Chapter 4

Vector Spaces Associated with Matrices

Section 4.1

Row Spaces and Column Spaces

Row spaces and column spaces (Definition 4.1.1)

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Let $\boldsymbol{A} = (a_{ij})$ be an $m \times n$ matrix.

Write $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \\ \vdots \\ \boldsymbol{r}_m \end{bmatrix}$ where $\boldsymbol{r}_i = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix}$

is the i^{th} row of $\boldsymbol{A}$.

The row space of $\boldsymbol{A}$ is the subspace of $\mathbb{R}^n$ spanned by the rows of $\boldsymbol{A}$,

i.e. the row space of $\boldsymbol{A}$ $= \text{span}\{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_m\} \subseteq \mathbb{R}^n$.

Row spaces and column spaces (Definition 4.1.1 & Remark 4.1.2)

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

On the other hand,

write $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{c}_1 & \boldsymbol{c}_2 & \cdots & \boldsymbol{c}_n \end{bmatrix}$

where $\boldsymbol{c}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$ is the j^{th} column of $\boldsymbol{A}$.

The column space of $\boldsymbol{A}$ is the subspace of $\mathbb{R}^m$ spanned by the column of $\boldsymbol{A}$,

i.e. the column space of $\boldsymbol{A}$ = span{ $\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_n$ } $\subseteq \mathbb{R}^m$.

Note that the column space of $\boldsymbol{A}$ = the row space of $\boldsymbol{A}^{\mathrm{T}}$

and the row space of $\boldsymbol{A}$ = the column space of $\boldsymbol{A}^{\mathrm{T}}$.

(In here, we identify the row vectors with the column vectors.)

Row and column vectors (Notation 4.1.5)

Recall that a vector in $\mathbb{R}^n$ can be identified as a matrix.

When a vector in $\mathbb{R}^n$ is written as $(u_1, u_2, ..., u_n)$, it is a row vector and is identified with a $1 \times n$ matrix

$$\begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix};$$

and if it is written as $(u_1, u_2, ..., u_n)^T$, it is a column vector and is identified with an $n \times 1$ matrix

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}.$$

An examples (Example 4.1.4.1)

Let $\boldsymbol{A} = \begin{bmatrix} 2 & -1 & 0 \\ 1 & -1 & 3 \\ -5 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Write $\boldsymbol{r}_1 = \begin{bmatrix} 2 & -1 & 0 \end{bmatrix}$, $\boldsymbol{r}_2 = \begin{bmatrix} 1 & -1 & 3 \end{bmatrix}$, $\boldsymbol{r}_3 = \begin{bmatrix} -5 & 1 & 0 \end{bmatrix}$, $\boldsymbol{r}_4 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$.

The row space of $\boldsymbol{A}$

$= \text{span}\{ \boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, \boldsymbol{r}_4 \}$

$= \{ a(2, -1, 0) + b(1, -1, 3) + c(-5, 1, 0) + d(1, 0, 1) \mid a, b, c, d \in \mathbb{R} \}$

$= \{ (2a + b - 5c + d,\ -a - b + c,\ 3b + d) \mid a, b, c, d \in \mathbb{R} \}$.

An examples (Example 4.1.4.1)

Write $\boldsymbol{c}_1 = \begin{bmatrix} 2 \\ 1 \\ -5 \\ 1 \end{bmatrix}$, $\boldsymbol{c}_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$, $\boldsymbol{c}_3 = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}$.

The column space of $\boldsymbol{A}$

$= \text{span}\{ \boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3 \}$

$= \{ a(2, 1, -5, 1)^T + b(-1, -1, 1, 0)^T + c(0, 3, 0, 1)^T \mid a, b, c \in \mathbb{R} \}$

$= \{ (2a - b, \; a - \mathrm{b} + 3c, \; -5a + b, \; a + c)^T \mid a, b, c, d \in \mathbb{R} \}$.

Row equivalent matrices (Discussion 4.1.6.1)

Recall that a matrix $\boldsymbol{B}$ is row equivalent to a matrix $\boldsymbol{A}$ if $\boldsymbol{B}$ can be obtain from $\boldsymbol{A}$ through a series of elementary operations.

Row equivalence is an equivalence relation on matrices:

(a) Any matrix is row equivalent to itself.

(b) If a matrix $\boldsymbol{B}$ is row equivalent to a matrix $\boldsymbol{A}$, then $\boldsymbol{A}$ is also row equivalent to $\boldsymbol{B}$.

(c) If a matrix $\boldsymbol{C}$ is row equivalent to a matrix $\boldsymbol{B}$ and $\boldsymbol{B}$ is row equivalent to another matrix $\boldsymbol{A}$, then $\boldsymbol{C}$ is also row equivalent to $\boldsymbol{A}$.

Row equivalent matrices (Discussion 4.1.6.2)

A matrix is row equivalent to its row-echelon form.

