Answers/Solutions of Exercise 2 (Version: October 3, 2016)

1. (a) $\begin{pmatrix} -6 & 6 & -6 \\ -6 & 6 & -6 \\ -6 & 6 & -6 \end{pmatrix}$ (b) $\begin{pmatrix} -4 & 2 & 5 & 8 \\ 1 & -5 & -5 & -8 \\ -1 & 2 & 2 & 8 \\ 1 & -2 & -5 & -11 \end{pmatrix}$ (c) Not possible

   (d) $\begin{pmatrix} 1 & 3 & 6 \\ 0 & 4 & 10 \\ 0 & 0 & 9 \end{pmatrix}$ (e) $\begin{pmatrix} -3 & 3 & -4 \\ 3 & -3 & 4 \\ -3 & 3 & -4 \\ 3 & -3 & 4 \end{pmatrix}$ (f) $\begin{pmatrix} 3 & -6 & -15 & -24 \\ 8 & -4 & -16 & -28 \\ 9 & 0 & -9 & -18 \end{pmatrix}$

   (g) Not possible (h) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ (i) $\begin{pmatrix} -38 \\ -28 \\ -18 \end{pmatrix}$

   (j) Not possible (k) $\begin{pmatrix} 16 & 0 \\ 22 & 2 \\ 6 & 0 \end{pmatrix}$ (l) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

   (m) $\begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 2 \\ -1 & -2 & 0 \end{pmatrix}$ (n) Not possible (o) $\begin{pmatrix} 1 & -1 & 3 & 2 \\ -1 & 1 & -3 & -2 \\ 3 & -3 & 9 & 6 \\ 2 & -2 & 6 & 4 \end{pmatrix}$

   (p) 15

2. $a = 0$, $b = -1$, $c = 2$, $d = -4$.

3. (a) (i) (3,4)-entry of $\boldsymbol{AB}$ (ii) (4,1)-entry of $\boldsymbol{AB}$

   (iii) (3,2)-entry of $\boldsymbol{BA}$ (iv) (2,5)-entry of $\boldsymbol{BA}$

   (b) (i) $\sum_{j=1}^{n} a_{3j}b_{j2}$ (ii) $\sum_{i=1}^{m} b_{4i}a_{i1}$

4. (a) $a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{ip}b_{pj} = \displaystyle\sum_{k=1}^{p} a_{ik}b_{kj}$

   (b) $c_{i1}c_{1j} + c_{i2}c_{2j} + c_{i3}c_{3j} + \cdots + c_{ip}c_{pj} = \displaystyle\sum_{k=1}^{p} c_{ik}c_{kj}$

   (c) $a_{i1}c_{j1} + a_{i2}c_{j2} + a_{i3}c_{j3} + \cdots + a_{ip}c_{jp} = \displaystyle\sum_{k=1}^{p} a_{ik}c_{jk}$

5. For example, $\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$.

   The general form of the matrix $\boldsymbol{A} = (a_{ij})_{3\times 3}$ is $a_{ii} = 0$ for $i = 1, 2, 3$ and $a_{ij} = -a_{ji}$ for all other values of $1 \le i, j \le 3$.

6. (a) For example, $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

   (b) For example, $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

   (c) For example, $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\boldsymbol{C} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

7. The matrix $\boldsymbol{A}$ can be $\begin{pmatrix} 2 & 3 & -1 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 2 & 3 & -1 \\ 2 & 3 & -1 \end{pmatrix}$, $\begin{pmatrix} 2 & 3 & -1 \\ 4 & 6 & -2 \end{pmatrix}$, etc.

8. (a) S is a straight line joining $(1, 0, 3)$ and $(0, -1, 3)$.

   (b) For example, $\boldsymbol{A} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

   The linear system consists of two planes which intersect at the line S.

9. If $\boldsymbol{Ax} = \boldsymbol{b}$ has a solution $\boldsymbol{x} = \boldsymbol{u}$, then $\boldsymbol{u} + \boldsymbol{v}$ is also a solution to $\boldsymbol{Ax} = \boldsymbol{b}$ for all solutions $\boldsymbol{x} = \boldsymbol{v}$ to $\boldsymbol{Ax} = \boldsymbol{0}$. Hence $\boldsymbol{Ax} = \boldsymbol{b}$ has either no solutions or infinitely many solutions.

10. (a) Let $\boldsymbol{x} = \boldsymbol{u}$ be any solution to the system $\boldsymbol{Bx} = \boldsymbol{0}$. Then $\boldsymbol{ABu} = \boldsymbol{A0} = \boldsymbol{0}$. The system $\boldsymbol{ABx} = \boldsymbol{0}$ has at least as many solutions as the system $\boldsymbol{Bx} = \boldsymbol{0}$. Thus it has infinitely many solutions.

    (b) No. For example, let $\boldsymbol{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and consider two cases (i) $\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and (ii) $\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Note that $\boldsymbol{Bx} = \boldsymbol{0}$ has only the trivial solution. For (i), $\boldsymbol{ABx} = \boldsymbol{0}$ has only the trivial solution while for (ii), $\boldsymbol{ABx} = \boldsymbol{0}$ has infinitely many solutions.

11. (a) (i) 2; (ii) -6; (iii) 16.

(b) $\mathrm{tr}(\boldsymbol{A}+\boldsymbol{B}) = (a_{11}+b_{11})+(a_{22}+b_{22})+\cdots+(a_{nn}+b_{nn})$
$= (a_{11}+a_{22}+\cdots+a_{nn})+(b_{11}+b_{22}+\cdots+b_{nn}) = \mathrm{tr}(\boldsymbol{A})+\mathrm{tr}(\boldsymbol{B})$.

(c) $\mathrm{tr}(c\boldsymbol{A}) = ca_{11}+...+ca_{nn} = c(a_{11}+...+a_{nn}) = c\,\mathrm{tr}(\boldsymbol{A})$.

(d) The (i,i)-entry of $\boldsymbol{CD} = c_{i1}d_{1i}+c_{i2}d_{2i}+...+c_{in}d_{ni}$. Thus,

$$\mathrm{tr}(\boldsymbol{CD}) = \sum_{i=1}^{m}(c_{i1}d_{1i}+c_{i2}d_{2i}+...+c_{in}d_{ni}) = \sum_{j=1}^{n}(c_{1j}d_{j1}+c_{2j}d_{j2}+...+c_{mj}d_{jm}).$$

But the (i,i)-entry of $\boldsymbol{DC} = d_{i1}c_{1i}+d_{i2}c_{2i}+...+d_{im}c_{mi}$. So the trace of $\boldsymbol{DC}$ is precisely the term on the right hand side above.

(e) By (d), $\mathrm{tr}(\boldsymbol{AB}) = \mathrm{tr}(\boldsymbol{BA})$. Then by (b) and (c), $\mathrm{tr}(\boldsymbol{AB}-\boldsymbol{BA}) = \mathrm{tr}(\boldsymbol{AB})-\mathrm{tr}(\boldsymbol{BA}) = 0$. However, $\mathrm{tr}(\boldsymbol{I})$=$n$. It is impossible to have square matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ such that $\boldsymbol{AB}-\boldsymbol{BA}=\boldsymbol{I}$.

12. (a) (i) is not orthogonal while (ii) is orthogonal.

(b) $(\boldsymbol{AB})(\boldsymbol{AB})^{\mathrm{T}} = \boldsymbol{ABB}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{AIA}^{\mathrm{T}} = \boldsymbol{I}$ and $(\boldsymbol{AB})^{\mathrm{T}}(\boldsymbol{AB}) = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{AB} = \boldsymbol{BIB}^{\mathrm{T}} = \boldsymbol{I}$ since both $\boldsymbol{A}$ and $\boldsymbol{B}$ are orthogonal. Thus $\boldsymbol{AB}$ is orthogonal.

