Chapter 7

Linear Transformations

In this chapter, all vectors are written as column vectors.

Section 7.1

Linear Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

Linear transformations (Definition 7.1.1)

A linear transformation is a mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ of the form

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

for $(x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$.

If $n = m$, then T is also called a linear operator on $\mathbb{R}^n$.

Linear transformations (Definition 7.1.1)

We can rewrite the formula of T as

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

The matrix $(a_{ij})_{m \times n}$ is called the standard matrix for T.

Examples (Example 7.1.2.1)

The identity mapping $I: \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$I(\boldsymbol{x}) = I\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \boldsymbol{x} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \boldsymbol{I}_n\boldsymbol{x}$$

for $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$.

I is a linear operator on $\mathbb{R}^n$

and the standard matrix for I is the identity matrix $\boldsymbol{I}_n$.

Examples (Example 7.1.2.2)

The zero mapping $O\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is defined by

$$O(\boldsymbol{x}) = O\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{0}_{m\times n}\boldsymbol{x}$$

for $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$.

O is a linear transformation

and the standard matrix for O is the zero matrix $\mathbf{0}_{m\times n}$.

Examples (Example 7.1.2.3)

Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ 2x \\ -3y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \quad \text{for} \quad \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2.$$

T is a linear transformation

and the standard matrix for T is $\begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & -3 \end{bmatrix}$.

An alternative definition (Remark 7.1.3)

Let V and W be vector spaces.

A mapping $T: V \to W$ is called a linear transformation if and only if

$T(c\boldsymbol{u} + d\boldsymbol{v}) = cT(\boldsymbol{u}) + dT(\boldsymbol{v})$ for all $\boldsymbol{u}, \boldsymbol{v} \in V$ and $c, d \in \mathbb{R}$.

(This is the definition of linear transformation in abstract linear algebra.)

The two definitions of linear transformations are the same if $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$. (See Question 7.4.)

Some basic properties (Theorem 7.1.4)

Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation.

1. $T(\mathbf{0}) = \mathbf{0}$.
2. If $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k} \in \mathbb{R}^n$ and $c_1, c_2, \ldots, c_k \in \mathbb{R}$, then

$$T(c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}) = c_1T(\boldsymbol{u_1}) + c_2T(\boldsymbol{u_2}) + \cdots + c_kT(\boldsymbol{u_k}).$$

Proof: Let $\boldsymbol{A}$ be the standard matrix for T,
i.e. $T(\boldsymbol{u}) = \boldsymbol{Au}$ for all $\boldsymbol{u} \in \mathbb{R}^n$.

1. $T(\mathbf{0}) = \boldsymbol{A}\mathbf{0} = \mathbf{0}$.

Some basic properties (Theorem 7.1.4)

2. $T(c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k})$

$$= \boldsymbol{A}(c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k})$$

$$= c_1\boldsymbol{Au_1} + c_2\boldsymbol{Au_2} + \cdots + c_k\boldsymbol{Au_k}$$

$$= c_1T(\boldsymbol{u_1}) + c_2T(\boldsymbol{u_2}) + \cdots + c_kT(\boldsymbol{u_k}).$$

Examples (Example 7.1.5.1)

Let $T_1 \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by

$$T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+1 \\ y+3 \end{bmatrix} \quad \text{for } \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2.$$

Since $T_1\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

(by Theorem 7.1.4.1) T_1 is not a linear transformation.

Examples (Example 7.1.5.2)

Let $T_2\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be defined by

$$T_2\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x^2 \\ yz \end{bmatrix} \quad \text{for} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3.$$

Note that $T_2\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (which satisfies Theorem 7.1.4.1).

But this does not means that T_2 is a linear transformation.

Examples (Example 7.1.5.2)

For example,

$$T_2\left(\begin{bmatrix}1\\0\\0\end{bmatrix}+\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=T_2\left(\begin{bmatrix}2\\0\\0\end{bmatrix}\right)=\begin{bmatrix}4\\0\end{bmatrix}$$

and

$$T_2\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)+T_2\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix}.$$

So (by Theorem 7.1.4.2) T_2 is not a linear transformation.

Bases for $\mathbb{R}^n$ (Discussion 7.1.6)

Let $\{ \boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_n} \}$ be a basis for $\mathbb{R}^n$.

Given any vector $\boldsymbol{v} \in \mathbb{R}^n$, we can write

$$\boldsymbol{v} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_n\boldsymbol{u_n}$$

for some $c_1, c_2, \ldots, c_n \in \mathbb{R}$.

For a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$, (by Theorem 7.1.4.2)

$$\begin{aligned} T(\boldsymbol{v}) &= T(c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_n}) \\ &= c_1T(\boldsymbol{u_1}) + c_2T(\boldsymbol{u_2}) + \cdots + c_nT(\boldsymbol{u_n}). \end{aligned}$$

The image $T(\boldsymbol{v})$ of $\boldsymbol{v}$ is completely determined by the images $T(\boldsymbol{u_1})$, $T(\boldsymbol{u_2})$, ..., $T(\boldsymbol{u_n})$ of the basis vectors $\boldsymbol{u_1}$, $\boldsymbol{u_2}$, ..., $\boldsymbol{u_n}$.

An examples (Example 7.1.7)

Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation such that

$$T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad T\left(\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ -1 \end{bmatrix}.$$

As $\{ (1, 1, 1)^T, (0, 1, 1)^T, (2, 0, -1)^T \}$ is a basis for $\mathbb{R}^3$, the image $T((x, y, z)^T)$ of every $(x, y, z)^T \in \mathbb{R}^3$ is completely determined by the images of $(1, 1, 1)^T$, $(0, 1, 1)^T$, and $(2, 0, -1)^T$.

An examples (Example 7.1.7.1)

For example, $\begin{bmatrix} -1 \\ 4 \\ 6 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}.$

$$T\left(\begin{bmatrix} -1 \\ 4 \\ 6 \end{bmatrix}\right) = 3T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) - 2T\left(\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}\right)$$

$$= 3\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 \\ 13 \end{bmatrix}.$$

An examples (Example 7.1.7.2)

In general, for any $(x, y, z)^T \in \mathbb{R}^3$, we first solve the vector equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

which gives us a solution

$c_1 = x - 2y + 2z$, $c_2 = -x + 3y - 2z$ and $c_3 = y - z$,

i.e. $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = (x - 2y + 2z) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (-x + 3y - 2z) \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + (y - z) \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}.$$

An examples (Example 7.1.7.2)

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = (x - 2y + 2z)T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) + (-x + 3y - 2z)T\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) + (y - z)T\left(\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}\right)$$

$$= (x - 2y + 2z)\begin{bmatrix} 1 \\ 3 \end{bmatrix} + (-x + 3y - 2z)\begin{bmatrix} -1 \\ 2 \end{bmatrix} + (y - z)\begin{bmatrix} 4 \\ -1 \end{bmatrix}.$$

$$= \begin{bmatrix} 2x - y \\ x - y + 3z \end{bmatrix}.$$

Standard matrices (Discussion 7.1.8)

Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation and $\boldsymbol{A} = (a_{ij})_{m \times n}$ be the standard matrix for T.

Take the standard basis $\{\, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \,\}$ for $\mathbb{R}^n$ where
$\mathbf{e}_1 = (1, 0, \ldots, 0)^T$, $\mathbf{e}_2 = (0, 1, 0, \ldots, 0)^T$, ..., $\mathbf{e}_n = (0, \ldots, 0, 1)^T$.

In particular, for $i = 1, 2, \ldots, n$,

$$T(\mathbf{e}_i) = \boldsymbol{A}\mathbf{e}_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix} = \text{the } i^{\text{th}} \text{ column of } \boldsymbol{A}.$$

So $\boldsymbol{A} = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{bmatrix}$.

An examples (Example 7.1.9)

Let $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be a linear transformation such that

$$T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad T\left(\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ -1 \end{bmatrix}.$$

Instead of computing the formula of T directly (as in Example 7.1.7.2), we find the standard matrix using images of basis vectors of the standard basis.

An examples (Example 7.1.9)

$$\left[\begin{array}{ccc|c|c|c} 1 & 0 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 1 & -2 & 2 \\ 0 & 1 & 0 & -1 & 3 & -2 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right]$$

Thus

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = -2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix},$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}.$$

An examples (Example 7.1.9)

Then

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) - T\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\3\end{bmatrix} - \begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}.$$

Similarly,

$$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\-1\end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\3\end{bmatrix}.$$

An examples (Example 7.1.9)

So the standard matrix for T is

$$\left[\; T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) \quad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) \quad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) \;\right]$$

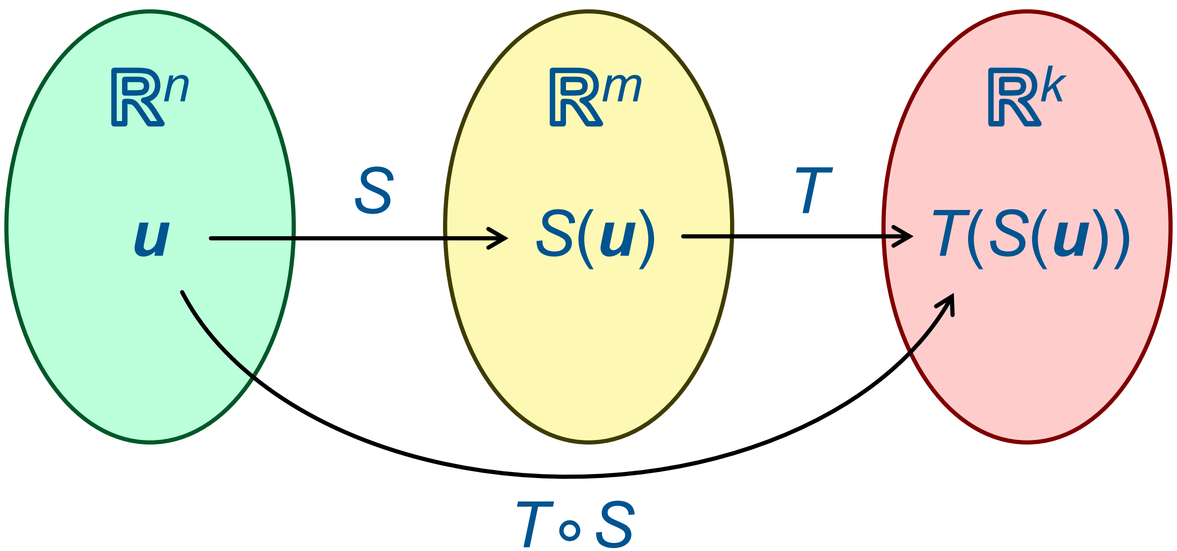
$$= \begin{bmatrix}2 & -1 & 0\\ 1 & -1 & 3\end{bmatrix}.$$

Compositions of mappings (Definition 7.1.10)

Let $S: \mathbb{R}^n \to \mathbb{R}^m$ and $T: \mathbb{R}^m \to \mathbb{R}^k$ be linear transformations.

