - 4(2v + 6) = -9 + 2(5v + 2)$
62. $98y - 100 = 2(y - 55) + 10$
63. $2.5x + 0.74 = 1.2x - 1.21$

A statement which is always true is called an *identity* while a statement which is always false is called a *contradiction*. The equations we've dealt with so far have been *conditional* statements, meaning that the equation is true on the condition that the variable(s) takes on a specific value(s). For example, the equation $12x = 96$ is true on the condition that $x$ is 8, and for all other values the equation is false. For each equation below, find the solution set.

64. $x + 2 = x - 1$
65. $x - 3 = -3 + x$
66. $2(x - 6) + 3 = 2x - 9$
67. $6x - 8 = 3 + 4(2x - 5) - 2x$
68. $2(x + 2) - 1 = 3x - (x - 3)$
69. $2(x + 2) = 2x + 3$

70. $2.5x + 0.8(2x + 5) = 5x - 0.9(x - 5)$

71. $7 - 3x = \frac{1}{2}(14 + 6x)$

### 4.4.3 Two Variables Linear Equations

Determine whether or not the given ordered pair is a solution to the given equation.

72. $2x - 3y = 7$; $(0, 2)$

73. $7x - 2y = 9$; $(1, -1)$

74. $4x + 3y = -7$; $(1, 1)$

75. $y = \frac{3}{2}x - 13$; $(4, 1)$

76. $y = 7$; $(7, 0)$

77. $x = 5$; $(5, 8)$

78. $y = -3x + 2$; $(0, 2)$

79. $-2x + 6y = 24$; $(-3, 5)$

80. $\frac{1}{2}x - \frac{1}{3}y = 0$; $(10, 15)$

81. $y = 13$; $(5, 13)$

82. $y = 0.6x$; $(0.72, 1.2)$

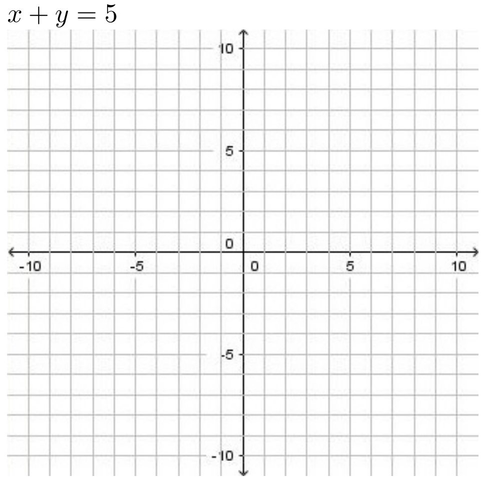
83. $\frac{1}{3}x + y = 20$; $(35, 8\frac{1}{3})$

84. $-8x - 16y = 30$; $(0.75, -2.25)$

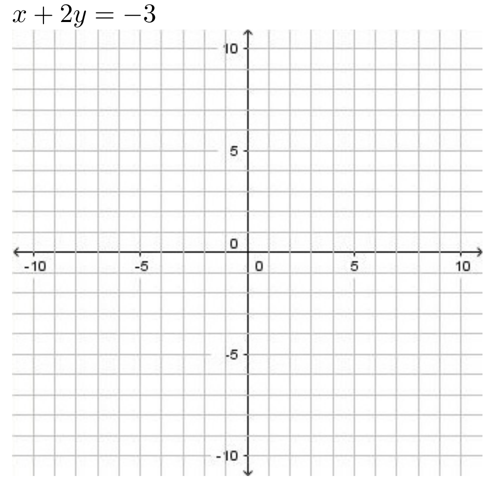
85. $y = -1.23x + 6.4$; $(-2.4, -9.352)$

Find three solutions of the given equation and use them to graph the solution set of the equation.

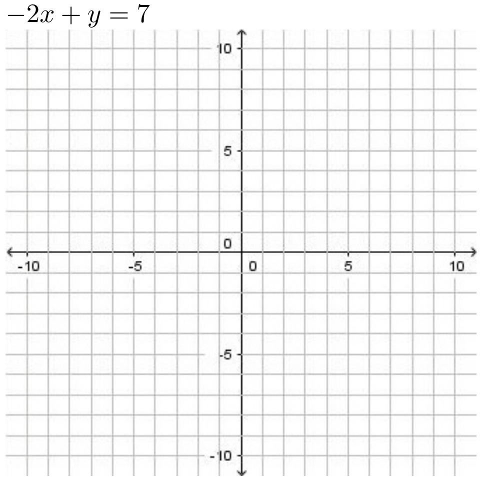
86. $x + y = 5$



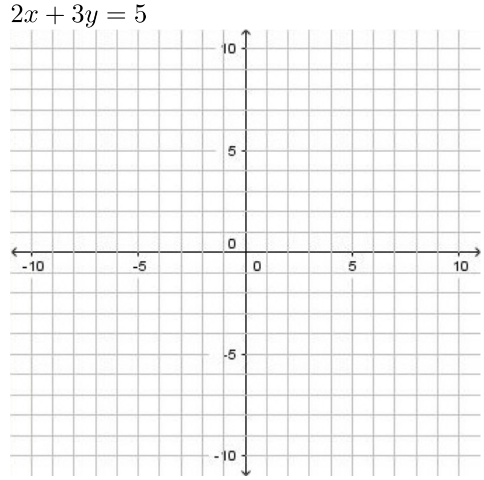

87. $x + 2y = -3$



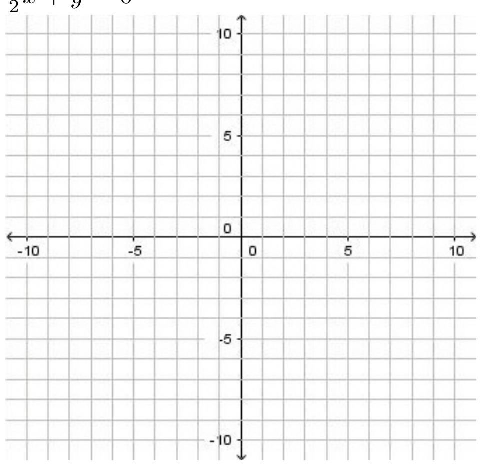
88. $-2x + y = 7$



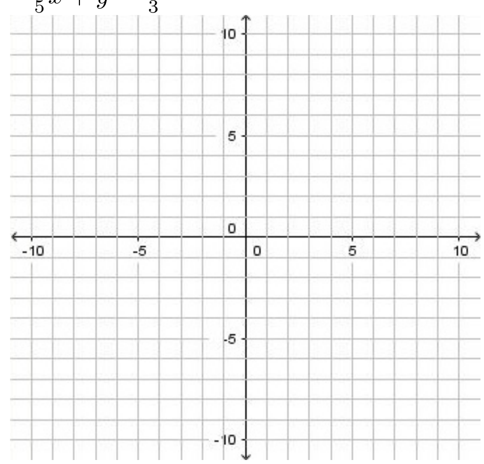

89. $2x + 3y = 5$



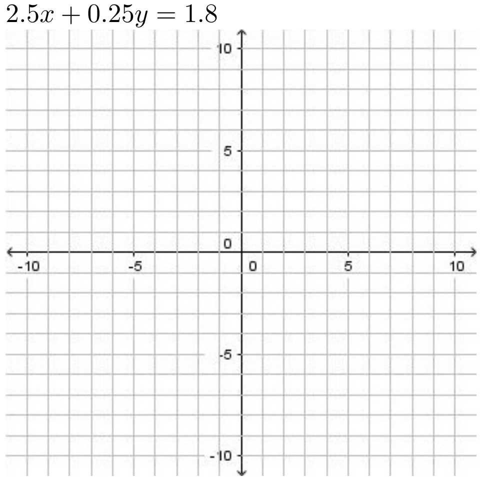

90. $\frac{1}{2}x + y = 0$



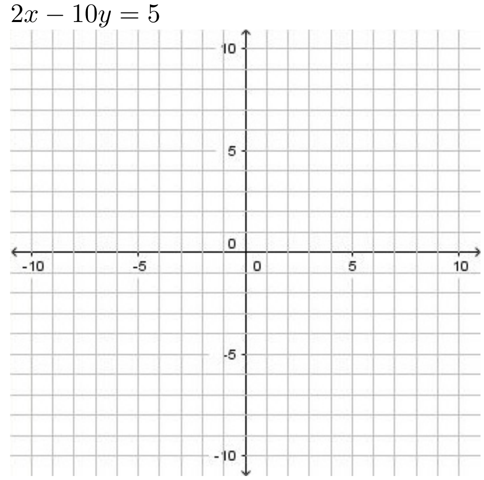

91. $-\frac{2}{5}x + y = \frac{1}{3}$




92. $2.5x + 0.25y = 1.8$




93. $2x - 10y = 5$



Identify the slope and $y$-intercept of the line described by the given equation.

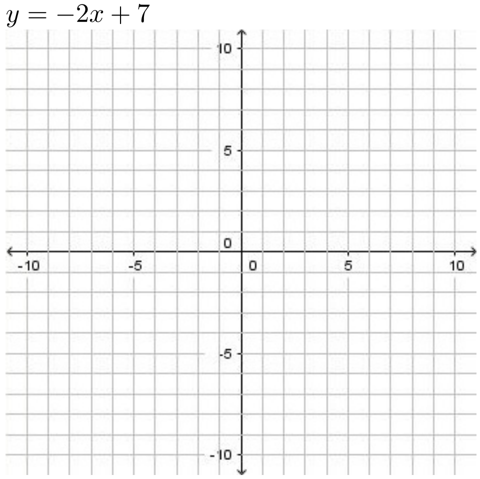
94. $y = 3x - 2$
95. $y = x + 8$
96. $y = -2.5x + 6$
97. $y = \frac{3}{2}x$
98. $y = 14$
99. $x = 9$
100. $3x - 2y = 24$
101. $12x + 8y = 16$
102. $y = \frac{4}{5}x - 2$
103. $x = -5$
104. $y = 2.3x - 1.5$
105. $5x + 6y = -60$

Give an equation of the line with the given characteristics.

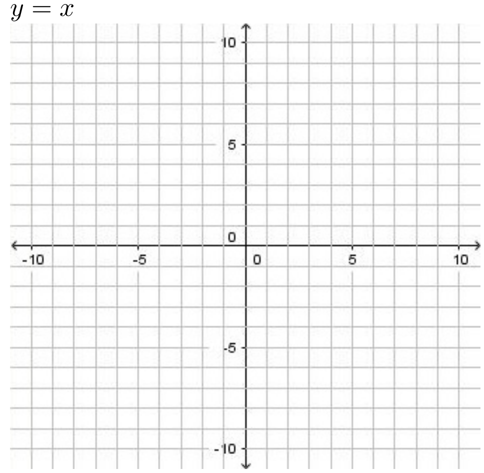
106. Slope of 5 and $y$-intercept of $(0, 6)$.
107. Slope of -2 and $y$-intercept of $(0, 12)$.
108. Slope of $\frac{3}{2}$ and $y$-intercept of $(0, -8)$.
109. Slope of 0 and containing $(2, 8)$.
110. Containing $(0, 8)$ and $(-4, 0)$.
111. Slope is undefined and passes through $(0, 0)$.
112. Slope of 0 with $y$-intercept $(0, -3)$.
113. Passes through $(12, 8)$ and $(0, 6)$.
114. Passes through $(5, 4)$ and has no $y$-intercept.
115. Slope of $-\frac{4}{5}$ and passing through the origin.

For each equation, use the $y$-intercept and the slope to graph the solution set.

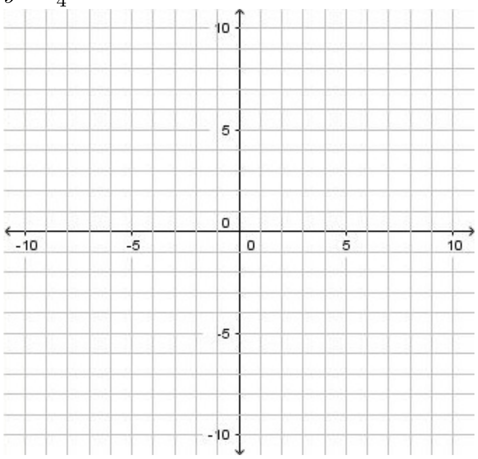
116. $y = -2x + 7$



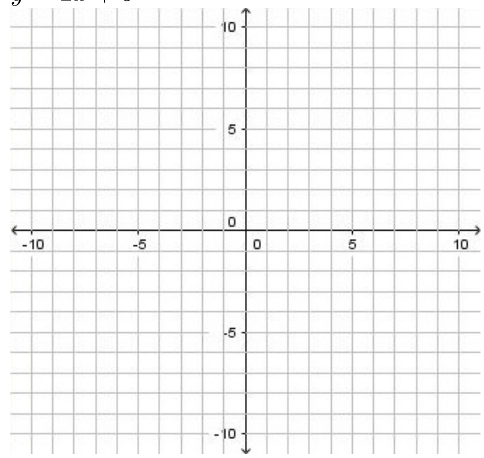

117. $y = x$



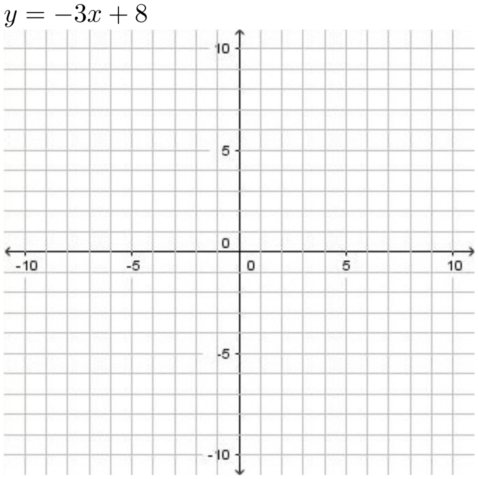
118. $y = \frac{3}{4}x - 1$



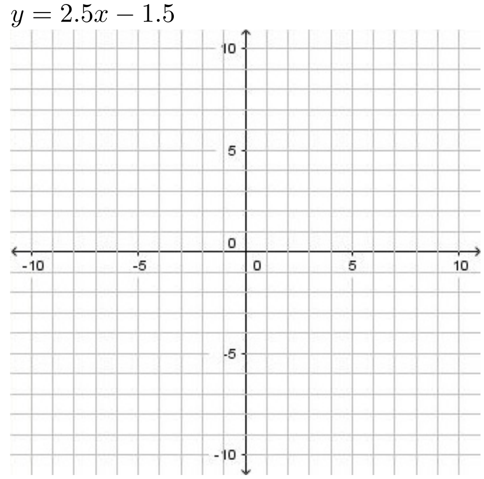

119. $y = 2x + 5$



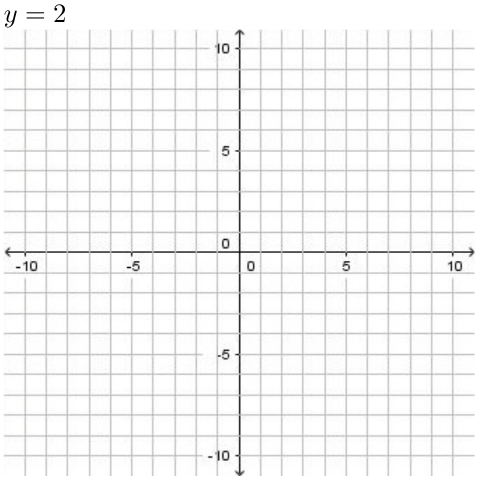

120. $y = -3x + 8$



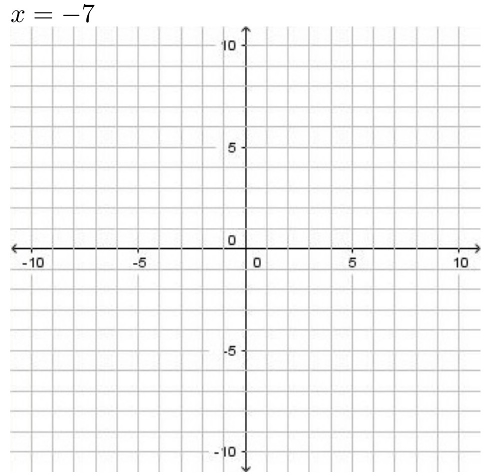

121. $y = 2.5x - 1.5$



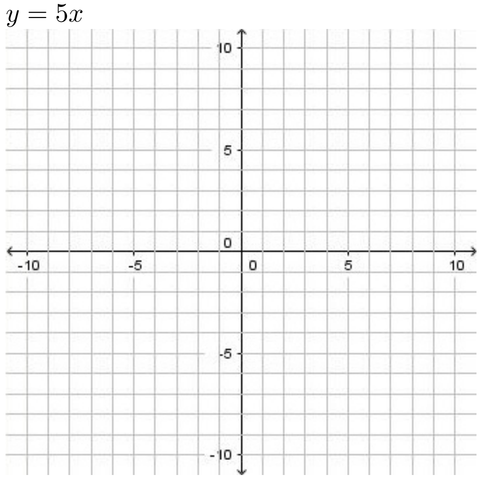

122. $y = 2$



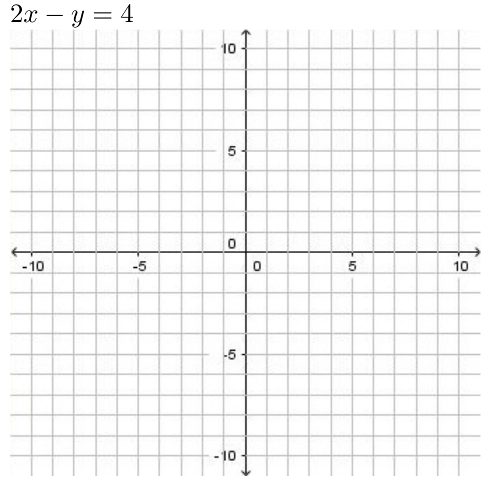

123. $x = -7$



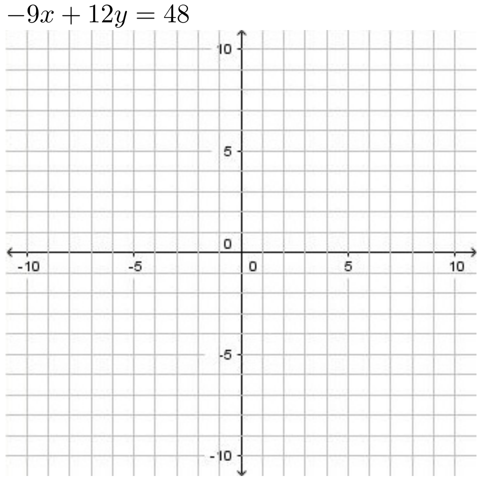
124. $y = 5x$



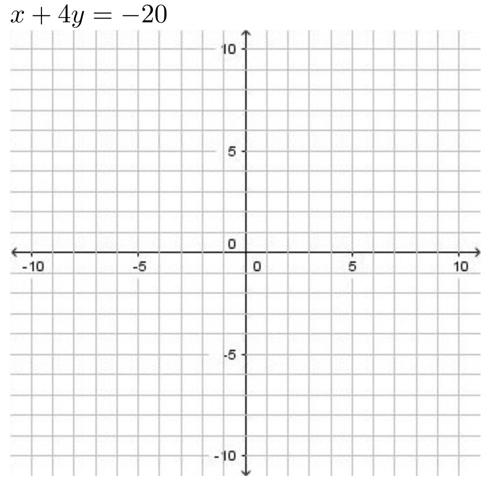
125. $2x - y = 4$



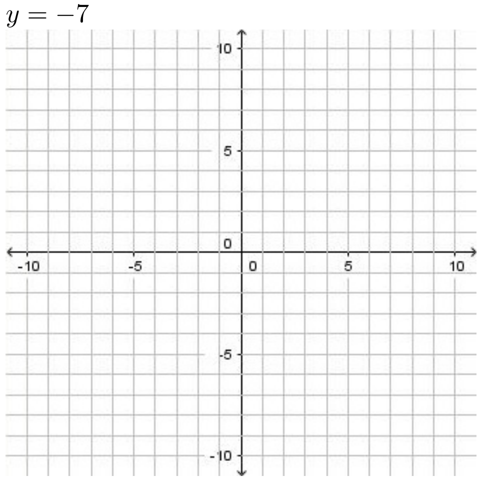
126. $-9x + 12y = 48$



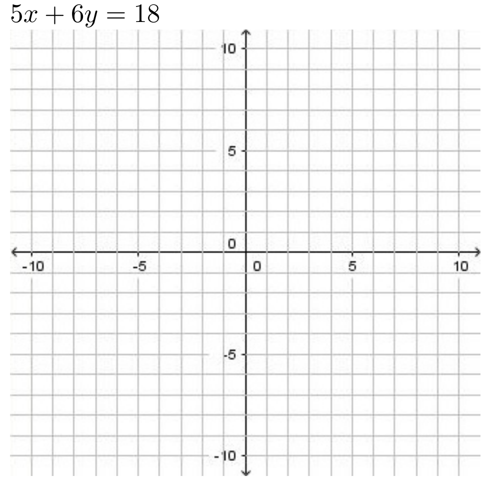
127. $x + 4y = -20$



128. $y = -7$



129. $5x + 6y = 18$



# Chapter 5

# Number Theory

## 5.1 Divisibility and Primes

Number theory is the study of numerical patterns and relationships between numbers, especially concerning the natural numbers. In this section we'll discuss the prime numbers and divisibility, including the Euclidean Algorithm. The first of these topics, prime numbers, is something most are already somewhat familiar with; a prime number is a natural number which is divisible by exactly two numbers: 1 and itself. Note that this excludes 1 from being a prime number since it is not divisible by two different numbers. The first twenty-five prime numbers, those less than 100, are given below.

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97$$

A critical result of number theory is the Fundamental Theorem of Arithmetic which states that all natural numbers greater than 1 can be written *uniquely* as a product of primes. For example, 36 can be written as $2^2 * 3^2$ and, aside from reordering the factors (writing $2 * 3 * 3 * 2$ or $3 * 2 * 3 * 2$ for example), there is no other way to factor 36 into primes. This is mentioned here because the prime factors of a number determines all factors of a number; if a number $x \neq 1$ is a factor of a second number $y$ then the prime factors of $x$ are also prime factors of $y$. Below, for example, we have the factors of 36 (other than 1 and 36) and their prime factorizations.

$$18 = 2^1 * 3^2 \qquad 9 = 3^2 \qquad 3 = 3^1$$
$$12 = 2^2 * 3^1 \qquad 6 = 2^1 * 3^1 \qquad 2 = 2^1$$

These factors of 36 (18, 12, 9, 6, 3, and 2) are also called *divisors* of 36; alternatively, 36 is divisible by each of the numbers just listed. A natural number $m$ is called a divisor of another natural number $n$ if there exists a third natural number $k$ such that $n = m * k$. For example, 18 is a divisor of 36 because $36 = 18 * 2$. One of the simplest questions from number theory, and one you may have investigated in an algebra course, asks how to find, given two numbers, the largest number which is a divisor of each. The goal here is called the greatest common divisor, or GCD. For example, the GCD of 36 and 24 is 12 because 12 is the largest number which is a divisor of both 36 and 24.

While the GCD of two numbers can be found by comparing the prime factorization of each, this is not always efficient. Instead, we can used what's known as the Euclidean Algorithm or Euclid's Algorithm.

## 5.2 Euclid's Algorithm

Given two numbers, say 234 and 182, we can find their greatest common divisor by taking the iterative steps of the Euclidean Algorithm. Below is an example of the Euclidean Algorithm being performed to find the GCD of 234 and 182.

$$234 = 1 * 182 + 52 \qquad \text{(the quotient of 234 and 182 is 1 with remainder 52)}$$
$$182 = 3 * 52 + 26 \qquad \text{(the quotient of 182 and 52 is 3 with remainder 26)}$$
$$52 = 2 * 26 + 0 \qquad \text{(the quotient of 52 and 26 is 2 with remainder 0)}$$

According to this process, 26 is the GCD of 234 and 182. To show that this is correct, consider the equations above, starting with the last.

- $52 = 2 * 26$: our last equation, this gives us an alternate way to write 52.

- $182 = 3 * 52 + 26$: we can replace 52 with $2 * 26$, according to the previous line, to get the equivalent equation $182 = 3 * 2 * 26 + 26$. This can be rewritten as follows.

$$\begin{aligned} 182 &= 3 * 2 * 26 + 26 \\ &= 6 * 26 + 26 \\ &= 7 * 26 \end{aligned}$$

  This shows that 182 is divisible by 26.

- $234 = 1 * 182 + 52$: based on the previous two lines we can perform more substitution to rewrite this equation.

$$\begin{aligned} 234 &= 1 * 182 + 52 \\ &= 1 * 7 * 26 + 2 * 26 \\ &= 7 * 26 + 2 * 26 \\ &= 9 * 26 \end{aligned}$$

Since $234 = 9 * 26$ and $182 = 7 * 26$, 26 is a factor of both 234 and 182. But is it the *greatest* common factor? If it isn't then there must be a larger number, divisible by 26, which itself divides both 234 and 182. If such a number existed then there would be a common factor between the numbers in these two equations as yet undiscussed: 9 and 7. However, 9 and 7 have no factors in common, so 26 is in fact the GCD of 234 and 182.

