Answers/Solutions of Exercise 6 (Version: October 22, 2016)

1. (a) The characteristic equation is $(\lambda+1)(\lambda-3)=0$; eigenvalues are -1 and 3; $\{(0,1)^{\mathrm{T}}\}$ is a basis for E_{-1} and $\{(1,2)^{\mathrm{T}}\}$ is a basis for E_3.

   (b) The characteristic equation is $(\lambda-2)^2=0$; the eigenvalue is 2; $\{(1,1)^{\mathrm{T}}\}$ is a basis for E_2.

   (c) The characteristic equation is $\lambda^2-4=0$; eigenvalues are -2 and 2; $\{(-2,1)^{\mathrm{T}}\}$ is a basis for E_{-2} and $\{(2,1)^{\mathrm{T}}\}$ is a basis for E_2 .

   (d) The characteristic equation is $\lambda^2=0$; the eigenvalue is 0; $\{(1,0),(0,1)^{\mathrm{T}}\}$ is a basis for E_0.

   (e) The characteristic equation is $\lambda(\lambda-2)^2=0$; eigenvalues are 0 and 2; $\{(-1,1,0)^{\mathrm{T}}\}$ is a basis for E_0 and $\{(1,1,0)^{\mathrm{T}}\}$ is a basis for E_2.

   (f) The characteristic equation is $(\lambda-2)(\lambda^2-9)=0$; eigenvalues are 2, -3 and 3; $\{(0,0,1)^{\mathrm{T}}\}$ is a basis for E_2, $\{(-1,3,0)^{\mathrm{T}}\}$ is a basis for E_{-3} and $\{(1,3,0)^{\mathrm{T}}\}$ is a basis for E_3.

   (g) The characteristic equation is $(\lambda-1)^3=0$; the eigenvalue is 1; $\{(0,0,1)^{\mathrm{T}}\}$ is a basis for E_1.

   (h) The characteristic equation is $(\lambda+1)(\lambda-1)^2=0$; eigenvalues are -1 and 1; $\{(-1,-1,1)^{\mathrm{T}}\}$ is a basis for E_{-1} and $\{(1,2,0)^{\mathrm{T}},(1,0,2)^{\mathrm{T}}\}$ is a basis for E_1.

   (i) The characteristic equation is $(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)=0$; eigenvalues are 1,2,3 and 4; $\{(0,0,0,1)^{\mathrm{T}}\}$is a bais for E_1, $\{(0,0,1,1)^{\mathrm{T}}\}$ is a basis for E_2, $\{(0,2,4,3)^{\mathrm{T}}\}$ is a basis for E_3 and $\{(3,9,12,8)^{\mathrm{T}}\}$ is a basis for E_4.

   (j) The characteristic equation is $\lambda^4-2\lambda^2+1=0$; eigenvalues are -1 and 1; $\{(-1,0,1,0)^{\mathrm{T}},(0,-1,0,1)^{\mathrm{T}}\}$ is a basis for E_{-1} and $\{(1,0,1,0)^{\mathrm{T}},(0,1,0,1)^{\mathrm{T}}\}$ is a basis for E_1.

2. (a) $\det(\lambda\boldsymbol{I}-\boldsymbol{A})=\begin{vmatrix}\lambda-a & -b\\ -c & \lambda-d\end{vmatrix}=\lambda^2+(-a-d)\lambda+(ad-bc)$

   Hence $m=-a-d=-\mathrm{tr}(\boldsymbol{A})$ and $n=\det(\boldsymbol{A})$.

   (b) Direct verification shows that $\boldsymbol{A}^2+m\boldsymbol{A}+n\boldsymbol{I}=\boldsymbol{0}$.

3. See Question 5 of Tutorial 10.

4. (a) Let $\boldsymbol{x}$ be an eigenvector of $\boldsymbol{A}$ associated with λ, i.e. $\boldsymbol{A}\boldsymbol{x}=\lambda\boldsymbol{x}$ and $\boldsymbol{x}$ is a nonzero vector. Then

$$\boldsymbol{A}^2=\boldsymbol{A} \quad\Rightarrow\quad \boldsymbol{A}^2\boldsymbol{x}=\boldsymbol{A}\boldsymbol{x} \quad\Rightarrow\quad \lambda^2\boldsymbol{x}=\lambda\boldsymbol{x} \quad\Rightarrow\quad \lambda(\lambda-1)\boldsymbol{x}=\boldsymbol{0}$$

   Since $\boldsymbol{x}$ is nonzero, $\lambda=0$ or 1.

(b) Since $\boldsymbol{A}$ has 2 distinct eigenvalues, it is diagonalizable. Let $\boldsymbol{P} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an invertible matrix such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} ad & -ab \\ cd & -cb \end{pmatrix} \quad \text{where } ad-bc \neq 0.$$

We can simplify the expression to $\boldsymbol{A} = \begin{pmatrix} r & s \\ t & 1-r \end{pmatrix}$ where $st = r(1-r)$.

5. (a) Let $\boldsymbol{x}$ be a nonzero eigenvector of $\boldsymbol{A}$ associated with λ, i.e. $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$.

$\boldsymbol{A}^2 = \boldsymbol{0} \quad \Rightarrow \quad \boldsymbol{A}^2\boldsymbol{x} = \boldsymbol{0}\boldsymbol{x} \quad \Rightarrow \quad \boldsymbol{A}(\lambda\boldsymbol{x}) = \boldsymbol{0} \quad \Rightarrow \quad \lambda^2\boldsymbol{x} = \boldsymbol{0}$

Since $\boldsymbol{x}$ is nonzero, $\lambda = 0$.

(b) No. Suppose $\boldsymbol{A}$ is diagonalizable. Then there exists invertible $\boldsymbol{P}$ such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{0}$. Then $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{0}\boldsymbol{P}^{-1} = \boldsymbol{0}$, a contradiction.

