Chapter 1

Linear Systems and Gaussian Elimination

Section 1.1

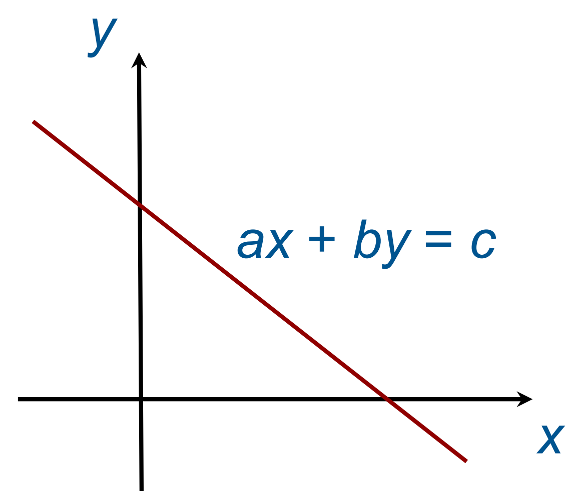
Linear Systems and Their Solutions

Lines in *xy*-plane (Discussion 1.1.1)

A line in the xy-plane can be represented algebraically by an equation of the form

$$ax + by = c$$

where a and b are not both zero.



An equation of this kind is known as a linear equation in the variables of x and y.

Linear equations in n variables (Definition 1.1.2)

A linear equation in n variables $x_1, x_2, \ldots, x_n$ has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n$ and b are real constants.

The variables in a linear equation are also called the unknowns.

If all $a_1, a_2, \ldots, a_n$ and b are zero, the equation is called a zero equation.

A linear equation is called non-zero if it is not a zero equation.

Examples (Example 1.1.3.1-2)

The following are linear equations:

$$x + 3y = 7,$$

$$x_1 + 2x_2 + 2x_3 + x_4 = x_5,$$

$$y = x - \frac{1}{2}z + 4.5,$$

$$x_1 + x_2 + \cdots + x_n = 1.$$

The following are not linear equations:

$$xy = 2,$$

$$\sin(\theta) + \cos(\phi) = 0.2,$$

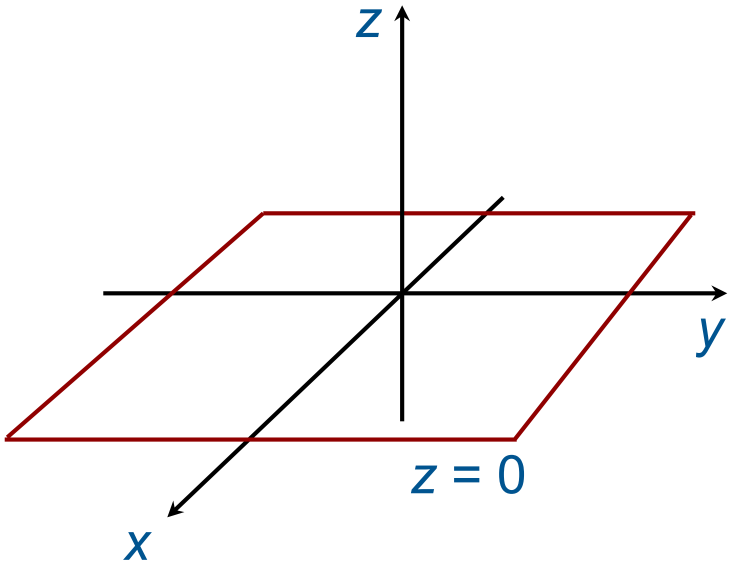
$$x_1^2 + x_2^2 + \cdots + x_n^2 = 1,$$

$$x = e^y.$$

Examples (Example 1.1.3.3)

The linear equation $ax + by + cz = d$, where a, b, c, d are constants and not all a, b, c are zero, represents a plane in the three dimensional space.

For example, $z = 0$ (i.e. $0x + 0y + z = 0$) is the xy-plane contained inside the three dimensional space.



Solutions of a linear equation (Definition 1.1.4)

Given n real numbers $s_1, s_2, \ldots, s_n$, we say that $x_1 = s_1,\ x_2 = s_2,\ \ldots,\ x_n = s_n$ is a solution to a linear equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$

if the equation is satisfied when we substitute the values into the equation accordingly,

i.e. $a_1s_1 + a_2s_2 + \cdots + a_ns_n = b$.

The set of all solutions to the equation is called the solution set of the equation.

An expression that gives us all the solutions to the equation is called the general solution of the equation.

Examples (Example 1.1.5.1)

Consider the linear equation $4x - 2y = 1$.

The general solution is

$$\begin{cases} x = t \\ y = 2t - \frac{1}{2} \end{cases}$$ where t is an arbitrary parameter.

We can also write the general solution as

$$\begin{cases} x = \frac{1}{2}s + \frac{1}{4} \\ y = s \end{cases}$$ where s is an arbitrary parameter.

These include solutions such as

$$\begin{cases} x = 1 \\ y = 1.5, \end{cases} \quad \begin{cases} x = 1.5 \\ y = 2.5, \end{cases} \quad \begin{cases} x = -1 \\ y = -2.5, \end{cases}$$

and infinitely many solutions.

Examples (Example 1.1.5.2)

Consider the linear equation $x_1 - 4x_2 + 7x_3 = 5$.

The general solution is

$$\begin{cases} x_1 = 5 + 4s - 7t \\ x_2 = s \\ x_3 = t \end{cases}$$

where s and t are arbitrary parameters.

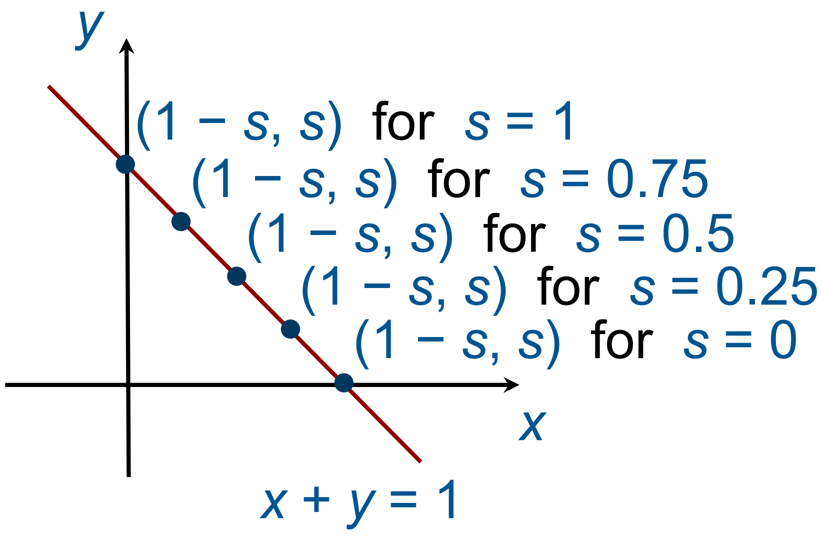
Geometrical interpretation (Example 1.1.5.3 (a))

In the xy-plane, the equation $x + y = 1$ represents the line shown below.

The solutions of the equation are points

$$(x, y) = (1 - s, s)$$

where s is any real number.



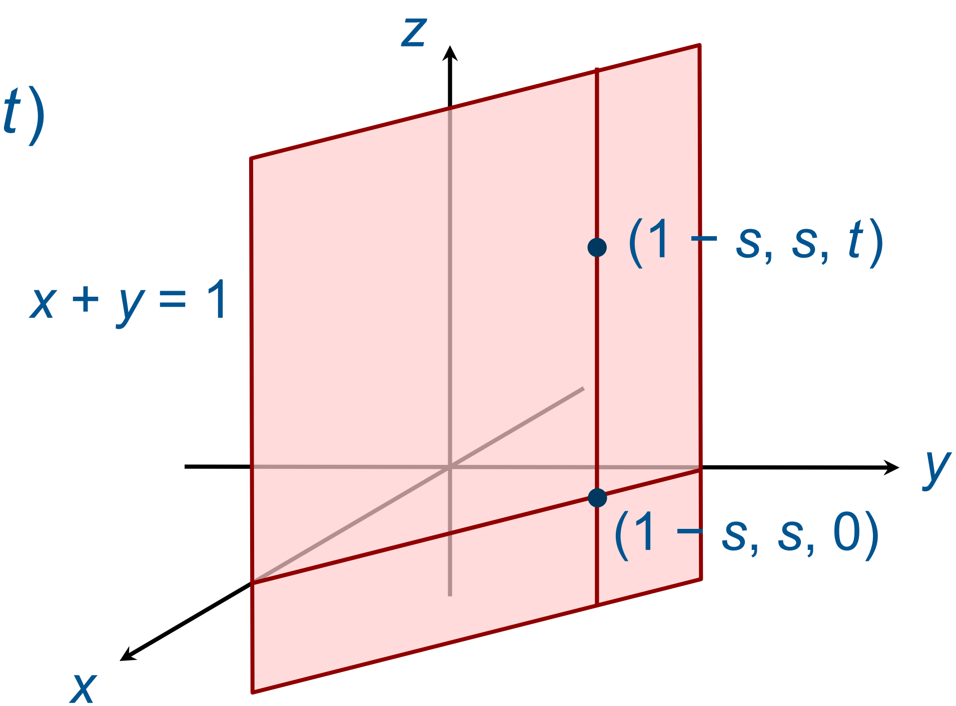
Geometrical interpretation (Example 1.1.5.3 (b))

In the xyz-space, the equation $x + y = 1$ (i.e. $x + y + 0z = 1$) represents the plane shown below.

The solutions of the equation are points

$$(x, y, z) = (1 - s, s, t)$$

where s and t are any real numbers.



Examples (Example 1.1.5.4-5)

Consider the zero linear equation

$$0x_1 + 0x_2 + \cdots + 0x_n = 0.$$

Any values of $x_1, x_2, \ldots, x_n$ give us a solution.

Thus the general solution is $x_1 = t_1,\ x_2 = t_2,\ \ldots,\ x_n = t_n$ where $t_1, t_2, \ldots, t_n$ are arbitrary parameters.

Consider the linear equation

$$0x_1 + 0x_2 + \cdots + 0x_n = b$$

where b is nonzero.