A more detailed explanation for this process using modular arithmetic will be considered in the next section. For now, note the steps taken in finding the greatest common divisor.

1. Given two numbers, $a$ and $b$, identify the quotient of $a$ and $b$ and the resulting remainder.

Call the quotient $q_1$ and the remainder $r_1$.

$$a = q_1 * b + r_1$$

2. Next, find the quotient of $b$ and the remainder from the previous step, $r_1$, as well as the remainder from this new quotient. Call the resulting quotient and remainder $q_2$ and $r_2$, respectively.

$$b = q_2 * r_1 + r_2$$

3. Repeat this process: find the quotient of $r_1$ and $r_2$ as well as the resulting remainder, with the new quotient labeled $q_3$ and the new remainder labeled $r_3$.

$$r_1 = q_2 * r_2 + r_3$$

4. Continue repeating this process until a remainder of 0 is eventually reached; do not continue past this point. If $r_k = 0$ then the GCD of $a$ and $b$ is $r_{k-1}$.

Relating this process back to our example, finding the GCD of 234 and 182 (written $GCD(234, 182)$), we saw three iterations before a remainder of 0 was found so $r_3 = 0$ in that example. Then according to this process $r_2$, or 26, is the greatest common divisor of 234 and 182.

Before moving on to modular arithmetic and an explanation for this algorithm, let's see a few more examples of the algorithm at work.

**Example 1:** Find the GCD of 210 and 165; find $GCD(210, 165)$.

$210 = 1 * 165 + 45$ (the quotient of 210 and 165 is 1 with remainder 45)

$165 = 3 * 45 + 30$ (the quotient of 165 and 45 is 3 with remainder 30)

$45 = 1 * 30 + 15$ (the quotient of 45 and 30 is 1 with remainder 15)

$30 = 2 * 15 + 0$ (the quotient of 30 and 15 is 2 with remainder 0)

The greatest common divisor of 210 and 165 is 15.

**Example 2:** Find the GCD of 1155 and 297.

$$1155 = 3 * 297 + 264 \quad \text{(the quotient of 1155 and 297 is 3 with remainder 264)}$$

$$297 = 1 * 264 + 33 \quad \text{(the quotient of 297 and 264 is 1 with remainder 33)}$$

$$264 = 8 * 33 + 0 \quad \text{(the quotient of 264 and 33 is 8 with remainder 0)}$$

The GCD of 1155 and 297 is 33.

In the examples so far, we've assumed that the variable $a$ from the Euclidean Algorithm is the larger of the two values we're handling and $b$ is the smaller. This is not necessary; the GCD will emerge through the same process regardless of how we assign the values to $a$ and $b$.

**Example 3:** Find the GCD of 882 and 70.

$$70 = 0 * 882 + 70 \quad \text{(the quotient of 70 and 882 is 0 with remainder 70)}$$

$$882 = 12 * 70 + 42 \quad \text{(the quotient of 882 and 70 is 12 with remainder 42)}$$

$$70 = 1 * 42 + 28 \quad \text{(the quotient of 70 and 42 is 1 with remainder 28)}$$

$$42 = 1 * 28 + 14 \quad \text{(the quotient of 42 and 28 is 1 with remainder 14)}$$

$$28 = 2 * 14 + 0 \quad \text{(the quotient of 28 and 14 is 2 with remainder 0)}$$

The GCD of 882 and 70 is 14.

If the GCD of two numbers is 1 then those numbers are *relatively* prime; they share no prime factors and, therefore, there are no factors common to each with the exception of 1.

**Example 4:** Find the GCD of 243 and 256.

$$256 = 1 * 243 + 13 \qquad \text{(the quotient of 256 and 243 is 1 with remainder 13)}$$

$$\begin{aligned} 243 &= 18 * 13 + 9 && \text{(the quotient of 243 and 13 is 18 with remainder 9)} \\ 13 &= 1 * 9 + 4 && \text{(the quotient of 13 and 9 is 1 with remainder 4)} \\ 9 &= 2 * 4 + 1 && \text{(the quotient of 9 and 4 is 2 with remainder 1)} \\ 4 &= 4 * 1 + 0 && \text{(the quotient of 4 and 1 is 4 with remainder 0)} \end{aligned}$$

The GCD of 243 and 256 is 1.

## 5.3 Modular Arithmetic

Modular arithmetic is something we use every day and learn to use from a very young age. Modular arithmetic fixes the results of our standard arithmetic operations to range from 0 to some specified number; arithmetic modulo $n$ would fix results to range from 0 to $n-1$. A 24-hour clock is a great example; the times on a 24-hour clock range from 00:00 to 23:59. A minute after 23:59 and the clock will return to 00:00 with two types of modular arithmetic occurring simultaneously. With the minutes counting from 0 to 59 then returning to 0, this is an example of addition *modulo 60*. With the hours turning from 23 back to 0 we have addition *modulo 24*.

For example, suppose the time is 15:00. What time will it be in six hours? In 12 hours? We can add 6 and 15 according to standard arithmetic just find to get 21:00. But six hours after that it won't be 27:00, the time will be 03:00. Reaching 24:00 is equivalent to reaching 00:00 and the count begins all over again.

We've even seen modulo arithmetic earlier in this text when there weren't enough bits to complete binary addition.

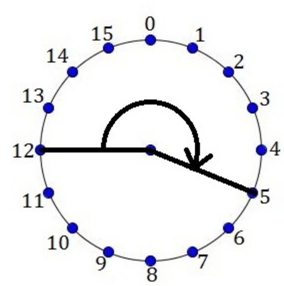
**Example 5:** Add the unsigned, four-bit numbers $1101_2$ and $0110_2$.

$$\begin{array}{r} 1101_2 \\ +0110_2 \\ \hline 10011_2 \end{array}$$

The decimal equivalent of $1101_2$ and $0110_2$ are 13 and 6 respectively. Ignoring the red digit in our sum, the result of $0011_2$ is equivalent to 3, in decimal, but 13+6 is 19, not 3. Except, by ignoring the carried fifth-bit we're working modulo 16 (since only 0 to 15 can be expressed in four bits). This means that 19 is equivalent to 3 modulo 16. This is denoted $19 \equiv 3 \mod 16$ or $19 \equiv_{16} 3$.

**Example 6:** Add 12 and 9 modulo 16, illustrating this addition on the 16-hour clock.

Adding 12 and 9 would normally give us 21, but we're working modulo 16. To see what this sum should be, and what 21 is equivalent to modulo 16, let's consider a 16-hour clock.

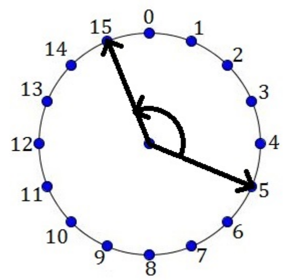



With a starting position of 12, we can cycle 9 positions clockwise to represent the addition by 9. Since we end on 5, $12 + 9 \equiv 5 \mod 16$.

Notice that 5 more than 16 is 21, or $21 - 16 = 5$. If we're looking for a sum modulo $n$ and we know what the standard sum is, we can subtract $n$ from the standard sum until we reach a non-negative number less than $n$; this result is the sum modulo $n$.

This modular clock can also help explain subtraction and negative numbers. For example, if addition is modeled on the clock by moving the hand in the clockwise direction then subtraction would be modeled by moving in the counter-clockwise direction. Following this line of thought,

$(5-6) \mod 16$ would look like this:




Even though $5-6=-1$, modulo 16 we get a result of 15; this is because $-1 \equiv 15 \mod 16$. Remember that, modulo $n$, we restrict results to values from 0 to $n-1$, so while the standard difference of 5 and 6 is $-1$ this value is out of the acceptable range.

Just as we subtracted 16 from 21 to find the equivalent of 21 modulo 16, we can add 16 to $-1$ to find its equivalent: $-1+16=15$. The equivalent of some negative values, modulo 16, are given below.

$-1 \equiv 15 \mod 16$ $\quad -3 \equiv 13 \mod 16$ $\quad -5 \equiv 11 \mod 16$ $\quad -7 \equiv 9 \mod 16$ $\quad -9 \equiv 7 \mod 16$

$-2 \equiv 14 \mod 16$ $\quad -4 \equiv 12 \mod 16$ $\quad -6 \equiv 10 \mod 16$ $\quad -8 \equiv 8 \mod 16$ $\quad -10 \equiv 6 \mod 16$

**Example 7:** Find the difference of 16 and 50 modulo 12.

As before, we can start by finding the standard difference: $16-50=-34$. From here we can add 12 repeatedly until we reach a non-negative value less than 12:

$-34+12=-22$ $\qquad -22+12=-10$ $\qquad -10+12=2$

Reaching such a value, we find that $16-50 \equiv 2 \mod 12$.

This repeated addition or subtraction, while it works, is not always efficient. Consider, for example, finding the sum of 1240 and 560 modulo 7. Following standard arithmetic the sum would be 1800. To find the equivalent value modulo 7 we could subtract 7 from 1800 until we reach an acceptable number, but this could take some time:

$$1800 - 7 = 1793,\ 1793 - 7 = 1786,\ \ldots,\ 15 - 7 = 8,\ 8 - 7 = 1$$

Fortunately, there is another arithmetic tool we can use here: division.

$$\begin{array}{r} 257 \\ 7\overline{)1800} \\ \underline{-14\phantom{0}} \\ 40 \\ \underline{-35} \\ 50 \\ \underline{-49} \\ 1 \end{array}$$

Notice that the remainder we get from longhand division is 1, we same result we'd see from repeated subtraction. This is because that same subtraction is embedded within the process of longhand division but performed in a more effective way, subtracting multiples of 7 rather than simply subtracting 7. Using division will greatly simplify modular arithmetic.

Another tool that will be useful in modular arithmetic is the commutative property of multiplication over addition, the property exemplified by the equation $a * (b + c) = a * b + a * b$. Division can be distributed across addition in the same way; instead of adding 1540 and 560 and then dividing that sum by 7, we could divide both 1540 and 560 by 7, identify the remainders, and find the sum of the remainders (modulo 7).

Taking this approach, we'd first find $1240 \div 7 = 177R1$ and $560 \div 7 = 80R0$. This means that $1240 \equiv 1 \mod 7$ and $560 \equiv 0 \mod 7$, so

$$\begin{aligned} 1240 + 560 &\equiv (1 + 0) \mod 7 \\ &\equiv 1 \mod 7 \end{aligned}$$

While this tool isn't always necessary, it can make some of our work significantly easier and leads to some other useful tools.

**Example 8:** Find the sum of 24, 28, and 40 modulo 9.

We'll approach this in two ways. For the first method, find the standard sum of the three numbers and then performing division.

First, the sum of 24, 28, and 40:

$$24 + 28 + 40 = 92$$

Dividing 92 by 9, we get $92 \div 9 = 10R2$. Therefore,

$$24 + 28 + 40 \equiv 2 \mod 9$$

For the second approach, find the equivalent of the three numbers modulo 9 and then operating on the remainders.

Looking for the modular equivalents of the 24, 28, and 40, we find $24 \div 9 = 2R6$, $28 \div 9 = 3R1$, and $40 \div 9 = 4R4$. Focusing on these remainders, we have

$$\begin{aligned} 24 + 28 + 40 &\equiv (6 + 1 + 4) \mod 9 \\ &\equiv 11 \mod 9 \\ &\equiv 2 \mod 9 \end{aligned}$$

**Example 9:** Find the difference of 256 and 302 modulo 8.

The standard difference here is $-46$. We can then apply division:

$$-46 \div 8 = -5R6$$

This implies that $-46 \equiv -6 \mod 8$. Recall that to find the positive modular equivalent of a negative value we can add; adding 8 to $-6$ gives us 2, so

$$256 - 302 \equiv 2 \mod 8$$

This modular subtraction could also be performed by first finding 256 and 302 modulo 8: $256 \equiv 0 \mod 8$ and $302 \equiv 6 \mod 8$. Then we can perform modular arithmetic with smaller numbers:

$$\begin{aligned} 256 - 302 &\equiv (0 - 6) \mod 8 \\ &\equiv -6 \mod 8 \\ &\equiv 2 \mod 8 \end{aligned}$$

Our last operation for modular arithmetic is multiplication. Much like finding sums and differences, to find the product of two numbers modulo $n$ we can first identify their standard product then find the equivalent value modulo $n$.

**Example 10:** Find the product of 7 and 9 modulo 12.

Since the standard product of 7 and 9 is 63, we can find the equivalent of 63 modulo 12. Again, we can perform division, $63 \div 12$, and the remainder will be our answer. Since $63 \div 12 = 5R3$, the product of 7 and 9 modulo 12 is 3.

$$7 * 9 \equiv 3 \mod 12$$

**Example 11:** Find the product of 160 and 204 modulo 9.

We could take the same approach as in the previous example, but there is another tool we can use here. Just as we were able to find the modular equivalent of terms before finding their modular sum (as in Example 8) we can find the equivalents of 160 and 204 modulo 9 before finding their modular product.

Note that $160 \equiv 7 \mod 9$ and $204 \equiv 6 \mod 9$. Then

$$\begin{aligned} 160 * 204 &\equiv (7 * 6) \mod 9 \\ &\equiv 42 \mod 9 \\ &\equiv 7 \mod 9 \end{aligned}$$

Again, this tool is not *necessary* for performing modular arithmetic, but it can make some problems simpler and helps explain or build other tools.

**Example 12:** Show that $m$ is a divisor of $n$ when $n \equiv 0 \mod m$.

We've already seen examples of this: $256 \equiv 0 \mod 8$ in Example 9 and $560 \equiv 0 \mod 7$ in Example 7. This example speaks of a more general rule and is implied by the definition of "divisor"; a natural number $m$ is called a divisor of another natural number $n$ if there exists a third natural number $k$ such that $n = m * k$.

So, assuming that $m$ is a divisor of $n$, there is $k$ such that $n = m * k$. Then $n \mod m$ and $(m * k) \mod m$ would have to be equal; however, $m$ itself must be equivalent to 0 modulo $m$, so $m * k \equiv (0 * k) \mod m$. The value of $k$ is irrelevant at this point since, no matter the value of $k$, the product $0 * k$ will be 0.

$$\begin{aligned} n &= m * k \\ &\equiv (m * k) \mod m \\ &\equiv (0 * k) \mod m \\ &\equiv 0 \mod m \end{aligned}$$

Now consider this statement from the other direction, assuming that $n \equiv 0 \mod m$ to show that $m$ must be a divisor of $n$. This is also relatively straight forward. If $n \equiv 0 \mod m$ then it follows that $n \div m = kR0$ for some number $k$. From the definition of division and divisor it follows that $n = m * k$, so $m$ is a divisor of $n$.

With this idea, we can give greater explanation for the Euclidean Algorithm. First, recall that the Euclidean Algorithm find the GCD of two numbers and that the greatest common divisor of two given numbers is the largest number that is a divisor of each given number. According to Example 12 then, the GCD of two numbers, say $a$ and $b$, is the largest number $x$ such that $a \equiv 0 \mod x$ and $b \equiv 0 \mod x$.

$$\begin{aligned} a &= q_1 * b + r_1 \\ b &= q_2 * r_1 + r_2 \\ r_1 &= q_3 * r_2 + r_3 \\ r_1 &= q_3 * r_2 + r_3 \\ &\vdots \\ r_{k-4} &= q_{k-2} * r_{k-3} + r_{k-2} \\ r_{k-3} &= q_{k-1} * r_{k-2} + r_{k-1} \\ r_{k-2} &= q_k * r_{k-1} + 0 \end{aligned}$$

So, suppose the Euclidean Algorithm takes $k$ steps with $r_{k-1}$ being the supposed GCD of two numbers, $a$ and $b$ (and $r_k = 0$). Then the steps taken would look something like what we see to the left.

Now, $r_{k-1}$ is a factor of both $a$ and $b$ and Example 12 can be used to argue this. Let's take a look at the last few steps of the Euclidean algorithm:

1. $r_{k-2} = q_k * r_{k-1} + 0 \equiv (q_k * 0) \mod r_{k-1} \equiv 0 \mod r_{k-1}$
2. $r_{k-3} = q_{k-1} * r_{k-2} + r_{k-1} \equiv (q_{k-1} * 0 + 0) \mod r_{k-1} \equiv 0 \mod r_{k-1}$
3. $r_{k-4} = q_{k-2} * r_{k-3} + r_{k-2} \equiv (q_{k-2} * 0 + 0) \mod r_{k-1} \equiv 0 \mod r_{k-1}$

In equation 1 on this list, since $r_{k-2} \equiv 0 \mod r_{k-1}$, it must be that $r_{k-1}$ is a divisor of $r_{k-2}$. In equation 2 the fact that $r_{k-2} \equiv 0 \mod r_{k-1}$ is used to show that $r_{k-3} \equiv 0 \mod r_{k-1}$, so, by Example 12, $r_{k-1}$ must be a divisor of $r_{k-3}$. In equation 3 we continue using the equivalences just shown: $r_{k-2} \equiv 0 \mod r_{k-1}$ and $r_{k-3} \equiv 0 \mod r_{k-1}$. From these it follows that $r_{k-4} \equiv 0 \mod r_{k-1}$ so that $r_{k-1}$ must be a divisor of $r_{k-4}$.

This process could continue back up through the sequence of equations resulting from the Euclidean Algorithm, with each step utilizing the results of the last two steps, until we reach $a \equiv 0 \mod r_{k-1}$ and $b \equiv 0 \mod r_{k-1}$.

This shows that that $r_{k-1}$ is a common factor to both $a$ and $b$. It must be the *greatest* common factor or a remainder of 0 would have appeared earlier in the sequence; essentially,

$$GCD(a,b) = GCD(b,r_1) = ... = GCD(r_{k-2}, r_{k-1})$$

## 5.4 Rules of Divisibility

Other useful results that comes from modular arithmetic are rules of divisibility; essentially, short-cut methods of determining whether or not one number is divisible by another. For example, if the sum of the digits of a number is divisible by three then the original number is also divisible by three.

**Example 13:** Show that a number is divisible by three when the sum of the number's digits is divisible by three.

For these types of rules we'll primarily use two tools: one is expanded decimal form, the other is the result of Example 12 from the previous section ($m$ is a divisor of $n$ when $n \equiv 0 \mod m$). Since we want to show that this rule works with *any* number, we'll need to use variables. We'll let the one's digit be represented by $a_0$, the ten's digit be represented by $a_1$, the hundred's digit by $a_2$, and so on. For example, if we were focusing on the number 12,305 then $a_0 = 5$, $a_1 = 0$, $a_2 = 3$, $a_4 = 2$, and $a_5 = 1$.

With this notation, the expanded form of a number with $k$ digits would look like this:

$$a_k * 10^k + a_{k-1} * 10^{k-1} + ... + a_2 * 10^2 + a_1 * 10^1 + a_0 * 10^0$$

Our next big step is to look at this expression's equivalent modulo 3. However, before we take that step, note that $10 \equiv 1 \mod 3$. This is also true for all powers of ten: $10^2 = 100 \equiv 1$

mod 3, $10^3 = 1000 \equiv 1 \mod 3$, et cetera. Since $10^0 = 1$, modulo 3 or not, we can say that all powers of ten are equivalent to one modulo 3. With this fact in mind we can simplify this expression quite a bit.

$$\begin{aligned}
&a_k * 10^k + a_{k-1} * 10^{k-1} + ... + a_2 * 10^2 + a_1 * 10^1 + a_0 * 10^0 \\
&\equiv (a_k * 1^k + a_{k-1} * 1^{k-1} + ... + a_2 * 1^2 + a_1 * 1^1 + a_0 * 1^0) \mod 3 \\
&\equiv (a_k + a_{k-1} + ... + a_2 + a_1 + a_0) \mod 3
\end{aligned}$$

This shows that any natural number is equivalent to the sum of its digits modulo 3. So by Example 12 of the previous section, if the sum of a number's digits is divisible by three (equivalent to 0 modulo 3) then the original number is divisible by three (equivalent to 0 modulo 3).

Another common rule of divisibility is that for twos: if a number's one's digit is 0, 2, 4, 6, or 8 then that number is divisible by two.

**Example 14:** Show that any number whose one's digit is 0, 2, 4, 6, or 8 is divisible by two.

Proving or deriving any rule of divisibility for any number, $n$, works in essentially the same way. Consider a generic number using the variable notation for digits seen in the last example and operate on this number modulo $n$. After operating modulo $n$, look for a pattern to expand into a rule of divisibility.

For this rule, we'll use the same generic number, namely

$$a_k * 10^k + a_{k-1} * 10^{k-1} + ... + a_2 * 10^2 + a_1 * 10^1 + a_0 * 10^0$$

Next, note that $10 \equiv 0 \mod 2$. This will be useful for all powers of 10 *except* $10^0$. Since

$10^0 = 1$, we'll have $10^0 = 1 \equiv 1 \mod 2$:

$$
\begin{aligned}
& a_k * 10^k + a_{k-1} * 10^{k-1} + ... + a_2 * 10^2 + a_1 * 10^1 + a_0 * 10^0 \\
& \equiv (a_k * 0 + a_{k-1} * 0 + ... + a_2 * 0 + a_1 * 0 + a_0 * 1) \mod 2 \\
& \equiv a_2 \mod 2
\end{aligned}
$$

This shows that any natural number is equivalent to its one's digit modulo 0. Since the only single digits that are divisible by two are 0, 2, 4, 6, and 8, a natural number is divisible by two when its one's digit is 0, 2, 4, 6, or 8.

The last rule of divisibility we'll show here is that for eleven.

**Example 15:** Show that a number is divisible by eleven when the alternating sum of its digits is divisible by eleven.

An alternating sum is an arithmetic expression in which terms are connected by either addition or subtraction, alternating between the two. For example, one alternating sum is $2 - 6 + 8 - 7 + 3 - 1 + 7$.

Given a number like 125408, the alternating sum of its digits would look like this: $1 - 2 + 5 - 4 + 0 - 8$.

So, again, suppose a number is $k$-digits long so that its expand form is

$$a_k * 10^k + a_{k-1} * 10^{k-1} + ... + a_2 * 10^2 + a_1 * 10^1 + a_0 * 10^0$$

To move on from here, investigate what powers of ten are equivalent to, modulo 11.

$$10^0 = 1 \equiv 1 \mod 11 \qquad 10^1 = 10 \equiv -1 \mod 11$$

$$10^2 = 100 \equiv 1 \mod 11 \qquad 10^3 = 1000 \equiv -1 \mod 11$$

While we usually want results in modular arithmetic to be nonnegative, noting the equivalence between positive and negative values can be very useful in deriving rules of divisibility. So, as seen above, instead of using the equivalence $10 \equiv 10 \mod 11$ we can use the fact that $10 \equiv -1 \mod 11$. Just as important, notice how powers of ten are equivalent to either 1 or $-1$ modulo 11, with the sign alternating from one power to the next; even powers give an equivalent of 1 while odd powers give an equivalent of $-1$. We'll use this fact as we consider our generic number modulo 11:

$$\begin{aligned}
&a_k * 10^k + ... + a_3 * 10^3 + a_2 * 10^2 + a_1 * 10^1 + a_0 * 10^0 \\
&\equiv (a_k * (-1)^k + ... + a_3 * (-1) + a_2 * 1 + a_1 * (-1) + a_0 * 1) \mod 11 \\
&\equiv (a_k * (-1)^k + ... - a_3 + a_2 - a_1 + a_0) \mod 11
\end{aligned}$$

See in the last several terms that we have an alternating sum. The front end of this sum would look like

$$(a_k - ... - a_3 + a_2 - a_1 + a_0) \mod 11$$

if $k$ is even, or

$$(-a_k + ... - a_3 + a_2 - a_1 + a_0) \mod 11$$

if $k$ is odd.

To see an example of this at work, consider the number 12,573. The alternating sum of this number's digits would look like this:

$$1 - 2 + 5 - 7 + 3 = 0$$

Since the alternating sum of the digits is divisible by eleven (0 is divisible by all numbers) the original number, 12,573, is itself divisible by eleven. However, we need not get an alternating sum of zero. For example, the number 2,091,727 has the following alternating sum:

$$2 - 0 + 9 - 1 + 7 - 2 + 7 = 22$$

Since the alternating sum, 22, is divisible by eleven the number 2,091,727 is also divisible by eleven.

An interesting result of the last example is that all palindromic number (numbers like 1331) with an even number of digits are divisible by eleven.

## 5.5 Exercises

### 5.5.1 Divisibility and Primes

Give the prime factorization of each number.

1. 8
2. 12
3. 15
4. 17
5. 18
6. 24
7. 26
8. 28
9. 32
10. 39
11. 42
12. 45
13. 48
14. 49
15. 51
16. 63
17. 64
18. 65
19. 72
20. 78
21. 81
22. 82
23. 83
24. 90
25. 97

Identify the indicated GCD using prime factorization; this can be approached strategically or by listing all divisors of the given numbers and identifying the largest number appearing in both lists of divisors.

26. $GCD(12, 16)$
27. $GCD(24, 36)$
28. $GCD(18, 42)$
29. $GCD(63, 105)$
30. $GCD(72, 175)$
31. $GCD(64, 256)$
32. $GCD(82, 26)$
33. $GCD(99, 121)$

### 5.5.2 Euclid's Algorithm

Identify the indicated GCD using the Euclidean Algorithm.