(c) Consider the vector equation

$$a\boldsymbol{u} + b\boldsymbol{A}\boldsymbol{u} = \boldsymbol{0}. \tag{$*$}$$

Pre-multiplying $\boldsymbol{A}$ to both side of $(*)$, we have

$$\boldsymbol{A}(a\boldsymbol{u} + \boldsymbol{A}\boldsymbol{u}) = \boldsymbol{A}\boldsymbol{0} \quad \Rightarrow \quad a\boldsymbol{A}\boldsymbol{u} = \boldsymbol{0}. \quad (\because\ \boldsymbol{A}^2 = \boldsymbol{0}.)$$

As $\boldsymbol{A}\boldsymbol{u} \neq \boldsymbol{0}$, $a = 0$. Substituting $a = 0$ into $(*)$, we have $b\boldsymbol{A}\boldsymbol{u} = \boldsymbol{0}$ and hence $b = 0$. Since $(*)$ has only the trivial solution, $\boldsymbol{u}$ and $\boldsymbol{A}\boldsymbol{u}$ are linearly independent.

(d) Let $\boldsymbol{P} = \begin{pmatrix} \boldsymbol{u} & \boldsymbol{A}\boldsymbol{u} \end{pmatrix}$. By (c), $\boldsymbol{P}$ is invertible. Since

$$\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} \boldsymbol{A}\boldsymbol{u} & \boldsymbol{A}^2\boldsymbol{u} \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}\boldsymbol{u} & \boldsymbol{0} \end{pmatrix}$$

and

$$\boldsymbol{P}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0\boldsymbol{u} + \boldsymbol{A}\boldsymbol{u} & 0\boldsymbol{u} + 0\boldsymbol{A}\boldsymbol{u} \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}\boldsymbol{u} & \boldsymbol{0} \end{pmatrix},$$

$\boldsymbol{A}\boldsymbol{P} = \boldsymbol{P}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ which implies $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

6. (a) Since $\det(-\boldsymbol{I} - \boldsymbol{A}) = 0$, -1 is an eigenvalue of $\boldsymbol{A}$.

(b) $\{(1,1,0)^{\mathrm{T}}, (0,0,1)^{\mathrm{T}}\}$ is a basis for E_{-1} and hence $\dim(E_{-1}) = 2$.

(c) For example, $\boldsymbol{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

7. (a) Since $\det(2\boldsymbol{I} - \boldsymbol{A}) = 0$, 2 is an eigenvalue of $\boldsymbol{A}$.

   (b) $\{(1, 2, 0)^{\mathrm{T}}, (-3, 0, 1)^{\mathrm{T}}\}$ is a basis for the eigenspace associated with 2.

   (c) Let E_2 be the eigenspace of $\boldsymbol{A}$ associated with 2 and let E'_λ be the eigenspace of $\boldsymbol{B}$ associated with λ.

   Since E_2 and E'_λ are subspaces of $\mathbb{R}^3$ and have dimension 2, they are two planes in $\mathbb{R}^3$ that contain the origin. So $E_2 \cap E'_\lambda$ is either a line through the origin or a plane containing the origin. In both cases, we can find a nonzero vector $\boldsymbol{u} \in E_2 \cap E'_\lambda$, i.e. $\boldsymbol{Au} = 2\boldsymbol{u}$ and $\boldsymbol{Bu} = \lambda\boldsymbol{u}$, such that

   $$(\boldsymbol{A} + \boldsymbol{B})\boldsymbol{u} = \boldsymbol{Au} + \boldsymbol{Bu} = 2\boldsymbol{u} + \lambda\boldsymbol{u} = (2 + \lambda)\boldsymbol{u}.$$

   So $2 + \lambda$ is an eigenvalue of $\boldsymbol{A} + \boldsymbol{B}$.

8. Note that for $i = 1, 2, \ldots, n$, $\boldsymbol{A}^n\boldsymbol{u_i} = \boldsymbol{A}^{n-1}\boldsymbol{u_{i+1}} = \cdots = \boldsymbol{A}^i\boldsymbol{u_n} = \boldsymbol{0}$.

   Let $\boldsymbol{v} \in \mathbb{R}^n$ be an eigenvector of $\boldsymbol{A}$ associated with eigenvalue λ, i.e. $\boldsymbol{Av} = \lambda\boldsymbol{v}$. Since $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_n}\}$ is a basis for $\mathbb{R}^n$,

   $$\boldsymbol{v} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_n\boldsymbol{u_n}$$

   for some $c_1, c_2, \ldots, c_n \in \mathbb{R}$. Then

   $$\boldsymbol{A}^n\boldsymbol{v} = c_1\boldsymbol{A}^n\boldsymbol{u_1} + c_2\boldsymbol{A}^n\boldsymbol{u_2} + \cdots + c_n\boldsymbol{A}^n\boldsymbol{u_n} = \boldsymbol{0}.$$

   From the proof of Question 6.3(a), $\boldsymbol{A}^n\boldsymbol{v} = \lambda^n\boldsymbol{v}$. Since $\boldsymbol{v} \neq \boldsymbol{0}$, $\lambda = 0$. Hence we have shown that $\boldsymbol{A}$ has only one eigenvalue 0.

   As $\lambda = 0$, we get $\boldsymbol{Av} = \boldsymbol{0}$. Then

   $$\boldsymbol{0} = \boldsymbol{Av} = c_1\boldsymbol{Au_1} + c_2\boldsymbol{Au_2} + \cdots + c_n\boldsymbol{Au_n} = c_1\boldsymbol{u_2} + c_2\boldsymbol{u_3} + \cdots + c_{n-1}\boldsymbol{u_n}.$$

   Since $\boldsymbol{u_2}, \boldsymbol{u_3}, \ldots, \boldsymbol{u_n}$ are linearly independent, $c_1 = 0$, $c_2 = 0$, $\ldots$, $c_{n-1} = 0$, i.e. $\boldsymbol{v} = c_n\boldsymbol{u_n}$. Hence all eigenvectors of $\boldsymbol{A}$ are scalar multiples of $\boldsymbol{u_n}$.

9. (a) Diagonalizable. Let $\boldsymbol{P} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{AP} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$.