Any values of $x_1, x_2, \ldots, x_n$ do not satisfy the equation.

Thus there is no solution.

Systems of linear equations (Definition 1.1.6)

A finite set of linear equations in the variables $x_1, x_2, \ldots, x_n$ is called a system of linear equations (or a linear system):

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \qquad\quad \vdots \qquad\qquad\qquad\qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

where $a_{11}, a_{12}, \ldots, a_{mn}$ and $b_1, b_2, \ldots, b_m$ are real constants.

Solutions of a linear system (Definition 1.1.6)

Given n real numbers $s_1, s_2, \ldots, s_n$, we say that $x_1 = s_1, \; x_2 = s_2, \; \ldots, \; x_n = s_n$ is a solution to the system if $x_1 = s_1, \; x_2 = s_2, \; \ldots, \; x_n = s_n$ is a solution to every equation in the system.

The set of all solutions to the system is called the solution set of the system.

An expression that gives us all the solutions to the system is called the general solution of the system.

Examples (Example 1.1.7 & Remark 1.1.8)

Consider the linear system $\begin{cases} 4x_1 - x_2 + 3x_3 = -1 \\ 3x_1 + x_2 + 9x_3 = -4. \end{cases}$

$x_1 = 1,\ x_2 = 2,\ x_3 = -1$
is a solution of the system.

$$\boxed{\begin{cases} 4\cdot 1 - 2 + 3\cdot(-1) = -1 \\ 3\cdot 1 + 2 + 9\cdot(-1) = -4. \end{cases}}$$

$x_1 = 1,\ x_2 = 8,\ x_3 = 1$
is not a solution of the system.

$$\boxed{\begin{cases} 4\cdot 1 - 8 + 3\cdot 1 = -1 \\ 3\cdot 1 + 8 + 9\cdot 1 = 20 \neq -4. \end{cases}}$$

Not all systems of linear equations have solutions.

For example, the system $\begin{cases} x + y = 4 \\ 2x + 2y = 6 \end{cases}$ has no solution.

Solutions of linear systems (Definition 1.1.9 & Remark 1.1.10)

A system of linear equations that has no solution is said to be inconsistent.

A system that has at least one solution is called consistent.

Every system of linear equations has either

(i) no solution, (inconsistent)

(ii) exactly one solution or

(iii) infinitely many solutions.

(consistent)

Geometrical interpretation (Discussion 1.1.11.1)

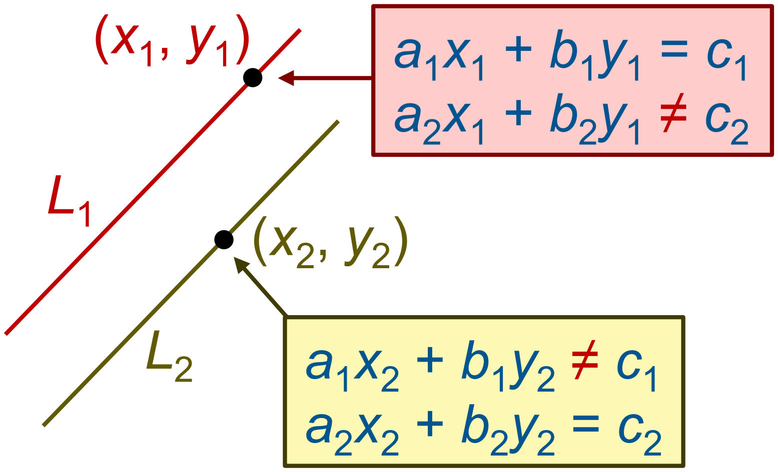
In the xy-plane, the two equations in the system

$$\begin{cases} a_1x + b_1y = c_1 & (L_1) \\ a_2x + b_2y = c_2, & (L_2) \end{cases}$$

where a_1, b_1 are not both zero and a_2, b_2 are not both zero, represents two straight lines.

A solution to the system is a point of intersection of the two lines.

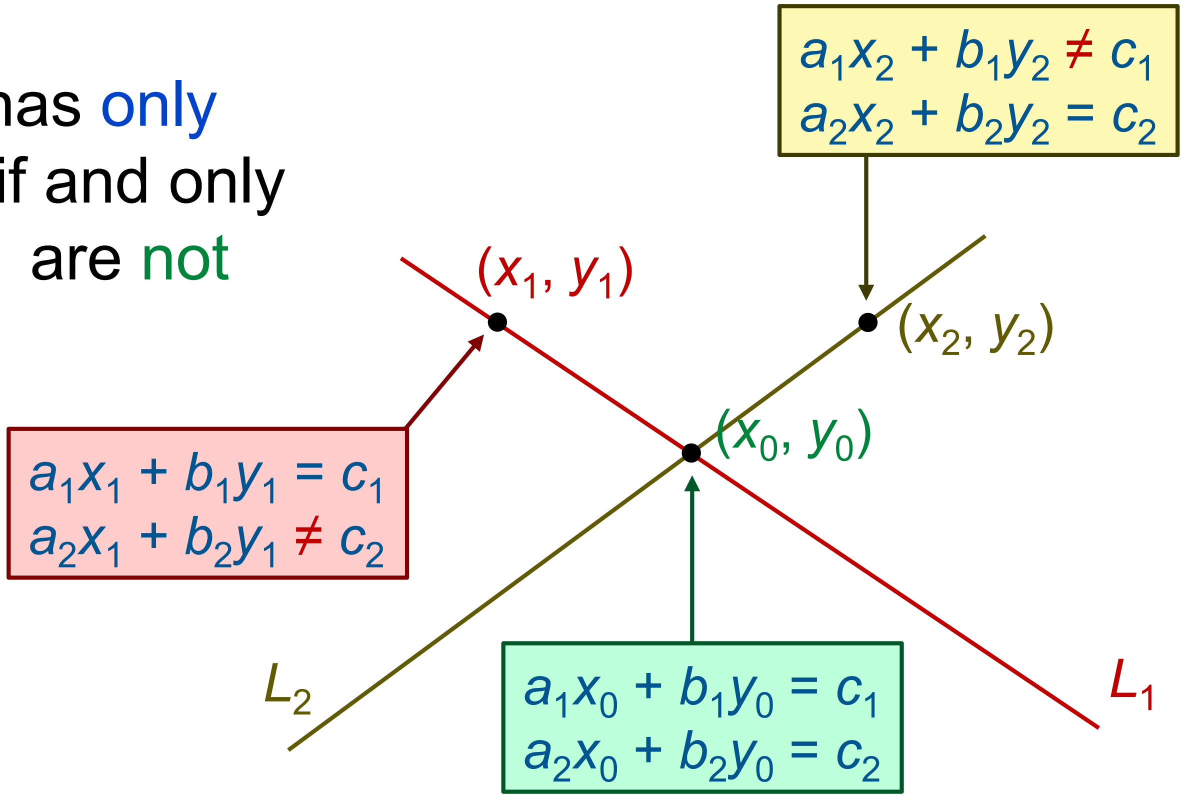
(a) The system has no solution if and only if L_1 and L_2 are different but parallel lines.



Geometrical interpretation (Discussion 1.1.11.1)

$$\begin{cases} a_1x + b_1y = c_1 & (L_1) \\ a_2x + b_2y = c_2, & (L_2) \end{cases}$$

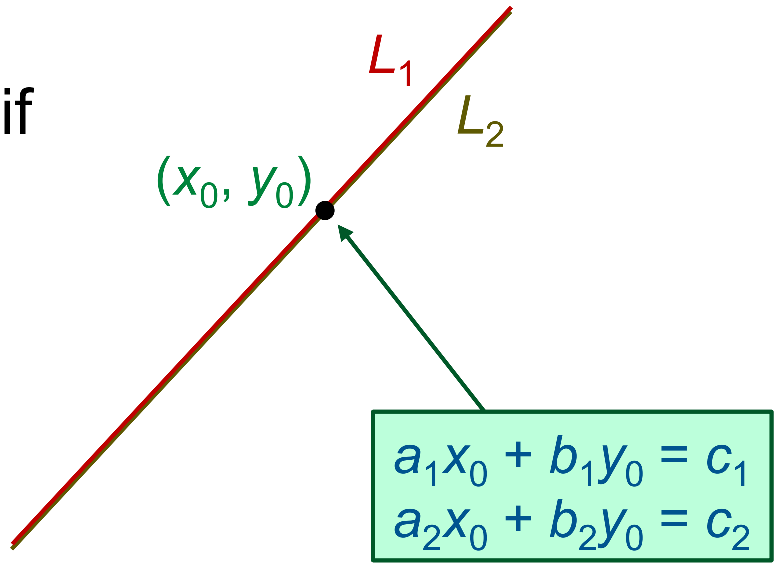
(b) The system has only one solution if and only if L_1 and L_2 are not parallel lines.



Geometrical interpretation (Discussion 1.1.11.1)

$$\begin{cases} a_1x + b_1y = c_1 & (L_1) \\ a_2x + b_2y = c_2, & (L_2) \end{cases}$$

(c) The system has infinitely many solutions if and only if L_1 and L_2 are the same line.



Geometrical interpretation (Discussion 1.1.11.2)

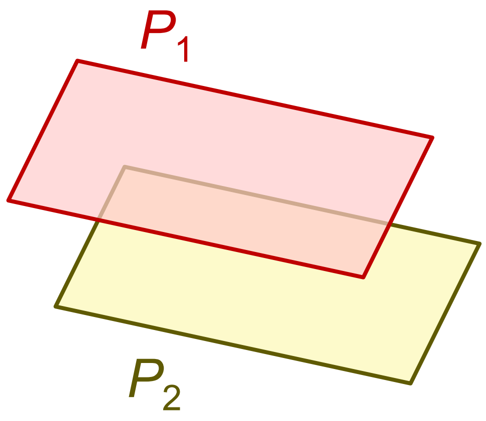
In the xyz-space, the two equations in the system

$$\begin{cases} a_1x + b_1y + c_1z = d_1 & (P_1) \\ a_2x + b_2y + c_2z = d_2, & (P_2) \end{cases}$$

where a_1, b_1, c_1 are not all zero and a_2, b_2, c_2 are not all zero, represents two planes.

A solution to the system is a point of intersection of the two planes.