In particular, if two matrices have a same row-echelon form, then they are row equivalent.

Since every matrix has a unique reduced row-echelon form, two matrices are row equivalent if and only if they have the same reduced row-echelon form.

Row spaces (Theorem 4.1.7)

$$\begin{array}{ccc} \boxed{\text{matrix } \boldsymbol{A}} & \xleftrightarrow[\text{equivalent}]{\text{row}} & \boxed{\text{matrix } \boldsymbol{B}} \\ \downarrow & & \downarrow \\ \boxed{\text{the row space of } \boldsymbol{A}} & = & \boxed{\text{the row space of } \boldsymbol{B}} \end{array}$$

Proof:

Write $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \\ \vdots \\ \boldsymbol{r}_m \end{bmatrix}$ where $\boldsymbol{r}_i = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix}$ is the i^{th} row of $\boldsymbol{A}$.

We need to show that each of the three types of elementary operations preserve the row space of $\boldsymbol{A}$.

Row spaces (Theorem 4.1.7)

In here, we only prove that the first type of elementary operations preserve the row space of $\boldsymbol{A}$ (the other two are similar).

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \\ \vdots \\ \boldsymbol{r}_m \end{bmatrix} \xrightarrow{kR_i} \boldsymbol{B} = \begin{bmatrix} \boldsymbol{r}_1 \\ \vdots \\ \boldsymbol{r}_{i-1} \\ k\boldsymbol{r}_i \\ \boldsymbol{r}_{i+1} \\ \vdots \\ \boldsymbol{r}_m \end{bmatrix} \quad \text{(where } k \neq 0\text{)}$$

Then the row space of $\boldsymbol{A}$ = span$\{ \boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_m \}$ and the row space of $\boldsymbol{B}$ = span$\{ \boldsymbol{r}_1, \ldots, \boldsymbol{r}_{i-1}, k\boldsymbol{r}_i, \boldsymbol{r}_{i+1}, \ldots, \boldsymbol{r}_m \}$.

Row spaces (Theorem 4.1.7)

Theorem 3.2.10: span($\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$) $\subseteq$ span($\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$) if and only if each $\boldsymbol{u}_i$ is a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$.

Since $k\boldsymbol{r}_i \in \text{span}\{\, \boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_m \,\}$, (by Theorem 3.2.10)

$$\text{span}\{\, \boldsymbol{r}_1, \ldots, \boldsymbol{r}_{i-1}, k\boldsymbol{r}_i, \boldsymbol{r}_{i+1}, \ldots, \boldsymbol{r}_m \,\} \subseteq \text{span}\{\, \boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_m \,\}.$$

Since $\boldsymbol{r}_i = \frac{1}{k}(k\boldsymbol{r}_i) \in \text{span}\{\, \boldsymbol{r}_1, \ldots, \boldsymbol{r}_{i-1}, k\boldsymbol{r}_i, \boldsymbol{r}_{i+1}, \ldots, \boldsymbol{r}_m \,\}$,

(by Theorem 3.2.10)

$$\text{span}\{\, \boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_m \,\} \subseteq \text{span}\{\, \boldsymbol{r}_1, \ldots, \boldsymbol{r}_{i-1}, k\boldsymbol{r}_i, \boldsymbol{r}_{i+1}, \ldots, \boldsymbol{r}_m \,\}.$$

So we have shown that

$$\text{span}\{\, \boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_m \,\} = \text{span}\{\, \boldsymbol{r}_1, \ldots, \boldsymbol{r}_{i-1}, k\boldsymbol{r}_i, \boldsymbol{r}_{i+1}, \ldots, \boldsymbol{r}_m \,\}$$

i.e. the row space of $\boldsymbol{A}$ = the row space of $\boldsymbol{B}$.

An example (Example 4.1.8)

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 4 \\ \frac{1}{2} & 1 & 2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \xrightarrow{2R_1} \xrightarrow{R_1 - R_2} \boldsymbol{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

Since $\boldsymbol{A}$ and $\boldsymbol{D}$ are row equivalent,

the row space of $\boldsymbol{A}$ = the row space of $\boldsymbol{D}$,

i.e. span{ (0, 0, 1), (0, 2, 4), ($\frac{1}{2}$, 1, 2) }

= span{ (1, 0, 0), (0, 2, 4), (0, 0, 1) }.

Row-echelon forms (Remark 4.1.9 & Example 4.1.8.2)

Let $\boldsymbol{A}$ be a matrix and $\boldsymbol{R}$ a row-echelon form of $\boldsymbol{A}$.

Since the nonzero rows of $\boldsymbol{R}$ are linearly independent (see Question 3.26), the nonzero rows of $\boldsymbol{R}$ form a basis for the row space of $\boldsymbol{R}$ and hence form a basis for the row space of $\boldsymbol{A}$.

