MA1101R AY19/20 Sem 1 Finals

Linear Algebra I (National University of Singapore)

Student Number: | | | | | | | | | |

Seat Number: | | | |

National University of Singapore

MA1101R Linear Algebra I

Semester I (2019 – 2020)

Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. Write down your student number and seat number clearly in the space provided at the top of this page. Do not write your name.
2. This booklet (and only this booklet) will be collected at the end of the examination.
3. This examination paper contains **SIX (6)** questions and comprises **FIFTEEN (15)** printed pages.
4. Answer **ALL** questions.
5. This is a **CLOSED BOOK** (with helpsheet) examination.
6. You are allowed to use one A4-size helpsheet.
7. You may use scientific calculators. However, you should lay out systematically the various steps in the calculations.

Examiner's Use Only	
Questions	**Marks**
1	
2	
3	
4	
5	
6	
Total	

Question 1 **[10 marks]**

Let $\boldsymbol{A} = \begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 0 & 2 & -2 & 0 \\ -1 & 1 & 1 & -1 & 1 \\ 0 & 0 & -1 & 1 & 0 \end{pmatrix}$ with reduced row echelon form $\boldsymbol{R} = \begin{pmatrix} 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

(i) Use $\boldsymbol{R}$ to find a basis for the column space V of $\boldsymbol{A}$.

(ii) Let $\boldsymbol{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\boldsymbol{u}_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$, $\boldsymbol{u}_3 = \begin{pmatrix} -12 \\ 0 \\ 9 \\ 11 \\ 0 \end{pmatrix}$.

Show that $S = \{\boldsymbol{A}\boldsymbol{u}_1, \boldsymbol{A}\boldsymbol{u}_2, \boldsymbol{A}\boldsymbol{u}_3\}$ is an orthogonal basis for V.

(iii) Find the coordinate vector $[\boldsymbol{w}]_S$ of $\boldsymbol{w} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \in V$ with respect to the basis S in part (ii).

(iv) Is it possible to find a one-dimensional subspace of V that does not contain any column of $\boldsymbol{A}$? Justify your answer.

Show your working below.

(i) Basis for V: $\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ -1 \\ 1 \end{pmatrix} \right\}$.

(Note that any three linearly independent columns of $\boldsymbol{A}$ also form a basis.)

(ii)

$$\boldsymbol{A}\boldsymbol{u}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \boldsymbol{A}\boldsymbol{u}_2 = \begin{pmatrix} 1 \\ 4 \\ 1 \\ -2 \end{pmatrix}, \boldsymbol{A}\boldsymbol{u}_3 = \begin{pmatrix} 10 \\ -4 \\ 10 \\ 2 \end{pmatrix}$$

Check the dot products: $\boldsymbol{A}\boldsymbol{u}_1 \cdot \boldsymbol{A}\boldsymbol{u}_2 = 0$, $\boldsymbol{A}\boldsymbol{u}_1 \cdot \boldsymbol{A}\boldsymbol{u}_3 = 0$, $\boldsymbol{A}\boldsymbol{u}_2 \cdot \boldsymbol{A}\boldsymbol{u}_3 = 0$.
This implies S is an orthogonal set, and hence is linearly independent.
Since $\dim V = 3$ (from part (i))
and $\boldsymbol{A}\boldsymbol{u}_1, \boldsymbol{A}\boldsymbol{u}_2, \boldsymbol{A}\boldsymbol{u}_3$ belongs to the column space V of $\boldsymbol{A}$,
so S is an orthogonal basis for V.

Continue on next page if you need more writing space.

More working space for Question 1.

(iii) Since S is orthogonal,

$$\begin{aligned}\boldsymbol{w} &= \frac{\boldsymbol{w}\cdot\boldsymbol{A}\boldsymbol{u}_1}{\|\boldsymbol{A}\boldsymbol{u}_1\|^2}\boldsymbol{A}\boldsymbol{u}_1 + \frac{\boldsymbol{w}\cdot\boldsymbol{A}\boldsymbol{u}_2}{\|\boldsymbol{A}\boldsymbol{u}_2\|^2}\boldsymbol{A}\boldsymbol{u}_2 + \frac{\boldsymbol{w}\cdot\boldsymbol{A}\boldsymbol{u}_3}{\|\boldsymbol{A}\boldsymbol{u}_3\|^2}\boldsymbol{A}\boldsymbol{u}_3 \\ &= \frac{1-1}{1^2+0^2+1^2+0^2}\boldsymbol{A}\boldsymbol{u}_1 + \frac{1+1}{1^2+4^2+1^2+2^2}\boldsymbol{A}\boldsymbol{u}_2 + \frac{10+10}{10^2+4^2+10^2+2^2}\boldsymbol{A}\boldsymbol{u}_3\end{aligned}$$