13. (a) $\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix}$ and $\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix}$.

(b) Since $\boldsymbol{AB}=\boldsymbol{BA}$, $(\boldsymbol{AB})^k = \boldsymbol{A}^k\boldsymbol{B}^k$ (you need to prove it by using the mathematical induction). Since $\boldsymbol{A}$ is nilpotent, $\boldsymbol{A}^k = 0$ for some positive integer k. Thus $(\boldsymbol{AB})^k = \boldsymbol{A}^k\boldsymbol{B}^k = \boldsymbol{0}$ and $\boldsymbol{AB}$ is nilpotent.

(c) No. For example, let $\boldsymbol{A} = \begin{pmatrix}0&1\\0&0\end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix}0&0\\1&0\end{pmatrix}$. Note that $\boldsymbol{A}$ is nilpotent and $\boldsymbol{AB} = \begin{pmatrix}1&0\\0&0\end{pmatrix} \neq \begin{pmatrix}0&0\\0&1\end{pmatrix} = \boldsymbol{BA}$. But $(\boldsymbol{AB})^k = \begin{pmatrix}1&0\\0&0\end{pmatrix}$ for all k and hence $\boldsymbol{AB}$ is not nilpotent. (For this case, $(\boldsymbol{AB})^k \neq \boldsymbol{A}^k\boldsymbol{B}^k$.)

14. (a) All except $\begin{pmatrix}1&1\\1&0\end{pmatrix}$ satisfy $(\star)$.

(b) Since $\boldsymbol{P},\boldsymbol{Q}\in\mathcal{B}$, $\boldsymbol{AP}=\boldsymbol{PA}$ and $\boldsymbol{AQ}=\boldsymbol{QA}$. Then,

$$\boldsymbol{A}(\boldsymbol{P}+\boldsymbol{Q}) = \boldsymbol{AP}+\boldsymbol{AQ} = \boldsymbol{PA}+\boldsymbol{QA} = (\boldsymbol{P}+\boldsymbol{Q})\boldsymbol{A}.$$

Hence $\boldsymbol{P}+\boldsymbol{Q}$ satisfies $(\star)$.

Likewise, $\boldsymbol{A}(\boldsymbol{PQ}) = \boldsymbol{APQ} = \boldsymbol{PAQ} = \boldsymbol{PQA} = (\boldsymbol{PQ})\boldsymbol{A}$ and hence$\boldsymbol{PQ}$ satisfies $(\star)$.

(c) $\boldsymbol{A}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}\boldsymbol{A} \;\Leftrightarrow\; \begin{pmatrix} p+r & q+s \\ r & s \end{pmatrix} = \begin{pmatrix} p & p+q \\ r & r+s \end{pmatrix}$

Thus the conditions are $r = 0$ and $s = p$.

15. (a) The statement is clearly true when $k = 1$. Assume that statement is true when $k = n$, i.e.

$$\boldsymbol{D}^n = \begin{pmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{pmatrix}.$$

Then $\boldsymbol{D}^{n+1} = \boldsymbol{D}\boldsymbol{D}^n$ ie.

$$\boldsymbol{D}^{n+1} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}\begin{pmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{pmatrix} = \begin{pmatrix} a^{n+1} & 0 & 0 \\ 0 & b^{n+1} & 0 \\ 0 & 0 & c^{n+1} \end{pmatrix}.$$

Thus the statement is true when $k = n+1$. By the mathematical induction the statement is true for all positive intergers k.

(b) $\boldsymbol{A} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

(c) There are 8 such diagonal matrices $\boldsymbol{B}$: $\begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 2 & 0 \\ 0 & 0 & \pm 3 \end{pmatrix}$.

16. (a) No. For example, $\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$.

(b) $\boldsymbol{ABC} = \boldsymbol{BAC} = \boldsymbol{BCA}$ and $\boldsymbol{ACB} = \boldsymbol{CAB} = \boldsymbol{CBA}$.

17. See Question 3 of Tutorial 2.

18. Let $\boldsymbol{A} = (a_{ij})_{m\times n}$ and $\boldsymbol{B} = (b_{ij})_{m\times n}$. Since all matrices in this question are of the same size, we only need to check the (i, j)-entries.

(a) For $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$,

$$\begin{aligned} \text{the } (i,j)\text{-entry of } \boldsymbol{A}+\boldsymbol{B} &= a_{ij} + b_{ij} \\ &= b_{ij} + a_{ij} \quad \text{(by a property of real numbers)} \\ &= \text{the } (i,j)\text{-entry of } \boldsymbol{B}+\boldsymbol{A}. \end{aligned}$$

Thus $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$.

(b) For $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$,

$$\begin{aligned}
\text{the } (i,j)\text{-entry of } c(\boldsymbol{A}+\boldsymbol{B}) &= c(\text{the } (i,j)\text{-entry of } \boldsymbol{A}+\boldsymbol{B}) \\
&= c(a_{ij}+b_{ij}) \\
&= ca_{ij}+cb_{ij} \quad \text{(by a property of real numbers)} \\
&= \text{the } (i,j)\text{-entry of } c\boldsymbol{A} + \text{the } (i,j)\text{-entry of } c\boldsymbol{B} \\
&= \text{the } (i,j)\text{-entry of } c\boldsymbol{A}+c\boldsymbol{B}.
\end{aligned}$$

Thus $c(\boldsymbol{A}+\boldsymbol{B}) = c\boldsymbol{A}+c\boldsymbol{B}$.

(c) For $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$,

$$\begin{aligned}
\text{the } (i,j)\text{-entry of } (c+d)\boldsymbol{A} &= (c+d)a_{ij} \\
&= ca_{ij}+da_{ij} \quad \text{(by a property of real numbers)} \\
&= \text{the } (i,j)\text{-entry of } c\boldsymbol{A} + \text{the } (i,j)\text{-entry of } d\boldsymbol{A} \\
&= \text{the } (i,j)\text{-entry of } c\boldsymbol{A}+d\boldsymbol{A}.
\end{aligned}$$

Thus $(c+d)\boldsymbol{A} = c\boldsymbol{A}+d\boldsymbol{A}$.

(d) For $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$,

$$\begin{aligned}
\text{the } (i,j)\text{-entry of } c(d\boldsymbol{A}) &= c(\text{the } (i,j)\text{-entry of } d\boldsymbol{A}) \\
&= c(da_{ij}) \\
&= (cd)a_{ij} \quad \text{(by a property of real numbers)} \\
&= \text{the } (i,j)\text{-entry of } (cd)\boldsymbol{A}.
\end{aligned}$$

Thus $c(d\boldsymbol{A}) = (cd)\boldsymbol{A}$ and hence $d(c\boldsymbol{A}) = (dc)\boldsymbol{A} = (cd)\boldsymbol{A}$ (where the last equality follows by $dc = cd$ which is a property of real number).

(e) For $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$,

$$\begin{aligned}
\text{the } (i,j)\text{-entry of } \mathbf{0}+\boldsymbol{A} &= 0+a_{ij} \\
&= a_{ij} \quad \text{(by a property of real numbers)} \\
&= \text{the } (i,j)\text{-entry of } \boldsymbol{A}.
\end{aligned}$$

an by (a), $\boldsymbol{A}+\mathbf{0} = \mathbf{0}+\boldsymbol{A} = \boldsymbol{A}$.

(f) For $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$,

$$\begin{aligned}
\text{the } (i,j)\text{-entry of } \boldsymbol{A}-\boldsymbol{A} &= a_{ij}-a_{ij} \\
&= 0 \quad \text{(by a property of real numbers)} \\
&= \text{the } (i,j)\text{-entry of } \mathbf{0}.
\end{aligned}$$

Thus $\boldsymbol{A}-\boldsymbol{A} = \mathbf{0}$.

