Chapter 3

Vector Spaces

Section 3.1

Euclidean *n*-Space

Euclid of Alexandria (about 320-260 B.C.)

Euclid of Alexandria is the most important mathematician of antiquity best known for his treatise on mathematics, The Elements, a textbook on plane geometry that summarized the works of the Golden Age of Greek Mathematics. However little is known of Euclid's life except that he taught at Alexandria in Egypt.

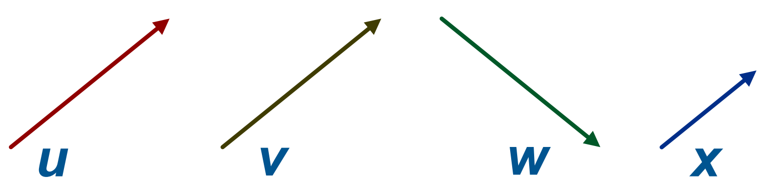
Geometric vectors (Discussion 3.1.1)

A (nonzero) vector can be represented geometrically by a directed line segment or an arrow.

The direction of the arrow specifies the direction of the vector and the length of the arrow describes its magnitude.

The zero vector, denoted by **0**, is represented by a point or a degenerated vector with zero length and no direction.

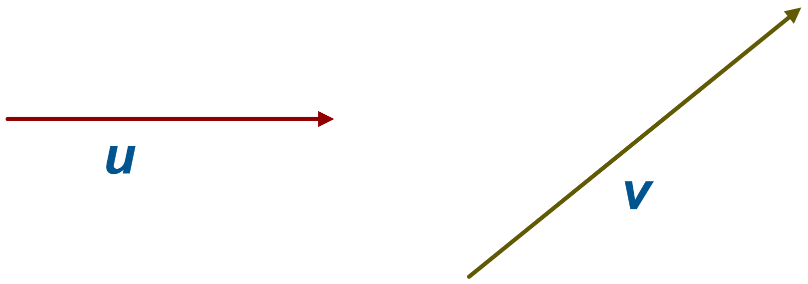
Two vectors are regarded as equal if they have the same length and direction.



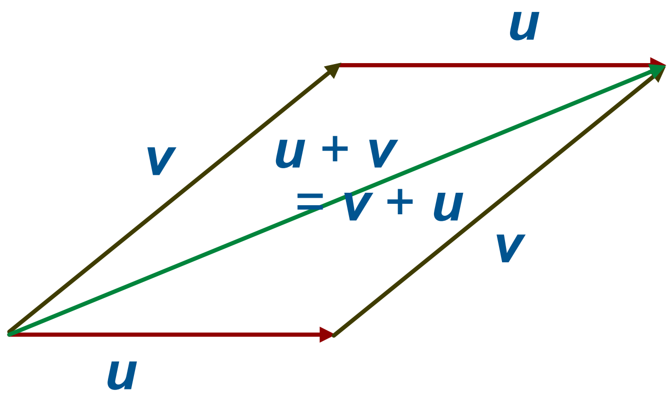
$\boldsymbol{u} = \boldsymbol{v}$, $\boldsymbol{u} \neq \boldsymbol{w}$ and $\boldsymbol{u} \neq \boldsymbol{x}$

Geometric vectors (Discussion 3.1.1)

(a) The addition $\boldsymbol{u} + \boldsymbol{v}$ of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$:

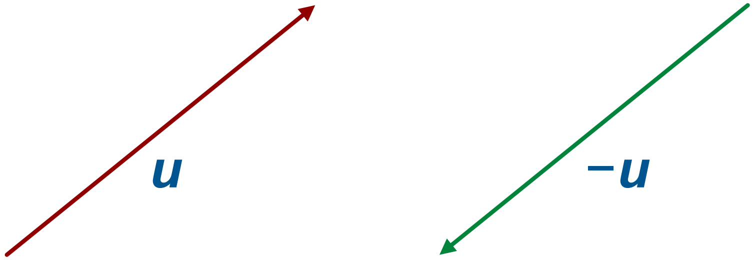


Note that $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$.



Geometric vectors (Discussion 3.1.1)

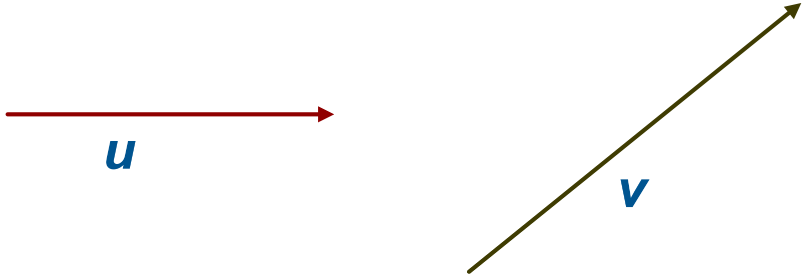
(b) The negative $-\boldsymbol{u}$ of a vector $\boldsymbol{u}$:

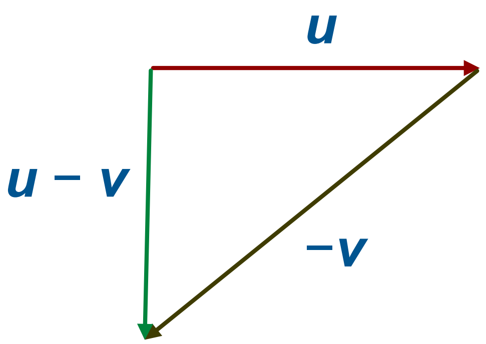


The vector $-\boldsymbol{u}$ has the same length as $\boldsymbol{u}$ but the reverse direction.

Geometric vectors (Discussion 3.1.1)

(c) The difference $\boldsymbol{u} - \boldsymbol{v}$ of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$:

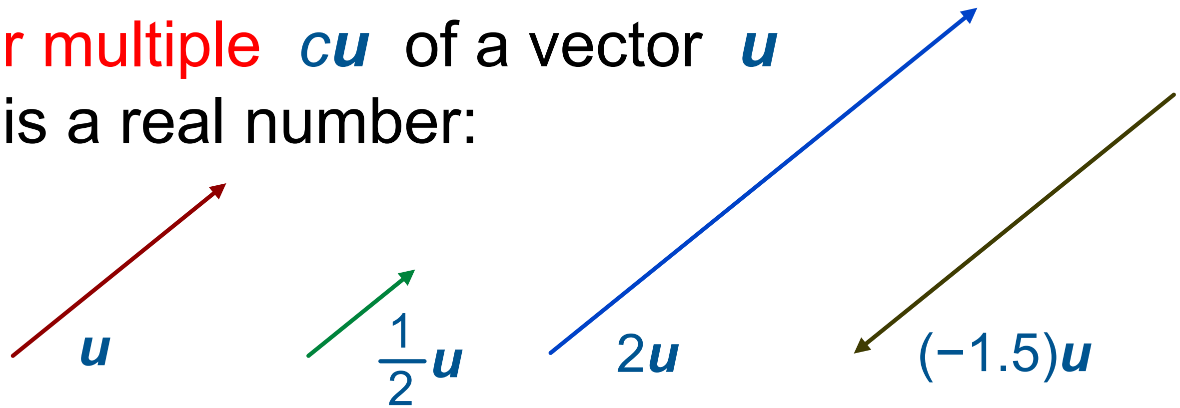




Note that $\boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u} + (-\boldsymbol{v})$.

Geometric vectors (Discussion 3.1.1)

(d) The scalar multiple $c\boldsymbol{u}$ of a vector $\boldsymbol{u}$ where c is a real number:



If c is positive, the vector $c\boldsymbol{u}$ has the same direction as $\boldsymbol{u}$ and its length is c times of the length of $\boldsymbol{u}$.

If c is negative, the vector $c\boldsymbol{u}$ has the reverse direction of $\boldsymbol{u}$ and its length is $|c|$ times of the length of $\boldsymbol{u}$.

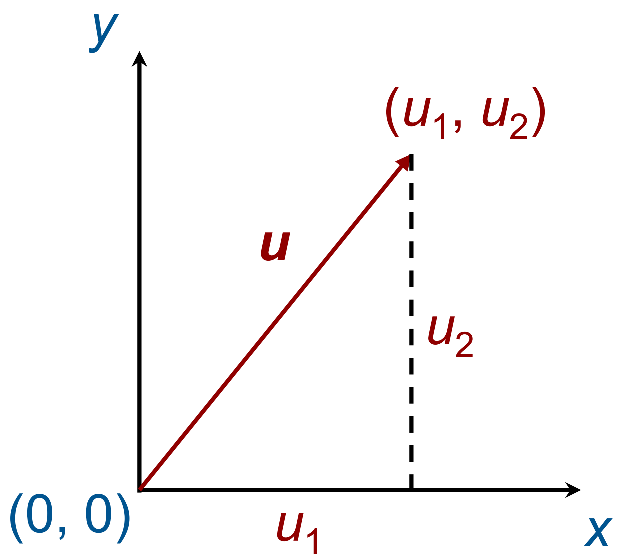
$0\boldsymbol{u} = \mathbf{0}$ is the zero vector.

$(-1)\boldsymbol{u} = -\boldsymbol{u}$ is the negative of $\boldsymbol{u}$.

Coordinate systems: *xy*-plane (Discussion 3.1.2.1)

Suppose we position a vector $\boldsymbol{u}$ in the *xy*-plane such that its initial point is at the origin $(0, 0)$.

The coordinates (u_1, u_2) of the end point of $\boldsymbol{u}$ are called the components of $\boldsymbol{u}$ and we write $\boldsymbol{u} = (u_1, u_2)$.

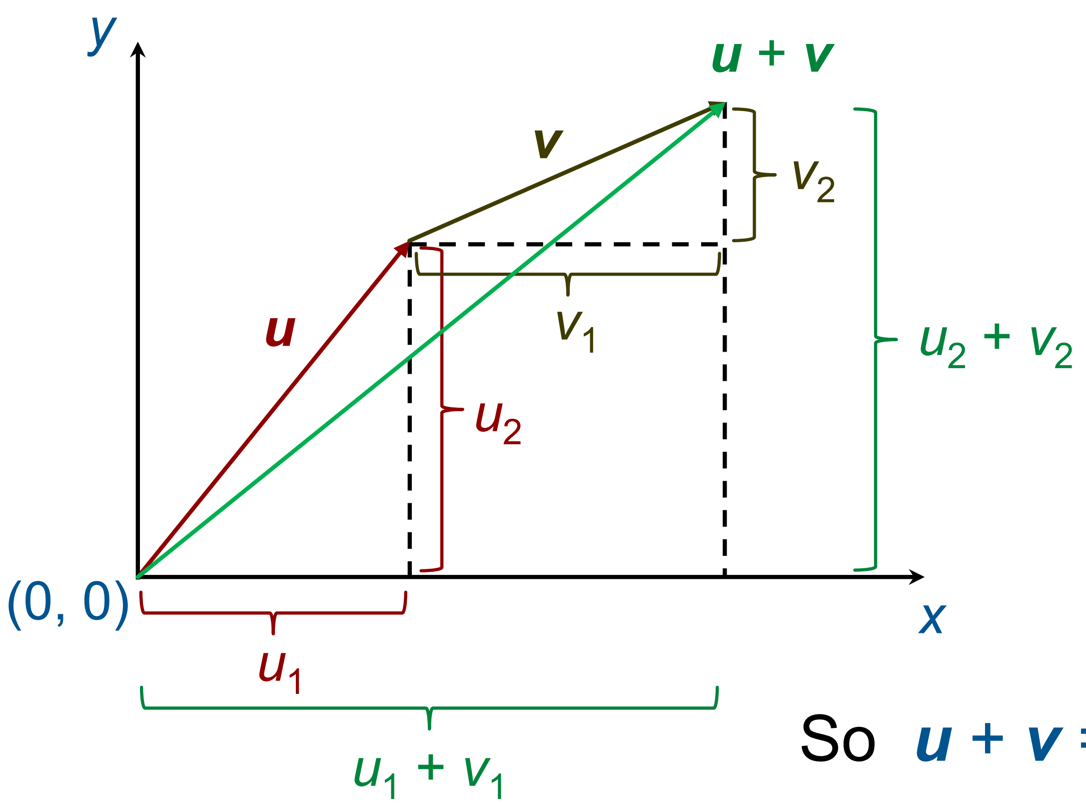


Algebraically, a vector in the *xy*-plane can be identified as a point on the plane.

Coordinate systems: *xy*-plane (Discussion 3.1.2.1)

(a) The addition of two vectors:

Let $\boldsymbol{u} = (u_1, u_2)$ and $\boldsymbol{v} = (v_1, v_2)$.

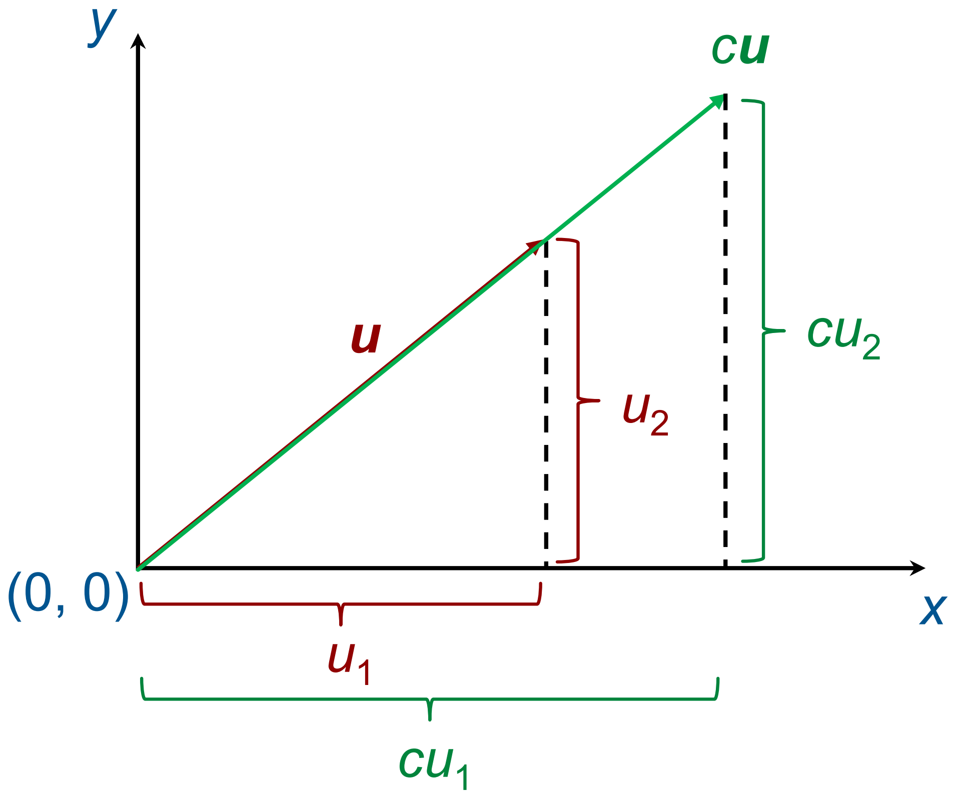


So $\boldsymbol{u} + \boldsymbol{v} = (u_1 + v_1,\ u_2 + v_2)$.

Coordinate systems: *xy*-plane (Discussion 3.1.2.1)

(b) The scalar multiple of a vector:

Let $\boldsymbol{u} = (u_1, u_2)$ and c a real constant.

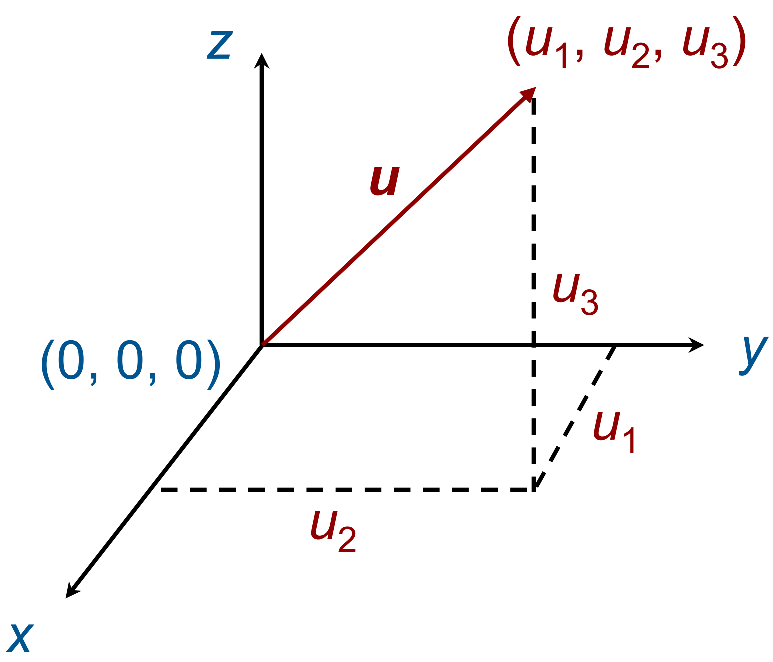


So $c\boldsymbol{u} = (cu_1, cu_2)$.

Coordinate systems: *xyz*-space (Discussion 3.1.2.2)

Suppose we position a vector $\boldsymbol{u}$ in the xyz-space such that its initial point is at the origin $(0, 0, 0)$.

The coordinates (u_1, u_2, u_3) of the end point of $\boldsymbol{u}$ are called the components of $\boldsymbol{u}$ and we write $\boldsymbol{u} = (u_1, u_2, u_3)$.



Algebraically, a vector in the xyz-space can be identified as a point in the space.

Coordinate systems: *xyz*-space (Discussion 3.1.2.2)

(a) The addition of two vectors:

Let $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$.