For example,

$$\boldsymbol{A} = \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{bmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \boldsymbol{R} = \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & \frac{3}{2} & -3 & \frac{3}{2} \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So $\{\, (2, 2, -1, 0, 1),\ (0, 0, \frac{3}{2}, -3, \frac{3}{2}),\ (0, 0, 0, 3, 0) \,\}$ is a basis for the row space of $\boldsymbol{A}$.

Column spaces (Discussion 4.1.10)

$$\boxed{\text{matrix } \boldsymbol{A}} \xleftrightarrow[\text{equivalent}]{\text{row}} \boxed{\text{matrix } \boldsymbol{B}}$$

Warning: In general,

the column space of $\boldsymbol{A} \neq$ the column space of $\boldsymbol{B}$.

For example,

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$\boldsymbol{A}$ and $\boldsymbol{B}$ are row equivalent

but the column space of $\boldsymbol{A} = \{ (0, y)^T \mid y \in \mathbb{R} \}$

while the column space of $\boldsymbol{B} = \{ (x, 0)^T \mid x \in \mathbb{R} \}$.

Column vectors (Theorem 4.1.11)

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_{i_1} & \cdots & \boldsymbol{a}_{i_2} & \cdots\cdots & \boldsymbol{a}_{i_k} & \cdots & \boldsymbol{a}_n \end{bmatrix}$$

row equivalent

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_{i_1} & \cdots & \boldsymbol{b}_{i_2} & \cdots\cdots & \boldsymbol{b}_{i_k} & \cdots & \boldsymbol{b}_n \end{bmatrix}$$

$\cdots$

$\cdots$

$S = \{ \boldsymbol{a}_{i_1}, \boldsymbol{a}_{i_2}, \ldots, \boldsymbol{a}_{i_k} \}$

$T = \{ \boldsymbol{b}_{i_1}, \boldsymbol{b}_{i_2}, \ldots, \boldsymbol{b}_{i_k} \}$

S is linearly independent. $\Leftrightarrow$ T is linearly independent.

S is a basis for the column space of $\boldsymbol{A}$. $\Leftrightarrow$ T is a basis for the column space of $\boldsymbol{B}$.

An example (Example 4.1.12.1)

$$A = \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{bmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} R = \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & \frac{3}{2} & -3 & \frac{3}{2} \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The three corresponding columns are linearly dependent. $\Leftarrow$ The three columns are linearly dependent.

Row-echelon forms (Remark 4.1.13 & Example 4.1.12.2)

Let $\boldsymbol{A}$ be a matrix and $\boldsymbol{R}$ a row-echelon form of $\boldsymbol{A}$.

A basis for the column space of $\boldsymbol{A}$ can be obtained by taking the columns of $\boldsymbol{A}$ that correspond to the pivot columns in $\boldsymbol{R}$.

$$\boldsymbol{A} = \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{bmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \boldsymbol{R} = \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & \frac{3}{2} & -3 & \frac{3}{2} \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The three corresponding columns form a basis for the column space of $\boldsymbol{A}$. $\Leftarrow$ The three pivot columns form a basis for the column space of $\boldsymbol{R}$.

Finding bases (Example 4.1.14.1)

Let $\boldsymbol{u}_1 = (1, 2, 2, 1)$, $\boldsymbol{u}_2 = (3, 6, 6, 3)$, $\boldsymbol{u}_3 = (4, 9, 9, 5)$, $\boldsymbol{u}_4 = (-2, -1, -1, 1)$, $\boldsymbol{u}_5 = (5, 8, 9, 4)$, $\boldsymbol{u}_6 = (4, 2, 7, 3)$.

Find a basis for $W = \text{span}\{ \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4, \boldsymbol{u}_5, \boldsymbol{u}_6 \}$.

Method 1:

Place the six vectors as row vectors to form a 6×4 matrix:

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 3 & 6 & 6 & 3 \\ 4 & 9 & 9 & 5 \\ -2 & -1 & -1 & 1 \\ 5 & 8 & 9 & 4 \\ 4 & 2 & 7 & 3 \end{bmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So

$\{ (1, 2, 2, 1), (0, 1, 1, 1), (0, 0, 1, 1) \}$ is a basis for W.

Finding bases (Example 4.1.14.1)

Method 2: Place the six vectors as column vectors to form a 4×6 matrix:

$$\begin{bmatrix} 1 & 3 & 4 & -2 & 5 & 4 \\ 2 & 6 & 9 & -1 & 8 & 2 \\ 2 & 6 & 9 & -1 & 9 & 7 \\ 1 & 3 & 5 & 1 & 4 & 3 \end{bmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{bmatrix} 1 & 3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns

So

$\{ (1, 2, 2, 1), (4, 9, 9, 5), (5, 8, 9, 4) \}$ is a basis for W.

