Answers/Solutions of Exercise 4 (Version: October 7, 2016)

Remark: Please note that bases for vector spaces are not unique. In the following, if a question asks for a basis, the answer given is only one of the possible answers.

1. In order to answer (iv), we obtain the reduced row-echelon form of each of the matrices. (To answer (i)-(iii), we only need a row-echelon form.)

   (a) Performing Gauss-Jordan elimination, we obtain the reduced row-echelon form of $\boldsymbol{A}$:
   $$\begin{pmatrix} 1 & 0 & 0 & 1 & -\frac{2}{7} \\ 0 & 1 & 0 & 1 & \frac{4}{7} \\ 0 & 0 & 1 & -1 & \frac{13}{7} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

   (i) $\{(1, 0, 0, 1, -\frac{2}{7}),\ (0, 1, 0, 1, \frac{4}{7})),\ (0, 0, 1, -1, \frac{13}{7})\}$ is a basis for the row space.
   $\{(1, 2, -1, 1)^{\mathrm{T}},\ (4, 1, 3, -1)^{\mathrm{T}},\ (0, 0, 0, 1)^{\mathrm{T}}\}$ is a basis for the column space.

   (ii) $\{(1, 0, 0, 1, -\frac{2}{7}),\ (0, 1, 0, 1, \frac{4}{7})),\ (0, 0, 1, -1, \frac{13}{7}),\ (0, 0, 0, 1, 0),\ (0, 0, 0, 0, 1)\}$ is a basis for $\mathbb{R}^5$.

   (iii) $\begin{pmatrix} 1 & 2 & -1 & 1 \\ 4 & 1 & 3 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & -7 & 7 & -6 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$
   So $\{(1, 2, -1, 1)^{\mathrm{T}},\ (4, 1, 3, -1)^{\mathrm{T}},\ (0, 0, 0, 1)^{\mathrm{T}},\ (0, 0, 1, 0)^{\mathrm{T}}\}$ is a basis for $\mathbb{R}^4$.

   (iv) $\{(-1, -1, 1, 1, 0)^{\mathrm{T}},\ (\frac{2}{7}, -\frac{4}{7}, -\frac{13}{7}, 0, 1)^{\mathrm{T}}\}$ is a basis for the nullspace.

   (v) $\text{rank}(\boldsymbol{A}) = 3$ and $\text{nullity}(\boldsymbol{A}) = 2$.
   Hence $\text{rank}(\boldsymbol{A}) + \text{nullity}(\boldsymbol{A}) = 3+2 = 5 =$ the number of column in $\boldsymbol{A}$.

   (vi) No.

   (b) Performing Gauss-Jordan elimination, we obtain the reduced row-echelon form of $\boldsymbol{B}$:
   $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

   (i) $\{1, 0, 0),\ (0, 1, 0),\ (0, 0, 1)\}$ is a basis for the row space.
   $\{(1, 0, -1, 2, 3)^{\mathrm{T}},\ (2, 1, 3, 1, 1)^{\mathrm{T}},\ (0, 1, 6, 0, -1)^{\mathrm{T}}\}$ is a basis for the column space.

   (ii) $\{1, 0, 0),\ (0, 1, 0),\ (0, 0, 1)\}$ is already a basis for $\mathbb{R}^3$.

(iii) $\begin{pmatrix} 1 & 0 & -1 & 2 & 3 \\ 2 & 1 & 3 & 1 & 1 \\ 0 & 1 & 6 & 0 & -1 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 1 & 0 & -1 & 2 & 3 \\ 0 & 1 & 5 & -3 & -5 \\ 0 & 0 & 1 & 3 & 4 \end{pmatrix}$.

So $\{(1,0,-1,2,3)^{\mathrm{T}},\ (2,1,3,1,1)^{\mathrm{T}},\ (0,1,6,0,-1)^{\mathrm{T}},\ (0,0,0,1,0)^{\mathrm{T}},\ (0,0,0,0,1)^{\mathrm{T}}\}$ is a basis for $\mathbb{R}^5$.

(iv) $\emptyset$ is the basis for the nullspace.