Then $\boldsymbol{u} + \boldsymbol{v} = (u_1 + v_1,\ u_2 + v_2,\ u_3 + v_3)$.

(b) The scalar multiple of a vector:

Let $\boldsymbol{u} = (u_1, u_2, u_3)$ and c a real constant.

Let $c\boldsymbol{u} = (cu_1, cu_2, cu_3)$.

n-vectors (Definition 3.1.3)

An n-vector or ordered n-tuple of real numbers has the form

$$(u_1, u_2, \ldots, u_i, \ldots, u_n)$$

where $u_1, u_2, \ldots, u_n$ are real numbers.

The number u_i in the i^{th} position of an n-vector is called the i^{th} component or the i^{th} coordinate of the n-vector.

Let $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ be two n-vectors.

1. $\boldsymbol{u} = \boldsymbol{v}$ if and only if $u_i = v_i$ for all $i = 1, 2, \ldots, n$.
2. The addition $\boldsymbol{u} + \boldsymbol{v}$ of $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined by
$$\boldsymbol{u} + \boldsymbol{v} = (u_1 + v_1,\ u_2 + v_2,\ \ldots,\ u_n + v_n).$$

n-vectors (Definition 3.1.3)

3. For a real number c, the scalar multiple $c\boldsymbol{u}$ of $\boldsymbol{u}$ is defined by
$$c\boldsymbol{u} = (cu_1, cu_2, \ldots, cu_n).$$
4. The n-vector $(0, 0, \ldots, 0)$ is called the zero vector and is denoted by $\boldsymbol{0}$.
5. The negative of $\boldsymbol{u}$ is defined by $(-1)\boldsymbol{u}$ and is denoted by $-\boldsymbol{u}$, i.e.
$$-\boldsymbol{u} = (-u_1, -u_2, \ldots, -u_n).$$
6. The subtraction $\boldsymbol{u} - \boldsymbol{v}$ of $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined by $\boldsymbol{u} + (-\boldsymbol{v})$, i.e.
$$\boldsymbol{u} - \boldsymbol{v} = (u_1 - v_1,\ u_2 - v_2,\ \ldots,\ u_n - v_n).$$

Examples (Example 3.1.4)

Let $\boldsymbol{u} = (1, 2, 3, 4)$ and $\boldsymbol{v} = (-1, -2, -3, 0)$. Then

$$\begin{aligned}\boldsymbol{u} + \boldsymbol{v} &= (1 + (-1),\ 2 + (-2),\ 3 + (-3),\ 4 + 0)\\ &= (0, 0, 0, 4),\end{aligned}$$

$$\begin{aligned}\boldsymbol{u} - \boldsymbol{v} &= (1 - (-1),\ 2 - (-2),\ 3 - (-3),\ 4 - 0)\\ &= (2, 4, 6, 4),\end{aligned}$$

$$3\boldsymbol{u} = (3 \cdot 1,\ 3 \cdot 2,\ 3 \cdot 3,\ 3 \cdot 4) = (3, 6, 9, 12),$$

$$\begin{aligned}&3\boldsymbol{u} + 4\boldsymbol{v}\\ &= (3 \cdot 1 + 4 \cdot (-1),\ 3 \cdot 2 + 3 \cdot (-2),\ 3 \cdot 3 + 3 \cdot (-3),\ 3 \cdot 4 + 3 \cdot 0)\\ &= (-1, -2, -3, 12).\end{aligned}$$

Row and column vectors (Notation 3.1.5)

The features in the definition of n-vectors are similar to those of matrices.

We can identify an n-vectors $(u_1, u_2, ..., u_n)$ with a $1 \times n$ matrix

$$\begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \quad \text{(a row vector)}$$

or an $n \times 1$ matrix

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{(a column vector).}$$

Warning: Do not use two different sets of notation for n-vectors within the same context.

Some basic properties (Theorem 3.1.6)

Let $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ be n-vectors and c, d real numbers.

1. $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$.
2. $\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$.
3. $\boldsymbol{u} + \mathbf{0} = \boldsymbol{u} = \mathbf{0} + \boldsymbol{u}$.
4. $\boldsymbol{u} + (-\boldsymbol{u}) = \mathbf{0}$.
5. $c(d\boldsymbol{u}) = (cd)\boldsymbol{u}$.
6. $c(\boldsymbol{u} + \boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$.
7. $(c + d)\boldsymbol{u} = c\boldsymbol{u} + d\boldsymbol{u}$.
8. $1\boldsymbol{u} = \boldsymbol{u}$.

Euclidean *n*-space (Definition 3.1.7)

The set of all n-vectors of real numbers is called the Euclidean n-space or simply n-space.

We use $\mathbb{R}$ to denote the set of all real numbers and $\mathbb{R}^n$ to denote the Euclidean n-space.

$\boldsymbol{u} \in \mathbb{R}^n$ if and only if $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ for some $u_1, u_2, \ldots, u_n \in \mathbb{R}$.

In set notation, we write

$$\mathbb{R}^n = \{ (u_1, u_2, \ldots, u_n) \mid u_1, u_2, \ldots, u_n \in \mathbb{R} \}.$$

Subsets (Example 3.1.8.1)

Let $B = \{ (u_1, u_2, u_3, u_4) \in \mathbb{R}^4 \mid u_1 = 0 \text{ and } u_2 = u_4 \}$
or simply $B = \{ (u_1, u_2, u_3, u_4) \mid u_1 = 0 \text{ and } u_2 = u_4 \}$.

It means that B is a subset of $\mathbb{R}^4$ such that

$(u_1, u_2, u_3, u_4) \in B$ if and only if $u_1 = 0$ and $u_2 = u_4$.

For example,

$$(0, 0, 0, 0), (0, 0, 10, 0), (0, 1, 3, 1), (0, \pi, \pi, \pi) \in B$$

and $(1, 2, 3, 4), (0, 10, 0, 0), (0, 1, 3, 2), (\pi, \pi, \pi, \pi) \notin B$.

Explicitly, we can write $B = \{ (0, a, b, a) \mid a, b \in \mathbb{R} \}$.

Solution sets (Example 3.1.8.2)

If a system of linear equation has n variables, then its solution set is a subset (may be empty) of $\mathbb{R}^n$.

For example, the general solution of the linear system

$$\begin{cases} x + y + z = 0 \\ x - y + 2z = 1 \end{cases}$$

can be expressed in vector form:

$$(x, y, z) = \left(\frac{1}{2} - \frac{3}{2}t,\ -\frac{1}{2} + \frac{1}{2}t,\ t\right) \text{ where } t \in \mathbb{R}.$$

The solution set can be written as

$$\{ (x, y, z) \mid x + y + z = 0 \text{ and } x - y + 2z = 1 \} \text{ (implicit)}$$

or $\left\{ \left(\frac{1}{2} - \frac{3}{2}t,\ -\frac{1}{2} + \frac{1}{2}t,\ t\right) \,\middle|\, t \in \mathbb{R} \right\}$ (explicit).

Lines in $\mathbb{R}^2$ (Example 3.1.8.3 (a))

A line in $\mathbb{R}^2$ can be expressed implicitly in set notation by

$$\{ (x, y) \mid ax + by = c \}$$

where a, b, c are real constants and a, b are not both zero.

Explicitly, the line can also be expressed as

$$\left\{ \left(\frac{c - bt}{a},\ t \right) \middle|\ t \in \mathbb{R} \right\} \quad \text{if } a \neq 0;$$

or

$$\left\{ \left(t,\ \frac{c - at}{b} \right) \middle|\ t \in \mathbb{R} \right\} \quad \text{if } b \neq 0.$$

Planes in $\mathbb{R}^3$ (Example 3.1.8.3 (b))

A plane in $\mathbb{R}^3$ can be expressed implicitly in set notation by

$$\{\,(x, y, z) \mid ax + by + cz = d\,\}$$

where a, b, c, d are real constants and a, b, c are not all zero.

Explicitly, the line can also be expressed as

$$\left\{ \left(\frac{d - bs - ct}{a},\ s,\ t \right) \,\middle|\, s, t \in \mathbb{R} \right\} \quad \text{if } a \neq 0;$$

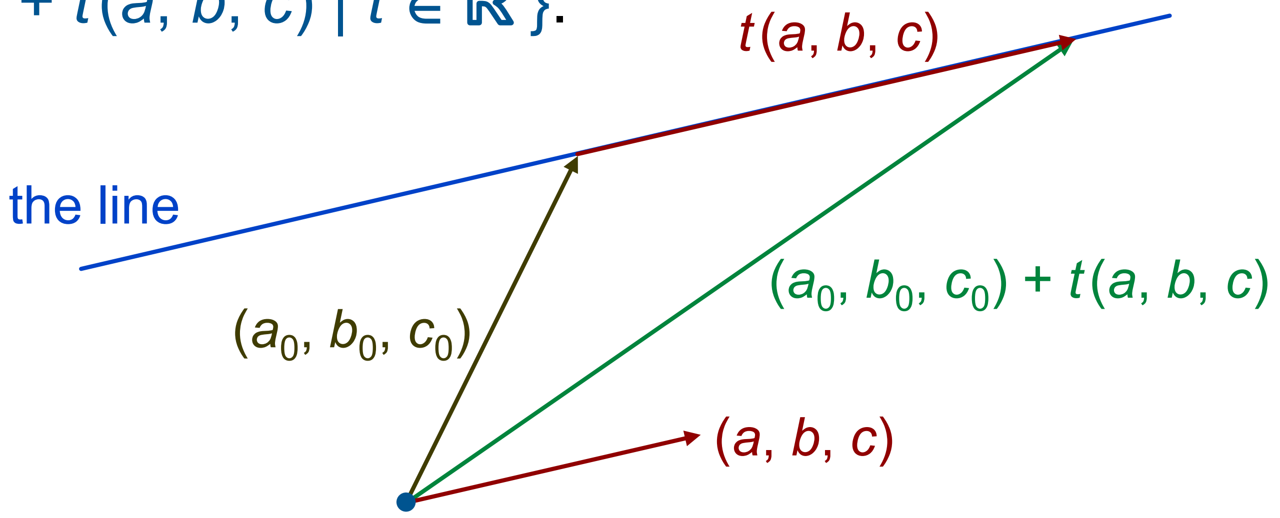
$$\left\{ \left(s,\ \frac{d - as - ct}{b},\ t \right) \,\middle|\, s, t \in \mathbb{R} \right\} \quad \text{if } b \neq 0;$$

or $$\left\{ \left(s,\ t,\ \frac{d - as - bt}{c} \right) \,\middle|\, s, t \in \mathbb{R} \right\} \quad \text{if } c \neq 0.$$

Lines in $\mathbb{R}^3$ (Example 3.1.8.3 (c))

A line in $\mathbb{R}^3$ is usually represented explicitly in set notation by

$$\{\ (a_0 + at,\ b_0 + bt,\ c_0 + ct)\ |\ t \in \mathbb{R}\ \}$$
$$= \{\ (a_0, b_0, c_0) + t(a, b, c)\ |\ t \in \mathbb{R}\ \}.$$



In here, (a_0, b_0, c_0) is a point on the line and (a, b, c) is the direction of the line.

Lines in $\mathbb{R}^3$ (Example 3.1.8.3 (c))

A line in $\mathbb{R}^3$ cannot be represented by a single equation as in the case of $\mathbb{R}^2$.

Instead, it can be regarded as the intersection of two non-parallel planes and hence written implicitly as

$$\{\, (x, y, z) \mid a_1x + b_1y + c_1z = d_1 \text{ and } a_2x + b_2y + c_2z = d_2 \,\}$$

for some suitable choice of constants a_1, b_1, c_1, d_1 and a_2, b_2, c_2, d_2.

Finite sets (Notation 3.1.9 & Example 3.1.10)

Let S be a finite set.

We use $|S|$ to denote the number of elements contained in S.

For example, let $S_1 = \{ 1, 2, 3, 4 \}$, $S_2 = \{ (1, 2, 3, 4) \}$ and $S_3 = \{ (1, 2, 3), (2, 3, 4) \}$.

Then $|S_1| = 4$, $|S_2| = 1$ and $|S_3| = 2$.

Section 3.2

Linear Combinations and Linear Spans

Linear combinations (Definition 3.2.1)

Let $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}$ be vectors in $\mathbb{R}^n$.

For any real numbers $c_1, c_2, \ldots, c_k$, the vector

$$c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}$$

is called a linear combination of $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}$.

Examples (Example 3.2.2.1 (a))

Let $\boldsymbol{u}_1 = (2, 1, 3)$, $\boldsymbol{u}_2 = (1, -1, 2)$ and $\boldsymbol{u}_3 = (3, 0, 5)$.

Is $\boldsymbol{v} = (3, 3, 4)$ a linear combination of $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$?

Write $\boldsymbol{v} = a\boldsymbol{u}_1 + b\boldsymbol{u}_2 + c\boldsymbol{u}_3$, i.e.

$$\begin{aligned}(3, 3, 4) &= a(2, 1, 3) + b(1, -1, 2) + c(3, 0, 5) \\ &= (2a + b + 3c,\ a - b,\ 3a + 2b + 5c).\end{aligned}$$

So $$\begin{cases} 2a + b + 3c = 3 \\ a - b = 3 \\ 3a + 2b + 5c = 4. \end{cases}$$

Examples (Example 3.2.2.1 (a))

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 1 & -1 & 0 & 3 \\ 3 & 2 & 5 & 4 \end{array}\right] \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left[\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 0 & -3/2 & -3/2 & 3/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

By using back-substitution, we obtain a general solution

$$(a, b, c) = (2 - t, \ -1 - t, \ t)$$

where t is an arbitrary parameter.

For example, we have particular solutions

$$(2, -1, 0), \ (1, -2, 1), \ \text{etc.}$$

So we can write $\boldsymbol{v}$ as linear combinations

$$\boldsymbol{v} = 2\boldsymbol{u}_1 - \boldsymbol{u}_2 + 0\boldsymbol{u}_3, \quad \boldsymbol{v} = \boldsymbol{u}_1 - 2\boldsymbol{u}_2 + \boldsymbol{u}_3, \quad \text{etc.}$$

Examples (Example 3.2.2.1 (b))

Let $\boldsymbol{u}_1 = (2, 1, 3)$, $\boldsymbol{u}_2 = (1, -1, 2)$ and $\boldsymbol{u}_3 = (3, 0, 5)$.

Is $\boldsymbol{w} = (1, 2, 4)$ a linear combination of $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$?

Write $\boldsymbol{w} = a\boldsymbol{u}_1 + b\boldsymbol{u}_2 + c\boldsymbol{u}_3$, i.e.

$$\begin{aligned}(1, 2, 4) &= a(2, 1, 3) + b(1, -1, 2) + c(3, 0, 5)\\ &= (2a + b + 3c,\ a - b,\ 3a + 2b + 5c).\end{aligned}$$

So
$$\begin{cases} 2a + b + 3c = 1 \\ a - b = 2 \\ 3a + 2b + 5c = 4. \end{cases}$$

Examples (Example 3.2.2.1 (b))

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 1 & -1 & 0 & 2 \\ 3 & 2 & 5 & 4 \end{array}\right] \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & -3/2 & -3/2 & 3/2 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

The system is inconsistent.

$\boldsymbol{w}$ is not a linear combination of $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$.

Examples (Example 3.2.2.2)

Let $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$ and $\mathbf{e}_3 = (0, 0, 1)$.

For any $(x, y, z) \in \mathbb{R}^3$,

$$(x, y, z) = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3.$$

So every vector in $\mathbb{R}^3$ is a linear combination of $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$.

Geometrically, $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are called the directional vectors of the x-axis, y-axis and z-axis, respectively, of $\mathbb{R}^3$.

Linear spans (Definition 3.2.3 & Example 3.2.4.1)

Let $S = \{ \boldsymbol{u_1}, \boldsymbol{u_2}, \dots, \boldsymbol{u_k} \}$ be a set of vectors in $\mathbb{R}^n$.

The set of all linear combinations of $\boldsymbol{u_1}, \boldsymbol{u_2}, \dots, \boldsymbol{u_k}$

$$\{ c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k} \mid c_1, c_2, \dots, c_k \in \mathbb{R} \}$$

is called a linear span of S (or the linear span of $\boldsymbol{u_1}, \boldsymbol{u_2}, \dots, \boldsymbol{u_k}$)

and is denoted by $\mathrm{span}(S)$ (or $\mathrm{span}\{ \boldsymbol{u_1}, \boldsymbol{u_2}, \dots, \boldsymbol{u_k} \}$).

Let $\boldsymbol{u_1} = (2, 1, 3)$, $\boldsymbol{u_2} = (1, -1, 2)$ and $\boldsymbol{u_3} = (3, 0, 5)$.

Then (by Example 3.2.2.1)

$$\boldsymbol{v} = (3, 3, 4) \in \mathrm{span}\{ \boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3} \}$$

and $\quad \boldsymbol{w} = (1, 2, 4) \notin \mathrm{span}\{ \boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3} \}$.

Examples (Example 3.2.4.2-3)

Let $S = \{ (1, 0, 0, -1), (0, 1, 1, 0) \} \subseteq \mathbb{R}^4$.

Every element in $\text{span}(S)$ is of the form

$$a(1, 0, 0, -1) + b(0, 1, 1, 0) = (a, b, b, -a)$$

where a and b are real numbers.

So $\text{span}(S) = \{ (a, b, b, -a) \mid a, b \in \mathbb{R} \}$.

Let $V = \{ (2a + b, a, 3b - a) \mid a, b \in \mathbb{R} \} \subseteq \mathbb{R}^3$.

