National University of Singapore
Department of Mathematics

Semester 2, 2019/2020

MA1101R Linear Algebra I

Time allowed: 2 hours and 30 minutes

Instructions to candidates:

1. Get ready a signed copy of the Exam declaration form for this exam.
2. Use A4 size paper and pen (blue or black ink) to write your answers.
3. Write down your student number clearly on the **top left of every page** of the answers. Do not write your name.
4. Write on one side of the paper only. Write the question number and page number on the **top right corner of each page** (e.g. Q1P1, Q1P2, . . . , Q2P1, . . .).
5. This examination paper consists of 7 questions, for a total of 90 points. Excluding the cover page, there are 2 pages.
6. Answer all 7 questions.
7. This is an **open book** examination.
8. You are permitted to use **any kind of calculator, including MATLAB**. However various steps in the calculations should be laid out systematically.
9. Join the Zoom conference, turn on the video setting and keep it on at all times during the exam. Adjust your camera such that your face and upper body including your hands are captured on Zoom.
10. You may go for a short toilet break (not more than 5 minutes) during the exam.
11. Do the following at the end of the exam:
    - scan or take pictures of your work (make sure the scans or images can be read clearly) together with the declaration form;
    - merge all your scans or images into one `pdf` file (arrange them in the order: declaration form, Q1 to Q7 in their ascending page sequence);
    - name the `pdf` file in the format ⟨Matric No⟩_MA1101R.`pdf` (e.g. **A1234567P_MA1101R**.`pdf`);
    - upload your `pdf` into the LumiNUS folder that has the name "Exam Submission".
12. This folder will close on **April 30, 2020 at 19:30**. After the folder is closed, exam answers that are not submitted will not be accepted.

Do not write below this box.

1. Let $\mathbf{u}_1 = (1, 0, 1, -2, 2)$, $\mathbf{u}_2 = (0, -1, -2, 1, 0)$, $\mathbf{v}_1 = (2, -3, -4, -1, 4)$, $\mathbf{v}_2 = (1, 1, 2, -2, 0)$, and $\mathbf{v}_3 = (2, 2, 3, -3, -2)$.
   (a) (5 points) Find a linearly independent set S of vectors in $\mathbb{R}^5$ such that the vectors in $\mathrm{span}(S)$ are precisely those in both $\mathrm{span}\{\mathbf{u}_1, \mathbf{u}_2\}$ and $\mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.
   (b) (5 points) Find a non-zero vector $\mathbf{w}$ in $\mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, all of whose coordinates are integers, that is orthogonal to $\mathrm{span}\{\mathbf{u}_1, \mathbf{u}_2\}$.

   You must **indicate how you obtained your final answers**. Writing only the final answers will **not** fetch points.

2. Let $\mathbf{v}_1 = (5, 2, 6, -4)$, $\mathbf{v}_2 = (-12, -3, -12, 6)$, and $\mathbf{v}_3 = (2a + 3, 8a + 3, -3a + 6, 2a - 6)$, where a is some unknown real number. Let $V = \mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.
   (a) (5 points) Transform $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ into an **orthonormal** basis for V by applying the **Gram-Schmidt Process**. Orthonormal bases obtained using a method different from the Gram-Schmidt Process will **not** be accepted.

   Hint: your solution will **depend on** the value of a. Also, all entries in the solution are **rational numbers**.
   (b) (2 points) What are the possible values for $\dim(V)$?
   (c) (4 points) Let T be the orthonormal basis for V found in Part (a). In the case when $\dim(V) = 3$, find the transition matrix from S to T, where $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

   Hint: all entries in the solution are **integers** or **integer multiples** of a.
   (d) (4 points) Find a **least squares solution** to

$$\begin{pmatrix} 5 & -12 & 5 \\ 2 & -3 & 11 \\ 6 & -12 & 3 \\ -4 & 6 & -4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 \\ 7 \\ -9 \\ -5 \end{pmatrix}.$$

3. Let $\mathbf{A} = \begin{pmatrix} 1 & -2 & 1 & 3 \\ 1 & -1 & 0 & 4 \\ 1 & 0 & -1 & 5 \end{pmatrix}$.
   (a) (6 points) Find a basis S for the row space of $\mathbf{A}$ and a basis T for the nullspace of $\mathbf{A}$.
   (b) (4 points) Find a matrix $\mathbf{B}$ whose nullspace is equal to the column space of $\mathbf{A}^{\mathrm{T}}$. You must **explain why your matrix works.**

