Stat 102C HW2: Answer Key

Muzhou Liang

May 7, 2014

Problem 1

(1)

$$E(\hat{I}) = \frac{1}{n}\sum_{i=1}^{n} E[h(X_i)] = \frac{1}{n} n E[h(X)] = E[h(X)] = I$$

(2)

$$\begin{aligned}
\mathrm{Cov}[h(X_i), h(X_j)] &= E[h(X_i)h(X_j)] - E[h(X_i)]E[h(X_j)] \\
&= \int\int h(X_i)h(X_j)f(x_i)f(x_j)\mathrm{d}x_i\mathrm{d}x_j - I^2 \\
&= \int h(x_i)f(x_i)\mathrm{d}x_i \int h(x_j)f(x_j)\mathrm{d}x_j - I^2 \\
&= E[h(X_i))E(h(X_j)] - I^2 = 0
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(\hat{I}) &= \frac{1}{n^2}\mathrm{Var}\big[\sum_{i=1}^{n} h(X_i)\big] \\
&= \frac{1}{n^2}\left[\sum_{i=1}^{n}\mathrm{Var}[h(X_i)] + \sum_{i\neq j}\mathrm{Cov}[h(X_i), h(X_j)]\right] \\
&= \frac{1}{n^2} n\mathrm{Var}[h(X)] = \frac{1}{n}\mathrm{Var}[h(X)]
\end{aligned}$$

Let $V = \mathrm{Var}[h(X)]$, $\hat{V} = \frac{1}{n}\sum_{i=1}^{n}[h(X_i) - \hat{I}]^2$ would be the estimator.

(3) $\hat{I}$ follows $\mathcal{N}(I, V/n)$ when sample size is large, according to the Central Limit Theorem (CLT). Hence, the 95% confidence interval is given as $[\hat{I} - z_{.025}\sqrt{\hat{V}/n}, \hat{I} + z_{.025}\sqrt{\hat{V}/n}]$, where $z_{.025} = 1.96$.

(4)
```
> h = function(x) x^4
> E = function(n) {
```

```
+   x = rnorm(n)
+   I = mean(h(x))
+   return(I)
+ }
> E(1e5)

[1] 3.003959
```

Problem 2

(1)

$$
\begin{aligned}
\mathrm{E}_g[h(X)W(X)] &= \int h(x)w(x)g(x)\mathrm{d}x \\
&= \int h(x)\frac{f(x)}{g(x)}g(x)\mathrm{d}x \\
&= \int h(x)f(x)\mathrm{d}x = \mathrm{E}_f[g(X)]
\end{aligned}
$$

(2)

$$
\mathrm{E}(\hat{I}) = \frac{1}{n}\sum_{i=1}^{n}\mathrm{E}[h(X_i)w_i] = \mathrm{E}_g[h(X)W(X)] = I
$$

Same as 1.2, $\mathrm{Cov}[h(X_i)w_i, h(X_j)w_j] = 0$.

$$
\begin{aligned}
\mathrm{Var}(\hat{I}) &= \frac{1}{n^2}\left[\sum_{i=1}^{n}\mathrm{Var}[h(X_i)w_i] + \sum_{i\neq j}\mathrm{Cov}[h(X_i)w_i, h(X_j)w_j]\right] \\
&= \frac{1}{n}\mathrm{Var}_g[h(X_i)w_i]
\end{aligned}
$$

(3) f is a truncated Normal density with $C = 0$; g is an Exponential density with $\lambda = 2$.

```
> f = function(x) sqrt(2/pi) * exp(-x^2/2)
> g = function(x) dexp(x, 2)
> E = function(n) {
+   x = rexp(n, 2)
+   w = f(x)/g(x)
+   I = mean(x * w)
+   return(I)
+ }
> E(1e5)

[1] 0.8000738
```

Problem 3

(1) We take g as the density of $\mathcal{N}(C, 1)$, $h(x) = \mathbf{1}_{\{X>C\}}$.

```
> f = function(x) dnorm(x)
> g = function(x, C) dnorm(x, mean = C)
> h = function(x, C) (x > C) # this function returns a vector
> # consists of TRUEs and FALSEs, which equal to 1 and 0 when
> # taken into computing
> E = function(n, C) {
+   x = rnorm(n, C)
+   w = f(x) / g(x, C)
+   I = mean(h(x, C) * w)
+   return(I)
+ }
> E(1e5, 2); 1-pnorm(2) #true value
```

```
[1] 0.02282523
```

```
[1] 0.02275013
```