So $[\boldsymbol{w}]_S = \left(0, \frac{1}{11}, \frac{1}{11}\right)$.

(iv) Yes.

We can take the linear span of a linear combination of the columns of $\boldsymbol{A}$:

e.g. $\operatorname{span}\left\{\begin{pmatrix}1\\0\\-1\\0\end{pmatrix} + \begin{pmatrix}0\\2\\1\\-1\end{pmatrix}\right\} = \operatorname{span}\left\{\begin{pmatrix}1\\2\\0\\-1\end{pmatrix}\right\}$.

This is a one dimensional subspace of V which does not contain any column of $\boldsymbol{A}$.

Continue on pages 14–15 if you need more writing space.

Question 2 **[10 marks]**

Let $\boldsymbol{A} = \begin{pmatrix} 3 & -2 & 1 \\ 0 & 4 & 0 \\ 1 & 2 & 3 \end{pmatrix}$ and $\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, $\boldsymbol{v}_4 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, $\boldsymbol{v}_5 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

(i) Determine which of the five vectors $\boldsymbol{v}_1$ to $\boldsymbol{v}_5$ are eigenvectors of $\boldsymbol{A}$.

(ii) Write down all the eigenvalues of $\boldsymbol{A}$. Justify your answers.

(iii) Write down a basis for each of the eigenspaces of $\boldsymbol{A}$.

(iv) Find an invertible matrix $\boldsymbol{P}$ and a diagonal matrix $\boldsymbol{D}$ such that $\boldsymbol{A}^3 = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$.

(v) Is $\boldsymbol{A}\boldsymbol{A}^T$ orthogonally diagonalizable? Why?

Show your working below.

(i)

$$\boldsymbol{A}\boldsymbol{v}_1 = \begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix} = 4\boldsymbol{v}_1,\ \ \boldsymbol{A}\boldsymbol{v}_2 = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} = 2\boldsymbol{v}_2,\ \ \boldsymbol{A}\boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 4 \\ 8 \end{pmatrix} = 4\boldsymbol{v}_3$$

$$\boldsymbol{A}\boldsymbol{v}_4 = \begin{pmatrix} 7 \\ 0 \\ 5 \end{pmatrix},\ \ \boldsymbol{A}\boldsymbol{v}_5 = \begin{pmatrix} -4 \\ 4 \\ 4 \end{pmatrix} = 4\boldsymbol{v}_5$$

So all except $\boldsymbol{v}_4$ are eigenvectors of $\boldsymbol{A}$.

(ii) From (i), we have two eigenvalues 2 and 4.
Since both $\boldsymbol{v}_1$ and $\boldsymbol{v}_3$ are linearly independent eigenvectors associated to 4, so the multiplicity of eigenvalue 4 is at least 2.
As $\boldsymbol{A}$ is a 3×3 matrix, we conclude that 2 and 4 are the only eigenvalues of $\boldsymbol{A}$.

(iii) We deduce from (i) that the eigenspace E_2 associated to 2 is one dimensional, and a basis is given by $\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.
Similarly, we deduce that the eigenspace E_4 associated to 4 is two dimensional, and a basis can be given by $\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.
(Other possible bases for E_4: the pair $\boldsymbol{v}_1, \boldsymbol{v}_5$ or the pair $\boldsymbol{v}_3, \boldsymbol{v}_5$.)

Continue on next page if you need more writing space.

More working space for Question 2.

(iv) The eigenvalues of $\boldsymbol{A}^3$ are 2^3 and 4^3 (repeated) with corresponding eigenvectors same as those of $\boldsymbol{A}$.