(a) The system has no solution if and only if P_1 and P_2 are different but parallel planes.

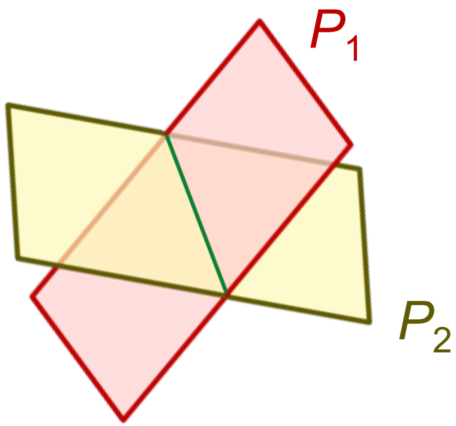


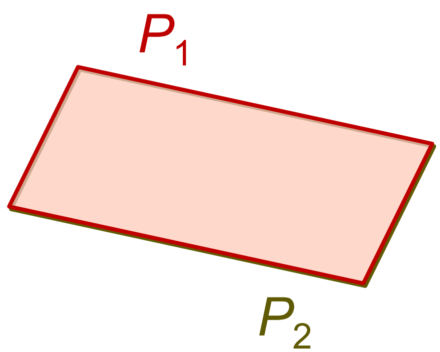
Geometrical interpretation (Discussion 1.1.11.2)

$$\begin{cases} a_1x + b_1y + c_1z = d_1 & (P_1) \\ a_2x + b_2y + c_2z = d_2, & (P_2) \end{cases}$$

(b) The system cannot have only one solution.

(c) The system has infinitely many solutions if and only if either P_1 and P_2 intersect at a line or P_1 and P_2 are the same planes.





An exercise (Question 1.8)

In the xyz-space, the three equations in the system

$$\begin{cases} a_1x + b_1y + c_1z = d_1 & (P_1) \\ a_2x + b_2y + c_2z = d_2 & (P_2) \\ a_3x + b_3y + c_3z = d_3, & (P_3) \end{cases}$$

where for each i, a_i, b_i, c_i are not all zero, represents three planes.

Discuss the relative positions of the three planes when the linear system

(a) has no solution;

(b) has only one solution;

(c) has infinitely many solution.

Section 1.2

Elementary Row Operations

Augmented matrices (Definition 1.2.1)

A system of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad\vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

can be represented by a rectangular array of numbers

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right].$$

This array is called the augmented matrix of the system.

An example (Example 1.2.2)

Consider the linear system

$$\begin{cases} x_1 + x_2 + 2x_3 = 9 \\ 2x_1 + 4x_2 - 3x_3 = 1 \\ 3x_1 + 6x_2 - 5x_3 = 0. \end{cases}$$

Its augment matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right].$$

Elementary row operations (Discussion 1.2.3 & Definition 1.2.4)

The following are the basic techniques for solving a system of linear equations:

1. Multiply an equation by a nonzero constant.
2. Interchange two equations.
3. Add a multiple of one equation to another equation.

In terms of augmented matrix, these correspond to:

1. Multiply a row by a nonzero constant.
2. Interchange two rows.
3. Add a multiple of one row to another row.

These three operations of the augmented matrix are known as elementary row operations.

An example (Example 1.2.5)

Consider the linear system and its augment matrix:

$$\begin{cases} x + y + 3z = 0 & (1) \\ 2x - 2y + 2z = 4 & (2) \\ 3x + 9y = 3 & (3) \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & -2 & 2 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right]$$

Add -2 times of Equation (1) to Equation (2).

Add -2 times of the first row to the second row.

$$\begin{cases} x + y + 3z = 0 & (1) \\ - 4y - 4z = 4 & (4) \\ 3x + 9y = 3 & (3) \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right]$$

An example (Example 1.2.5)

$$\left\{\begin{array}{rrrrl} x & + \; y & + \; 3z & = 0 & \quad (1) \\ & -\; 4y & -\; 4z & = 4 & \quad (4) \\ 3x & + \; 9y & & = 3 & \quad (3) \end{array}\right. \qquad \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right]$$

$\downarrow$ Add -3 times of Equation (1) to Equation (3).

$\downarrow$ Add -3 times of the first row to the third row.

$$\left\{\begin{array}{rrrrl} x & + \; y & + \; 3z & = 0 & \quad (1) \\ & -\; 4y & -\; 4z & = 4 & \quad (4) \\ & 6y & -\; 9z & = 3 & \quad (5) \end{array}\right. \qquad \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 6 & -9 & 3 \end{array}\right]$$

An example (Example 1.2.5)

$$\left\{\begin{array}{rcrcrcl} x & + & y & + & 3z & = & 0 \quad (1)\\ & - & 4y & - & 4z & = & 4 \quad (4)\\ & & 6y & - & 9z & = & 3 \quad (5)\end{array}\right.$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0\\ 0 & -4 & -4 & 4\\ 0 & 6 & -9 & 3\end{array}\right]$$

Add 6/4 times of Equation (4) to Equation (5).

Add 6/4 times of the second row to the third row.

$$\left\{\begin{array}{rcrcrcl} x & + & y & + & 3z & = & 0 \quad (1)\\ & - & 4y & - & 4z & = & 4 \quad (4)\\ & & & & -15z & = & 9 \quad (6)\end{array}\right.$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0\\ 0 & -4 & -4 & 4\\ 0 & 0 & -15 & 9\end{array}\right]$$

An example (Example 1.2.5)

$$\left\{\begin{array}{rcrcrcl} x & + & y & + & 3z & = & 0 \quad (1) \\ & - & 4y & - & 4z & = & 4 \quad (4) \\ & & & & -15z & = & 9 \quad (6) \end{array}\right. \qquad \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 0 & -15 & 9 \end{array}\right]$$

By Equation (6), $z = -\dfrac{9}{15} = -\dfrac{3}{5}$.

Substituting $z = -\dfrac{3}{5}$ into Equation (4),

$$-4y - 4\left(-\frac{3}{5}\right) = 4 \text{ which gives us } y = -\frac{2}{5}.$$

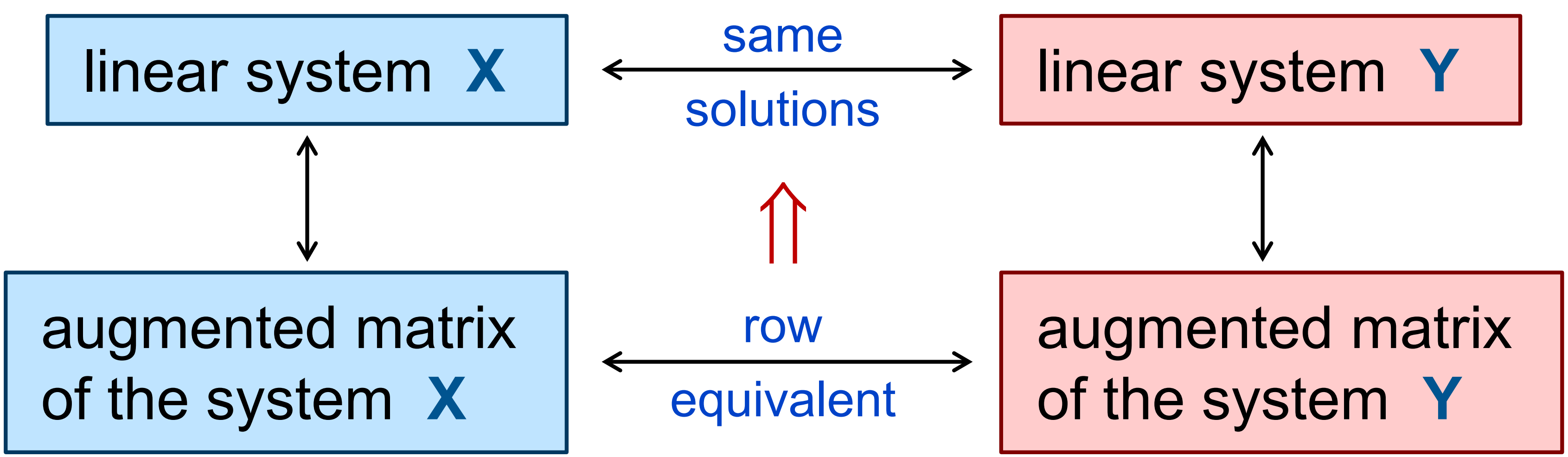
Substituting $y = -\dfrac{2}{5}$ and $z = -\dfrac{3}{5}$ into Equation (1),

$$x + \left(-\frac{2}{5}\right) + 3\left(-\frac{3}{5}\right) = 0 \text{ which gives us } x = \frac{11}{5}.$$

Row equivalent matrices (Definition 1.2.6 & Theorem 1.2.7)

Two augmented matrices are said to be row equivalent if one can be obtained from the other by a series of elementary row operations.

If augmented matrices of two systems of linear equations are row equivalent, then the two systems have the same set of solutions. (See Remark 2.4.6 for a proof.)



An example (Example 1.2.8)

Consider the linear systems in Example 1.2.5:

$$\left\{\begin{array}{rrrl} x & + \;\; y & + 3z & = 0 \\ 2x & - 2y & + 2z & = 4 \\ 3x & + 9y & & = 3 \end{array}\right. \quad \xleftrightarrow[\left\{\begin{array}{l} x = 11/5 \\ y = -2/5 \\ z = -3/5 \end{array}\right.]{\text{same solution}} \quad \left\{\begin{array}{rrrl} x & + \;\; y & + \;\; 3z & = 0 \\ & - 4y & - \;\; 4z & = 4 \\ & & -15z & = 9 \end{array}\right.$$

$$\updownarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad \updownarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & -2 & 2 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right] \quad \xleftrightarrow[\text{equivalent}]{\text{row}} \quad \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 0 & -15 & 9 \end{array}\right]$$

Section 1.3

Row-Echelon Forms

Row-echelon forms (Definition 1.3.1)

An augmented matrix is said to be in row-echelon form if it has the following Properties 1 and 2:

1. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.

$$\begin{bmatrix} * & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & * \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} * \\ \vdots \\ * \end{matrix}}\right\} \text{non-zero rows} \\ \left.\vphantom{\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}}\right\} \text{zero rows} \end{matrix}$$

Row-echelon forms (Definition 1.3.1)

2. In any two successive rows that do not consist entirely of zeros, the first nonzero number in the lower row occurs farther to the right than the first nonzero number in the higher row.