Finding bases (Example 4.1.14.2)

Let $S = \{ (1, 4, -2, 5, 1), (2, 9, -1, 8, 2), (2, 9, -1, 9, 3) \}$.

S is linearly independent.

Extend S to a basis for $\mathbb{R}^5$.

Solution:

In the following, we present an algorithm that extends a linearly independent subset S of $\mathbb{R}^n$ to a basis for $\mathbb{R}^n$.

Finding bases (Example 4.1.14.2)

$$\boldsymbol{A} = \begin{bmatrix} 1 & 4 & -2 & 5 & 1 \\ 2 & 9 & -1 & 8 & 2 \\ 2 & 9 & -1 & 9 & 3 \end{bmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \boldsymbol{R} = \begin{bmatrix} 1 & 4 & -2 & 5 & 1 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

non-pivot columns

Step 1: Form a matrix $\boldsymbol{A}$ using the vectors in S as rows.

Step 2: Reduce $\boldsymbol{A}$ to a row-echelon form $\boldsymbol{R}$.

Step 3: Identify the non-pivot columns in $\boldsymbol{R}$.

For this example, the 3rd and 5th columns are non-pivot columns.

Finding bases (Example 4.1.14.2)

$$\boldsymbol{A} = \begin{bmatrix} 1 & 4 & -2 & 5 & 1 \\ 2 & 9 & -1 & 8 & 2 \\ 2 & 9 & -1 & 9 & 3 \end{bmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \boldsymbol{R} = \begin{bmatrix} 1 & 4 & -2 & 5 & 1 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

non-pivot columns

Step 4: For each non-pivot column, get a vector such that the leading entry of the vector is at that column.

For this example, we need two vectors of the form $(0, 0, x, *, *)$ and $(0, 0, 0, 0, y)$ where x and y are nonzero.

In particular, we take $(0, 0, 1, 0, 0)$ and $(0, 0, 0, 0, 1)$.

Finding bases (Example 4.1.14.2)

$$\boldsymbol{A} = \begin{bmatrix} 1 & 4 & -2 & 5 & 1 \\ 2 & 9 & -1 & 8 & 2 \\ 2 & 9 & -1 & 9 & 3 \end{bmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \boldsymbol{R} = \begin{bmatrix} 1 & 4 & -2 & 5 & 1 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

non-pivot columns

Step 5: Finally,

S ∪ (the set of vectors choose in Step 4)

is a basis for $\mathbb{R}^n$.

For this example,

$\{ (1, 4, -2, 5, 1),\ (2, 9, -1, 8, 2),\ (2, 9, -1, 9, 3),\ (0, 0, 1, 0, 0),\ (0, 0, 0, 0, 1) \}$

is a basis for $\mathbb{R}^5$.

Linear Systems (Discussion 4.1.15)

$$\begin{cases} 2x - y = -1 \\ x - y + 3z = 4 \\ -5x + y = -2 \\ x + z = 3 \end{cases} \qquad \begin{bmatrix} 2 & -1 & 0 \\ 1 & -1 & 3 \\ -5 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -2 \\ 3 \end{bmatrix}$$

Rewrite the system as

$$x \begin{bmatrix} 2 \\ 1 \\ -5 \\ 1 \end{bmatrix} + y \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -2 \\ 3 \end{bmatrix}.$$

Linear Systems (Discussion 4.1.15)

For example,

$(x, y, z) = (1, 3, 2)$

is a solution to the system.

$$1\begin{bmatrix} 2 \\ 1 \\ -5 \\ 1 \end{bmatrix} + 3\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -2 \\ 3 \end{bmatrix}$$

Let $\boldsymbol{A} = \begin{bmatrix} 2 & -1 & 0 \\ 1 & -1 & 3 \\ -5 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} -1 \\ 4 \\ -2 \\ 3 \end{bmatrix}$.

Then a solution to the system is a way of writing the vector $\boldsymbol{b}$ as a linear combination of the column vectors of $\boldsymbol{A}$.

Linear systems (Theorem 4.1.16)

Let $\boldsymbol{A}$ be an $m \times n$ matrix.

The column space of $\boldsymbol{A} = \{ \boldsymbol{Au} \mid \boldsymbol{u} \in \mathbb{R}^n \}$.

A linear system $\boldsymbol{Ax} = \boldsymbol{b}$ is consistent if and only if $\boldsymbol{b}$ lies in the column space of $\boldsymbol{A}$.

Proof:

Write $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{c}_1 & \boldsymbol{c}_2 & \cdots & \boldsymbol{c}_n \end{bmatrix}$

where $\boldsymbol{c}_j$ is the j^{th} column of $\boldsymbol{A}$.