   (b) Not diagonalizable.

   (c) Diagonalizable. Let $\boldsymbol{P} = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{AP} = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$.

   (d) Diagonalizable. Let $\boldsymbol{P} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{AP} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

   (e) Not diagonalizable.

(f) Diagonalizable. Let $\boldsymbol{P} = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 3 & 3 \\ 1 & 0 & 0 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

(g) Not diagonalizable.

(h) Diagonalizable. Let $\boldsymbol{P} = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(i) Diagonalizable. Let $\boldsymbol{P} = \begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 9 \\ 0 & 1 & 4 & 12 \\ 1 & 1 & 3 & 8 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$.

(j) Diagonalizable. Let $\boldsymbol{P} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

10. (a) Eigenvalues are $-i$ and i.

Let $\boldsymbol{P} = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$.

(b) Eigenvalues are $2-i$ and $2+i$.

Let $\boldsymbol{P} = \begin{pmatrix} 1+i & 1-i \\ 2 & 2 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 2-i & 0 \\ 0 & 2+i \end{pmatrix}$.

(c) Eigenvalues are 0, $2-i$ and $2+i$.

Let $\boldsymbol{P} = \begin{pmatrix} 1 & 1+3i & 1-3i \\ 0 & 5i & -5i \\ 0 & 5 & 5 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2-i & 0 \\ 0 & 0 & 2+i \end{pmatrix}$.

11. (a) Let $\boldsymbol{P} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

(b) $\boldsymbol{A}^{10} = \begin{pmatrix} 1 & 0 & 4^{10}-1 \\ 0 & 4^{10} & 0 \\ 0 & 0 & 4^{10} \end{pmatrix}$

(c) For example, let $\boldsymbol{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and $\boldsymbol{B} = \boldsymbol{P}\boldsymbol{C}\boldsymbol{P}^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Then $\boldsymbol{B}^2 = \boldsymbol{A}$.

12. Let $\boldsymbol{P} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$ and $\boldsymbol{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Then the matrix $\boldsymbol{PDP}^{-1} =$

$\begin{pmatrix} -1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ has the required eigenvalues and eigenvectors.

13. The matrix is diagonalizable if and only if $a \neq b$.

14. (a) The eigenvalues are 2, 0, 1 and -1.

    (b) $\boldsymbol{u_1}$ is an eigenvector associated with 2.
    $\boldsymbol{u_2}$ is an eigenvector associated with 0.
    $\boldsymbol{u_3} + \boldsymbol{u_4}$ is an eigenvector associated with 1.
    $\boldsymbol{u_3} - \boldsymbol{u_4}$ is an eigenvector associated with -1.

    (c) Note that $\boldsymbol{u_1}$, $\boldsymbol{u_2}$, $\boldsymbol{u_3}$, $\boldsymbol{u_3} + \boldsymbol{u_4}$, $\boldsymbol{u_3} + \boldsymbol{u_4}$ are linearly independent eigenvectors. By Theorem 6.2.3, $\boldsymbol{B}$ is diagonalizable.

    Alternative Solution: Since $\boldsymbol{B}$ has 4 distinct eigenvalues, by Theorem 6.2.7, $\boldsymbol{B}$ is diagonalizable.

15. (a) (i) $\boldsymbol{B} = \boldsymbol{P}^{-1}\boldsymbol{AP} \Rightarrow \boldsymbol{B}^n = \underbrace{(\boldsymbol{P}^{-1}\boldsymbol{AP})(\boldsymbol{P}^{-1}\boldsymbol{AP})\cdots(\boldsymbol{P}^{-1}\boldsymbol{AP})}_{n \text{ times}} = \boldsymbol{P}^{-1}\boldsymbol{A}^n\boldsymbol{P}$

    So $\boldsymbol{A}^n$ is similar to $\boldsymbol{B}^n$.

    (ii) $\boldsymbol{B} = \boldsymbol{P}^{-1}\boldsymbol{AP} \Rightarrow \boldsymbol{B}^{-1} = (\boldsymbol{P}^{-1}\boldsymbol{AP})^{-1} = \boldsymbol{P}^{-1}\boldsymbol{A}^{-1}\boldsymbol{P}$
    So $\boldsymbol{A}^{-1}$ is similar to $\boldsymbol{B}^{-1}$.

    (iii) Suppose there exists an invertible matrix $\boldsymbol{Q}$ such that $\boldsymbol{Q}^{-1}\boldsymbol{AQ}$ is a diagonal matrix. Let $\boldsymbol{R} = \boldsymbol{P}^{-1}\boldsymbol{Q}$. Then $\boldsymbol{R}$ is invertible and $\boldsymbol{R}^{-1}\boldsymbol{BR} = \boldsymbol{Q}^{-1}\boldsymbol{PBP}^{-1}\boldsymbol{Q} = \boldsymbol{Q}^{-1}\boldsymbol{AQ}$ is a diagonal matrix.

    (b) Since $\boldsymbol{A}$ is a triangular matrix, its eigenvalues are 0, 1 and -1. Also it is easy to find from the characteristic equation of $\boldsymbol{B}$ that the eigenvalues of $\boldsymbol{B}$ are 0, 1 and -1. By Theorem 6.2.7, both $\boldsymbol{A}$ and $\boldsymbol{B}$ are diagonalizable. So there exist invertible matrices $\boldsymbol{R}$ and $\boldsymbol{Q}$ such that

$$\boldsymbol{R}^{-1}\boldsymbol{AR} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \boldsymbol{Q}^{-1}\boldsymbol{BQ}.$$

    Let $\boldsymbol{P} = \boldsymbol{RQ}^{-1}$. Then $\boldsymbol{P}$ is invertible matrix and $\boldsymbol{P}^{-1}\boldsymbol{AP} = \boldsymbol{QR}^{-1}\boldsymbol{ARQ}^{-1} = \boldsymbol{B}$.

16. (a) Let $\boldsymbol{A} = (a_{ij})_{n\times n}$. Then $a_{1i} + a_{2i} + \cdots + a_{ni} = 1$ for $i = 1, 2, \ldots, n$.