A more efficient alternative: We take g as the density of the exponential distribution $Exp(1)$ truncated at C, $h(x) = \mathbf{1}_{\{X>C\}}$ still. But note that h(x) will always equal to 1 since the generated sample is always bigger than C. Therefore, we can just ignore $h(x)$ here.

$$g(y) = e^{-y} / \int_C^\infty e^{-x} \mathrm{d}x = e^{-(y-C)}$$

$$\hat{I} = \frac{1}{n} \sum_{i=1}^{n} \frac{f(X_i)}{g(X_i)}$$

```
> f = function(x) dnorm(x)
> g = function(x, C) dexp(x-C)
> E = function(n, C) {
+   x = rexp(n) + C
+   w = f(x)/g(x, C)
+   I = mean(w)
+   return(I)
+ }
> E(1e5, 2); 1-pnorm(2) #true value
```

```
[1] 0.02271789
```

```
[1] 0.02275013
```

(2) We can generate (X_1, X_2, X_3) from $\mathcal{N}(\alpha, 1)$, where α is a positive number slightly smaller to C. Thus, $P(M > C)$ would be much larger. So g would be the joint density of three normal distributions $\mathcal{N}(\alpha, 1)$, and f is the joint density of three $\mathcal{N}(\mu, 1)$. $h(X) = \mathbf{1}_{\{M>C\}}$.

```
> f = function(x1, x2, x3, mu) {
+   dnorm(x1, mu, 1) * dnorm(x2, mu, 1) * dnorm(x3, mu, 1)
+ }
> g = function(x1, x2, x3, alpha) {
+   dnorm(x1, alpha, 1) * dnorm(x2, alpha, 1) * dnorm(x3, alpha, 1)
+ }
> h = function(M, C) (M > C)
> E = function(N, mu, alpha, C) {
+   X = cbind(
+     x1 = rnorm(N, alpha, 1),
+     x2 = rnorm(N, alpha, 1),
+     x3 = rnorm(N, alpha, 1)
+   )
+   M = apply(X, 1, FUN = function(x) {max(x[1], x[1]+x[2],
+             x[1]+x[2]+x[3])})
+   w = apply(X, 1, FUN = function(x) {f(x[1], x[2], x[3], mu) /
+             g(x[1], x[2], x[3], alpha)})
+   I = mean(w * h(M, C))
+   return(I)
+ }
> E(1e5, -1, 0.5, 1)

[1] 0.03921958
```

Problem 4

(1)

$$f_{R,\Theta}(r,\theta) = f_{X,Y}(r\cos\theta, r\sin\theta) | \begin{vmatrix} \frac{\partial r\cos\theta}{\partial r} & \frac{\partial r\cos\theta}{\partial\theta} \\ \frac{\partial r\sin\theta}{\partial r} & \frac{\partial r\sin\theta}{\partial\theta} \end{vmatrix} | = \frac{r}{2\pi}e^{-r^2/2}$$

(2)

$$f_{R,\Theta}(r,\theta) = \frac{1}{2\pi}\mathrm{d}\theta r e^{-r^2/2}\mathrm{d}r$$

Since R and Θ are independent, we can get the two densities as below

$$g_R(r) = re^{-r^2/2}\mathrm{d}r = e^{-r^2/2}\mathrm{d}\frac{r^2}{2} = e^{-t}\mathrm{d}t \ \ (\text{let } T = \frac{R^2}{2})$$

$$h_\Theta(\theta) = \frac{1}{2\pi}\mathrm{d}\theta$$

Thus T follows $\mathcal{E}xp(1)$. Remember that an Exponential sample can be obtained by a uniform generator using inverse sampling method. Hence

$$T = -\log U$$
$$R = \sqrt{2T}$$
$$\Theta = 2\pi V$$

```
> normsamp = function(n) {
+   u = runif(n)
+   v = runif(n)
+   t = -log(u)
+   r = sqrt(2*t)
+   theta = 2*pi*v
+   x = r*cos(theta)
+   y = r*sin(theta)
+   res = list("X"=x, "Y"=y)
+   return(res)
+ }
> par(mfrow=c(1,2))
> sample = normsamp(1e4)
> hist(sample$X, freq=FALSE)
> curve(dnorm(x), col = 'red', lwd=2,add = T)
> hist(sample$Y, freq=FALSE)
> curve(dnorm(x), col = 'red', lwd=2,add = T)
```