Linear systems (Theorem 4.1.16)

For any $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)^{\mathrm{T}} \in \mathbb{R}^n$,

$$\boldsymbol{Au} = \begin{bmatrix} \boldsymbol{c_1} & \boldsymbol{c_2} & \cdots & \boldsymbol{c_n} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

$$= u_1\boldsymbol{c_1} + u_2\boldsymbol{c_2} + \cdots + u_n\boldsymbol{c_n} \in \text{span}\{\, \boldsymbol{c_1}, \boldsymbol{c_2}, \ldots, \boldsymbol{c_n} \,\}$$

$=$ the column space of $\boldsymbol{A}$.

Thus $\{\, \boldsymbol{Au} \mid \boldsymbol{u} \in \mathbb{R}^n \,\} \subseteq$ the column space of $\boldsymbol{A}$.

Linear systems (Theorem 4.1.16)

On the other hand, suppose

$$\boldsymbol{b} \in \text{span}\{\, \boldsymbol{c_1}, \boldsymbol{c_2}, \ldots, \boldsymbol{c_n} \,\} = \text{ the column space of } \boldsymbol{A}.$$

This means that there exists $u_1, u_2, \ldots, u_n \in \mathbb{R}$ such that

$$\boldsymbol{b} = u_1\boldsymbol{c_1} + u_2\boldsymbol{c_2} + \cdots + u_n\boldsymbol{c_n} = \boldsymbol{A}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \in \{\, \boldsymbol{Au} \mid \boldsymbol{u} \in \mathbb{R}^n \,\}.$$

Thus the column space of $\boldsymbol{A} \subseteq \{\, \boldsymbol{Au} \mid \boldsymbol{u} \in \mathbb{R}^n \,\}$

and hence we have shown that

$$\text{the column space of } \boldsymbol{A} = \{\, \boldsymbol{Au} \mid \boldsymbol{u} \in \mathbb{R}^n \,\}.$$

Linear systems (Theorem 4.1.16)

Finally,

the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ is consistent

$\Leftrightarrow$ there exists $\boldsymbol{u} \in \mathbb{R}^n$ such that $\boldsymbol{Au} = \boldsymbol{b}$

$\Leftrightarrow$ $\boldsymbol{b} \in \{\, \boldsymbol{Au} \mid \boldsymbol{u} \in \mathbb{R}^n \,\} = \text{span}\{\, \boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_n \,\}$

$=$ the column space of $\boldsymbol{A}$.

Section 4.2

Ranks

Row spaces and column spaces (Theorem 4.2.1)

The row space and column space of a matrix has the same dimension.

Proof: Let $\boldsymbol{A}$ be a matrix and $\boldsymbol{R}$ a row-echelon form of $\boldsymbol{A}$.

$$\boldsymbol{R} = \begin{bmatrix} 0 & \otimes & * & \cdots & \cdots & \cdots & * \\ 0 & 0 & \otimes & * & \cdots & \cdots & * \\ \vdots & & & \ddots & & & \vdots \\ 0 & \cdots & \cdots & 0 & \otimes & * & * \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

the number of nonzero rows
= the number of leading entries
= the number of pivot columns

pivot columns

Row spaces and column spaces (Theorem 4.2.1)

The nonzero rows in $\boldsymbol{R}$ form a basis for the row space of $\boldsymbol{A}$ (see Remark 4.1.9).

So dim(the row space of $\boldsymbol{A}$)
= the number of nonzero rows in $\boldsymbol{R}$.

The columns of $\boldsymbol{A}$ corresponding to the pivot columns in $\boldsymbol{R}$ form a basis for the column space of $\boldsymbol{A}$ (see Remark 4.1.13).

So dim(the column space of $\boldsymbol{A}$)
= the number of pivot columns in $\boldsymbol{R}$.
= the number of nonzero rows in $\boldsymbol{R}$ (see the previous slide)
= dim(the row space of $\boldsymbol{A}$).

An example (Example 4.2.2)

$$\boldsymbol{C} = \begin{bmatrix} 2 & 0 & 3 & -1 & 8 \\ 2 & 1 & 1 & -2 & 5 \\ -4 & -3 & 0 & 5 & -7 \end{bmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{bmatrix} 2 & 0 & 3 & -1 & 8 \\ 0 & 1 & -2 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\{ (2, 0, 3, -1, 8), (0, 1, -2, -1, -3) \}$ is a basis for the row space of $\boldsymbol{C}$.

$\{ (2, 2, -4)^T, (0, 1, -3)^T \}$ is a basis for the column space of $\boldsymbol{C}$.

The dimension of both the row space and column space of $\boldsymbol{C}$ is 2.

Ranks (Definition 4.2.3 & Example 4.2.4.1 & Remark 4.2.5)

The rank of a matrix is the dimension of its row space (and its column space).

We denote the rank of a matrix $\boldsymbol{A}$ by $\text{rank}(\boldsymbol{A})$.

- $\text{rank}(\mathbf{0}) = 0$ and $\text{rank}(\boldsymbol{I}_n) = n$.
- For an $m \times n$ matrix $\boldsymbol{A}$, $\text{rank}(\boldsymbol{A}) \leq \min\{ m, n \}$.