(g) For $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$,

$$\begin{aligned}\text{the } (i,j)\text{-entry of } 0\boldsymbol{A} &= 0 \cdot a_{ij} \\ &= 0 \quad \text{(by a property of real numbers)} \\ &= \text{the } (i,j)\text{-entry of } \mathbf{0}.\end{aligned}$$

Thus $0\boldsymbol{A} = \mathbf{0}$.

19. It is easier to use the summation notation $\sum$ to do this question.

(a) Let $\boldsymbol{A} = (a_{ij})_{m\times p}$, $\boldsymbol{B} = (b_{ij})_{p\times q}$ and $\boldsymbol{C} = (c_{ij})_{q\times n}$.

(i) The size of $\boldsymbol{BC}$ is $p \times n$ and hence the size of $\boldsymbol{A}(\boldsymbol{BC})$ is $m \times n$. On the other hand, the size of $\boldsymbol{AB}$ is $m \times q$ and hence the size of $(\boldsymbol{AB})\boldsymbol{C}$ is $m \times n$. So the sizes of $\boldsymbol{A}(\boldsymbol{BC})$ and $(\boldsymbol{AB})\boldsymbol{C}$ are the same.

(ii) For $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$,

$$\begin{aligned}&\text{the } (i,j)\text{-entry of } \boldsymbol{A}(\boldsymbol{BC}) \\ &= \sum_{k=1}^{p} a_{ik}(\text{the } (k,j)\text{-entry of } \boldsymbol{BC}) \\ &= \sum_{k=1}^{p} a_{ik}(b_{k1}c_{1j} + b_{k2}c_{2j} + \cdots + b_{kq}c_{qj}) \\ &= \sum_{k=1}^{p} (a_{ik}b_{k1}c_{1j} + a_{ik}b_{k2}c_{2j} + \cdots + a_{ik}b_{kq}c_{qj}) \\ &= \sum_{k=1}^{p}\sum_{r=1}^{q} a_{ik}b_{kr}c_{rj}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\text{the } (i,j)\text{-entry of } (\boldsymbol{AB})\boldsymbol{C} \\ &= \sum_{r=1}^{q} (\text{the } (i,r)\text{-entry of } \boldsymbol{AB})c_{r,j} \\ &= \sum_{r=1}^{q} (a_{i1}b_{1r} + a_{i2}b_{2r} + \cdots + a_{iq}b_{qr})c_{r,j} \\ &= \sum_{r=1}^{q} (a_{i1}b_{1r}c_{rj} + a_{i2}b_{2r}c_{rj} + \cdots + a_{iq}b_{qr}c_{rj}) \\ &= \sum_{r=1}^{q}\sum_{k=1}^{p} a_{ik}b_{kr}c_{rj} = \sum_{k=1}^{p}\sum_{r=1}^{q} a_{ik}b_{kr}c_{rj}.\end{aligned}$$

Thus the (i,j)-entries of $\boldsymbol{A}(\boldsymbol{BC})$ and $(\boldsymbol{AB})\boldsymbol{C}$ are the same.

By (i) and (ii), $\boldsymbol{A}(\boldsymbol{BC}) = (\boldsymbol{AB})\boldsymbol{C}$.

(b) Let $\boldsymbol{A} = (a_{ij})_{p\times n}$, $\boldsymbol{C_1} = (c_{ij})_{m\times p}$ and $\boldsymbol{C_2} = (d_{ij})_{m\times p}$.

(i) The size of $\boldsymbol{C_1} + \boldsymbol{C_2}$ is $m \times p$ and hence the size of $(\boldsymbol{C_1} + \boldsymbol{C_2})\boldsymbol{A}$ is $m\times n$. On the other hand, the sizes of both $\boldsymbol{C_1 A}$ and $\boldsymbol{C_2 A}$ are $m\times n$ and hence the size of $\boldsymbol{C_1 A}+\boldsymbol{C_2 A}$ is $m\times n$. So the sizes of $(\boldsymbol{C_1}+\boldsymbol{C_2})\boldsymbol{A}$ and $\boldsymbol{C_1 A} + \boldsymbol{C_2 A}$ are the same.

(ii) For $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$,

$$\begin{aligned}
&\text{the } (i,j)\text{-entry of } (\boldsymbol{C_1} + \boldsymbol{C_2})\boldsymbol{A} \\
= &\sum_{k=1}^{p} (\text{the } (i,k)\text{-entry of } \boldsymbol{C_1} + \boldsymbol{C_2}) a_{kj} \\
= &\sum_{k=1}^{p} (c_{ik} + d_{ik}) a_{kj} \\
= &\sum_{k=1}^{p} (c_{ik}a_{kj} + d_{ik}a_{kj}) \\
= &\sum_{k=1}^{p} c_{ik}a_{kj} + \sum_{k=1}^{p} d_{ik}a_{kj} \\
= &(\text{the } (i,j)\text{-entry of } \boldsymbol{C_1 A}) + (\text{the } (i,j)\text{-entry of } \boldsymbol{C_2 A}).
\end{aligned}$$

By (i) and (ii), $(\boldsymbol{C_1} + \boldsymbol{C_2})\boldsymbol{A} = \boldsymbol{C_1 A} + \boldsymbol{C_2 A}$.

(c) Let $\boldsymbol{A} = (a_{ij})_{m\times p}$ and $\boldsymbol{B} = (b_{ij})_{p\times n}$.

(i) The sizes of all the three matrices are $m \times n$.

(ii) For $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$,

$$\text{the } (i,j)\text{-entry of } c(\boldsymbol{AB}) = c\sum_{k=1}^{p} a_{ik}b_{kj} = \sum_{k=1}^{p} ca_{ik}b_{kj},$$

$$\text{the } (i,j)\text{-entry of } (c\boldsymbol{A})\boldsymbol{B} = \sum_{k=1}^{p} (\text{the } (i,k)\text{-entry of } c\boldsymbol{A})b_{kj} = \sum_{k=1}^{p} (ca_{ik})b_{kj},$$

$$\text{the } (i,j)\text{-entry of } \boldsymbol{A}(c\boldsymbol{B}) = \sum_{k=1}^{p} a_{ik}(\text{the } (k,j)\text{-entry of } c\boldsymbol{B}) = \sum_{k=1}^{p} a_{ik}(cb_{kj}).$$

Thus the (i,j)-entries of all the three matrices are the same.

By (i) and (ii), $c(\boldsymbol{AB}) = (c\boldsymbol{A})\boldsymbol{B} = \boldsymbol{A}(c\boldsymbol{B})$.

(d) Let $\boldsymbol{A} = (a_{ij})_{m\times p}$ and let $\delta_{ij} = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } i \neq j. \end{cases}$

(i) The size of $\boldsymbol{A}\mathbf{0}_{n\times q}$ is $m \times q$ which is equal to the size of $\mathbf{0}_{m\times q}$; the size of $\mathbf{0}_{p\times m}\boldsymbol{A}$ is $p \times n$ which is equal to the size of $\mathbf{0}_{p\times n}$; and finally, all three matrices $\boldsymbol{A}\boldsymbol{I}_n$, $\boldsymbol{I}_m\boldsymbol{A}$ and $\boldsymbol{A}$ are $m \times n$.

