AM 213A HW1

Joseph Moore

Winter 2022

1. We can write A as $QUQ^{-1} = QUQ^T$ where Q and U are real matrices. If we remember that $Ax = \lambda x$

$$Q^t AQ = \begin{pmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_m^T \end{pmatrix} \begin{pmatrix} Aq_1 & Aq_2 & ... & Aq_m \end{pmatrix}$$

$$= \begin{pmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_m^T \end{pmatrix} \begin{pmatrix} \lambda_1 q_1 & \lambda_2 q_2 & ... & \lambda_m q_m \end{pmatrix}$$

$$= \begin{pmatrix} \lambda & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & ... & \lambda_m \end{pmatrix}$$

2. Using $\epsilon = 1e - 17$ and $c = 1e18$ resulted in $x = 0.00000$ and $y = 1.00000$, which is wrong. We should get $(1, 1)$. After multiplying the row by a large c, the algorithm pivots this row to the top and we get

$$\left(\begin{array}{cc|c} c\epsilon & c & c \\ 1 & 1 & 2 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} c\epsilon & c & c \\ 1 - \frac{c\epsilon}{c\epsilon} & 1 - \frac{c}{c\epsilon} & 2 - \frac{c}{c\epsilon} \end{array}\right) = \left(\begin{array}{cc|c} c\epsilon & c & c \\ 0 & 1 - \frac{1}{\epsilon} & 2 - \frac{1}{\epsilon} \end{array}\right)$$

performing back substitution gets us $y = 1$ from the bottom row and now the top row will give us $x = 0$. If we don't multiply by some very large constant then ϵ won't be pivoted to the top and we won't run into this problem. By leaving ϵ on the bottom row we can zero it out by subtracting a very small multiply of the first row from the second row. This will leave all the other elements at play virtually unchanged.

3. We know that $x^T Ax > 0$ for all vectors. Then it must be the case that $e_i^T Ae_i > 0$ for each i. But $e_i^T Ae_i$ is just the i'th diagonal element of A and therefore each diagonal element of A must be positive.

4.

(a) We need to show that $A_{21} - A_{21}A_{11}^{-1}A_{11} = 0$. Which is obvious since $A_{21} - A_{21}A_{11}^{-1}A_{11} = A_{21} - A_{21}I = A_{21} - A_{21} = 0$

(b) A can be upper triagularized by applying a lower triangular matrix to A or $L^-1A = U$. From this we can write

$$L^{-1} = \begin{pmatrix} Y & 0 \\ X & W \end{pmatrix}$$

From this we know that $YA_{11} = U_{11}$, where U_{11} is A_{11} upper triagularized and thus $Y = L_{11}^{-1}$ and

$$L^{-1} = \begin{pmatrix} L_{11}^{-1} & 0 \\ X & W \end{pmatrix}$$

$$L^{-1}A = \begin{pmatrix} U_{11} & L_{11}^{-1}A_{12} \\ XA_{11} + WA_{21} & XA_{12} + WA_{22} \end{pmatrix}$$

Now we set $XA_{11} + WA_{21} = 0$ and solving for X we get $X = -WA_{21}A_{11}^{-1}$. From this we have

$$XA_{12} + WA_{22} = -WA_{21}A_{11}^{-1}A_{12} + WA_{22}$$

Which is exactly what we want if we set $W = I$.

5.

(a) If we decompose A into $A_1 + iA_2$ then we have

$$Ax = (A_1 + iA_2)x = A_1x + iA_2x = b_1 + ib_2$$

Which becomes

$$A_1x = b_1$$

$$A_2x = b_2$$

Which can be written as in matrix form

$$\begin{pmatrix} A_1 & A_2 \end{pmatrix} x = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Which is'nt square but looks correct.

(b) Performing Gaussian elimination on a square $2m$ matrix is $O(\frac{8m^3}{3})$. For complex algebra, each multiplication will be quadrupled resulting in a $O(\frac{4m^3}{3})$, which is slightly cheaper.