For any $a, b \in \mathbb{R}$,

$$(2a + b, a, 3b - a) = a(2, 1, -1) + b(1, 0, 3).$$

So $V = \text{span}\{ (2, 1, -1), (1, 0, 3) \}$.

Examples (Example 3.2.4.4)

Show that $\text{span}\{ (1, 0, 1), (1, 1, 0), (0, 1, 1) \} = \mathbb{R}^3$.

Solution: We need to verify that for any $(x, y, z) \in \mathbb{R}^3$, there exists real numbers a, b, c such that

$$a(1, 0, 1) + b(1, 1, 0) + c(0, 1, 1) = (x, y, z).$$

This is equivalent to show that the linear system

$$\begin{cases} a + b \phantom{{}+c} = x \\ \phantom{a+{}} b + c = y \\ a \phantom{{}+b} + c = z \end{cases}$$

where a, b, c are variables

is consistent for all $x, y, z \in \mathbb{R}$.

Examples (Example 3.2.4.4)

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & x \\ 0 & 1 & 1 & y \\ 1 & 0 & 1 & z \end{array}\right] \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & x \\ 0 & 1 & 1 & y \\ 0 & 0 & 2 & z-x+y \end{array}\right]$$

The system is consistent regardless of the values of x, y, z.

So we have shown that

$$\text{span}\{\, (1, 0, 1),\ (1, 1, 0),\ (0, 1, 1)\,\} = \mathbb{R}^3.$$

Examples (Example 3.2.4.5)

Show that

$$\text{span}\{ (1, 1, 1), (1, 2, 0), (2, 1, 3), (2, 3, 1) \} \neq \mathbb{R}^3.$$

Solution: For any $(x, y, z) \in \mathbb{R}^3$, we solve the vector equation

$$a(1, 1, 1) + b(1, 2, 0) + c(2, 1, 3) + d(2, 3, 1) = (x, y, z),$$

where a, b, c, d are variables.

The linear system is

$$\begin{cases} a + b + 2c + 2d = x \\ a + 2b + c + 3d = y \\ a + 3c + d = z. \end{cases}$$

Examples (Example 3.2.4.5)

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & x \\ 1 & 2 & 1 & 3 & y \\ 1 & 0 & 3 & 1 & z \end{array}\right] \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & x \\ 0 & 1 & -1 & 1 & y - x \\ 0 & 0 & 0 & 0 & z + x - 2y \end{array}\right]$$

The system is inconsistent if $z + x - 2y \neq 0$.

For example, if $(x, y, z) = (1, 0, 0)$, then $z + x - 2y \neq 0$,
i.e. $a(1, 1, 1) + b(1, 2, 0) + c(2, 1, 3) + d(2, 3, 1) = (1, 0, 0)$
has no solution and hence

$$(1, 0, 0) \notin \text{span}\{ (1, 1, 1), (1, 2, 0), (2, 1, 3), (2, 3, 1) \}.$$

So $\text{span}\{ (1, 1, 1), (1, 2, 0), (2, 1, 3), (2, 3, 1) \} \neq \mathbb{R}^3$.

When span(S) = $\mathbb{R}^n$ (Discussion 3.2.5)

Let $S = \{ \boldsymbol{u_1}, \boldsymbol{u_2}, ..., \boldsymbol{u_k} \} \subseteq \mathbb{R}^n$

where $\boldsymbol{u_i} = (a_{i1}, a_{i2}, ..., a_{in})$ for $i = 1, 2, ..., k$.

For any $\boldsymbol{v} = (v_1, v_2, ..., v_n) \in \mathbb{R}^n$, $\boldsymbol{v} \in \text{span}(S)$ if and only if the vector equation

$$c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k} = \boldsymbol{v}$$

has solution for c_1, c_2, ..., c_k,

i.e. the following linear system is consistent:

$$\begin{cases} a_{11}c_1 + a_{21}c_2 + \cdots + a_{k1}c_k = v_1 \\ a_{12}c_1 + a_{22}c_2 + \cdots + a_{k2}c_k = v_2 \\ \qquad \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{1n}c_1 + a_{2n}c_2 + \cdots + a_{kn}c_k = v_n . \end{cases}$$

When span(S) = $\mathbb{R}^n$ (Discussion 3.2.5)

Let $A = \begin{array}{c} \begin{array}{cccc} \boldsymbol{u}_1 & \boldsymbol{u}_2 & & \boldsymbol{u}_k \end{array} \\ \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{k1} \\ a_{12} & a_{22} & \cdots & a_{k2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{kn} \end{pmatrix} \end{array}$.

1. If a row-echelon form of $\boldsymbol{A}$ has no zero row, then the linear system is always consistent regardless the values of v_1, v_2, ..., v_n and hence span(S) = $\mathbb{R}^n$.
2. If a row-echelon form of $\boldsymbol{A}$ has at least one zero row, then the linear system is not always consistent and hence span(S) $\neq \mathbb{R}^n$.

(See Question 43 of Exercise 2 of the textbook.)

Examples (Example 3.2.4.6)

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

So span$\{ (1, 0, 1), (1, 1, 0), (0, 1, 1) \} = \mathbb{R}^3$.

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 0 & 3 & 1 \end{bmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So span$\{ (1, 1, 1), (1, 2, 0), (2, 1, 3), (2, 3, 1) \} \neq \mathbb{R}^3$.

When span(S) = $\mathbb{R}^n$ (Theorem 3.2.7 & Example 3.2.8)

Let $S = \{ \boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k \} \subseteq \mathbb{R}^n$. If $k < n$, span(S) $\neq \mathbb{R}^n$.

$$\begin{array}{cccc} \boldsymbol{u}_1 & \boldsymbol{u}_2 & & \boldsymbol{u}_k \\ \end{array}$$

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{k1} \\ \vdots & \vdots & & \vdots \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{kn} \end{bmatrix} \xrightarrow{\text{Gaussian Elimination}} \begin{bmatrix} * & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & * \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

the number of nonzero rows
= the number of leading entries
= the number of pivot columns
$\leq k < n$

In particular,

1. one vector cannot span $\mathbb{R}^2$;
2. one vector or two vectors cannot span $\mathbb{R}^3$.

Some basic results (Theorem 3.2.9)

Let $S = \{ \boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k} \} \subseteq \mathbb{R}^n$.

1. $\mathbf{0} \in \text{span}(S)$.
2. For any $\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_r} \in \text{span}(S)$ and $c_1, c_2, \ldots, c_r \in \mathbb{R}$,

$$c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_r\boldsymbol{v_r} \in \text{span}(S).$$

Proof:

1. $\mathbf{0} = 0\boldsymbol{u_1} + 0\boldsymbol{u_2} + \cdots + 0\boldsymbol{u_k} \in \text{span}(S)$.
2. Write

$$\begin{aligned}
\boldsymbol{v_1} &= a_{11}\boldsymbol{u_1} + a_{12}\boldsymbol{u_2} + \cdots + a_{1k}\boldsymbol{u_k}, \\
\boldsymbol{v_2} &= a_{21}\boldsymbol{u_1} + a_{22}\boldsymbol{u_2} + \cdots + a_{2k}\boldsymbol{u_k}, \\
&\vdots \\
\boldsymbol{v_r} &= a_{r1}\boldsymbol{u_1} + a_{r2}\boldsymbol{u_2} + \cdots + a_{rk}\boldsymbol{u_k}.
\end{aligned}$$

Some basic results (Theorem 3.2.9)

Then $c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_r\boldsymbol{v_r}$

$$\begin{aligned}
&= c_1(a_{11}\boldsymbol{u_1} + a_{12}\boldsymbol{u_2} + \cdots + a_{1k}\boldsymbol{u_k}) \\
&\qquad + c_2(a_{21}\boldsymbol{u_1} + a_{22}\boldsymbol{u_2} + \cdots + a_{2k}\boldsymbol{u_k}) \\
&\qquad\qquad + \cdots + c_r(a_{r1}\boldsymbol{u_1} + a_{r2}\boldsymbol{u_2} + \cdots + a_{rk}\boldsymbol{u_k}) \\
&= (c_1a_{11} + c_2a_{21} + \cdots + c_ra_{r1})\boldsymbol{u_1} \\
&\qquad + (c_1a_{12} + c_2a_{22} + \cdots + c_ra_{r2})\boldsymbol{u_2} \\
&\qquad\qquad + \cdots + (c_1a_{1k} + c_2a_{2k} + \cdots + c_ra_{rk})\boldsymbol{u_k}
\end{aligned}$$

which is a linear combination of $\boldsymbol{u_1}$, $\boldsymbol{u_2}$, ..., $\boldsymbol{u_k}$.

Hence $c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_r\boldsymbol{v_r} \in \text{span}(S)$.

When $\text{span}(S_1) \subseteq \text{span}(S_2)$ (Theorem 3.2.10)

Let $S_1 = \{ \boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k} \}$ and $S_2 = \{ \boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_m} \}$ are subsets of $\mathbb{R}^n$.

Then $\text{span}(S_1) \subseteq \text{span}(S_2)$ if and only if each $\boldsymbol{u_i}$ is a linear combination of $\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_m}$.

(Please read our textbook for a proof of the result.)

Examples (Example 3.2.11.1)

Let $\boldsymbol{u}_1 = (1, 0, 1)$, $\boldsymbol{u}_2 = (1, 1, 2)$, $\boldsymbol{u}_3 = (-1, 2, 1)$ and $\boldsymbol{v}_1 = (1, 2, 3)$, $\boldsymbol{v}_2 = (2, -1, 1)$.

Show that $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} = \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

Solution: (For two sets A and B, if we want to show that $A = B$, we need to show that $A \subseteq B$ and $B \subseteq A$.)

To show $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \subseteq \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$:

It suffices to show that $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{u}_3$ are linear combinations of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$. (See Theorem 3.2.10.)

Examples (Example 3.2.11.1)

$\boldsymbol{u}_1$ is a linear combination of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$

$\Leftrightarrow$ $\boldsymbol{u}_1 = a\boldsymbol{v}_1 + b\boldsymbol{v}_2$ for some real numbers a and b

$\Leftrightarrow$ $(1, 0, 1) = a(1, 2, 3) + b(2, -1, 1)$ for some real numbers a and b

$\Leftrightarrow$ $(1, 0, 1) = (a + 2b,\ 2a - b,\ 3a + b)$ for some real numbers a and b

$\Leftrightarrow$ the linear system $\begin{cases} a + 2b = 1 \\ 2a - b = 0 \\ 3a + b = 1 \end{cases}$ is consistent.

Examples (Example 3.2.11.1)

Similarly,

$\boldsymbol{u}_2$ is a linear combination of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$

$\Leftrightarrow$ the linear system $\begin{cases} a + 2b = 1 \\ 2a - b = 1 \\ 3a + b = 2 \end{cases}$ is consistent;

and

$\boldsymbol{u}_3$ is a linear combination of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$

$\Leftrightarrow$ the linear system $\begin{cases} a + 2b = -1 \\ 2a - b = 2 \\ 3a + b = 1 \end{cases}$ is consistent.

Examples (Example 3.2.11.1)

$$\begin{cases} a + 2b = 1 \\ 2a - b = 0 \\ 3a + b = 1 \end{cases} \qquad \begin{cases} a + 2b = 1 \\ 2a - b = 1 \\ 3a + b = 2 \end{cases} \qquad \begin{cases} a + 2b = -1 \\ 2a - b = 2 \\ 3a + b = 1 \end{cases}$$

The row operations required to solve the three systems are the same, we can work them out together:

$$\left[\begin{array}{cc|c|c|c} 1 & 2 & 1 & 1 & -1 \\ 2 & -1 & 0 & 1 & 2 \\ 3 & 1 & 1 & 2 & 1 \end{array}\right] \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left[\begin{array}{cc|c|c|c} 1 & 2 & 1 & 1 & -1 \\ 0 & -5 & -2 & -1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The three systems are consistent, i.e. all $\boldsymbol{u}_i$ are linear combinations of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

So $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \subseteq \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

Examples (Theorem 3.2.11.1)

To show span{ $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ } $\subseteq$ span{ $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{u}_3$ }:

To check that $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ are linear combinations of $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{u}_3$, we need to show the following two linear systems are consistent.

$$\begin{cases} a + b - c = 1 \\ b + 2c = 2 \\ a + 2b + c = 3 \end{cases} \qquad \begin{cases} a + b - c = 2 \\ b + 2c = -1 \\ a + 2b + c = 1 \end{cases}$$

$$\left[\begin{array}{ccc|c|c} 1 & 1 & -1 & 1 & 2 \\ 0 & 1 & 2 & 2 & -1 \\ 1 & 2 & 1 & 3 & 1 \end{array}\right] \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left[\begin{array}{ccc|c|c} 1 & 1 & -1 & 1 & 2 \\ 0 & 1 & 2 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Examples (Example 3.2.11.1)

The two systems are consistent, i.e. all $\boldsymbol{v}_i$ are linear combinations of $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{u}_3$.

So $\text{span}\{ \boldsymbol{v}_1, \boldsymbol{v}_2 \} \subseteq \text{span}\{ \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3 \}$.

Since we have shown $\text{span}\{ \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3 \} \subseteq \text{span}\{ \boldsymbol{v}_1, \boldsymbol{v}_2 \}$ and $\text{span}\{ \boldsymbol{v}_1, \boldsymbol{v}_2 \} \subseteq \text{span}\{ \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3 \}$,

$$\text{span}\{ \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3 \} = \text{span}\{ \boldsymbol{v}_1, \boldsymbol{v}_2 \}.$$

Examples (Example 3.2.11.2)

Let $\boldsymbol{u}_1 = (1, 0, 0, 1)$, $\boldsymbol{u}_2 = (0, 1, -1, 2)$, $\boldsymbol{u}_3 = (2, 1, -1, 4)$,
$\boldsymbol{v}_1 = (1, 1, 1, 1)$, $\boldsymbol{v}_2 = (-1, 1, -1, 1)$, $\boldsymbol{v}_3 = (-1, 1, 1, -1)$.

To check whether $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \subseteq \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$:

$$\left[\begin{array}{ccc|c|c|c} 1 & -1 & -1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & -1 & 1 & 0 & -1 & -1 \\ 1 & 1 & -1 & 1 & 2 & 4 \end{array}\right] \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left[\begin{array}{ccc|c|c|c} 1 & -1 & -1 & 1 & 0 & 2 \\ 0 & 2 & 2 & -1 & 1 & -1 \\ 0 & 0 & 2 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

All three systems are consistent, i.e. all $\boldsymbol{u}_i$ are linear combinations of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$.

So $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \subseteq \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.

Examples (Example 3.2.11.2)

$\boldsymbol{u}_1 = (1, 0, 0, 1)$, $\boldsymbol{u}_2 = (0, 1, -1, 2)$, $\boldsymbol{u}_3 = (2, 1, -1, 4)$,
$\boldsymbol{v}_1 = (1, 1, 1, 1)$, $\boldsymbol{v}_2 = (-1, 1, -1, 1)$, $\boldsymbol{v}_3 = (-1, 1, 1, -1)$.

To check whether $\text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\} \subseteq \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$:

$$\left[\begin{array}{ccc|c|c|c} 1 & 0 & 2 & 1 & -1 & -1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 1 & -1 & 1 \\ 1 & 2 & 4 & 1 & 1 & -1 \end{array}\right] \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left[\begin{array}{ccc|c|c|c} 1 & 0 & 2 & 1 & -1 & -1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since not all systems are consistent, some $\boldsymbol{v}_i$ are not linear combinations of $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{u}_3$.

So $\text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\} \not\subseteq \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$.

Redundant vectors (Theorem 3.2.12 & Example 3.2.13)

Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k \in \mathbb{R}^n$.

If $\boldsymbol{u}_k$ is a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}$, then

$$\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}\} = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}, \boldsymbol{u}_k\}.$$

(Please read our textbook for a proof of the result.)

$\boldsymbol{u}_k$ is a redundant vector.

For example, let $\boldsymbol{u}_1 = (1, 1, 0, 2)$, $\boldsymbol{u}_2 = (1, 0, 0, 1)$, $\boldsymbol{u}_3 = (0, 1, 0, 1)$.

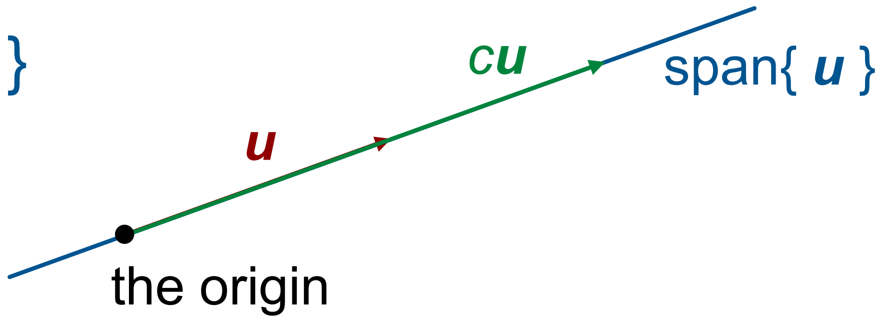
It is easy to check that $\boldsymbol{u}_3 = \boldsymbol{u}_1 - \boldsymbol{u}_2$.

So $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$.

Geometrical interpretation (Discussion 3.2.14.1)

Let $\boldsymbol{u}$ be a nonzero vector in $\mathbb{R}^2$ or $\mathbb{R}^3$.

Then $\text{span}\{\, \boldsymbol{u} \,\} = \{\, c\boldsymbol{u} \mid c \in \mathbb{R} \,\}$ is a line through the origin.