Hence $\boldsymbol{P} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 2 \end{pmatrix}$ (depending on the choices of eigenvectors in (iii)),

and $\boldsymbol{D} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 64 & 0 \\ 0 & 0 & 64 \end{pmatrix}$.

(v) Yes.

$\boldsymbol{A}\boldsymbol{A}^T$ is a symmetric matrix, and hence is orthogonally diagonalizable.

Continue on page 14–15 if you need more writing space.

Question 3 **[10 marks]**

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

(i) Show that the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ is inconsistent.

(ii) Find the least squares solution of the system in (i).

(iii) Find the projection $\boldsymbol{p}$ of $\boldsymbol{b}$ onto the column space of $\boldsymbol{A}$.

(iv) Find the smallest possible value of $\|\boldsymbol{Av} - \boldsymbol{b}\|$ among all vectors $\boldsymbol{v} \in \mathbb{R}^3$.

(v) Note that the three columns of $\boldsymbol{A}$ form an orthogonal set. Extend this set to an orthogonal basis for $\mathbb{R}^4$.

Show your working below.

(i)

$$(\boldsymbol{A} \mid \boldsymbol{b}) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right) \xrightarrow{G.E.} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

So $\boldsymbol{Ax} = \boldsymbol{b}$ is inconsistent.

(ii) $\boldsymbol{A}^T\boldsymbol{A} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and $\boldsymbol{A}^T\boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

So the solution of $\boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{b}$ is $\boldsymbol{x} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$

which gives the least squares solution for $\boldsymbol{Ax} = \boldsymbol{b}$.

(iii) The projection is given by $\boldsymbol{p} = \boldsymbol{A} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \\ 0 \end{pmatrix}$.

(iv) The samllest possible value of $\|\boldsymbol{Av} - \boldsymbol{b}\|$ is given by

$$\|\boldsymbol{p} - \boldsymbol{b}\| = \left\| \begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \right\| = \sqrt{\frac{1}{4} + \frac{1}{4}} = \frac{\sqrt{2}}{2}.$$

Continue on next page if you need more writing space.

More working space for Question 3.

(v) We need to add one more vector to the set. This vector can be given by

$$\boldsymbol{p} - \boldsymbol{b} = \begin{pmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

which is orthogonal to the column space of $\boldsymbol{A}$, and hence to the three columns of $\boldsymbol{A}$.

Continue on pages 14–15 if you need more writing space.

Question 4 [10 marks]

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$T\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right) = \boldsymbol{v}_1, \quad T\left(\begin{pmatrix}0\\1\\1\end{pmatrix}\right) = \boldsymbol{v}_2, \quad T\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right) = \boldsymbol{v}_3$$

where $\boldsymbol{v}_1, \boldsymbol{v}_2$ and $\boldsymbol{v}_3$ are non-zero vectors.

(i) Find $T\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right)$ and $T\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\right)$ as linear combinations of $\boldsymbol{v}_1, \boldsymbol{v}_2$ and $\boldsymbol{v}_3$.

(ii) Find the standard matrix $\boldsymbol{A}$ for T in terms of $\boldsymbol{v}_1, \boldsymbol{v}_2$ and $\boldsymbol{v}_3$.

(iii) Suppose $\boldsymbol{v}_1, \boldsymbol{v}_2$ and $\boldsymbol{v}_3$ are linearly independent. Show that $\ker(T) = \{\boldsymbol{0}\}$.

(iv) Suppose $T(\boldsymbol{v}_1) = 2\boldsymbol{v}_1$, $T(\boldsymbol{v}_2) = 3\boldsymbol{v}_2$, $T(\boldsymbol{v}_3) = 5\boldsymbol{v}_3$. Find $\boldsymbol{v}_1, \boldsymbol{v}_2$ and $\boldsymbol{v}_3$.

Show your working below.