The composition of T with S, denoted by $T \circ S$, is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^k$ defined by

$$(T \circ S)(\boldsymbol{u}) = T(S(\boldsymbol{u})) \quad \text{for } \boldsymbol{u} \in \mathbb{R}^n.$$



Compositions of mappings (Theorem 7.1.11)

Suppose $S: \mathbb{R}^n \to \mathbb{R}^m$ and $T: \mathbb{R}^m \to \mathbb{R}^k$ are linear transformations.

Then $T \circ S$ is also a linear transformation.

Furthermore, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are standard matrices for S and T respectively, then $\boldsymbol{BA}$ is the standard matrix for $T \circ S$.

Proof: For all $\boldsymbol{u} \in \mathbb{R}^n$,

$$(T \circ S)(\boldsymbol{u}) = T(S(\boldsymbol{u})) = T(\boldsymbol{Au}) = \boldsymbol{BAu}.$$

So $T \circ S$ is a linear transformation and $\boldsymbol{BA}$ is the standard matrix for $T \circ S$.

An example (Example 7.1.12)

Let $S\colon \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation defined by

$$S\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ z \end{bmatrix} \quad \text{for} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$$

and $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ a linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ y \\ x \end{bmatrix} \quad \text{for} \quad \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2.$$

An example (Example 7.1.12)

Then $T \circ S: \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$(T \circ S)\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = T\left(S\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)\right) = T\left(\begin{bmatrix} x+y \\ z \end{bmatrix}\right)$$

$$= \begin{bmatrix} z \\ z \\ x+y \end{bmatrix} \quad \text{for} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3.$$

An example (Example 7.1.12)

The standard matrices for S, T and $T \circ S$ are

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \text{respectively.}$$

Note that

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Section 7.2

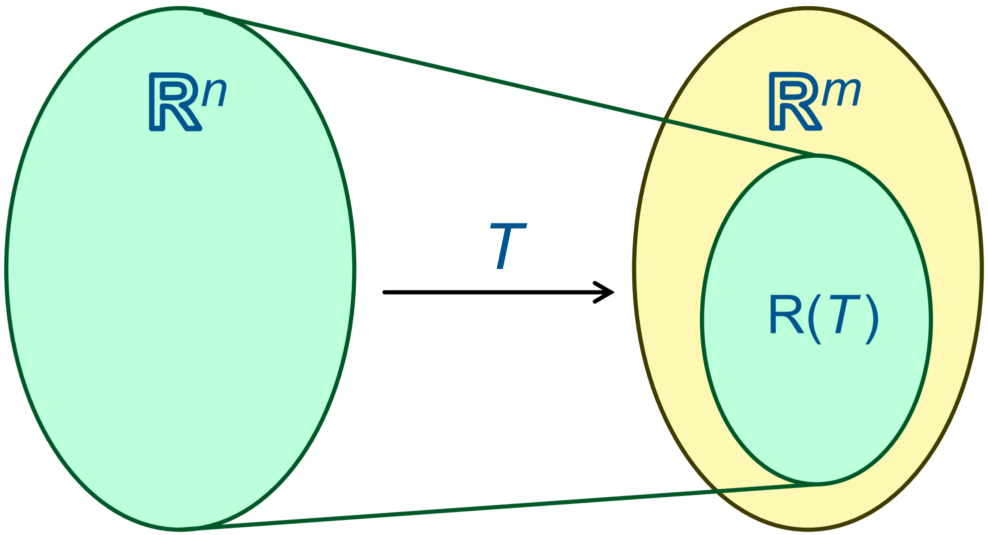
Ranges and Kernels

Ranges (Definition 7.2.1)

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

The range of T, which is denoted by $R(T)$, is the set of images of T,

i.e. $R(T) = \{ T(\boldsymbol{u}) \mid \boldsymbol{u} \in \mathbb{R}^n \} \subseteq \mathbb{R}^m$.



An example (Example 7.2.2)

Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ y \\ x \end{bmatrix} \quad \text{for} \quad \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2.$$

Then the range of T is the set of vectors

$$\mathrm{R}(T) = \left\{ x\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \,\middle|\, x, y \in \mathbb{R} \right\}$$

$$= \operatorname{span}\{ (1, 0, 1)^{\mathrm{T}}, (1, 1, 0)^{\mathrm{T}} \}.$$

Bases for $\mathbb{R}^n$ (Discussion 7.2.3)

Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation and $\{ \boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_n \}$ a basis for $\mathbb{R}^n$.

Given any vector $\boldsymbol{v} \in \mathbb{R}^n$, (by Discussion 7.1.6.) $T(\boldsymbol{v})$ is a linear combinations of the images $T(\boldsymbol{u}_1)$, $T(\boldsymbol{u}_2)$, ..., $T(\boldsymbol{u}_n)$ of the basis vectors $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, ..., $\boldsymbol{u}_n$,
i.e. $T(\boldsymbol{v}) \in \text{span}\{ T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), ..., T(\boldsymbol{u}_n) \}$.

Hence $\text{R}(T) \subseteq \text{span}\{ T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), ..., T(\boldsymbol{u}_n) \}$.

On the other hand, $T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), ..., T(\boldsymbol{u}_n) \in \text{R}(T)$ and hence $\text{span}\{ T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), ..., T(\boldsymbol{u}_n) \} \subseteq \text{R}(T)$.

So we have $\text{R}(T) = \text{span}\{ T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), ..., T(\boldsymbol{u}_n) \}$.

Ranges and column spaces (Theorem 7.2.4)

Let $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation and $\boldsymbol{A}$ the standard matrix for T.

Then $R(T)$ = the column space of $\boldsymbol{A}$.

Proof: Take the standard basis $\{\, \boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n \,\}$ for $\mathbb{R}^n$.

Then (by Discussion 7.1.8), $\boldsymbol{A} = \begin{bmatrix} T(\boldsymbol{e}_1) & T(\boldsymbol{e}_2) & \cdots & T(\boldsymbol{e}_n) \end{bmatrix}$.

Hence

$R(T) = \text{span}\{\, T(\boldsymbol{e}_1), T(\boldsymbol{e}_2), \ldots, T(\boldsymbol{e}_n) \,\}$ (by Discussion 7.2.3)

$\quad\quad$ = the column space of $\boldsymbol{A}$.

Ranks (Definition 7.2.5)

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

The rank of T, which is denoted by $\text{rank}(T)$, is the dimension of $R(T)$.

If $\boldsymbol{A}$ is the standard matrix for T, then

$$\begin{aligned}\text{rank}(T) &= \dim(R(T)) \\ &= \dim(\text{the column space of } \boldsymbol{A}) \\ &= \text{rank}(\boldsymbol{A}).\end{aligned}$$

By Theorem 7.2.4,
$R(T)$ = the column space of $\boldsymbol{A}$.

An example (Example 7.2.6)

Let $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation defined by

$$T\left(\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 2y + z \\ x + 3y \\ x + 4y - z \\ y - z \end{bmatrix} \quad \text{for} \quad \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \in \mathbb{R}^4.$$

The standard matrix for T is $\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$.

An example (Example 7.2.6)

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So (see Remark 4.1.13 and Method 2 of Example 4.1.14.1)

$\{ (1, 1, 1, 0)^T, (2, 3, 4, 1)^T \}$ is a basis for the column space of $\boldsymbol{A}$

and hence is a basis for $R(T)$.

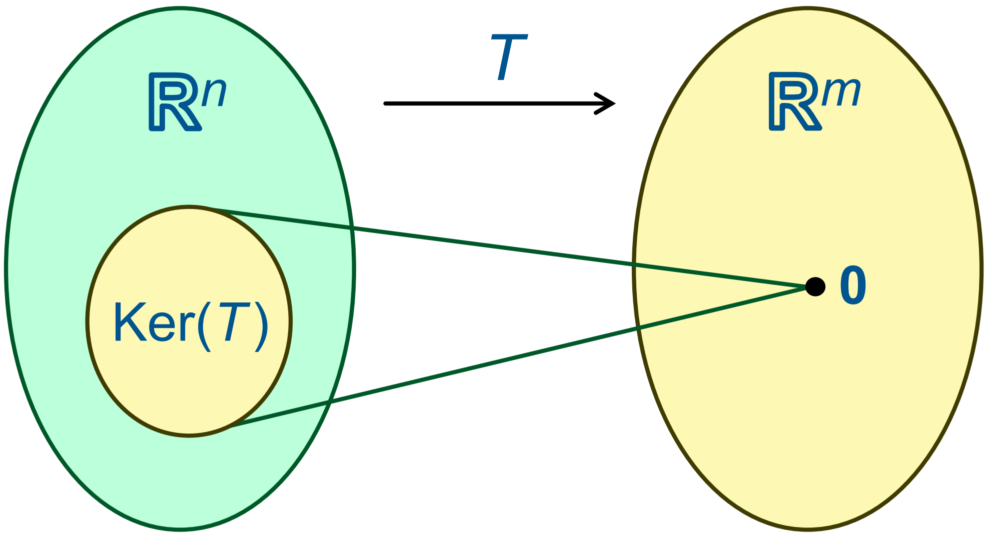
Then $\text{rank}(T) = \dim(R(T)) = 2$.