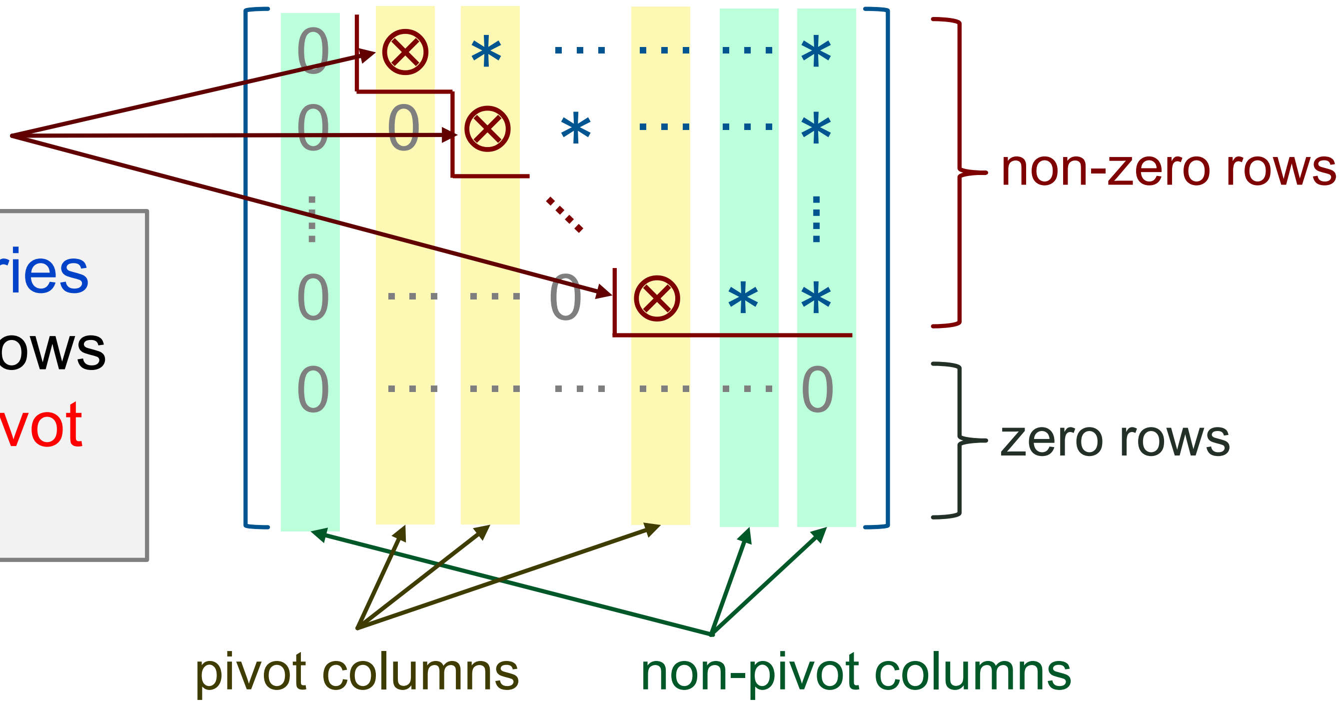
$$\begin{bmatrix} \\ 0 \cdots 0 \; \otimes \; * \; \cdots\cdots \; * \\ 0 \; \cdots\cdots \; 0 \; \otimes \; * \; \cdots \; * \\ \\ \end{bmatrix} \quad \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{two successive rows}$$

leading entries

The first nonzero number in a row is called the leading entry of the row.

Row-echelon forms (Definition 1.3.1)

The following is the general form of a row-echelon form:



Leading entries of nonzero rows are called pivot points.

A column contains a pivot point is called a pivot column. Otherwise, it is called a non-pivot column.

Reduced row-echelon forms (Definition 1.3.1)

An augmented matrix is said to be in reduced row-echelon form if it is in row-echelon form and has the following Properties 3 and 4:

3. The leading entry of every nonzero row is 1.

$$\begin{bmatrix} 0 & 1 & * & \cdots & \cdots & \cdots & * \\ 0 & 0 & 1 & * & \cdots & \cdots & * \\ \vdots & & & \ddots & & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & * & * \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & & & & & & \end{bmatrix}$$

non-zero rows

zero rows

(Remark 1.3.2)

In some textbook, row-echelon forms are required to have Properties 1, 2 and 3.

Reduced row-echelon forms (Definition 1.3.1)

4. In each pivot column, except the pivot point, all other entries are zero.

$$
\begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & \cdots & * \\
0 & 0 & 1 & * & 0 & \cdots & * \\
\vdots & & & \ddots & \vdots & & \vdots \\
0 & \cdots & \cdots & 0 & 1 & * & * \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0
\end{bmatrix}
\begin{array}{l}
\left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} \text{non-zero rows} \\
\left.\begin{array}{l} \\ \\ \end{array}\right\} \text{zero rows}
\end{array}
$$

pivot columns

Examples (Example 1.3.3.1)

The following augmented matrices are in reduced row-echelon form:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \end{array}\right] \qquad \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{cccc|c} 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The pivot points are marked by ◯.

Examples (Example 1.3.3.2)

The following augmented matrices are in row-echelon form but not in reduced row-echelon form:

$$\left[\begin{array}{cc|c} 3 & 2 & 1 \end{array}\right] \qquad \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} -1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 3 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{cccc|c} 0 & 1 & 2 & 8 & 1 \\ 0 & 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The pivot points are marked by $\bigcirc$.

Examples (Example 1.3.3.3)

The following augmented matrices are not in row-echelon form:

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \end{array}\right] \qquad \left[\begin{array}{cc|c} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Examples (Discussion 1.3.4 & Example 1.3.5.1)

If the augmented matrix of a system of linear equations is in row-echelon form or reduced row-echelon form, we can get the solutions to the system easily.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \longleftrightarrow \begin{cases} x_1 = 1 \\ x_2 = 2 \\ x_3 = 3 \end{cases}$$

The system has only one solution:

$$\begin{cases} x_1 = 1 \\ x_2 = 2 \\ x_3 = 3 \end{cases}$$

Examples (Example 1.3.5.2)

$$\left[\begin{array}{ccccc|c} 0 & 2 & 2 & 1 & -2 & 2 \\ 0 & 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{array}\right] \longleftrightarrow \left\{\begin{array}{r} 2x_2 + 2x_3 + x_4 - 2x_5 = 2 \\ x_3 + x_4 + x_5 = 3 \\ 2x_5 = 4 \end{array}\right.$$

The coefficients of x_1 are zero in all the three equations and this means that x_1 is arbitrary.

By the 3rd equation, $x_5 = 2$.

Substituting $x_5 = 2$ into the 2nd equation, $x_3 + x_4 + 2 = 3$ which gives us $x_3 = 1 - x_4$.

Substituting $x_3 = 1 - x_4$ and $x_5 = 2$ into the 1st equation, $2x_2 + 2(1 - x_4) + x_4 - 2 \cdot 2 = 2$ which gives us $x_2 = 2 + \frac{1}{2}x_4$.

The method we used here is called the back substitution.

Examples (Example 1.3.5.2)

$$\left[\begin{array}{ccccc|c} 0 & 2 & 2 & 1 & -2 & 2 \\ 0 & 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{array}\right] \longleftrightarrow \left\{\begin{aligned} 2x_2 + 2x_3 + x_4 - 2x_5 &= 2 \\ x_3 + x_4 + x_5 &= 3 \\ 2x_5 &= 4 \end{aligned}\right.$$

The system has infinitely many solutions:

$$\left\{\begin{aligned} x_1 &= s \\ x_2 &= 2 + \frac{1}{2}t \\ x_3 &= 1 - t \\ x_4 &= t \\ x_5 &= 2 \end{aligned}\right.$$

where s and t are arbitrary parameters.

Examples (Example 1.3.5.3)

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 3 & -2 \\ 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longleftrightarrow \begin{cases} x_1 = -2 - (-x_2) - 3x_4 \\ x_3 = 5 - 2x_4 \end{cases}$$

The system has infinitely many solutions:

$$\begin{cases} x_1 = -2 + s - 3t \\ x_2 = s \\ x_3 = 5 - 2t \\ x_4 = t \end{cases}$$

where s and t are arbitrary parameters.

Examples (Example 1.3.5.4)

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longleftrightarrow \begin{cases} 0x_1 + 0x_2 + 0x_3 = 0 \\ 0x_1 + 0x_2 + 0x_3 = 0 \end{cases}$$

The system has infinitely many solutions:

$$\begin{cases} x_1 = r \\ x_2 = s \\ x_3 = t \end{cases}$$

where r, s and t are arbitrary parameters.

Examples (Example 1.3.5.5)

$$\left[\begin{array}{cc|c} 3 & 1 & 4 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{array}\right] \longleftrightarrow \begin{cases} 3x_1 + x_2 = 4 \\ 0x_1 + 2x_2 = 1 \\ 0x_1 + 0x_2 = 1 \end{cases}$$

The system has a no solution.

Section 1.4

Gaussian Elimination

Friedrich Carl Gauss (1777-1855)

Friedrich Carl Gauss, a German scientist known as the Prince of Mathematicians, is generally regarded as one of the greatest mathematicians of all time for his contributions to number theory, geometry, probability theory, geodesy, planetary astronomy, the theory of functions, and potential theory.

Row echelon forms (Definition 1.4.1)

Let ***A*** and ***R*** be row-equivalent augmented matrices i.e. ***R*** can be obtained from ***A*** by a series of elementary row operations.

If ***R*** is in row-echelon form, then
R is called a row-echelon form of ***A***;
and ***A*** is said to have a row-echelon form ***R***.

If ***R*** is in reduced row-echelon form, then
R is called a reduced row-echelon form of ***A***;
and ***A*** is said to have a reduced row-echelon form ***R***.

Gaussian Elimination (Algorithm 1.4.2)

Gaussian Elimination is an algorithm that reduces an augmented matrix to a row-echelon form by using elementary operations.

