Answers/Solutions of Questions 1-14 of Exercise 3 (Version: October 3, 2016)

1. $\boldsymbol{u} = (1,\ \sqrt{3})$, $\boldsymbol{v} = (-\sqrt{3},\ -1)$, $\boldsymbol{u}+\boldsymbol{v} = (1-\sqrt{3},\ -1+\sqrt{3})$,
   $3\boldsymbol{u} - 2\boldsymbol{v} = (3+2\sqrt{3},\ 2+3\sqrt{3})$.

2. See Question 1 of Tutorial 5.

3. $A = B = C = F$ and A, D, E are all different.

4. (a) U and V contains the origin but W does not.

   (b) $$\left\{\begin{array}{r} 2x - y + 3z = 0 \\ 3x + 2y - z = 0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} x = -\frac{5}{7}t \\ y = \frac{11}{7}t \\ z = t \end{array}\right. \quad \text{where } t \in \mathbb{R}$$

   So $U \cap V = \{\, (-\frac{5}{7}t,\ \frac{11}{7}t,\ t) \mid t \in \mathbb{R} \,\}$.

   $$\left\{\begin{array}{r} 3x + 2y - z = 0 \\ x - 3y - 2z = 1 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} x = \frac{1}{11}(2+7t) \\ y = \frac{1}{11}(-3-5t) \\ z = t \end{array}\right. \quad \text{where } t \in \mathbb{R}$$

   So $V \cap W = \{\, (\frac{2+7t}{11},\ \frac{-3-5t}{11},\ t) \mid t \in \mathbb{R} \,\}$.

5. (a) A is a line joining the points $(1, 1, 1)$ and $(2, 3, 4)$.

   (b) Let $B = \{(x, y, z) \mid x+y-z = 1 \text{ and } x-2y+z = 0\}$. Since $x+y-z = 1$ and $x+y-z = 1$ are two non-parallel lines, B is the line of intersection of the two planes. To show that $A = B$, it suffices to show that the line A lies on both planes. This is true because $(1+t)+(1+2t)-(1+3t) = 1$ and $(1+t)-2(1+2t)+(1+3t) = 0$ for all $t \in \mathbb{R}$.

   (c) For example, $\boldsymbol{M} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

6. Since

   $$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ a & b & c & d \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ b & c & d \end{vmatrix} - 0 + \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ a & b & d \end{vmatrix} - 0 = a + b - d - c,$$

   $V = \{(a, b, c, d) \mid a+b-d-c = 0\} = \{(x, y, z, w) \mid x+y-z-w = 0\} = T$.

   On the other hand, $S \neq T$ because $(1, -1, 0, 0) \in T$ but $(1, -1, 0, 0) \notin S$.

7. (a) For example, $P = \{(1+s-t,\ s, t) \mid s, t \in \mathbb{R}\}$.
   (b) A lies in P because $a - a + 1 = 1$. Since both B and C pass through $(0,0,0)$ and $(0,0,0) \notin P$, B and C does not lies in P.
   (c) B intersects P at one point, $(1,0,0)$.
   (d) The plane $x - y + z = 0$ contains C but not A and B.
   (e) No. By Discussion 1.4.11, the solution set of a consistent nonzero linear system in three variables represents a point, a line or a plane in $\mathbb{R}^3$. Suppose we have a nonzero linear system whose solution set contains both B and C. Then the solution set must be a plane. However, the plane containing both B and C is the xz-plane which does not contain A. So the solution set cannot contain A.

8. $(2,3,-7,3)$, $(0,0,0,0)$ and $(-4,6,-13,4)$ are vectors in $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ while $(1,1,1,1)$ is not.

9. S_4 and S_6 span $\mathbb{R}^3$ while S_1, S_2, S_3 and S_5 do not span $\mathbb{R}^3$.

10. (a) Since $(1,1,0)$ and $(5,2,3)$ satisfy the equation $x-y-z=0$, $(1,1,0), (5,2,3) \in V$ and hence $\text{span}(S) \subseteq V$.

    Note that a general solution of $x - y - z = 0$ is $x = s + t$, $y = s$, $z = t$ where $s, t \in \mathbb{R}$. Let $(s+t, s, t)$ be any vector in V. Consider the following equation:

$$a(1,1,0) + b(5,2,3) = (s+t, s, t) \quad \Leftrightarrow \quad \begin{cases} a + 5b = s + t \\ a + 2b = s \\ \quad\ \ 3b = t. \end{cases}$$

    Since

$$\left(\begin{array}{cc|c} 1 & 5 & s+t \\ 1 & 2 & s \\ 0 & 3 & t \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{cc|c} 1 & 5 & s+t \\ 0 & 3 & t \\ 0 & 0 & 0 \end{array}\right),$$

    the system is consistent for all $s, t \in \mathbb{R}$. So $V \subseteq \text{span}(S)$.