Kernels (Definition 7.2.7)

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

The kernel of T, which is denoted by $\mathrm{Ker}(T)$, is the set of vectors in $\mathbb{R}^n$ whose image is the zero vector in $\mathbb{R}^m$,

i.e. $\mathrm{Ker}(T) = \{\, \boldsymbol{u} \mid T(\boldsymbol{u}) = \mathbf{0} \,\} \subseteq \mathbb{R}^n$.



Examples (Example 7.2.8.1)

Let $T_1 \colon \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation defined by

$$T_1\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x - y \\ x - y + 3z \\ -5x + y \\ x - z \end{bmatrix} \quad \text{for} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3.$$

Then

$$\operatorname{Ker}(T_1) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \,\middle|\, \begin{bmatrix} 2 & -1 & 0 \\ 1 & -1 & 3 \\ -5 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Examples (Example 7.2.8.1)

Solving
$$\begin{bmatrix} 2 & -1 & 0 \\ 1 & -1 & 3 \\ -5 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

we get only the trivial solution $x = 0$, $y = 0$, $z = 0$.

Thus $\mathrm{Ker}(T_1) = \{ (0, 0, 0)^T \}$.

Examples (Example 7.2.8.2)

Let $T_2\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a linear transformation defined by

$$T_2\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} z-y \\ 0 \\ x \end{bmatrix} \quad \text{for} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3.$$

Solving $\begin{bmatrix} z-y \\ 0 \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, we get $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ t \end{bmatrix}$ for $t \in \mathbb{R}$.

So $\mathrm{Ker}(T_2) = \mathrm{span}\{\, (0, 1, 1)^T \,\}$.

Kernels and nullspaces (Theorem 7.2.9)

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and $\boldsymbol{A}$ the standard matrix for T.

Then Ker(T) = the nullspace of $\boldsymbol{A}$.

Proof:
$$\begin{aligned}\text{Ker}(T) &= \{\, \boldsymbol{u} \mid T(\boldsymbol{u}) = \boldsymbol{0} \,\} \\ &= \{\, \boldsymbol{u} \mid \boldsymbol{A}\boldsymbol{u} = \boldsymbol{0} \,\} \\ &= \text{the nullspace of } \boldsymbol{A}.\end{aligned}$$

Nullities (Definition 7.2.10)

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

The nullity of T, which is denoted by nullity(T), is the dimension of Ker(T).

If $\boldsymbol{A}$ is the standard matrix for T, then

$$\begin{aligned} \text{nullity}(T) &= \dim(\text{Ker}(T)) \\ &= \dim(\text{the nullspace of } \boldsymbol{A}) \\ &= \text{nullity}(\boldsymbol{A}). \end{aligned}$$

By Theorem 7.2.9,
Ker(T) = the nullspace of $\boldsymbol{A}$.

Examples (Example 7.2.8.1 & Example 7.2.11.1)

Let $T_1: \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation defined by

$$T_1\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x - y \\ x - y + 3z \\ -5x + y \\ x - z \end{bmatrix} \quad \text{for} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3.$$

Since $\mathrm{Ker}(T_1) = \{ (0, 0, 0)^T \}$,

$$\mathrm{nullity}(T_1) = \dim(\mathrm{Ker}(T_1)) = 0.$$

Examples (Example 7.2.8.2 & Example 7.2.11.1)

Let $T_2\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by

$$T_2\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} z - y \\ 0 \\ x \end{bmatrix} \quad \text{for} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3.$$

So $\mathrm{Ker}(T_2) = \mathrm{span}\{ (0, 1, 1)^T \}$,

$\mathrm{nullity}(T_2) = \dim(\mathrm{Ker}(T_2)) = 1.$

Examples (Example 7.2.6 & Example 7.2.11.2)

Let $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation defined by

$$T\left(\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 2y + z \\ x + 3y \\ x + 4y - z \\ y - z \end{bmatrix} \quad \text{for} \quad \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \in \mathbb{R}^4.$$

$$T\left(\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Examples (Example 7.2.6 & Example 7.2.11.2)

$$\Leftrightarrow \quad \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -3 \\ 1 \\ 1 \end{bmatrix}$$

for $s, t \in \mathbb{R}$.

So $\{ (1, 0, 0, 0)^T, (0, -3, 1, 1)^T \}$ is a basis for $\mathrm{Ker}(T)$ and $\mathrm{nullity}(T) = \dim(\mathrm{Ker}(T)) = 2$.

Dimension Theorem for Linear Transformation (Theorem 7.2.12)

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation then

$$\text{rank}(T) + \text{nullity}(T) = n.$$

Proof: Let $\boldsymbol{A}$ be the standard matrix for T.

By the Dimension Theorem for Matrices (Theorem 4.3.4),

$$\text{rank}(\boldsymbol{A}) + \text{nullity}(\boldsymbol{A}) = \text{the number of columns of } \boldsymbol{A}.$$

$\text{rank}(T) = \text{rank}(\boldsymbol{A})$.

$\text{nullity}(T) = \text{nullity}(\boldsymbol{A})$.

Then

$$\begin{aligned}\text{rank}(T) + \text{nullity}(T) &= \text{rank}(\boldsymbol{A}) + \text{nullity}(\boldsymbol{A}) \\ &= \text{the number of columns of } \boldsymbol{A} \\ &= n.\end{aligned}$$

$\boldsymbol{A}$ is an $m \times n$ matrix.

Section 7.3

Geometric Linear Transformations

Geometric transformations (Discussion 7.3.1)

Several well-known geometric transformations on $\mathbb{R}^2$ and $\mathbb{R}^3$ such as

- scalings,
- reflections about lines and planes through the origin,
- rotations about the origin

are linear transformations.

Scalings in $\mathbb{R}^2$ (Example 7.3.2)

Suppose $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that

$$S\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \lambda_1 \\ 0 \end{bmatrix} \quad \text{and} \quad S\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ \lambda_2 \end{bmatrix}$$

for some positive real numbers λ_1 and λ_2.

The standard matrix for S is $\begin{bmatrix} S(\mathbf{e_1}) & S(\mathbf{e_2}) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

(see Discussion 7.1.8)

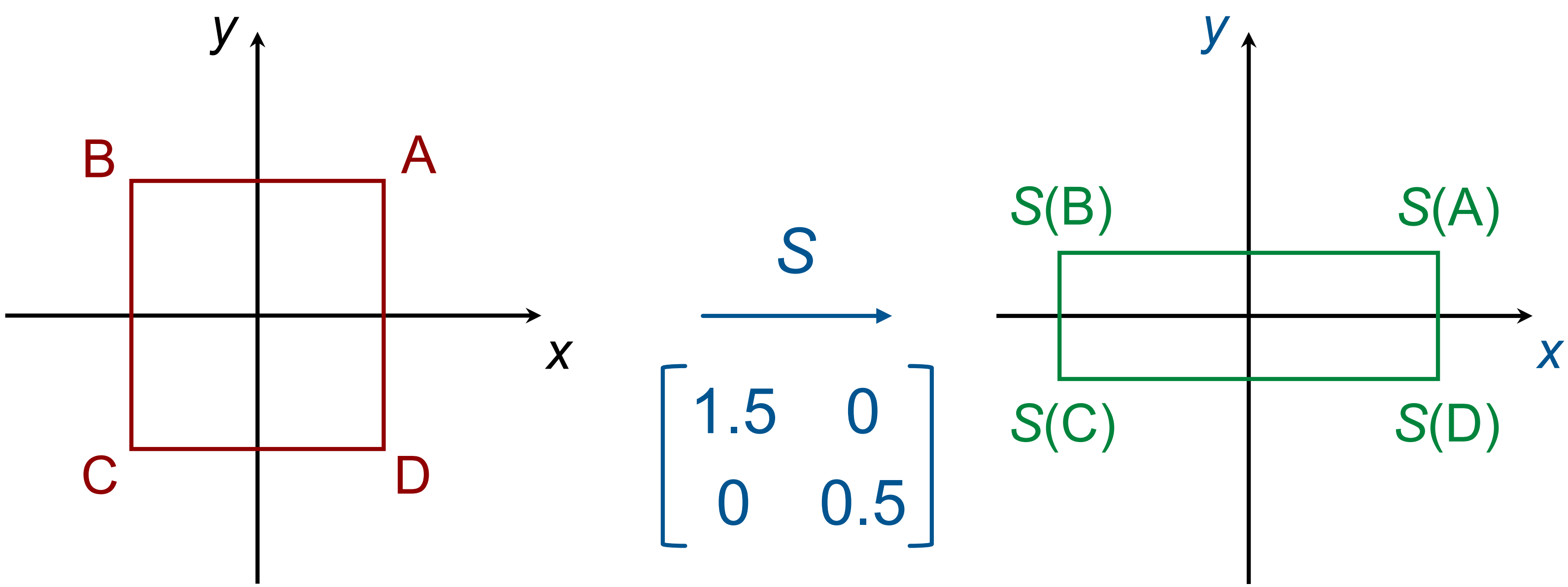
and $S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda_1 x \\ \lambda_2 y \end{bmatrix}$ for $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$.

Scalings in $\mathbb{R}^2$ (Example 7.3.1)

For example, consider the square with vertices

$$A = (1, 1)^T,\ B = (-1, 1)^T,\ C = (-1, -1)^T,\ D = (1, -1)^T.$$

We apply S, with $\lambda_1 = 1.5$ and $\lambda_2 = 0.5$, to the square ABCD.



Scalings in $\mathbb{R}^2$ (Example 7.3.1)

The effect of the linear transformation S is to scale by a factor λ_1 along the x-axis and by a factor λ_2 along the y-axis.