(v) $\text{rank}(\boldsymbol{B}) = 3$ and $\text{nullity}(\boldsymbol{B}) = 0$.
Hence $\text{rank}(\boldsymbol{B})+\text{nullity}(\boldsymbol{B}) = 3+0 = 3 =$ the number of column in $\boldsymbol{B}$.

(vi) Yes.

(c) Performing Gauss-Jordan elimination, we obtain the reduced row-echelon form of $\boldsymbol{C}$:

$$\begin{pmatrix} 1 & \frac{1}{2} & 0 & \frac{5}{6} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{6} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(i) $\{(1, \frac{1}{2}, 0, \frac{5}{6}, \frac{1}{3}),\ (0, 0, 1, -\frac{1}{6}, \frac{1}{3})\}$ is a basis for the row space.
$\{(2,4,2,6)^{\mathrm{T}},\ (4,2,-2,6)^{\mathrm{T}}\}$ is a basis for the column space.

(ii) $\{(1, \frac{1}{2}, 0, \frac{5}{6}, \frac{1}{3}),\ (0, 0, 1, -\frac{1}{6}, \frac{1}{3}), (0,1,0,0,0),\ (0,0,0,1,0),\ (0,0,0,0,1)\}$ is a basis for $\mathbb{R}^5$.

(iii) $\begin{pmatrix} 2 & 4 & 2 & 6 \\ 4 & 2 & -2 & 6 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 2 & 4 & 2 & 6 \\ 0 & -6 & -6 & -6 \end{pmatrix}$.

So $\{(2,4,2,6)^{\mathrm{T}},\ (4,2,-2,6)^{\mathrm{T}},\ (0,0,1,0)^{\mathrm{T}},\ (0,0,0,1)^{\mathrm{T}}\}$ is a basis for $\mathbb{R}^4$.

(iv) $\{(-\frac{1}{2}, 1, 0, 0, 0)^{\mathrm{T}},\ (-\frac{5}{6},\ 0,\ \frac{1}{6},\ 1, 0)^{\mathrm{T}},\ (-\frac{1}{3}, 0, -\frac{1}{3}, 0, 1)^{\mathrm{T}}\}$ is the basis for the nullspace.

(v) $\text{rank}(\boldsymbol{C}) = 2$ and $\text{nullity}(\boldsymbol{C}) = 3$.
Hence $\text{rank}(\boldsymbol{C})+\text{nullity}(\boldsymbol{C}) = 2+3 = 5 =$ the number of column in $\boldsymbol{C}$.

(vi) No.

(d) Performing Gauss-Jordan elimination, we obtain the reduced row-echelon form of $\boldsymbol{D}$:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(i) $\{(1,0,1,0),\ (0,1,1,0),\ (0,0,0,1)\}$ is a basis for the row space.
$\{(1,-1,2)^{\mathrm{T}},\ (4,4,0)^{\mathrm{T}},\ (8,0,1)^{\mathrm{T}}\}$ is a basis for the column space.

(ii) $\{(1,0,1,0),\ (0,1,1,0),\ (0,0,0,1),\ (0,0,1,0)\}$ is a basis for $\mathbb{R}^4$.

(iii) $\{(1,-1,2)^{\mathrm{T}},\ (4,4,0)^{\mathrm{T}},\ (8,0,1)^{\mathrm{T}}\}$ is already a basis for $\mathbb{R}^3$.

(iv) $\{(-1,-1,1,0)^{\mathrm{T}}\}$ is a basis for the nullspace.

(v) $\mathrm{rank}(\boldsymbol{D}) = 3$ and $\mathrm{nullity}(\boldsymbol{D}) = 1$.
Hence $\mathrm{rank}(\boldsymbol{D})+\mathrm{nullity}(\boldsymbol{D}) = 3+1 = 4 =$ the number of column in $\boldsymbol{D}$.

(vi) Yes.

2. (a) $$\begin{pmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 1 & 15 & 8 & 6 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & -1 & -3 & -2 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$
$\{(1,-2,0,0,3),\ (0,-1,-3,-2,0),\ (0,0,0,-2,0)\}$ is a basis for W.