  If $\text{rank}(\boldsymbol{A}) = \min\{ m, n \}$, then $\boldsymbol{A}$ is said to have full rank.
- A square matrix $\boldsymbol{A}$ is of full rank if and only if $\det(\boldsymbol{A}) \neq 0$.
- $\text{rank}(\boldsymbol{A}) = \text{rank}(\boldsymbol{A}^{\mathrm{T}})$.

Linear systems (Remark 4.2.6 & Example 4.2.7)

A linear system $\boldsymbol{Ax} = \boldsymbol{b}$ is consistent if and only if $\boldsymbol{A}$ and $(\,\boldsymbol{A} \mid \boldsymbol{b}\,)$ have the same rank.

$$\underbrace{\begin{bmatrix} 2 & -1 & 0 \\ 1 & -1 & 3 \\ -5 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\boldsymbol{b}} \qquad \underbrace{\left[\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ 1 & -1 & 3 & 0 \\ -5 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]}_{(\,\boldsymbol{A} \mid \boldsymbol{b}\,)}$$

$$\xrightarrow{\text{Gaussian Elimination}} \left[\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ 0 & -1/2 & 3 & -1/2 \\ 0 & 0 & -9 & 4 \\ 0 & 0 & 0 & 7/9 \end{array}\right]$$

For this example, $\text{rank}(\boldsymbol{A}) = 3$
while $\text{rank}((\,\boldsymbol{A} \mid \boldsymbol{b}\,)) = 4$.

The system is inconsistent.

Ranks of product of matrices (Theorem 4.2.8)

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $m \times n$ and $m \times p$ matrices respectively. Then

$$\operatorname{rank}(\boldsymbol{AB}) \le \min\{ \operatorname{rank}(\boldsymbol{A}),\ \operatorname{rank}(\boldsymbol{B}) \}.$$

(Please read our textbook for a proof of the result.)

Section 4.3

Nullspaces and Nullities

Nullspaces (Definition 4.3.1 & Notation 4.3.2)

Let $\boldsymbol{A}$ be an $m \times n$ matrix.

The solution space of the homogeneous linear system $\boldsymbol{Ax} = \boldsymbol{0}$ is known as the nullspace of $\boldsymbol{A}$.

The dimension of the nullspace of $\boldsymbol{A}$ is called the nullity of $\boldsymbol{A}$ and is denoted by $\text{nullity}(\boldsymbol{A})$.

Since the nullspace is a subspace of $\mathbb{R}^n$,

$$\begin{aligned}\text{nullity}(\boldsymbol{A}) &= \dim(\text{ the nullspace of } \boldsymbol{A}\text{ }) \\ &\leq \dim(\mathbb{R}^n) = n.\end{aligned}$$

From now on, vectors in nullspaces, as well as solution sets of linear systems, will always written as column vectors. (See Notation 4.1.5.)

Examples (Example 4.3.3.1)

Find a basis for the nullspace of $\boldsymbol{A} = \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{bmatrix}$.

Solution:

$$\left[\begin{array}{ccccc|c} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \end{array}\right] \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Examples (Example 4.3.3.1)

The reduced row-echelon form of $\boldsymbol{A}$:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The general solution to the homogeneous linear system $\boldsymbol{Ax} = \boldsymbol{0}$ is

$$\boldsymbol{x} = \begin{bmatrix} -s-t \\ s \\ -t \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

where s, t are arbitrary parameters.

Thus $\{ (-1, 1, 0, 0, 0)^T, \ (-1, 0, -1, 0, 1)^T \}$ is a basis for the nullspace of $\boldsymbol{A}$.

Here $\text{nullity}(\boldsymbol{A}) = 2$.

Examples (Example 4.3.3.2)

Determine the nullity of $\boldsymbol{B} = \begin{bmatrix} 2 & 1 & -5 & 1 \\ -1 & -1 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix}$.

Solution:

$$\left[\begin{array}{cccc|c} 2 & 1 & -5 & 1 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 \end{array}\right] \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -7/9 & 0 \\ 0 & 1 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & -4/9 & 0 \end{array}\right]$$

Examples (Example 4.3.3.2)

The reduced row-echelon form of $\boldsymbol{B}$:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -7/9 & 0 \\ 0 & 1 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & -4/9 & 0 \end{array}\right]$$

The general solution to the homogeneous linear system $\boldsymbol{Bx} = \boldsymbol{0}$ is

$$\boldsymbol{x} = \begin{bmatrix} \frac{7}{9}t \\ -\frac{1}{3}t \\ \frac{4}{9}t \\ t \end{bmatrix} = t\begin{bmatrix} \frac{7}{9} \\ -\frac{1}{3} \\ \frac{4}{9} \\ 1 \end{bmatrix}$$

where t is a arbitrary parameter.