Step 1: Locate the leftmost column that does not consist entirely of zeros.

$$\begin{bmatrix} 0 & 3 & \cdots & \cdots \\ 1 & -2 & \cdots & \cdots \\ 4 & 0 & \cdots & \cdots \end{bmatrix} \qquad \begin{bmatrix} 0 & 3 & 9 & \cdots \\ 0 & 2 & -3 & \cdots \\ 0 & -1 & 6 & \cdots \end{bmatrix}$$

the leftmost nonzero column

the leftmost nonzero column

Gaussian Elimination (Algorithm 1.4.2)

Step 2: Interchange the top row with another row, if necessary, to bring a nonzero entry to the top of the column found in Step 1.

$$\begin{bmatrix} 1 & -2 & \cdots & \cdots \\ 0 & 3 & \cdots & \cdots \\ 4 & 0 & \cdots & \cdots \end{bmatrix} \qquad \begin{bmatrix} 0 & 3 & 9 & \cdots \\ 0 & 2 & -3 & \cdots \\ 0 & -1 & 6 & \cdots \end{bmatrix}$$

Interchange the 1st row and the 2nd row.

No action is needed.

Gaussian Elimination (Algorithm 1.4.2)

Step 3: For each row below the top row, add a suitable multiple of the top row to it so that the entry below the leading entry of the top row becomes zero.

$$\begin{bmatrix} 1 & -2 & \cdots & \cdots \\ 0 & 3 & \cdots & \cdots \\ 0 & 8 & \cdots & \cdots \end{bmatrix} \qquad \begin{bmatrix} 0 & 3 & 9 & \cdots \\ 0 & 0 & -9 & \cdots \\ 0 & 0 & 9 & \cdots \end{bmatrix}$$

Add -4 times of the 1st row to the 3rd row so that the entry 4 becomes 0.

Add $-2/3$ times of the 1st row to the 2nd row so that the entry 2 becomes 0.

Add $1/3$ times of the 1st row to the 3rd row so that the entry -1 becomes 0.

Gaussian Elimination (Algorithm 1.4.2)

Step 4: Now cover the top row in the matrix.

$$\begin{bmatrix} 1 & -2 & \cdots & \cdots \\ 0 & 3 & \cdots & \cdots \\ 0 & 8 & \cdots & \cdots \end{bmatrix} \qquad \begin{bmatrix} 0 & 3 & 9 & \cdots \\ 0 & 0 & -9 & \cdots \\ 0 & 0 & 9 & \cdots \end{bmatrix}$$

Begin again with Step 1 applied to the submatrix that remains.

Continue in this way until the entire matrix is in row-echelon form.

An example (Example 1.4.4.1)

$$\left[\begin{array}{ccccc|c} 0 & 0 & 2 & 4 & 2 & 8 \\ 1 & 2 & 4 & 5 & 3 & -9 \\ -2 & -4 & -5 & -4 & 3 & 6 \end{array}\right]$$

Step 1: The 1st column is the leftmost nonzero column.

Step 2: Interchange the 1st and 2nd rows.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 4 & 5 & 3 & -9 \\ 0 & 0 & 2 & 4 & 2 & 8 \\ -2 & -4 & -5 & -4 & 3 & 6 \end{array}\right]$$

Step 3: Add 2 times of the 1st row to the 3rd row.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 4 & 5 & 3 & -9 \\ 0 & 0 & 2 & 4 & 2 & 8 \\ 0 & 0 & 3 & 6 & 9 & -12 \end{array}\right]$$

An example (Example 1.4.4.1)

$$\left[\begin{array}{ccccc|c} 1 & 2 & 4 & 5 & 3 & -9 \\ 0 & 0 & 2 & 4 & 2 & 8 \\ 0 & 0 & 3 & 6 & 9 & -12 \end{array}\right]$$

Step 4: Cover the 1st row and begin again with Step 1.

Step 1: The 3rd column is the leftmost nonzero column.

Step 2: No action is needed.

Step 3: Add $-3/2$ times of the 2nd row to the 3rd row.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 4 & 5 & 3 & -9 \\ 0 & 0 & 2 & 4 & 2 & 8 \\ 0 & 0 & 0 & 0 & 6 & -24 \end{array}\right]$$

The augmented matrix is already in row-echelon form:

$$\left[\begin{array}{ccccc|c} 1 & 2 & 4 & 5 & 3 & -9 \\ 0 & 0 & 2 & 4 & 2 & 8 \\ 0 & 0 & 0 & 0 & 6 & -24 \end{array}\right]$$

Gauss-Jordan Elimination (Algorithm 1.4.3)

Gauss-Jordan Elimination is an algorithm that reduces an augmented matrix to the reduced row-echelon form by using elementary operations.

First, use the Gaussian Elimination (Algorithm 1.4.2) to reduce the augmented matrix to a row-echelon form.

$$\begin{bmatrix} 0 & 3 & \cdots & \cdots \\ 1 & -2 & \cdots & \cdots \\ 4 & 0 & \cdots & \cdots \end{bmatrix} \xrightarrow{\text{Gaussian Elimination}} \begin{bmatrix} 1 & -2 & \cdots & \cdots \\ 0 & 3 & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots \end{bmatrix}$$

$$\begin{bmatrix} 0 & 3 & 9 & \cdots \\ 0 & 2 & -3 & \cdots \\ 0 & -1 & 6 & \cdots \end{bmatrix} \xrightarrow{\text{Gaussian Elimination}} \begin{bmatrix} 0 & 3 & 9 & \cdots \\ 0 & 0 & -9 & \cdots \\ 0 & 0 & 0 & \cdots \end{bmatrix}$$

Gauss-Jordan Elimination (Algorithm 1.4.3)

Step 5: Multiply a suitable constant to each row so that all the leading entries becomes 1.

$$\begin{bmatrix} 1 & -2 & \cdots & \cdots \\ 0 & 1 & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 3 & \cdots \\ 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & \cdots \end{bmatrix}$$

No action is needed for the 1st row.

Multiply the 2nd row by $1/3$ so that the entry 3 become 1.

Multiply the 1st row by $1/3$ so that the entry 3 become 1.

Multiply the 2nd row by $-1/9$ so that the entry -9 become 1.

Gauss-Jordan Elimination (Algorithm 1.4.3)

Step 6: Beginning with the last pivot column and working backward, add a suitable multiples of each row to the rows above to introduce zeros above each pivot point.

$$\begin{bmatrix} 0 & 1 & 4 & \cdots & \cdots & -4 & \cdots & \cdots \\ 0 & 0 & 1 & \cdots & \cdots & 3 & \cdots & \cdots \\ \vdots & & & \ddots & \vdots & \vdots & & \vdots \\ \vdots & & & & 1 & 2 & & \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

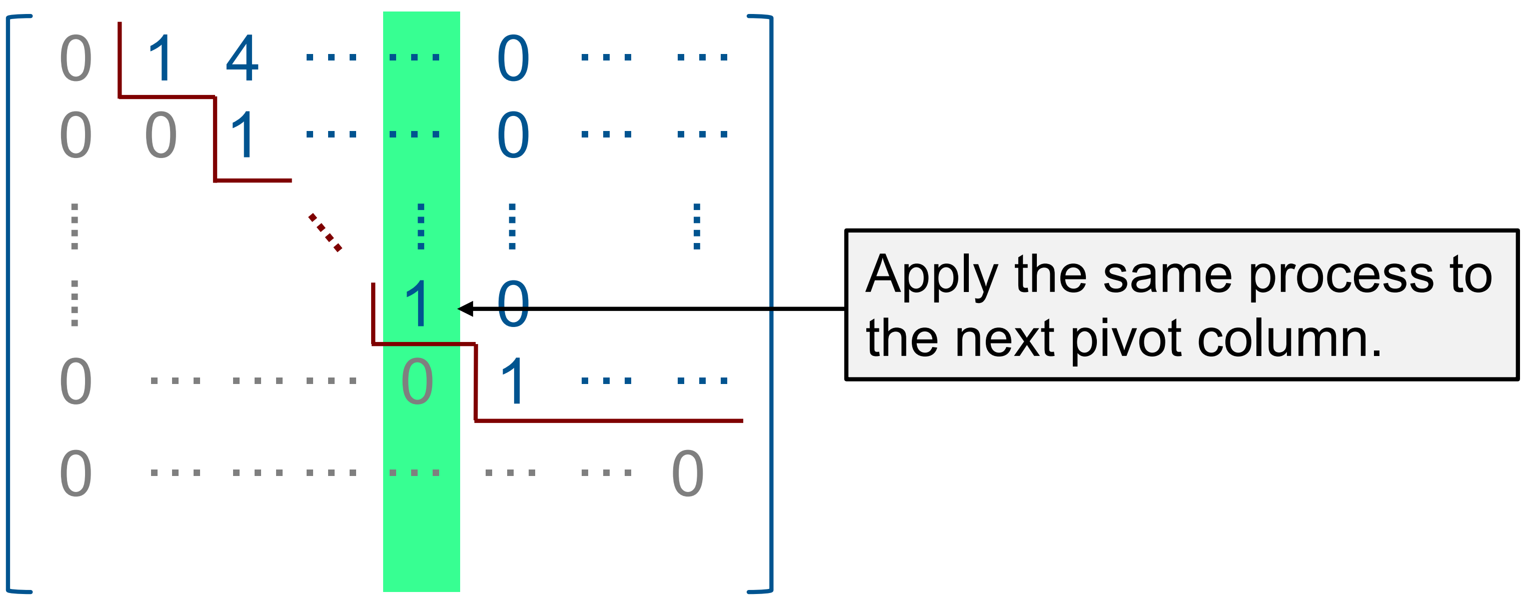
Add 4 times of the last nonzero row to this row so that the entry -4 becomes 0.

Add -3 times of the last nonzero row to this row so that the entry 3 becomes 0.

Add -2 times of the last nonzero row to this row so that the entry 2 becomes 0.

Gauss-Jordan Elimination (Algorithm 1.4.3)

Step 6: Beginning with the last pivot column and working backward, add a suitable multiples of each row to the rows above to introduce zeros above each pivot point.