(i)

$$T\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right) = T\left[\begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}0\\1\\1\end{pmatrix}\right] = T\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right) - T\left(\begin{pmatrix}0\\1\\1\end{pmatrix}\right) = \boldsymbol{v}_1 - \boldsymbol{v}_2.$$

$$T\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\right) = T\left[\begin{pmatrix}0\\1\\1\end{pmatrix} - \begin{pmatrix}0\\0\\1\end{pmatrix}\right] = T\left(\begin{pmatrix}0\\1\\1\end{pmatrix}\right) - T\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right) = \boldsymbol{v}_2 - \boldsymbol{v}_3.$$

(ii) From (i) we have:

$$\boldsymbol{A}\begin{pmatrix}1\\0\\0\end{pmatrix} = \boldsymbol{v}_1 - \boldsymbol{v}_2 \text{ and } \boldsymbol{A}\begin{pmatrix}0\\1\\0\end{pmatrix} = \boldsymbol{v}_2 - \boldsymbol{v}_3.$$

And from the given condition, we have:

$$\boldsymbol{A}\begin{pmatrix}0\\0\\1\end{pmatrix} = \boldsymbol{v}_3.$$

These are the three columns of $\boldsymbol{A}$.

Hence $\boldsymbol{A} = (\boldsymbol{v}_1 - \boldsymbol{v}_2 \mid \boldsymbol{v}_2 - \boldsymbol{v}_3 \mid \boldsymbol{v}_3)$.

Continue on next page if you need more writing space.

More working space for Question 4.

(iii) We shall show $S = \{\boldsymbol{v}_1 - \boldsymbol{v}_2, \boldsymbol{v}_2 - \boldsymbol{v}_3, \boldsymbol{v}_3\}$ is linearly independent.
Set up the vector equation

$$c_1(\boldsymbol{v}_1 - \boldsymbol{v}_2) + c_2(\boldsymbol{v}_2 - \boldsymbol{v}_3) + c_3\boldsymbol{v}_3 = \boldsymbol{0}$$

Rearranging the terms gives:
$c_1\boldsymbol{v}_1 + (c_2 - c_1)\boldsymbol{v}_2 + (c_3 - c_2)\boldsymbol{v}_3 = \boldsymbol{0}$.
Since $\boldsymbol{v}_1, \boldsymbol{v}_2$ and $\boldsymbol{v}_3$ are linearly independent, this implies:
$c_1 = 0, c_2 - c_1 = 0, c_3 - c_2 = 0,$
which will further give

$$c_1 = c_2 = c_3 = 0.$$

So S is linearly independent.
Since the standard matrix $\boldsymbol{A}$ of T has three linearly independent columns, it is invertible.
This implies the nullspace of $\boldsymbol{A} = \mathrm{Ker}(T) = \{\boldsymbol{0}\}$.

(iv) From the given information, we have

$$\boldsymbol{A}\boldsymbol{v}_1 = 2\boldsymbol{v}_1,\ \boldsymbol{A}\boldsymbol{v}_2 = 3\boldsymbol{v}_2,\ \boldsymbol{A}\boldsymbol{v}_3 = 5\boldsymbol{v}_3\ (*)$$

Since $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are non-zero vectors, they are eigenvectors of $\boldsymbol{A}$ with eigenvalues 2, 3, 5 respectively.
Since all the eigenvalues are non-zero, $\boldsymbol{A}$ is invertible.
Hence the linear systems $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{v}_1,\ \boldsymbol{A}\boldsymbol{x} = \boldsymbol{v}_2,\ \boldsymbol{A}\boldsymbol{x} = \boldsymbol{v}_3$ all have unique solutions.
From the given conditions of T, we have

$$\boldsymbol{A}\begin{pmatrix}1\\1\\1\end{pmatrix} = \boldsymbol{v}_1,\quad \boldsymbol{A}\begin{pmatrix}0\\1\\1\end{pmatrix} = \boldsymbol{v}_2,\quad \boldsymbol{A}\begin{pmatrix}0\\0\\1\end{pmatrix} = \boldsymbol{v}_3$$

On the other hand, by $(*)$, we have $\boldsymbol{A}(\frac{1}{2}\boldsymbol{v}_1) = \boldsymbol{v}_1,\ \boldsymbol{A}(\frac{1}{3}v_2) = \boldsymbol{v}_2,\ \boldsymbol{A}\frac{1}{5}v_3) = \boldsymbol{v}_3$.
By comparison, we have:

$$\tfrac{1}{2}\boldsymbol{v}_1 = \begin{pmatrix}1\\1\\1\end{pmatrix} \Rightarrow \boldsymbol{v}_1 = \begin{pmatrix}2\\2\\2\end{pmatrix};$$

$$\tfrac{1}{3}\boldsymbol{v}_2 = \begin{pmatrix}0\\1\\1\end{pmatrix} \Rightarrow \boldsymbol{v}_2 = \begin{pmatrix}0\\3\\3\end{pmatrix};$$

$$\tfrac{1}{5}\boldsymbol{v}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix} \Rightarrow \boldsymbol{v}_3 = \begin{pmatrix}0\\0\\5\end{pmatrix}.$$

Continue on pages 14–15 if you need more writing space.