(ii) For $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, q$,

$$\text{the } (i,j)\text{-entry of } \boldsymbol{A}\mathbf{0}_{n\times q} = \sum_{k=1}^{n} a_{ik}0 = 0 = \text{the } (i,j)\text{-entry of } \mathbf{0}_{m\times q}.$$

For $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, n$,

$$\text{the } (i,j)\text{-entry of } \mathbf{0}_{p\times m}\boldsymbol{A} = \sum_{k=1}^{m} 0a_{kj} = 0 = \text{the } (i,j)\text{-entry of } \mathbf{0}_{p\times n}.$$

For $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$,

$$\text{the } (i,j)\text{-entry of } \boldsymbol{A}\boldsymbol{I}_n = \sum_{k=1}^{n} a_{ik}\delta_{kj} = a_{ij} = \text{the } (i,j)\text{-entry of } \boldsymbol{A}.$$

For $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$,

$$\text{the } (i,j)\text{-entry of } \boldsymbol{I}_m\boldsymbol{A} = \sum_{k=1}^{m} \delta_{ik}a_{kj} = a_{ij} = \text{the } (i,j)\text{-entry of } \boldsymbol{A}.$$

Thus $\boldsymbol{A}\mathbf{0}_{n\times q} = \mathbf{0}_{m\times q}$, $\mathbf{0}_{p\times m}\boldsymbol{A} = \mathbf{0}_{p\times n}$ and $\boldsymbol{A}\boldsymbol{I}_n = \boldsymbol{I}_m\boldsymbol{A} = \boldsymbol{A}$.

20. (a) (i) The size of $\boldsymbol{A}^{\mathrm{T}}$ is $n \times m$ and hence the size of $(\boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}}$ is $m \times n$ which is equal to the size of $\boldsymbol{A}$.

(ii) For $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$,

$$\text{the } (i,j)\text{-entry of } (\boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}} = \text{the } (j,i)\text{-entry of } \boldsymbol{A}^{\mathrm{T}} = \text{the } (i,j)\text{-entry of } \boldsymbol{A}.$$

By (i) and (ii), $(\boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}} = \boldsymbol{A}$.

(b) Let $\boldsymbol{A} = (a_{ij})_{m\times n}$ and $\boldsymbol{B} = (b_{ij})_{m\times n}$.

(i) The sizes of the two matrices are $n \times m$.

(ii) For $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$,

$$\begin{aligned}
&\text{the } (i,j)\text{-entry of } (\boldsymbol{A}+\boldsymbol{B})^{\mathrm{T}} \\
&= \text{the } (j,i)\text{-entry of } \boldsymbol{A}+\boldsymbol{B} \\
&= a_{ji} + b_{ji} \\
&= \text{the } (i,j)\text{-entry of } \boldsymbol{A}^{\mathrm{T}} + \text{the } (i,j)\text{-entry of } \boldsymbol{B}^{\mathrm{T}} \\
&= \text{the } (i,j)\text{-entry of } \boldsymbol{A}^{\mathrm{T}} + \boldsymbol{B}^{\mathrm{T}}.
\end{aligned}$$

By (i) and (ii), $(\boldsymbol{A}+\boldsymbol{B})^{\mathrm{T}} = \boldsymbol{A}^{\mathrm{T}} + \boldsymbol{B}^{\mathrm{T}}$.

(c) (i) The sizes of the two matrices are $n \times m$.

(ii) For $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$,

$$\begin{aligned}
&\text{the } (i,j)\text{-entry of } (c\boldsymbol{A})^{\mathrm{T}} \\
&= \text{the } (j,i)\text{-entry of } c\boldsymbol{A} \\
&= c\,(\text{the } (j,i)\text{-entry of } \boldsymbol{A}) \\
&= c\,(\text{the } (i,j)\text{-entry of } \boldsymbol{A}^{\mathrm{T}}) \\
&= \text{the } (i,j)\text{-entry of } c\boldsymbol{A}^{\mathrm{T}}.
\end{aligned}$$

By (i) and (ii), $(c\boldsymbol{A})^{\mathrm{T}} = c\boldsymbol{A}^{\mathrm{T}}$.

21. $\boldsymbol{X} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

22. See Question 5 of Tutorial 2.

23. See Question 4 of Tutorial 2.

24. (a) True. Let $\boldsymbol{A} = (a_{ij})_{n\times n}$ and $\boldsymbol{B} = (b_{ij})_{n\times n}$. Since $a_{ij} = b_{ij} = 0$ for $i \neq j$, the (i,j)-entry of $\boldsymbol{AB}$ is equal to

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \begin{cases} a_{ii}b_{ii} & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Likewise, the (i,j)-entry of $\boldsymbol{BA}$ is equal to

$$b_{i1}a_{1j} + b_{i2}a_{2j} + \cdots + b_{in}a_{nj} = \begin{cases} b_{ii}a_{ii} & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Thus $\boldsymbol{AB} = \boldsymbol{BA}$.

(b) True. Let $\boldsymbol{D} = \frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}})$.

$$\boldsymbol{D}^{\mathrm{T}} = \left[\tfrac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}})\right]^{\mathrm{T}} = \tfrac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}} = \tfrac{1}{2}(\boldsymbol{A}^{\mathrm{T}} + (\boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}}) = \tfrac{1}{2}(\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{A}) = \boldsymbol{D}.$$

Thus $\boldsymbol{D}$ is symmetric.

(c)-(g) See Question 6 of Tutorial 2.

25. (a) $\boldsymbol{A}^2 = \begin{pmatrix} 4 & -6 & -6 \\ 0 & 10 & 6 \\ 0 & 6 & 10 \end{pmatrix}$, $-6\boldsymbol{A} = \begin{pmatrix} -12 & 6 & 6 \\ 0 & -18 & -6 \\ 0 & -6 & -18 \end{pmatrix}$, $8\boldsymbol{I} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}$.

It is easy to be checked that $\boldsymbol{A}^2 - 6\boldsymbol{A} + 8\boldsymbol{I} = \boldsymbol{0}$.

(b) By (a), $\boldsymbol{A}^2 = 6\boldsymbol{A} - 8\boldsymbol{I}$. Since

$$\boldsymbol{A}\left[\tfrac{1}{8}(6\boldsymbol{I} - \boldsymbol{A})\right] = \tfrac{1}{8}\boldsymbol{A}(6\boldsymbol{I} - \boldsymbol{A}) = \tfrac{1}{8}(6\boldsymbol{A} - \boldsymbol{A}^2) = \tfrac{1}{8}(6\boldsymbol{A} - 6\boldsymbol{A} + 8\boldsymbol{I}) = \boldsymbol{I},$$

$\boldsymbol{A}^{-1} = \tfrac{1}{8}(6\boldsymbol{I} - \boldsymbol{A})$.

26. See Question 2 of Tutorial 3.

27. (a) For example, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

(b) Since $(\boldsymbol{I} + \boldsymbol{A})\left[\tfrac{1}{2}(2\boldsymbol{I} - \boldsymbol{A})\right] = \tfrac{1}{2}(\boldsymbol{I} + \boldsymbol{A})(2\boldsymbol{I} - \boldsymbol{A}) = \tfrac{1}{2}(2\boldsymbol{I} + \boldsymbol{A} - \boldsymbol{A}^2) = \boldsymbol{I}$, $\boldsymbol{I} + \boldsymbol{A}$ is invertible and its inverse is $\tfrac{1}{2}(2\boldsymbol{I} - \boldsymbol{A})$.

28. (a) False. For example, let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

(b) False. For example, let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

29. (Since we cannot assume that $\boldsymbol{A}^{-1} + \boldsymbol{B}^{-1}$ is invertible at the beginning, we cannot prove $(\boldsymbol{A} + \boldsymbol{B})^{-1} = \boldsymbol{A}^{-1}(\boldsymbol{A}^{-1} + \boldsymbol{B}^{-1})^{-1}\boldsymbol{B}^{-1}$ directly. Instead, we first prove the equivalent form $(\boldsymbol{A}(\boldsymbol{A} + \boldsymbol{B})^{-1}\boldsymbol{B})^{-1} = \boldsymbol{A}^{-1} + \boldsymbol{B}^{-1}$.)