(i) $$\boldsymbol{A}^{\mathrm{T}} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} a_{11} + a_{21} + \cdots + a_{n1} \\ a_{12} + a_{22} + \cdots + a_{n2} \\ \vdots \\ a_{1n} + a_{2n} + \cdots + a_{nn} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Thus 1 is an eigenvalue of $\boldsymbol{A}^{\mathrm{T}}$. By Question 6.3(c), 1 is an eigenvalue of $\boldsymbol{A}$.

(ii) By Question 6.3(c), λ is an eigenvalue of $\boldsymbol{A}^{\mathrm{T}}$.

Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ be a eigenvector of $\boldsymbol{A}^{\mathrm{T}}$ associated with the eigenvalue λ, i.e. $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{x} = \lambda\boldsymbol{x}$. Choose $k \in \{1, 2, \ldots, n\}$ such that $|x_k| = \max\{\, |x_i| \mid i = 1, 2, \ldots, n\,\}$, i.e. $|x_k| \geq |x_i|$ for $i = 1, 2, \ldots, n$. Since $\boldsymbol{x}$ is a nonzero vector, $|x_k| > 0$.

By comparing the kth coordinate of both sides of $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{x} = \lambda\boldsymbol{x}$, we have

$$\begin{aligned} & a_{1k}x_1 + a_{2k}x_2 + \cdots + a_{nk}x_n = \lambda x_k \\ \Rightarrow\ & |\lambda|\,|x_k| = |a_{1k}x_1 + a_{2k}x_2 + \cdots + a_{nk}x_n| \\ & \qquad \leq |a_{1k}x_1| + |a_{2k}x_2| + \cdots + |a_{nk}x_n| \\ & \qquad \leq a_{1k}|x_1| + a_{2k}|x_2| + \cdots + a_{nk}|x_n| \quad (\because a_{ij} \geq 0 \text{ for all } i, j) \\ & \qquad \leq (a_{1k} + a_{2k} + \cdots + a_{nk})|x_k| \\ & \qquad = |x_k| \\ \Rightarrow\ & |\lambda| \leq 1. \end{aligned}$$

(b) (i) Yes.

(ii) Let $\boldsymbol{P} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.95 & 0 \\ 0 & 0 & 0.9 \end{pmatrix}$.

17. Let a_n (respectively, b_n) be the number of customers who pay late (respectively, early) in month n. Then for $n = 1, 2, \ldots$,

$$\begin{cases} a_n = \frac{1}{2}a_{n-1} + \frac{2}{10}b_{n-1} \\ b_n = \frac{1}{2}a_{n-1} + \frac{8}{10}b_{n-1}. \end{cases}$$

Let $\boldsymbol{x}_n = \begin{pmatrix} a_n \\ b_n \end{pmatrix}$ and $\boldsymbol{A} = \begin{pmatrix} \frac{1}{2} & \frac{1}{5} \\ \frac{1}{2} & \frac{4}{5} \end{pmatrix}$. Then $\boldsymbol{x_n} = \boldsymbol{A}\boldsymbol{x}_{n-1} = \cdots = \boldsymbol{A}^{n-1}\boldsymbol{x_1}$ where $\boldsymbol{x_1} = \begin{pmatrix} 0 \\ 10000 \end{pmatrix}$.

By Algorithm 6.2.4, we find a matrix $\boldsymbol{P} = \begin{pmatrix} 2 & 1 \\ 5 & -1 \end{pmatrix}$ such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0.3 \end{pmatrix}$. Then

$$\boldsymbol{x_n} = \boldsymbol{P} \begin{pmatrix} 1 & 0 \\ 0 & 0.3^{n-1} \end{pmatrix} \boldsymbol{P}^{-1}\boldsymbol{x_1} = \frac{10000}{7} \begin{pmatrix} 2 - 2(0.3)^{n-1} \\ 5 + 2(0.3)^{n-1} \end{pmatrix}.$$

So the number of customers that will pay on time in April is $b_4 = \frac{10000}{7}[5 + 2(0.3)^3] = 7220$.

The number of customers that will pay on time will stabilize in the long run and $\lim_{n\to\infty} b_n = \frac{50000}{7} \approx 7143$.

18. Let a_n, b_n and c_n be the percentage of customers choosing brand A, B and C, respectively, after n months. Then for $n = 1, 2, ...,$

$$\begin{cases} a_n = 0.97a_{n-1} + 0.01b_{n-1} + 0.02c_{n-1} \\ b_n = 0.01a_{n-1} + 0.97b_{n-1} + 0.02c_{n-1} \\ c_n = 0.02a_{n-1} + 0.02b_{n-1} + 0.96c_{n-1}. \end{cases}$$

Let $\boldsymbol{x_n} = \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}$ and $\boldsymbol{A} = \begin{pmatrix} 0.97 & 0.01 & 0.02 \\ 0.01 & 0.97 & 0.02 \\ 0.02 & 0.02 & 0.96 \end{pmatrix}$.

Then $\boldsymbol{x_n} = \boldsymbol{A}\boldsymbol{x}_{n-1} = \cdots = \boldsymbol{A}^n\boldsymbol{x_0}$ where $\boldsymbol{x_0} = \begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix}$.

By Algorithm 6.2.4, we find $\boldsymbol{P} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & 0 & 2 \end{pmatrix}$ such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.96 & 0 \\ 0 & 0 & 0.94 \end{pmatrix}$.

Then

$$\boldsymbol{x_n} = \boldsymbol{P} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.96^n & 0 \\ 0 & 0 & 0.94^n \end{pmatrix} \boldsymbol{P}^{-1}\boldsymbol{x_0} = \frac{50}{3} \begin{pmatrix} 2 + 3 \cdot 0.96^n + 0.94^n \\ 2 - 3 \cdot 0.96^n + 0.94^n \\ 2 - 2 \cdot 0.94^n \end{pmatrix}.$$

The present market shares are $\frac{50}{3}[2 + 3 \cdot 0.96^4 + 0.94^4]\% \approx 88.8\%$, $\frac{50}{3}[2 - 3 \cdot 0.96^4 + 0.94^4]\% \approx 3.9\%$ and $\frac{50}{3}[2 - 2 \cdot 0.94^4]\% \approx 7.3\%$ for brand A, B and C, respectively.