In $\mathbb{R}^2$, if $\boldsymbol{u} = (u_1, u_2)$, then

$$\text{span}\{\, \boldsymbol{u} \,\} = \{\, (cu_1, cu_2) \mid c \in \mathbb{R} \,\} = \{\, (x, y) \mid u_2x - u_1y = 0 \,\}.$$

In $\mathbb{R}^3$, if $\boldsymbol{u} = (u_1, u_2, u_3)$, then

$$\text{span}\{\, \boldsymbol{u} \,\} = \{\, (cu_1, cu_2, cu_3) \mid c \in \mathbb{R} \,\}.$$

Geometrical interpretation (Discussion 3.2.14.2)

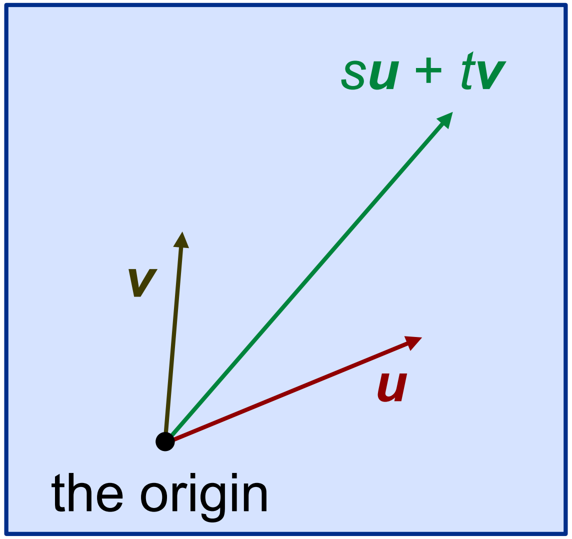
Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be a nonzero vector in $\mathbb{R}^2$ or $\mathbb{R}^3$.

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are not parallel, then

$$\text{span}\{\, \boldsymbol{u}, \boldsymbol{v} \,\} = \{\, s\boldsymbol{u} + t\boldsymbol{v} \mid s, t \in \mathbb{R} \,\}$$

is a plane containing the origin.

In $\mathbb{R}^2$, then $\text{span}\{\, \boldsymbol{u}, \boldsymbol{v} \,\} = \mathbb{R}^2$.



span($\boldsymbol{u}$, $\boldsymbol{v}$)

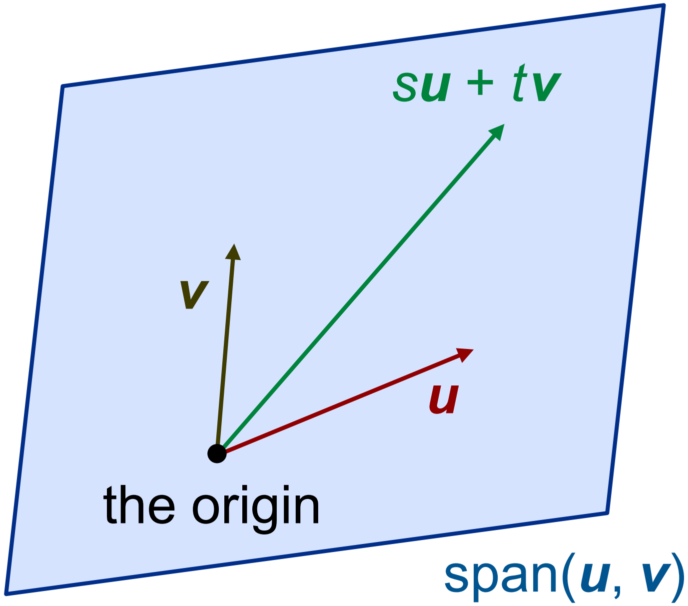
Geometrical interpretation (Discussion 3.2.14.2)

In $\mathbb{R}^3$, if $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$, then

$$\begin{aligned}\text{span}\{\boldsymbol{u}, \boldsymbol{v}\} &= \{ (su_1 + tv_1,\ su_2 + tv_2,\ su_3 + tv_3) \mid s, t \in \mathbb{R} \} \\ &= \{ (x, y, z) \mid ax + by + cz = 0 \}\end{aligned}$$

where (a, b, c) is (any) one non-trivial solution of the linear system

$$\begin{cases} u_1 a + u_2 b + u_3 c = 0 \\ v_1 a + v_2 b + v_3 c = 0. \end{cases}$$



See Example 5.2.11 for the geometrical interpretation of the vector (a, b, c).

Lines in $\mathbb{R}^2$ and $\mathbb{R}^3$ (Discussion 3.2.15.1)

Let L be a line in $\mathbb{R}^2$ or $\mathbb{R}^3$.

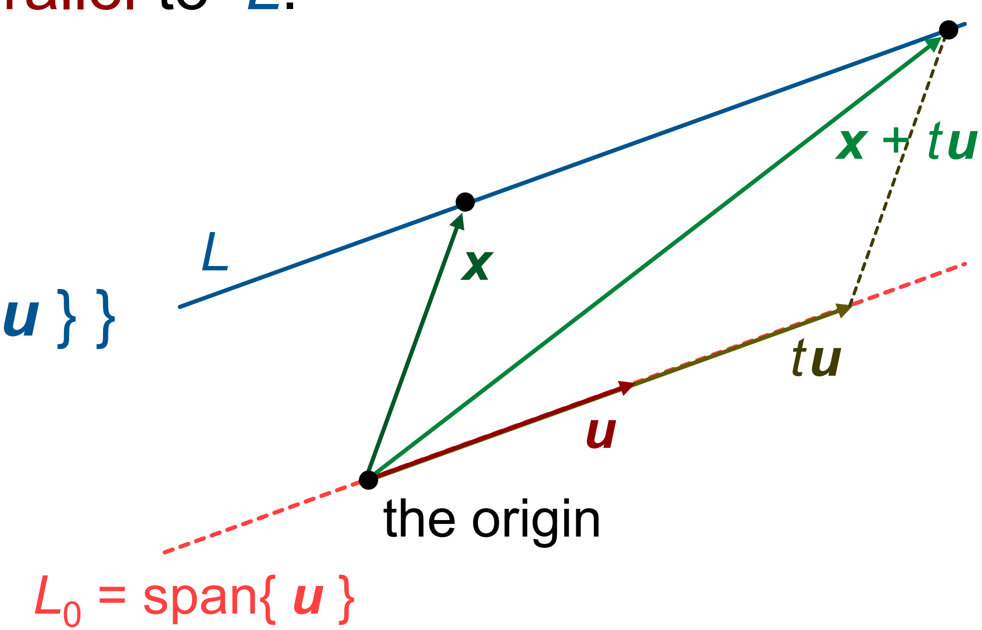
Pick a point $\boldsymbol{x}$ on L

(where $\boldsymbol{x}$ is regarded as a vector joining the origin to the point)

and a nonzero vector $\boldsymbol{u}$ such that span{ $\boldsymbol{u}$ } is a line L_0 through the origin and parallel to L.

Explicitly,

$$
\begin{aligned}
L &= \{\, \boldsymbol{x} + \boldsymbol{w} \mid \boldsymbol{w} \in L_0 \,\} \\
&= \{\, \boldsymbol{x} + \boldsymbol{w} \mid \boldsymbol{w} \in \operatorname{span}\{\, \boldsymbol{u} \,\} \,\} \\
&= \{\, \boldsymbol{x} + t\boldsymbol{u} \mid t \in \mathbb{R} \,\}.
\end{aligned}
$$



Planes in $\mathbb{R}^3$ (Discussion 3.2.15.2)

Let P be a plane in $\mathbb{R}^3$.

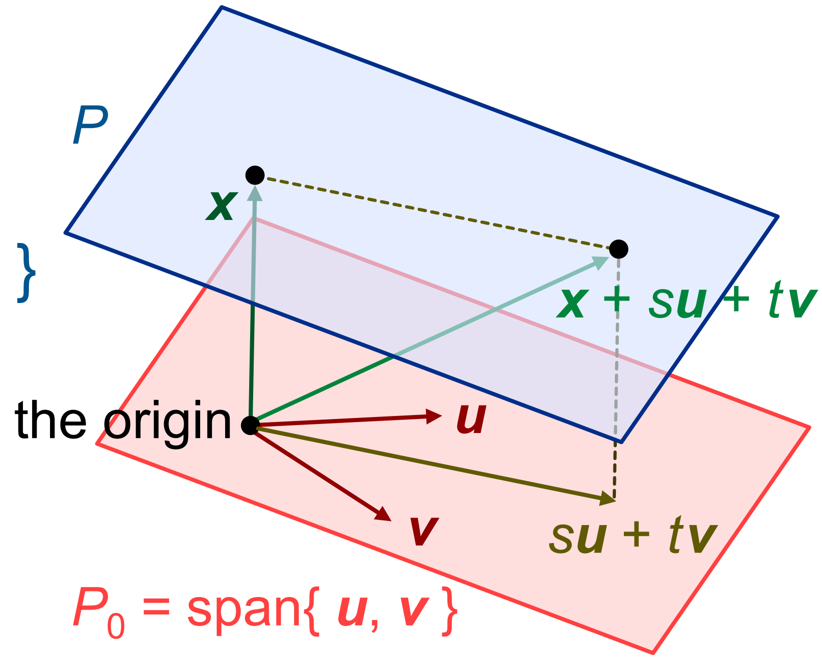
Pick a point $\boldsymbol{x}$ on P

(where $\boldsymbol{x}$ is regarded as a vector joining the origin to the point)

and two nonzero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ such that $\text{span}\{\boldsymbol{u}, \boldsymbol{v}\}$ is a plane P_0 containing the origin and parallel to P.

Explicitly,

$$\begin{aligned} P &= \{\boldsymbol{x} + \boldsymbol{w} \mid \boldsymbol{w} \in P_0\} \\ &= \{\boldsymbol{x} + \boldsymbol{w} \mid \boldsymbol{w} \in \text{span}\{\boldsymbol{u}, \boldsymbol{v}\}\} \\ &= \{\boldsymbol{x} + s\boldsymbol{u} + t\boldsymbol{v} \mid s, t \in \mathbb{R}\}. \end{aligned}$$



Lines and planes in $\mathbb{R}^n$ (Discussion 3.2.15)

We can generalize the idea of lines and planes to $\mathbb{R}^n$:

Take $\boldsymbol{x}, \boldsymbol{u} \in \mathbb{R}^n$ where $\boldsymbol{u}$ is a nonzero vector.

The set $L = \{\, \boldsymbol{x} + \boldsymbol{w} \mid \boldsymbol{w} \in \text{span}\{\, \boldsymbol{u} \,\}\,\}$ is called a line in $\mathbb{R}^n$.

Take $\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ where $\boldsymbol{u}, \boldsymbol{v}$ is a nonzero vector and $\boldsymbol{u}$ is not a multiple of $\boldsymbol{v}$.

The set $P = \{\, \boldsymbol{x} + \boldsymbol{w} \mid \boldsymbol{w} \in \text{span}\{\, \boldsymbol{u}, \boldsymbol{v} \,\}\,\}$ is called a plane in $\mathbb{R}^n$.

Lines and planes in $\mathbb{R}^n$ (Discussion 3.2.15)

Take $\boldsymbol{x}, \boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_r \in \mathbb{R}^n$.

The set $Q = \{ \boldsymbol{x} + \boldsymbol{w} \mid \boldsymbol{w} \in \text{span}\{ \boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_r \} \}$ is called a k-plane in $\mathbb{R}^n$

where k is the "dimension" of span$\{ \boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_r \}$ which will be studied in Section 3.6.

Section 3.3

Subspaces

Subspaces (Discussion 3.3.1)

At the end of last section, we have learnt that linear spans are used to define lines and planes in $\mathbb{R}^n$.

Furthermore, a linear span of vectors in $\mathbb{R}^n$ always behaves like a smaller "space" sitting insider in $\mathbb{R}^n$.

For example, in $\mathbb{R}^3$, span$\{ (1, 0, 0), (0, 1, 0) \}$ is the xy-plane which behaves like an $\mathbb{R}^2$ sitting inside $\mathbb{R}^3$.

To make it easier for us to refer to such sets of vectors, we give a new name to describe them.

Subspaces (Definition 3.3.2)

Let V be a subset of $\mathbb{R}^n$.

V is called a subspace of $\mathbb{R}^n$ if $V = \text{span}(S)$ where $S = \{ \boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k \}$ for some vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k \in \mathbb{R}^n$.

More precisely, we say that

V is a subspace spanned by S;

or V is a subspace spanned by $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$.

We also say that

S spans V.

or $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ spans V.

Trivial subspaces (Remark 3.3.3.1-2)

Let $\mathbf{0}$ be the zero vector of $\mathbb{R}^n$.

The set $\{\mathbf{0}\} = \text{span}\{\mathbf{0}\}$ is a subspace of $\mathbb{R}^n$ and is known as the zero space.

Let $\mathbf{e_1} = (1, 0, ..., 0)$, $\mathbf{e_2} = (0, 1, 0, ..., 0)$, ..., $\mathbf{e}_n = (0, ..., 0, 1)$ be vectors in $\mathbb{R}^n$.

Any vector $\boldsymbol{u} = (u_1, u_2, ..., u_n) \in \mathbb{R}^n$ can be written as

$$\boldsymbol{u} = u_1\mathbf{e_1} + u_2\mathbf{e_2} + \cdots + u_n\mathbf{e}_n.$$

Thus $\mathbb{R}^n = \text{span}\{\mathbf{e_1}, \mathbf{e_2}, ..., \mathbf{e}_n\}$ is a subspace of $\mathbb{R}^n$.

Examples (Example 3.3.4.1-2)

Let $V_1 = \{ (a + 4b, \ a) \mid a, b \in \mathbb{R} \} \subseteq \mathbb{R}^2$.

For any $a, b \in \mathbb{R}$, $(a + 4b, \ a) = a(1, 1) + b(4, 0)$.

So $V_1 = \text{span}\{ (1, 1), \ (4, 0) \}$ is a subspace of $\mathbb{R}^2$.

Let $V_2 = \{ (x, y, z) \mid x + y - z = 0 \} \subseteq \mathbb{R}^3$.

The equation $x + y - z = 0$ has a general solution

$$(x, y, z) = (-s + t, \ s, \ t) = s(-1, 1, 0) + t(1, 0, 1)$$

where s and t are arbitrary parameters.

So $V_2 = \text{span}\{ (-1, 1, 0), \ (1, 0, 1) \}$ is a subspace of $\mathbb{R}^3$.

How to determine a subset is a subspace?

By Definition 3.2.3, if $S = \{ \boldsymbol{u_1}, \boldsymbol{u_2}, ..., \boldsymbol{u_k} \}$, then $\text{span}(S)$ is a set containing all the vectors written in the form $c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}$.

If all the vectors in a subset V of $\mathbf{R}^n$ can be written as

$$c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}$$

where $c_1, c_2, ..., c_k$ are arbitrary parameters

and $\boldsymbol{u_1}, \boldsymbol{u_2}, ..., \boldsymbol{u_k}$ are constant vectors

(their coordinates do not consist of arbitrary parameters),

then V is a subspace of $\mathbf{R}^n$.

(This is what we have done in the previous slide.)

How to determine a subset is a subspace?

By the definition, to show that a subset W is not a subspace, one need to show that not all vectors in W can be written as $c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}$ for some constant vectors $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}$.

But it is always difficult to show that something cannot be expressed in certain form.

Instead, we try to show that W does not satisfy some known properties of subspaces.

How to determine a subset is a subspace?

We replace span(S) in Theorem 3.2.9 by a subspace V:

Let V be a subspace of $\mathbb{R}^n$.

1. $\mathbf{0} \in V$.
2. For any $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r \in V$ and $c_1, c_2, \ldots, c_r \in \mathbb{R}$,
   $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_r\boldsymbol{v}_r \in V$.

For example, if a subset W does not contain the zero vector, then W is not a subspace;

or if there exists $\boldsymbol{v} \in W$ and $c \in \mathbb{R}$ such that $c\boldsymbol{v} \notin W$, then W is not a subspace;

or if there exists $\boldsymbol{u}, \boldsymbol{v} \in W$ such that $\boldsymbol{u} + \boldsymbol{v} \notin W$, then W is not a subspace.

Examples (Example 3.3.4.3-4)

Let $V_3 = \{ (1, a) \mid a \in \mathbb{R} \} \subseteq \mathbb{R}^2$.

As $(0, 0) \neq (1, a)$ for any $a \in \mathbb{R}$, the zero vector is not contained in V_3.

So V_3 is not a subspace of $\mathbb{R}^2$.

Let $V_4 = \{ (x, y, z) \mid x^2 \leq y^2 \leq z^2 \} \subseteq \mathbb{R}^3$.

Observe that $(1, 1, 2), (1, 1, -2) \in V_4$.

Note that $(1, 1, 2) + (1, 1, -2) = (2, 2, 0)$,
but $(x, y, z) = (2, 2, 0)$ does not satisfy $x^2 \leq y^2 \leq z^2$.

This means $(1, 1, 2) + (1, 1, -2) \notin V_4$.

So V_4 is not a subspace of $\mathbb{R}^3$.

Geometrical interpretation (Remark 3.3.5.1)

The following are all the subspaces of $\mathbb{R}^2$:

(a) the zero space $\{ (0, 0) \}$;

(b) lines through the origin;

(c) $\mathbb{R}^2$.

The following are all the subspaces of $\mathbb{R}^3$:

(a) the zero space $\{ (0, 0, 0) \}$;

(b) lines through the origin;

(c) planes containing the origin;

(d) $\mathbb{R}^3$.

Solution spaces (Theorem 3.3.6)

The solution set of a homogenous system of linear equations in n variables is a subspace of $\mathbb{R}^n$.

Proof: If the homogenous system has only the trivial solution, then the solution set is $\{ \mathbf{0} \}$ which is the zero space.