Thus $\{ (7, -3, 4, 9)^T \}$ is a basis for the nullspace of $\boldsymbol{B}$ and $\text{nullity}(\boldsymbol{B}) = 1$.

Dimension Theorem for matrices (Theorem 4.3.4)

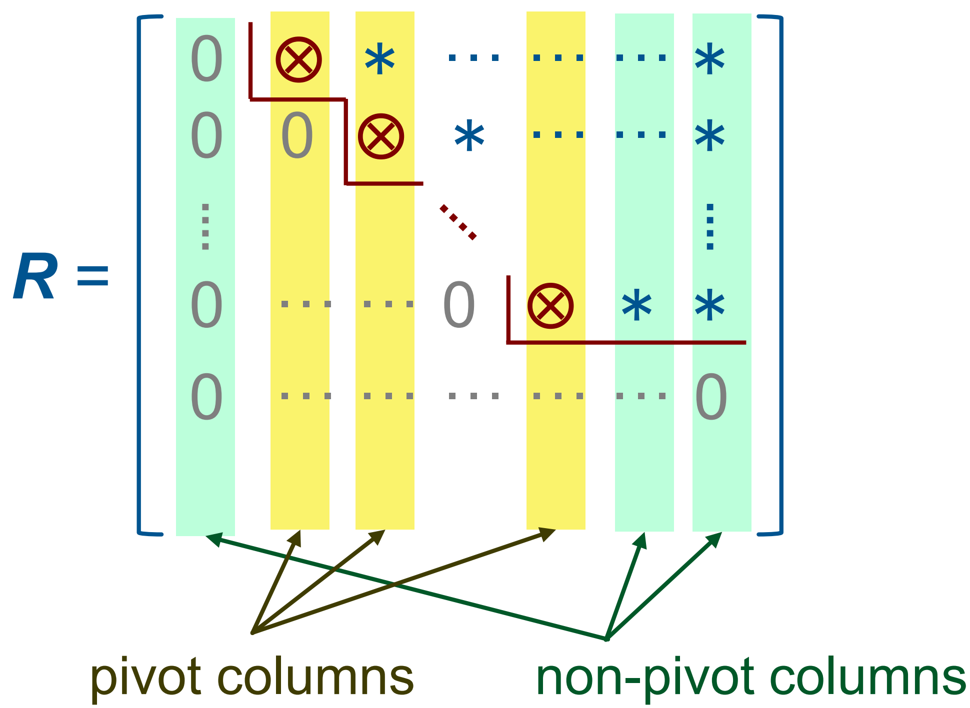
Let A be a matrix.

Then rank(A) + nullity(A) = the number of columns of A.

Proof: Let R be a row-echelon form of A.

The columns of R can be classified into two types:

- pivot columns,
- non-pivot columns.



Dimension Theorem for matrices (Theorem 4.3.4)

Then (by Remark 4.1.13)

rank($\boldsymbol{A}$) = dim(the column space of $\boldsymbol{A}$)

= the number of pivot-columns in $\boldsymbol{R}$

and (by Discussion 3.6.5)

nullity($\boldsymbol{A}$) = dim(the solution space of $\boldsymbol{Ax} = \mathbf{0}$)

= the number of non-pivot-columns in $\boldsymbol{R}$.

So rank($\boldsymbol{A}$) + nullity($\boldsymbol{A}$)

= the number of pivot-columns in $\boldsymbol{R}$

+ the number of non-pivot-columns in $\boldsymbol{R}$

= the number of columns of $\boldsymbol{R}$.

= the number of columns of $\boldsymbol{A}$.

Examples (Example 4.3.5.2)

In each of the following cases, find $\text{rank}(\boldsymbol{A})$, $\text{nullity}(\boldsymbol{A})$ and $\text{nullity}(\boldsymbol{A}^T)$. (Recall that $\text{rank}(\boldsymbol{A}) = \text{rank}(\boldsymbol{A}^T)$.)

(a) $\boldsymbol{A}$ is a 3×4 matrix and $\text{rank}(\boldsymbol{A}) = 3$.

Answer: $\text{nullity}(\boldsymbol{A}) = 4 - \text{rank}(\boldsymbol{A}) = 1$,

$\text{nullity}(\boldsymbol{A}^T) = 3 - \text{rank}(\boldsymbol{A}^T) = 3 - \text{rank}(\boldsymbol{A}) = 0$.

(b) $\boldsymbol{A}$ is a 7×5 matrix and $\text{nullity}(\boldsymbol{A}) = 3$.

Answer: $\text{rank}(\boldsymbol{A}) = 5 - \text{nullity}(\boldsymbol{A}) = 2$,

$\text{nullity}(\boldsymbol{A}^T) = 7 - \text{rank}(\boldsymbol{A}^T) = 7 - \text{rank}(\boldsymbol{A}) = 5$.