34. $GCD(16, 14)$
35. $GCD(32, 18)$
36. $GCD(27, 6)$
37. $GCD(18, 0)$
38. $GCD(1, 36)$
39. $GCD(35, 63)$
40. $GCD(92, 108)$
41. $GCD(56, 98)$
42. $GCD(74, 30)$
43. $GCD(84, 25)$
44. $GCD(240, 60)$
45. $GCD(128, 72)$

46. $GCD(196, 92)$
47. $GCD(112, 154)$
48. $GCD(350, 140)$
49. $GCD(280, 56)$
50. $GCD(143, 65)$
51. $GCD(297, 162)$
52. $GCD(147, 246)$
53. $GCD(320, 420)$
54. $GCD(294, 154)$
55. $GCD(234, 156)$
56. $GCD(610, 255)$
57. $GCD(425, 357)$
58. $GCD(378, 105)$

### 5.5.3 Modular Arithmetic

What is $a$ equivalent to modulo $n$? (Give the smallest, nonnegative equivalent.)

59. $a = 17$, $n = 3$
60. $a = 24$, $n = 10$
61. $a = 13$, $n = 13$
62. $a = 15$, $n = 16$
63. $a = 17$, $n = 2$
64. $a = 10$, $n = 9$
65. $a = 25$, $n = 6$
66. $a = 18$, $n = 6$
67. $a = 36$, $n = 24$
68. $a = 36$, $n = 16$
69. $a = 42$, $n = 15$
70. $a = 29$, $n = 3$
71. $a = 34$, $n = 14$
72. $a = 39$, $n = 9$
73. $a = 49$, $n = 14$
74. $a = 52$, $n = 7$

Perform the indicated operation modulo $n$.

75. $16 + 24$, $n = 3$
76. $42 - 35$, $n = 6$
77. $15 - 27$, $n = 20$
78. $6 * 12$, $n = 8$
79. $11 + 34$, $n = 4$
80. $7 * 12$, $n = 9$
81. $4 * 35$, $n = 4$
82. $18 + 44$, $n = 7$
83. $34 - 106$, $n = 16$
84. $24 - 52$, $n = 7$
85. $15 * 16$, $n = 11$
86. $13 - 34$, $n = 15$
87. $8 * 9$, $n = 13$
88. $96 - 54$, $n = 9$
89. $12 + 43$, $n = 6$
90. $62 + 27$, $n = 9$
91. $52 + 64$, $n = 4$
92. $37 - 92$, $n = 13$
93. $42 - 43$, $n = 6$
94. $13 * 12$, $n = 5$
95. $15 * 16$, $n = 13$
96. $46 + 103$, $n = 25$
97. $63 + 42$, $n = 7$
98. $43 - 35$, $n = 10$

### 5.5.4 Rules of Divisibility

Use the rules of divisibility already covered to determine whether the given statement is true or false.

99. 256 is divisible by three.
100. 574 is divisible by eleven.
101. 108 is divisible by two.
102. 264 is divisible by three.
103. 10,207 is divisible by eleven.
104. 150,051 is divisible by eleven.
105. 479 is divisible by two.
106. 11,904 is divisible by three.
107. 7,590 is divisible by two.
108. 20,687 is divisible by two.
109. 13,531 is divisible by eleven.
110. 197 is divisible by three.
111. 108 is divisible by three.
112. 290 is divisible by eleven.
113. 513 is divisible by two.
114. 240 is divisible by two.
115. 10,154 is divisible by three.
116. 5,006 is divisible by eleven.

Find a rule of divisibility for the given number. (Describe how you can tell, without performing division, whether or not any natural number is divisible by the given number.)

117. 4
118. 5
119. 6
120. 9
121. 10
122. 7*

*Hint - Find the equivalent of *at least* the first seven powers of ten, modulo 7: $10^0$, $10^1$,..., $10^6$.

# Chapter 6

# Introduction to Linear Algebra

## 6.1 Addition, Subtraction, and Scalar Multiplication

In linear algebra a *matrix* is an ordered, rectangular array of objects such as numbers, expressions, or functions. Before investigating matrix arithmetic, consider the matrices below.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{bmatrix}$$

$2 \times 3$ matrix $\qquad$ $3 \times 2$ matrix $\qquad$ $3 \times 3$ matrix $\qquad$ $m \times n$ matrix

As we can see from the examples above, an $m \times n$ matrix has $m$ rows and $n$ columns, and the entry $x_{ij}$ is the entry in the $i^{th}$ row and $j^{th}$ column of matrix $X$. For example, in the $2 \times 3$ matrix

$$A = \begin{bmatrix} 2 & -4 & 6 \\ 0 & 9 & -1 \end{bmatrix}$$

$a_{11} = 2$, $a_{12} = -4$, $a_{13} = 6$, $a_{21} = 0$, $a_{22} = 9$, and $a_{23} = -1$.

We will focus on some of the basic matrix operations including addition, subtraction, multiplication, and scalar multiplication. The first of these, addition, is performed by adding corresponding entries; symbolically, if $X$, $Y$, and $Z$ are matrices with $X + Y = Z$ then $x_{ij} + y_{ij} = z_{ij}$.

**Example 1:** Perform the matrix addition, if possible.

(a) $\begin{bmatrix} 2 & -4 & 6 \\ 0 & 9 & -1 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 7 \\ -10 & 5 & 13 \end{bmatrix}$

Since addition of matrices is performed by adding corresponding entries, the first step is to identify which entries of the two matrices correspond to one another. For example, in the first row and first column of the two matrices the entries are 2 and 3; these are corresponding entries and will be added together, and the sum of these entries will be the entry of the first row and first column in the resulting matrix.

$$\begin{bmatrix} 2 & -4 & 6 \\ 0 & 9 & -1 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 7 \\ -10 & 5 & 13 \end{bmatrix} = \begin{bmatrix} 2+3 & -4+1 & 6+7 \\ 0+(-10) & 9+5 & -1+13 \end{bmatrix}$$

Simplifying the addition of entries, the result is

$$\begin{bmatrix} 5 & -3 & 13 \\ -10 & 14 & 12 \end{bmatrix}$$

(b) $\begin{bmatrix} 1 & 4 \\ 6 & 0 \end{bmatrix} + \begin{bmatrix} 8 & 1 & 5 \\ -10 & 2 & -3 \end{bmatrix}$

Notice that these matrices are different sizes, or dimensions; the first is a $2 \times 2$ matrix and the second is a $2 \times 3$ matrix. In a situation like this the addition is undefined because an entry of one matrix has no corresponding entry in the other. For example, the entries in the third column of the second matrix, 5 and $-3$, have no corresponding entries in the first matrix. As a result, this addition cannot be performed.

**Note:** For two matrices $A$ and $B$, their sum $A + B$ is defined only when $A$ and $B$ have the same dimensions; the number of rows of $A$ must be equal to the number of rows of $B$, and the number of columns of $A$ must be equal to the number of columns of $B$.

Given how matrix addition is defined, matrix subtraction works exactly as one might think; to find the difference $A - B$ the entries of $B$ are subtracted from corresponding entries of $A$. Matrix subtraction, like addition, is defined only when the matrices have the same dimensions.

**Example 2:** Perform the subtraction below, if possible.

(a) $\begin{bmatrix} 1.5 & 3 \\ 0 & 0.9 \end{bmatrix} - \begin{bmatrix} 3 & 1.5 \\ -4 & 2 \end{bmatrix}$

Since these matrices have the same dimensions the subtraction can be performed. Subtracting corresponding entries yields the following.

$$\begin{aligned} \begin{bmatrix} 1.5 & 3 \\ 0 & 0.9 \end{bmatrix} - \begin{bmatrix} 3 & 1.5 \\ -4 & 2 \end{bmatrix} &= \begin{bmatrix} 1.5 - 3 & 3 - 1.5 \\ 0 - (-4) & 0.9 - 2 \end{bmatrix} \\ &= \begin{bmatrix} -1.5 & 1.5 \\ 4 & -1.1 \end{bmatrix} \end{aligned}$$

(b) $\begin{bmatrix} 1.5 & 3 \\ 0 & 0.9 \end{bmatrix} - \begin{bmatrix} 3 & 1 & 7 \\ -10 & 2 & 13 \end{bmatrix}$

Because the matrices do not have the same dimensions (one matrix has two columns while the other has three) the subtraction is undefined; the subtraction cannot be performed.

(c) $\begin{bmatrix} 1 & 4 & 9 & 0 \\ -6 & -2 & 0 & -1 \\ 4 & -3 & 6 & 8 \end{bmatrix} - \begin{bmatrix} 12 & -9 & 1 & 7 \\ 5 & 0 & -10 & 6 \\ 12 & 10 & 3 & -4 \end{bmatrix}$

Because the dimensions of these matrices are the same the subtraction can be performed.

$$\begin{bmatrix} 1 & 4 & 9 & 0 \\ -6 & -2 & 0 & -1 \\ 4 & -3 & 6 & 8 \end{bmatrix} - \begin{bmatrix} 12 & -9 & 1 & 7 \\ 5 & 0 & -10 & 6 \\ 12 & 10 & 3 & -4 \end{bmatrix} = \begin{bmatrix} 1-12 & 4-(-9) & 9-1 & 0-7 \\ -6-5 & -2-0 & 0-(-10) & -1-6 \\ 4-12 & -3-10 & 6-3 & 8-(-4) \end{bmatrix}$$

$$= \begin{bmatrix} -11 & 13 & 8 & -7 \\ -11 & -2 & 10 & -7 \\ -8 & -13 & 3 & 12 \end{bmatrix}$$

(d) $\begin{bmatrix} 2 & -4 & 6 \\ 0 & 5 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 0 & -10 \\ 2 & 0 \end{bmatrix}$

This subtraction cannot be performed since the matrices have different dimensions.

The next operation we look at will be scalar multiplication, or multiplication of a matrix by a number (the scalar). We can define scalar multiplication just as we define multiplication of two numbers; in terms of addition. For example, $5 * 6 = 6 + 6 + 6 + 6 + 6$. In the same way, if $A$ is a matrix then $5 * A = A + A + A + A + A$.

**Example 3:** Simplify the scalar multiplication.

(a) $3 * \begin{bmatrix} -3 & 4 \\ 5 & 0 \end{bmatrix}$

Using the approach described above, we can rewrite the given expression as addition then perform the addition:

$$\begin{aligned} 3 * \begin{bmatrix} -3 & 4 \\ 5 & 0 \end{bmatrix} &= \begin{bmatrix} -3 & 4 \\ 5 & 0 \end{bmatrix} + \begin{bmatrix} -3 & 4 \\ 5 & 0 \end{bmatrix} + \begin{bmatrix} -3 & 4 \\ 5 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -3+(-3)+(-3) & 4+4+4 \\ 5+5+5 & 0+0+0 \end{bmatrix} \\ &= \begin{bmatrix} -9 & 12 \\ 15 & 0 \end{bmatrix} \end{aligned} \tag{2}$$

Notice that this scalar multiplication can be computed a bit faster. Rather than writing out this repeated addition of entries, as in line (2) above, each entry can simply be multiplied by the scalar:

$$3 * \begin{bmatrix} -3 & 4 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 3*(-3) & 3*4 \\ 3*5 & 3*0 \end{bmatrix}$$
$$= \begin{bmatrix} -9 & 12 \\ 15 & 0 \end{bmatrix}$$

In general, for a scalar $k$ and a matrix $X$,

$$k * X = \begin{bmatrix} k*x_{11} & k*x_{12} & \dots & k*x_{1n} \\ k*x_{21} & k*x_{22} & \dots & k*x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ k*x_{m1} & k*x_{m2} & \dots & k*x_{mn} \end{bmatrix}$$

(b) $5 * \begin{bmatrix} 1 & 0 \\ 0 & 3 \\ -1 & 6 \end{bmatrix}$

$$5 * \begin{bmatrix} 1 & 0 \\ 0 & 3 \\ -1 & 6 \end{bmatrix} = \begin{bmatrix} 5*1 & 5*0 \\ 5*0 & 5*3 \\ 5*(-1) & 5*6 \end{bmatrix}$$
$$= \begin{bmatrix} 5 & 0 \\ 0 & 15 \\ -5 & 30 \end{bmatrix}$$

## 6.2 Vectors, the Dot Product, and Matrix Multiplication

Before we look at multiplication of two matrices it will help to look at the *dot product* of *vectors*. We can think of *vectors* as ordered $n$-tuples (like ordered pairs, ordered triplets, etc.). If $\mathbf{v}$ and $\mathbf{w}$

are vectors with the same number of entries, then the *dot product* $\mathbf{v} \cdot \mathbf{w}$ is computed by finding the sum of the products of corresponding entries. For example, if $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ then the dot product of $\mathbf{v}$ and $\mathbf{w}$ is

$$\mathbf{v} \cdot \mathbf{w} = v_1 * w_1 + v_2 * w_2 + v_3 * w_3$$

**Example 4:** Compute the dot product of each pair of vectors.

(a) $\mathbf{v} = (1, -2, 5)$, $\mathbf{w} = (0, 6, 3)$

$$\begin{aligned} \mathbf{v} \cdot \mathbf{w} &= (1, -2, 5) \cdot (0, 6, 3) \\ &= 1 * 0 + (-2) * 6 + 5 * 3 \\ &= 0 - 12 + 15 \\ &= 3 \end{aligned}$$

(b) $\mathbf{v} = (7, 2, -2, 1)$, $\mathbf{w} = (6, 4, 0, 7)$

$$\begin{aligned} \mathbf{v} \cdot \mathbf{w} &= (7, 2, -2, 1) \cdot (6, 4, 0, 7) \\ &= 7 * 6 + 2 * 4 + (-2) * 0 + 1 * 7 \\ &= 42 + 8 + 0 + 7 \\ &= 57 \end{aligned}$$

**Note:** The dot product of two vectors is defined only when the vectors have the same number of entries.

The reason we look at the dot product before discussing matrix multiplication is because we define matrix multiplication as a sequence of dot products between the rows of the first matrix and the columns of the second. Letting $A$ be an $l \times m$ matrix and $B$ be an $m \times n$ matrix, we can view the rows of $A$ as vectors and the columns of $B$ as vectors. With this perspective we define the product $AB$ as shown below.

$$AB = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{l1} & a_{l2} & \dots & a_{lm} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_l \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_n \end{bmatrix}$$

Writing the $i^{th}$ row of $A$ as vector $\mathbf{a}_i$ and the $j^{th}$ column of $B$ as vector $\mathbf{b}_j$.

$$= \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{b}_1 & \mathbf{a}_1 \cdot \mathbf{b}_2 & \dots & \mathbf{a}_1 \cdot \mathbf{b}_n \\ \mathbf{a}_2 \cdot \mathbf{b}_1 & \mathbf{a}_2 \cdot \mathbf{b}_2 & \dots & \mathbf{a}_2 \cdot \mathbf{b}_n \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_l \cdot \mathbf{b}_1 & \mathbf{a}_l \cdot \mathbf{b}_2 & \dots & \mathbf{a}_l \cdot \mathbf{b}_n \end{bmatrix}$$

**Note:** The product of matrices $A$ and $B$ is defined when the number of columns of $A$ is equal to the number of rows of $B$. Furthermore, the product of an $l \times m$ matrix and an $m \times n$ matrix is an $l \times n$ matrix.

**Example 5:** Find the product $AB$.

(a) $A = \begin{bmatrix} 5 & 3 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 6 \\ 5 & 3 \end{bmatrix}$

$$\begin{bmatrix} 5 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 6 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} (5,3) \cdot (-2,5) & (5,3) \cdot (6,3) \\ (0,1) \cdot (-2,5) & (0,1) \cdot (6,3) \end{bmatrix}$$

$$= \begin{bmatrix} (-10+15) & (30+9) \\ (0+5) & (0+3) \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 39 \\ 5 & 3 \end{bmatrix}$$

For part (a), the rows of $A$ and the columns of $B$ are colored to help illustrate how rows and columns are combined through dot products to find the product of $A$ and $B$. For example, the dot product of the first row of $A$, $(5,3)$, and the first column of $B$, $(-2,5)$, provides the entry in the first row and first column of $AB$, 5.

(b) $A = \begin{bmatrix} -2 & 6 \\ 5 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 3 \\ 0 & 1 \end{bmatrix}$

$$\begin{aligned} \begin{bmatrix} -2 & 6 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} (-2,6)\cdot(5,0) & (-2,6)\cdot(3,1) \\ (5,3)\cdot(5,0) & (5,3)\cdot(3,1) \end{bmatrix} \\ &= \begin{bmatrix} (-10+0) & (-6+6) \\ (25+0) & (15+3) \end{bmatrix} \\ &= \begin{bmatrix} -10 & 0 \\ 25 & 18 \end{bmatrix} \end{aligned}$$

The difference between part (b) and part (a) is the order in which we're multiplying the matrices; the definitions of $A$ and $B$ have been swapped. Because of this change the resulting product is also different. In short, *matrix multiplication is* **not** *commutative.*

(c) $A = \begin{bmatrix} 2 & -4 & 6 \\ 0 & 5 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 0 & -10 \\ 2 & 0 \end{bmatrix}$

$$\begin{aligned} \begin{bmatrix} 2 & -4 & 6 \\ 0 & 5 & -1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & -10 \\ 2 & 0 \end{bmatrix} &= \begin{bmatrix} (2,-4,6)\cdot(3,0,2) & (2,-4,6)\cdot(1,-10,0) \\ (0,5,-1)\cdot(3,0,2) & (0,5,-1)\cdot(1,-10,0) \end{bmatrix} \\ &= \begin{bmatrix} (6+0+12) & (2+40+0) \\ (0+0-2) & (0-50+0) \end{bmatrix} \\ &= \begin{bmatrix} 18 & 42 \\ -2 & -50 \end{bmatrix} \end{aligned}$$

Notice that the matrices in part (c) have different dimensions: $A$ is a $2 \times 3$ matrix while $B$ is a $3 \times 2$ matrix. However, since the number of rows of columns of $A$ (3) is equal to the number or rows of $B$ (also 3) the multiplication between $A$ and $B$ is defined. Also, the product the $2 \times 3$ matrix and the $3 \times 2$ matrix results in a $2 \times 2$ matrix. If the order is changed, as in part (d), the result will instead be a $3 \times 3$ matrix.

(d) $A = \begin{bmatrix} 3 & 1 \\ 0 & -10 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -4 & 6 \\ 0 & 5 & -1 \end{bmatrix}$

$$\begin{bmatrix} 3 & 1 \\ 0 & -10 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & -4 & 6 \\ 0 & 5 & -1 \end{bmatrix} = \begin{bmatrix} (3,1)\cdot(2,0) & (3,1)\cdot(-4,5) & (3,1)\cdot(6,-1) \\ (0,-10)\cdot(2,0) & (0,-10)\cdot(-4,5) & (0,-10)\cdot(6,-1) \\ (2,0)\cdot(2,0) & (2,0)\cdot(-4,5) & (2,0)\cdot(6,-1) \end{bmatrix}$$

$$= \begin{bmatrix} (6+0) & (-12+5) & (18-1) \\ (0+0) & (0-50) & (0+10) \\ (4+0) & (-8+0) & (12+0) \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -7 & 17 \\ 0 & -50 & 10 \\ 4 & -8 & 12 \end{bmatrix}$$

(e) $A = \begin{bmatrix} 1 & 4 \\ 6 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 8 & 1 \\ 5 & -10 \\ 2 & -3 \end{bmatrix}$

In this case $A$ is a $2 \times 2$ matrix while $B$ is a $3 \times 2$ matrix; since the number of columns in $A$ does not match the number of rows in $B$ the product $AB$ does not exist.

(f) $A = \begin{bmatrix} 8 & 1 \\ 5 & -10 \\ 2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 \\ 6 & 0 \end{bmatrix}$

$$\begin{bmatrix} 8 & 1 \\ 5 & -10 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 6 & 0 \end{bmatrix} = \begin{bmatrix} (8,1)\cdot(1,6) & (8,1)\cdot(4,0) \\ (5,-10)\cdot(1,6) & (5,-10)\cdot(4,0) \\ (2,-3)\cdot(1,6) & (2,-3)\cdot(4,0) \end{bmatrix}$$

$$= \begin{bmatrix} (8+6) & (32+0) \\ (5-60) & (20+0) \\ (2-18) & (8+0) \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 32 \\ -55 & 20 \\ -16 & 8 \end{bmatrix}$$

The difference between parts (e) and (f) is a matter of order, but this simple change is important. This time $A$ is a $3 \times 2$ matrix and $B$ is a $2 \times 2$ matrix. Since the number of columns of $A$ matches the number of rows in $B$, the product $AB$ is defined and will be a $3 \times 2$ matrix.

(g) $A = \begin{bmatrix} 6 & -4 & 0 \\ 5 & 1 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

In part (g) we see a special type of matrix: an identity matrix. A square matrix like $B$, with 1's along the diagonal and 0's elsewhere, is called an $n \times n$ *identity matrix* and denoted $I_n$. In this case, $B$ is a $3 \times 3$ identity matrix, so $B = I_3$. Such matrices are special because for a matrix $X$, if $XI_n$ is defined then $XI_n = X$. Similarly, if $I_nX$ is defined then $I_nX = X$.

In this case, we have

$$\begin{bmatrix} 6 & -4 & 0 \\ 5 & 1 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -4 & 0 \\ 5 & 1 & 7 \end{bmatrix}$$

| **Properties of Matrix Multiplication** | |
|---|---|
| Let $A$ be a $k \times l$ matrix and $B$ be an $m \times n$ matrix. | |
| 1. | If $l = m$ then the product $AB$ is defined and is a $k \times n$ matrix. |
| 2. | If $l \neq m$ then the product $AB$ does not exist. |
| 3. | If $k = n$ then the product $BA$ is defined and is a $m \times l$ matrix. |
| 4. | If $k \neq n$ then the product $BA$ does not exist. |
| 5. | A square matrix with 1's on the diagonal and 0's elsewhere is called an identity matrix; when such a matrix has $n$ rows it is denoted $I_n$. |
| 6. | For any matrix $X$, if $XI_n$ is defined then $XI_n = X$. If $I_nX$ is defined then $I_nX = X$. |
| 7. | Matrix multiplication *is not* commutative. (Compare Example 5 parts (a) and (b).) |
| 8. | Matrix multiplication *is* associative. (See exercises on associativity.) |

## 6.3 Matrices and Linear Systems

A *linear system* is a collection of linear equations of $n$ variables. For example, below are systems of linear equations of $n = 2$, $n = 3$, and $n = 4$ variables.

| $\underline{n=2}$ | $\underline{n=3}$ | $\underline{n=4}$ |
|---|---|---|
| $x+2y=7$ | $x+y+z=7$ | $x_1-x_2=8$ |
| $5x-3y=2$ | $5x-y=4$ | $3x_1+x_2-x_3-2x_4=-3$ |
| | $x-7y+2z=-1$ | $2x_1+x_3-x_4=0$ |
| | | $3x_2+4x_3+9x_4=-10$ |

Notice in two of these examples that not all variables need to be used in all equations. Also note that in each of these examples the number of equations in the system is equal to the number of variables; while most of the systems considered in this section will share this quality, it is not necessary. The system below, for example, has only three equations of four variables.

$$3x_2-2x_2+5x_3+x_4=7$$

$$2x_1+x_3-2x_4=8$$

$$5x_1+x_4=3$$

While not all variables need to be explicitly used in each equation, understand that the missing variable terms *can* be included; the missing terms in the previous example can be included as shown below.

$$3x_2-2x_2+5x_3+x_4=7$$

$$2x_1+0x_2+x_3-2x_4=8$$

$$5x_1+0x_2+0x_3+x_4=3$$

This is particularly useful when representing a linear system as a matrix equation of the form

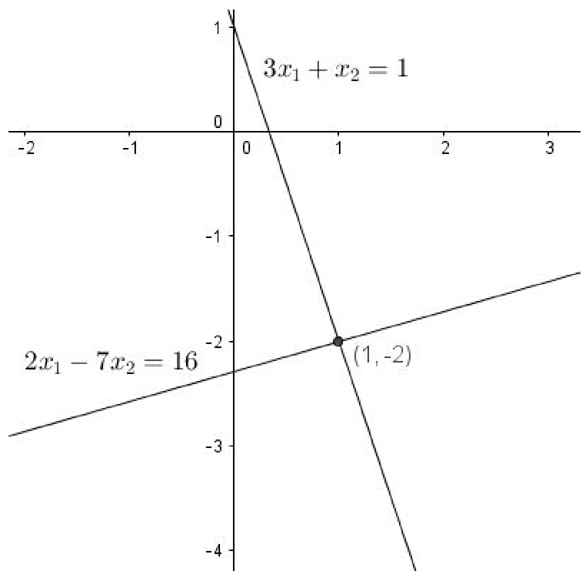
$A\mathbf{x} = \mathbf{b}$, where $A$ is the *coefficient matrix*, $\mathbf{x}$ is a column vector with the system's variables as entries, and $\mathbf{b}$ is the column vector with the system's constant terms as entries. For example, below is a linear system of three variables and the associated matrix equation.

$$\begin{aligned} 3x_1 + 5x_2 - 3x_3 &= 10 \\ x_1 + 4x_2 + 0x_3 &= 7 \\ 2x_1 - x_2 + x_3 &= 15 \end{aligned} \qquad \begin{bmatrix} 3 & 5 & -3 \\ 1 & 4 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 7 \\ 15 \end{bmatrix}$$

Matrices will be used in this section not only to represent linear systems, but also to *solve* linear systems. The *solution set of a system* of $n$ variables consists of all ordered $n$-tuples which are solutions to each equation of the system; graphically, a solution is a point where the graphs of all equations in the system intersect. Below we have yet another example of a system, this time of two variables, and a graph depicting the lines described by each equation.

$$\begin{aligned} 2x_1 - 7x_2 &= 16 \\ 3x_1 + x_2 &= 1 \end{aligned}$$

In the graph to the right the two lines described by $3x_1 + x_2 = 1$ and $2x_1 - 7x_2 = 16$ are depicted. Their point of intersection, $(1, -2)$, is also the one ordered pair which is a solution to both equations and therefore the solution of the system.