Suppose the homogeneous system has infinitely many solutions.

Let x_1, x_2, ..., x_n be the variables of the system.

Solution spaces (Theorem 3.3.6)

By solving the system using Gauss-Jordan Elimination, a general solution can be expressed in the form

$$\begin{cases} x_1 = r_{11}t_1 + r_{12}t_2 + \cdots + r_{1k}t_k \\ x_2 = r_{21}t_1 + r_{22}x_2 + \cdots + r_{2k}t_k \\ \vdots \qquad\qquad\qquad\quad \vdots \\ x_n = r_{n1}t_1 + r_{n2}t_2 + \cdots + r_{nk}t_k \end{cases}$$

for some arbitrary parameters t_1, t_2, ..., t_k, where r_{11}, r_{12}, …, r_{nk} are real numbers,

Solution spaces (Theorem 3.3.6)

We can rewrite the general solution as

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = t_1 \begin{bmatrix} r_{11} \\ r_{21} \\ \vdots \\ r_{n1} \end{bmatrix} + t_2 \begin{bmatrix} r_{12} \\ r_{22} \\ \vdots \\ r_{n2} \end{bmatrix} + \cdots + t_n \begin{bmatrix} r_{1k} \\ r_{2k} \\ \vdots \\ r_{nk} \end{bmatrix}.$$

So the solution set is

span$\{ (r_{11}, r_{21}, \ldots, r_{n1}),\ (r_{12}, r_{22}, \ldots, r_{n2}),\ \ldots,\ (r_{1k}, r_{2k}, \ldots, r_{nk}) \}$

and hence is a subspace of $\mathbb{R}^n$.

Examples (Example 3.3.7.1)

The linear system $\begin{cases} x - 2y + 3z = 0 \\ 2x - 4y + 6z = 0 \\ 3x - 6y + 9z = 0 \end{cases}$

has a general solution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2s - 3t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

where s, t are arbitrary parameters.

The solution set is $\{\, (2s - 3t,\ s,\ t) \mid s, t \in \mathbb{R} \,\}$
$= \mathrm{span}\{\, (2, 1, 0),\ (-3, 0, 1) \,\}$.

It is a plane in $\mathbb{R}^3$ containing the origin.

Examples (Example 3.3.7.2)

The linear system
$$\begin{cases} x - 2y + 3z = 0 \\ -3x + 7y - 8z = 0 \\ -2x + 4y - 6z = 0 \end{cases}$$

has a general solution
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5t \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} -5 \\ -1 \\ 1 \end{bmatrix}$$
where t is an arbitrary parameter.

The solution set is $\{ (-5t, -t, t) \mid t \in \mathbb{R} \}$
$= \operatorname{span}\{ (-5, -1, 1) \}$.

It is a line in $\mathbb{R}^3$ through the origin.

Examples (Example 3.3.7.3)

The linear system

$$\begin{cases} x - 2y + 3z = 0 \\ -3x + 7y - 8z = 0 \\ 4x + y + 2z = 0 \end{cases}$$

has a general solution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The solution set is $\{ (0, 0, 0) \}$, the zero space.

An alternative definition (Remark 3.3.8)

Let V be a non-empty subset of $\mathbb{R}^n$.

V is called a subspace of $\mathbb{R}^n$ if and only if

for all $\boldsymbol{u}, \boldsymbol{v} \in V$ and $c, d \in \mathbb{R}$, $c\boldsymbol{u} + d\boldsymbol{v} \in V$.

(This is the definition of subspaces in abstract linear algebra.)

The two definitions of subspaces are the same. (See Question 3.31.)

This definition sometimes gives us a neater way to show when a set of vectors is a subspace.

For example, we give another proof for Theorem 3.3.6, i.e. the solution set of a homogenous system of linear equations in n variables is a subspace of $\mathbb{R}^n$.

An alternative definition (Remark 3.3.8)

A subset V of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if and only if
(i) V is non-empty and (ii) for all $\boldsymbol{u}, \boldsymbol{v} \in V$ and $c, d \in \mathbb{R}$, $c\boldsymbol{u} + d\boldsymbol{v} \in V$.

Let $\boldsymbol{Ax} = \boldsymbol{0}$ be the homogeneous linear system and $V \subseteq \mathbb{R}^n$ be the solution set of the system.

Since $\boldsymbol{x} = \boldsymbol{0}$ is a solution to $\boldsymbol{Ax} = \boldsymbol{0}$, $\boldsymbol{0} \in V$ and hence V is non-empty.

Take any $\boldsymbol{u}, \boldsymbol{v} \in V$, i.e. $\boldsymbol{Au} = \boldsymbol{0}$ and $\boldsymbol{Av} = \boldsymbol{0}$.

For any $c, d \in \mathbb{R}$,

$$\boldsymbol{A}(c\boldsymbol{u} + d\boldsymbol{v}) = \boldsymbol{A}(c\boldsymbol{u}) + \boldsymbol{A}(d\boldsymbol{v}) = c\boldsymbol{Au} + d\boldsymbol{Av} = c\boldsymbol{0} + d\boldsymbol{0} = \boldsymbol{0}.$$

Thus $\boldsymbol{x} = c\boldsymbol{u} + d\boldsymbol{v}$ is a solution to $\boldsymbol{Ax} = \boldsymbol{0}$ and hence $c\boldsymbol{u} + d\boldsymbol{v} \in V$.

So we have shown that V is a subspace of $\mathbb{R}^n$.

Section 3.4

Linear Independence

Redundant vectors (Discussion 3.4.1)

Let $\boldsymbol{u}_1 = (1, 1, 0, 2)$, $\boldsymbol{u}_2 = (1, 0, 0, 1)$, $\boldsymbol{u}_3 = (0, 1, 0, 1)$.

Since $\boldsymbol{u}_3 = \boldsymbol{u}_1 - \boldsymbol{u}_2$, $\text{span}\{\, \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3 \,\} = \text{span}\{\, \boldsymbol{u}_1, \boldsymbol{u}_2 \,\}$.

$\boldsymbol{u}_3$ is a redundant vector.

Given a subspace $V = \text{span}\{\, \boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k \,\}$, how do we know whether there are redundant vectors among $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, ..., $\boldsymbol{u}_k$?

We can answer the question by using the concept of linear dependence (or independence).

Linear independence (Definition 3.4.2)

Let $S = \{ \boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k} \} \subseteq \mathbb{R}^n$.

Consider the vector equation

$$c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k} = \boldsymbol{0} \qquad (*)$$

where $c_1, c_2, \ldots, c_k$ are variables.

Note that $c_1 = 0, c_2 = 0, \ldots, c_n = 0$ satisfies $(*)$ and hence is a solution to $(*)$. This solution is called the trivial solution.

1. S is called a linearly independent set and $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}$ are said to be linearly independent if $(*)$ has only the trivial solution.

Linear independence (Definition 3.4.2)

2. S is called a linearly dependent set and $\boldsymbol{u_1}$, $\boldsymbol{u_2}$, ..., $\boldsymbol{u_k}$ are said to be linearly dependent if $(*)$ has non-trivial solutions,

   i.e. there exists real numbers a_1, a_2, ..., a_k, not all of them are zero, such that $a_1\boldsymbol{u_1} + a_2\boldsymbol{u_2} + \cdots + a_k\boldsymbol{u_k} = \boldsymbol{0}$.

We shall learn that in span$\{ \boldsymbol{u_1}, \boldsymbol{u_2}, ..., \boldsymbol{u_k} \}$, there are no redundant vectors among $\boldsymbol{u_1}$, $\boldsymbol{u_2}$, ..., $\boldsymbol{u_k}$ if and only if $\boldsymbol{u_1}$, $\boldsymbol{u_2}$, ..., $\boldsymbol{u_k}$ are linearly independent.

(See Theorem 3.4.4 and Remark 3.4.5.)

Examples (Example 3.4.3.1)

Determine whether the vectors $(1, -2, 3)$, $(5, 6, -1)$, $(3, 2, 1)$ are linearly independent.

Solution:

$$c_1(1, -2, 3) + c_2(5, 6, -1) + c_3(3, 2, 1) = (0, 0, 0)$$

$$\Leftrightarrow \begin{cases} c_1 + 5c_2 + 3c_3 = 0 \\ -2c_1 + 6c_2 + 2c_3 = 0 \\ 3c_1 - c_2 + c_3 = 0. \end{cases}$$

By Gaussian Elimination, we find that there are infinitely many solutions.

So the vectors are linearly dependent.

Examples (Example 3.4.3.2)

Determine whether the vectors $(1, 0, 0, 1)$, $(0, 2, 1, 0)$, $(1, -1, 1, 1)$ are linearly independent.

Solution:

$$c_1(1, 0, 0, 1) + c_2(0, 2, 1, 0) + c_3(1, -1, 1, 1) = (0, 0, 0, 0)$$

$$\Leftrightarrow \begin{cases} c_1 + c_3 = 0 \\ 2c_2 - c_3 = 0 \\ c_2 + c_3 = 0 \\ c_1 + c_3 = 0. \end{cases}$$

By Gaussian Elimination, we find that there is only the trivial solution.

So the vectors are linearly independent.

Examples (Example 3.4.3.3)

Let $S = \{ \boldsymbol{u} \} \subseteq \mathbb{R}^n$.

S is linearly dependent means that there exists a real number $a \neq 0$ such that $a\boldsymbol{u} = \boldsymbol{0}$.

For any $a \neq 0$, $a\boldsymbol{u} = \boldsymbol{0} \Leftrightarrow \boldsymbol{u} = a^{-1}\boldsymbol{0} = \boldsymbol{0}$.

So S is linearly dependent if and only if $\boldsymbol{u} = \boldsymbol{0}$.

Examples (Example 3.4.3.4)

Let $S = \{ \boldsymbol{u}, \boldsymbol{v} \} \subseteq \mathbb{R}^n$.

S is linearly dependent means that there exist real numbers c, d, not both are zero, such that $c\boldsymbol{u} + d\boldsymbol{v} = \boldsymbol{0}$.

When $c \neq 0$, $c\boldsymbol{u} + d\boldsymbol{v} = \boldsymbol{0} \Leftrightarrow \boldsymbol{u} = -c^{-1}d\boldsymbol{v}$.

When $d \neq 0$, $c\boldsymbol{u} + d\boldsymbol{v} = \boldsymbol{0} \Leftrightarrow \boldsymbol{v} = -d^{-1}c\boldsymbol{u}$.

So S is linearly dependent if and only if

$\boldsymbol{u} = a\boldsymbol{v}$ for some real number a

or $\boldsymbol{v} = b\boldsymbol{u}$ for some real number b.

Examples (Example 3.4.3.5)

Let $S = \{ \boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k} \} \subseteq \mathbb{R}^n$.

Suppose $\boldsymbol{0} \in S$, say, $\boldsymbol{u_i} = \boldsymbol{0}$ for some $i = 1, 2, \ldots, k$.

Then $c_1 = 0$, $\ldots$, $c_{i-1} = 0$, $c_i = 1$, $c_{i+1} = 0$, $\ldots$, $c_k = 0$ is a non-trivial solution to the equation

$$c_1\boldsymbol{u_1} + \cdots + c_{i-1}\boldsymbol{u_{i-1}} + c_i\boldsymbol{u_i} + c_{i+1}\boldsymbol{u_{i+1}} + \cdots + c_k\boldsymbol{u_k} = \boldsymbol{0}.$$

So S is linearly dependent.

Linear independence (Theorem 3.4.4)

Let $S = \{ \boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k} \} \subseteq \mathbb{R}^n$ where $k \geq 2$.

1. S is linearly dependent if and only if at least one vector $\boldsymbol{u_i} \in S$ can be written as a linear combination of other vectors in S,

   i.e. $\boldsymbol{u_i} = a_1\boldsymbol{u_1} + \cdots + a_{i-1}\boldsymbol{u_{i-1}} + a_{i+1}\boldsymbol{u_{i+1}} + \cdots + a_k\boldsymbol{u_k}$ for some real numbers $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_k$.

2. S is linearly independent if and only if no vector in S can be written as a linear combination of other vectors in S.

(The statements 1 and 2 are logically equivalent. We only need to prove one of them.)

Linear independence (Theorem 3.4.4)

In the following, we prove statement 1:

($\Rightarrow$) Suppose S is linearly dependent, i.e. there exists real numbers b_1, b_2, ..., b_k, not all of them are zero, such that $b_1\boldsymbol{u_1} + b_2\boldsymbol{u_2} + \cdots + b_k\boldsymbol{u_k} = \boldsymbol{0}$.

Let b_i be one of the nonzero coefficients.

Then

$$b_i\boldsymbol{u_i} = -(b_1\boldsymbol{u_1} + \cdots + b_{i-1}\boldsymbol{u_{i-1}} + b_{i+1}\boldsymbol{u_{i+1}} + \cdots + b_k\boldsymbol{u_k})$$

implies

$$\begin{aligned}\boldsymbol{u_i} &= -b_i^{-1}(b_1\boldsymbol{u_1} + \cdots + b_{i-1}\boldsymbol{u_{i-1}} + b_{i+1}\boldsymbol{u_{i+1}} + \cdots + b_k\boldsymbol{u_k}) \\ &= a_1\boldsymbol{u_1} + \cdots + a_{i-1}\boldsymbol{u_{i-1}} + a_{i+1}\boldsymbol{u_{i+1}} + \cdots + a_k\boldsymbol{u_k}\end{aligned}$$

where $a_j = -b_i^{-1}b_j$ for $j = 1, ..., i-1, j+1, ..., k$.

Linear independence (Theorem 3.4.4)

(⇐) Suppose there exists $\boldsymbol{u_i}$ such that

$$\boldsymbol{u_i} = a_1\boldsymbol{u_1} + \cdots + a_{i-1}\boldsymbol{u_{i-1}} + a_{i+1}\boldsymbol{u_{i+1}} + \cdots + a_k\boldsymbol{u_k}$$

for some real numbers a_1, ..., a_{i-1}, a_{i+1}, ..., a_k.

Then $c_1 = a_1$, ..., $c_{i-1} = a_{i-1}$, $c_i = -1$, $c_{i+1} = a_{i+1}$, ..., $c_k = a_k$ is a non-trivial solution to the equation

$$c_1\boldsymbol{u_1} + \cdots + c_{i-1}\boldsymbol{u_{i-1}} + c_i\boldsymbol{u_i} + c_{i+1}\boldsymbol{u_{i+1}} + \cdots + c_k\boldsymbol{u_k} = \boldsymbol{0}.$$

So S is linearly dependent.

Redundant vectors (Remark 3.4.5)

1. If a set of vectors is linearly dependent, then there exists at least one redundant vector in the set.
2. If a set of vectors is linearly independent, then there is no redundant vector in the set.

Examples (Example 3.4.6.1)

Let $S_1 = \{ (1, 0), (0, 4), (2, 4) \} \subseteq \mathbb{R}^2$.

S_1 is linearly dependent.

(The equation $c_1(1, 0) + c_2(0, 4) + c_1(2, 4) = (0, 0)$ has non-trivial solution.)

We see that $(2, 4) = 2(1, 0) + (0, 4)$,

i.e. $(2, 4)$ can be expressed as a linear combination of $(1, 0)$ and $(0, 4)$.

Examples (Example 3.4.6.2)

Let $S_2 = \{ (-1, 0, 0), (0, 3, 0), (0, 0, 7) \} \subseteq \mathbb{R}^3$.

S_2 is linearly independent.

(The equation $c_1(-1, 0, 0) + c_2(0, 3, 0) + c_1(0, 0, 7) = (0, 0, 0)$ has only the trivial solution.)

$(-1, 0, 0)$ cannot be expressed as a linear combination of $(0, 3, 0)$ and $(0, 0, 7)$.

$(0, 3, 0)$ cannot be expressed as a linear combination of $(-1, 0, 0)$ and $(0, 0, 7)$.

$(0, 0, 7)$ cannot be expressed as a linear combination of $(-1, 0, 0)$ and $(0, 3, 0)$.

Linear independence (Theorem 3.4.7 & Example 3.4.9)

Let $S = \{ \boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k \} \subseteq \mathbb{R}^n$.

If $k > n$, then S is linearly dependent.

In particular,

1. In $\mathbb{R}^2$, a set of three or more vectors must be linearly dependent;
2. In $\mathbb{R}^3$, a set of four or more vectors must be linearly dependent.

Proof of the theorem (Theorem 3.4.7)

Proof: Let $\boldsymbol{u_i} = (a_{i1}, a_{i2}, \ldots, a_{in})$ for $i = 1, 2, \ldots, k$.