Question 5 **[10 marks]**

Suppose $\boldsymbol{A}$ is a 3×5 matrix with row space given by $\text{span}\{(1,2,3,4,5)\}$.

(i) What are the rank and nullity of $\boldsymbol{A}$?

(ii) Write down the reduced row echelon form of $\boldsymbol{A}$.

(iii) Find a basis for the nullspace of $\boldsymbol{A}$.

(iv) Find the general solution of the non-homogeneous system $\boldsymbol{Ax} = \boldsymbol{b}$ where $\boldsymbol{b}$ is the first column of $\boldsymbol{A}$.

(v) Suppose the first column of $\boldsymbol{A}$ is $\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$. Do we have enough information to determine the matrix $\boldsymbol{A}$? Why?

Show your working below.

(i) $\text{rank}(\boldsymbol{A}) = 1$ (since the row space is spanned by one non-zero vector).
$\text{nullity}(\boldsymbol{A}) = 5 - 1 = 4$ (by Dimension Theorem)

(ii) $\text{rref} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

(iii) Let the variables of $\boldsymbol{Ax} = \boldsymbol{0}$ be x_1, x_2, x_3, x_4, x_5.
Then set $x_2 = s, x_3 = t, x_4 = u, x_5 = v$ where s, t, u, v are parameters.
Then $x_1 = -2s - 3t - 4u - 5v$.
So a general solution of the system is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2s - 3t - 4u - 5v \\ s \\ t \\ u \\ v \end{pmatrix} = s\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + u\begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + v\begin{pmatrix} -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

So a basis for the nullspace of $\boldsymbol{A}$ is given by:

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

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More working space for Question 5.

(iv) Since $\boldsymbol{b}$ is the first column of $\boldsymbol{A}$, a solution of $\boldsymbol{Ax} = \boldsymbol{b}$ is $\boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

So a general solution of $\boldsymbol{Ax} = \boldsymbol{b}$ = (general solution of $\boldsymbol{Ax} = \boldsymbol{0}$) + $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 - 2s - 3t - 4u - 5v \\ s \\ t \\ u \\ v \end{pmatrix}.$$

(v) Yes.

Since $\text{rank}(\boldsymbol{A}) = 1$, all columns of $\boldsymbol{A}$ are scalar multiples of the first column.

Also, since the row space of $\boldsymbol{A}$ is $\text{span}\{(1, 2, 3, 4, 5)\}$, all the rows of $\boldsymbol{A}$ are scalar multiples of $(1, 2, 3, 4, 5)$.

Hence we must have

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ -2 & -4 & -6 & -8 & -10 \end{pmatrix}$$

Continue on pages 14–15 if you need more writing space.

Question 6 **[10 marks]**

Prove the following statements.

(a) If $\boldsymbol{A}$ is an $n \times n$ matrix such that $\boldsymbol{A}^2 = \boldsymbol{I}$, then $\text{rank}(\boldsymbol{I}+\boldsymbol{A}) + \text{rank}(\boldsymbol{I}-\boldsymbol{A}) = n$.
(Hint: $\text{rank}(\boldsymbol{M}+\boldsymbol{N}) \leq \text{rank}(\boldsymbol{M}) + \text{rank}(\boldsymbol{N})$)

(b) There are no orthogonal matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ (of the same order) such that $\boldsymbol{A}^2 - \boldsymbol{B}^2 = \boldsymbol{AB}$.
(Hint: Prove by contradiction. Recall that the product of two orthogonal matrices is an orthogonal matrix.)

Show your working below.