An example (Example 1.4.4.2)

$$\left[\begin{array}{ccccc|c} 0 & 0 & 2 & 4 & 2 & 8 \\ 1 & 2 & 4 & 5 & 3 & -9 \\ -2 & -4 & -5 & -4 & 3 & 6 \end{array}\right] \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left[\begin{array}{ccccc|c} 1 & 2 & 4 & 5 & 3 & -9 \\ 0 & 0 & 2 & 4 & 2 & 8 \\ 0 & 0 & 0 & 0 & 6 & -24 \end{array}\right]$$

Step 5: No action is needed for the 1st row.

Multiply the 2nd row by $1/2$.

Multiply the 3rd row by $1/6$.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 4 & 5 & 3 & -9 \\ 0 & 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{array}\right]$$

Step 6: Add -3 times of the 3rd row to the 1st row.

Add -1 times of the 3rd row to the 2nd row.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 4 & 5 & 0 & 3 \\ 0 & 0 & 1 & 2 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{array}\right]$$

An example (Example 1.4.4.2)

$$\left[\begin{array}{ccccc|c} 1 & 2 & 4 & 5 & 0 & 3 \\ 0 & 0 & 1 & 2 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{array}\right]$$

Step 6: Add -4 times of the 2nd row to the 1st row.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & -3 & 0 & -29 \\ 0 & 0 & 1 & 2 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{array}\right]$$

The augmented matrix is already in reduced row-echelon form.

Some remarks (Remark 1.4.5)

1. Every matrix has a unique reduced row-echelon form but can have many different row-echelon forms.

2. In the actual implementation of the algorithm, steps mentioned in Gaussian Elimination (Algorithm 1.4.2) and Gauss-Jordan Elimination (Algorithm 1.4.3) are usually modified to avoid the round-off errors during the computation.

Solving linear systems (Discussion 1.4.6)

$$
\begin{tikzcd}
\text{a given linear system} \arrow[rr, leftrightarrow, "\text{same}", "\text{solutions}"'] \arrow[d, leftrightarrow] & & \text{the linear system corresponding to the row-echelon form} \arrow[d, leftrightarrow] \\
\text{augmented matrix of the system} \arrow[rr, "\text{Gaussian or Gauss-Jordan}", "\text{Elimination}"'] & & \text{a (reduced) row-echelon form of the augmented matrix}
\end{tikzcd}
$$

By solving the system corresponding to the augmented matrix in a row-echelon form or the reduced row-echelon form, we can find the solutions of the original system easily.

An example (Example 1.4.7)

Consider the linear system:

$$\begin{cases} \phantom{x_1 + 2x_2 + 4x_3 + {}} 2x_3 + 4x_4 + 2x_5 = 8 \\ x_1 + 2x_2 + 4x_3 + 5x_4 + 3x_5 = -9 \\ -2x_1 - 4x_2 - 5x_3 - 4x_4 + 3x_5 = 6. \end{cases}$$

The augmented matrix is

$$\left[\begin{array}{ccccc|c} 0 & 0 & 2 & 4 & 2 & 8 \\ 1 & 2 & 4 & 5 & 3 & -9 \\ -2 & -4 & -5 & -4 & 3 & 6 \end{array}\right].$$

We solve the system in two different ways.

An example (Example 1.4.7)

Method 1: Use the Gaussian Elimination.

$$\left[\begin{array}{ccccc|c} 0 & 0 & 2 & 4 & 2 & 8 \\ 1 & 2 & 4 & 5 & 3 & -9 \\ -2 & -4 & -5 & -4 & 3 & 6 \end{array}\right] \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 2 & 4 & 5 & 3 & -9 \\ 0 & 0 & 2 & 4 & 2 & 8 \\ 0 & 0 & 0 & 0 & 6 & -24 \end{array}\right] \end{array}$$

To get the general solution from an augmented matrix in row-echelon form, we first set the variable corresponding to non-pivot columns to be arbitrary.

Then equate the other variables using the method of back substitution (see Example 1.3.5.2).

(The remark at the end of Example 1.4.7)

An example (Example 1.4.7)

$$\begin{cases} x_1 + 2x_2 + 4x_3 + 5x_4 + 3x_5 = -9 \\ \qquad\qquad\quad 2x_3 + 4x_4 + 2x_5 = 8 \\ \qquad\qquad\qquad\qquad\quad 6x_5 = -24. \end{cases}$$

We set $x_2 = s$ and $x_4 = t$ where s and t are arbitrary parameters.

By the 3rd equation, $x_5 = -4$.

Substituting $x_5 = -4$ into the 2nd equation,

$$2x_3 + 4t - 2 \cdot 4 = 8 \text{ which gives us } x_3 = 8 - 2t.$$

Substituting $x_3 = 8 - 2t$ and $x_5 = -4$ into the 1st equation,

$$x_1 + 2s + 4(8 - 2t) + 5t - 3 \cdot 4 = -9$$

which gives us $x_1 = -29 - 2s + 3t$.

An example (Example 1.4.7)

Method 2: Use the Gauss-Jordan Elimination.

$$\left[\begin{array}{ccccc|c} 0 & 0 & 2 & 4 & 2 & 8 \\ 1 & 2 & 4 & 5 & 3 & -9 \\ -2 & -4 & -5 & -4 & 3 & 6 \end{array}\right] \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 2 & 0 & -3 & 0 & -29 \\ 0 & 0 & 1 & 2 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{array}\right] \end{array}$$

To get the general solution from an augmented matrix in row-echelon form, we first set the variable corresponding to non-pivot columns to be arbitrary.

The other variables can be equated directly.

(The remark at the end of Example 1.4.7)

An example (Example 1.4.7)

$$\begin{cases} x_1 + 2x_2 - 3x_4 = -29 \\ x_3 + 2x_4 = 8 \\ x_5 = -4. \end{cases}$$

We set $x_2 = s$ and $x_4 = t$ where s and t are arbitrary parameters.

By the 1st equation, $x_1 = -29 - 2s + 3t$.

By the 2nd equation, $x_3 = 8 - 2t$.

By the 3rd equation, $x_5 = -4$.

An example (Example 1.4.7)

For both method, the general solution of the system is:

$$\begin{cases} x_1 = -29 - 2s + 3t \\ x_2 = s \\ x_3 = 8 - 2t \\ x_4 = t \\ x_5 = -4 \end{cases}$$

where s and t are arbitrary parameters.

A question from some students

$$\begin{cases} \qquad\qquad\qquad\quad 2x_3 + 4x_4 + 2x_5 = \ \ 8 \\ \ \ x_1 + 2x_2 + 4x_3 + 5x_4 + 3x_5 = -9 \\ -2x_1 - 4x_2 - 5x_3 - 4x_4 + 3x_5 = \ \ 6. \end{cases}$$

$$\left[\begin{array}{ccccc|c} 0 & 0 & 2 & 4 & 2 & 8 \\ 1 & 2 & 4 & 5 & 3 & -9 \\ -2 & -4 & -5 & -4 & 3 & 6 \end{array}\right] \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 2 & 0 & -3 & 0 & -29 \\ 0 & 0 & 1 & 2 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{array}\right] \end{array}$$

Instead of reducing the augmented matrix to reduced-row echelon form, some students claim that they can solve the linear system by other ways.

For example, by a series of elementary operations, we get ...

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 2 & 1.5 & 0 & 0 & -17 \\ 0 & 0 & 0.5 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{array}\right] \end{array}$$

A question from some students

Gaussian and Gauss-Jordan Eliminations are just methods for solving systems of linear equations. There are many different methods to do the job.

However, as beginners, you are advised to understand and familiarize with the Gaussian and Gauss-Jordan Eliminations by following the algorithms to work out the exercises of the textbook.

A question from some students

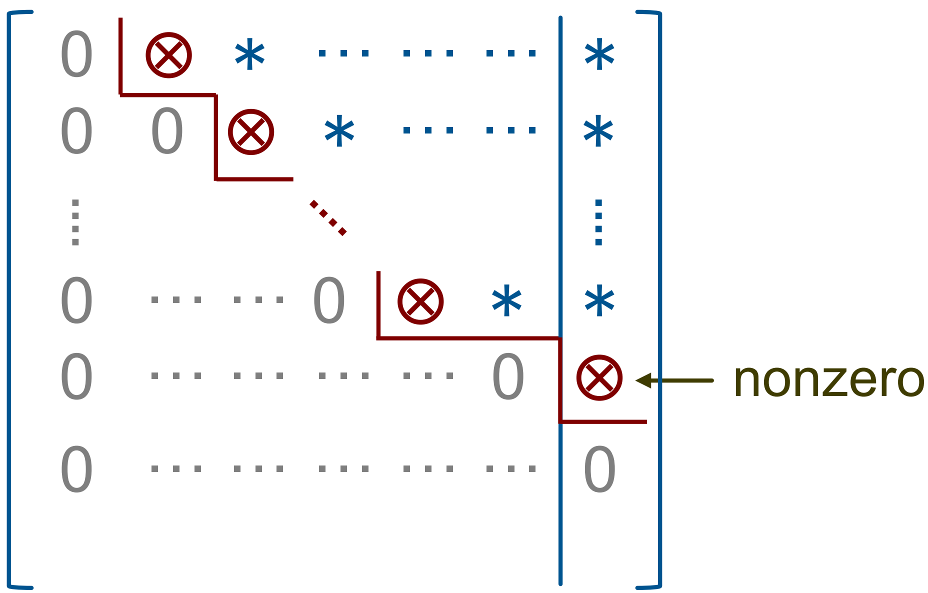
By the way, the alternative method shown previously is still Gauss-Jordan Elimination if we change the setting of our system.

$$\begin{cases} 4x_4 + 2x_3 + 2x_5 = 8 \\ x_1 + 2x_2 + 5x_4 + 4x_3 + 3x_5 = -9 \\ -2x_1 - 4x_2 - 4x_4 - 5x_3 + 3x_5 = 6. \end{cases}$$

$$\left[\begin{array}{ccccc|c} 0 & 0 & 4 & 2 & 2 & 8 \\ 1 & 2 & 5 & 4 & 3 & -9 \\ -2 & -4 & -4 & -5 & 3 & 6 \end{array}\right] \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_4 & x_3 & x_5 \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 1.5 & 0 & -17 \\ 0 & 0 & 1 & 0.5 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{array}\right] \end{array}$$

No solution (Remark 1.4.8.1)

A linear system is inconsistent, i.e. has no solution, if the last column of a row-echelon form of the augmented matrix is a pivot column.