S is called a scaling along the x and y-axes by factors of λ_1 and λ_2 respectively.

For the special case when $\lambda_1 = \lambda_2 = \lambda$,
S is known as a dilation if $\lambda > 1$ and
S is known as a contraction if $\lambda < 1$.

Diagonalizable matrices (Remark 7.3.3)

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $T(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$ for $\boldsymbol{x} \in \mathbb{R}^2$ where $\boldsymbol{A}$ is a 2×2 matrix.

Suppose $\boldsymbol{A}$ is diagonalizable, i.e. there exists a 2×2 invertible matrix $\boldsymbol{P} = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix}$ such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ for some positive real numbers λ_1 and λ_2.

Then $T(\boldsymbol{u}_1) = \boldsymbol{A}\boldsymbol{u}_1 = \lambda_1\boldsymbol{u}_1$ and $T(\boldsymbol{u}_2) = \boldsymbol{A}\boldsymbol{u}_2 = \lambda_2\boldsymbol{u}_2$.

Thus T can be regarded as a scaling that scales along axes in the directions of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ by factors λ_1 and λ_2 respectively.

(In here, the new axes may not be perpendicular to each other.)

Diagonalizable matrices (Example 7.3.4)

Suppose $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation with a standard matrix $\begin{bmatrix} 1 & 1 \\ 0.25 & 1 \end{bmatrix}$.

Note that $\begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 0.25 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1.5 & 0 \\ 0 & 0.5 \end{bmatrix}$.

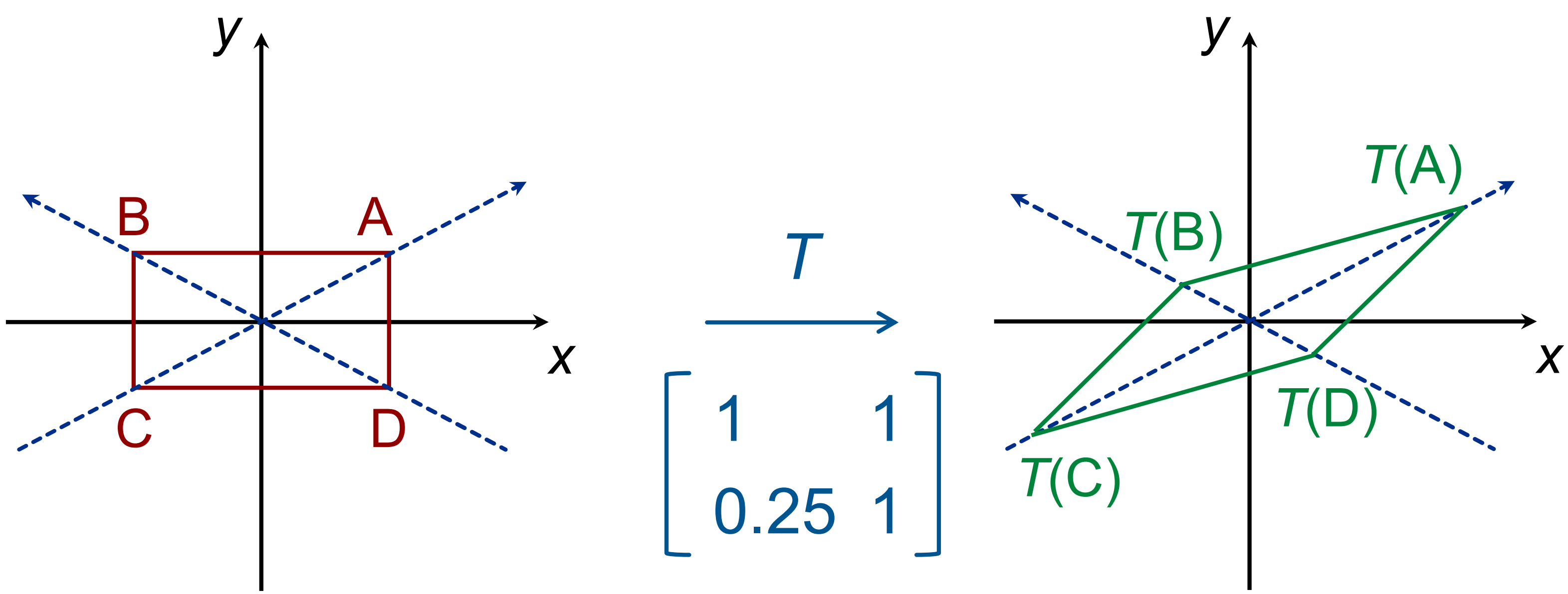
Thus T can be regarded as a scaling that scales along axes in the directions of $(2, 1)^T$ and $(-2, 1)^T$ by factors 1.5 and 0.5 respectively.

Scalings in $\mathbb{R}^2$ (Example 7.3.1)

Consider the rectangle with vertices

$A = (2, 1)^T$, $B = (-2, 1)^T$, $C = (-2, -1)^T$, $D = (2, -1)^T$.

We apply T to the rectangle ABCD.



Scalings in $\mathbb{R}^3$ (Example 7.3.5)

The standard matrix for the scaling along the x, y, z-axes in $\mathbb{R}^3$ by factors λ_1, λ_2, λ_2, respectively, is

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

For the special case when $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$,
the scaling is known as a dilation if $\lambda > 1$ and
the scaling is known as a contraction if $\lambda < 1$.

Reflections in $\mathbb{R}^2$ (Example 7.3.6.1)

Let $F_1 \colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that

$$F_1\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad F_1\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

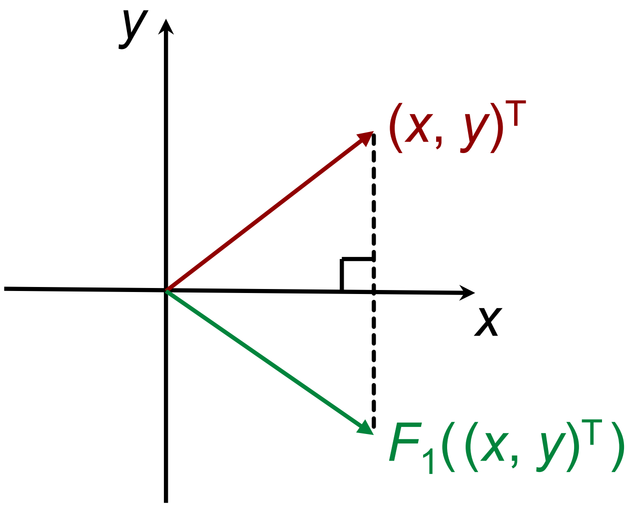
The standard matrix for F_1 is $\begin{bmatrix} F_1(\mathbf{e}_1) & F_1(\mathbf{e}_2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(see Discussion 7.1.8)

and $F_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$ for $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$.

Reflections in $\mathbb{R}^2$ (Example 7.3.6.1)

$$F_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ -y \end{bmatrix} \quad \text{for } \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2.$$



F_1 is the reflection about the x-axis.

Similarly, the reflection $F_2 \colon \mathbb{R}^2 \to \mathbb{R}^2$ about the y-axis

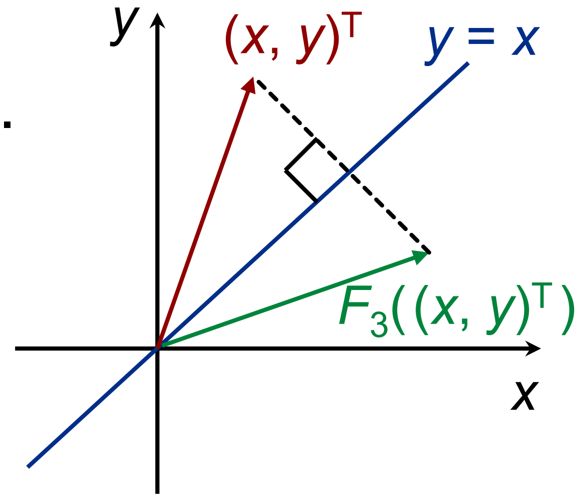
has the standard matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

and $F_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$ for $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$.

Reflections in $\mathbb{R}^2$ (Example 7.3.6.2)

Let $F_3: \mathbb{R}^2 \to \mathbb{R}^2$ be the refection about the line $y = x$.

$$F_3\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } F_3\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$



The standard matrix for F_3 is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

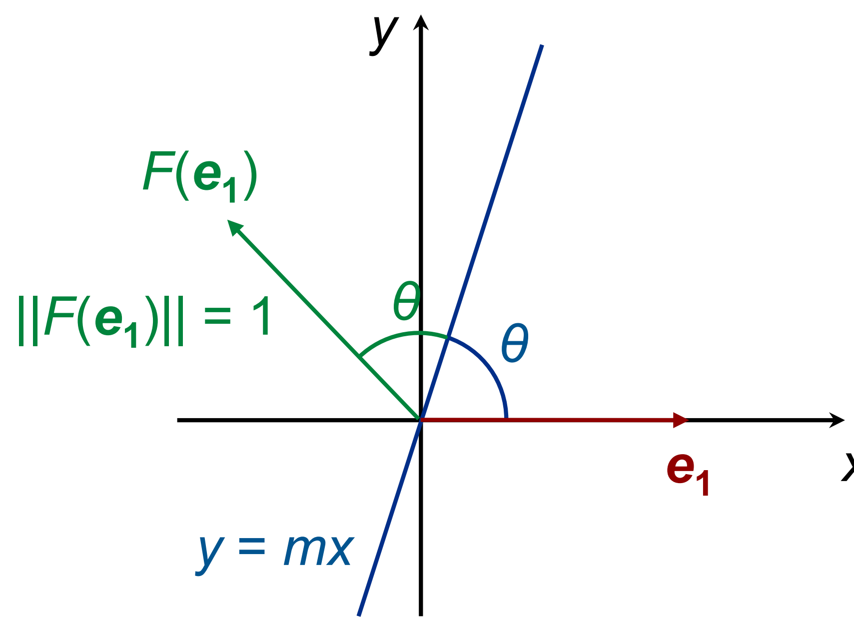
and $F_3\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$ for $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$.