The market shares will stabilize after a long run and $\lim_{n\to\infty} \boldsymbol{x_n} = \begin{pmatrix} \frac{100}{3} \\ \frac{100}{3} \\ \frac{100}{3} \end{pmatrix}$.

19. Note that $e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$ for $x \in \mathbb{R}$.

(a) Since $\boldsymbol{A}^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{pmatrix}$ for $n = 1, 2, \ldots,$

$$e^{\boldsymbol{A}} = \begin{pmatrix} 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots & 0 & 0 \\ 0 & 1 + \frac{1}{1!}2 + \frac{1}{2!}2^2 + \cdots & 0 \\ 0 & 0 & 1 + \frac{1}{1!}3 + \frac{1}{2!}3^2 + \cdots \end{pmatrix} = \begin{pmatrix} e & 0 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & e^3 \end{pmatrix}.$$

(b) Let $\boldsymbol{P} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$. Since $\boldsymbol{A}^n = \boldsymbol{P}\begin{pmatrix} 2^n & 0 \\ 0 & 4^n \end{pmatrix}\boldsymbol{P}^{-1}$ for $n = 1, 2, \ldots,$

$$e^{\boldsymbol{A}} = \boldsymbol{P}\begin{pmatrix} 1 + \frac{1}{1!}2 + \frac{1}{2!}2^2 + \cdots & 0 \\ 0 & 1 + \frac{1}{1!}4 + \frac{1}{2!}4^2 + \cdots \end{pmatrix}\boldsymbol{P}^{-1} = \frac{1}{2}\begin{pmatrix} e^4 + e^2 & e^4 - e^2 \\ e^4 - e^2 & e^4 + e^2 \end{pmatrix}.$$

(c) Let $\boldsymbol{P} = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Since $\boldsymbol{A}^n = \boldsymbol{P}\begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\boldsymbol{P}^{-1}$ for $n = 1, 2, \ldots,$

$$\begin{aligned} e^{\boldsymbol{A}} &= \boldsymbol{P}\begin{pmatrix} 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots & 0 & 0 \\ 0 & 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots & 0 \\ 0 & 0 & 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots \end{pmatrix}\boldsymbol{P}^{-1} \\ &= \begin{pmatrix} e^{-1} & \frac{1}{2}(e - e^{-1}) & \frac{1}{2}(e - e^{-1}) \\ -e + e^{-1} & \frac{1}{2}(3e - e^{-1} & \frac{1}{2}(e - e^{-1}) \\ e - e^{-1} & \frac{1}{2}(-e + e^{-1}) & \frac{1}{2}(e + e^{-1}) \end{pmatrix}. \end{aligned}$$

20. In the following, we use the procedure discussed in Example 6.2.11.2.

(a) Let $\boldsymbol{x_n} = \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}$ and $\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$. Then $\boldsymbol{x_n} = \boldsymbol{A}\boldsymbol{x_{n-1}} = \cdots = \boldsymbol{A}^n\boldsymbol{x_0}$.

Let $\boldsymbol{P} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. Thus

$$\begin{aligned} \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix} = \boldsymbol{x_n} &= \boldsymbol{P}\begin{pmatrix} 2^n & 0 \\ 0 & 1 \end{pmatrix}\boldsymbol{P}^{-1}\boldsymbol{x_0} \\ &= \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2^n & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2^n - 1 \\ 2^{n+1} - 1 \end{pmatrix}. \end{aligned}$$

Thus $a_n = 2^n - 1$.

(b) Let $\boldsymbol{x_n} = \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}$ and $\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$. Then $\boldsymbol{x_n} = \boldsymbol{A}\boldsymbol{x_{n-1}} = \cdots = \boldsymbol{A}^n\boldsymbol{x_0}$.

Let $\boldsymbol{P} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$. Thus

$$\begin{aligned}\begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix} &= \boldsymbol{x_n} = \boldsymbol{P}\begin{pmatrix} 2^n & 0 \\ 0 & (-1)^n \end{pmatrix}\boldsymbol{P}^{-1}\boldsymbol{x_0} \\ &= \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2^n & 0 \\ 0 & (-1)^n \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}[2^n + 2(-1)^n] \\ \frac{1}{3}[2^{n+1} - 2(-1)^n] \end{pmatrix}.\end{aligned}$$

Thus $a_n = \frac{1}{3}[2^n + 2(-1)^n]$.

21. Use cofactor expansion along the first row:

$$d_n = \begin{vmatrix} 3 & 1 & & & 0 & \\ 1 & 3 & 1 & & & \\ & 1 & 3 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 3 & 1 \\ & 0 & & & 1 & 3 \end{vmatrix}_{n\times n}$$

$$= 3\begin{vmatrix} 3 & 1 & & 0 & \\ 1 & 3 & \ddots & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & 3 & 1 \\ & 0 & & 1 & 3 \end{vmatrix}_{(n-1)\times(n-1)} - \begin{vmatrix} 1 & 1 & & 0 & \\ 0 & 3 & \ddots & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & 3 & 1 \\ & 0 & & 1 & 3 \end{vmatrix}_{(n-1)\times(n-1)}.$$

The first determinant above is d_{n-1}. By using cofactor expansion along the first column, we find that the second determinant is d_{n-2}. So

$$d_n = 3d_{n-1} - d_{n-2}.$$

Note that $d_1 = 3$ and $d_2 = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} = 8$.

