AM 213A HW1

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1. $A^{-1} = A^*$ and $A^*A = I$. Thus the i, j components of A^*A will be zero when $i \neq j$ and 1 when $i = j$. Now looking at A,

$$A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1m} \\ 0 & a_{22} & ... & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & ... & a_{mm} \end{bmatrix}$$

We then notice that $(A^*A)_{11} = \bar{a}_{11}a_{11} = 1$ as the rest of the elements of the first column of A are zero. Plugging in these zeros and working our way down the diagonal reveals that the same must be true of $(A^*A)_{22}$ as the rest of the elements of the second column of A are now zero as well. We can work our way down the diagonal like this until all non-diagonal elements are zero, thus leaving a diagonal matrix.

2.

a) Taking $\lambda \neq 0$ as the eigenvalue of A we have

$$Ax = \lambda x \Rightarrow A^{-1}Ax = A^{-1}\lambda x \Rightarrow x = A^{-1}\lambda x \Rightarrow A^{-1}x = \frac{1}{\lambda}x.$$

From this form we can conclude that $\frac{1}{\lambda}$ is the eigenvalue of A^{-1}.

b) If we let λ be the eigenvalue of AB and $Bx = y$ then we can formulate the following

$$(AB)x = \lambda x \Rightarrow Ay = \lambda x \Rightarrow BAy = B\lambda x \Rightarrow BAy = \lambda Bx \Rightarrow (BA)y = \lambda y$$

In this form we can see that λ is also the eigenvalue of BA.

c) The eigenvalues of A are given by $det(A - \lambda I)$. Now using the following

$$(A - \lambda I)^T = A^T - \lambda I^T = A^T - \lambda I$$

and that $det(M) = det(M^T)$ we can conclude that $det(A^T - \lambda x) = det(A - \lambda I)$ and thus A^T will have the same eigenvalues of A.

3.

a) Given that A is Hermitian we have that $A^* = A$. With $Ax = \lambda x$ we have

$$\bar{\lambda}x^*x = (\lambda x)^*x = (Ax)^*x = x^*A^*x = x^*Ax = x^*\lambda x = \lambda x^*x$$

Thus $\lambda = \bar{\lambda}$.

b) Let $Ax = \lambda_1 x$ and $Ay = \lambda_2 y$ then it follows that

$$x^*Ay = x^*A^*y = (Ax)^*y = (\lambda_1 x)^*y = \bar{\lambda_1}x^*y = \lambda_1 x^*y$$

But also

$$x^*Ay = x^*\lambda_2 y = \lambda_2 x^*y$$

Which is a contradiction unless $x^*y = 0$, which is another way of saying the two vector are orthogonal.

4. We can compose a matrix of the eigenvectors of A and call P. Then we can decompose A into PDP^{-1}, where D is the diagonal matrix of eigenvalues. Next, we can rewrite x as a linear combination of these eigenvectors, $x = \sum_{i=1}^m a_i u_i$. From this we can derive the following from the inner-product of (Ax, x)

$$x^tAx = (a_1u_1^T + a_2u_2^T + ... + a_mu_m^T)PDP^*(a_1u_1 + a_2u_2 + ... + a_mu_m)$$

Because u_i are orthogonal $xu_j = \sum_{i=1}^m a_iu_i^Tu_j = a_ju_j^Tu_j = a_j||u_j||_2^2$ and we get

$$\begin{pmatrix} a_1||u_1||_2^2 & a_2||u_2||_2^2 & ... & a_m||u_m||_2^2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & ... & \lambda_m \end{pmatrix} \begin{pmatrix} a_1||u_1||_2^2 \\ a_2||u_2||_2^2 \\ \vdots \\ a_m||u_m||_2^2 \end{pmatrix}$$

u_i form an orthonormal bases so their 2-norm is always one. Thus

$$x^TAx = \sum_{i=1}^m \lambda_i|a_i|^2$$

If all $\lambda_i > 0$ then A will meet the qualifications for being positive definite.