Reflections in $\mathbb{R}^2$ (Example 7.3.6.3)

Let $F\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the refection about the line $y = mx$ with $m = \tan(\theta)$ where θ is the angle between the x-axis and the line.

$$F\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = F(\mathbf{e}_1)$$

$$= \begin{bmatrix} \cos(2\theta) \\ \sin(2\theta) \end{bmatrix}$$

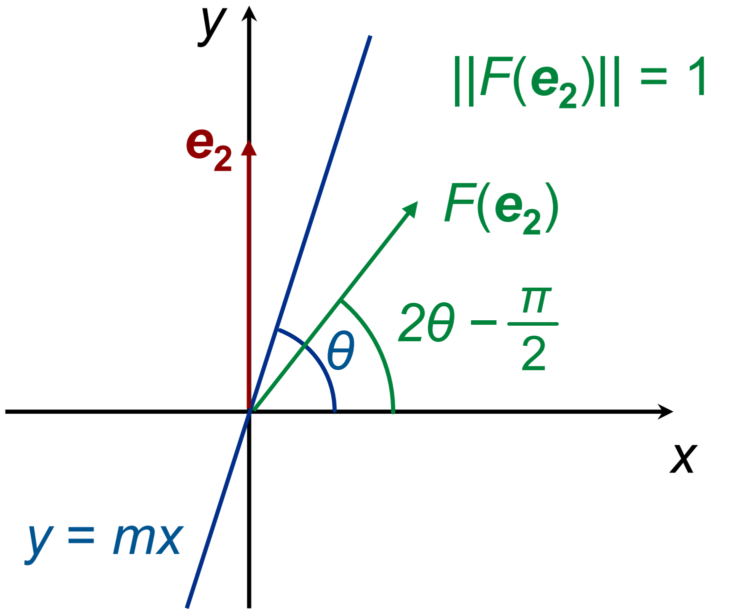


Reflections in $\mathbb{R}^2$ (Example 7.3.6.3)

$$F\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = F(\mathbf{e}_2)$$

$$= \begin{bmatrix} \cos\left(2\theta - \frac{\pi}{2}\right) \\ \sin\left(2\theta - \frac{\pi}{2}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \sin(2\theta) \\ -\cos(2\theta) \end{bmatrix}$$



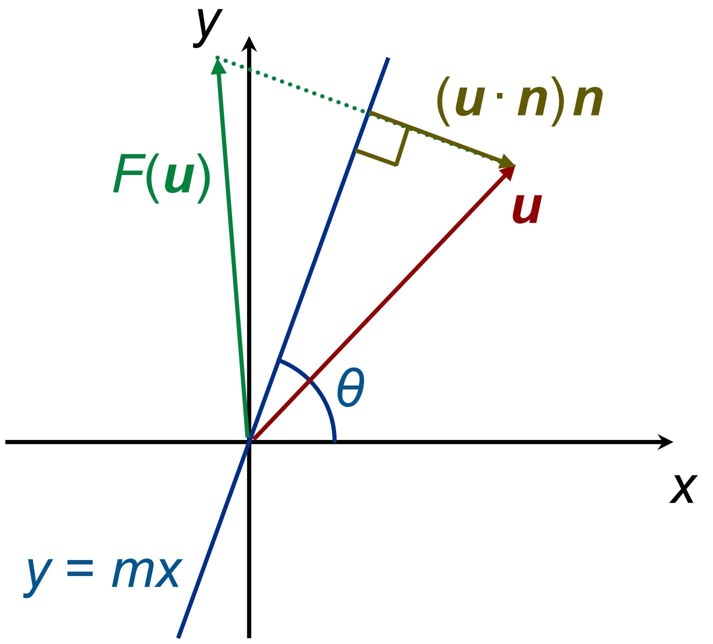
Reflections in $\mathbb{R}^2$ (Example 7.3.6.3 & Remark 7.3.7)

So the standard matrix for F is $\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$.

The formula of the reflection F can also be written as

$$F(\boldsymbol{u}) = \boldsymbol{u} - 2(\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{n} \quad \text{for} \quad \boldsymbol{u} \in \mathbb{R}^2,$$

where $\boldsymbol{n} = (\sin(\theta), -\cos(\theta))^T$.



Reflections in $\mathbb{R}^3$ (Example 7.3.8 & Question 7.26)

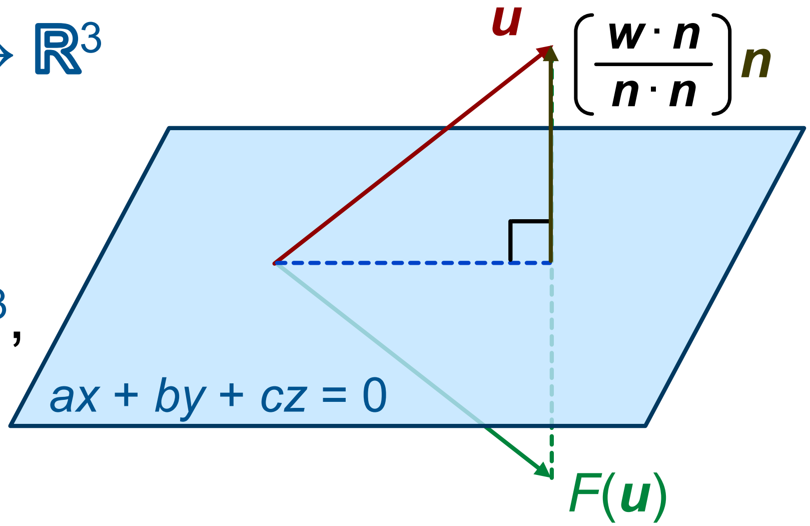
The standard matrices for reflections about the *xy*-plane, *xz*-plane, *yz*-plane in $\mathbb{R}^3$ are

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ respectively.}$$

In general, the reflection $F: \mathbb{R}^3 \to \mathbb{R}^3$ about the plane $ax + by + cz = 0$ can be formulate as

$$F(\boldsymbol{u}) = \boldsymbol{u} - 2\left(\frac{\boldsymbol{w} \cdot \boldsymbol{n}}{\boldsymbol{n} \cdot \boldsymbol{n}}\right)\boldsymbol{n} \text{ for } \boldsymbol{u} \in \mathbb{R}^3,$$

where $\boldsymbol{n} = (a, b, c)^T$.

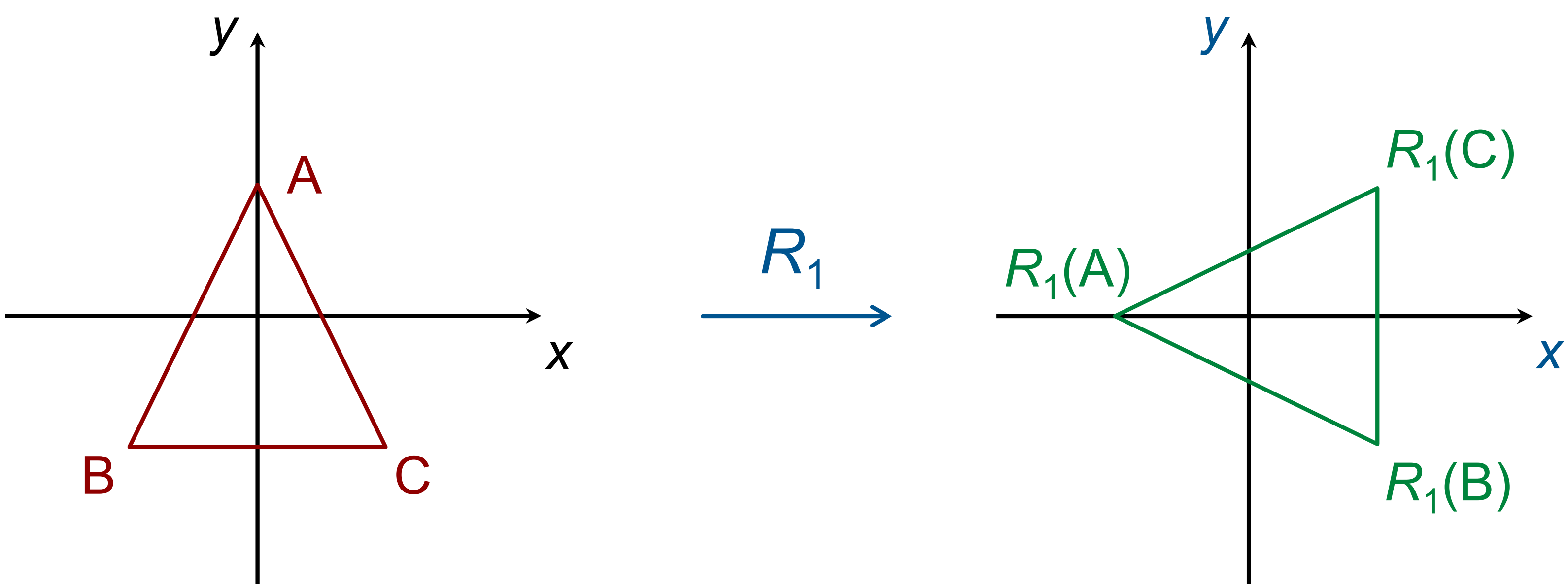


Rotations in $\mathbb{R}^2$ (Example 7.3.9.1)

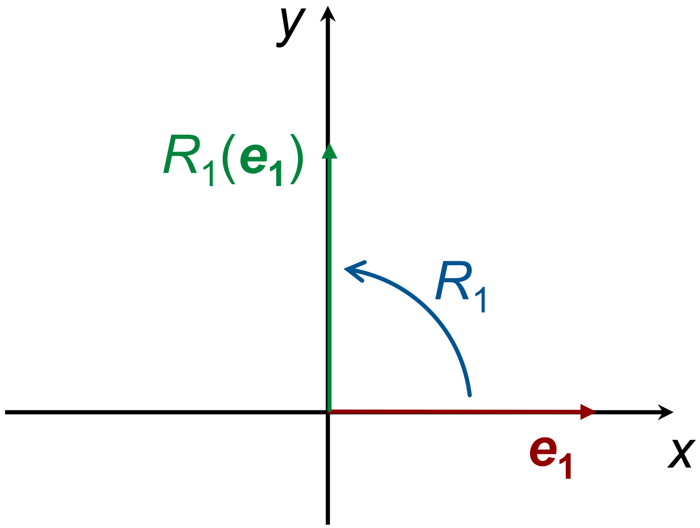
Let $R_1: \mathbb{R}^2 \to \mathbb{R}^2$ be the anti-clockwise rotation about the origin through an angle $\frac{\pi}{2}$.