(b) $\dim(W) = 3$

(c) $\{(1,-2,0,0,3),\ (0,-1,-3,-2,0),\ (0,0,0,-2,0),\ (0,0,1,0,0),\ (0,0,0,0,1)\}$ is a basis for $\mathbb{R}^5$.

3. (a) $$\begin{pmatrix} 1 & 2 & -1 & 0 & 3 \\ 0 & -1 & 3 & 1 & -1 \\ 1 & 0 & 5 & 2 & 1 \\ 3 & 1 & 12 & 5 & 4 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 1 & 2 & -1 & 0 & 3 \\ 0 & -1 & 3 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$
So $S' = \{(1,0,1,3),\ (2,-1,0,1)\}$ is a basis for V.

(b) $$\begin{pmatrix} 1 & 2 & -3 & 0 & 0 \\ 0 & 1 & -2 & 5 & 4 \\ 1 & -2 & 5 & 2 & 6 \\ 3 & 1 & 1 & 1 & 6 \\ 4 & 0 & 4 & 1 & 9 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 1 & 2 & -3 & 0 & 0 \\ 0 & 1 & -2 & 5 & 4 \\ 0 & 0 & 0 & 22 & 22 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$
So $S' = \{(1,0,1,3,4),\ (2,1,-2,1,0),\ (0,5,2,1,1)\}$ is a basis for V.

4. Since
$$\begin{aligned} &(a+b+3c+3d,\ b+2c+d,\ a+c+2d,\ -a-b-3c-3d,\ a+c+2d) \\ =\,&a(1,0,1,-1,1)+b(1,1,0,-1,0)+c(3,2,1,-3,1)+d(3,1,2,-3,2), \end{aligned}$$
$V = \mathrm{span}\{(1,0,1,-1,1),\ (1,1,0,-1,0),\ (3,2,1,-3,1),\ (3,1,2,-3,2)\}$. By
$$\begin{pmatrix} 1 & 0 & 1 & -1 & 1 \\ 1 & 1 & 0 & -1 & 0 \\ 3 & 2 & 1 & -3 & 1 \\ 3 & 1 & 2 & -3 & 2 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$
So $\{(1,0,1,-1,1),\ (0,1,-1,0,-1)\}$ is a basis for V.

5. (a) $\boldsymbol{A} \xrightarrow{R_2 - R_1} \xrightarrow{R_3 + R_1} \xrightarrow{R_1 + R_3} \xrightarrow{R_2 - 3R_3} \boldsymbol{R}$

   (b) (i) Note that span(S) and span(T) are the row spaces of $\boldsymbol{B}$ and $\boldsymbol{R}$ respectively. Since $\boldsymbol{B}$ and $\boldsymbol{R}$ are row equivalent, span(S) = span(T). Also, dim(span(T)) = rank(R) = 3. So by Theorem 3.6.7, S is a basis for span(T).

   (ii) $$\left(\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & 2 \\ 1 & -1 & 0 & 1 & 0 & -1 \\ 2 & -1 & 1 & 1 & 3 & 0 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

   (You do not need to really do any computations to claim the result on the RHS. Why?)

   So the transition matrix from S to T is $\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ -1 & 2 & 2 \end{pmatrix}$.

6. $$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & a & a & a & a \\ 1 & a & a^2 & a & a^2 \\ 1 & a^3 & a & 2a - a^3 & a \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & a-1 & a-1 & a-1 & a-1 \\ 0 & 0 & a^2 - a & 0 & a^2 - a \\ 0 & 0 & 0 & 2a - 2a^3 & 0 \end{pmatrix}$$

   - If $a = 1$, then $\{(1, 1, 1, 1, 1)\}$ is a basis for V and $\dim(V) = 1$.
   - If $a = 0$, then $\{(1, 1, 1, 1, 1), (0, 1, 1, 1, 1)\}$ is a basis for V and $\dim(V) = 2$.
   - If $a = -1$, then $\{(1, 1, 1, 1, 1), (0, -2, -2, -2, -2), (0, 0, 2, 0, 2)\}$ is a basis for V and $\dim(V) = 3$.
   - If $a \notin \{1, 0, -1\}$, then $\{(1, 1, 1, 1, 1), (0, a-1, a-1, a-1, a-1), (0, 0, a^2 - a, 0, a^2 - a), (0, 0, 0, 2a - 2a^3, 0)\}$ is a basis for V and $\dim(V) = 4$.