Equivalently, $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is the solution of the matrix equation $\begin{bmatrix} 2 & -7 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 16 \\ 1 \end{bmatrix}$.

The focus of this section is <u>how</u> such a linear system can be solved. This is done by performing *row operations* on a *partitioned matrix* until the matrix is in *reduced row echelon form* in which the first nonzero entry of each row is a 1 is the only nonzero entry in its column.

The aforementioned row operations include:

R1: swapping rows

R2: multiplying a row by a scalar

R3: replacing a row with the sum of itself an another row

These operations can be combined or performed simultaneously.

**Example 5:** Solve $\begin{bmatrix} 3 & 5 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -8 \end{bmatrix}$

.

First we'll write the associated augmented partitioned matrix, which combines the coefficient matrix and the column vector $\begin{bmatrix} -1 \\ -8 \end{bmatrix}$.

$$\left[\begin{array}{cc|c} 3 & 5 & -1 \\ -2 & 1 & -8 \end{array}\right]$$

To solve the original equation we can perform row operations R1, R2, and R3 until the matrix is in reduced row echelon form; the beauty behind these row operations is that they allow us to change the appearance of the augmented partitioned matrix with each new matrix representing an equivalent system (an system with the same solution set).

$$\left[\begin{array}{cc|c} 3 & 5 & -1 \\ -2 & 1 & -8 \end{array}\right] \xrightarrow{R3: row1+row2=row1} \left[\begin{array}{cc|c} 1 & 6 & -9 \\ -2 & 1 & -8 \end{array}\right] \xrightarrow{R2: 2*row1=row1} \left[\begin{array}{cc|c} 2 & 12 & -18 \\ -2 & 1 & -8 \end{array}\right] \ldots$$

In the first step rows 1 and 2 are added together, with the result replacing row 1. In the resulting matrix, row 1 is then multiplied by 2. This is done so that the first entries in rows 1 and 2 are opposites. In the third step, shown below, rows 1 and 2 are added together with the result replacing row 2; notice that the first entry in row 2 is now 0.

$$\xrightarrow{R3: row2+row1=row2} \left[\begin{array}{cc|c} 2 & 12 & -18 \\ 0 & 13 & -26 \end{array}\right] \xrightarrow[R2: \frac{1}{13}*row2=row2]{R2: \frac{1}{2}*row1=row1} \left[\begin{array}{cc|c} 1 & 6 & -9 \\ 0 & 1 & -2 \end{array}\right] \ldots$$

In the fourth step, above, row 1 is multiplied by $\frac{1}{2}$ and row 2 is multiplied by $\frac{1}{13}$; now the first nonzero entry in each row is a 1. The final step, shown below, is the first time that two row operations have been used simultaneously. Multiplying row 2 by $-6$ and adding the result to row 1 results in the second entry of row 1 being 0, bringing our augmented partitioned matrix into *reduced row echelon form*.

$$\xrightarrow{R2/R3: row1-6*row2=row1} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right]$$

At each stage in this process a new augmented partitioned matrix was found, and each of these matrices represents a system with the same solution set as the original. Then the original matrix equation can be rewritten using our final augmented partitioned matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Simplifying the left side of this equation we have our solution:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

**Example 6:** A company's analyst team finds that the quantity supplied and the quantity demanded for a particular product are related to the retail price by the equations below

$$demand: \quad 2q + 3p = 340$$

$$supply: \quad -75q + 100p = 0$$

where $p$ is the retail price and $q$ is the quantity. Solve this system to find the equilibrium price and quantity; in other words, find the point at which supply meets demand with no surplus or shortage.

This system is represented by the augmented partitioned matrix

$$\left[\begin{array}{cc|c} 2 & 3 & 340 \\ -75 & 100 & 0 \end{array}\right]$$

From here, solving the system means writing the augmented partitioned matrix above in reduced row echelon form.

$$\begin{array}{rcll}
\left[\begin{array}{cc|c} 2 & 3 & 340 \\ -75 & 100 & 0 \end{array}\right] & \xrightarrow{R2} & \left[\begin{array}{cc|c} 2 & 3 & 340 \\ -3 & 4 & 0 \end{array}\right] & \frac{1}{25} * row2 = row2 \\
& \xrightarrow{R3} & \left[\begin{array}{cc|c} -1 & 7 & 340 \\ -3 & 4 & 0 \end{array}\right] & row1 + row2 = row1 \\
& \xrightarrow{R2} & \left[\begin{array}{cc|c} 1 & -7 & -340 \\ -3 & 4 & 0 \end{array}\right] & -1 * row1 = row1 \\
& \xrightarrow{R2/R3} & \left[\begin{array}{cc|c} 1 & -7 & -340 \\ 0 & -17 & -1020 \end{array}\right] & row2 + 3 * row1 = row2 \\
& \xrightarrow{R2} & \left[\begin{array}{cc|c} 1 & -7 & -340 \\ 0 & 1 & 60 \end{array}\right] & -\frac{1}{17} * row2 = row2 \\
& \xrightarrow{R2/R3} & \left[\begin{array}{cc|c} 1 & 0 & 80 \\ 0 & 1 & 60 \end{array}\right] & row1 + 7 * row2 = row1
\end{array}$$

Our solution here is $\begin{bmatrix} 80 \\ 60 \end{bmatrix}$, or $q = 80$ and $p = 60$. In the context of this problem, this means that at a price of \$60 per unit, supply will meet demand exactly.

In the examples we've seen so far each system has had exactly one solution; as we'll see in the next example, this is not always the case.

**Example 7:** Solve the system below.

$$2x_1 + x_3 = 2$$

$$x_1 + 3x_2 + 2x_3 = -1$$

$$4x_1 + 6x_2 + 5x_3 = 0$$

As before, this system can be represented by an augmented partitioned matrix and solved by writing the augmented partitioned matrix in reduced row echelon form.

$$\left[\begin{array}{ccc|c} 2 & 0 & 1 & 2 \\ 1 & 3 & 2 & -1 \\ 4 & 6 & 5 & 0 \end{array}\right] \xrightarrow{R1} \left[\begin{array}{ccc|c} 1 & 3 & 2 & -1 \\ 2 & 0 & 1 & 2 \\ 4 & 6 & 5 & 0 \end{array}\right] \quad \text{swap } row1 \text{ and } row2$$

$$\xrightarrow{R2/R3} \left[\begin{array}{ccc|c} 1 & 3 & 2 & -1 \\ 0 & -6 & -3 & 4 \\ 0 & -6 & -3 & 4 \end{array}\right] \quad \begin{array}{c} row2 - 2 * row1 = row2 \\ row3 - 4 * row1 = row3 \end{array}$$

$$\xrightarrow{R3} \left[\begin{array}{ccc|c} 1 & 3 & 2 & -1 \\ 0 & -6 & -3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad row3 - row2 = row3$$

$$\xrightarrow{R3} \left[\begin{array}{ccc|c} 1 & 3 & 2 & -1 \\ 0 & 1 & \frac{1}{2} & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array}\right] \quad -\frac{1}{6} * row2 = row2$$

$$\xrightarrow{R2/R3} \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & -3 \\ 0 & 1 & \frac{1}{2} & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array}\right] \quad row1 - 3 * row2 = row1$$

Translating the last augmented partitioned matrix above into a matrix equation, we get

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ -\frac{3}{2} \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 1x_1 + 0x_2 + \frac{1}{2}x_3 \\ 0x_1 + 1x_2 + \frac{1}{2}x_3 \\ 0x_1 + 0x_2 + 0x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ -\frac{3}{2} \\ 0 \end{bmatrix}$$

This last matrix equation translates into three linear equations: $x_1 + \frac{1}{2}x_3 = -3$, $x_2 + \frac{1}{2}x_3 = \frac{2}{3}$, and 0=0. Of these three, the last is an identity and does not help us find specific solutions of the system, so we can focus on the first two equations. Solving the first and second of these equations for $x_1$ and $x_2$ respectively yields

$$x_1 = -\frac{1}{2}x_3 - 3$$

and

$$x_2 = -\frac{1}{2}x_3 + \frac{2}{3}$$

In this situation $x_3$ is called a *free variable*; any value may be chosen for $x_3$ and for each values of $x_3$ there exist associated values for $x_1$ and $x_2$. So the solutions of this system are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}k - 3 \\ -\frac{1}{2}k + \frac{2}{3} \\ k \end{bmatrix}$$

for any number $k$.

In the last example there were an infinite number of solutions - each value of $k$ yields a different solution to the original system. However, if the equations we started with were just slightly different then a very different result would emerge. Consider the system below, almost identical to the last system; only the constant term in the third equation has been changed.

$$2x_1 + x_3 = 2$$

$$x_1 + 3x_2 + 2x_3 = -1$$

$$4x_1 + 6x_2 + 5x_3 = 1$$

Attempting to solve this system would lead to the augmented partitioned matrix below.

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & -3 \\ 0 & 1 & \frac{1}{2} & \frac{2}{3} \\ 0 & 0 & 0 & 1 \end{array}\right]$$

In this case the third row translates into the equation $0 = 1$; this is a contradiction. Since this equation can never be true, this new system has no solution.

## 6.4 Exercises

### 6.4.1 Addition, Subtraction, and Scalar Multiplication

1. Can any two matrices be added together? Why or why not?
2. Let $A$ be an $m \times n$ matrix. Describe a matrix $B$ such that $A + B = A$.
3. Let $A$ and $B$ be as described in problem 2. Describe a matrix $C$ such that $A + C = B$.
4. Is matrix addition commutative? Is it associative?
5. Is matrix subtraction commutative? Is it associative?
6. Does scalar multiplication distribute across matrix addition?

Find the sum, if it exists.

7. $\begin{bmatrix} 1 & 2 \\ 5 & -9 \end{bmatrix} + \begin{bmatrix} 8 & 9 \\ 0 & 6 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 3 & 2 \end{bmatrix} + \begin{bmatrix} -5 & 0 & 2 \end{bmatrix}$

9. $\begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix} + \begin{bmatrix} 12 \\ 9 \\ -3 \end{bmatrix}$

10. $\begin{bmatrix} 12 & 0 \\ 1 & -16 \end{bmatrix} + \begin{bmatrix} -2 & 5 \\ 0 & 8 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 0 & 3 \\ 6 & \frac{9}{2} & 2 \\ 0 & 5 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 6 \\ 2 & -5 & \frac{4}{3} \\ 3 & -7 & 0 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 5 & 6 & 3 & -5 \\ 2 & 0 & 4 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 6 & 11 & 5 \\ -3 & 4 & 0 & 2 \\ 5 & 1 & 1 & 9 \end{bmatrix}$

13. $\begin{bmatrix} \frac{2}{3} & 0 \\ 1 & 9 \end{bmatrix} + \begin{bmatrix} 1 \\ 8 \end{bmatrix}$

14. $\begin{bmatrix} 1 & \frac{2}{5} & 6 \\ 3 & 0 & 4 \\ -2 & 5 & -6 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

15. $\begin{bmatrix} 2 & 0 \\ 6 & -5 \\ -13 & 7 \end{bmatrix} + \begin{bmatrix} 5 & 2 \\ 11 & 7 \\ 50 & -14 \end{bmatrix}$

16. $\begin{bmatrix} 6 & 1 & 9 & 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \\ 0 \\ 8 \end{bmatrix}$

Find the difference, if it exists.

17. $\begin{bmatrix} 5 \\ -2 \\ -6 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ -7 \end{bmatrix}$

18. $\begin{bmatrix} 0 & -3 & 7 \end{bmatrix} - \begin{bmatrix} 8 & 3 & 10 \end{bmatrix}$

19. $\begin{bmatrix} 24 & 6 \\ 7 & 12 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 7 & 5 \end{bmatrix}$

20. $\begin{bmatrix} 4 & 20 \\ 14 & 6 \end{bmatrix} - \begin{bmatrix} -2 & 15 \\ 8 & 8 \end{bmatrix}$

21. $\begin{bmatrix} 18 & -7 \\ -4 & -3 \end{bmatrix} - \begin{bmatrix} -4 & \frac{1}{7} & 2 \\ 6 & 1 & 8 \end{bmatrix}$

22. $\begin{bmatrix} \frac{1}{2} & \frac{3}{2} & 9 \\ 1 & -2 & 0 \end{bmatrix} - \begin{bmatrix} 3 & \frac{3}{4} & 0 \\ 0 & \frac{1}{2} & 6 \end{bmatrix}$

23. $\begin{bmatrix} 7 & 10 & 23 \\ -7 & 12 & -4 \\ 1 & -9 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 2 & -3 \\ -4 & 5 & 8 \\ 6 & -9 & 1 \end{bmatrix}$

24. $\begin{bmatrix} 1 & 3 & 6 \\ 5 & 3 & 12 \\ 0 & -6 & 20 \\ 0 & 8 & 1 \end{bmatrix} - \begin{bmatrix} 24 & -16 & 7 \\ 15 & 0 & 13 \\ 1 & 9 & 23 \\ -8 & -10 & 4 \end{bmatrix}$

25. $\begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 5 & 8 \end{bmatrix} - \begin{bmatrix} 13 & 21 \\ 34 & 55 \\ 84 & 139 \end{bmatrix}$

26. $\begin{bmatrix} \frac{2}{3} & \frac{5}{6} \\ -2 & 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & -1 \\ \frac{1}{3} & \frac{1}{6} \end{bmatrix}$

Perform the scalar multiplication

27. $2\begin{bmatrix} 6 \\ 3 \end{bmatrix}$

28. $\frac{2}{3}\begin{bmatrix} 9 & 0 \\ 12 & -60 \end{bmatrix}$

29. $3\begin{bmatrix} \frac{2}{3} & \frac{5}{6} \\ -2 & 0 \end{bmatrix}$

30. $-3\begin{bmatrix} 2 & -7 & 1.6 \end{bmatrix}$

31. $2.6\begin{bmatrix} 1 & 0 \\ -3 & 5.2 \\ 30 & 7 \end{bmatrix}$

32. $-4\begin{bmatrix} \frac{5}{3} & 2 & 6 \\ -12 & 14 & \frac{7}{2} \\ 0 & -5 & -1 \end{bmatrix}$

33. $6\begin{bmatrix} 1 & 3 & 6 \\ 5 & 3 & 12 \\ 0 & -6 & 20 \\ 0 & 8 & 1 \end{bmatrix}$

34. $2.6\begin{bmatrix} 1 & 0 \\ -3 & 5.2 \\ 30 & 7 \end{bmatrix}$

35. $-4\begin{bmatrix} \frac{5}{3} & 2 & 6 \\ -12 & 14 & \frac{7}{2} \\ 0 & -5 & -1 \end{bmatrix}$

Simplify.

36. $3\begin{bmatrix} 1 & 2 \\ 5 & -9 \end{bmatrix} - \begin{bmatrix} 8 & 9 \\ 0 & 6 \end{bmatrix}$

37. $-\begin{bmatrix} 1 & 3 \\ 5 & -7 \end{bmatrix} + 4\begin{bmatrix} 1 & -8 \\ 2 & 3 \end{bmatrix}$

38. $\begin{bmatrix} 10 \\ 14 \end{bmatrix} - (-2)\begin{bmatrix} -7 \\ 12 \end{bmatrix}$

39. $4\begin{bmatrix} 8 & -2 & 3 \\ 4 & 0 & 6 \end{bmatrix} + 2\begin{bmatrix} 12 & 8 & -2 \\ 0 & -11 & 13 \end{bmatrix}$

40. $\frac{1}{2}\begin{bmatrix} 1 & 6 & -8 \\ 4 & \frac{6}{5} & 0 \end{bmatrix} + \begin{bmatrix} 7 & -3 & 5 \\ -1 & 2 & 1 \end{bmatrix}$

41. $-8\begin{bmatrix} 0 & 2.7 & 4 & 3.2 \\ -9 & 1 & 0.6 & 6 \end{bmatrix} + 7\begin{bmatrix} 0.3 & 2 & 1.1 & 2.5 \\ 0 & -4 & 0.6 & 0 \end{bmatrix}$

42. $6\begin{bmatrix} \frac{2}{3} & -\frac{2}{8} \\ 1 & \frac{1}{2} \\ 3 & 9 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} 12 & 8 \\ 9 & 4 \\ -8 & 48 \end{bmatrix}$

43. $-2.5\begin{bmatrix} 1 & 2 & -3 \\ 0 & 14 & 5 \\ 0 & -1 & 7 \end{bmatrix} - 1.8\begin{bmatrix} 0 & 1 & -6 \\ 12 & -5 & 4 \\ 20 & 6 & -2 \end{bmatrix}$

44. $2\begin{bmatrix} 4 & 0 \\ 1 & -3 \end{bmatrix} - 4\begin{bmatrix} 1 & -4 \\ 7 & -2 \end{bmatrix} + 5\begin{bmatrix} 0 & 3 \\ 1 & 8 \end{bmatrix}$

45. $2\begin{bmatrix} 1 & -2 \\ 3 & 6 \end{bmatrix} - \left(2\begin{bmatrix} 4 & 7 \\ -3 & 5 \end{bmatrix} + \begin{bmatrix} 6 & 10 \\ -2 & 11 \end{bmatrix}\right)$

46. $5\left(\begin{bmatrix} 5 & -2 \\ 0 & -7 \end{bmatrix} + 3\begin{bmatrix} 6 & 2 \\ 0 & -8 \end{bmatrix}\right) - 2\left(4\begin{bmatrix} 10 & -3 \\ -6 & 4 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ 0 & -15 \end{bmatrix}\right)$

47. $3\begin{bmatrix} 50 & 6.2 \\ 3 & 9 \\ -1.2 & 30 \\ 45 & 11 \end{bmatrix} + 5.1\begin{bmatrix} 2 & 6 \\ 9.5 & 4 \\ 30 & -1 \\ 6 & 24 \end{bmatrix} - 12\begin{bmatrix} 8.7 & -8 \\ 4 & 16 \\ -8 & 0 \\ 3 & 10.2 \end{bmatrix}$

### 6.4.2 Vectors, the Dot Product, and Matrix Multiplication

48. Can any two matrices be multiplied together? Why or why not?

49. Is matrix multiplication commutative? Is it associative?

50. Let $A$ be an $m \times n$ matrix. Describe a matrix $B$ such that $AB = A$. Describe a matrix $C$ such that $CA = A$. Under what conditions will $B$ and $C$ be the same matrix (when will $B = C$)?

Find the dot product $\mathbf{u}\cdot\mathbf{v}$, if it exists.

51. $\mathbf{u}=(3,0)$, $\mathbf{v}=(1,6)$

52. $\mathbf{u}=(-6)$, $\mathbf{v}=(5)$

53. $\mathbf{u}=(3,2,5)$, $\mathbf{v}=(4,7)$

54. $\mathbf{u}=(1,6,-3)$, $\mathbf{v}=(5,2,0)$

55. $\mathbf{u}=(-6,8,4)$, $\mathbf{v}=(0,9)$

56. $\mathbf{u}=(6,0)$, $\mathbf{v}=(-2,5)$

57. $\mathbf{u}=(1,0,8)$, $\mathbf{v}=(7,-10,-9)$

58. $\mathbf{u}=(2,2,5,1)$, $\mathbf{v}=(6,11,-2,4)$

59. $\mathbf{u}=(3,5,-4,-1)$, $\mathbf{v}=(0,5,-7,9)$

60. $\mathbf{u}=(10,8,-6,7)$, $\mathbf{v}=(4,-3,-2,-5)$

61. $\mathbf{u}=(3,5,-4,-1)$, $\mathbf{v}=(1,0,0,0)$

62. $\mathbf{u}=(3,5,-4,-1)$, $\mathbf{v}=(0,1,0,0)$

Find the product, if it exists.

63. $\begin{bmatrix}2 & -3\end{bmatrix}\begin{bmatrix}1 & 5\\ 2 & 0\end{bmatrix}$

64. $\begin{bmatrix}1 & 6 & 3\end{bmatrix}\begin{bmatrix}-4 & 0 & 5\\ 4 & 0 & 3\end{bmatrix}$

65. $\begin{bmatrix}2 & 3\\ 2 & 2\end{bmatrix}\begin{bmatrix}-1 & 1.5\\ 1 & -1\end{bmatrix}$

66. $\begin{bmatrix}2 & 1\\ -4 & -2\end{bmatrix}\begin{bmatrix}3 & -0.5\\ -6 & 1\end{bmatrix}$

67. $\begin{bmatrix}3 & -0.5\\ -6 & 1\end{bmatrix}\begin{bmatrix}2 & 1\\ -4 & -2\end{bmatrix}$

68. $\begin{bmatrix}2 & 3\\ 2 & 2\end{bmatrix}\begin{bmatrix}-1 & 1.5\\ 1 & -1\end{bmatrix}$

69. $\begin{bmatrix}-1 & 1.5\\ 1 & -1\end{bmatrix}\begin{bmatrix}2 & 3\\ 2 & 2\end{bmatrix}$

70. $\begin{bmatrix}7\\ 6\\ -2\end{bmatrix}\begin{bmatrix}1 & 3 & 5\end{bmatrix}$

71. $\begin{bmatrix}2 & -2 & 3\end{bmatrix}\begin{bmatrix}-6\\ -12\\ 1\end{bmatrix}$

72. $\begin{bmatrix}7 & 10 & 23\\ -7 & 12 & -4\\ 1 & -9 & 0\end{bmatrix}\begin{bmatrix}1 & 2 & -3\\ -4 & 5 & 8\end{bmatrix}$

73. $\begin{bmatrix}1 & 2 & -3\\ 0 & 14 & 5\end{bmatrix}\begin{bmatrix}0 & 1 & -6 & 0 & 1\\ 12 & -5 & 4 & 2 & -1\\ 20 & 6 & -2 & 0 & 0\end{bmatrix}$

74. $\begin{bmatrix}1 & 3 & 6\\ 5 & 3 & 12\\ 0 & -6 & 20\\ 0 & 8 & 1\end{bmatrix}\begin{bmatrix}24 & 15 & 1 & -8\\ -16 & 0 & 9 & -10\\ 7 & 13 & 23 & 4\end{bmatrix}$

75. $\begin{bmatrix}24 & 15 & 1 & -8\\ -16 & 0 & 9 & -10\\ 7 & 13 & 23 & 4\end{bmatrix}\begin{bmatrix}1 & 3 & 6\\ 5 & 3 & 12\\ 0 & -6 & 20\\ 0 & 8 & 1\end{bmatrix}$

76. $\begin{bmatrix}2 & -4 & 6\\ 0 & 5 & -1\end{bmatrix}\begin{bmatrix}3 & 1\\ 0 & -10\\ 2 & 0\end{bmatrix}$

77. $\begin{bmatrix}3 & 1\\ 0 & -10\\ 2 & 0\end{bmatrix}\begin{bmatrix}2 & -4 & 6\\ 0 & 5 & -1\end{bmatrix}$

78. $\begin{bmatrix} 1 & 4 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} 8 & 1 \\ 5 & -10 \\ 2 & -3 \end{bmatrix}$

79. $\begin{bmatrix} 8 & 1 \\ 5 & -10 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 6 & 0 \end{bmatrix}$

80. $\begin{bmatrix} 6 & -4 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

81. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & -4 \\ 5 & 1 \end{bmatrix}$

82. $\left( \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \right) \begin{bmatrix} 0 & 3 \\ 2 & 2 \end{bmatrix}$

83. $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \left( \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 2 & 2 \end{bmatrix} \right)$

### 6.4.3 Matrices and Linear Systems

For problems 84-93,

(a) What matrix equation, and what augmented partitioned matrix, models the given system of equations?

(b) Write the augmented partitioned matrix, found in part (a), in reduced row echelon form.

(c) Identify the solution(s) of the system.

84. $\begin{aligned} x + y &= 2 \\ 2x - y &= 0 \end{aligned}$

85. $\begin{aligned} 2x - 5y &= 0 \\ x + 4.5y &= 7 \end{aligned}$

86. $\begin{aligned} 0.3x_1 + 1.2x_2 &= 5.1 \\ 100x_1 + 80x_2 &= 580 \end{aligned}$

87. $\begin{aligned} \tfrac{1}{2}x_1 - x_2 &= 70 \\ \tfrac{1}{3}x_1 + x_2 &= 20 \end{aligned}$

88. $\begin{aligned} 0.06x_1 + 0.025x_2 &= 2800 \\ x_1 - x_2 &= 24000 \end{aligned}$

89. $\begin{aligned} 15x_1 + 24x_2 &= 133.2 \\ -2x_1 + x_2 &= -6 \end{aligned}$

90. $\begin{aligned} x + y - z &= 0 \\ 3x + z &= 1 \\ x - y &= -1 \end{aligned}$

91. $\begin{aligned} 3x_1 + 2x_2 + x_3 &= 12 \\ -x_1 - x_2 + 5x_2 &= 10 \\ x_2 + 28x_3 &= 24 \end{aligned}$

92. $$\begin{aligned} 5x_1 + 6x_2 + x_3 &= 0 \\ 2x_1 - x_3 &= 0 \end{aligned}$$

93. $$\begin{aligned} x_1 + x_2 &= 32 \\ \tfrac{2}{5}x_1 + \tfrac{3}{4}x_2 &= 15.95 \\ 2x_1 - x_2 &= 10 \end{aligned}$$

For each of the augmented partitioned matrices:

(a) Give system of linear equations the matrix represents.