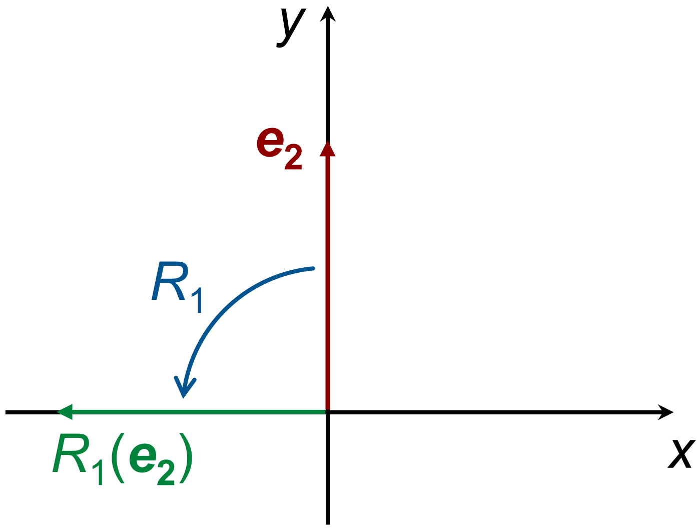
Consider the triangle with vertices

$$A = (0, 1)^T, \quad B = (-1, -1)^T, \quad C = (1, -1)^T.$$



Rotations in $\mathbb{R}^2$ (Example 7.3.9.1)





$$R_1\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \qquad R_1\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

The standard matrix for R_1 is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Rotations in $\mathbb{R}^2$ (Example 7.3.9.1)

For the triangle with vertices

$$A = (0, 1)^T, \ B = (-1, -1)^T, \ C = (1, -1)^T,$$

the image of A is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$;

the image of B is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$;

the image of C is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

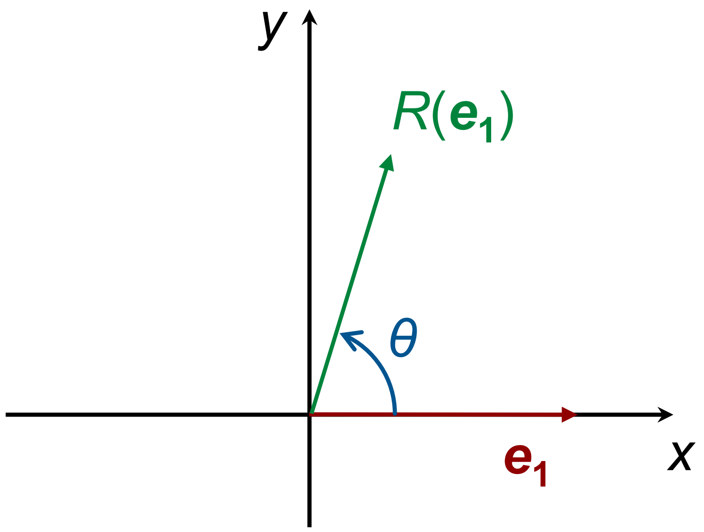
Rotations in $\mathbb{R}^2$ (Example 7.3.9.1)

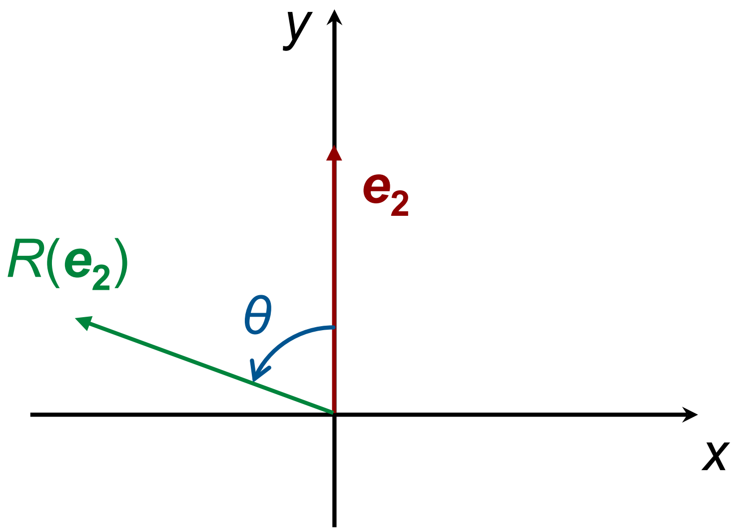
Similarly, the standard matrices for anti-clockwise rotations about the origin through angles π and $\dfrac{3\pi}{2}$ are

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \text{ respectively.}$$

Rotations in $\mathbb{R}^2$ (Example 7.3.9.2)

Let $R\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the anti-clockwise rotation about the origin through an angle θ.





$$R\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \qquad R\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$$

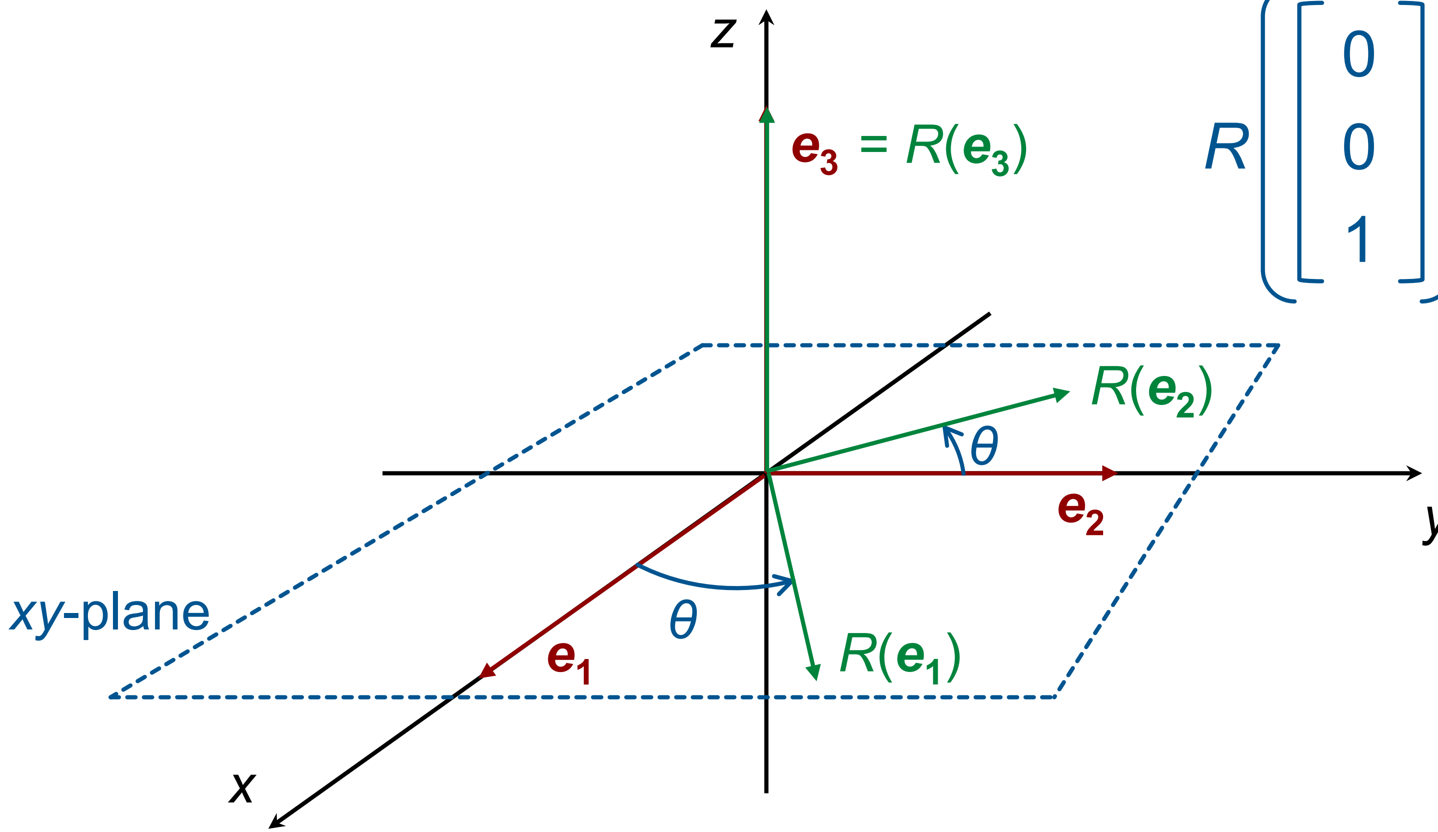
Rotations in $\mathbb{R}^2$ (Example 7.3.9.2)