7. See Question 4 of Tutorial 3.

8. (a) We can choose $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

   (b) We can choose $\boldsymbol{B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

(c) V is the solution space of the linear equation $2x_1 - x_2 - x_3 + 0x_4 = 0$.

We can choose $\boldsymbol{C} = \begin{pmatrix} 2 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

9. See Question 1 of Tutorial 8.

10. (a) Let $\boldsymbol{R}$ be the reduced row-echelon form of $\boldsymbol{A}$. Since $\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}$ are linearly independent, the first three columns of $\boldsymbol{R}$ are linearly independent. Thus the first three columns of $\boldsymbol{R}$ must be $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Together with the information given for the fourth and fifth columns, $\boldsymbol{R} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

(b) $\{(1,0,0,1,0),\ (0,1,0,-2,1),\ (0,0,1,1,1)\}$ is a basis for the row space of $\boldsymbol{A}$; and $\{\boldsymbol{a_1},\ \boldsymbol{a_2},\ \boldsymbol{a_3}\}$ is a basis for the column space of $\boldsymbol{A}$.

11. (a) $\boldsymbol{x} = (2, -1, 3)$ is the solution to the linear system.

Thus $\begin{pmatrix} 16 \\ 13 \\ -4 \\ 7 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 1 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 5 \\ 4 \\ -1 \\ 2 \end{pmatrix}$.

(b) $\boldsymbol{x} = (-3+s+t,\ 13-3s-2t,\ 1-t,\ s,\ t)$, where $s, t \in \mathbb{R}$, is a general solution for the linear system.

In particular, $\begin{pmatrix} -1 \\ 9 \\ 4 \end{pmatrix} = -3\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + 13\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

12. (a) For example, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$.

(b) No. By Theorem 4.2.1, the dimensions of the row space and column space of a matrix must be the same.

(c) For example, $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$.

(d) For example, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

13.

	largest possible rank	smallest possible nullity
(a)	5	0
(b)	4	2
(c)	3	0

14. (a) $a = b = c = d = 0$.

(b) $ad - bc \neq 0$.

(c) $ad - bc = 0$ but not all a, b, c, d are zero.

15. (a) If $a = 1$, $\operatorname{rank}(\boldsymbol{A}) = 1$.
If $a = -2$, $\operatorname{rank}(\boldsymbol{A}) = 2$.
If $a \neq 1$ and $a \neq -2$, $\operatorname{rank}(\boldsymbol{A}) = 3$.

(b) If $b = c = d = e = f = 0$, $\operatorname{rank}(\boldsymbol{B}) = 0$.
If either (i) $b = c = 0$ and not all d, e, f are zero or (ii) $d = e = 0$ and not all b, c, f are zero, $\operatorname{rank}(\boldsymbol{B}) = 1$.
If not all b, c are zero and not all d, e are zero, $\operatorname{rank}(\boldsymbol{B}) = 2$.

16. (a) $\boldsymbol{X_1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 - R_1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

So $\operatorname{rank}(\boldsymbol{X_1}) = 2$ and $\operatorname{nullity}(\boldsymbol{X_1}) = 3 - 2 = 1$.

$$\boldsymbol{X_2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_5 - R_1} \xrightarrow{R_4 - R_2} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So $\operatorname{rank}(\boldsymbol{X_2}) = 3$ and $\operatorname{nullity}(\boldsymbol{X_2}) = 5 - 3 = 2$.