(b) Write the matrix in reduced row echelon form.

(c) Identify the solution(s) of the system found in part (a).

94. $\left[\begin{array}{cc|c} -2 & 20 & -8 \\ 5 & 10 & -40 \end{array}\right]$

95. $\left[\begin{array}{cc|c} 1 & 5 & 0 \\ 3 & -6 & 63 \end{array}\right]$

96. $\left[\begin{array}{cc|c} 6 & 2 & 54 \\ 9 & 3 & 81 \end{array}\right]$

97. $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & 1 & 1 & -3 \\ 1 & 0 & 1 & 0 \end{array}\right]$

98. $\left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 5 & -6 & 2 & 1 \\ 6 & 18 & 3 & 6 \end{array}\right]$

99. $\left[\begin{array}{cc|c} 24 & 15 & 1 \\ -16 & 9 & -7 \\ 8 & 24 & -9 \end{array}\right]$

100. $\left[\begin{array}{cc|c} 8 & 6 & 12 \\ 12 & -10 & 0 \\ 5 & -1 & 3 \end{array}\right]$

101. $\left[\begin{array}{ccc|c} 11 & 5 & 0 & 9 \\ 0 & 6 & 2 & 0 \\ 33 & 3 & -4 & 27 \end{array}\right]$

102. $\left[\begin{array}{ccc|c} 2 & 6 & 4 & -4 \\ 7 & -3 & 2 & -2 \\ \frac{1}{4} & 1 & -2 & 2 \end{array}\right]$

103. $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 10000 \\ 6 & 2 & 0 & 0 & 7000 \\ 0 & 1 & 0 & 8 & 15000 \\ 0.01 & 0.03 & 0.065 & 0.02 & 417.5 \end{array}\right]$

104. $\left[\begin{array}{cccc|c} 2 & 3 & 5 & 9 & 42 \\ 1 & -1 & 1 & -1 & -1 \\ 0 & 2 & -1 & 0 & 0 \\ 1 & 2 & 4 & 8 & 35 \end{array}\right]$

For problems 105-108:

(a) Set up a system of linear equations to model the situation.

(b) Solve the system by writing the associated augmented partitioned matrix in reduced row echelon form.

(c) Describe the solution(s) in the context of the problem.

105. A manufacturing company has a fixed monthly cost of \$2520 and materials for manufacturing cost \$1.05 per unit. The company brings in \$3.30 per unit sold. Let $y$ represent the cost (revenue) of manufacturing (selling) $x$ units.

106. A shop owner wants to sell three new products: products A, B, and C. The shop only has 100 units of space on the "New Items" shelf and product A takes up 2 units of space, product B takes up 4 units of space, and product C takes up 5 units of space. Product A costs the shop owner \$3 per unit, product B costs \$7, and product C costs \$5; after placing orders for the shop's regular products, there is \$133 left to stock up on the new products. Considering how much product C costs to order compared to how much space it will take up on the shelves, the shop owner only wants to order 10 units of product C. How many units of products A and B should the shop owner order?

107. The height of a projectile launched into the air can be found by a quadratic equation,

$$h = at^2 + bt + c$$

where $h$ is the height of the object (in feet above sea level) $t$ seconds after it was launched into the air. After 2 seconds the object's height is 376 feet, after ten seconds the object is 600 feet above sea level, and after 13 seconds the projectile is 39 feet above sea level. Find the values of $a$, $b$, and $c$.

108. What restrictions should be placed on $a$, $b$, $c$, $d$, and $e$ such that

$$y = ax^4 + bx^3 + cx^2 + dx + c$$

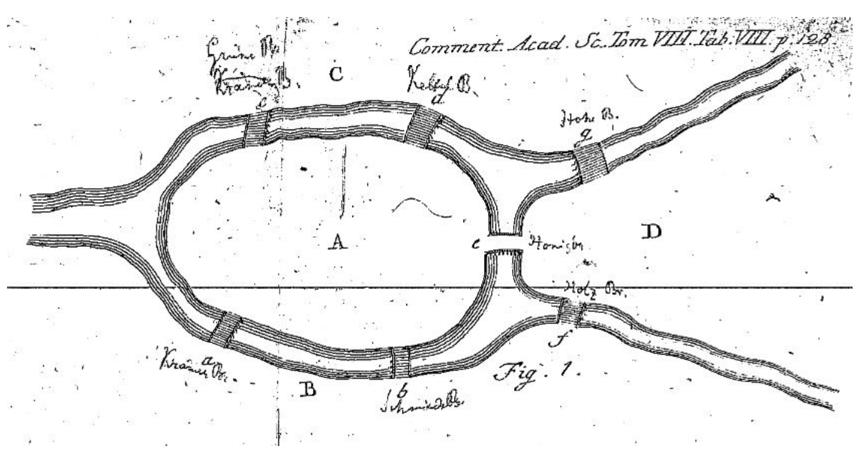
has solutions $(-1, -0.3)$, $(1, -1.1)$, $(2, -1.8)$, and $(4, 2.2)$?

# Chapter 7

# Introduction to Graph Theory

## 7.0 The Origins of Graph Theory

Graph theory arose from a problem posed to famous mathematician Leonard Euler. Known as the Seven Bridges of Königsberg or the Königsberg Bridge problem, it asks if it is possible to walk across each of Königsberg's seven bridges exactly once.



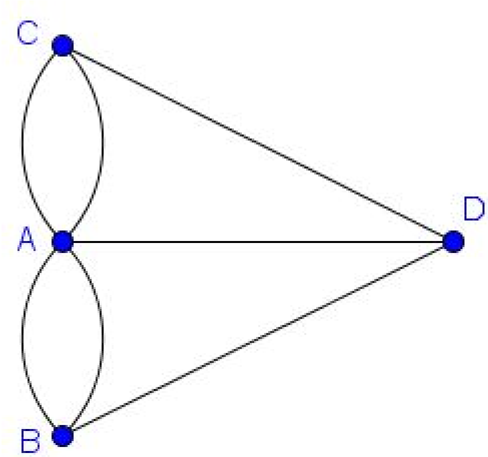

Leonard Euler's map of the Königsberg bridges from *Solutio problematis ad geometriam situs pertinentis**.

The land masses $A$, $B$, $C$, and $D$ are connected by seven bridges.

*Available at http://eulerarchive.maa.org/

From this simple question Euler began the development of graph theory, a study of mathematics with applications including communication networks and website structure, route planning, gaming schedules, and even linking people by relationships and shared connections.

Returning to the bridges of Königsberg, rather than focusing on a map of the area we can consider a simpler visual: a graph.

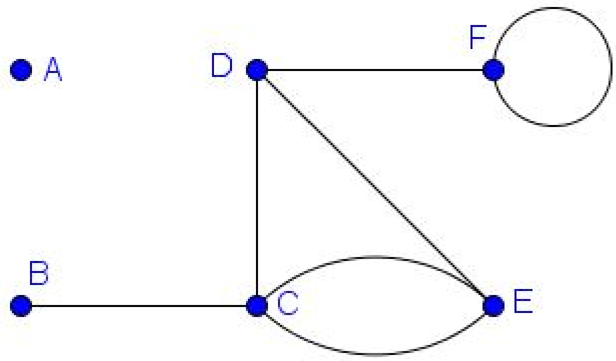



A graph in graph theory is not like graphs seen in previous chapters; a graph is a collection of dots and curves representing objects and the connections between them or, in this case, locations and paths connecting them. In the graph to the left, each land mass is represented by a dot and each bridge is represented by a curve. After some investigation of our graph one comes to the conclusion that a trip through Königsberg crossing each bridge exactly once is impossible, as Euler did.

The Königsberg Bridge problem is an example of a Euler circuit problem, which we will revisit later. First, the terminology of graph theory must be explained.

## 7.1 The Language of Graph Theory

As previously mentioned, a *graph* is a collection of dots, called *vertices* or, singularly, a *vertex*, and curves called *edges*. In a graph, such as the one below, a vertex is a dot and an edge is a curve connecting two vertices or a vertex to itself. In a directed graph each edge has an associated direction; in such a graph an edge may only be traversed in the given direction (think one-way streets). However, the graphs seen in this chapter are all undirected and can be crossed from either direction.




Referring to the graph above, the notation for edges and vertices is as follows:

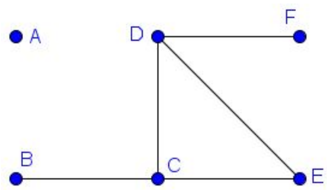
- The vertices in the graph are $A$, $B$, $C$, $D$, $E$, and $F$.
- The edges in the graph are $BC$, $CD$, $CE$, $DE$, $DF$ and $FF$.

Here, $BC$ is the edge connecting vertices $B$ and $C$, while $CE$ represents *both* of the edges connecting $C$ and $E$, and $FF$ represents the edge connecting $F$ to itself; $CE$ is an example of *multiple edges* and the edge $FF$ is also called a *loop*. With this basic terminology of graph theory, some of the properties of a graph and its parts can now be discussed.

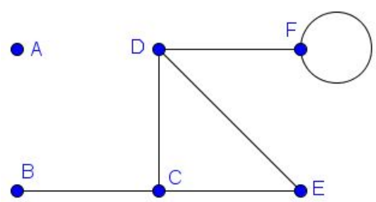
The first property to cover is the *degree of a vertex*; the degree of a vertex $X$ is the number of ways one might exit $X$. Usually this amounts to counting the number of edges in a graph for which $X$ is an endpoint. For example, in the graph above, the vertex $C$ has degree 4 because there are four edges for which $C$ is an endpoint. Similarly, $A$ has degree 0 and $B$ has degree 1. However, loops are a special case; while $F$ is an endpoint for only two edges, $F$ has degree 3 because there are three ways to exit $F$: one direction along $FD$ and two directions along $FF$ (clockwise and counter-clockwise).

Another term, one which applies to a graph in its entirety, is simple. A graph is called *simple* when no two vertices are connected by more than one edge and no edge links a vertex to itself. The graph above is not simple because of the multiple edges $CE$ and the loop $FF$; note that the existence of just one of these, multiple edges or a loop, in a graph would mean the graph in question is not simple.

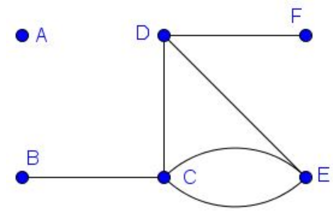
Below we have three different graphs attained by removing certain edges from the graph above. One of these (left) is simple while the other two (center and right) are not.




Simple

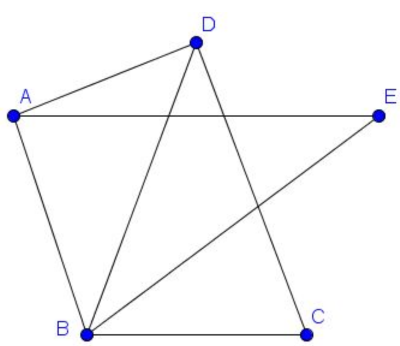



Not Simple

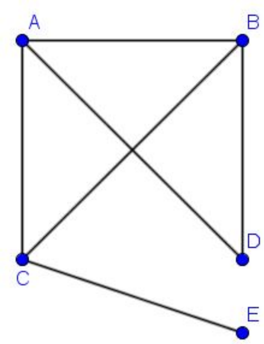



Not Simple

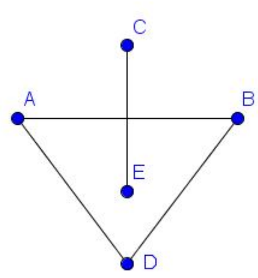
The next quality of a graph to discuss is connectedness. A graph is called *connected* when all pairs of vertices are linked by a sequence of edges. Of the graphs we've seen so far, only the graph modeling the Königberg Bridges problem is connected; in each of the others, $A$ is not linked to any other vertex. Below are further examples of graphs that are or are not connected.



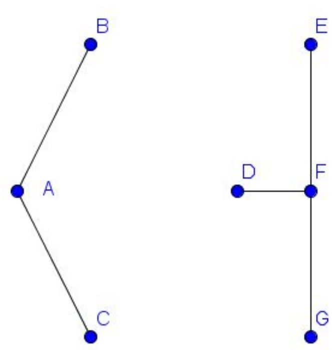

Connected




Connected



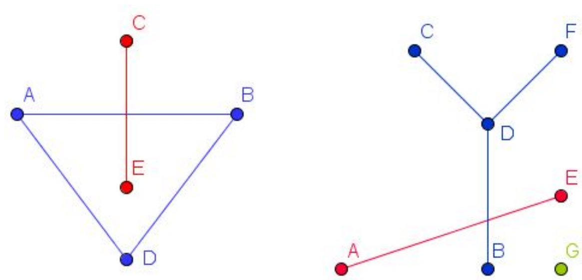

Not Connected




Not Connected

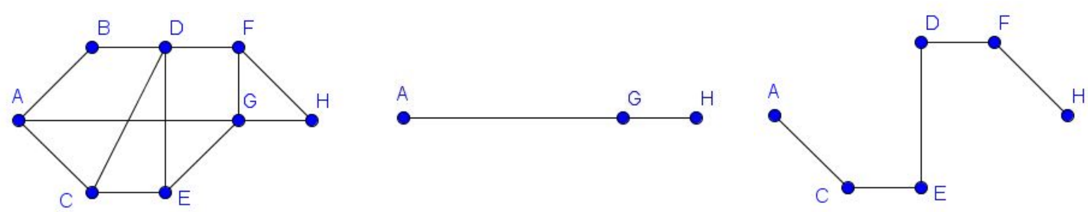
Notice that in the bottom-left graph, edges $CE$ and $AB$ appear to cross; however, these edges are not linked by a vertex and so no sequence of edges links vertices $C$ and $A$ (among other combinations). This is why the graph in question is not connected.

A graph that is connected is comprised of a single *component*, or collection of connected edges and vertices. A graph that is not connected, on the other hand, is comprised of multiple components. The graphs below are comprised of 2 components (left) and 3 components (right); to help distinguish between disconnected components of a graph, the different components are colored.



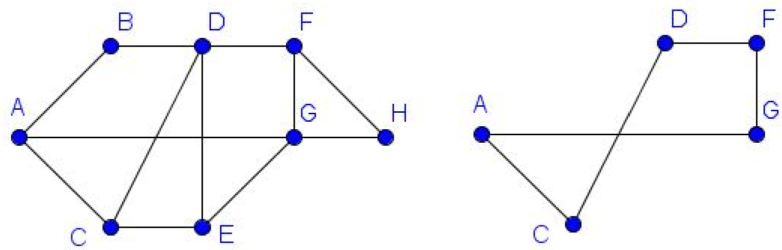

2 Components 3 Components

Within a component of a graph, a *path* is a sequence of vertices, with sequential pairs connected by an edge, with no edge being used more than once. For example, in the connected graph below one path from $A$ to $H$ is $A, G, H$. Another path from $A$ to $H$ is $A, C, E, D, F, H$.



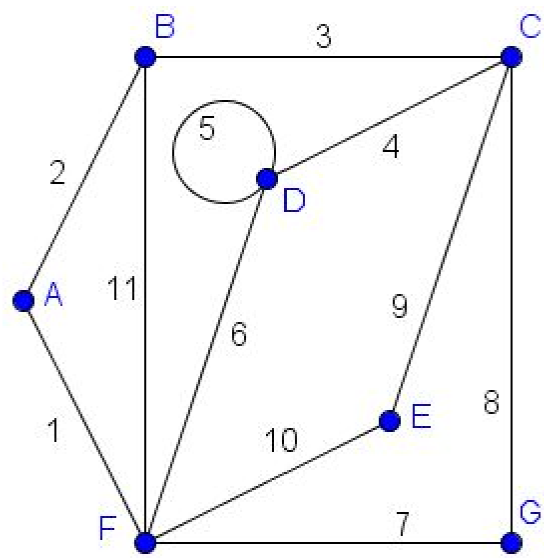

Graph Path: $A, G, H$ Path: $A, C, E, D, F, H$

A path which begins and ends at the same vertex is called a *circuit*. For example, in this same graph one circuit is $A, C, D, F, G, A$.

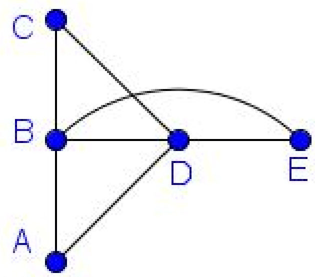



Graph Circuit $A, C, D, F, G, A$

A special case of these paths and circuits are Euler paths and Euler circuits, each of which involves *every* edge of the graph. In the graph below, one Euler path is $F, A, B, C, D, D, F, G, C, E, F, B$. To help see this Euler path more clearly, the edges are numbered in the order they are used in the path.




Notice that we're starting at $F$ and ending at $B$; because the vertices at which this path begins and ends are not the same, this Euler path is not an Euler circuit. In fact, the graph above has no Euler circuits at all. The next graph, on the other hand, has Euler circuit $A, B, C, D, E, B, D, A$.



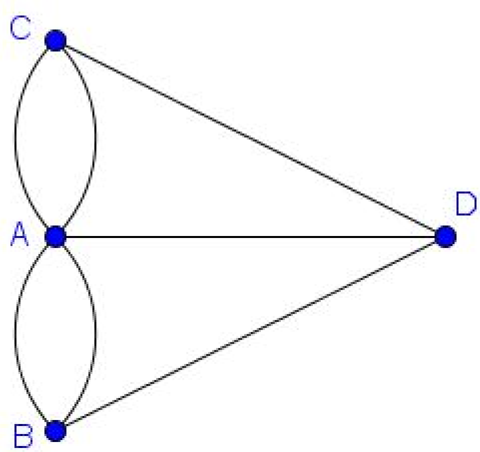

A graph that contains an Euler circuit is called an *Eulerian graph* while a graph containing an Euler path (which is not a circuit) is called *semi-Eulerian*.

To help determine whether or not a graph has an Euler path or an Euler circuit one can focus on the *odd vertices*, or vertices of an odd degree.

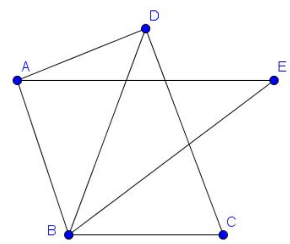
## Theorem: Euler Paths and Circuits

| | |
|---|---|
| 1. | If a graph has an Euler circuit then there are no odd vertices. |
| 2. | If a graph has an Euler path then it contains at most 2 odd vertices. |

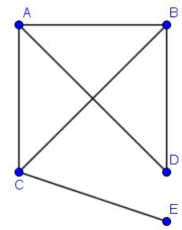
Briefly returning to the Königsberg Bridges problem, recall that the goal was to travel through Königsberg by crossing each bridge exactly once; in other words, the goal was to find an Euler path in the graph representing the city and its bridges. This graph, shown again below, has four odd vertices and so no Euler path exists.



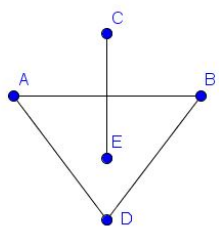
**Example 1:** Can the given graph contain any Euler paths or Euler circuits?

(a) 

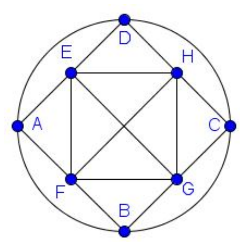

The degree of the vertices $A$ through $E$ are 3, 4, 2, 3, and 2; since there are two odd vertices there may be a Euler path but no Euler circuit can exist. In fact this graph contains multiple Euler paths; for example, $A, D, C, B, A, E, B, D$.

(b) 

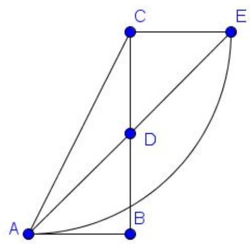

In this graph $A$, $B$, $C$, and $E$ are odd vertices; with four odd vertices no Euler paths are possible.

(c) 

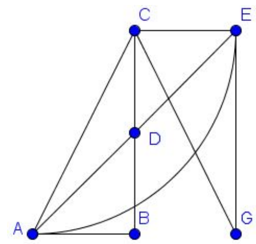

Note that this graph is not connected; Euler paths are possible only in connected graphs.

(d) 


In this graph the vertices $E$, $F$, $G$, and $H$ are odd, so no Euler path is possible.

(e) 

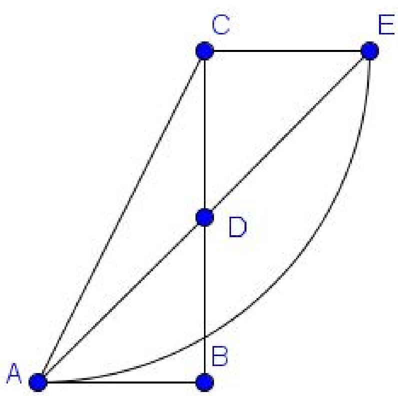

The only odd vertices of this connected graph are $C$ and $E$; since only two vertices are odd there may be an Euler path but no Euler circuits. For example, $C, D, E, C, A, B, D, A, E$ is an Euler path of this graph.

(f) 


This graph, a modified version of the graph in part (c), is connected and has no odd vertices. As such, there may be an Euler circuit. $A, C, D, C, A, D, E, G, C, E, A$ is one such circuit.

## 7.2 Identifying Euler Circuits and Paths

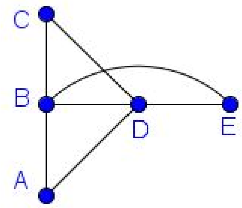
After determining whether an Euler path or circuit may exist, the next natural question is how such a path or circuit can be identified. For example, consider the graph below.




From the previous example we know of one Euler path, but there are others; some Euler paths of this graph are given below.

$C, D, E, C, A, B, D, A, E$
$C, E, D, C, A, B, D, A, E$
$E, C, D, E, A, B, D, A, C$
$E, D, C, E, A, B, D, A, C$
$E, C, A, E, D, B, A, D, C$
$C, A, D, C, E, D, B, A, E$

Notice that each of these paths begins and ends at odd vertices; whenever the Euler path being identified is not a circuit this will be the case. On the other hand, when identifying an Euler circuit one can start at *any* vertex. For example, several Euler circuits are listed for the graph below.

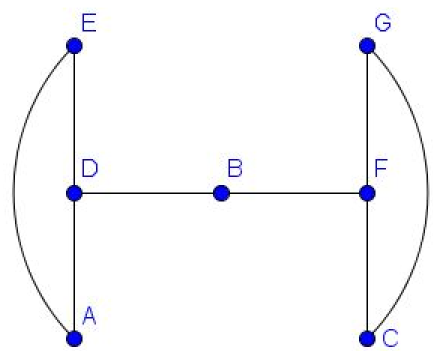



| | | | |
|---|---|---|---|
| $A, B, C, D, B, E, D, A$ | $A, B, D, C, B, E, D, A$ | $A, B, E, D, B, C, D, A$ | $A, D, E, B, C, D, B, A$ |
| $B, A, D, B, C, D, E, B$ | $B, C, D, A, B, D, E, B$ | $B, D, E, B, A, D, C, B$ | $B, E, D, B, A, D, C, B$ |
| $C, B, A, D, E, B, D, C$ | $C, B, E, D, A, B, D, C$ | $C, D, E, B, D, A, B, C$ | $C, D, A, B, E, D, B, C$ |
| $D, A, B, C, D, B, E, D$ | $D, B, C, D, A, B, E, D$ | $D, C, B, E, D, A, B, D$ | $D, E, B, D, A, B, C, D$ |
| $E, D, B, C, D, A, B, E$ | $E, D, A, B, D, C, B, E$ | $E, B, A, D, C, B, D, E$ | $E, B, D, C, B, A, D, E$ |

The steps taken to identify these Euler paths and circuits come from *Fleury's Algorithm*, described below.

| Fleury's Algorithm |
|---|
| 1. Check that the graph is connected and...<br>(a) for Euler circuits, that all vertices are even.<br>(b) for Euler paths, that there are exactly two odd vertices. |
| 2. Choose a starting vertex:<br>(a) any vertex will do for an Euler circuit.<br>(b) the vertex must be odd for an Euler path. |
| 3. Traverse edges from one vertex to another; when there is more than one option, choose an edge whose removal would not make the graph disconnected. |
| 4. Repeat step 3 as long as possible. With all edges traversed...<br>(a) the Euler circuit will end at the starting vertex.<br>(b) the Euler path will end at an odd vertex different from the starting vertex. |

**Example 2:** Does the graph below contain any Euler paths or circuits? If so, identify an Euler path/circuit.




First note that this graph is connected. Since the graph is connected and contains two odd vertices, $D$ and $F$, an Euler path exists. We can start the path at either of the odd vertices, so let's start at $D$.