Since $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{A} + \boldsymbol{B}$ are invertible, $\boldsymbol{A}(\boldsymbol{A} + \boldsymbol{B})^{-1}\boldsymbol{B}$ is invertible and

$$\left(\boldsymbol{A}(\boldsymbol{A} + \boldsymbol{B})^{-1}\boldsymbol{B}\right)^{-1} = \boldsymbol{B}^{-1}(\boldsymbol{A} + \boldsymbol{B})\boldsymbol{A}^{-1} = (\boldsymbol{B}^{-1}\boldsymbol{A} + \boldsymbol{I})\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1} + \boldsymbol{B}^{-1}.$$

Hence $\boldsymbol{A}^{-1} + \boldsymbol{B}^{-1}$ is invertible and $\boldsymbol{A}(\boldsymbol{A} + \boldsymbol{B})^{-1}\boldsymbol{B} = \left(\boldsymbol{A}^{-1} + \boldsymbol{B}^{-1}\right)^{-1}$ which implies $(\boldsymbol{A} + \boldsymbol{B})^{-1} = \boldsymbol{A}^{-1}(\boldsymbol{A}^{-1} + \boldsymbol{B}^{-1})^{-1}\boldsymbol{B}^{-1}$.

30. (a) $(c\boldsymbol{A})(\tfrac{1}{c}\boldsymbol{A}^{-1}) = (c\tfrac{1}{c})\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}$ and $(\tfrac{1}{c}\boldsymbol{A}^{-1})(c\boldsymbol{A}) = (\tfrac{1}{c}c)\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$.

So $c\boldsymbol{A}$ is invertible and $(c\boldsymbol{A})^{-1} = \tfrac{1}{c}\boldsymbol{A}^{-1}$.

(b) $\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$ and $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}$.

So $\boldsymbol{A}^{-1}$ is invertible and $(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}$.

(c) $(\boldsymbol{A}\boldsymbol{B})(\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}) = \boldsymbol{A}\boldsymbol{B}\boldsymbol{B}^{-1}\boldsymbol{A}^{-1} = \boldsymbol{A}\boldsymbol{I}\boldsymbol{A}^{-1} = \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}$ and $(\boldsymbol{B}^{-1}\boldsymbol{A}^{-1})(\boldsymbol{A}\boldsymbol{B}) = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}\boldsymbol{A}\boldsymbol{B} = \boldsymbol{B}^{-1}\boldsymbol{I}\boldsymbol{B} = \boldsymbol{B}^{-1}\boldsymbol{B} = \boldsymbol{I}$.

So $\boldsymbol{A}\boldsymbol{B}$ is invertible and $(\boldsymbol{A}\boldsymbol{B})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$.

31. See Question 3 of Tutorial 3.

32. $\boldsymbol{A} = \begin{pmatrix} 5 & -2 & 6 & 0 \\ -2 & 1 & 3 & 1 \end{pmatrix} \xrightarrow{R_2 + \frac{2}{5}R_1} \begin{pmatrix} 5 & -2 & 6 & 0 \\ 0 & \frac{1}{5} & \frac{27}{5} & 1 \end{pmatrix} \xrightarrow{R_1 + 10R_2} \begin{pmatrix} 5 & 0 & 60 & 10 \\ 0 & \frac{1}{5} & \frac{27}{5} & 1 \end{pmatrix}$

$\xrightarrow{\frac{1}{5}R_1} \begin{pmatrix} 1 & 0 & 12 & 2 \\ 0 & \frac{1}{5} & \frac{27}{5} & 1 \end{pmatrix} \xrightarrow{5R_2} \begin{pmatrix} 1 & 0 & 12 & 2 \\ 0 & 1 & 27 & 5 \end{pmatrix} = \boldsymbol{R}$

So $\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 10 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{2}{5} & 1 \end{pmatrix} \boldsymbol{A} = \begin{pmatrix} 1 & 0 & 12 & 2 \\ 0 & 1 & 27 & 5 \end{pmatrix}$ and hence

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ -\frac{2}{5} & 1 \end{pmatrix} \begin{pmatrix} 1 & -10 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{5} \end{pmatrix} \begin{pmatrix} 1 & 0 & 12 & 2 \\ 0 & 1 & 27 & 5 \end{pmatrix}.$$

33. (a) $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$

(b) $\boldsymbol{A}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$

34. (a) $\boldsymbol{B} \xrightarrow{R_1 - R_3} \xrightarrow{R_1 \leftrightarrow R_3} \xrightarrow{R_3 + 2R_2} \xrightarrow{2R_3} \boldsymbol{A}$

(b) Yes. Since $\boldsymbol{B} = \boldsymbol{E_4}^{-1}\boldsymbol{E_3}^{-1}\boldsymbol{E_2}^{-1}\boldsymbol{E_1}^{-1}\boldsymbol{A}$, if $\boldsymbol{A}$ is invertible, $\boldsymbol{B}$ is invertible.

35. Since $\boldsymbol{E_1}\boldsymbol{E_2}\boldsymbol{A} = \boldsymbol{E_3}\boldsymbol{E_4}\boldsymbol{B}$, we have $\boldsymbol{E_4}^{-1}\boldsymbol{E_3}^{-1}\boldsymbol{E_1}\boldsymbol{E_2}\boldsymbol{A} = \boldsymbol{B}$. Thus $\boldsymbol{B}$ can be obtained from $\boldsymbol{A}$ by the following elementary row operations.

$$\boldsymbol{A} \xrightarrow{R_2 - R_1} \xrightarrow{2R_3} \xrightarrow{R_1 - 2R_3} \xrightarrow{R_1 \leftrightarrow R_4} \boldsymbol{B}$$

36. (a) Since $ac \neq 0$, we have $a \neq 0$ and $c \neq 0$.

$$\boldsymbol{A} \xrightarrow{\frac{1}{a}R_1} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & c \end{pmatrix} \xrightarrow{\frac{1}{c}R_2} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix} \xrightarrow{R_1 - \frac{b}{a}R_2} \boldsymbol{I}_2$$

So $\boldsymbol{A} = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}.$

(b) $\boldsymbol{B} \xrightarrow{R_3+R_1} \xrightarrow{R_3-R_2} \xrightarrow{R_2-3R_3} \xrightarrow{R_1-2R_2} \boldsymbol{I}_3$

So $\boldsymbol{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

37. (a) $\left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right)$

Hence the matrix is invertible and its inverse is $\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$.

(b) $\left(\begin{array}{ccc|ccc} -1 & 3 & -4 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ -4 & 2 & -9 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{ccc|ccc} -1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 10 & -7 & 2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{array}\right)$

Hence the matrix is not invertible.

(c) $\left(\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{6} & -\frac{1}{3} \\ 0 & 1 & 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 0 & \frac{1}{3} & \frac{2}{3} \end{array}\right)$

Hence the matrix is invertible and its inverse is $\begin{pmatrix} \frac{1}{2} & -\frac{1}{6} & -\frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$.

(d) $\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & 3 & 4 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{8} & \frac{1}{8} \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -\frac{1}{4} & \frac{1}{4} \end{array}\right)$

Hence the matrix is invertible and its inverse is $\begin{pmatrix} 1 & 0 & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{8} & \frac{1}{8} \\ 0 & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{4} & \frac{1}{4} \end{pmatrix}$.

(e) $\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 2 & 6 & 3 & 0 & 1 & 0 & 0 \\ 1 & -2 & -6 & -4 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 3 & 7 & 4 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right)$

Hence the matrix is not invertible.

(f) $$\left(\begin{array}{cccc|cccc} 1 & 3 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 5 & 2 & 2 & 0 & 1 & 0 & 0 \\ 1 & 3 & 8 & 9 & 0 & 0 & 1 & 0 \\ 1 & 3 & 2 & 2 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -4 & 3 & 0 & -1 \\ 0 & 1 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -7 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 & 6 & 0 & 1 & -7 \end{array}\right)$$

Hence the matrix is invertible and its inverse is $\begin{pmatrix} -4 & 3 & 0 & -1 \\ 2 & -1 & 0 & 0 \\ -7 & 0 & -1 & 8 \\ 6 & 0 & 1 & -7 \end{pmatrix}$.