Then

$$c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k} = \mathbf{0}$$

$$\Leftrightarrow \begin{cases} a_{11}c_1 + a_{21}c_2 + \cdots + a_{k1}c_k = 0 \\ a_{12}c_1 + a_{22}c_2 + \cdots + a_{k2}c_k = 0 \\ \qquad \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{1n}c_1 + a_{2n}c_2 + \cdots + a_{kn}c_k = 0. \end{cases}$$

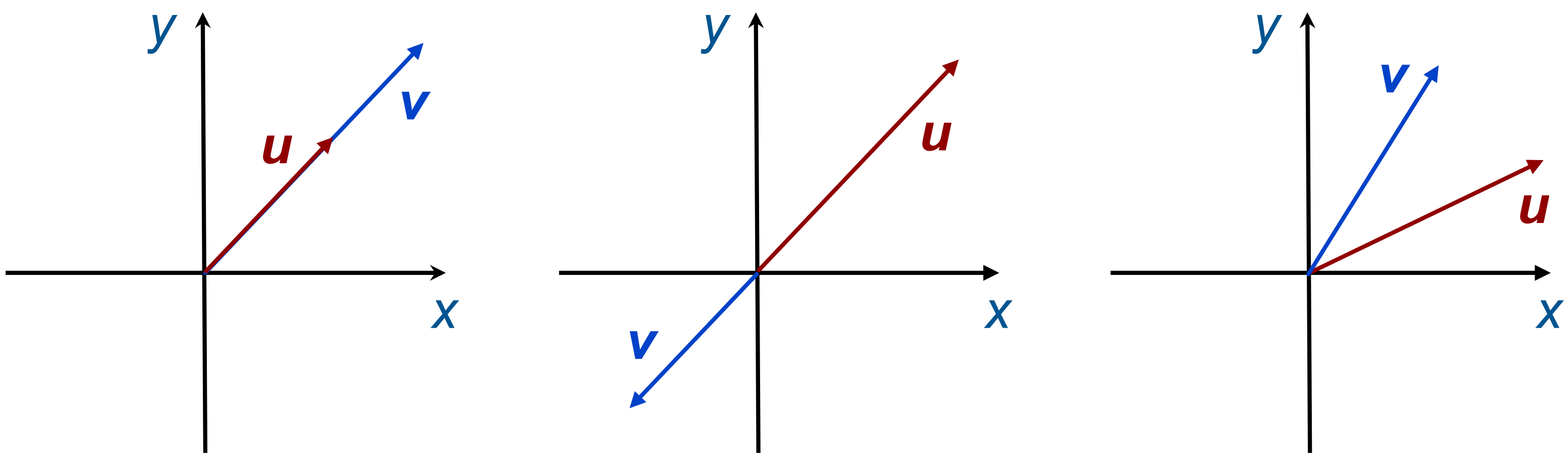
The system has k unknowns and n equations.

Since $k > n$, (by Remark 1.5.4.2) the system has non-trivial solutions.

So S is linearly dependent.

Geometrical interpretation (Discussion 3.4.9.1)

In $\mathbb{R}^2$ or $\mathbb{R}^3$, two vectors $\boldsymbol{u}$, $\boldsymbol{v}$ are linearly dependent if and only if they lie on the same line (when they are placed with their initial points at the origin).



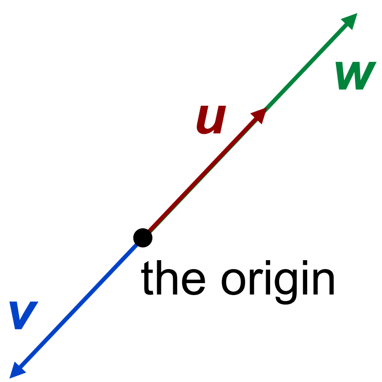
$\boldsymbol{u}$, $\boldsymbol{v}$ are linearly dependent.

$\boldsymbol{u}$, $\boldsymbol{v}$ are linearly dependent.

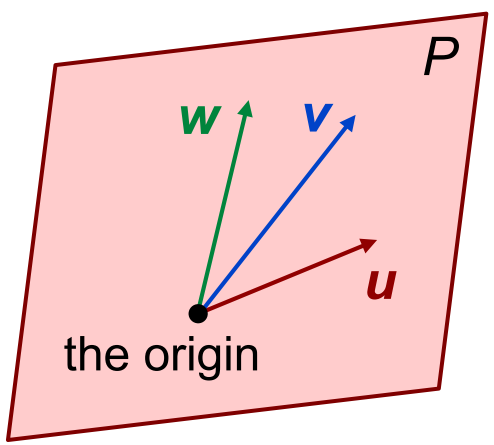
$\boldsymbol{u}$, $\boldsymbol{v}$ are linearly independent.

Geometrical interpretation (Discussion 3.4.9.1)

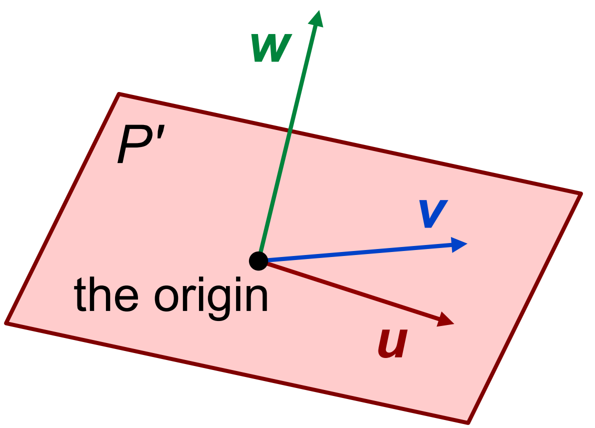
In $\mathbb{R}^3$, three vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ are linearly dependent if and only if they lie on the same line or the same plane (when they are placed with their initial points at the origin).



$\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ are linearly dependent.



$\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ are linearly dependent ($P = \text{span}\{\, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \,\}$).



$\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ are linearly independent ($P' = \text{span}\{\, \boldsymbol{u}, \boldsymbol{v} \,\}$).

Add an independent vector (Theorem 3.4.10)

Let $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, ..., $\boldsymbol{u}_k$ be linearly independent vectors in $\mathbb{R}^n$.

If $\boldsymbol{u}_{k+1}$ is not a linear combination of $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, ..., $\boldsymbol{u}_k$, then $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, ..., $\boldsymbol{u}_k$, $\boldsymbol{u}_{k+1}$ are linearly independent.

A new vector is added.

(Please read our textbook for a proof of the result.)

Section 3.5

Bases

Vector spaces and subspaces (Discussion 3.5.1)

We adopt the following conventions:

1. A set V is called a vector space if

   either $V = \mathbb{R}^n$ or V is a subspace of $\mathbb{R}^n$.

2. Let W be a vector space, say, $W = \mathbb{R}^n$ or W is a subspace of $\mathbb{R}^n$.

   A set V is called a subspace of W if

   V is a vector space and $V \subseteq W$,

   i.e. V is a subspace of $\mathbb{R}^n$ which lies completely inside W.

Examples (Example 3.5.2)

Let $U = \text{span}\{ (1, 1, 1) \}$, $V = \text{span}\{ (1, 1, -1) \}$
and $W = \text{span}\{ (1, 0, 0), (0, 1, 1) \}$.

Since U, V, W are subspace of $\mathbb{R}^3$, they are vector spaces.

As $(1, 1, 1) = (1, 0, 0) + (0, 1, 1)$, (by Theorem 3.2.10)

$$U = \text{span}\{ (1, 1, 1) \} \subseteq \text{span}\{ (1, 0, 0), (0, 1, 1) \} = W.$$

So U is a subspace of W.

As $(1, 1, -1) \notin \text{span}\{ (1, 0, 0), (0, 1, 1) \} = W$, $V \not\subseteq W$.

So V is not a subspace of W.

Bases (Definition 3.5.4 & Discussion 3.5.3)

Let V be a vector space
and $S = \{ \boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_k \}$ a subset of V.

Then S is called a basis (plural bases) for V if

1. S is linearly independent and
2. S spans V.

We shall learn that a basis for V can be used to build a coordinate system for V. (See Theorem 3.5.7 and Definition 3.5.6.)

Examples (Example 3.5.5.1)

Show that $S = \{ (1, 2, 1), (2, 9, 0), (3, 3, 4) \}$ is a basis for $\mathbb{R}^3$.

Solution:

(a) $c_1(1, 2, 1) + c_2(2, 9, 0) + c_3(3, 3, 4) = (0, 0, 0)$

$$\Leftrightarrow \begin{cases} c_1 + 2c_2 + 3c_3 = 0 \\ 2c_1 + 9c_2 + 3c_3 = 0 \\ c_1 + 4c_3 = 0. \end{cases}$$

The system has only the trivial solution.

So S is linearly independent.

Examples (Example 3.5.5.1)

(b) $$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left.\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & -3 \\ 0 & 0 & -1/5 \end{bmatrix}\right\}$$ There is no zero rows.

Thus (by Discussion 3.2.5) $\text{span}(S) = \mathbb{R}^3$.

By (a) and (b), S is a basis for $\mathbb{R}^3$.

Examples (Example 3.5.5.2)

Let $V = \text{span}\{ (1, 1, 1, 1), (1, -1, -1, 1), (1, 0, 0, 1) \}$
and $S = \{ (1, 1, 1, 1), (1, -1, -1, 1) \}$.
Show that S is a basis for V.

Solution:

(a) $c_1(1, 1, 1, 1) + c_2(1, -1, -1, 1) = (0, 0, 0, 0)$

$$\Leftrightarrow \quad \begin{cases} c_1 + c_2 = 0 \\ c_1 - c_2 = 0 \\ c_1 - c_2 = 0 \\ c_1 + c_2 = 0. \end{cases}$$

The system has only the trivial solution.
So S is linearly independent.

Examples (Example 3.5.5.2)

$V = \text{span}\{\, (1, 1, 1, 1),\ (1, -1, -1, 1),\ (1, 0, 0, 1) \,\}$,
$S = \{\, (1, 1, 1, 1),\ (1, -1, -1, 1) \,\}$.

(b) Since $(1, 0, 0, 1) = \frac{1}{2}(1, 1, 1, 1) + \frac{1}{2}(1, -1, -1, 1)$,

(by Theorem 3.2.12) $\text{span}(S) = V$.

By (a) and (b), S is a basis for V.

Examples (Example 3.5.5.3-4)

Is $S = \{ (1, 1, 1, 1), (0, 0, 1, 2), (-1, 0, 0, 1) \}$ is a basis for $\mathbb{R}^4$?

Solution: Since three vectors cannot span $\mathbb{R}^4$ (see Theorem 3.2.7), S is not a basis for $\mathbb{R}^4$.

Let $V = \text{span}(S)$ with $S = \{ (1, 1, 1), (0, 0, 1), (1, 1, 0) \}$.

Is S a basis for V?

Solution: S is linearly dependent.

So S is not a basis for V.

$\boxed{(1, 1, 1) = (0, 0, 1) + (1, 1, 0)}$

Some remarks (Remark 3.5.6)

1. A basis for V is a set of the smallest size that can span V. (See Theorem 3.6.1.2.)

2. For convenience, the empty set, $\varnothing$, is defined to be the basis for the zero space.

3. Except the zero space, any vector space has infinitely many different bases.

Coordinate systems (Theorem 3.5.7)

Let $S = \{ \boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k} \}$ be a basis for a vector space V.

Then every vector $\boldsymbol{v} \in V$ can be expressed in the form

$$\boldsymbol{v} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}$$

in exactly one way, where c_1, c_2, ..., c_k are real numbers.

Proof: Since S spans V, every vector $\boldsymbol{v} \in V$ can be expressed in the form

$$\boldsymbol{v} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}$$

for some real numbers c_1, c_2, ..., c_k.

It remains to show that the expression is unique.

Coordinate systems (Theorem 3.5.7)

Suppose $\boldsymbol{v}$ can be expressed in two ways

$$\boldsymbol{v} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}, \quad \boldsymbol{v} = d_1\boldsymbol{u_1} + d_2\boldsymbol{u_2} + \cdots + d_k\boldsymbol{u_k}$$

where c_1, c_2, ..., c_k, d_1, d_2, ..., d_k are real numbers.

$$\begin{aligned}(c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}) - (d_1\boldsymbol{u_1} + d_2\boldsymbol{u_2} + \cdots + d_k\boldsymbol{u_k}) \\ = \boldsymbol{v} - \boldsymbol{v} = \boldsymbol{0}\end{aligned}$$

$$\Rightarrow \quad (c_1 - d_1)\boldsymbol{u_1} + (c_2 - d_2)\boldsymbol{u_2} + \cdots + (c_k - d_k)\boldsymbol{u_k} = \boldsymbol{0}. \qquad (\#)$$

Since $\boldsymbol{u_1}$, $\boldsymbol{u_2}$, ..., $\boldsymbol{u_k}$ are linearly independent, (#) can only have the trivial solution

$$c_1 - d_1 = 0, \quad c_2 - d_2 = 0, \quad \ldots, \quad c_k - d_k = 0,$$

i.e. $c_1 = d_1$, $c_2 = d_2$, ..., $c_k = d_k$.

So the expression is unique.

Coordinate systems (Definition 3.5.8)

Let $S = \{ \boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k \}$ be a basis for a vector space V.

A vector $\boldsymbol{v} \in V$ can be expressed uniquely in the form

$$\boldsymbol{v} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k .$$

The coefficients c_1, c_2, ..., c_k are called the coordinates of $\boldsymbol{v}$ relative to the basis S.

The vector

$$(\boldsymbol{v})_S = (c_1, c_2, \ldots, c_k) \in \mathbb{R}^k$$

is called the coordinate vector of $\boldsymbol{v}$ relative to S.

(In here, we assume that the vectors of S are in a fixed order so that $\boldsymbol{u}_1$ is the 1st vector, $\boldsymbol{u}_2$ is the 2nd vector, etc.

In some textbooks, such a basis is called an ordered basis.)

Examples (Example 3.5.9.1 (a))

Let $S = \{ (1, 2, 1), (2, 9, 0), (3, 3, 4) \}$ which is a basis for $\mathbb{R}^3$.

Find the coordinate vector of $\boldsymbol{v} = (5, -1, 9)$ relative to S.

Solution: Solving

$$a(1, 2, 1) + b(2, 9, 0) + c(3, 3, 4) = (5, -1, 9),$$

we obtain only one solution $a = 1$, $b = -1$, $c = 2$,

i.e. $\boldsymbol{v} = (1, 2, 1) - (2, 9, 0) + 2(3, 3, 4)$.

The coordinate vector of $\boldsymbol{v}$ relative to S is

$$(\boldsymbol{v})_S = (1, -1, 2).$$

Examples (Example 3.5.9.1 (b))

Let $S = \{ (1, 2, 1), (2, 9, 0), (3, 3, 4) \}$ which is a basis for $\mathbb{R}^3$.

Find a vector $\boldsymbol{w}$ such that $(\boldsymbol{w})_S = (-1, 3, 2)$.

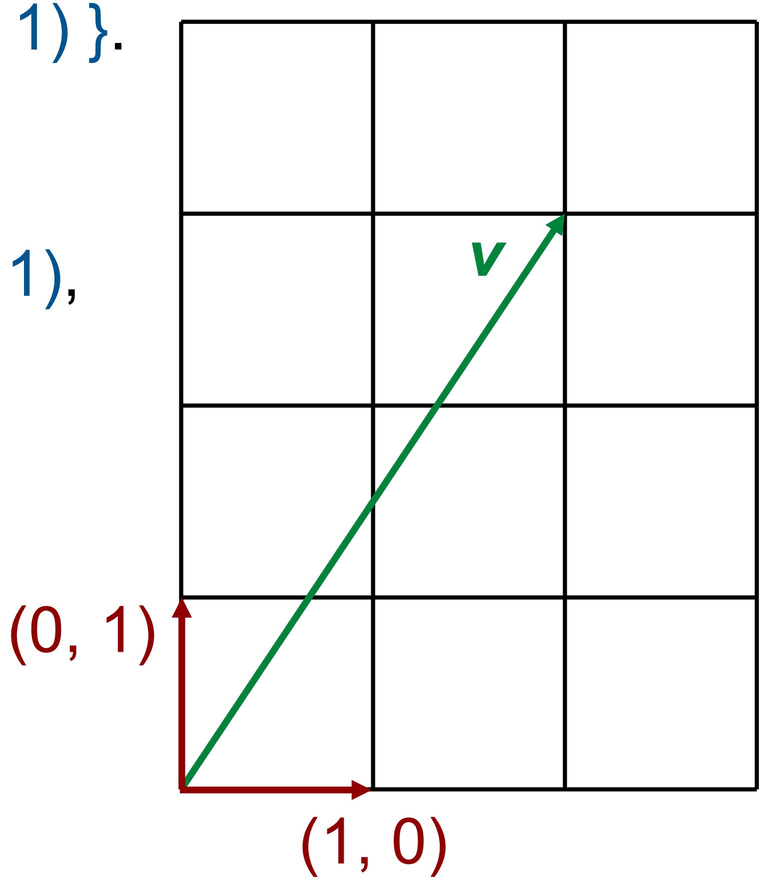
Solution: Since $(\boldsymbol{w})_S = (-1, 3, 2)$,

$$\begin{aligned} \boldsymbol{w} &= -(1, 2, 1) + 3(2, 9, 0) + 2(3, 3, 4) \\ &= (11, 31, 7). \end{aligned}$$

Examples (Example 3.5.9.2 (a))

Let $\boldsymbol{v} = (2, 3) \in \mathbb{R}^2$
and $S_1 = \{ (1, 0), (0, 1) \}$.

Since
$(2, 3) = 2(1, 0) + 3(0, 1)$,
$(\boldsymbol{v})_{S_1} = (2, 3)$.