(b) $\boldsymbol{X_n} \xrightarrow{R_{2n+1}-R_1} \xrightarrow{R_{2n}-R_2} \cdots\cdots \xrightarrow{R_{n+2}-R_n} \begin{pmatrix} 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 \\ 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 1 & 0 \\ & & \cdot & & & & \cdot & & \\ & & & \cdot & & \cdot & & & \\ 0 & \cdot & \cdot & 0 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & 0 & 0 & \cdot & \cdot & 0 \\ & & \cdot & & & & \cdot & & \\ & \cdot & & & & & & \cdot & \\ 0 & \cdot & \cdot & 0 & 0 & 0 & \cdot & \cdot & 0 \end{pmatrix}$

So $\text{rank}(\boldsymbol{X_n}) = n+1$ and $\text{nullity}(\boldsymbol{X_n}) = (2n+1)-(n+1) = n$.

17. When the rank is 0, the solution set is the entire $\mathbb{R}^3$.

When the rank is 1, the solution set is a plane in $\mathbb{R}^3$ that passes through the origin.

When the rank is 2, the solution set is a line in $\mathbb{R}^3$ that passes through the origin.

When the rank is 3, the solution set is $\{\boldsymbol{0}\}$.

18. Let $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a_1} & \boldsymbol{a_2} & \cdots & \boldsymbol{a_n} \end{pmatrix}$ and $\boldsymbol{B}$ be $m \times n$ matrices where $\boldsymbol{a_i}$ is the ith column of $\boldsymbol{A}$. Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are row equivalent, i.e. there exists elementary matrices $\boldsymbol{E_1}$, $\boldsymbol{E_2}$, ..., $\boldsymbol{E_k}$ such that

$$\boldsymbol{B} = \boldsymbol{E_k} \cdots \boldsymbol{E_2}\boldsymbol{E_1}\boldsymbol{A}.$$

Define $\boldsymbol{P} = \boldsymbol{E_k} \cdots \boldsymbol{E_2}\boldsymbol{E_1}\boldsymbol{A}$. Then $\boldsymbol{B} = \boldsymbol{P}\boldsymbol{A} = \begin{pmatrix} \boldsymbol{P}\boldsymbol{a_1} & \boldsymbol{P}\boldsymbol{a_2} & \cdots & \boldsymbol{P}\boldsymbol{a_n} \end{pmatrix}$ where $\boldsymbol{P}\boldsymbol{a_i}$ is the ith column of $\boldsymbol{B}$. By Theorem 2.4.7, $\boldsymbol{P}$ is invertible.

Let $S_1 = \{\boldsymbol{a_{i_1}}, \boldsymbol{a_{i_2}}, \ldots, \boldsymbol{a_{i_r}}\}$ be a set of columns of $\boldsymbol{A}$. Note that $S_2 = \{\boldsymbol{P}\boldsymbol{a_{i_1}}, \boldsymbol{P}\boldsymbol{a_{i_2}}, \ldots, \boldsymbol{P}\boldsymbol{a_{i_r}}\}$ is the set of corresponding columns of $\boldsymbol{B}$.

(a) Since $\boldsymbol{P}$ is invertible, by Question 3.30, S_1 is linearly independent if and only if S_2 is linearly independent.

(b) Suppose S_1 is a basis for the column space of $\boldsymbol{A}$. We want to show that S_2 is a basis for the column space of $\boldsymbol{B}$:

(i) By (a), S_2 is linearly independent.

(ii) It is obvious that $\text{span}(S_2) \subseteq$ the column space of $\boldsymbol{B}$.

Take any $\boldsymbol{u} \in$ the column space of $\boldsymbol{B}$, i.e. for some $c_1, c_2, \ldots, c_n \in \mathbb{R}$,

$$\boldsymbol{u} = c_1\boldsymbol{P}\boldsymbol{a_1} + c_2\boldsymbol{P}\boldsymbol{a_2} + \cdots + c_n\boldsymbol{P}\boldsymbol{a_n}.$$

Since $\text{span}(S_1) =$ the column space of $\boldsymbol{A}$,

$$\boldsymbol{a_1}, \boldsymbol{a_2}, \ldots, \boldsymbol{a_n} \in \text{span}(S_1) = \text{span}\{\boldsymbol{a_{i_1}}, \boldsymbol{a_{i_2}}, \ldots, \boldsymbol{a_{i_r}}\}$$

and hence

$$\boldsymbol{Pa_1}, \boldsymbol{Pa_2}, \ldots, \boldsymbol{Pa_n} \in \operatorname{span}\{\boldsymbol{Pa_{i_1}},\ \boldsymbol{Pa_{i_2}},\ \ldots,\ \boldsymbol{Pa_{i_r}}\} = \operatorname{span}(S_2).$$

By Theorem 3.2.9.2, $\boldsymbol{u} \in \operatorname{span}(S_2)$. So the column space of $\boldsymbol{B} \subseteq \operatorname{span}(S_2)$.