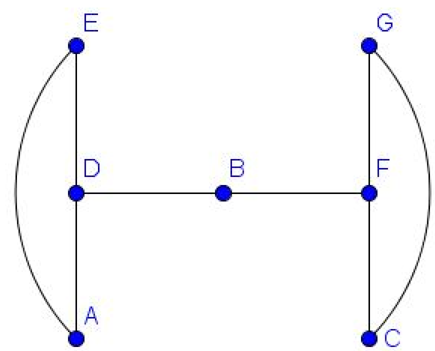
From $D$ there are three options on where to travel next: $E$, $A$, or $B$. Notice that traveling first to $B$ would disconnect the remaining edges, so instead choose $E$ or $A$.

If we travel from $D$ to $A$, our only option is to then travel to $E$ then back to $D$. From here we must travel to $B$, then $F$ where we have two options. We can traverse $FG$ or $FC$; neither option disconnects the remaining edges so each will result in a viable path. From $F$ we may travel to $C$, then to $G$, and finally to $F$.

The Euler path we've found above is as follows:

$$D, A, E, D, B, F, C, G, F$$

**Example 3:** Does the graph below contain any Euler paths or circuits? If so, identify an Euler path/circuit.




First note that this graph is connected. Since the graph is connected and contains two odd vertices, $D$ and $F$, an Euler path exists. We can start the path at either of the odd vertices, so let's start at $D$.

From $D$ there are three options on where to travel next: $E$, $A$, or $B$. Notice that traveling first to $B$ would disconnect the remaining edges, so instead choose $E$ or $A$.

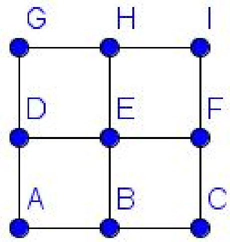
If we travel from $D$ to $A$, our only option is to then travel to $E$ and then back to $D$. From here we must travel to $B$, then $F$ where we have two options. We can traverse $FG$ or $FC$; neither option disconnects the remaining edges so each will result in a viable path. From $F$ we may travel to $C$, then to $G$, and finally to $F$.

The Euler path we've found above is as follows:

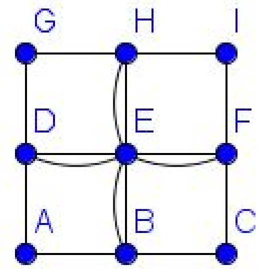
$$D, A, E, D, B, F, C, G, F$$

## 7.3 Eulerization and Routing Problems

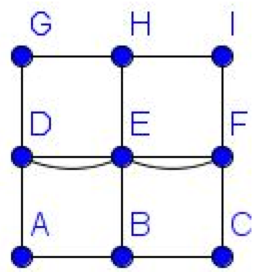
Given a non-Eulerian graph, *Eulerization* of the graph is the inclusion of copies of existing edges so that an Euler circuit exists. For example, consider the graph below.




Since there are four odd vertices, $B$, $D$, $F$, and $H$, no Euler circuit exists in this graph. However, copying four of the edges results in an Eulerian graph. To the right is one Eulerization of the previous graph.

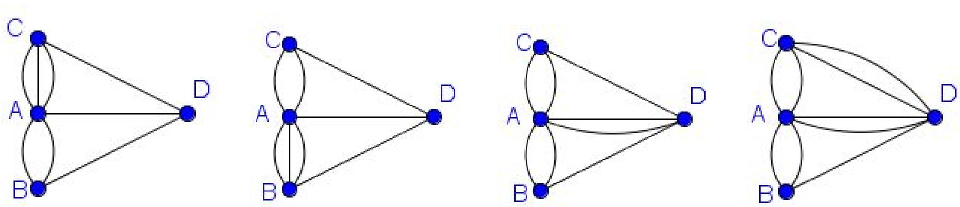



Similarly, *semi-Eulerization* of a graph is the inclusion of copies of edges so that the result is a semi-Euler graph (a graph containing a Euler path). Below is one semi-Eulerization of the graph above.




**Example 4:** Semi-Eulerize the graph for the Königsberg Bridges problem and identify an Euler path. Eulerize the same graph and identify an Euler circuit.

The graph for the Königsberg Bridges problem has four odd vertices; below are some examples of semi-Eulerizations of the graph.

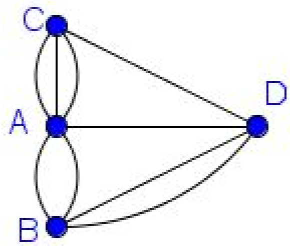



(a) (b) (c) (d)

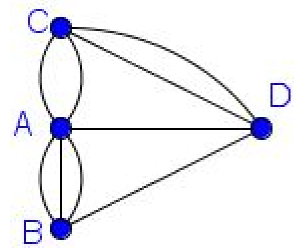
The semi-Eulerizations seen in (a), (b), and (c) copy only a single edge while the semi-

Eulerization of (d) copies two edges. Each of these is a legitimate semi-Eulerization of the original graph, but in many applications it's desirable to copy the fewest number of edges possible; such a Eulerization, or semi-Eulerization, is called *optimal*. In the optimal semi-Eulerization seen in (a) above, one Euler path is $B, A, C, A, D, C, A, B, D$.

A Eulerization of the Königsberg graph requires all vertices be even. While this can be done by copying three or more edges (ex: copying $AB$, $AC$, and $AD$ would Eulerize the graph), an optimal Eulerization requires only two copied edges.



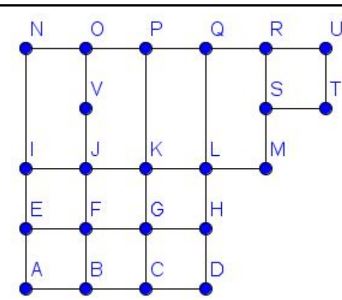
$AC$ and $BD$ copied



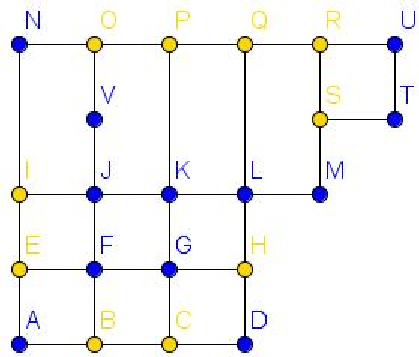
$AB$ and $CD$ copied

Above are two optimal Eulerizations of the Königsberg graph. If $AC$ and $BD$ are copied, then one circuit is $A, B, D, C, A, B, D, A, C, A$.

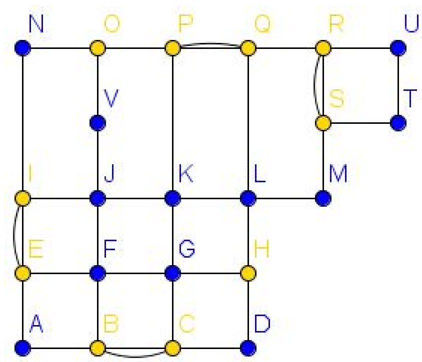
**Example 5:** The route for a paperboy is described by the graph to the right; streets where deliveries are made are represented by edges and vertices $A$ through $U$ represent street intersections. The last vertex, $V$, is where the paperboy lives. If the paperboy wants to begin and end his route at his home, revisiting as few streets as possible, what route could he take?



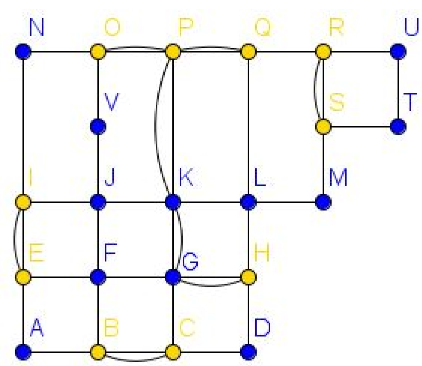

This problem is looking for an Euler circuit in an optimal Eulerization of the given graph. To help, the paperboy's graph is given again below with odd vertices highlighted.



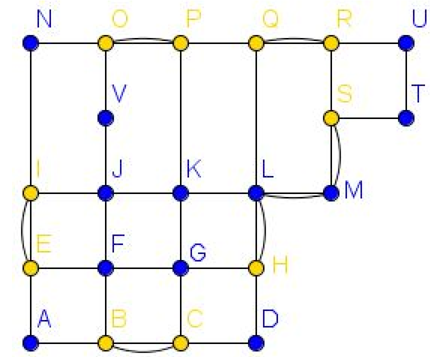

As usual, multiple Eulerizations of this graph exist, but how many copied edges are required for an *optimal Eulerization*? While answering this question requires a bit of brute force, one strategy that often (not always) works is to start by copying select edges with odd vertices as endpoints. For example, one might copy $RS$, $QP$, $IE$, and $BC$ as illustrated to the right.



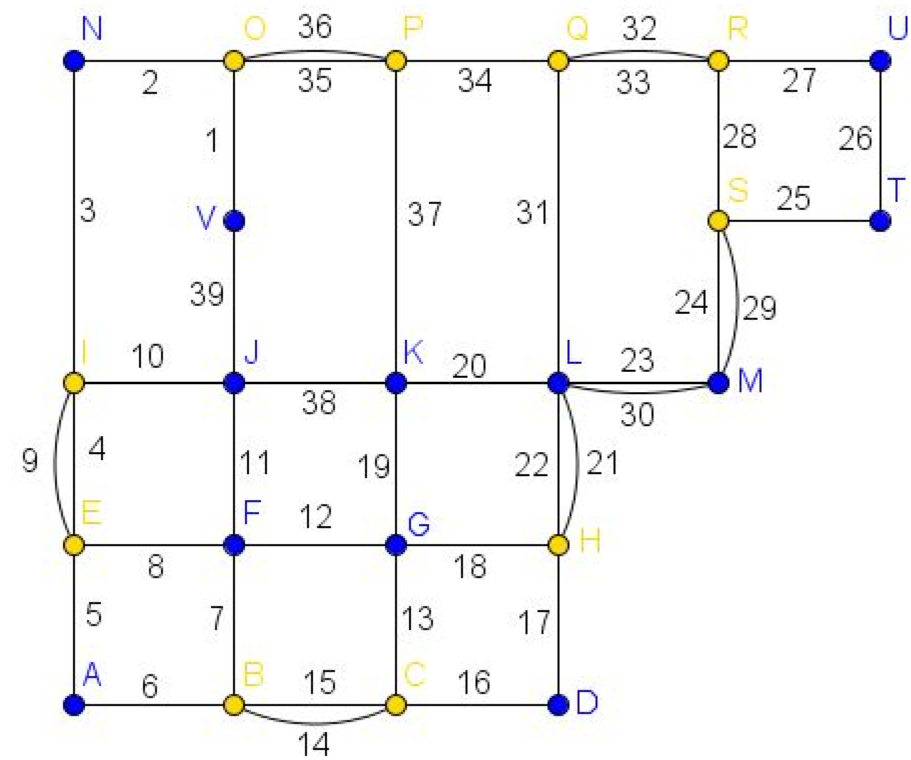

With this approach, the only odd vertices remaining are $O$ and $H$; to connect these two vertices at least four more edges must be copied, bringing the total number of copied edges to 8. Since any path from $O$ to $H$ contains at least four edges, one might look for a pair of vertices that are closer but not necessarily adjacent. For example, $S$ and $H$ can be connected by a path containing only three edges, and all other odd vertices are adjacent to some other odd vertex.




Connecting these other odd vertices ($O$ and $P$, $Q$ and $R$, $I$ and $E$, and $B$ and $C$) would require copying four edges. Then linking $S$ and $H$ brings to total number of copied edges to 7; for this graph, this is the minimal number of copied edges required, giving an optimal Eulerization (below).




Our next task is to find an Euler circuit starting at $V$. Following Fleury's Algorithm, one might find several Euler circuits for the paperboy. One such circuit is given illustrated below, with the edges labeled by the order it is crossed within the route.

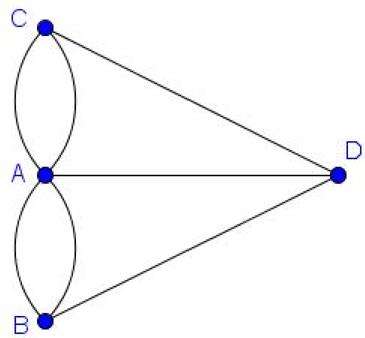



Circuit:

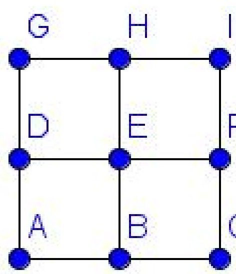
$V, O, N, I, E, A, B, F, E, I, J, F, G, C, B, C, D, H, G, K, L, H, L, M, S, T, U, R, S, M, L, Q, R, Q, P, O, P, K, J, V$

## 7.4 Hamilton Paths and Circuits

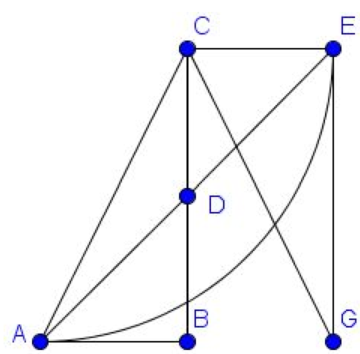
Another classification of paths and circuits is Hamiltonian. A Hamiltonian path (resp. circuit) is a path (resp. circuit) that passes through each vertex exactly once; note that a Hamiltonian circuit will still begin and end at the same vertex. For example, one Hamiltonian circuit of Königsberg's graph is $A - B - D - C - A$.




Below are two other graphs, the first containing Hamilton paths but no circuits and the second containing both circuits and paths which are not circuits.

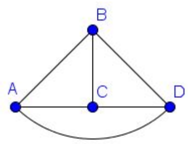



There are many Hamiltonian paths, for example $A - B - C - F - E - D - G - H - I$, but no Hamiltonian circuits.

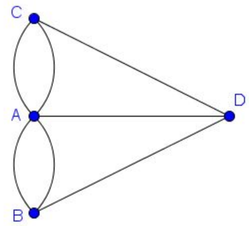



There are both Hamiltonian circuits, like $D - B - A - C - G - E - D$, and Hamiltonian paths which aren't circuits, like $G - E - C - A - B - D$.

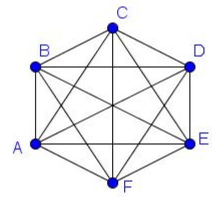
When looking for Hamiltonian circuits one type of graph is especially nice to work with: complete graphs. A graph is called *complete* when there is exactly one edge between any two vertices.



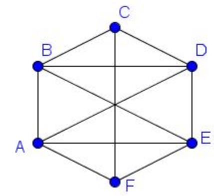

Complete




Not Complete



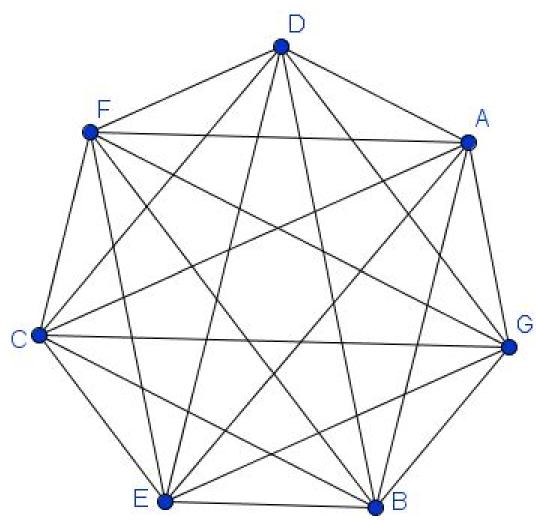

Complete




Not Complete

These graphs are particularly nice because in a complete graph with three *or more* vertices there exists a Hamiltonian circuit.

**Example 6:** Is the graph below complete? Identify a Hamilton circuit, if possible, and identify an Euler circuit, if possible.

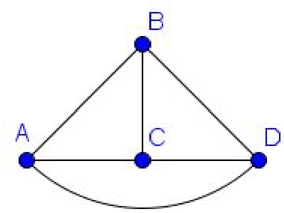



This graph is complete because every pair of vertices is connected by exactly one edge. Since it is complete and includes seven vertices there must be at least one Hamilton circuit. For example, $A{-}G{-}B{-}E{-}C{-}F{-}D{-}A$ is one such circuit; another is $A{-}B{-}C{-}D{-}E{-}F{-}G{-}A$.

Since every vertex is even there *might* be an Euler circuit. In this case, the graph does have an Euler circuit. One example is the circuit below.

$$A, C, G, F, B, D, E, A, F, E, G, D, C, B, A, G, B, E, C, G, D, A$$

**Example 7:** The graph below is complete and has four vertices. Identify a Hamilton circuit and identify an Euler circuit, if possible.

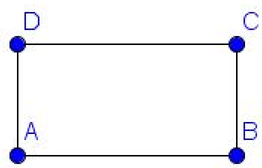


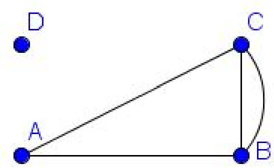

Many Hamilton circuits exist; one such circuit is $A - B - C - D - A$. However, all four vertices are odd so no Euler circuit exists. (The connection between complete graphs and Euler circuits will be investigated in the exercises for this section.)

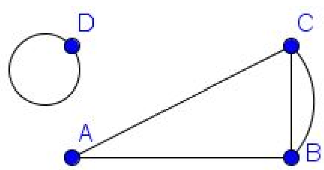
## 7.5 Exercises

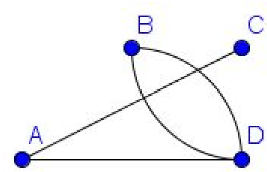
### 7.5.1 The Language of Graph Theory

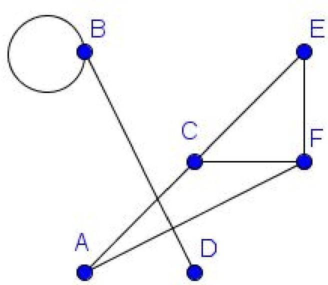
For problems 1-17: (a) list all vertices, (b) list all edges, (c) list all multiple edges, (d) list all loops, (e) identify the degree of each vertex, (f) determine whether the graph is simple, and (g) determine whether the graph is connected.

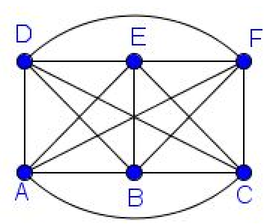
1.


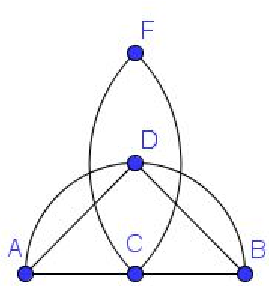

2.


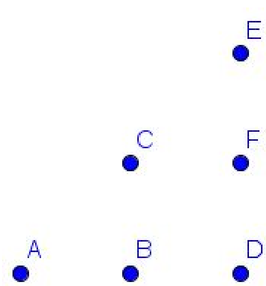

3.


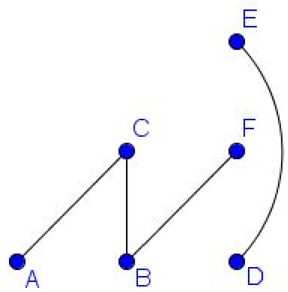

4.


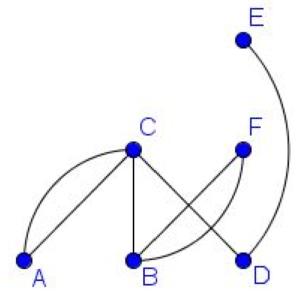

5.


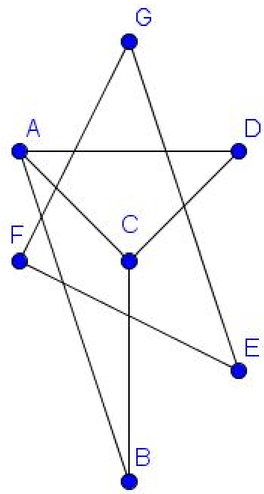

6.


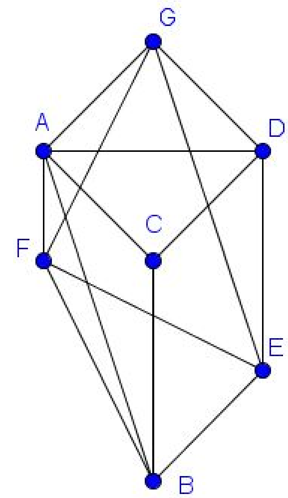

7.


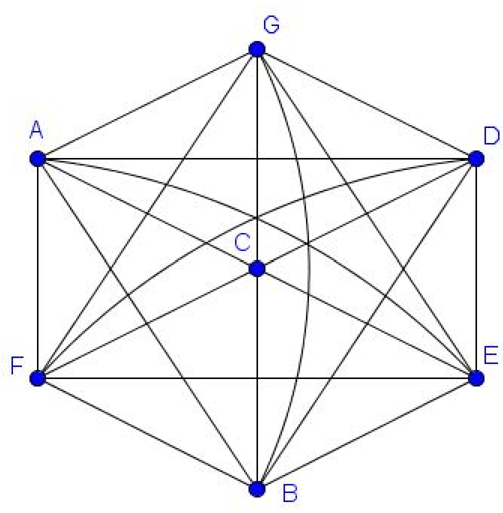

8.


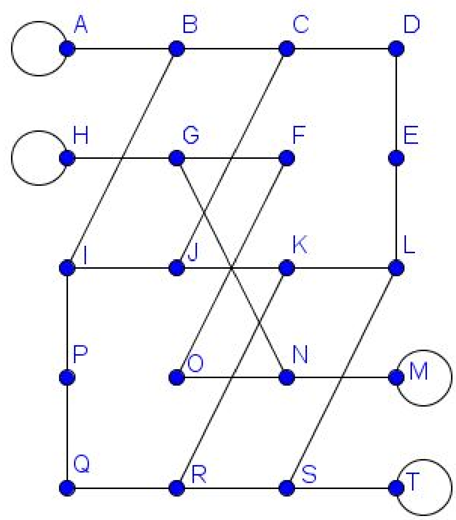
9.


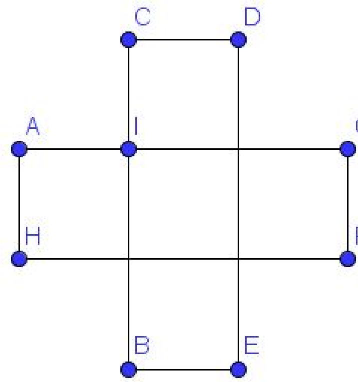

10.


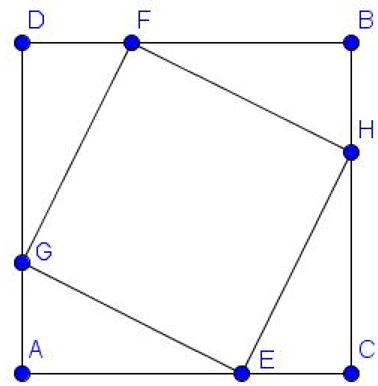

11.


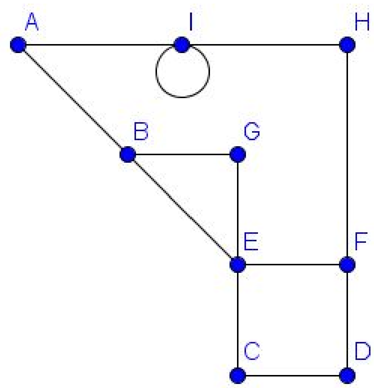

12.



13.



14.


15. 


16. 


17. 


18. Is it possible for a graph to have a exactly one odd vertex (a vertex whose degree is odd)? Is it possible for a graph to have exactly three odd vertices?

19. What is the difference between a path and a circuit?

20. Is every path a circuit? Is every circuit a path?

21. What makes a circuit an Euler circuit?

22. A graph that is not connected cannot contain an Euler circuit; explain why.

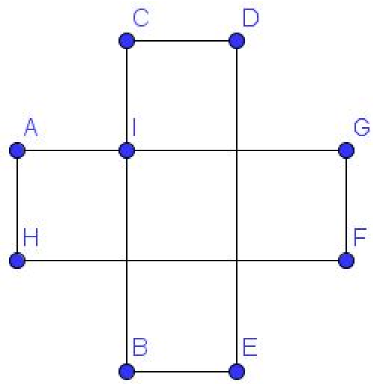
23. Explain why a graph that has exactly two odd vertices cannot contain an Euler circuit.

24. It was stated that an Euler path (that is not a circuit) must begin at an odd vertex; why can't such a circuit begin at an even vertex?

25. A graph containing an Euler circuit cannot also contain an Euler path which is not a circuit; why is this?

## 7.5.2 Identifying Euler Path and Circuits

For problems 26-35: According to our theorem on Euler paths and circuits (page 165), could the given graph contain an Euler circuit? An Euler path that is not a circuit?