Examples (Example 3.5.9.2 (b))

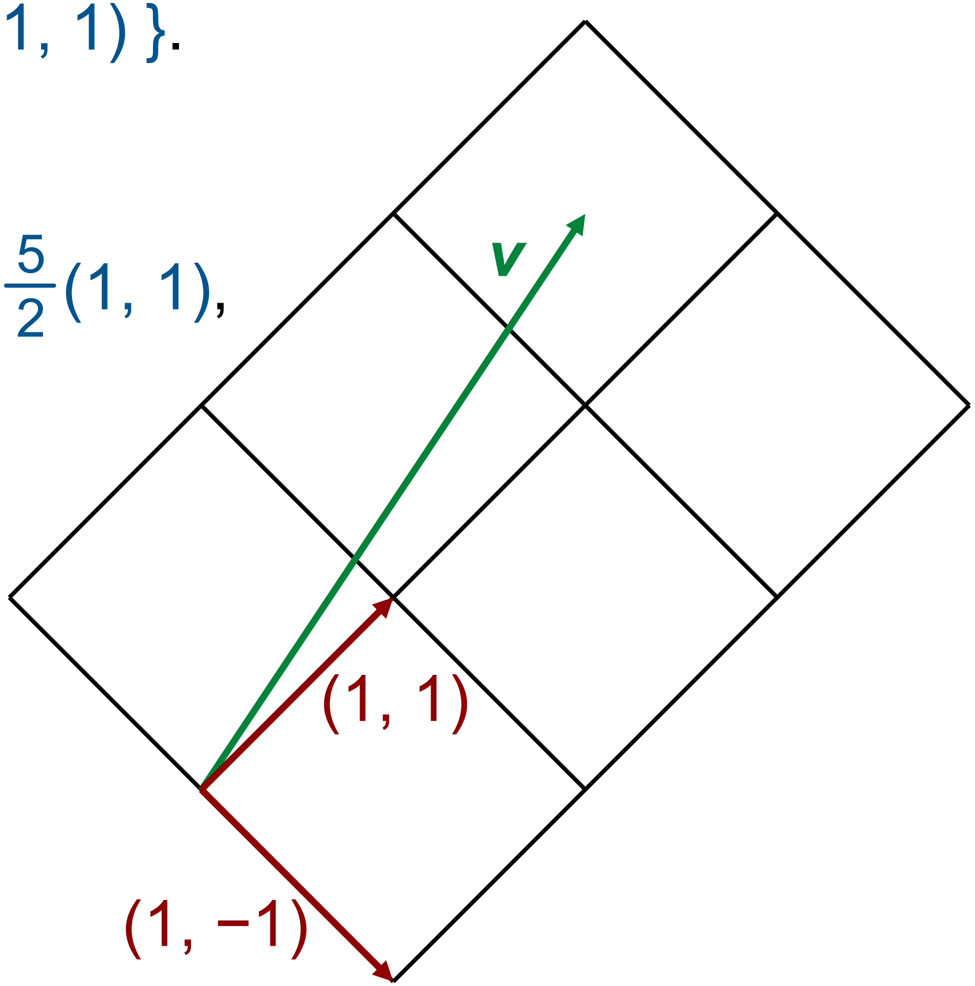
Let $\boldsymbol{v} = (2, 3) \in \mathbb{R}^2$

and $S_2 = \{ (1, -1),\ (1, 1) \}$.

Since

$(2, 3) = -\frac{1}{2}(1, -1) + \frac{5}{2}(1, 1)$,

$(\boldsymbol{v})_{S_2} = \left(-\frac{1}{2}, \frac{5}{2}\right)$.



Examples (Example 3.5.9.2 (c))

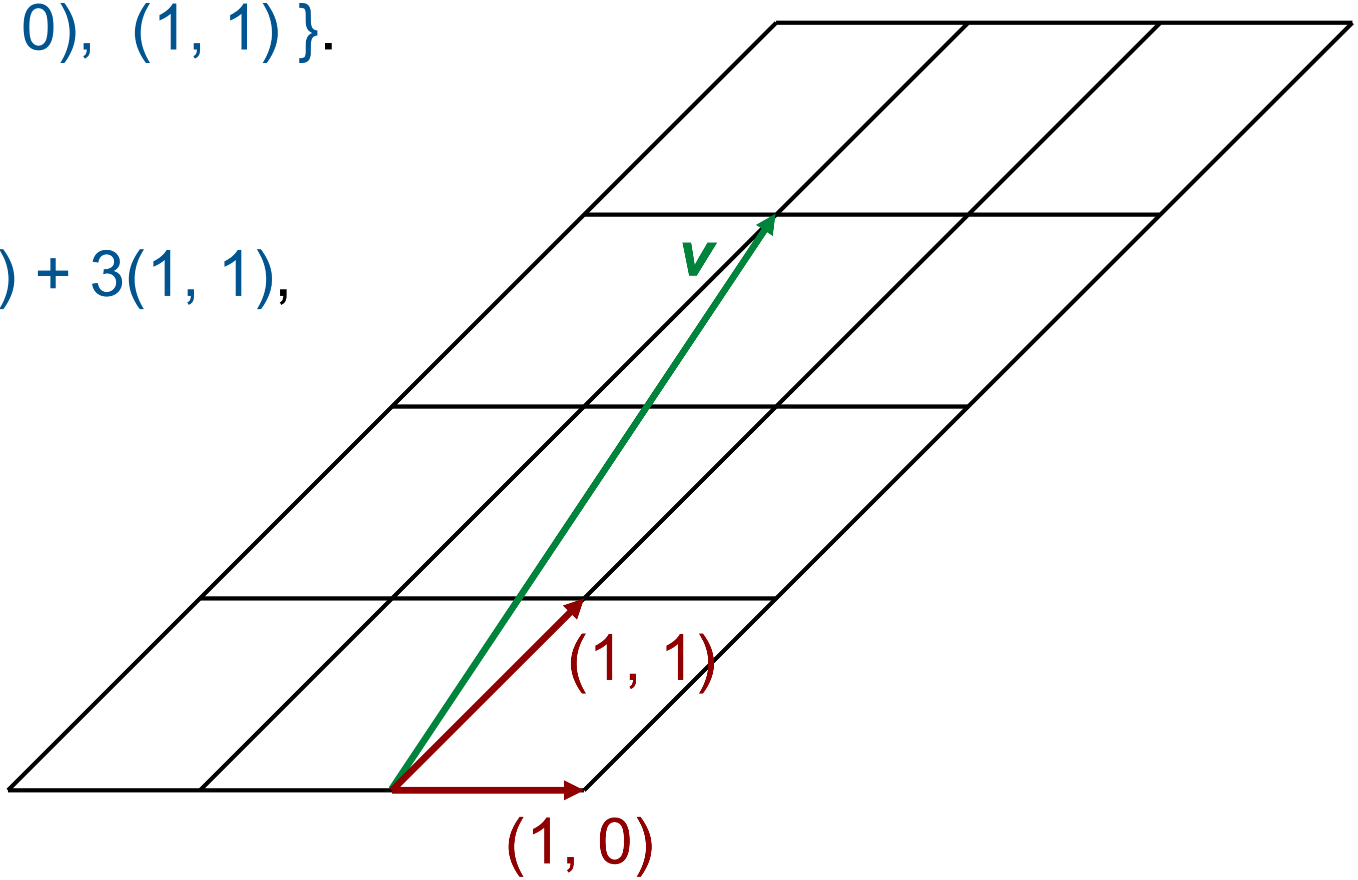
Let $\boldsymbol{v} = (2, 3) \in \mathbb{R}^2$

and $S_3 = \{ (1, 0), \ (1, 1) \}$.

Since

$(2, 3) = -(1, 0) + 3(1, 1)$,

$(\boldsymbol{v})_{S_3} = (-1, 3)$.



Standard basis for $\mathbb{R}^n$ (Example 3.5.9.3)

Let $E = \{ \boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n \}$

where $\boldsymbol{e}_1 = (1, 0, \ldots, 0)$, $\boldsymbol{e}_2 = (0, 1, 0, \ldots, 0)$, ...,

$\boldsymbol{e}_n = (0, \ldots, 0, 1)$ are vectors in $\mathbb{R}^n$.

(Recall Remark 3.3.2.2.)

For any $\boldsymbol{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$,

$$\boldsymbol{u} = u_1\boldsymbol{e}_1 + u_2\boldsymbol{e}_2 + \cdots + u_n\boldsymbol{e}_n.$$

Thus $\mathbb{R}^n = \mathrm{span}(E)$ and hence E spans $\mathbb{R}^n$.

Standard basis for $\mathbb{R}^n$ (Example 3.5.9.3)

On the other hand,

$$c_1\mathbf{e_1} + c_2\mathbf{e_2} + \cdots + c_n\mathbf{e_n} = \mathbf{0} \qquad (\dagger)$$

$$\Rightarrow \quad (c_1, c_2, \ldots, c_n) = (0, 0, \ldots, 0)$$

$$\Rightarrow \quad c_1 = 0, \; c_2 = 0, \; \ldots, \; c_n = 0.$$

Since the vector equation $(\dagger)$ has only the trivial solution, E is linearly independent.

Thus E is a basis for $\mathbb{R}^n$ which is known as the standard basis for $\mathbb{R}^n$.

For $\boldsymbol{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$,

$$(\boldsymbol{u})_E = (u_1, u_2, \ldots, u_k) = \boldsymbol{u}.$$

Coordinate systems (Remark 3.5.10)

Let S be a basis for a vector space V.

1. For any $\boldsymbol{u}, \boldsymbol{v} \in V$, $\boldsymbol{u} = \boldsymbol{v}$ if and only if $(\boldsymbol{u})_S = (\boldsymbol{v})_S$.
2. For any $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r \in V$ and $c_1, c_2, \ldots, c_r \in \mathbb{R}$,

$$(c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_r\boldsymbol{v}_r)_S = c_1(\boldsymbol{v}_1)_S + c_2(\boldsymbol{v}_2)_S + \cdots + c_n(\boldsymbol{v}_r)_S.$$

Coordinate systems (Theorem 3.5.11)

Let S be a basis for a vector space V where $|S| = k$.

Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r \in V$.

1. $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r$ are linearly dependent vectors in V if and only if $(\boldsymbol{v}_1)_S, (\boldsymbol{v}_2)_S, \ldots, (\boldsymbol{v}_r)_S$ are linearly dependent vectors in $\mathbb{R}^k$;

   equivalently, $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r$ are linearly independent vectors in V if and only if $(\boldsymbol{v}_1)_S, (\boldsymbol{v}_2)_S, \ldots, (\boldsymbol{v}_r)_S$ are linearly independent vectors in $\mathbb{R}^k$.

2. $\text{span}\{ \boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r \} = V$ if and only if $\text{span}\{ (\boldsymbol{v}_1)_S, (\boldsymbol{v}_2)_S, \ldots, (\boldsymbol{v}_r)_S \} = \mathbb{R}^k$.

(Please read our textbook for a proof of the result.)

Section 3.6

Dimensions

Size of bases (Theorem 3.6.1 & Remark 3.6.2)

Let V be a vector space which has a basis with k vectors.

1. Any subset of V with more than k vectors is always linearly dependent.
2. Any subset of V with less than k vectors cannot spans V.

This means that every basis for V have the same size k.

Proof of the theorem (Theorem 3.6.1)

Proof: Let S be a basis for V and $|S| = k$.

1. Let $T = \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r \} \subseteq V$ where $r > k$.

   Since $(\mathbf{v}_1)_S$, $(\mathbf{v}_2)_S$, ..., $(\mathbf{v}_r)_S$ are vectors in $\mathbb{R}^k$, (by Theorem 3.4.7) $(\mathbf{v}_1)_S$, $(\mathbf{v}_2)_S$, ..., $(\mathbf{v}_r)_S$ are linearly dependent.

   Then (by Theorem 3.5.11.1) T is linearly dependent.

2. Let $T' = \{ \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_t \} \subseteq V$ where $t < k$.

   Since $(\mathbf{w}_1)_S$, $(\mathbf{w}_2)_S$, ..., $(\mathbf{w}_t)_S$ are vectors in $\mathbb{R}^k$, (by Theorem 3.2.7) $(\mathbf{w}_1)_S$, $(\mathbf{w}_2)_S$, ..., $(\mathbf{w}_t)_S$ cannot span $\mathbb{R}^k$.

   Then (by Theorem 3.5.11.2) T' cannot span V.

Dimensions (Definition 3.6.3 & Example 3.6.4.1-3)

The dimension of a vector space V, denoted by $\dim(V)$, is defined to be the number of vectors in a basis for V.

The dimension of the zero space is defined to be 0.

$\dim(\mathbb{R}^n) = n$.

(Note that the standard basis $E = \{ \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \}$ for $\mathbb{R}^n$ has n vectors.)

Except $\{ 0 \}$ and $\mathbb{R}^2$, subspaces of $\mathbb{R}^2$ are lines through the origin which are of dimension 1.

Except $\{ 0 \}$ and $\mathbb{R}^3$, subspaces of $\mathbb{R}^3$ are either lines through the origin, which are of dimension 1, or planes containing the origin, which are of dimension 2.

An example (Example 3.6.4.4)

Find a basis for and determine the dimension of the subspace $W = \{ (x, y, z) \mid y = z \}$ of $\mathbb{R}^3$.

Solution: Every vector of W is of the form

$$(x, y, y) = x(1, 0, 0) + y(0, 1, 1).$$

So $W = \text{span}\{ (1, 0, 0), (0, 1, 1) \}$.

On the other hand,

$$c_1(1, 0, 0) + c_2(0, 1, 1) = (0, 0, 0) \Rightarrow c_1 = 0, \ c_2 = 0.$$

Thus $\{ (1, 0, 0), (0, 1, 1) \}$ is linearly independent.

So $\{ (1, 0, 0), (0, 1, 1) \}$ is a basis for W and $\dim(W) = 2$.

Solution spaces (Discussion 3.6.5)

How do we determine the dimension of the solution space of a homogeneous linear system $\boldsymbol{Ax} = \mathbf{0}$?

$$(\,\boldsymbol{A} \mid \mathbf{0}\,) \xrightarrow[\text{Elimination}]{\text{Gaussian or Gauss-Jordan}} (\,\boldsymbol{R} \mid \mathbf{0}\,) \quad \text{i.e.} \quad \boldsymbol{A} \xrightarrow[\text{Elimination}]{\text{Gaussian or Gauss-Jordan}} \boldsymbol{R}$$

$$\boldsymbol{R} = \begin{bmatrix} 0 & \otimes & * & \cdots & \cdots & \cdots & * \\ 0 & 0 & \otimes & * & \cdots & \cdots & * \\ \vdots & & & \ddots & & & \vdots \\ 0 & \cdots & \cdots & 0 & \otimes & * & * \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

non-pivot columns

First, set the variable corresponding to non-pivot columns of $\boldsymbol{R}$ to be arbitrary.

Equate the other variable accordingly.

Solution spaces (Discussion 3.6.5)

Then a general solution to the system can be written as

$$\boldsymbol{x} = t_1\boldsymbol{u_1} + t_2\boldsymbol{u_2} + \cdots + t_k\boldsymbol{u_k}$$

where t_1, t_2, ..., t_k are arbitrary parameters and $\boldsymbol{u_1}$, $\boldsymbol{u_2}$, ..., $\boldsymbol{u_k}$ are fixed vectors.

In this way, the vectors $\boldsymbol{u_1}$, $\boldsymbol{u_2}$, ..., $\boldsymbol{u_k}$ are always linearly independent.

So $\{ \boldsymbol{u_1}, \boldsymbol{u_2}, ..., \boldsymbol{u_k} \}$ is a basis for the solution space.

Note that the dimension of the solution space is equal to

k = the number of arbitrary parameters

= the number of non-pivot columns in $\boldsymbol{R}$.

An example (Example 3.6.6)

Consider the homogeneous linear system

$$\begin{cases} 2v + 2w - x + z = 0 \\ -v - w + 2x - 3y + z = 0 \\ x + y + z = 0 \\ v + w - 2x - z = 0. \end{cases}$$

$$\left[\begin{array}{ccccc|c} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \end{array}\right] \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

An example (Example 3.6.6)

We have a general solution

$$\begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -s-t \\ s \\ -t \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

where s, t are arbitrary parameters.

So $\{ (-1, 1, 0, 0, 0),\ (-1, 0, -1, 0, 1) \}$ is a basis for the solution space

and the dimension of the solution space is 2.

A useful result (Theorem 3.6.7)

Let V be a vector space of dimension k and S a subset of V. The following are equivalent:

1. S is a basis for V,
   i.e. S is linearly independent and S spans V.
2. S is linearly independent and $|S| = k$.
3. S spans V and $|S| = k$.

(Please read our textbook for a proof of the result.)

If we want to check that S is a basis for V, we only need to check any two of the three conditions:

(i) S is linearly independent; (ii) S spans V;
(iii) $|S| = k$.

An example (Example 3.6.8)

Let $\boldsymbol{u_1} = (2, 0, -1)$, $\boldsymbol{u_2} = (4, 0, 7)$ and $\boldsymbol{u_3} = (-1, 1, 4)$.
Show that $\{ \boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3} \}$ is a basis for $\mathbb{R}^3$.

Solution:

$$c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + c_3\boldsymbol{u_3} = \boldsymbol{0}$$
$$\Rightarrow \quad c_1(2, 0, -1) + c_2(4, 0, 7) + c_3(-1, 1, 4) = (0, 0, 0)$$
$$\Rightarrow \quad c_1 = 0, \ c_2 = 0, \ c_3 = 0.$$

So $\{ \boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3} \}$ is linearly independent.

Since $\dim(\mathbb{R}^3) = 3$, (by Theorem 3.6.7) $\{ \boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3} \}$ is a basis for $\mathbb{R}^3$.

Dimensions of subspaces (Theorem 3.6.9)

Let U be a subspace of a vector space V.

Then $\dim(U) \leq \dim(V)$.

Furthermore, if $U \neq V$, $\dim(U) < \dim(V)$.

Proof: Let S be a basis for U.

Since $U \subseteq V$, S is a linearly independent subset of V.

Then $\dim(U) = |S| \leq \dim(V)$.

By Theorem 3.6.1.1, a subset T of V with $|T| > \dim(V)$ must be linearly dependent.

Assume $\dim(U) = \dim(V)$.

As S is linearly independent and $|S| = \dim(V)$, (by Theorem 3.6.7) S is a basis for V.

But then $U = \text{span}(S) = V$.