38. The inverse of $\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 3 & 2 \end{pmatrix}$ is $\frac{1}{7}\begin{pmatrix} 4 & -1 & -1 \\ -2 & -3 & 4 \\ 1 & 5 & -2 \end{pmatrix}$. So

$$\boldsymbol{X} = \frac{1}{7}\begin{pmatrix} 4 & -1 & -1 \\ -2 & -3 & 4 \\ 1 & 5 & -2 \end{pmatrix}\begin{pmatrix} 2 & 3 & 4 & 1 \\ 1 & 0 & 3 & 7 \\ 2 & 1 & 1 & 2 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 5 & 11 & 12 & -5 \\ 1 & -2 & -13 & -15 \\ 3 & 1 & 17 & 32 \end{pmatrix}.$$

39. (a) Let x_1, x_2, x_3 denote the number of chairs of type A, B, C manufactured respectively. We have the linear system

$$\begin{cases} 4x_1 + 4x_2 + 3x_3 = 260 \\ \phantom{4x_1 + {}} x_2 + 2x_3 = 60 \\ 2x_1 + 4x_2 + 5x_3 = 240, \end{cases}$$

or

$$\begin{pmatrix} 4 & 4 & 3 \\ 0 & 1 & 2 \\ 2 & 4 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 260 \\ 60 \\ 240 \end{pmatrix}.$$

The inverse of the data matrix is $\begin{pmatrix} \frac{3}{2} & 4 & -\frac{5}{2} \\ -2 & -7 & 4 \\ 1 & 4 & -2 \end{pmatrix}$ and hence

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & 4 & -\frac{5}{2} \\ -2 & -7 & 4 \\ 1 & 4 & -2 \end{pmatrix}\begin{pmatrix} 260 \\ 60 \\ 240 \end{pmatrix} = \begin{pmatrix} 30 \\ 20 \\ 20 \end{pmatrix}.$$

That is, 30 chairs of type A, 20 chairs of type B and 20 chairs of type C should be manufactured.

(b) Since $10 \times$ (the (3,1)-entry of the inverse of the data matrix) $= 10$, the number of chairs of type C is increased by 10.

40. $\begin{pmatrix} 1 & 0 & a \\ 0 & a & 1 \\ a & 1 & 0 \end{pmatrix} \xrightarrow{R_3 - aR_1} \xrightarrow{R_2 \leftrightarrow R_3} \xrightarrow{R_3 - aR_2} \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & -a^2 \\ 0 & 0 & 1+a^3 \end{pmatrix}$

The matrix is invertible if and only if $a \neq -1$. The inverse is $\dfrac{1}{1+a^3}\begin{pmatrix} 1 & -a & a^2 \\ -a & a^2 & 1 \\ a^2 & 1 & -a \end{pmatrix}$.

41. (a) $\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix} \xrightarrow[R_3 - a^2R_1]{R_2 - aR_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{pmatrix} \xrightarrow{R_3-(b+a)R_2}$

$\begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & (c-a)(c-b) \end{pmatrix}$

The homogeneous linear system has nontrivial solution if and only if $(b-a)=0$ or $(c-a)(c-b)=0$, i.e. $a=b$ or $a=c$ or $b=c$.

(b) The matrix is invertible if and only if the homogeneous system in (a) has only the trivial solution, i.e. $a \neq b$ and $a \neq c$ and $b \neq c$.

42. Assume $\boldsymbol{AB}$ is invertible. Let $\boldsymbol{C}$ be the inverse of $\boldsymbol{AB}$. Then $(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{I}$ and hence $\boldsymbol{A}(\boldsymbol{BC}) = \boldsymbol{I}$. By Theorem 2.4.12, $\boldsymbol{A}$ is invertible which contradicts that $\boldsymbol{A}$ is singular.

Assume $\boldsymbol{BC}$ is invertible. Let $\boldsymbol{D}$ be the inverse of $\boldsymbol{AB}$. Then $\boldsymbol{D}(\boldsymbol{BC}) = \boldsymbol{I}$ and hence $(\boldsymbol{DB})\boldsymbol{A} = \boldsymbol{I}$. By Theorem 2.4.12, $\boldsymbol{A}$ is invertible which contradicts that $\boldsymbol{A}$ is singular.

43. Suppose $\boldsymbol{A} = \boldsymbol{E_k} \cdots \boldsymbol{E_1} \begin{pmatrix} \boldsymbol{R} \\ 0 \; \cdots \; 0 \end{pmatrix}$ for some elementary matrices $\boldsymbol{E_1}, \ldots, \boldsymbol{E_k}$. Let $\boldsymbol{b} = \boldsymbol{E_k} \cdots \boldsymbol{E_1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$. (This is only an example of many possible choices of $\boldsymbol{b}$.)

Then

$$\boldsymbol{Ax} = \boldsymbol{b} \quad \Leftrightarrow \quad \begin{pmatrix} \boldsymbol{R} \\ 0 \; \cdots \; 0 \end{pmatrix} \boldsymbol{x} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

which is inconsistent, see Remark 1.4.8.1.

44. See Question 6 of Tutorial 3.

45. For $i = 1, 2, \ldots, n$, let $\boldsymbol{E_i}$ be the elementary matrix associated with the row operation $\mathcal{R}_i$ (and the column operation $\mathcal{C}_i$). Since $\boldsymbol{A}$ is reduced to $\boldsymbol{I}$ by the row operations $\mathcal{R}_1$, $\mathcal{R}_2$, ... $\mathcal{R}_n$, we have

$$\boldsymbol{E_n} \cdots \boldsymbol{E_2} \boldsymbol{E_1} \boldsymbol{A} = \boldsymbol{I}.$$

By Theorem 2.4.12, $\boldsymbol{A}$ is invertible and $\boldsymbol{A}^{-1} = \boldsymbol{E_n} \cdots \boldsymbol{E_2} \boldsymbol{E_1}$. So

$$\boldsymbol{A} \boldsymbol{E_n} \cdots \boldsymbol{E_2} \boldsymbol{E_1} = \boldsymbol{I}.$$

Thus $\boldsymbol{A} \xrightarrow{\mathcal{C}_n} \xrightarrow{\mathcal{C}_{n-1}} \cdots \xrightarrow{\mathcal{C}_1} \boldsymbol{I}$.

46. See Question 1 of Tutorial 4.

47. (a) (i) $0 - \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 2$

(ii) $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \xrightarrow{R_3 - R_1} \xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix}$

So $\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{vmatrix} = 2.$

(iii) $\dfrac{1}{2} \begin{pmatrix} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \end{pmatrix}^{\mathrm{T}} = \dfrac{1}{2} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$

(b) (i) $(-1) \begin{vmatrix} 4 & 1 \\ 2 & -9 \end{vmatrix} - 3 \begin{vmatrix} 2 & 1 \\ -4 & -9 \end{vmatrix} + (-4) \begin{vmatrix} 2 & 4 \\ -4 & 2 \end{vmatrix} = 0$

(ii) $\begin{pmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{pmatrix} \xrightarrow{R_2 + 2R_1} \xrightarrow{R_3 - 4R_1} \xrightarrow{R_3 + R_2} \begin{pmatrix} -1 & 3 & -4 \\ 0 & 10 & -7 \\ 0 & 0 & 0 \end{pmatrix}$

So $\begin{vmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{vmatrix} = \begin{vmatrix} -1 & 3 & -4 \\ 0 & 10 & -7 \\ 0 & 0 & 0 \end{vmatrix} = 0.$

(iii) The matrix is not invertible.