5.

a) We start $Ax = \lambda x$ to get

$$(Ax)^*Ax = (Ax)^*\lambda x$$

$$x^*A^*Ax = x^*\lambda^*\lambda x$$

$$x^*Ix = x^*|\lambda|^2x$$

$$||x||^2 = |\lambda|^2||x||^2$$

$$1 = |\lambda|^2 \Rightarrow \lambda = 1$$

b)

$$||A||_F = \sqrt{trace(A^*A)} = \sqrt{trace(I)}$$

The sum of the diagonals will always be greater than 1 and thus $||A||_F$ can never be equal to one.

6.

a) We start with $Ax = \lambda x$ and apply x^* to it to get

$$x^*Ax = x^*\lambda x \Rightarrow x^*Ax = \lambda x^*x \Rightarrow \lambda = \frac{x^*Ax}{x^*x}$$

taking the conjugate transpose gives us

$$\lambda^* = \frac{(x^*Ax)^*}{(x^*x)^*} \Rightarrow \bar{\lambda} = \frac{x^*A^*x}{x^*x} \Rightarrow \bar{\lambda} = -\frac{x^*Ax}{x^*x} \Rightarrow$$

Thus $\bar{\lambda} = -\lambda$ which must mean that λ is purely imaginary.

b) Applying $I - A$ to x will yield $1 - \lambda$. Because λ is purely imaginary this will never be equal to zero and thus $I - A$ is non-singular.

7. Assume that u is eigenvector of A such that $||u|| = 1$. Then we have that

$$||A|| \geq ||Au|| = ||\lambda u|| = |\lambda|$$

and all eigenvalues must be equal or less than $||A||$.

8.

a) If we note that vv^* is a rank one matrix with the largest eigenvalue given by v^*v then it follows that

$$||A||_2 = (\sigma(A^*A))^{1/2} = (\sigma(vu^*uv^*))^{1/2} = (u^*u)^{1/2}\sigma(vv^*)^{1/2} = (u^*u)^{1/2}(v^*v)^{1/2} = ||u||_2||v||_2$$

b)

$$||A||_F = \sqrt{trace(A^*A)} = \sqrt{trace(vu^*uv^*)} = \sqrt{trace(v^*vu^*u)} = \sqrt{vec(v^*v)vec(u^*u)}$$
$$= \sqrt{trace(v^*v)}\sqrt{trace(u^*u)} = ||v||_F||u||_F$$

9.

a) First we start by establishing

$$||Qx||_2 = \sqrt{(Qx)^*(Qx)} = \sqrt{x^*Q^*Qx} = \sqrt{x^*x} = ||x||_2$$

so that

$$||AQ||_2 = sup\frac{||AQx||_2}{||x||_2} = sup\frac{||Ax||_2}{||x||_2} = ||A||_2$$

b)

$$||AQ||_F = \sqrt{trace((AQ)^*AQ)} = \sqrt{trace((Q^*A^*AQ)} = \sqrt{trace(QQ^*A^*A)}$$
$$= \sqrt{trace(A^*A)} = ||A||_F$$

10.

a) If we write B as $U\Sigma V^*$ then we can derive the following

$$A = QBQ^* = QU\Sigma V^*Q^* = U'\Sigma V'^*$$

Since the product of unitary matrices are themselves unitary then the result is the SVD of A and thus A and B share the same singular values in Σ.

b) ?

11.

a)

$$\kappa = \frac{||J||_\infty ||x||_\infty}{||f(x)||} = \frac{2 \cdot max\{|x_1|, |x_2|\}}{|x_1 + x_2|}$$

The quantity is vary large as $|x_1 + x_2| -> 0$ so κ is ill-conditioned when $x_1 \approx -x_2$.

b)

$$\kappa = \frac{||J||_\infty ||x||_\infty}{||f(x)||} = \frac{(|x_1| + |x_2|) \cdot max\{|x_1|, |x_2|\}}{|x_1 x_2|}$$

When we split this fraction we notice that κ becomes ill-conditioned when $x_1 >> x_2$ or $x_2 >> x_1$.

c)

$$\kappa = \frac{||J||_\infty ||x||_\infty}{||f(x)||} = \frac{9|x-2|^8 \cdot |x|}{|x-2|^9} \approx \frac{|x|^9}{|x|^9}$$

κ should always be relatively well-conditioned.

12. On GitHub.