For example, if the augmented matrix of a linear system has a row-echelon form

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right],$$

then the system is inconsistent.

One solution (Remark 1.4.8.2)

A consistent linear system has only one solution if except the last column, every column of a row-echelon form of the augmented matrix is a pivot column.

$$\left[\begin{array}{ccccc|c} \otimes & * & \cdots & \cdots & & * \\ 0 & \otimes & * & \cdots & & * \\ \vdots & & \ddots & & & \vdots \\ 0 & \cdots & 0 & \otimes & & * \\ 0 & \cdots & \cdots & \cdots & & 0 \\ & & & & & \end{array}\right]$$

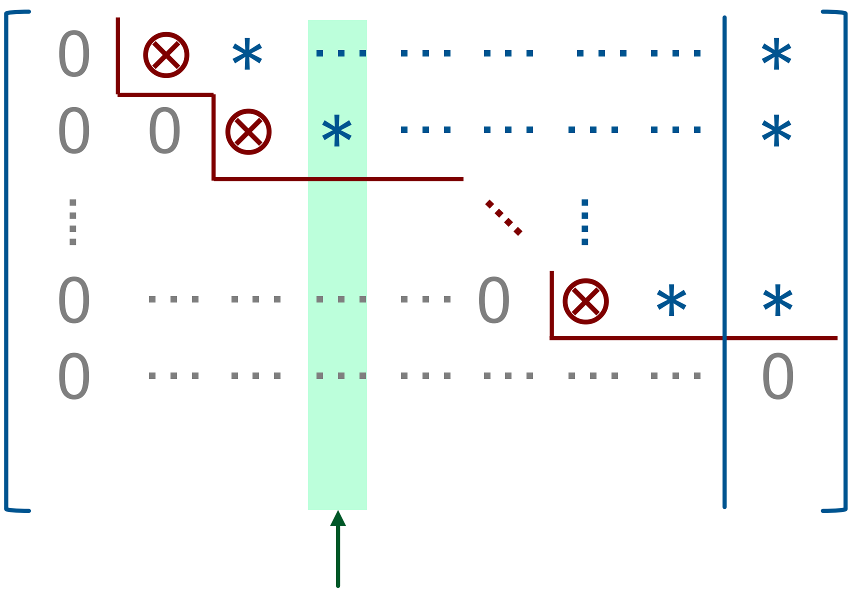
For example, if the augmented matrix of a linear system has a row-echelon form

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & -1 & 2 \end{array}\right],$$

then the system has only one solution.

Infinitely many solutions (Remark 1.4.8.3)

A consistent linear system has infinitely many solutions if apart from the last column, a row-echelon form of the augmented matrix has at least one more non-pivot column.



non-pivot columns

For example, if the augmented matrix of a linear system has a row-echelon form

$$\left[\begin{array}{cccc|c} 5 & 1 & 2 & 3 & 4 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right],$$

then the system has infinitely many solutions.

Notation (Notation 1.4.9)

When doing elementary row operations, we adopt the following notation:

1. The symbol cR_i means

   "multiply the i^{th} row by the constant c".

2. The symbol $R_i \leftrightarrow R_j$ means

   "interchange the i^{th} and the j^{th} rows".

3. The symbol $R_i + cR_j$ means

   "add c times of the j^{th} row to the i^{th} row".

Examples (Example 1.4.10.1)

Find the condition on a, b and c so that the linear system has at least one solution:

$$\begin{cases} x+2y-3z=a \\ 2x+6y-11z=b \\ x-2y+7z=c. \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 2 & 6 & -11 & b \\ 1 & -2 & 7 & c \end{array}\right] \xrightarrow[R_3-R_1]{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & 2 & -5 & b-2a \\ 0 & -4 & 10 & c-a \end{array}\right]$$

$$\xrightarrow{R_3+2R_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & 2 & -5 & b-2a \\ 0 & 0 & 0 & 2b+c-5a \end{array}\right]$$

Examples (Example 1.4.10.1)

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & 2 & -5 & b-2a \\ 0 & 0 & 0 & 2b+c-5a \end{array}\right]$$

The system has either no solution or infinitely many solutions.

It has (infinitely many) solutions if $2b + c - 5a = 0$.

Examples (Example 1.4.10.2)

Determine the values of b so that the linear system

$$\begin{cases} x + 2y + z = 1 \\ 2x + by + 2z = 2 \\ 4x + 8y + b^2 z = 2b. \end{cases}$$

has (a) no solution,

(b) a unique solution, and

(c) infinitely many solutions.

Examples (Example 1.4.10.2)

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & b & 2 & 2 \\ 4 & 8 & b^2 & 2b \end{array}\right] \xrightarrow[R_3 - 4R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & b-4 & 0 & 0 \\ 0 & 0 & b^2-4 & 2b-4 \end{array}\right]$$

(a) The system has no solution
if $b^2 - 4 = 0$ and $2b - 4 \neq 0$; i.e. $b = -2$.

(b) The system has a unique solution
if $b - 4 \neq 0$ and $b^2 - 4 \neq 0$; i.e. $b \neq 4$ and $b \neq \pm 2$.

(c) The system has infinitely many solutions if
(i) $b - 4 = 0$ or (ii) $b^2 - 4 = 0$ and $2b - 4 = 0$;
i.e. $b = 4$ or $b = 2$.

Examples (Example 1.4.10.3)

Determine the values of a and b so that the linear system

$$\begin{cases} ax + y \phantom{{}+az} = a \\ x + y + z = 1 \\ y + az = b. \end{cases}$$

has (a) no solution, (b) a unique solution, and (c) infinitely many solutions.

In doing elementary row operations,

(i) you cannot multiply a row by 0 or $\frac{1}{0}$; and

(ii) you cannot add $\frac{1}{0}$ times of a row to another row.

Examples (Example 1.4.10.3)

$$\left[\begin{array}{ccc|c} a & 1 & 0 & a \\ 1 & 1 & 1 & 1 \\ 0 & 1 & a & b \end{array}\right]$$

Can we add $-\frac{1}{a}$ times of the 1st row to the 2nd row?

We cannot do it if $a = 0$.

Case 1: $a = 0$.

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & b \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & b \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & b \end{array}\right]$$

Under the assumption that $a = 0$,

the system has no solution if $b \neq 0$; and

the system has infinitely many solutions if $b = 0$.

Examples (Example 1.4.10.3)

Case 2: $a \neq 0$.

$$\left[\begin{array}{ccc|c} a & 1 & 0 & a \\ 1 & 1 & 1 & 1 \\ 0 & 1 & a & b \end{array}\right] \xrightarrow{R_2 - \frac{1}{a}R_1} \left[\begin{array}{ccc|c} a & 1 & 0 & a \\ 0 & \frac{a-1}{a} & 1 & 0 \\ 0 & 1 & a & b \end{array}\right]$$

Can we add $-\dfrac{a}{a-1}$ times of the 2nd row to the 3rd row?

We cannot do it if $a = 1$.

Case 2a: $a = 1$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & b \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & b \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Under the assumption that $a = 1$,
the system has a unique solution.

Examples (Example 1.4.10.3)

Case 2b: $a \neq 0$ and $a \neq 1$.

$$\left[\begin{array}{ccc|c} a & 1 & 0 & a \\ 0 & \frac{a-1}{a} & 1 & 0 \\ 0 & 1 & a & b \end{array}\right] \xrightarrow{R_3 - \frac{a}{a-1}R_2} \left[\begin{array}{ccc|c} a & 1 & 0 & a \\ 0 & \frac{a-1}{a} & 1 & 0 \\ 0 & 0 & \frac{a^2-2a}{a-1} & b \end{array}\right]$$

nonzero

Under the assumption that $a \neq 0$ and $a \neq 1$,

the system has no solution if $\frac{a^2-2a}{a-1} = 0$ and $b \neq 0$, i.e. $a = 2$ and $b \neq 0$.

the system has a unique solution if $\frac{a^2-2a}{a-1} \neq 0$, i.e. $a \neq 2$.

the system has infinitely many solutions if $\frac{a^2-2a}{a-1} = 0$ and $b = 0$, i.e. $a = 2$ and $b = 0$.

Examples (Example 1.4.10.3)

(a) The system has no solution
if (Case 1) $a = 0$ and $b \neq 0$
or (Case 2b) $a = 2$ and $b \neq 0$;
i.e. $b \neq 0$ and $a = 0$ or 2.

(b) The system has a unique solution
if (Case 2a) $a = 1$,
or (Case 2b) $a \neq 0$ and $a \neq 1$ and $a \neq 2$;
i.e. $a \neq 0$ and $a \neq 2$.

(c) The system has infinitely many solutions if
if (Case 1) $a = 0$ and $b = 0$
or (Case 2b) $a = 2$ and $b = 0$;
i.e. $b = 0$ and $a = 0$ or 2.

The system can be solved easily if we rearrange the rows in a suitable order. See the remark at the end of Example 1.4.10.3.

A question from some students

Some students think that by doing this row operation, they can avoid dividing a row by 0.

$$\left[\begin{array}{ccc|c} a & 1 & 0 & a \\ 1 & 1 & 1 & 1 \\ 0 & 1 & a & b \end{array}\right] \xrightarrow[\text{to } aR_2 - R_1]{\text{Change } R_2} \left[\begin{array}{ccc|c} a & 1 & 0 & a \\ 0 & a-1 & a & 0 \\ 0 & 1 & a & b \end{array}\right]$$

This row operation is equivalent to the sequence of elementary operations shown below:

$$\xrightarrow{aR_2} \left[\begin{array}{ccc|c} a & 1 & 0 & a \\ a & a & a & a \\ 0 & 1 & a & b \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|c} a & 1 & 0 & a \\ 0 & a-1 & a & 0 \\ 0 & 1 & a & b \end{array}\right]$$

What will happen if $a = 0$?

The information of the second equation, $x + y + z = 1$, is completely wiped out when $a = 0$.

Examples (Example 1.4.10.4)

Given a cubic curve with equation

$$y = a + bx + cx^2 + dx^3,$$

where a, b, c, d are real constants, that passes through the points $(0, 10)$, $(1, 7)$, $(3, -11)$ and $(4, -14)$, find a, b, c, d.