By the procedure discussed in Example 6.2.11.2, we obtain

$$d_n = \left(\frac{5+3\sqrt{5}}{10}\right)\left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{5-3\sqrt{5}}{10}\right)\left(\frac{3-\sqrt{5}}{2}\right)^n.$$

22. Consider the vector equation

$$a_1\boldsymbol{u_1} + a_2\boldsymbol{u_2} + \cdots + a_m\boldsymbol{u_m} + b_1\boldsymbol{v_1} + b_2\boldsymbol{v_2} + \cdots + b_p\boldsymbol{v_p} = \boldsymbol{0}. \tag{1}$$

Pre-multiplying $\boldsymbol{A}$ to both side of (1), we have

$$a_1\lambda_1\boldsymbol{u_1} + a_2\lambda_2\boldsymbol{u_2} + \cdots + a_m\lambda_m\boldsymbol{u_m} + b_1\mu\boldsymbol{v_1} + b_2\mu\boldsymbol{v_2} + \cdots + b_p\mu\boldsymbol{v_p} = \boldsymbol{0}. \tag{2}$$

Subtracting (2) by μ times of (1), we obtain

$$a_1(\lambda_1 - \mu)\boldsymbol{u_1} + a_2(\lambda_2 - \mu)\boldsymbol{u_2} + \cdots + a_m(\lambda_m - \mu)\boldsymbol{u_m} = \boldsymbol{0}.$$

Since $\boldsymbol{u_1}$, $\boldsymbol{u_2}$, $\ldots$, $\boldsymbol{u_m}$ are linearly independent, $a_1(\lambda_1 - \mu) = 0$, $a_2(\lambda_2 - \mu) = 0$, $\ldots$, $a_m(\lambda_m - \mu) = 0$. As $\lambda_i \neq \mu$ for $i = 1, 2, \ldots, m$, we have $a_1 = 0$, $a_2 = 0$, $\ldots$, $a_m = 0$.

Substituting $a_1 = 0$, $a_2 = 0$, $\ldots$, $a_m = 0$ into (2), we have

$$b_1\boldsymbol{v_1} + b_2\boldsymbol{v_2} + \cdots + b_p\boldsymbol{v_p} = \boldsymbol{0}.$$

Since $\boldsymbol{v_1}$, $\boldsymbol{v_2}$, $\ldots$, $\boldsymbol{v_p}$ are linearly independent, $b_1 = 0$, $b_2 = 0$, $\ldots$, $b_p = 0$.

We have shown that the vector equation (1) has only the trivial solution. Thus $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_m}, \boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_p}\}$ is linearly independent.

23. See Question 6 of Tutorial 10.

24. (a) Let $\boldsymbol{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$. Then $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{AP} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$.

(b) Let $\boldsymbol{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$. Then $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{AP} = \begin{pmatrix} -2 & 0 \\ 0 & 4 \end{pmatrix}$.

(c) Let $\boldsymbol{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. Then $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{AP} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2+\sqrt{2} & 0 \\ 0 & 0 & 2-\sqrt{2} \end{pmatrix}$.

(d) Let $\boldsymbol{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$. Then $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{AP} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

(e) Let $\boldsymbol{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}$. Then $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{AP} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$.

(f) Let $\boldsymbol{P} = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. Then $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$.

(g) Let $\boldsymbol{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$. Then $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

(h) Let $\boldsymbol{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{12}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{12}} & \frac{1}{2} \\ 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{12}} & \frac{1}{2} \\ 0 & 0 & \frac{3}{\sqrt{12}} & \frac{1}{2} \end{pmatrix}$. Then $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$.

25. (a) Since $(\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})^{\mathrm{T}} = \boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$, $\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$ is symmetric. Hence $\boldsymbol{I} - \boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$ is also symmetric and thus is orthogonally diagonalizable.

(b) When $\boldsymbol{u} = (1, -1, 1)^{\mathrm{T}}$, $\boldsymbol{I} - \boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$.

Let $\boldsymbol{P} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$. Then $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$.

26. By the given conditions, we have $\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A}$, $\boldsymbol{A}\boldsymbol{u} = \lambda\boldsymbol{u}$ and $\boldsymbol{A}\boldsymbol{v} = \mu\boldsymbol{v}$. We compute $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{u}$ in two ways:

$$\begin{aligned} \boldsymbol{v}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{u} &= \boldsymbol{v}^{\mathrm{T}}(\lambda\boldsymbol{u}) = \lambda\,\boldsymbol{v}^{\mathrm{T}}\boldsymbol{u} = \lambda(\boldsymbol{v}\cdot\boldsymbol{u}), \\ \boldsymbol{v}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{u} &= \boldsymbol{v}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{u} = (\boldsymbol{A}\boldsymbol{v})^{\mathrm{T}}\boldsymbol{u} = (\mu\boldsymbol{v})\boldsymbol{u} = \mu\,\boldsymbol{v}^{\mathrm{T}}\boldsymbol{u} = \mu(\boldsymbol{v}\cdot\boldsymbol{u}). \end{aligned}$$

Thus $\lambda(\boldsymbol{v}\cdot\boldsymbol{u}) = \mu(\boldsymbol{v}\cdot\boldsymbol{u})$ which implies $(\lambda-\mu)(\boldsymbol{v}\cdot\boldsymbol{u}) = 0$. Since $\lambda \neq \mu$, we have $\boldsymbol{v}\cdot\boldsymbol{u} = 0$.

27. Since

$$E_1 = \{(x, y, z)^{\mathrm{T}} \mid x + y - z = 0\} = \mathrm{span}\{(-1, 1, 0)^{\mathrm{T}},\ (1, 0, 1)^{\mathrm{T}}\},$$

$\{(-1, 1, 0)^{\mathrm{T}},\ (1, 0, 1)^{\mathrm{T}}\}$ is a basis for E_1.

Let $\boldsymbol{u}$ be an eigenvector associated with -1. Since $\boldsymbol{A}$ is symmetric, by Question 6.26, $\boldsymbol{u}$ is orthogonal to E_1, i.e. $\boldsymbol{u}$ is perpendicular to $x + y - z = 0$. Hence $\boldsymbol{u}$ is a scalar multiple of $(1, 1, -1)^{\mathrm{T}}$. This means

$$E_{-1} = \mathrm{span}\{(1, 1, -1)^{\mathrm{T}}\}$$

and $\{(1,1,-1)^{\mathrm{T}}\}$ is a basis for E_{-1}.