We have shown that $\operatorname{span}(S_2) =$ the column space of $\boldsymbol{B}$.

By (i) and (ii), S_2 is a basis for the column space of $\boldsymbol{B}$.

Similarly, follow the arguments above by replacing $\boldsymbol{a_i}$ by $\boldsymbol{Pa_i}$ and $\boldsymbol{P}$ by $\boldsymbol{P}^{-1}$. We conclude that if S_2 is a basis for the column space of $\boldsymbol{B}$, then S_1 is a basis for the column space of $\boldsymbol{A}$.

19. (a) $$\left(\begin{array}{cccc|c|c|c} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{cccc|c|c|c} 1 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right)$$

(i) $\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ t \\ 1-t \\ -1 \end{pmatrix}$ where $t \in \mathbb{R}$.

(ii) $\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ t \\ -1-t \\ 1 \end{pmatrix}$ where $t \in \mathbb{R}$.

(iii) $\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ t \\ -t \\ 1 \end{pmatrix}$ where $t \in \mathbb{R}$.

If $\boldsymbol{u_1}$ is a solution of (i), $\boldsymbol{u_2}$ a solution of (ii) and $\boldsymbol{u_3}$ a solution of (iii), then $\begin{pmatrix} \boldsymbol{u_1} & \boldsymbol{u_2} & \boldsymbol{u_3} \end{pmatrix}$ is a right inverse of $\boldsymbol{B}$. The answer is certainly not unique.

For example, $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$ is a right inverse of $\boldsymbol{B}$.

(b) For example, $\boldsymbol{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ has no right inverse.

(c)

$$\begin{aligned}
&\boldsymbol{B} \text{ has a right inverse}\\
\Leftrightarrow\ &\text{the column space of } \boldsymbol{B} = \mathbb{R}^m \quad \text{(see Question 6 of Tutorial 7)}\\
\Leftrightarrow\ &\dim(\text{the column space of } \boldsymbol{B}) = m\\
\Leftrightarrow\ &\operatorname{rank}(\boldsymbol{B}) = m.
\end{aligned}$$

20. Let $\boldsymbol{B} = \begin{pmatrix} \boldsymbol{b_1} & \cdots & \boldsymbol{b_n} \end{pmatrix}$ where b_j is the jth column of $\boldsymbol{B}$.

$$\boldsymbol{AB} = \boldsymbol{0} \quad \Rightarrow \quad \begin{pmatrix} \boldsymbol{Ab_1} & \cdots & \boldsymbol{Ab_n} \end{pmatrix} = \boldsymbol{0} \quad \Rightarrow \quad \boldsymbol{Ab_j} = \boldsymbol{0} \text{ for all } j,$$

i.e. $\boldsymbol{b_1}, \ldots, \boldsymbol{b_n}$ are contained in the nullspace of $\boldsymbol{A}$.
So the column space of $\boldsymbol{B} = \operatorname{span}\{\boldsymbol{b_1}, \ldots, \boldsymbol{b_n}\} \subseteq$ the nullspace of $\boldsymbol{A}$.

21. Let $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a_1} \\ \vdots \\ \boldsymbol{a_n} \end{pmatrix}$ be a matrix where $\boldsymbol{a_i}$ is the ith row of $\boldsymbol{A}$.