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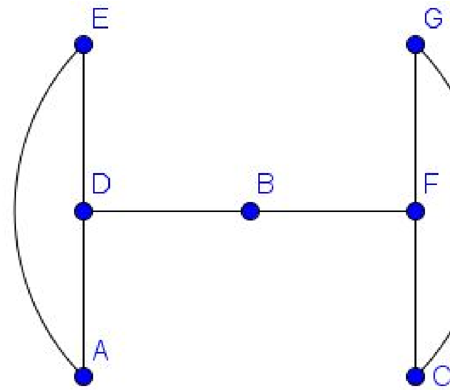


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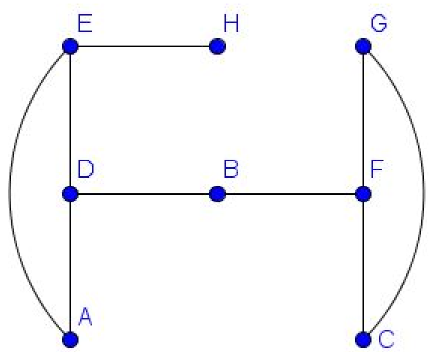


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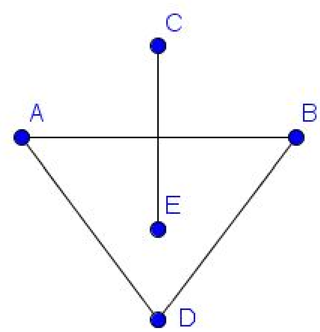


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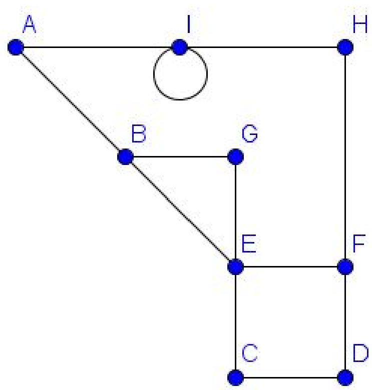


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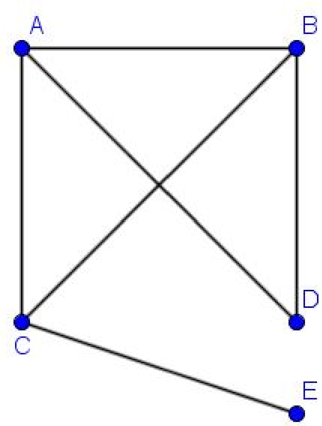


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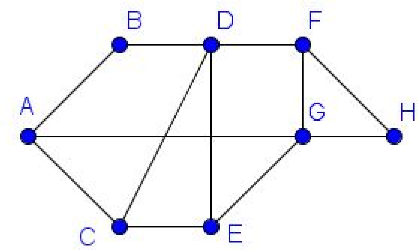


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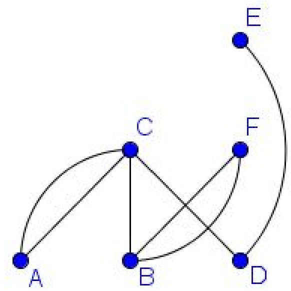


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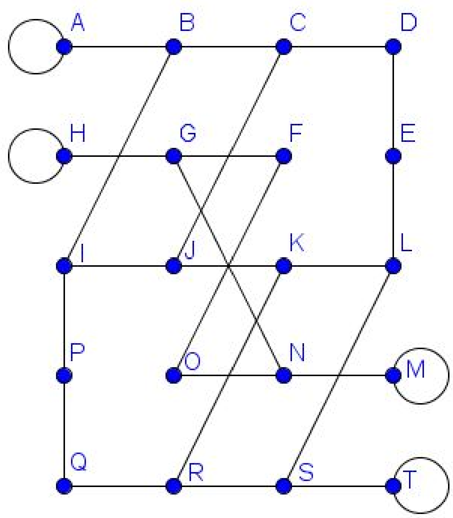


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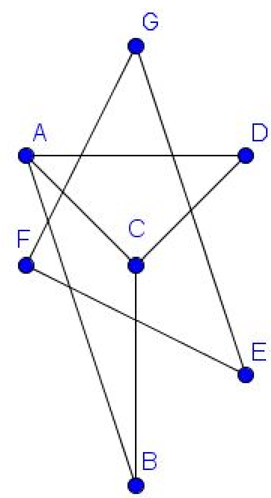
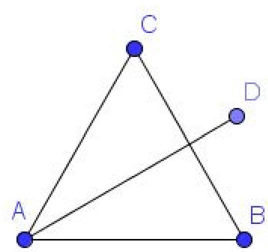



For problems 36-45, identify an Euler path.

36.
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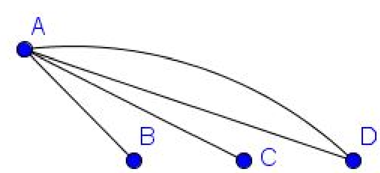
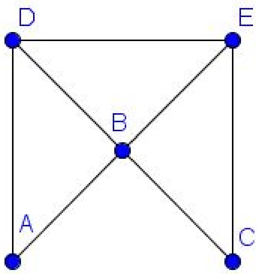
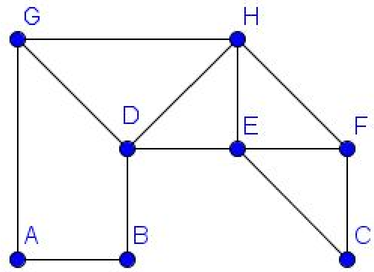
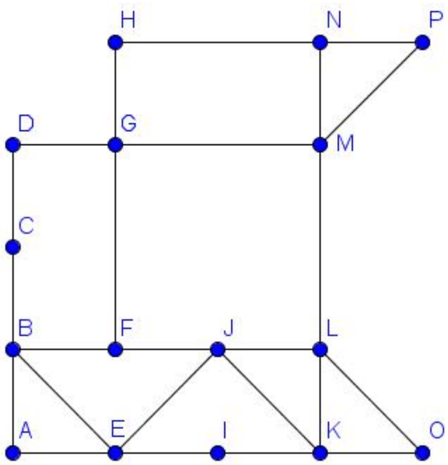
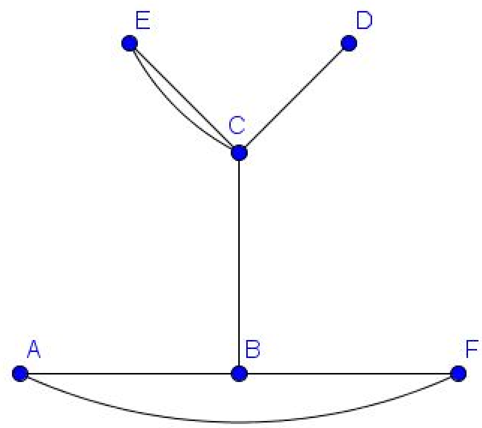
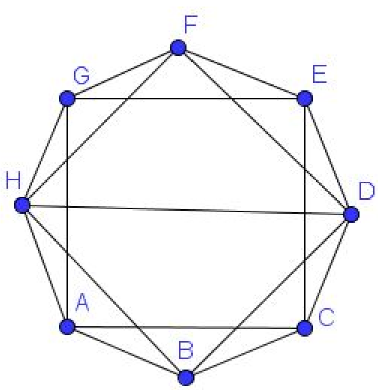


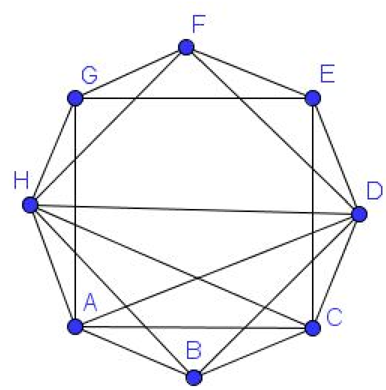
38. 

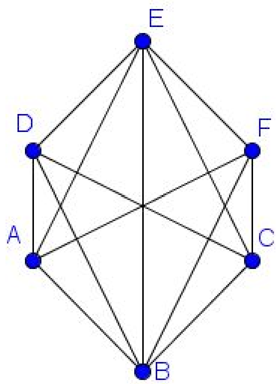

39. 

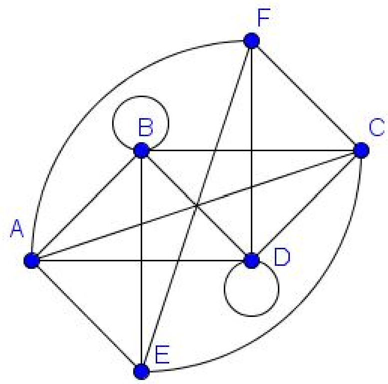

40. 


41. 

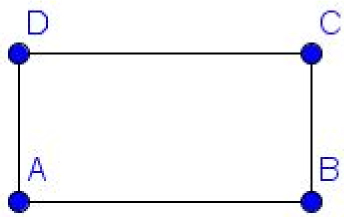

42. 

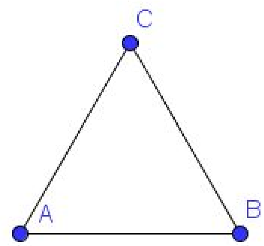

43. 

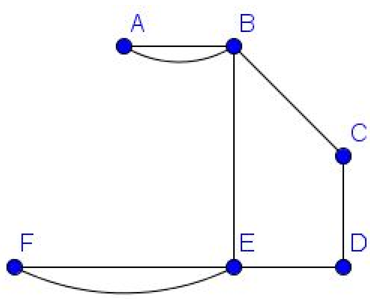
44. 

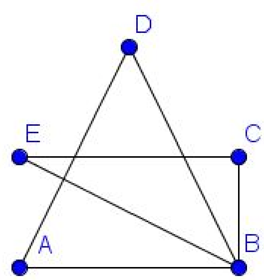

45. 

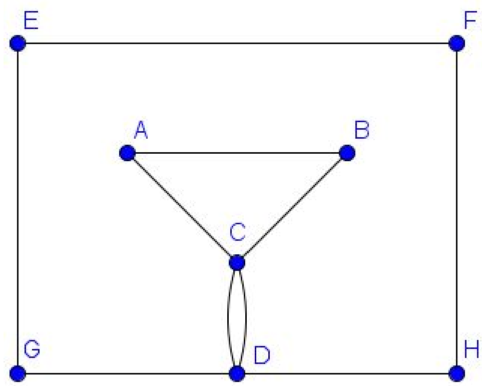

For problems 46-55, identify an Euler circuit.

46. 


47. 


48. 


49. 

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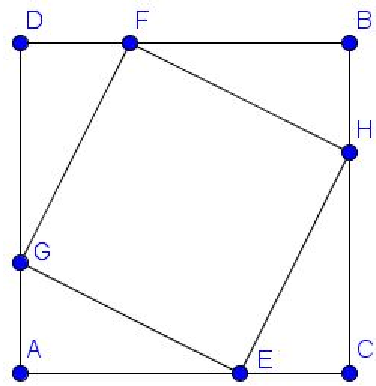


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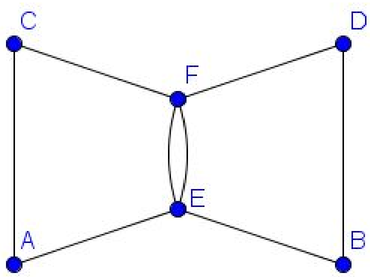


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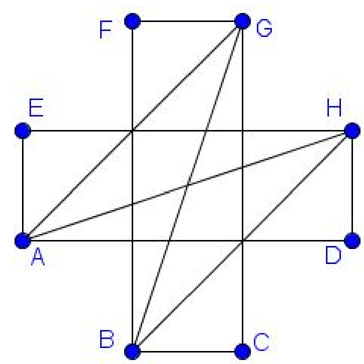


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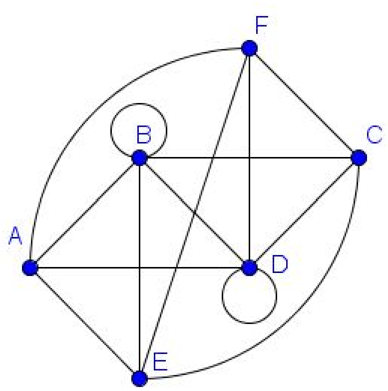


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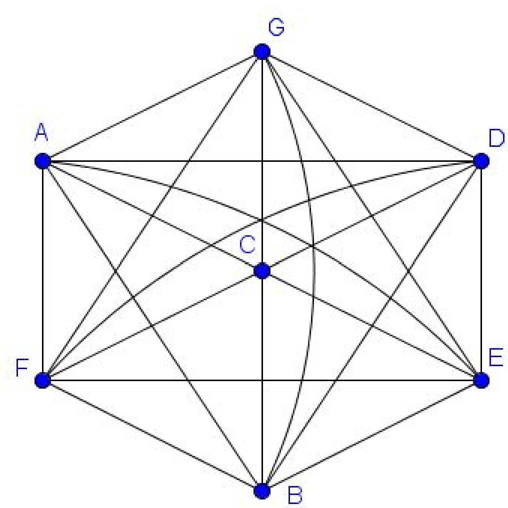



55.

### 7.5.3 Eulerization and Routing Problems

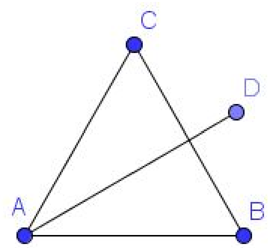
56. What does it mean to Eulerize a graph?

57. What does it mean to semi-Eulerize a graph?

For problems 58-73:

(a) If the graph is not already semi-Euler then semi-Eulerize the graph, if possible; if it is not possible, explain why.

(b) Eulerize the given graph if possible; if it's not possible, explain why.



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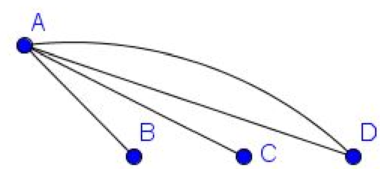

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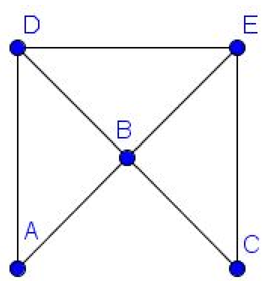

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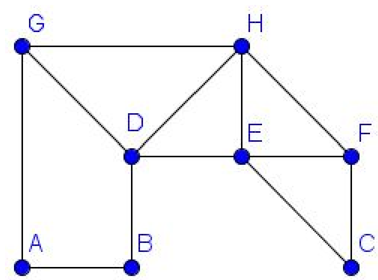

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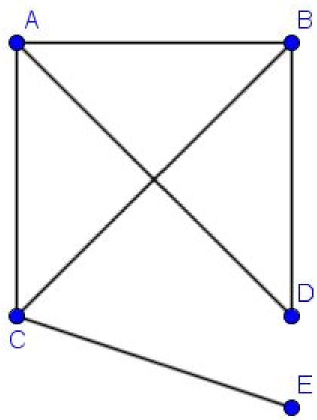


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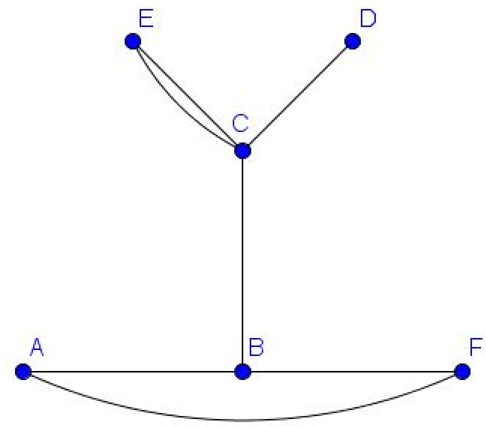
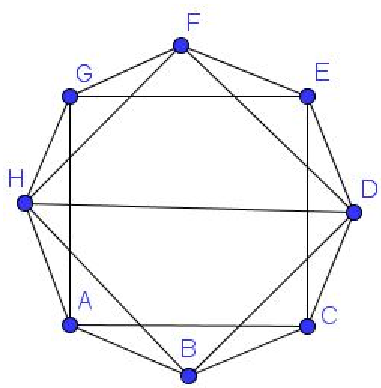
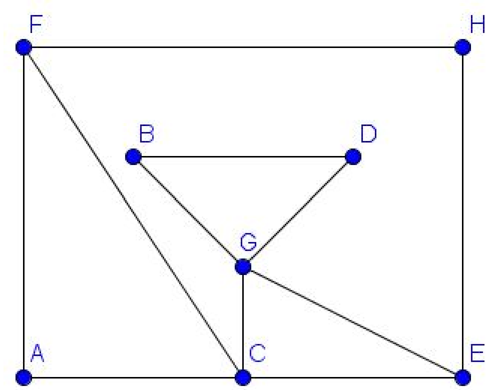


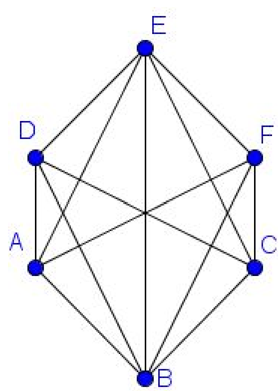

63.



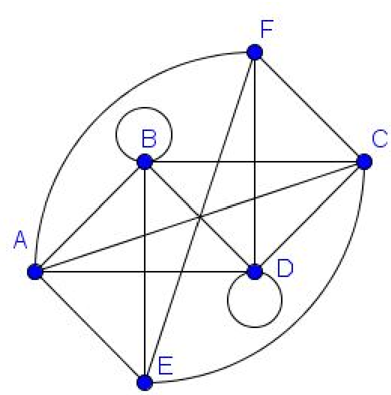

64.



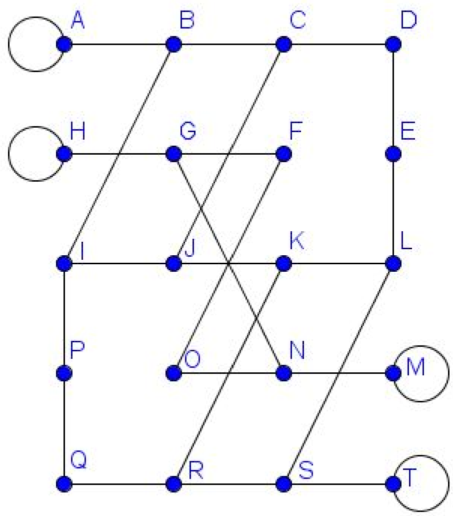

65.




66.




67.

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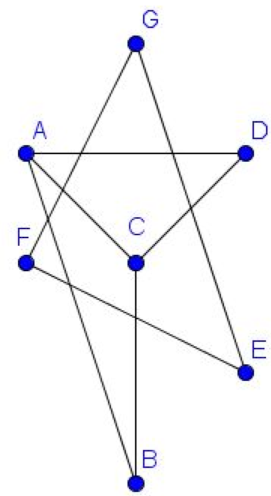


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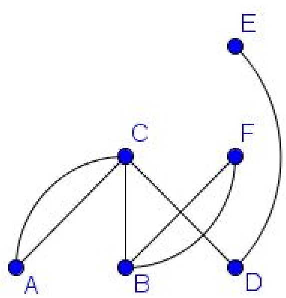


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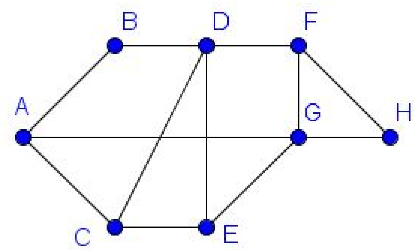
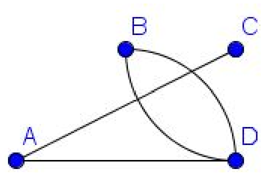
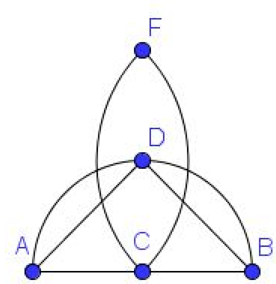



71.

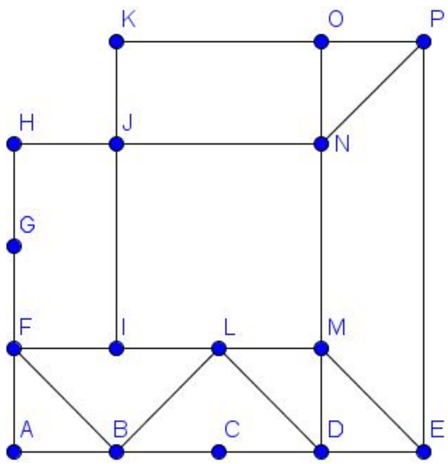



72.




73.

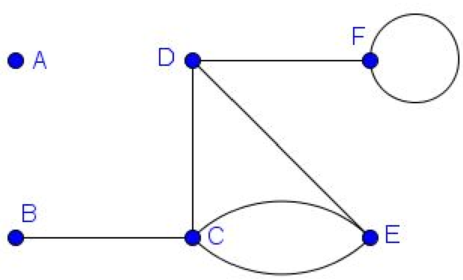
74. The graph below represents a delivery route. Find an optimal Eulerization of the graph so that the delivery driver can begin and end her deliveries at the warehouse located at vertex $A$.



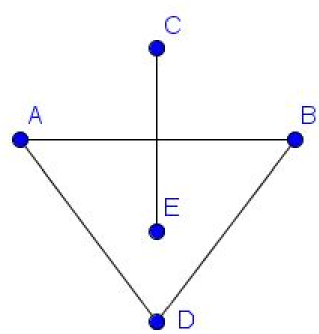

### 7.5.4 Hamilton Paths and Circuits

If possible, (a) identify a Hamilton path (which cannot be extended into a circuit) and (b) a Hamilton circuit in the given graph.

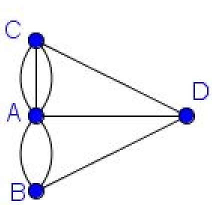
75.




76.




77.

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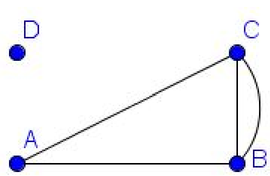

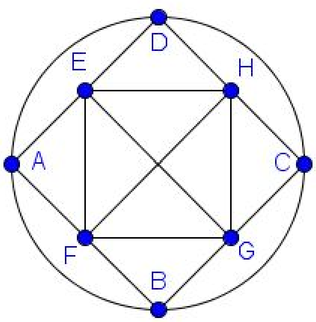


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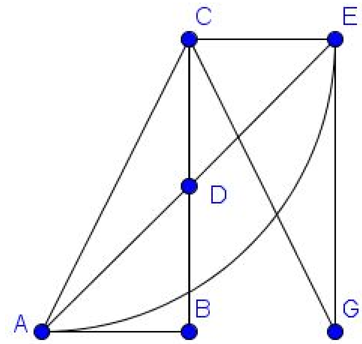
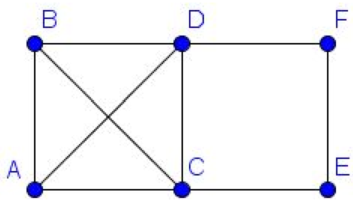
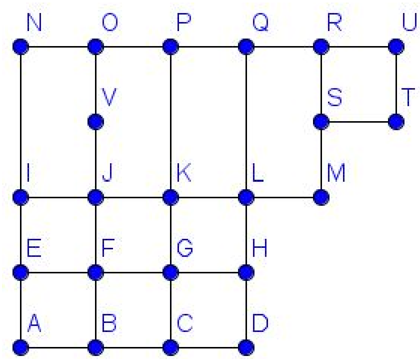



80.




81.

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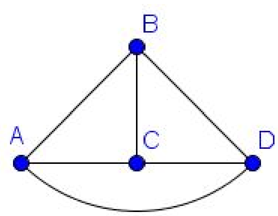


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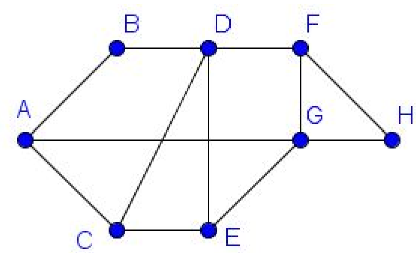


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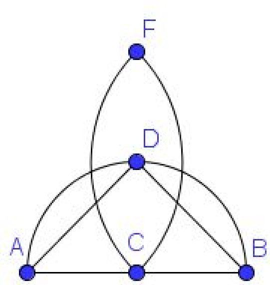
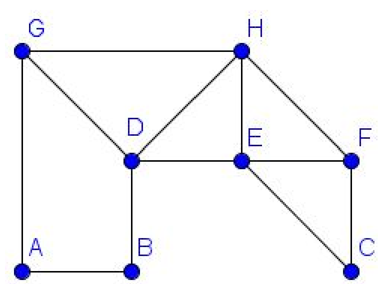
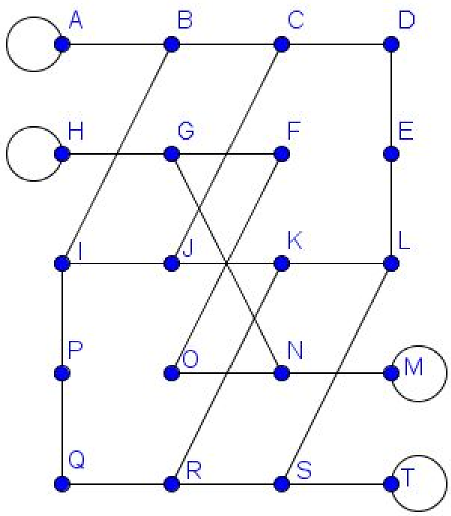
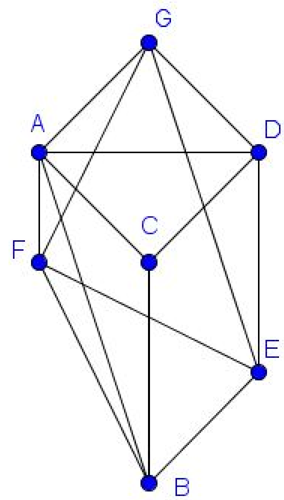


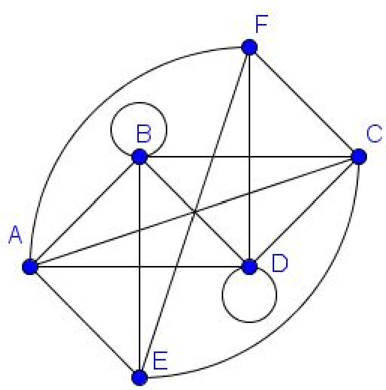

85.