Let $\boldsymbol{P} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Hence

$$\boldsymbol{A} = \boldsymbol{P}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\boldsymbol{P}^{-1} = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

28. Suppose the eigenvalues associated with the eigenspaces span$\{(1,0,1,0)^{\mathrm{T}}, (1,1,1,1)^{\mathrm{T}}\}$ and span$\{(1,1,-1,-1)^{\mathrm{T}}, (1,-1,-1,1)^{\mathrm{T}}\}$ are λ and μ respectively.

Let $\boldsymbol{P} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 0 & 1 & -1 & 1 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{P}\boldsymbol{A} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix}$. So

$$\boldsymbol{A} = \boldsymbol{P}\begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix}\boldsymbol{P}^{-1} = \begin{pmatrix} \frac{1}{2}(\lambda+\mu) & 0 & \frac{1}{2}(\lambda-\mu) & 0 \\ 0 & \frac{1}{2}(\lambda+\mu) & 0 & \frac{1}{2}(\lambda-\mu) \\ \frac{1}{2}(\lambda-\mu) & 0 & \frac{1}{2}(\lambda+\mu) & 0 \\ 0 & \frac{1}{2}(\lambda-\mu) & 0 & \frac{1}{2}(\lambda+\mu) \end{pmatrix}$$

which is a symmetric matrix.

Alternative Solution: Since

$$\begin{aligned} (1,0,1,0)\cdot(1,1,-1,-1) &= 0, \\ (1,0,1,0)\cdot(1,-1,-1,1) &= 0, \\ (1,1,1,1)\cdot(1,1,-1,-1) &= 0, \\ (1,1,1,1)\cdot(1,-1,-1,1) &= 0, \end{aligned}$$

any vector from span$\{(1,0,1,0)^{\mathrm{T}}, (1,1,1,1)^{\mathrm{T}}\}$ is orthogonal to any vector from span$\{(1,1,-1,-1)^{\mathrm{T}}, (1,-1,-1,1)^{\mathrm{T}}\}$.

Take any orthonormal bases $\{\boldsymbol{u_1}, \boldsymbol{u_2}\}$ and $\{\boldsymbol{v_1}, \boldsymbol{v_2}\}$ for span$\{(1,0,1,0)^{\mathrm{T}}, (1,1,1,1)^{\mathrm{T}}\}$ and span$\{(1,1,-1,-1)^{\mathrm{T}}, (1,-1,-1,1)^{\mathrm{T}}\}$ respectively. By the observation above, $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{v_1}, \boldsymbol{v_2}\}$ is orthonormal. Let $\boldsymbol{P} = \begin{pmatrix} \boldsymbol{u_1} & \boldsymbol{u_2} & \boldsymbol{v_1} & \boldsymbol{v_2} \end{pmatrix}$. Then $\boldsymbol{P}$ is an orthogonal matrix that diagonalizes $\boldsymbol{A}$. By Theorem 6.3.4, $\boldsymbol{A}$ is symmetric.

29. (a) Since $\boldsymbol{A}\boldsymbol{u} = 4\boldsymbol{u}$, $\boldsymbol{u}$ is an eigenvector of $\boldsymbol{A}$ associated with the eigenvalue 4.

(b) $\boldsymbol{v}\cdot\boldsymbol{u} = 0 \quad \Rightarrow \quad a+b+c+d = 0$.

Thus $\boldsymbol{A}\boldsymbol{v} = \boldsymbol{0} = 0\boldsymbol{v}$, $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{A}$ associated with the eigenvalue 0.

(c) Since $\boldsymbol{P}$ is an orthogonal matrix, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \cdot (a_i, b_i, c_i, d_i) = 0$ for $i = 1, 2, 3$. By (a), the first column of $\boldsymbol{P}$ is the eigenvector of $\boldsymbol{A}$ associated with the eigenvalue 4. By (b), the other four columns of $\boldsymbol{P}$ are eigenvectors of $\boldsymbol{A}$ associated with the eigenvalue 0. So

$$\boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

30. See Question 6 of Tutorial 10.

31. (a) (i) $Q_1(x, y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

(ii) Let $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} x \\ y \end{pmatrix}$. Then

$$Q_1(x, y) = 3x'^2 + 7y'^2 = \tfrac{3}{2}(x + y)^2 + \tfrac{7}{2}(x - y)^2.$$

(b) (i) $Q_2(x, y, z) = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 7 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

(ii) Let $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Then

$$\begin{aligned} Q_2(x, y, z) &= 3x'^2 + 6y'^2 + 9z'^2 \\ &= \tfrac{1}{3}(-x - 2y + 2z)^2 + \tfrac{2}{3}(2x + y + 2z)^2 + (-2x + 2y + z)^2. \end{aligned}$$

32. (a) (i) With $(x_1, x_2, x_3) = (1, 0, 0)$, we have $x_1^2 + x_2^2 + x_3^2 = 1$ and $\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 = \lambda_1$. So $\min\{\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 \mid x_1^2 + x_2^2 + x_3^2 = 1\} \leq \lambda_1$.

On the other hand, for any x_1, x_2, x_3 satisfying $x_1^2 + x_2^2 + x_3^2 = 1$,

$$\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 \geq \lambda_1 x_1^2 + \lambda_1 x_2^2 + \lambda_1 x_3^2 = \lambda_1(x_1^2 + x_2^2 + x_3^2) = \lambda_1.$$

So $\min\{\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 \mid x_1^2 + x_2^2 + x_3^2 = 1\} = \lambda_1$.

(ii) The proof is similar to Part (i) above.

(b) (i) Let $\boldsymbol{u} = (x_1, x_2, x_3)^{\mathrm{T}}$. Then $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{u} = x_1^2 + 2x_2^2 + 3x_3^2$ and $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u} = x_1^2 + x_2^2 + x_3^2$. Thus by (a), the minimum value is 1 and the maximum value is 3.