    We have shown that $\text{span}\{(1,1,0),\ (5,2,3)\} = V$.

    (b) Since

$$\begin{pmatrix} 1 & 5 & 0 \\ 1 & 2 & 0 \\ 0 & 3 & 1 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 1 & 5 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

    by Discussion 3.2.5, $\text{span}\{(1,1,0),\ (5,2,3),\ (0,0,1)\} = \mathbb{R}^3$.

11. (a) $\left(\begin{array}{cc|c} 1 & 0 & 2 \\ -1 & 1 & -2 \\ -5 & 1 & 0 \end{array}\right) \underset{\text{Elimination}}{\overset{\text{Gaussian}}{\longrightarrow}} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{array}\right)$

Since $\boldsymbol{u_2} \notin \text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}\}$, $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\} \neq \text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}\}$.

(b) $\left(\begin{array}{ccc|c|c} 1 & 2 & -1 & 1 & 0 \\ 6 & 4 & 2 & -2 & 8 \\ 4 & -1 & 5 & -5 & 9 \end{array}\right) \underset{\text{Elimination}}{\overset{\text{Gaussiann}}{\longrightarrow}} \left(\begin{array}{ccc|c|c} 1 & 2 & -1 & 1 & 0 \\ 0 & -8 & 8 & -8 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

The systems are consistent and thus $\text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}\} \subseteq \text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$.

$\left(\begin{array}{cc|c|c|c} 1 & 0 & 1 & 2 & -1 \\ -2 & 8 & 6 & 4 & 2 \\ -5 & 9 & 4 & -1 & 5 \end{array}\right) \underset{\text{Elimination}}{\overset{\text{Gaussian}}{\longrightarrow}} \left(\begin{array}{cc|c|c|c} 1 & 0 & 1 & 2 & -1 \\ 0 & 8 & 8 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

The systems are consistent and thus $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\} \subseteq \text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}\}$.

So $\text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}\} = \text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$.

12. (a) $\left(\begin{array}{cccc|c} -1 & 3 & 0 & -4 & 1 \\ 2 & 1 & 1 & 3 & 0 \\ 1 & 4 & 1 & -1 & 2 \\ 0 & 0 & 3 & 6 & 5 \end{array}\right) \underset{\text{Elimination}}{\overset{\text{Gaussian}}{\longrightarrow}} \left(\begin{array}{cccc|c} -1 & 3 & 0 & -4 & 1 \\ 0 & 7 & 1 & -5 & 2 \\ 0 & 0 & 3 & 6 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$

Since $\boldsymbol{u_2} \notin \text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\}$, $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\} \not\subseteq \text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\}$.

(b) $\left(\begin{array}{cccc|c|c|c|c} 2 & 1 & 0 & 1 & -1 & 3 & 0 & -4 \\ 0 & 0 & 3 & 1 & 2 & 1 & 1 & 3 \\ 2 & 2 & 6 & 2 & 1 & 4 & 1 & -1 \\ 0 & 5 & 9 & -1 & 0 & 0 & 3 & 6 \end{array}\right) \underset{\text{Elimination}}{\overset{\text{Gaussian}}{\longrightarrow}} \left(\begin{array}{cccc|c|c|c|c} 2 & 1 & 0 & 1 & -1 & 3 & 0 & -4 \\ 0 & 1 & 6 & 1 & 2 & 1 & 1 & 3 \\ 0 & 0 & 3 & 1 & 2 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & 4 & 2 & 5 & 12 \end{array}\right)$

The systems are consistent and thus $\text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\} \subseteq \text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\}$.

(c) $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\} = \mathbb{R}^4$.

(d) $\text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\} \neq \mathbb{R}^4$.

13. For S_1, S_2 and S_3, see Question 5 of Tutorial 5.

Both S_4 and S_5 span $\mathbb{R}^3$.

14. (a) True. Let $\boldsymbol{u} = (u)$ for $u \neq 0$. Then for any $(c) \in \mathbb{R}^1$, $(c) = \frac{c}{u}\boldsymbol{u}$.

(b) False. For example, let $\boldsymbol{u} = (1, 1)$, $\boldsymbol{v} = (2, 2)$.

(c) False. For example, let $S_1 = \{(1, 0), (0, 1)\}$, $S_2 = \{(1, 0), (0, 2)\}$.

(d) False. For example, let $S_1 = \{(1, 0)\}$, $S_2 = \{(0, 1)\}$.