(c) (i) $2\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} - 0 + 0 = 6$

(ii) $$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \overset{R_3 + \frac{1}{2}R_2}{\longrightarrow} \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & \frac{3}{2} \end{pmatrix}$$

So $\begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & \frac{3}{2} \end{vmatrix} = 6.$

(iii) $$\frac{1}{6}\begin{pmatrix} \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} & -\begin{vmatrix} 0 & -1 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 0 & -1 \end{vmatrix} \\ -\begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 0 & -1 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} \end{pmatrix}^{\mathrm{T}} = \frac{1}{6}\begin{pmatrix} 3 & -1 & -2 \\ 0 & 4 & 2 \\ 0 & 2 & 4 \end{pmatrix}$$

(d) (i) $$\begin{vmatrix} 2 & 0 & 0 \\ 2 & 3 & 0 \\ 2 & 3 & 4 \end{vmatrix} - 0 + 0 - \begin{vmatrix} 1 & 2 & 0 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{vmatrix} = \left[2\begin{vmatrix} 3 & 0 \\ 3 & 4 \end{vmatrix} - 0 + 0\right] - \left[\begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} + 0\right]$$
$$= 24$$

(ii) $$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 4 \end{pmatrix} \underset{R_4 - R_1}{\overset{\substack{R_2 - R_1 \\ R_3 - R_1}}{\longrightarrow}} \underset{R_4 - R_2}{\overset{R_3 - R_2}{\longrightarrow}} \overset{R_4 - R_3}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

So $\begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{vmatrix} = 24.$

(iii) $$\frac{1}{24}\begin{pmatrix} \begin{vmatrix}2&0&0\\2&3&0\\2&3&4\end{vmatrix} & -\begin{vmatrix}1&0&0\\1&3&0\\1&3&4\end{vmatrix} & \begin{vmatrix}1&2&0\\1&2&0\\1&2&4\end{vmatrix} & -\begin{vmatrix}1&2&0\\1&2&3\\1&2&3\end{vmatrix} \\ -\begin{vmatrix}0&0&1\\2&3&0\\2&3&4\end{vmatrix} & \begin{vmatrix}1&0&1\\1&3&0\\1&3&4\end{vmatrix} & -\begin{vmatrix}1&0&1\\1&2&0\\1&2&4\end{vmatrix} & \begin{vmatrix}1&0&0\\1&2&3\\1&2&3\end{vmatrix} \\ \begin{vmatrix}0&0&1\\2&0&0\\2&3&4\end{vmatrix} & -\begin{vmatrix}1&0&1\\1&0&0\\1&3&4\end{vmatrix} & \begin{vmatrix}1&0&1\\1&2&0\\1&2&4\end{vmatrix} & -\begin{vmatrix}1&0&0\\1&2&0\\1&2&3\end{vmatrix} \\ -\begin{vmatrix}0&0&1\\2&0&0\\2&3&0\end{vmatrix} & \begin{vmatrix}1&0&1\\1&0&0\\1&3&0\end{vmatrix} & -\begin{vmatrix}1&0&1\\1&2&0\\1&2&0\end{vmatrix} & \begin{vmatrix}1&0&0\\1&2&0\\1&2&3\end{vmatrix} \end{pmatrix}^{\mathrm{T}}$$

$$= \frac{1}{24}\begin{pmatrix} 24 & 0 & 6 & -6 \\ -12 & 12 & -3 & 3 \\ 0 & -8 & 8 & 0 \\ 0 & 0 & -6 & 6 \end{pmatrix}$$

48. (a) $x = 1$, $y = -1$.

    (b) $x = \frac{1}{2}$, $y = \frac{1}{2}$, $z = \frac{3}{2}$.

    (c) $x = 1$, $y = 0$, $z = -2$.

    (d) $w = 0$, $x = 0$, $y = 0$, $z = -1$.

49. (a) abc

    (b) $\boldsymbol{A}$ is invertible if and only if $a \neq 0$, $b \neq 0$ and $c \neq 0$.

    $$\boldsymbol{A}^{-1} = \begin{pmatrix} \frac{1}{a} & -\frac{1}{a} & 0 \\ 0 & \frac{1}{b} & -\frac{1}{b} \\ 0 & 0 & \frac{1}{c} \end{pmatrix}.$$

50. (a) $\det(\boldsymbol{C}) = 0$.

    (b) Since $\boldsymbol{C}$ is singular, by Theorem 2.4.14, $\boldsymbol{AC}$ is singular. Thus by Theorem 2.4.7, $\boldsymbol{ACx} = \boldsymbol{0}$ has infinitely many solutions.

51. (a) Since $\det(\boldsymbol{A}) = (\lambda-2)(\lambda+4)+5 = (\lambda+3)(\lambda-1)$, $\det(\boldsymbol{A}) = 0$ if and only if $\lambda = -3$ or 1.

    (b) Since $\det(\boldsymbol{A}) = (\lambda-1)(\lambda^2-\lambda-6) = (\lambda-4)(\lambda-3)(\lambda+2)$, $\det(\boldsymbol{A}) = 0$ if and only if $\lambda = 4$, 3 or -2.

(c) $\begin{pmatrix} 1 & \lambda & \lambda & \lambda \\ 2 & \lambda & \lambda & \lambda \\ \lambda+1 & 1 & 2 & 0 \\ 4 & 0 & 1 & 2\lambda \end{pmatrix} \xrightarrow{R_2 - R_1} \begin{pmatrix} 1 & \lambda & \lambda & \lambda \\ 1 & 0 & 0 & 0 \\ \lambda+1 & 1 & 2 & 0 \\ 4 & 0 & 1 & 2\lambda \end{pmatrix}$

$$\det(\boldsymbol{A}) = \begin{vmatrix} 1 & \lambda & \lambda & \lambda \\ 1 & 0 & 0 & 0 \\ \lambda+1 & 1 & 2 & 0 \\ 4 & 0 & 1 & 2 \end{vmatrix} = -\begin{vmatrix} \lambda & \lambda & \lambda \\ 1 & 2 & 0 \\ 0 & 1 & 2\lambda \end{vmatrix} = 2\lambda^2 + \lambda.$$

Hence $\det(\boldsymbol{A}) = 0$ if and only if $\lambda = 0$ or $-\frac{1}{2}$.

(d) $\begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 2-\lambda^2 & 2 & 3 \\ 2 & 3 & 1 & 5 \\ 2 & 3 & 1 & 9-\lambda^2 \end{pmatrix} \xrightarrow{R_2 - R_1} \xrightarrow{R_4 - R_3} \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1-\lambda^2 & 0 & 0 \\ 2 & 3 & 1 & 5 \\ 0 & 0 & 0 & 4-\lambda^2 \end{pmatrix}$

$$\det(\boldsymbol{A}) = \begin{vmatrix} 1 & 1 & 2 & 3 \\ 0 & 1-\lambda^2 & 0 & 0 \\ 2 & 3 & 1 & 5 \\ 0 & 0 & 0 & 4-\lambda^2 \end{vmatrix} = (1-\lambda^2)\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 5 \\ 0 & 0 & 4-\lambda^2 \end{vmatrix}$$

$$= (1-\lambda^2)(4-\lambda^2)\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3(1-\lambda^2)(4-\lambda^2).$$

Hence $\det(\boldsymbol{A}) = 0$ if and only if $\lambda = \pm 1$ or ± 2.