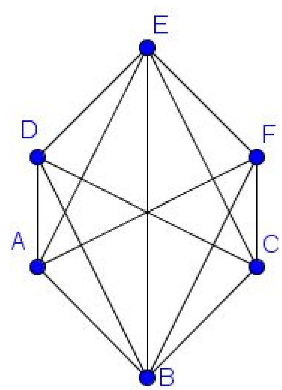



86.

87. 


88. 


89. 


90. 


Draw the complete graph containing the indicated number of vertices.

91. 2 92. 3 93. 4 94. 5 95. 6 96. 7

Does a complete graph containing the given number of vertices contain any Hamilton circuits? Euler circuits?

97. 2 98. 3 99. 4 100. 5 101. 6 102. 7

103. A complete graph containing $n$ vertices contains an Euler circuit; what can you say about $n$?

# References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3. *How to Solve It* by G. Polya. Princeton University Press, 1945.

4. https://www.top500.org/lists/2015/11/, accessed May 2016

5. *Solutio problematis ad geometriam situs pertinentis* by Leonard Euler, written 1735, published 1741. Source: http://eulerarchive.maa.org/docs/originals/E053.pdf, accessed 7 July, 2016.

6. http://spacemath.gsfc.nasa.gov, accessed 23 June, 2016.

7. *Excursions in Modern Mathematics,* $7^{th}$ *Edition* by Peter Tannenbaum. Pearson, 2009.

8. *Mathematical Ideas,* $13^{th}$ *Edition* by Miller, Heeren, Hornsby, and Heeren. Pearson, 2015.

# Solutions to Selected Exercises

Chapter 1

1. -12
3. -36
5. 28
7. 4
9. -72
11. 13
13. $\frac{3}{8}$ or 0.325
15. 2
17. $\frac{2}{27}$
19. -14.62
21. 3.33165
23. 10.243
25. $\frac{1}{9}$
27. $\frac{1}{125}$ or 0.008
29. 1
31. 225
33. $\frac{243}{2}$ or 121.5
35. $x^{15}$
37. $v^3$
39. $\frac{1}{y^5}$
41. 4
43. 3
45. -1
47. $x^3$
49. $7s^4$
51. $\frac{4c^3}{z^{10}}$
53. $2.675 * 10^{-1}$
55. $2.504 * 10^{13}$
57. $6 * 10^7$
59. $1 * 10^0$
61. 321
63. 25100000000000
65. 0.0000000478
67. $1.2 * 10^{-12}$
69. $1.1872 * 10^{-35}$
71. $5.1 * 10^{14}$
73. 10
75. 11
77. 1
79. 0
81. (a) 1, (b) 0, (c) not possible
83. $4.17 * 10^4$ CD-roms

Chapter 2

1. $1 * 10^1 + 2 * 10^0$
3. $4 * 10^1 + 2 * 10^0$
5. $1 * 10^1 + 5 * 10^0$
7. $1 * 10^2 + 0 * 10^1 + 0 * 10^0$
9. $2 * 10^2 + 0 * 10^1 + 3 * 10^0$
11. $7 * 10^2 + 5 * 10^1 + 3 * 10^0$

13. $1 * 10^3 + 4 * 10^2 + 8 * 10^1 + 0 * 10^0$

15. $1 * 10^4 + 2 * 10^3 + 5 * 10^2 + 2 * 10^1 + 8 * 10^0$

17. MSD: 1, weight: thousands, LSD: 6, weight: ones

19. MSD: 5, weight: tens, LSD: 9, weight: ones

21. MSD: 3, weight: hundreds, LSD: 2, weight: ones

23. $1 * 2^1 + 1 * 2^0$

25. $1 * 2^3 + 1 * 2^2 + 1 * 2^1 + 1 * 2^0$

27. $0 * 2^3 + 1 * 2^2 + 0 * 2^1 + 1 * 2^0$

29. $0 * 2^7 + 1 * 2^6 + 0 * 2^5 + 1 * 2^4 + 1 * 3^1 + 1 * 2^2 + 0 * 2^1 + 0 * 2^0$

31. 1

33. 2

35. 0

37. 4

39. 11

41. 22

43. 103

45. 255

47. $0_2$

49. $1000_2$

51. $1100_2$

53. $1111_2$

55. $10001_2$

57. $110010_2$

59. $1100001_2$

61. $11.1_2$

63. 1.5

65. 2.75

67. 10.01

69. 0.875

71. $0.11_2$

73. $1.\overline{0011}$

75. $5_8$, $5_{16}$

77. $15_8$, $D_{16}$

79. $30_8$, $18_{16}$

81. $44_8$, $24_{16}$

83. $200_8$, $80_{16}$

85. $3_8$, $3_{16}$

87. $13_8$, $B_{16}$

89. $11_8$, $9_{16}$

91. $30_8$, $18_{16}$

93. $172_8$, $7A_{16}$

95. 9, $1001_2$

97. 48, $110000_2$

99. 100, $1100100_2$

101. 197, $11000101_2$

103. 52, $110100_2$

109. $1100_2$

111. $10010_2$

113. $01111_2$

115. $101000_2$

117. $0100000_2$

119. -6

121. 3

123. -60

125. 88

127. -27

129.(a) 12+7

129.(b) $12 = 001100_2$
$7 = 000111_2$

129.(c) $010011_2$

129.(d) 19

131.(a) 5+2

131.(b) $5 = 0101_2$
$2 = 0010_2$

131.(c) $0111_2$

131.(d) 7

133.(a) 1+7

133.(b) $1 = 00001_2$

$7 = 00111_2$

133.(c) $01000_2$

133.(d) 8

135.(a) 14+12

135.(b) $14 = 001110_2$

$12 = 001100_2$

135.(c) $011010_2$

135.(d) 26

137.(a) $-3 + 14$

137.(b) $-3 = 11101_2$

$14 = 01110_2$

137.(c) $01011_2$

137.(d) 11

139.(a) $-7 + (-15)$

139.(b) $-7 = 111001_2$

$-15 = 110001_2$

139.(c) $101010_2$

139.(d) $-22$

Chapter 3

1. $\{10, 11, 12, 13, 14, 15, 16, 17, 18, 19\}$

3. $\{2, 3, 5, 7\}$

5. $\{10, 11, 12, 14, 15, 16, 17, 18, 19\}$

7. $\{0, 1, 2, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18\}$

9. $\{\}$ or $\varnothing$

11. $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 18\}$

13. $\{0, 1, 2, 3, 5, 8\}$

15. $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ or $B$

17. Companies not based in Washington state.

19. Companies that have more than 20 employees and are based in Washington state.

21. Companies that have more than 20 employees, are based in Washington state, and whose annual revenue is at least $1,000,000.

23. Companies based in Washington state or have both an annual revenue of at least $1,000,000 and more than 20 employees.

25. Companies which are based in Washington state or do not have both an annual revenue greater than $1,000,000 and more than 20 employees.

27.

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 0 | 0 |
| 0 | 1 | 0 |
| 0 | 0 | 0 |

29.

| $p$ | $\neg p$ | $\neg(\neg p)$ |
|---|---|---|
| 1 | 0 | 1 |
| 0 | 1 | 0 |

31.

| $p$ | $q$ | $r$ | $p \vee q$ | $(p \vee q) \vee r$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 |
| 0 | 0 | 1 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 |

33.

| $p$ | $q$ | $r$ | $p \wedge q$ | $(p \wedge q) \wedge r$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 | 0 |
| 1 | 0 | 1 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 |

35.

| $p$ | $q$ | $r$ | $q \wedge r$ | $p \vee (q \wedge r)$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 0 | 1 |
| 1 | 0 | 0 | 0 | 1 |
| 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 |

37.

| $p$ | $q$ | $p \wedge q$ | $\neg(p \wedge q)$ |
|---|---|---|---|
| 1 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 0 | 0 | 1 |

39.

| $p$ | $q$ | $\neg p$ | $\neg q$ | $(\neg p) \vee (\neg q)$ |
|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 |
| 1 | 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 0 | 1 |
| 0 | 0 | 1 | 1 | 1 |

41.

| $p$ | $\neg p$ | $p \wedge (\neg p)$ | $\neg(p \wedge (\neg p))$ |
|---|---|---|---|
| 1 | 0 | 0 | 1 |
| 0 | 1 | 0 | 1 |

43.

| $p$ | $q$ | $\neg p$ | $(\neg p) \wedge q$ | $\neg q$ | $p \wedge (\neg q)$ | $((\neg p) \wedge q) \wedge (p \wedge (\neg q))$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 1 | 1 | 0 |
| 0 | 1 | 1 | 1 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 1 | 0 | 0 |

45. The house costs more than \$180,000.

49. The house has less than three bedrooms or less than two full bathrooms, and the house costs more than \$180,000.

51. 1, 4, 6, 8, & 9

53. No such value exists.

55. Tautology

| $p$ | $p \Rightarrow p$ |
|---|---|
| 1 | 1 |
| 0 | 1 |

57.

| $p$ | $q$ | $p \Rightarrow q$ |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 0 | 0 |
| 0 | 1 | 1 |
| 0 | 0 | 1 |

59.

| $p$ | $q$ | $p \Leftrightarrow q$ |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 0 | 0 |
| 0 | 1 | 0 |
| 0 | 0 | 1 |

61. Tautology

| $p$ | $q$ | $\neg q$ | $p \wedge (\neg q)$ | $(p \wedge (\neg q)) \Rightarrow (\neg q)$ |
|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 | 1 |

63.

| $p$ | $q$ | $p \Rightarrow q$ | $(p \Rightarrow q) \Rightarrow p$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 |
| 0 | 1 | 1 | 0 |
| 0 | 0 | 1 | 0 |

65.

| $p$ | $\neg p$ | $p \Rightarrow (\neg p)$ | $(p \Rightarrow (\neg p)) \Rightarrow p$ |
|---|---|---|---|
| 1 | 0 | 0 | 1 |
| 0 | 1 | 1 | 0 |

67.

| $p$ | $q$ | $r$ | $p \Rightarrow r$ | $p \Rightarrow q$ | $(p \Rightarrow q) \Rightarrow r$ | $(p \Rightarrow r) \Rightarrow ((p \Rightarrow q) \Rightarrow r)$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 | 1 | 0 | 1 |
| 1 | 0 | 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 | 0 | 0 |
| 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 0 | 0 | 0 | 1 | 1 | 0 | 0 |

69. Tautology

| $p$ | $q$ | $p \Rightarrow q$ | $\neg q$ | $\neg p$ | $(\neg q) \Rightarrow (\neg p)$ | $(p \Rightarrow q) \Leftrightarrow ((\neg q) \Rightarrow (\neg p))$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 | 0 | 1 |
| 0 | 1 | 1 | 0 | 1 | 1 | 1 |
| 0 | 0 | 1 | 1 | 1 | 1 | 1 |

71.

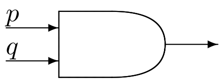
| $p$ | $q$ | $r$ | $p \wedge q$ | $(p \wedge q) \wedge r$ | $q \wedge r$ | $p \wedge (q \wedge r)$ | $((p \wedge q) \wedge r) \Leftrightarrow (p \wedge (q \wedge r))$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 | 0 | 0 | 0 | 1 |
| 1 | 0 | 1 | 0 | 0 | 0 | 0 | 1 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

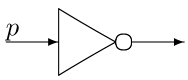
73. Tautology

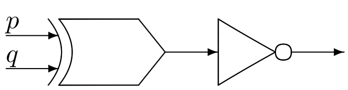
| | | | | | $A$ | | $B$ | $A \Leftrightarrow B$ |
|---|---|---|---|---|---|---|---|---|
| $p$ | $q$ | $r$ | $p \Rightarrow q$ | $p \Rightarrow r$ | $(p \Rightarrow q) \wedge (p \Rightarrow r)$ | $q \wedge r$ | $p \Rightarrow (q \wedge r)$ | |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 1 |
| 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 1 |
| 0 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 1 |
| 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 1 |

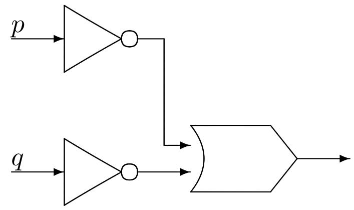
75.

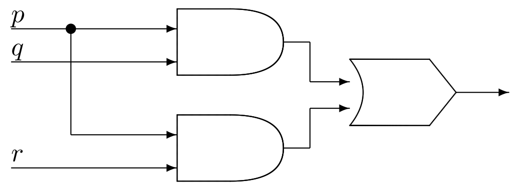
| | | | | $A$ | | | $B$ | |
|---|---|---|---|---|---|---|---|---|
| $p$ | $q$ | $r$ | $q \wedge r$ | $p \vee (q \wedge r)$ | $p \wedge q$ | $p \wedge r$ | $(p \wedge q) \vee (p \wedge r)$ | $A \Leftrightarrow B$ |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

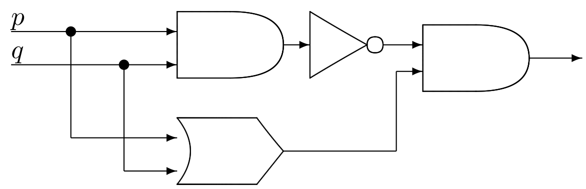
77. 


79. 


81. 


83. 


85. 


87. 


Chapter 4

1. $24 + 18 = 42$
3. $\frac{84}{2.1} = 40$
5. $5\frac{6}{10} + 4\frac{4}{10} = 10$
7. $(-20 - 18) + 12 = -26$
9. $0 \div 18 + 1 = 1$
11. $-(8 + (-12)) = 4$
13. $n + 6 = 28$
15. $16 * 12 = n$
17. $\frac{1}{3} * n = n - 9$

19. $r$: number of red marbles,
    $b$: number of blue marbles,
    $r + b = 32$

21. $c$: average cost of a paperback book,
    $5c = 32.2$

23. $p$: pounds of produce,
    $0.98p = 5.56$

25. $h$: number of hardback books,
    $7.1 * 3 + 15.7h = 99.8$

27. $r$: tax rate,
    $450 + 450r = 497.25$

29. $x$: height (in feet) of first book shelf,
    $y$: height (in feet) of second book shelf,
    $x + y = 6$

31. $x = -7$
33. $t = 59$
35. $m = 3.89$
37. $x = -.07$
39. $w = -50$
41. $m = 6.95$

43. $m = -18$

45. $x = -2.86$

47. $n = \frac{153}{32}$ or $4\frac{25}{32}$

49. $x = -5$

51. $m = -5$

53. $m = -4.5$

55. $x = -\frac{2}{13}$

57. $x = -\frac{9}{23}$

59. $y = 3.1$

61. $v = -1$

63. $x = -1.5$

65. All real numbers, or $\mathbb{R}$

67. No solution, or $\varnothing$

69. No solution, or $\varnothing$

71. $x = 0$

73. yes

75. no

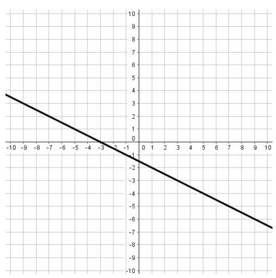
77. yes

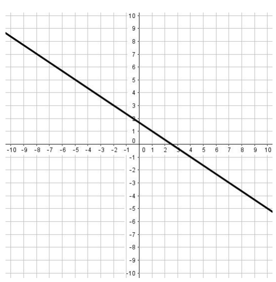
79. no

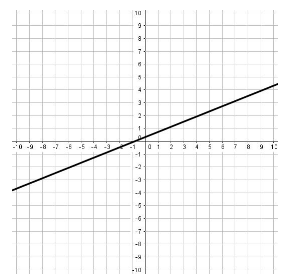
81. yes

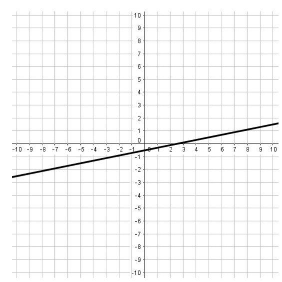
83. yes

85. no

87. 

89. 

91. 

93. 

95. $m = 1$, $(0, 8)$

97. $m = \frac{3}{2}$, $(0, 0)$

99. undefined slope, no $y$-intercept

101. $m = -\frac{3}{2}$, $(0, 2)$

103. undefined slope, no $y$-intercept

105. $m = -\frac{5}{6}$, $(0, -10)$

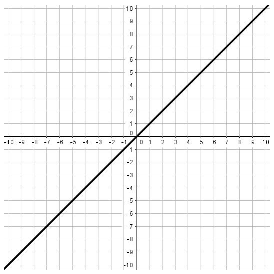
107. $y = -2x + 12$

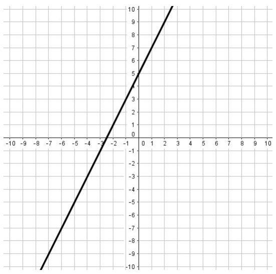
109. $y = 8$

111. $x = 0$

113. $y = \frac{1}{6}x + 6$

115. $y = -\frac{4}{5}x$


117.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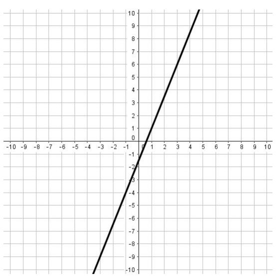
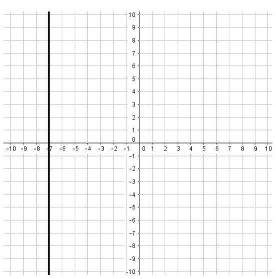
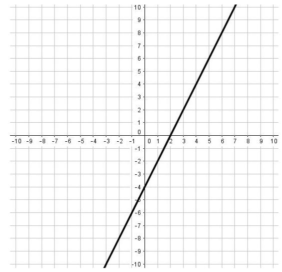
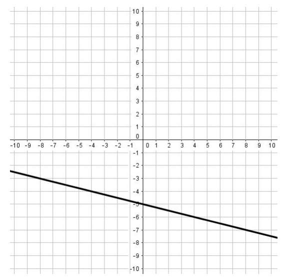
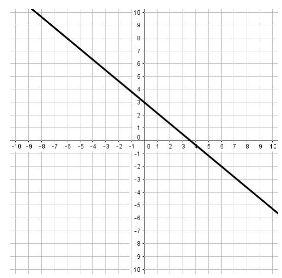

121.


123.


125.


127.


129.

Chapter 5

1. $2^3$
3. $3 * 5$
5. $2 * 3^2$
7. $2 * 13$
9. $2^5$
11. $2 * 3 * 7$
13. $2^4 * 3$
15. $3 * 17$
17. $2^6$
19. $2^3 * 3^2$
21. $3^4$
23. 83
25. 97
27. 12
29. 21
31. 64
33. 11
35. 2
39. 7
41. 14
43. 1
45. 8
47. 14
49. 56
51. 27

53. 20
55. 78
57. 17
59. 2
61. 0
63. 1
65. 1
67. 12
69. 12
71. 6
73. 7
75. 1
77. 8
79. 1
81. 0
83. 8
85. 9
87. 7
89. 1
91. 0
93. 5
95. 6
97. 0
99. false
101. true
103. false
105. false
107. true
109. false
111. true
113. false
115. false

Chapter 6

Chapter 7

1. (a) A, B, C, & D
   (b) AB, BC, CD, & DA
   (c) none
   (d) none
   (e) A, B, C, & D have degree 2
   (f) yes
   (g) yes

3. (a) A, B, C, & D
   (b) AB, BC, CA, & DD
   (c) BC
   (d) DD
   (e) A & D have degree 2, B & C have degree 3
   (f) no
   (g) no

5. (a) A, B, C, D, E, & F
   (b) AC, AF, CF, CE, EF, BB, & BD
   (c) none
   (d) BB
   (e) A & E have degree 2, D has degree 1, B, C, & F have degree 3
   (f) no
   (g) no

7. (a) A, B, C, D, & F
   (b) AD, AC, BD, BC, & CF
   (c) AD, BD, & CF
   (d) none
   (e) F has degree 2, A & B have degree 3, C & D have degree 4
   (f) no
   (g) yes

9. (a) A, B, C, D, E, F
   (b) AC, BC, BF, DE
   (c) none
   (d) none
   (e) A, F, D, & E have degree 1, B & C have degree 2
   (f) yes
   (g) no

11. (a) A, B, C, D, E, F, G
    (b) AB, AC, AD, BC, CD, EF, FG, GE
    (c) none

(d) none

(e) B, D, E, F, & G have degree 2, A & C have degree 3

(f) yes

(g) no

13. (a) A, B¡ C¡ D, E, F, G

(b) AB, AC, AD, AE, AF, AG, BC, BD, BE, BF, BG, CD, CE, CF, CG, DE, DE, DG, EF, EG, FG

(c) none

(d) none

(e) A, B, C, D, E, F & G have degree 6

(f) yes

(g) yes

15. (a) A, B, C, D, E, F, G¡ H, I

(b) AH, AI, CD, CI, BE, BI, GF, GI, FH, DE

(c) none

(d) none

(e) A, B, C, D, E, F, G, & H have degree 2, I has degree 4

(f) yes

(g) yes

17. (a) A, B, C, D, E, F, G, H, I

(b) AB, AI, BG, BE, EG, EF, EC, DC, DF, FH, II, IH

(c) none

(d) II

(e) A, H, G, C, & D have degree 2, B & F have degree 3, I & E have degree 4

(f) no

(g) yes

19. A circuit must begin and end at the same vertex while this is not required of a path.

21. A circuit is Euler when it crosses every edge.

23. Answers will vary.

25. Answers will vary.

27. No. Yes.

29. No. No.

31. No. No.

33. No. Yes.

35. No. No.

**For 37-55 there may be many correct answers, only one is provided as an example of a correct solution.**

37. B,A,D,A,C

39. G,A,B,D,G,H,D,E,H,F,E,C,F

41. D,C,E,C,B,A,F,B

43. A,B,C,D,E,F,G,H,A,C,E,G,A,D,F,H,B,D,H,C

45. A,B,B,E,A,D,D,B,C,A,F,C,E,F,D,C

47. A,B,C,A

49. B,C,E,B,A,D,B

51. G,A,E,C,H,B,F,D,G,E,H,F,G

53. A,E,H,D,A,G,F,B,C,G,B,H,A

55. A,F,B,E,D,G,A,B,D,A,C,G,F,E,G,B,C,F,D,C,E,A

57. To copy edges of a graph until the graph is semi-Euler.

    **for 59-73, there may be multiple correct ways to Eulerize or semi-Eulerize a graph; only one solution will be provided as an example.**

59. (a) Already semi-Euler

    (b) Copy AB & AC

61. (a) Already semi-Euler

    (b) Copy GH & HF

63. (a) Already semi-Euler

    (b) Copy CD & BC

65. (a) Already semi-EUler

    (b) Copy FC & CE

67. (a) Already semi-Euler

    (b) Copy AC

69. Not possible - the graph isn't connected.

71. (a) Copy AC

    (b) Copy AC, EG, & GF

73. (a) Already semi-Euler

    (b) Copy AC & CB

    **For 75-89 there may be many correct answers, only one is provided as an example of a correct solution.**

75. Not possible, the grpah isn't connected.

77. (a) B-D-A-C

    (b) B-D-C-A-B

79. (a) A-E-D-H-F-B-G-C

    (b) A-E-D-H-C-G-B-F-A

81. (a) A-B-C-D-F-E

    (b) A-B-D-F-E-C-A

83. (a) Not possible

    (b) A-B-C-D-A

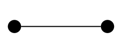
85. (a) A-D-B-C-F

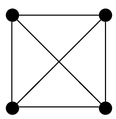
(b) Not possible

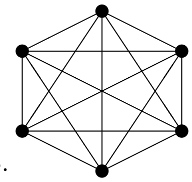
87. Not possible - the graph isn't connected.

89. (a) E-A-B-C-F-D

(b) A-B-C-D-F-E-A

91. 

93. 

95. 

97. No. No.

99. Yes. No.

101. Yes. No.

103. $n$ is odd.

# Index