The standard matrix for R is given by

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

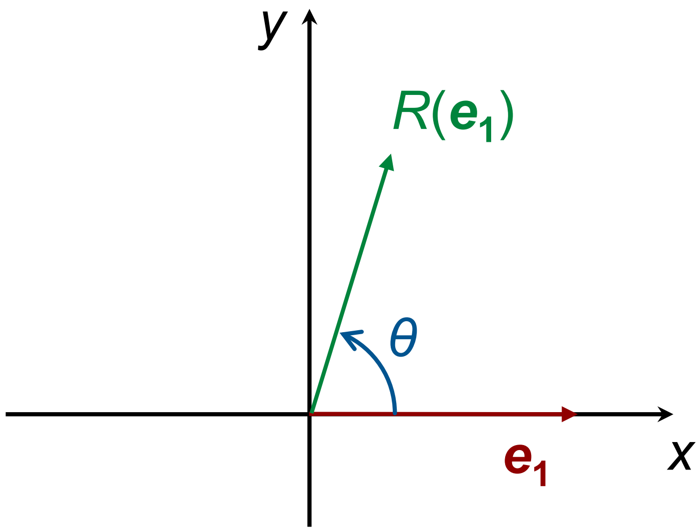
Rotations in $\mathbb{R}^3$ (Example 7.3.10)

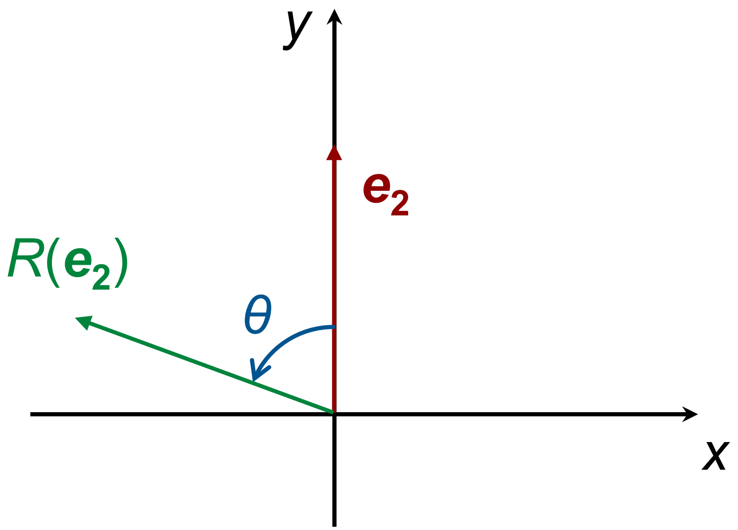
Let R: $\mathbb{R}^3 \to \mathbb{R}^3$ be the anti-clockwise rotation about the z-axis through an angle θ.



$$R\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Rotations in $\mathbb{R}^3$ (Example 7.3.10)





$$R\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}\cos(\theta)\\\sin(\theta)\\0\end{bmatrix}$$

$$R\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}-\sin(\theta)\\\cos(\theta)\\0\end{bmatrix}$$

Rotations in $\mathbb{R}^3$ (Example 7.3.10)

The standard matrix for R is given by

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Similarly, the standard matrices for anti-clockwise rotations about the x-axis and y-axis through an angle θ are

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \text{ and } \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}, \text{ respectively.}$$

Translations in $\mathbb{R}^2$ (Example 7.3.11.1)

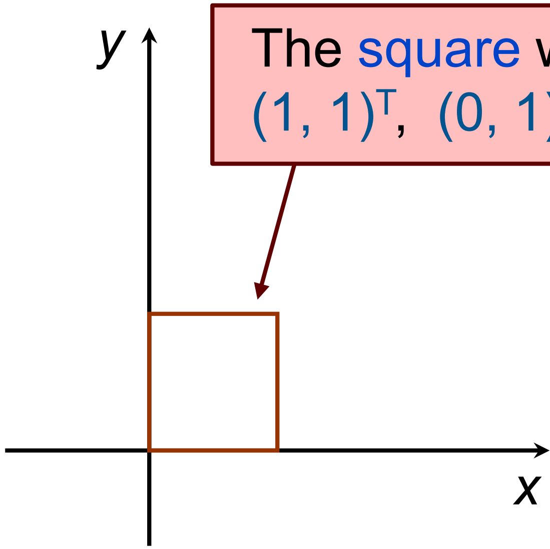
Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the translation such that

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + a \\ y + b \end{bmatrix} \quad \text{for } \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2,$$

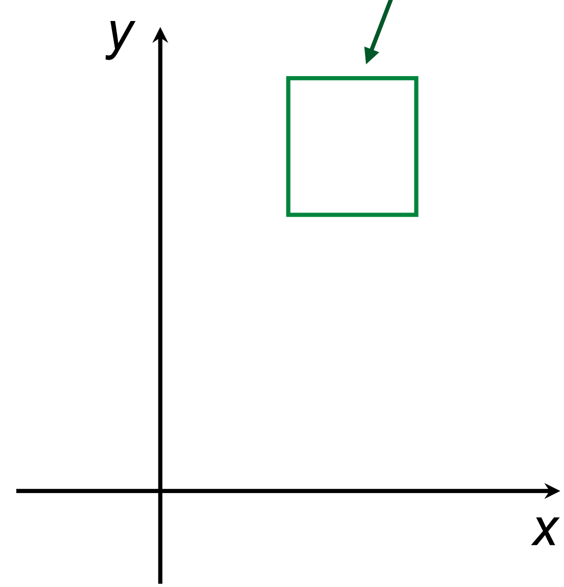
where a and b are real constants.

The square with vertices $T((1, 1)^T)$, $T((0, 1)^T)$, $T((0, 0)^T)$, $T((1, 0)^T)$.

The square with vertices $(1, 1)^T$, $(0, 1)^T$, $(0, 0)^T$, $(1, 0)^T$.



$$\xrightarrow[\substack{a = 1 \\ b = 2}]{T}$$



Translations in $\mathbb{R}^2$ (Example 7.3.11.1)

Except $a = b = 0$, T is not a linear transformation.

Proof: If a and b are not both zero,

$$T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} a \\ b \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Hence (by Theorem 7.1.4.1) T is not a linear transformation.

Translations in $\mathbb{R}^3$ (Example 7.3.11.2)

Let $T': \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the translation such that

$$T'\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + a \\ y + b \\ z + c \end{bmatrix} \quad \text{for} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3,$$

where a, b and c are real constants.

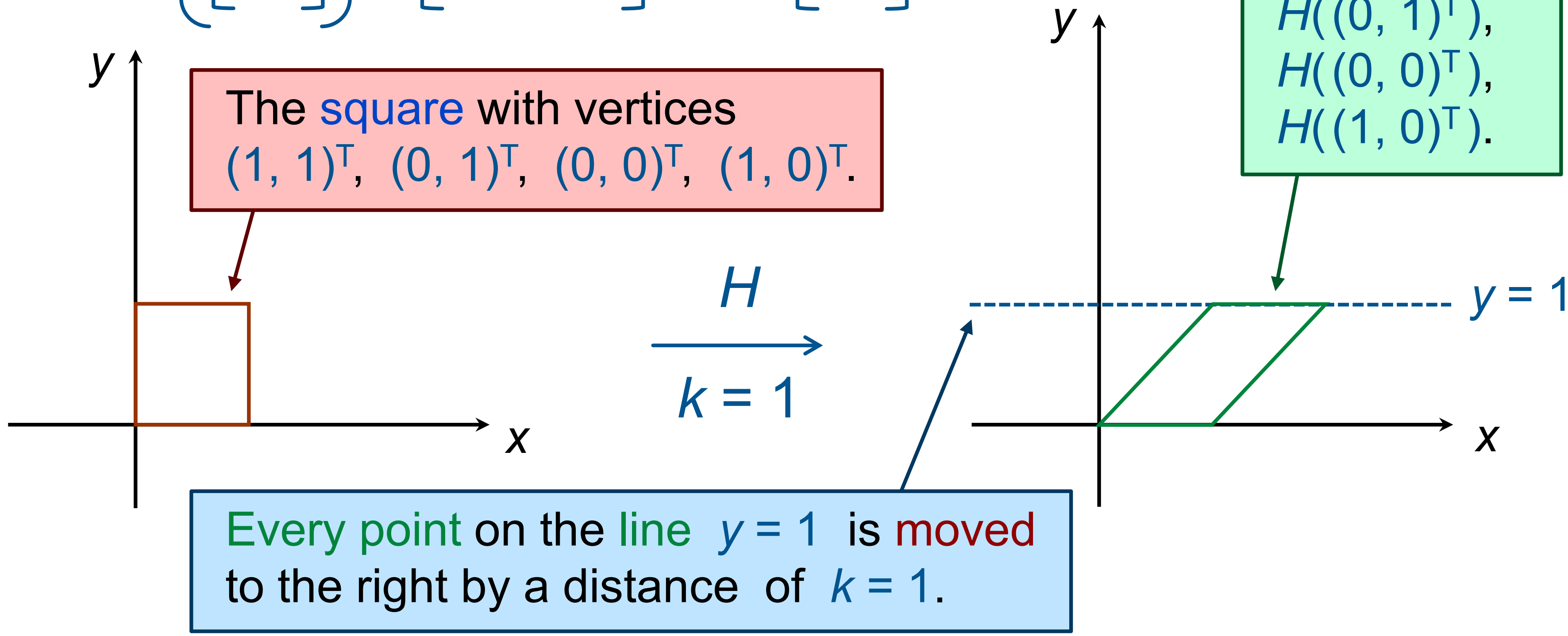
Except $a = b = c = 0$, T' is not a linear transformation.

Shears in $\mathbb{R}^2$ (Example 7.3.12.1)

A mapping $H\colon \mathbb{R}^2 \to \mathbb{R}^2$ is called a shear in the x-direction by a factor of k if

$$H\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + ky \\ y \end{bmatrix} \quad \text{for} \quad \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2.$$

The square with vertices $H((1,1)^T)$, $H((0,1)^T)$, $H((0,0)^T)$, $H((1,0)^T)$.



Shears in $\mathbb{R}^2$ (Example 7.3.12.1)

Observe that

$$H\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + ky \\ y \end{bmatrix} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \text{ for } \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2.$$

Thus H is a linear transformation

and the standard matrix for H is $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$.

Shears in $\mathbb{R}^3$ (Example 7.3.12.2)

A mapping $H'\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a shear in the x-direction by a factor of k_1 and in the y-direction by a factor of k_2.