Let $\boldsymbol{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n$ such that $\boldsymbol{u}^{\mathrm{T}}$ is a vector in the nullspace of $\boldsymbol{A}$. Then

$$\boldsymbol{Au}^{\mathrm{T}} = \boldsymbol{0} \quad \Rightarrow \quad \begin{pmatrix} \boldsymbol{a_1 u}^{\mathrm{T}} \\ \vdots \\ \boldsymbol{a_n u}^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{a_i u}^{\mathrm{T}} = 0 \ \text{ for all } i.$$

Assume that $\boldsymbol{u}$ is also contained in the row space of $\boldsymbol{A}$, i.e. $\boldsymbol{u} = c_1\boldsymbol{a_1} + \cdots + c_n\boldsymbol{a_n}$ for some $c_1, \ldots, c_n \in \mathbb{R}$. We have

$$\boldsymbol{uu}^{\mathrm{T}} = c_1\boldsymbol{a_1 u}^{\mathrm{T}} + \cdots + c_n\boldsymbol{a_n u}^{\mathrm{T}} = 0.$$

On the other hand, $\boldsymbol{uu}^{\mathrm{T}} = u_1^2 + \cdots + u_n^2$. So $u_1^2 + \cdots + u_n^2 = 0$ which implies $u_1 = 0, \ \ldots, \ u_n = 0$, i.e. $\boldsymbol{u}$ is the zero vector.

22. (a) Since $\boldsymbol{P}$ is invertible, we can write $\boldsymbol{P} = \boldsymbol{E_n} \cdots \boldsymbol{E_1}$ where $\boldsymbol{E_i}$ are elementary matrices. So $\boldsymbol{PA} = \boldsymbol{E_n} \cdots \boldsymbol{E_1 A}$ and $\boldsymbol{A}$ are row-equivalent matrices. They have the same row space. Thus

$$\begin{aligned}
\operatorname{rank}(\boldsymbol{PA}) &= \dim(\text{the row space of } \boldsymbol{PA})\\
&= \dim(\text{the row space of } \boldsymbol{A}) = \operatorname{rank}(\boldsymbol{A}).
\end{aligned}$$

(b) For example, $\boldsymbol{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(c) No. For example, let $\boldsymbol{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

23. Let $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a_1} & \cdots & \boldsymbol{a_n} \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} \boldsymbol{b_1} & \cdots & \boldsymbol{b_n} \end{pmatrix}$ where $\boldsymbol{a_j}$ is the jth column of $\boldsymbol{A}$ and $\boldsymbol{b_j}$ is the jth column of $\boldsymbol{B}$. Let $\{\boldsymbol{a'_1}, \dots, \boldsymbol{a'_r}\}$ be a basis for the column space of $\boldsymbol{A}$ and let $\{\boldsymbol{b'_1}, \dots, \boldsymbol{b'_s}\}$ be a basis for the column space of $\boldsymbol{B}$. Then

$$\begin{aligned}\text{the column space of } \boldsymbol{A}+\boldsymbol{B} &= \operatorname{span}\{\, \boldsymbol{a_1}+\boldsymbol{b_1},\ \dots,\ \boldsymbol{a_n}+\boldsymbol{b_n} \,\} \\ &\subseteq \operatorname{span}\{\boldsymbol{a'_1}, \dots, \boldsymbol{a'_r}, \boldsymbol{b'_1}, \dots, \boldsymbol{b'_s}\}.\end{aligned}$$

So

$$\operatorname{rank}(\boldsymbol{A}+\boldsymbol{B}) = \dim(\text{the column space of } \boldsymbol{A}+\boldsymbol{B}) \le r+s = \operatorname{rank}(\boldsymbol{A})+\operatorname{rank}(\boldsymbol{B}).$$

24. Since $\boldsymbol{Ax} = \boldsymbol{b}$ is consistent for all $\boldsymbol{b} \in \mathbb{R}^m$, by Theorem 4.1.16, the column space of $\boldsymbol{A}$ is $\mathbb{R}^m$, i.e. $\operatorname{rank}(\boldsymbol{A}) = m$. Hence

$$\operatorname{nullity}(\boldsymbol{A}^{\mathrm{T}}) = m - \operatorname{rank}(\boldsymbol{A}^{\mathrm{T}}) = m - \operatorname{rank}(\boldsymbol{A}) = 0.$$

It means that the linear system $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}$ has only the trivial solution.

