Student Number: | | | | | | | | | |

Seat Number: | | | |

National University of Singapore

MA1101R Linear Algebra I

Semester I (2019 – 2020)

Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. Write down your student number and seat number clearly in the space provided at the top of this page. Do not write your name.
2. This booklet (and only this booklet) will be collected at the end of the examination.
3. This examination paper contains **SIX (6)** questions and comprises **FIFTEEN (15)** printed pages.
4. Answer **ALL** questions.
5. This is a **CLOSED BOOK** (with helpsheet) examination.
6. You are allowed to use one A4-size helpsheets.
7. You may use scientific calculators. However, you should lay out systematically the various steps in the calculations.

Examiner's Use Only	
Questions	**Marks**
1	
2	
3	
4	
5	
6	
Total	

Question 1 [10 marks]

Let $\boldsymbol{A} = \begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 0 & 2 & -2 & 0 \\ -1 & 1 & 1 & -1 & 1 \\ 0 & 0 & -1 & 1 & 0 \end{pmatrix}$ with reduced row echelon form $\boldsymbol{R} = \begin{pmatrix} 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

(i) Use $\boldsymbol{R}$ to find a basis for the column space V of $\boldsymbol{A}$.

(ii) Let $\boldsymbol{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\boldsymbol{u}_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$, $\boldsymbol{u}_3 = \begin{pmatrix} -12 \\ 0 \\ 9 \\ 11 \\ 0 \end{pmatrix}$.

Show that $S = \{\boldsymbol{A}\boldsymbol{u}_1, \boldsymbol{A}\boldsymbol{u}_2, \boldsymbol{A}\boldsymbol{u}_3\}$ is an orthogonal basis for V.

(iii) Find the coordinate vector $[\boldsymbol{w}]_S$ of $\boldsymbol{w} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \in V$ with respect to the basis S in part (ii).

(iv) Is it possible to find a one-dimensional subspace of V that does not contain any column of $\boldsymbol{A}$? Justify your answer.

Show your working below.

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More working space for Question 1.

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Question 2 **[10 marks]**

Let $\boldsymbol{A} = \begin{pmatrix} 3 & -2 & 1 \\ 0 & 4 & 0 \\ 1 & 2 & 3 \end{pmatrix}$ and $\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, $\boldsymbol{v}_4 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, $\boldsymbol{v}_5 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

(i) Determine which of the five vectors $\boldsymbol{v}_1$ to $\boldsymbol{v}_5$ are eigenvectors of $\boldsymbol{A}$.

(ii) Write down all the eigenvalues of $\boldsymbol{A}$. Justify your answers.

(iii) Write down a basis for each of the eigenspaces of $\boldsymbol{A}$.

(iv) Find an invertible matrix $\boldsymbol{P}$ and a diagonal matrix $\boldsymbol{D}$ such that $\boldsymbol{A}^3 = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$.

(v) Is $\boldsymbol{A}\boldsymbol{A}^T$ orthogonally diagonalizable? Why?

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Question 3 **[10 marks]**

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

(i) Show that the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ is inconsistent.

(ii) Find the least squares solution of the system in (i).

(iii) Find the projection $\boldsymbol{p}$ of $\boldsymbol{b}$ onto the column space of $\boldsymbol{A}$.

(iv) Find the smallest possible value of $\|\boldsymbol{Av} - \boldsymbol{b}\|$ among all vectors $\boldsymbol{v} \in \mathbb{R}^3$.

(v) Note that the three columns of $\boldsymbol{A}$ form an orthogonal set. Extend this set to an orthogonal basis for $\mathbb{R}^4$.

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Question 4 **[10 marks]**

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$T\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right) = \boldsymbol{v}_1, \quad T\left(\begin{pmatrix}0\\1\\1\end{pmatrix}\right) = \boldsymbol{v}_2, \quad T\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right) = \boldsymbol{v}_3$$

where $\boldsymbol{v}_1, \boldsymbol{v}_2$ and $\boldsymbol{v}_3$ are non-zero vectors.

(i) Find $T\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right)$ and $T\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\right)$ as linear combinations of $\boldsymbol{v}_1, \boldsymbol{v}_2$ and $\boldsymbol{v}_3$.

(ii) Find the standard matrix $\boldsymbol{A}$ for T in terms of $\boldsymbol{v}_1, \boldsymbol{v}_2$ and $\boldsymbol{v}_3$.

(iii) Suppose $\boldsymbol{v}_1, \boldsymbol{v}_2$ and $\boldsymbol{v}_3$ are linearly independent. Show that $\ker(T) = \{\boldsymbol{0}\}$.

(iv) Suppose $T(\boldsymbol{v}_1) = 2\boldsymbol{v}_1$, $T(\boldsymbol{v}_2) = 3\boldsymbol{v}_2$, $T(\boldsymbol{v}_3) = 5\boldsymbol{v}_3$. Find $\boldsymbol{v}_1, \boldsymbol{v}_2$ and $\boldsymbol{v}_3$.

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Question 5 **[10 marks]**

Suppose $\boldsymbol{A}$ is a 3×5 matrix with row space given by $\text{span}\{(1, 2, 3, 4, 5)\}$.

(i) What are the rank and nullity of $\boldsymbol{A}$?

(ii) Write down the reduced row echelon form of $\boldsymbol{A}$.

(iii) Find a basis for the nullspace of $\boldsymbol{A}$.

(iv) Find the general solution of the non-homogeneous system $\boldsymbol{Ax} = \boldsymbol{b}$ where $\boldsymbol{b}$ is the first column of $\boldsymbol{A}$.

(v) Suppose the first column of $\boldsymbol{A}$ is $\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$. Do we have enough information to determine the matrix $\boldsymbol{A}$? Why?

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Question 6 **[10 marks]**

Prove the following statements.

(a) If $\boldsymbol{A}$ is an $n \times n$ matrix such that $\boldsymbol{A}^2 = \boldsymbol{I}$, then $\text{rank}(\boldsymbol{I} + \boldsymbol{A}) + \text{rank}(\boldsymbol{I} - \boldsymbol{A}) = n$.
(Hint: $\text{rank}(\boldsymbol{M} + \boldsymbol{N}) \leq \text{rank}(\boldsymbol{M}) + \text{rank}(\boldsymbol{N})$)

(b) There are no orthogonal matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ (of the same order) such that $\boldsymbol{A}^2 - \boldsymbol{B}^2 = \boldsymbol{AB}$.
(Hint: Prove by contradiction. Recall that the product of two orthogonal matrices is an orthogonal matrix.)

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