By substituting $(x, y) = (0, 10)$, $(1, 7)$, $(3, -11)$ and $(4, -14)$ into the equation of the cubic curve, we obtain

$$\begin{cases} a & & & & = 10 \\ a + b + c + d = 7 \\ a + 3b + 9c + 27d = -11 \\ a + 4b + 16c + 64d = -14. \end{cases}$$

Examples (Example 1.4.10.4)

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 10 \\ 1 & 1 & 1 & 1 & 7 \\ 1 & 3 & 9 & 27 & -11 \\ 1 & 4 & 16 & 64 & -14 \end{array}\right] \xrightarrow[R_4 - R_1]{\substack{R_2 - R_1 \\ R_3 - R_1}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 10 \\ 0 & 1 & 1 & 1 & -3 \\ 0 & 3 & 9 & 27 & -21 \\ 0 & 4 & 16 & 64 & -24 \end{array}\right]$$

$$\xrightarrow[R_4 - 4R_2]{R_3 - 3R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 10 \\ 0 & 1 & 1 & 1 & -3 \\ 0 & 0 & 6 & 24 & -12 \\ 0 & 0 & 12 & 60 & -12 \end{array}\right] \xrightarrow{R_4 - 2R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 10 \\ 0 & 1 & 1 & 1 & -3 \\ 0 & 0 & 6 & 24 & -12 \\ 0 & 0 & 0 & 12 & 12 \end{array}\right]$$

Examples (Example 1.4.10.4)

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 10 \\ 0 & 1 & 1 & 1 & -3 \\ 0 & 0 & 6 & 24 & -12 \\ 0 & 0 & 0 & 12 & 12 \end{array}\right] \xrightarrow[\frac{1}{12}R_4]{\frac{1}{6}R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 10 \\ 0 & 1 & 1 & 1 & -3 \\ 0 & 0 & 1 & 4 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 - 4R_4]{R_2 - R_4} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 10 \\ 0 & 1 & 1 & 0 & -4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_2 - R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

So $a = 10$, $b = 2$, $c = -6$ and $d = 1$.

The equation of the cubic curve is $y = 10 + 2x - 6x^2 + x^3$.

Geometrical interpretation (Discussion 1.4.11)

Consider a system of linear equations in variables x, y and z:

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ \quad\vdots \qquad\qquad\qquad \vdots \\ a_mx + b_my + c_mz = d_m \end{cases}$$

$$\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ \vdots & \vdots & \vdots & \vdots \\ a_m & b_m & c_m & d_m \end{array}\right] \xrightarrow[\text{Elimination}]{\text{Gaussian}} \boldsymbol{R}$$

A row echelon form

Geometrical interpretation (Discussion 1.4.11)

If the last non-zero row of $\boldsymbol{R}$ is of the form $(0\ 0\ 0\ *)$ where $*$ is a nonzero number, then the system is inconsistent, i.e. there is no solution.

Suppose the system is consistent:

$\boldsymbol{R}$ has at most 3 non-zero rows.

row-echelon form $\boldsymbol{R}$	the general solution	the solution set in *xyz*-space
3 non-zero rows	0 arbitrary parameter	a point
2 non-zero rows	1 arbitrary parameter	a line
1 non-zero row	2 arbitrary parameters	a plane
0 non-zero row	3 arbitrary parameters	the whole space

(Read Discussion 1.4.11 and Example 1.4.12 for more details.)

Section 1.5

Homogeneous Linear Systems

Homogeneous linear systems (Definition 1.5.1)

A system of linear equations is said to be homogeneous if it has the form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \quad\vdots \qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{cases}$$

where $a_{11}, a_{12}, \ldots, a_{mn}$ are real constants.

Note that $x_1 = 0,\ x_2 = 0,\ \ldots,\ x_n = 0$ is always a solution to the homogeneous system and it is called the trivial solution.

Any solution other than the trivial solution is called a non-trivial solution.

An example (Example 1.5.2)

Given a quadric surface with equation

$$ax^2 + by^2 + cz^2 = d,$$

where a, b, c, d are real constants, that passes through the points $(1, 1, -1)$, $(1, 3, 3)$ and $(-2, 0, 2)$, find a formula for the quadric surface.

By substituting $(x, y, z) = (1, 1, -1)$, $(1, 3, 3)$ and $(-2, 0, 2)$ into the equation of the quadric surface, we obtain

$$\begin{cases} a + b + c - d = 0 \\ a + 9b + 9c - d = 0 \\ 4a + 4c - d = 0. \end{cases}$$

An example (Example 1.5.2)

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 1 & 9 & 9 & -1 & 0 \\ 4 & 0 & 4 & -1 & 0 \end{array}\right] \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -3/4 & 0 \\ 0 & 0 & 1 & 3/4 & 0 \end{array}\right]$$

There are infinitely many solutions:

$$\begin{cases} a = t \\ b = \frac{3}{4}t \\ c = -\frac{3}{4}t \\ d = t \end{cases}$$

where t is an arbitrary parameter.

An example (Example 1.5.2)

$$\begin{cases} a = t \\ b = \dfrac{3}{4}t \\ c = -\dfrac{3}{4}t \\ d = t \end{cases}$$ where t is an arbitrary parameter.

Any one of the nontrivial solutions gives us a formula for the quadric surface.

For example,

$$x^2 + \frac{3}{4}y^2 - \frac{3}{4}z^2 = 1 \quad \text{and} \quad 4x^2 + 3y^2 - 3z^2 = 4$$

are two formulae which represent the same quadric surface.

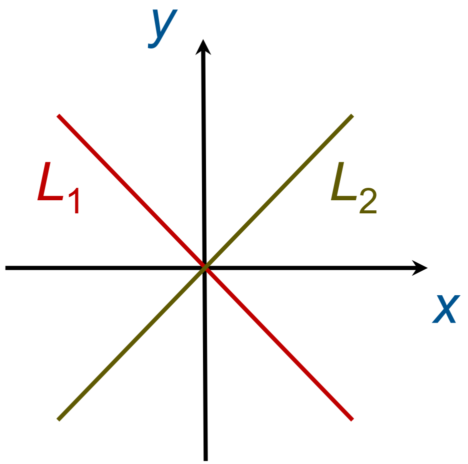
Geometrical interpretation (Discussion 1.5.3.1)

In the xy-plane, the two equations in the system

$$\begin{cases} a_1x + b_1y = 0 & (L_1) \\ a_2x + b_2y = 0, & (L_2) \end{cases}$$

where a_1, b_1 are not both zero and a_2, b_2 are not both zero, represents two straight lines through the origin (i.e. the point $(0, 0)$).

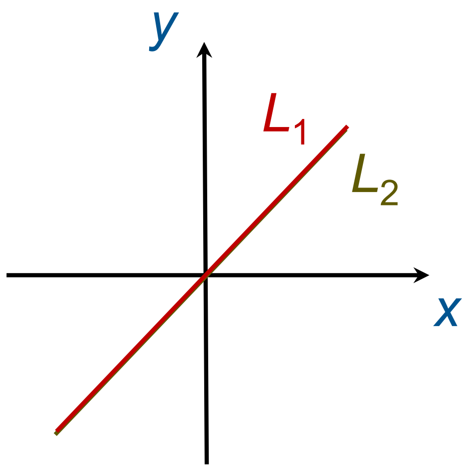
(a) The system has only one solution if and only if L_1 and L_2 are not parallel lines.



Geometrical interpretation (Discussion 1.5.3.1)

$$\begin{cases} a_1 x + b_1 y = 0 & (L_1) \\ a_2 x + b_2 y = 0, & (L_2) \end{cases}$$

(b) The system has infinitely many solutions if and only if L_1 and L_2 are the same line.



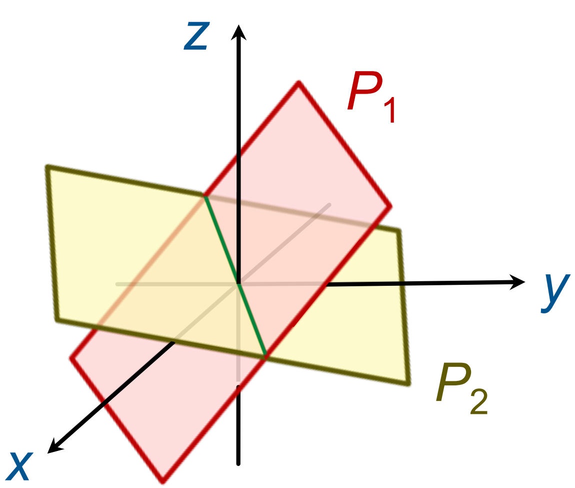
Geometrical interpretation (Discussion 1.5.3.2)

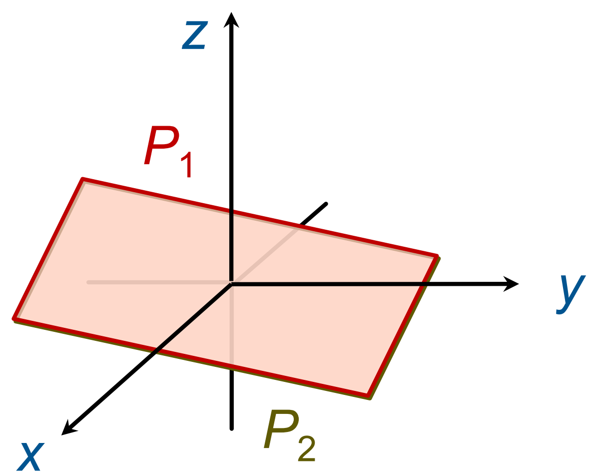
In the xyz-space, the two equations in the system

$$\begin{cases} a_1x + b_1y + c_1z = 0 & (P_1) \\ a_2x + b_2y + c_2z = 0, & (P_2) \end{cases}$$

where a_1, b_1, c_1 are not all zero and a_2, b_2, c_2 are not all zero, represents two planes containing the origin.

The system has infinitely many solutions:





Solutions of homogenous system (Remark 1.5.4)

1. A homogeneous system of linear equations has either only the trivial solution or infinitely many solutions in addition to the trivial solution.

2. A homogeneous system of linear equations with more unknowns than equations has infinitely many solutions.