(c) $\boldsymbol{A}$ is a 3×2 matrix and $\text{nullity}(\boldsymbol{A}^T) = 3$.

Answer: $\text{rank}(\boldsymbol{A}) = \text{rank}(\boldsymbol{A}^T) = 3 - \text{nullity}(\boldsymbol{A}^T) = 0$,

$\text{nullity}(\boldsymbol{A}) = 2 - \text{rank}(\boldsymbol{A}) = 2$.

Linear systems (Theorem 4.3.6 & Remark 4.3.7)

Suppose $\boldsymbol{x} = \boldsymbol{v}$ is a solution to a linear system $\boldsymbol{Ax} = \boldsymbol{b}$.

Then the solution set of the system $\boldsymbol{Ax} = \boldsymbol{b}$ is given by

$$\{\, \boldsymbol{u} + \boldsymbol{v} \mid \boldsymbol{u} \in \text{the nullspace of } \boldsymbol{A} \,\},$$

i.e. the system $\boldsymbol{Ax} = \boldsymbol{b}$ has a general solution

$$\boldsymbol{x} = (\text{a general solution for } \boldsymbol{Ax} = \boldsymbol{0}) + (\text{one particular solution to } \boldsymbol{Ax} = \boldsymbol{b}).$$

By this result, a consistent linear system $\boldsymbol{Ax} = \boldsymbol{b}$ has only one solution if and only if the nullspace of $\boldsymbol{A}$ is $\{\, \boldsymbol{0} \,\}$.

Proof of the theorem (Theorem 4.3.6)

Proof: Let $M = \{ \boldsymbol{u} + \boldsymbol{v} \mid \boldsymbol{u} \in \text{the nullspace of } \boldsymbol{A} \}$.

Since $\boldsymbol{x} = \boldsymbol{v}$ is a solution to the system $\boldsymbol{Ax} = \boldsymbol{b}$, $\boldsymbol{Av} = \boldsymbol{b}$.

Let $\boldsymbol{x} = \boldsymbol{w}$ be any solution to the system $\boldsymbol{Ax} = \boldsymbol{b}$,

i.e. $\boldsymbol{Aw} = \boldsymbol{b}$.

Write $\boldsymbol{u} = \boldsymbol{w} - \boldsymbol{v}$.

Then $\boldsymbol{Au} = \boldsymbol{A}(\boldsymbol{w} - \boldsymbol{v}) = \boldsymbol{Aw} - \boldsymbol{Av} = \boldsymbol{b} - \boldsymbol{b} = \boldsymbol{0}$.

This means that $\boldsymbol{u} \in$ the nullspace of $\boldsymbol{A}$
and hence $\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v} \in M$.

We have shown that

$$\text{the solution set of the system} \subseteq M.$$

Proof of the theorem (Theorem 4.3.6)

On the other hand, take any $\boldsymbol{w} \in M$,
i.e. $\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v}$ for some $\boldsymbol{u} \in$ the nullspace of $\boldsymbol{A}$.

(Remind that $\boldsymbol{Au} = \boldsymbol{0}$ and $\boldsymbol{Av} = \boldsymbol{b}$.)

Then $\boldsymbol{Aw} = \boldsymbol{A}(\boldsymbol{u} + \boldsymbol{v}) = \boldsymbol{Au} + \boldsymbol{Av} = \boldsymbol{0} + \boldsymbol{b} = \boldsymbol{b}$.

Thus $\boldsymbol{x} = \boldsymbol{w}$ is a solution to the system $\boldsymbol{Ax} = \boldsymbol{b}$.

We have shown that

$M \subseteq$ the solution set of the system.

So the solution set of the system $= M$.

An examples (Example 4.3.8)

Consider the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ where

$$\boldsymbol{A} = \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ -3 \end{bmatrix}.$$

We know that (by Example 4.3.3.1)

$$\text{the nullspace of } \boldsymbol{A} = \left\{ s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; s, t \in \mathbb{R} \right\}$$

An examples (Example 4.3.8)

It can be checked easily that $(1, -1, 1, 1, 1)^T$ is a solution to $\boldsymbol{Ax} = \boldsymbol{b}$.

Thus the system has a general solution

$$\boldsymbol{x} = s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

where s, t are arbitrary parameters.

A related result (Remark 4.3.9)

Consider the ordinary differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x).$$

where a, b, c are real constants.

The general solution for the equation is usually written as

$$y = \left[\text{a general solution for } a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0\right]$$

$$+ \left[\text{one particular solution for } a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)\right].$$

A related result (Remark 4.3.9)

The form looks similar to the solutions to $\boldsymbol{Ax} = \boldsymbol{b}$ discussed in Theorem 4.3.6.

This is not a coincidence. To explain the relation between these two different types of equations, we need the concept of "abstract" vector spaces that will be covered in MA2101 Linear Algebra II.