4. Let $\mathbf{A}$ be the matrix $\begin{pmatrix} \frac{5}{2} & 1 & -2 \\ 1 & 1 & -1 \\ 2 & 1 & -\frac{3}{2} \end{pmatrix}$.
   (a) (3 points) Write down a matrix whose determinant is equal to the characteristic polynomial of $\mathbf{A}$. Find the characteristic polynomial of $\mathbf{A}$ and verify that the eigenvalues of $\mathbf{A}$ are $\lambda = 1$ and $\lambda = \frac{1}{2}$.
   (b) (3 points) Write down a homogeneous linear system whose solution space is equal to the eigenspace E_1 of $\mathbf{A}$ corresponding to the eigenvalue $\lambda = 1$. By solving this linear system, find a basis for E_1.
   (c) (3 points) Write down a homogeneous linear system whose solution space is equal to the eigenspace $E_{\frac{1}{2}}$ of $\mathbf{A}$ corresponding to the eigenvalue $\lambda = \frac{1}{2}$. By solving this linear system, find a basis for $E_{\frac{1}{2}}$.
   (d) (6 points) Write down an invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$. Calculate $\lim_{n\to\infty}\mathbf{D}^n$, and use this answer to calculate $\lim_{n\to\infty}\mathbf{A}^n$.

5. All vectors are column vectors in this question.

Define

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \\ 0 \end{pmatrix} \middle| \; x, y, z \in \mathbb{R} \right\}.$$

(a) (3 points) Verify that V is a subspace of $\mathbb{R}^4$ and find a basis for V.

(b) (2 points) Write down a 4×3 matrix $\mathbf{A}$ whose column space is equal to V.

(c) (2 points) What is the rank and the nullity of $\mathbf{A}$? **Justify your answer.**

(d) (3 points) Find a matrix $\mathbf{B}$ such that $\mathbf{BA} = \mathbf{I}_3$. Is $\mathbf{B}$ unique? **Justify your answer**.

(e) (5 points) Show that there is no matrix $\mathbf{D}$ such that $\mathbf{AD} = \mathbf{I}_4$.

6. Let

$$\mathbf{A} = \begin{pmatrix} 2 & a & b \\ 0 & c & d \\ 0 & 0 & e \end{pmatrix},$$

where a, b, c, d, and e are real numbers.

(a) (5 points) Show that 2 is an eigenvalue of $\mathbf{A}$ and find an eigenvector associated with the eigenvalue 2.

(b) (5 points) Assuming that 9 is also an eigenvalue of $\mathbf{A}$ and that $\lambda^3 + f\lambda^2 + g\lambda + 18$ is the characteristic polynomial of $\mathbf{A}$, where f and g are some real numbers, prove that $\mathbf{A}$ is diagonalizable.

7. Let $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be a set of column vectors. Suppose S is an orthonormal basis for a subspace V of $\mathbb{R}^n$. For a vector $\mathbf{v} \in V$, $(\mathbf{v})_S$ denotes the coordinates of $\mathbf{v}$ relative to S. Let $\mathbf{A}$ be an $m \times n$ matrix. Define $M = ||\mathbf{Au}_1|| + \cdots + ||\mathbf{Au}_k||$.

(a) (3 points) Prove that for any $\mathbf{v} \in V$, if $(\mathbf{v})_S = (c_1, \ldots, c_k)$, then $||\mathbf{v}|| \le |c_1| + \cdots + |c_k|$.

(b) (4 points) Prove that for any $\mathbf{v} \in V$, if $(\mathbf{v})_S = (c_1, \ldots, c_k)$ and $||\mathbf{v}|| \le 1$, then $|c_i| \le 1$, for each $1 \le i \le k$.

(c) (4 points) Prove that for any $\mathbf{v} \in V$, if $||\mathbf{v}|| \le 1$, then $||\mathbf{Av}|| \le M$.

(d) (4 points) Prove that for any vector $\mathbf{v} \in V$, $||\mathbf{Av}|| \le M\,||\mathbf{v}||$.