Alternative Solution: Let $\boldsymbol{e_1}, \dots, \boldsymbol{e_m}$ be the standard basis for $\mathbb{R}^m$ and let $\boldsymbol{u_1}, \dots, \boldsymbol{u_m}$ be vectors in $\mathbb{R}^n$ such that $\boldsymbol{Au_i} = \boldsymbol{e_i}$ for each i. (In here, all the vectors are column vectors.) Suppose $\boldsymbol{v} = (v_1, \dots, v_m)^{\mathrm{T}}$ is a solution to the system $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{v} = \boldsymbol{0}$. Then for $i = 1, \dots, m$,

$$v_i = \boldsymbol{e_i}^{\mathrm{T}}\boldsymbol{v} = (\boldsymbol{Au_i})^{\mathrm{T}}\boldsymbol{v} = \boldsymbol{u_i}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{v} = \boldsymbol{u_i}\boldsymbol{0} = 0.$$

So $\boldsymbol{v} = \boldsymbol{0}$. That is, the system $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}$ has only the trivial solution.

25. (a) Let $\boldsymbol{u}$ be any vector in the nullspace of $\boldsymbol{A}$, i.e. $\boldsymbol{Au} = \boldsymbol{0}$. Then $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Au} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{0} = \boldsymbol{0}$. So $\boldsymbol{u}$ is also a vector in the nullspace of $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$.
We have shown that the nullspace of $\boldsymbol{A}$ is a subspace of the nullspace of $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$.

Let $\boldsymbol{v}$ be any vector in the nullspace of $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$, i.e. $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Av} = \boldsymbol{0}$. Suppose $\boldsymbol{Av} = (b_1, b_2, \dots, b_m)^{\mathrm{T}}$. Then

$$\begin{aligned} & (\boldsymbol{Av})^{\mathrm{T}}(\boldsymbol{Av}) = \boldsymbol{v}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Av} = \boldsymbol{v}^{\mathrm{T}}\boldsymbol{0} = 0 \\ \Rightarrow\quad & b_1^2 + b_2^2 + \cdots + b_m^2 = 0 \\ \Rightarrow\quad & b_1 = b_2 = \cdots = b_m = 0. \end{aligned}$$

That is, $\boldsymbol{Av} = \boldsymbol{0}$. So $\boldsymbol{v}$ is also a vector in the nullspace of $\boldsymbol{A}$.
We have shown that the nullspace of $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$ is a subspace of the nullspace of $\boldsymbol{A}$.

Hence the nullspace of $\boldsymbol{A}$ is equal to the nullspace of $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$.

(b) By (a), nullity($\boldsymbol{A}$) = nullity($\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$).

Since $\boldsymbol{A}$ is an $m \times n$ matrix, $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$ is an $n \times n$ matrix. By the Dimension Theorem for Matrices (Theorem 4.3.4),

$$\mathrm{rank}(\boldsymbol{A}) = n - \mathrm{nullity}(\boldsymbol{A}) = n - \mathrm{nullity}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}) = \mathrm{rank}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}).$$

(c) No. For example, let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

(d) Yes. By (b) and Remark 4.2.5.3, rank($\boldsymbol{A}$) = rank($\boldsymbol{A}^{\mathrm{T}}$) = rank($(\boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}$) = rank($\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}}$).

26. (a) False. For example, let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

(b) True. By Theorem 4.1.7, the row space of $\boldsymbol{A}$ and the the row space of $\boldsymbol{B}$ are the same. Hence the column space of $\boldsymbol{A}^{\mathrm{T}}$ and the column space of $\boldsymbol{B}^{\mathrm{T}}$ are the same.

(c) False. For example, let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

(d) False. For example, let $\boldsymbol{A} = \boldsymbol{B} = \boldsymbol{I}_2$.

(e) False. For example, let $\boldsymbol{A} = \boldsymbol{B} = \boldsymbol{0}_{2\times 2}$.

(f) False. For example, let $\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

(g) False. For example, let $\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.