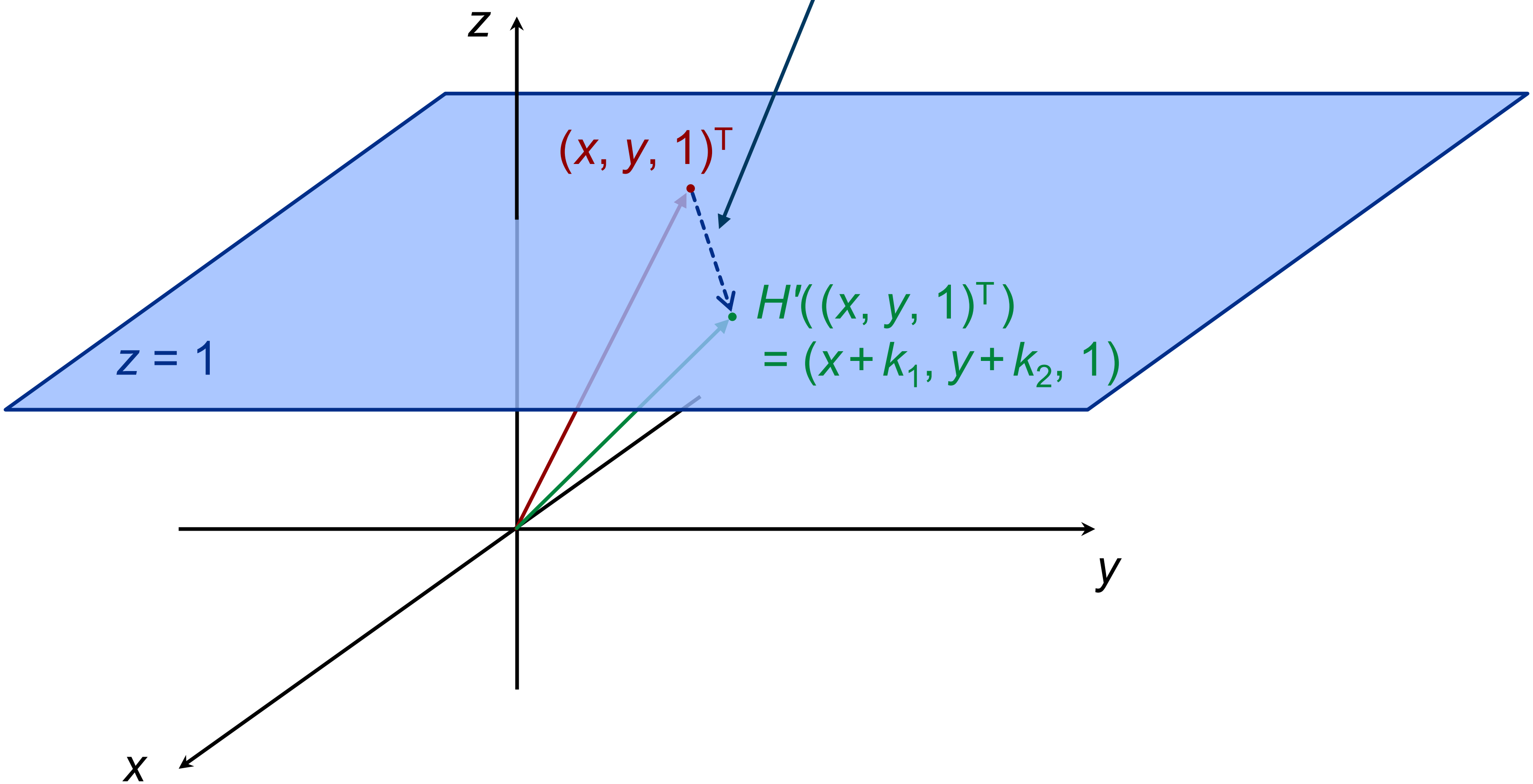
$$H'\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + k_1 z \\ y + k_2 z \\ z \end{bmatrix} \quad \text{for} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3.$$

H' is a linear transformation

and the standard matrix for H' is $\begin{bmatrix} 1 & 0 & k_1 \\ 0 & 1 & k_2 \\ 0 & 0 & 1 \end{bmatrix}$.

Shears in $\mathbb{R}^3$ (Example 7.3.12.2)

Every point on the plane $z = 1$ is translated by k_1 in the x-direction and by k_2 in the y-direction.



2D computer graphics (Discussion 7.3.13)

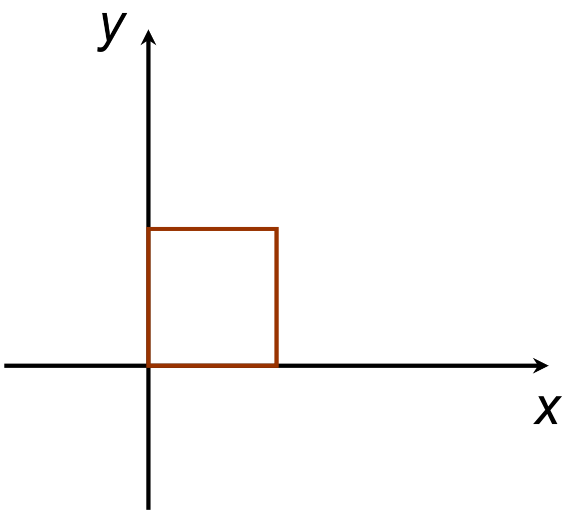
In 2D (dimension two) computer graphic, a figure is drawn by connecting a set of points by lines.

If a figure is drawn by connecting n points, we can store it by a $2 \times n$ matrix.

For example, the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

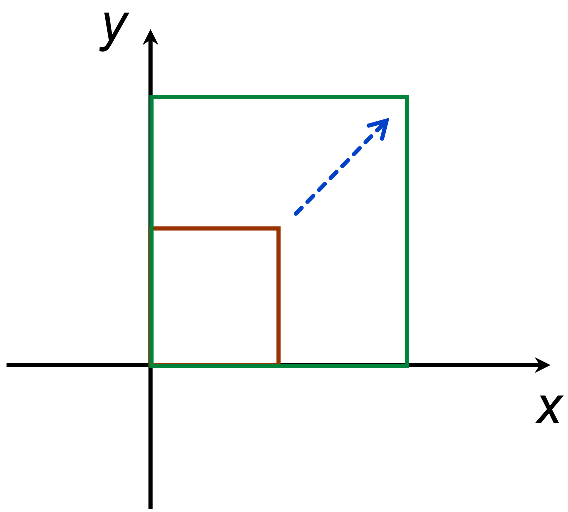
gives us the square with vertices $(1, 1)^T$, $(0, 1)^T$, $(0, 0)^T$, $(1, 0)^T$.



We can transform a figure by changing the positions of the vertices and then redrawing the figure.

2D computer graphics (Discussion 7.3.13)

If the transformation is linear, it can be carried out by pre-multiplying the standard matrix for the transformation to the matrix representing the figure.



For example, if we want to double both the width and the height of the square in the previous slide,

we only need to pre-multiply $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

to $\boldsymbol{A}$:

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 2 & 2 \\ 2 & 2 & 0 & 0 & 2 \end{bmatrix}.$$

2D computer graphics (Discussion 7.3.13)

There are four primary geometric transformations that are used in 2D computer graphics.

- scalings,
- reflections,
- rotations,
- translations.

We know that scalings, reflections and rotations are linear transformations

but translations are not.

The problem can be solved by using the homogeneous coordinates.

2D computer graphics (Discussion 7.3.13)

The homogeneous coordinates is formed by equaling each vector in $\mathbb{R}^2$ with a vector in $\mathbb{R}^3$ having the same first two coordinates and having 1 as its third coordinate.

For example, the matrix $\boldsymbol{A}$ representing the square with vertices $(1, 1)^T$, $(0, 1)^T$, $(0, 0)^T$, $(1, 0)^T$ becomes

$$\boldsymbol{A}' = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

If we want to draw the figure, we simply ignore the third coordinate.

2D computer graphics (Discussion 7.3.13)

Suppose $\boldsymbol{P}$ is the standard matrix for a geometric linear transformation on $\mathbb{R}^2$ such as a scaling, a reflection or a rotation.

The matrix $\boldsymbol{P}' = \begin{bmatrix} \multicolumn{2}{c}{\multirow{2}{*}{$\boldsymbol{P}$}} & 0 \\ & & 0 \\ 0 & 0 & 1 \end{bmatrix}$ will transform $\boldsymbol{A}'$ accordingly.

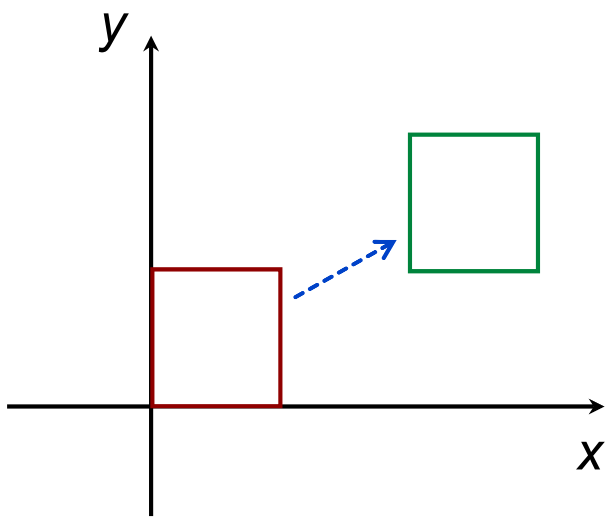
For example, to double both the width and the height of the square,

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 2 & 2 \\ 2 & 2 & 0 & 0 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

2D computer graphics (Discussion 7.3.13)

To do a translation, we need to use a shear defined in $\mathbb{R}^3$.

For example, to translate the square by a distance of 2 in the x-direction and by a distance of 1 in the y-direction, the shear with the standard matrix



$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$ will do the job:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 2 & 3 & 3 \\ 2 & 2 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$