52. $\begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix} \overset{R_2 - R_1}{\underset{R_3 - R_1}{\longrightarrow}} \begin{pmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{pmatrix}$

So $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = (b-a)(c^2-a^2) - (c-a)(b^2-a^2)$

$$= (b-a)(c-a)(c-b).$$

53. (a) $3^4 \cdot 9 = 729$ (b) $\frac{1}{9}$ (c) $3^4 \cdot \frac{1}{9} = 9$ (d) $\frac{1}{729}$

54. (a) $\boldsymbol{B} \xrightarrow{R_4 + R_2} \xrightarrow{R_2 \leftrightarrow R_3} \xrightarrow{R_1 - R_2} \xrightarrow{3R_2} \xrightarrow{R_3 + 2R_1} \boldsymbol{A}$

(b) $\det(\boldsymbol{A}) = 1 \cdot 2 \cdot 3 \cdot (-1) = -6$ and hence $\det(\boldsymbol{B}) = (-1) \cdot \frac{1}{3} \cdot \det(\boldsymbol{A}) = 2$.

55. (a) $\boldsymbol{B} \xrightarrow{R_1 - 2R_3} \xrightarrow{R_3 - 3R_2} \xrightarrow{\frac{1}{2}R_2} \xrightarrow{R_1 \leftrightarrow R_2} \boldsymbol{A}$

(b) $\det(\boldsymbol{B}) = (-1)\cdot 2\cdot \det(\boldsymbol{A}) = -8$

56. $\det(\boldsymbol{A}) = aei + bfg + cdh - afh - bdi - ceg.$

If all $a, b, c, d, e, f, g, h, i$ are 1, then $\det(\boldsymbol{A}) = 0$.

Suppose at least one of $a, b, c, d, e, f, g, h, i$ is 0, say $a = 0$ (other cases are similar). Then $\det(\boldsymbol{A}) = bfg + cdh - bdi - ceg$. As b, c, d, e, f, g, h, i can only be 0 and 1, $-2 \le \det(\boldsymbol{A}) \le 2$.

Note that $\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 2$ and $\begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = -2$.

The maximum possible value of $\det(\boldsymbol{A})$ is 2 and the minimum is -2.

57. (a) Since $\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{I}$ and $\det(\boldsymbol{A}) = \det(\boldsymbol{A}^{\mathrm{T}})$, we have $\det(\boldsymbol{A})^2 = \det(\boldsymbol{I}) = 1$. Thus $\det(\boldsymbol{A}) = \pm 1$.

(b) Let $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since $\boldsymbol{A}$ is orthogonal, $\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A}^{-1}$, i.e.

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \frac{1}{\det(\boldsymbol{A})}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

So $a = d$ and $b = -c$. Furthermore, $\det(\boldsymbol{A}) = 1$ implies $a^2 + c^2 = ad - bc = 1$. Let $a = \cos(\theta)$ and $c = \sin(\theta)$. Then

$$\boldsymbol{A} = \begin{pmatrix} a & -c \\ c & a \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

(c) Similar to (b) except now $a = -d$ and $b = c$.

58. (a) Let $\boldsymbol{A}$ be a 2×2 matrix with two identical rows, say, $\boldsymbol{A} = \begin{pmatrix} a & b \\ a & b \end{pmatrix}$. Then $\det(\boldsymbol{A}) = ab - ab = 0$.

Assume that the determinant of any $k \times k$ matrices with two identical rows is zero where $k \ge 2$.

Let $\boldsymbol{A}$ be a $(k+1) \times (k+1)$ matrices with two identical rows, say, the ith and jth row of $\boldsymbol{A}$ are identical. Take $m = 1, 2, \ldots, k+1$ such that $m \ne i, j$. Then by Theorem 2.5.6,

$$\det(\boldsymbol{A}) = a_{m1}A_{m1} + a_{m2}A_{m2} + \cdots + a_{i,k+1}A_{m,k+1}$$

where $A_{mr} = (-1)^{m+r}\det(\boldsymbol{M_{mr}})$. Each $\boldsymbol{M_{mr}}$ is a $k \times k$ matrix obtained from $\boldsymbol{A}$ by deleting the mth row and the rth column of $\boldsymbol{A}$. Since the

ith and jth row of $\boldsymbol{A}$ are identical, the corresponding rows of $\boldsymbol{M_{mr}}$ are identical. By the inductive assumption, $\det(M_{mr}) = 0$, i.e. $A_{mr} = 0$, for every r. This means $\det(\boldsymbol{A}) = 0$.

By mathematical induction, the determinant of any square matrix with two identical row is zero.

(b) If $\boldsymbol{A}$ is a square matrix with two identical columns, then $\boldsymbol{A}^{\mathrm{T}}$ has two identical rows. By (a), $\det(\boldsymbol{A}^{\mathrm{T}}) = 0$. So $\det(\boldsymbol{A}) = \det(\boldsymbol{A}^{\mathrm{T}}) = 0$.

59. Since Theorem 2.5.15.3 has been proved, we can use it in the following proofs.

(a) Let $\boldsymbol{A} = (a_{ij})_{n\times n}$. Suppose $\boldsymbol{B}$ is obtained from $\boldsymbol{A}$ by multiplying the mth row of $\boldsymbol{A}$ by k. Observe that for all j, the (m, j)-cofactor of $\boldsymbol{B}$ is the equal to the (m, j)-cofactor of $\boldsymbol{A}$; and the (m, j)-entry of $\boldsymbol{B}$ is ka_{mj}. Thus by Theorem 2.5.6,

$$\begin{aligned}\det(\boldsymbol{B}) &= ka_{i1}A_{i1} + ka_{i2}A_{i2} + \cdots + ka_{i,n}A_{i,n}\\ &= k(a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{i,n}A_{i,n}) = k\det(\boldsymbol{A}).\end{aligned}$$

(b) Suppose $\boldsymbol{B}$ is obtained from $\boldsymbol{A}$ by interchanging the ith and jth rows of $\boldsymbol{A}$. Observe that

$$\boldsymbol{A} \xrightarrow{R_i + R_j} \xrightarrow{R_j - R_i} \xrightarrow{R_i + R_j} \xrightarrow{-R_j} \boldsymbol{B}.$$

By (a) and Theorem 2.5.15.3, $\det(\boldsymbol{B}) = -\det(\boldsymbol{A})$.

(c) (i) Suppose $\boldsymbol{E}$ is the elementary matrix defined in Discussion 2.4.2.1. Note that $\det(\boldsymbol{E}) = k$. Since $\boldsymbol{EA}$ can be obtained from $\boldsymbol{A}$ by multiplying the ith row by k, by (a), $\det(\boldsymbol{EA}) = k\det(\boldsymbol{A}) = \det(\boldsymbol{E})\det(\boldsymbol{A})$.

(ii) Suppose $\boldsymbol{E}$ is the elementary matrix defined in Discussion 2.4.2.2. Note that $\boldsymbol{E}$ can be obtained form $\boldsymbol{I}$ by interchanging the ith and jth rows of $\boldsymbol{I}$. By (b), $\det(\boldsymbol{E}) = -\det(\boldsymbol{I}) = -1$. Since $\boldsymbol{EA}$ can be obtained from $\boldsymbol{A}$ by interchanging the ith and jth rows of $\boldsymbol{A}$, by (b) again, $\det(\boldsymbol{EA}) = -\det(\boldsymbol{A}) = \det(\boldsymbol{E})\det(\boldsymbol{A})$.

(iii) Suppose $\boldsymbol{E}$ is the elementary matrix defined in Discussion 2.4.2.3. Note that $\det(\boldsymbol{E}) = 1$. Since $\boldsymbol{EA}$ can be obtained from $\boldsymbol{A}$ by adding k times of the ith row of $\boldsymbol{A}$ to the jth row, by Theorem 2.5.15.3, $\det(\boldsymbol{EA}) = \det(\boldsymbol{A}) = \det(\boldsymbol{E})\det(\boldsymbol{A})$.

60. See Question 6 of Tutorial 4.

61. See Question 5 of Tutorial 4.