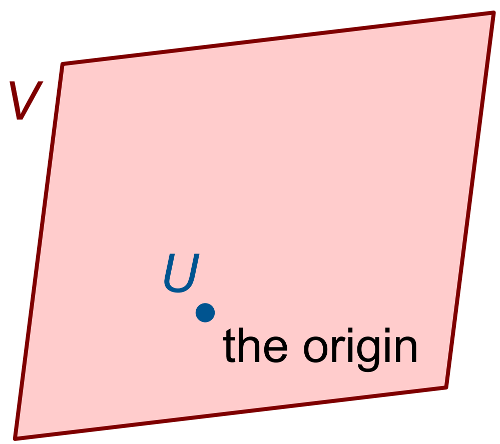
Hence if $U \neq V$, $\dim(U) < \dim(V)$.

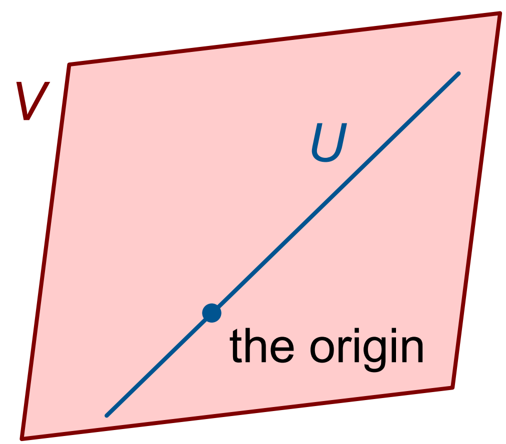
An example (Example 3.6.10)

Let V be a plane in $\mathbb{R}^3$ containing the origin.
V is a vector space of dimension 2.

Suppose U is a subspace of V such that $U \neq V$.
Then (by Theorem 3.6.9) $\dim(U) < 2$.

U is either the zero space $\{ (0, 0, 0) \}$ (of dimension 0)
or a line through the origin (of dimension 1).





Invertible matrices (Theorem 3.6.11)

Let $\boldsymbol{A}$ be an $n \times n$ matrix. The following statements are equivalent:

1. $\boldsymbol{A}$ is invertible.
2. The linear system $\boldsymbol{Ax} = \mathbf{0}$ has only the trivial solution.
3. The reduced row-echelon form of $\boldsymbol{A}$ is an identity matrix.
4. $\boldsymbol{A}$ can be expressed as a product of elementary matrices.
5. $\det(\boldsymbol{A}) \neq 0$.
6. The rows of $\boldsymbol{A}$ form a basis for $\mathbb{R}^n$.
7. The columns of $\boldsymbol{A}$ form a basis for $\mathbb{R}^n$.

Invertible matrices (Theorem 3.6.11)

We already learn that statements 1-5 are equivalent (see Theorem 2.4.7 and Theorem 2.5.19).

To prove $7 \Leftrightarrow 1$:

Let $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n \end{bmatrix}$ where $\boldsymbol{a}_i$ is the i^{th} column of $\boldsymbol{A}$.

$\{ \boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n \}$ is a basis for $\mathbb{R}^n$

$\Leftrightarrow$ $\text{span}\{ \boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n \} = \mathbb{R}^n$ (by Theorem 3.6.7)

$\Leftrightarrow$ a row echelon form of $\boldsymbol{A}$ has no zero row (by Discussion 3.2.5)

$\Leftrightarrow$ $\boldsymbol{A}$ is invertible. (by Remark 2.4.10)

Invertible matrices (Theorem 3.6.11)

The rows of $\boldsymbol{A}$ is the columns of $\boldsymbol{A}^{\mathrm{T}}$.

Since $\boldsymbol{A}$ is invertible if and only if $\boldsymbol{A}^{\mathrm{T}}$ is invertible (see Theorem 2.3.9) "1 $\Leftrightarrow$ 6" follows from "1 $\Leftrightarrow$ 7".

Examples (Example 3.6.12)

1. Let $\boldsymbol{u}_1 = (1, 1, 1)$, $\boldsymbol{u}_2 = (-1, 1, 2)$ and $\boldsymbol{u}_3 = (1, 0, 1)$.

$$\begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{vmatrix} = 3 \neq 0 \quad \Rightarrow \quad \{ \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3 \} \text{ is a basis for } \mathbb{R}^3.$$

2. Let $\boldsymbol{u}_1 = (1, 1, 1, 1)$, $\boldsymbol{u}_2 = (1, -1, 1, -1)$, $\boldsymbol{u}_3 = (0, 1, -1, 0)$ and $\boldsymbol{u}_4 = (2, 1, 1, 0)$.

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 2 & 1 & 1 & 0 \end{vmatrix} = 0 \quad \Rightarrow \quad \{ \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4 \} \text{ is not a basis for } \mathbb{R}^4.$$

Section 3.7

Transition Matrices

Coordinate vectors (Notation 3.7.1)

Let $S = \{ \boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k \}$ be a basis for a vector space V.

For $\boldsymbol{v} \in V$, recall that if $\boldsymbol{v} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k$, then the row vector

$$(\boldsymbol{v})_S = (c_1, c_2, \ldots, c_k)$$

is called the coordinate vector of $\boldsymbol{v}$ relative to S.

From now on, we also define the column vector

$$[\boldsymbol{v}]_S = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$$

to be the coordinate vector of $\boldsymbol{v}$ relative to S.

Transition matrices (Discussion 3.7.2)

Let $S = \{ \boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k} \}$ and $T = \{ \boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_k} \}$ be two bases for a vector space V.

Take any vector $\boldsymbol{w} \in V$.

Since S is a basis for V,

$$\boldsymbol{w} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}$$

for some real constants $c_1, c_2, \ldots, c_k$,

i.e. $[\boldsymbol{w}]_S = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$.

Transition matrices (Discussion 3.7.2)

Since T is a basis for V, we can write

$$
\begin{aligned}
\boldsymbol{u_1} &= a_{11}\boldsymbol{v_1} + a_{21}\boldsymbol{v_2} + \cdots + a_{k1}\boldsymbol{v_k},\\
\boldsymbol{u_2} &= a_{12}\boldsymbol{v_1} + a_{22}\boldsymbol{v_2} + \cdots + a_{k2}\boldsymbol{v_k},\\
&\;\;\vdots \qquad\qquad\qquad \vdots\\
\boldsymbol{u_k} &= a_{1k}\boldsymbol{v_1} + a_{2k}\boldsymbol{v_2} + \cdots + a_{kk}\boldsymbol{v_k}
\end{aligned}
$$

for some real constants a_{11}, a_{12}, ..., a_{kk},

i.e. $[\boldsymbol{u_1}]_T = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{k1} \end{bmatrix}$, $[\boldsymbol{u_2}]_T = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{k2} \end{bmatrix}$, ..., $[\boldsymbol{u_k}]_T = \begin{bmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{kk} \end{bmatrix}$.

Transition matrices (Discussion 3.7.2)

Then
$$
\begin{aligned}
\boldsymbol{w} &= c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k} \\
&= c_1(a_{11}\boldsymbol{v_1} + a_{21}\boldsymbol{v_2} + \cdots + a_{k1}\boldsymbol{v_k}) \\
&\qquad + c_2(a_{12}\boldsymbol{v_1} + a_{22}\boldsymbol{v_2} + \cdots + a_{k2}\boldsymbol{v_k}) \\
&\qquad\qquad + \cdots + c_k(a_{1k}\boldsymbol{v_1} + a_{2k}\boldsymbol{v_2} + \cdots + a_{kk}\boldsymbol{v_k}) \\
&= (c_1a_{11} + c_2a_{12} + \cdots + c_ka_{1k})\boldsymbol{v_1} \\
&\qquad + (c_1a_{21} + c_2a_{22} + \cdots + c_ka_{2k})\boldsymbol{v_2} \\
&\qquad\qquad + \cdots + (c_1a_{k1} + c_2a_{k2} + \cdots + c_ka_{kk})\boldsymbol{v_k},
\end{aligned}
$$

i.e.
$$
[\boldsymbol{w}]_T = \begin{bmatrix} c_1a_{11} + c_2a_{12} + \cdots + c_ka_{1k} \\ c_1a_{21} + c_2a_{22} + \cdots + c_ka_{2k} \\ \vdots \\ c_1a_{k1} + c_2a_{k2} + \cdots + c_ka_{kk} \end{bmatrix}.
$$

Question:

How is $[\boldsymbol{w}]_T$ related to $[\boldsymbol{w}]_S$?

Transition matrices (Discussion 3.7.2 & Definition 3.7.3)

$$[\boldsymbol{w}]_T = \begin{bmatrix} c_1a_{11} + c_2a_{12} + \cdots + c_ka_{1k} \\ c_1a_{21} + c_2a_{22} + \cdots + c_ka_{2k} \\ \vdots \\ c_1a_{k1} + c_2a_{k2} + \cdots + c_ka_{kk} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$$

$$= \begin{bmatrix} [\boldsymbol{u_1}]_T & [\boldsymbol{u_2}]_T & \cdots & [\boldsymbol{u_k}]_T \end{bmatrix} [\boldsymbol{w}]_S.$$

The matrix $\boldsymbol{P}$ is called the transition matrix from S to T.

Let $\boldsymbol{P} = \begin{bmatrix} [\boldsymbol{u_1}]_T & [\boldsymbol{u_2}]_T & \cdots & [\boldsymbol{u_k}]_T \end{bmatrix}$. Then $[\boldsymbol{w}]_T = \boldsymbol{P}[\boldsymbol{w}]_S$.

Examples (Example 3.7.4.1 (a))

Let $S = \{ \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3 \}$,

where $\boldsymbol{u}_1 = (1, 0, -1)$, $\boldsymbol{u}_2 = (0, -1, 0)$, $\boldsymbol{u}_3 = (1, 0, 2)$,

and $T = \{ \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \}$,

where $\boldsymbol{v}_1 = (1, 1, 1)$, $\boldsymbol{v}_2 = (1, 1, 0)$, $\boldsymbol{v}_3 = (-1, 0, 0)$.

Both S and T are bases for $\mathbb{R}^3$.

Find the transition matrix from S to T.

Solution: First we need to find a_{11}, a_{12}, ..., a_{33} such that

$$a_{11}\boldsymbol{v}_1 + a_{21}\boldsymbol{v}_2 + a_{31}\boldsymbol{v}_3 = \boldsymbol{u}_1,$$

$$a_{12}\boldsymbol{v}_1 + a_{22}\boldsymbol{v}_2 + a_{32}\boldsymbol{v}_3 = \boldsymbol{u}_2,$$

$$a_{13}\boldsymbol{v}_1 + a_{23}\boldsymbol{v}_2 + a_{33}\boldsymbol{v}_3 = \boldsymbol{u}_3.$$

Examples (Example 3.7.4.1 (a))

$$\begin{aligned} a_{11}\boldsymbol{v}_1 + a_{21}\boldsymbol{v}_2 + a_{31}\boldsymbol{v}_3 &= \boldsymbol{u}_1, \\ a_{12}\boldsymbol{v}_1 + a_{22}\boldsymbol{v}_2 + a_{32}\boldsymbol{v}_3 &= \boldsymbol{u}_2, \\ a_{13}\boldsymbol{v}_1 + a_{23}\boldsymbol{v}_2 + a_{33}\boldsymbol{v}_3 &= \boldsymbol{u}_3. \end{aligned}$$

$$\begin{array}{c} \begin{array}{cccccc} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 \end{array} \\ \left[\begin{array}{ccc|c|c|c} 1 & 1 & -1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 2 \end{array}\right] \end{array} \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{array}\right]$$

We have

$$\begin{aligned} \boldsymbol{u}_1 &= -\boldsymbol{v}_1 + \boldsymbol{v}_2 - \boldsymbol{v}_3, \\ \boldsymbol{u}_2 &= \phantom{-\boldsymbol{v}_1} - \boldsymbol{v}_2 - \boldsymbol{v}_3, \\ \boldsymbol{u}_3 &= 2\boldsymbol{v}_1 - 2\boldsymbol{v}_2 - \boldsymbol{v}_3. \end{aligned}$$

So the transition matrix from S to T is

$$\boldsymbol{P} = \begin{bmatrix} -1 & 0 & 2 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{bmatrix}.$$

Examples (Example 3.7.4.1 (b))

$S = \{ \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3 \}$,

where $\boldsymbol{u}_1 = (1, 0, -1)$, $\boldsymbol{u}_2 = (0, -1, 0)$, $\boldsymbol{u}_3 = (1, 0, 2)$,

$T = \{ \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \}$,

where $\boldsymbol{v}_1 = (1, 1, 1)$, $\boldsymbol{v}_2 = (1, 1, 0)$, $\boldsymbol{v}_3 = (-1, 0, 0)$.

Let $\boldsymbol{w}$ such that $(\boldsymbol{w})_S = (2, -1, 2)$. Find $(\boldsymbol{w})_T$.

Solution:

$\boldsymbol{P}$ is the transition matrix from S to T.

$$[\boldsymbol{w}]_T = \boldsymbol{P}[\boldsymbol{w}]_S = \begin{bmatrix} -1 & 0 & 2 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$$

So $(\boldsymbol{w})_T = (2, -1, -3)$.

Examples (Example 3.7.4.2)

Let $S = \{ \boldsymbol{u}_1, \boldsymbol{u}_2 \}$, where $\boldsymbol{u}_1 = (1, 1)$, $\boldsymbol{u}_2 = (1, -1)$, and $T = \{ \boldsymbol{v}_1, \boldsymbol{v}_2 \}$, where $\boldsymbol{v}_1 = (1, 0)$, $\boldsymbol{v}_2 = (1, 1)$.

We have
$$\begin{aligned} \boldsymbol{u}_1 &= \phantom{2\boldsymbol{v}_1 -{}} \boldsymbol{v}_2, \\ \boldsymbol{u}_2 &= 2\boldsymbol{v}_1 - \boldsymbol{v}_2. \end{aligned}$$

Thus the transition matrix from S to T is $\boldsymbol{P} = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}$.

On the other hand,
$$\begin{aligned} \boldsymbol{v}_1 &= \frac{1}{2}\boldsymbol{u}_1 + \frac{1}{2}\boldsymbol{u}_2, \\ \boldsymbol{v}_2 &= \phantom{\frac{1}{2}} \boldsymbol{u}_1. \end{aligned}$$

Thus the transition matrix from T to S is $\boldsymbol{Q} = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$.

Note that $\boldsymbol{Q} = \boldsymbol{P}^{-1}$.

Transition matrices (Theorem 3.7.5)

Let S and T be two bases for a vector space and let $\boldsymbol{P}$ be the transition matrix from S to T.

1. $\boldsymbol{P}$ is invertible.
2. $\boldsymbol{P}^{-1}$ is the transition matrix from T to S.

Proof: Let $\boldsymbol{Q}$ be the transition matrix from T to S.

If we can show that $\boldsymbol{QP} = \boldsymbol{I}$, then (by Theorem 2.4.12) $\boldsymbol{P}$ is invertible and $\boldsymbol{P}^{-1} = \boldsymbol{Q}$.

Transition matrices (Theorem 3.7.5)

An observation:

Given a matrix $\boldsymbol{A} = (a_{ij})_{m\times n}$ and let $\boldsymbol{e_i} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. ← the i^{th} entry

$$\boldsymbol{Ae_i} = \begin{bmatrix} a_{11} & \cdots & a_{1,i-1} & a_{1i} & a_{1,i+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,i-1} & a_{2i} & a_{2,i+1} & \cdots & a_{2n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{m,i-1} & a_{mi} & a_{m,i+1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix}$$ = the i^{th} column of $\boldsymbol{A}$.

Transition matrices (Theorem 3.7.5)

Suppose $S = \{ \boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k \}$.

$$[\boldsymbol{u}_1]_S = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \boldsymbol{e}_1, \quad [\boldsymbol{u}_2]_S = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \boldsymbol{e}_2, \quad \ldots, \quad [\boldsymbol{u}_k]_S = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix} = \boldsymbol{e}_k.$$

For $i = 1, 2, \ldots, k$,

the i^{th} column of $\boldsymbol{QP} = \boldsymbol{QPe}_i$

$$= \boldsymbol{QP}[\boldsymbol{u}_i]_S = \boldsymbol{Q}[\boldsymbol{u}_i]_T = [\boldsymbol{u}_i]_S = \boldsymbol{e}_i.$$

$\boldsymbol{P}$ is the transition matrix from S to T.

$\boldsymbol{Q}$ is the transition matrix from T to S.

So $\boldsymbol{QP} = \begin{bmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \cdots & \boldsymbol{e}_k \end{bmatrix} = \boldsymbol{I}.$

An example (Example 3.7.6)

Let $S = \{ \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3 \}$,

where $\boldsymbol{u}_1 = (1, 0, -1)$, $\boldsymbol{u}_2 = (0, -1, 0)$, $\boldsymbol{u}_3 = (1, 0, 2)$,

and $T = \{ \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \}$,

where $\boldsymbol{v}_1 = (1, 1, 1)$, $\boldsymbol{v}_2 = (1, 1, 0)$, $\boldsymbol{v}_3 = (-1, 0, 0)$.

The transition matrix from S to T is

$$\boldsymbol{P} = \begin{bmatrix} -1 & 0 & 2 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{bmatrix}$$

(see Example 3.7.4.1).

An example (Example 3.7.6)

Then the transition matrix from T to S is

$$P^{-1} = \begin{bmatrix} -1 & 0 & 2 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -1 & -1 & 0 \\ \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \end{bmatrix}.$$

If $(\boldsymbol{w})_T = (2, -1, -3)$, then

$$[\boldsymbol{w}]_S = \boldsymbol{P}^{-1}[\boldsymbol{w}]_T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -1 & -1 & 0 \\ \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}.$$