(a)

$$\begin{aligned}
& \boldsymbol{A}^2 = \boldsymbol{I} \\
\Rightarrow\ & \boldsymbol{I} - \boldsymbol{A}^2 = \boldsymbol{0} \\
\Rightarrow\ & (\boldsymbol{I}-\boldsymbol{A})(\boldsymbol{I}+\boldsymbol{A}) = \boldsymbol{0} \\
\Rightarrow\ & \text{column space of } \boldsymbol{I}+\boldsymbol{A} \subseteq \text{nullspace of } \boldsymbol{I}-\boldsymbol{A} \\
\Rightarrow\ & \text{rank}(\boldsymbol{I}+\boldsymbol{A}) \leq \text{nullity}(\boldsymbol{I}-\boldsymbol{A}) = n - \text{rank}(\boldsymbol{I}-\boldsymbol{A}) \\
\Rightarrow\ & \text{rank}(\boldsymbol{I}+\boldsymbol{A}) + \text{rank}(\boldsymbol{I}-\boldsymbol{A}) \leq n - - - - - - (1)
\end{aligned}$$

On the other hand,
$\text{rank}(\boldsymbol{I}+\boldsymbol{A}) + \text{rank}(\boldsymbol{I}-\boldsymbol{A}) \geq \text{rank}[(\boldsymbol{I}+\boldsymbol{A}) + (\boldsymbol{I}-\boldsymbol{A})] = \text{rank}(2\boldsymbol{I}) = n - - - - - - (2)$.
By (1) and (2), we have $\text{rank}(\boldsymbol{I}+\boldsymbol{A}) + \text{rank}(\boldsymbol{I}-\boldsymbol{A}) = n$.

(b) Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are orthogonal matrices such that $\boldsymbol{A}^2 - \boldsymbol{B}^2 = \boldsymbol{AB}$.

$$\boldsymbol{A}^2 - \boldsymbol{AB} = \boldsymbol{B}^2 \Rightarrow \boldsymbol{A}(\boldsymbol{A}-\boldsymbol{B}) = \boldsymbol{B}^2 \Rightarrow \boldsymbol{A}-\boldsymbol{B} = \boldsymbol{A}^{-1}\boldsymbol{B}^2 = \boldsymbol{A}^T\boldsymbol{B}^2.$$

$$\boldsymbol{AB} + \boldsymbol{B}^2 = \boldsymbol{A}^2 \Rightarrow (\boldsymbol{A}+\boldsymbol{B})\boldsymbol{B} = \boldsymbol{A}^2 \Rightarrow \boldsymbol{A}+\boldsymbol{B} = \boldsymbol{A}^2\boldsymbol{B}^{-1} = \boldsymbol{A}^2\boldsymbol{B}^T.$$

Since product of orthogonal matrices is orthogonal, $\boldsymbol{A}-\boldsymbol{B}$ and $\boldsymbol{A}+\boldsymbol{B}$ are both orthogonal. So

$$(\boldsymbol{A}-\boldsymbol{B})^{-1} = (\boldsymbol{A}-\boldsymbol{B})^T = \boldsymbol{A}^T - \boldsymbol{B}^T \text{ and } (\boldsymbol{A}+\boldsymbol{B})^{-1} = (\boldsymbol{A}+\boldsymbol{B})^T = \boldsymbol{A}^T + \boldsymbol{B}^T$$

Then

$$\boldsymbol{I} = (\boldsymbol{A}^T - \boldsymbol{B}^T)(\boldsymbol{A}-\boldsymbol{B}) = 2\boldsymbol{I} - \boldsymbol{A}^T\boldsymbol{B} - \boldsymbol{B}^T\boldsymbol{A} - - - (1)$$
$$\boldsymbol{I} = (\boldsymbol{A}^T + \boldsymbol{B}^T)(\boldsymbol{A}+\boldsymbol{B}) = 2\boldsymbol{I} + \boldsymbol{A}^T\boldsymbol{B} + \boldsymbol{B}^T\boldsymbol{A} - - - (2)$$

Adding (1) and (2): $2\boldsymbol{I} = 4\boldsymbol{I}$, which is a contradiction.
Hence such orthogonal matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ do not exist.

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More working space for Question 6.

Continue on pages 14–15 if you need more writing space.

More working spaces. Please indicate the question numbers clearly.

More working spaces. Please indicate the question numbers clearly.