(ii) The eigenvalues of $\boldsymbol{Q}$ are $2-\sqrt{2}$, 2 and $2+\sqrt{2}$. There exists an orthogonal matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{P} = \begin{pmatrix} 2-\sqrt{2} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2+\sqrt{2} \end{pmatrix}$.

Let $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{u} = (x_1, x_2, x_3)^{\mathrm{T}}$. Then

$$\begin{aligned} \boldsymbol{u}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{u} &= \boldsymbol{u}^{\mathrm{T}}(\boldsymbol{P}\boldsymbol{P}^{\mathrm{T}})\boldsymbol{Q}(\boldsymbol{P}\boldsymbol{P}^{\mathrm{T}})\boldsymbol{u} \\ &= (\boldsymbol{P}^{\mathrm{T}}\boldsymbol{u})^{\mathrm{T}}(\boldsymbol{P}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{P})(\boldsymbol{P}^{\mathrm{T}}\boldsymbol{u}) \\ &= (2-\sqrt{2})x_1^2 + 2x_2^2 + (2+\sqrt{2})x_3^2 \end{aligned}$$

and

$$\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u} = \boldsymbol{u}^{\mathrm{T}}(\boldsymbol{P}\boldsymbol{P}^{\mathrm{T}})\boldsymbol{u} = (\boldsymbol{P}^{\mathrm{T}}\boldsymbol{u})^{\mathrm{T}}(\boldsymbol{P}^{\mathrm{T}}\boldsymbol{u}) = x_1^2 + x_2^2 + x_3^2.$$

Thus by (a), the minimum value is $2-\sqrt{2}$ and the maximum value is $2+\sqrt{2}$.

33. (a) The quadratic form is $\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. Then

$$x^2 + 2y^2 - 2x + 8y + 8 = 0 \quad \Leftrightarrow \quad (x-1)^2 + \frac{(y+2)^2}{1/2} = 1.$$

The conic is an ellipse.

(b) The quadratic form is $\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. Then

$$x^2 - 4x + 4y + 4 = 0 \quad \Leftrightarrow \quad (x-2)^2 = -4y.$$

The conic is a parabola.

(c) The quadratic form is $\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 2 & -2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

Let $\boldsymbol{P} = \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$. Then $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$. Define $\begin{pmatrix} x' \\ y' \end{pmatrix} = \boldsymbol{P}^{\mathrm{T}} \begin{pmatrix} x \\ y \end{pmatrix}$. Then

$$2x^2 - 4xy - y^2 + 8 = 0 \quad \Leftrightarrow \quad -\frac{x'^2}{8/3} + \frac{y'^2}{4} = 1.$$

The conic is a hyperbola.

(d) The quadratic form is $\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

Let $\boldsymbol{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$. Then $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}$. Define $\begin{pmatrix} x' \\ y' \end{pmatrix} = \boldsymbol{P}^{\mathrm{T}} \begin{pmatrix} x \\ y \end{pmatrix}$.
Then

$$x^2 + xy + y^2 = 6 \quad \Leftrightarrow \quad \frac{x'^2}{12} + \frac{y'^2}{4} = 1.$$

The conic is an ellipse.

(e) The quadratic form is $\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 11 & 12 \\ 12 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

Let $\boldsymbol{P} = \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix}$. Then $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 20 & 0 \\ 0 & -5 \end{pmatrix}$. Define $\begin{pmatrix} x' \\ y' \end{pmatrix} = \boldsymbol{P}^{\mathrm{T}} \begin{pmatrix} x \\ y \end{pmatrix}$.
Then

$$11x^2 + 24xy + 4y^2 - 15 = 0 \quad \Leftrightarrow \quad \frac{x'^2}{3/4} - \frac{y'^2}{3} = 1.$$

The conic is a hyperbola.

(f) The quadratic form is $\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

Let $\boldsymbol{P} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$. Then $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix}$. Define $\begin{pmatrix} x' \\ y' \end{pmatrix} = \boldsymbol{P}^{\mathrm{T}} \begin{pmatrix} x \\ y \end{pmatrix}$. Then

$$9x^2 - 4xy + 6y^2 - 10x - 20y - 5 = 0 \quad \Leftrightarrow \quad \frac{(x' - \sqrt{5})^2}{6} + \frac{y'^2}{3} = 1.$$

The conic is an ellipse.

(g) The quadratic form is $\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

Let $\boldsymbol{P} = \begin{pmatrix} -\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix}$. Then $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 0 & 0 \\ 0 & 10 \end{pmatrix}$. Define $\begin{pmatrix} x' \\ y' \end{pmatrix} = \boldsymbol{P}^{\mathrm{T}} \begin{pmatrix} x \\ y \end{pmatrix}$. Then

$$9x^2 + 6xy + y^2 - 10\sqrt{10}x + 10\sqrt{10}y + 90 = 0 \quad \Leftrightarrow \quad (y' - 1)^2 = -4(x' + 2).$$

The conic is a parabola.

34. Since $\boldsymbol{A}$ is a symmetric matrix with eigenvalues 1 and 4, there exists an orthogonal matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$. Define $\begin{pmatrix} x' \\ y' \end{pmatrix} = \boldsymbol{P}^{\mathrm{T}} \begin{pmatrix} x \\ y \end{pmatrix}$, i.e. $\begin{pmatrix} x \\ y \end{pmatrix} = \boldsymbol{P} \begin{pmatrix} x' \\ y' \end{pmatrix}$. Then

$$\begin{pmatrix} x & y \end{pmatrix} \boldsymbol{A} \begin{pmatrix} x \\ y \end{pmatrix} = 8 \quad \Leftrightarrow \quad \begin{pmatrix} x' & y' \end{pmatrix} \boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{P} \begin{pmatrix} x' \\ y' \end{pmatrix} = 8 \quad \Leftrightarrow \quad \frac{x'^2}{8} + \frac{y'^2}{2} = 1.$$

The conic is an ellipse.