Student Number: ______________________

NATIONAL UNIVERSITY OF SINGAPORE

MA1101R - Linear Algebra I

(Semester 2 : AY2015/2016)

Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation/student number clearly in the space provided at the top of this page.** This booklet (and only this booklet) will be collected at the end of the examination.

2. Please write your matriculation/student number only. Do not write your name.

3. This examination paper contains **SIX** questions and comprises **FIFTEEN** printed pages.

4. Answer **ALL** questions.

5. This is a CLOSED BOOK (with helpsheet) examination.

6. You are allowed to use two A4 size helpsheets.

7. You may use scientific calculators. However, you should lay out systematically the various steps in the calculations)

Examiner's Use Only	
Questions	Marks
1	
2	
3	
4	
5	
6	
Total	

Question 1 [10 marks]

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ be a subset of $\mathbb{R}^3$ where $\boldsymbol{u}_1 = (1,1,1), \boldsymbol{u}_2 = (1,0,2), \boldsymbol{u}_3 = (2,1,0)$.

(i) [2 marks] Show that S forms a basis for $\mathbb{R}^3$.

(ii) [2 marks] Let $\boldsymbol{w} = (7,3,-1)$. Find the coordinate vector of $\boldsymbol{w}$ with respect to S.

(iii) [4 marks] Let $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ be another basis for $\mathbb{R}^3$ where $\boldsymbol{v}_1 = (0,3,1), \boldsymbol{v}_2 = (2,2,2), \boldsymbol{v}_3 = (1,2,3)$. Find the transition matrix from T to S.

(iv) [2 marks] Suppose the coordinate vector of $\boldsymbol{x}$ with respect to T is (a,b,c). Find the coordinate vector of $\boldsymbol{x}$ with respect to S.

Show your working below.

(i) There are many ways to show that $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ is a basis.

Here we use $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ to form a 3×3 matrix $\boldsymbol{A}$ such that

$$\det(\boldsymbol{A}) = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{vmatrix} = 3 \neq 0$$

Hence $\boldsymbol{A}$ is invertible, which means $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ form a basis for $\mathbb{R}^3$.

(ii) We need to express $(7,3,-1) = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + c_3\boldsymbol{u}_3$.

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 7 \\ 1 & 0 & 1 & 3 \\ 1 & 2 & 0 & -1 \end{array}\right) \xrightarrow[R_3+(-1)R_1]{R_2+(-1)R_1} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 7 \\ 0 & -1 & -1 & -4 \\ 0 & 1 & -2 & -8 \end{array}\right) \xrightarrow{R_3+R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 7 \\ 0 & -1 & -1 & -4 \\ 0 & 0 & -3 & -12 \end{array}\right)$$

So we can solve for $c_1 = -1, c_2 = 0, c_3 = 4$.

Hence the coordinate vector $(\boldsymbol{w})_S = (c_1, c_2, c_3) = (-1, 0, 4)$.

(iii) To find the transition matrix $\boldsymbol{P}$ from T to S, express $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ in terms of the basis S:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 2 & 1 \\ 1 & 0 & 1 & 3 & 2 & 2 \\ 1 & 2 & 0 & 1 & 2 & 3 \end{array}\right) \xrightarrow[R_3+(-1)R_1]{R_2+(-1)R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 2 & 1 \\ 0 & -1 & -1 & 3 & 0 & 1 \\ 0 & 1 & -2 & 1 & 0 & 2 \end{array}\right) \xrightarrow{R_3+R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 2 & 1 \\ 0 & -1 & -1 & 3 & 0 & 1 \\ 0 & 0 & -3 & 4 & 0 & 3 \end{array}\right)$$

$$\xrightarrow[-R_2]{-\frac{1}{3}R_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 2 & 1 \\ 0 & 1 & 1 & -3 & 0 & -1 \\ 0 & 0 & 1 & -\frac{4}{3} & 0 & -1 \end{array}\right) \xrightarrow[R_1+(-2)R_3]{R_2+(-1)R_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{8}{3} & 2 & 3 \\ 0 & 1 & 0 & -\frac{5}{3} & 0 & 0 \\ 0 & 0 & 1 & -\frac{4}{3} & 0 & -1 \end{array}\right)$$

$$\xrightarrow{R_1+(-1)R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{13}{3} & 2 & 3 \\ 0 & 1 & 0 & -\frac{5}{3} & 0 & 0 \\ 0 & 0 & 1 & -\frac{4}{3} & 0 & -1 \end{array}\right).$$

So $\boldsymbol{P} = \begin{pmatrix} \frac{13}{3} & 2 & 3 \\ -\frac{5}{3} & 0 & 0 \\ -\frac{4}{3} & 0 & -1 \end{pmatrix}$.

Question 1

Continue your working below.

(iv)

$$[x]_S = \boldsymbol{P}\,[x]_T = \begin{pmatrix} \frac{13}{3} & 2 & 3 \\ -\frac{5}{3} & 0 & 0 \\ -\frac{4}{3} & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \frac{13}{3}a + 2b + 3c \\ -\frac{5}{3}a \\ -\frac{4}{3}a - c \end{pmatrix}$$

Continue on page 14 and 15 if you need more writing space.

Question 2 [10 marks]

Let $\boldsymbol{A} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$.

(i) [2 marks] Find the eigenvalues for $\boldsymbol{A}$. Justify your answer.

(ii) [8 marks] Show that $\boldsymbol{A}$ is diagonalizable and find an invertible matrix $\boldsymbol{P}$ and a diagonal matrix $\boldsymbol{D}$ such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$.

Show your working below.

(i) The eigenvalues are the diagonal entries of this lower triangular matrix, namely 1,2 and 3.

(Note: There is no need to solve the characteristic equation of $\boldsymbol{A}$.)

(ii) $$3\boldsymbol{I} - \boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 2 \end{pmatrix} \xrightarrow[R_4+R_1]{\substack{R_2+R_1 \\ R_3+R_2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$(3\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0} \Rightarrow \boldsymbol{x} = (0, -s+2t, s, t) = s(0, -1, 1, 0) + t(0, 2, 0, 1)$.

So a basis for the eigenspace for eigenvalue 3 is $(0, -1, 1, 0), (0, 2, 0, 1)$.

$$2\boldsymbol{I} - \boldsymbol{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ -1 & -1 & -1 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_4} \begin{pmatrix} -1 & -1 & -1 & 1 \\ -1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[R_3+(-1)R_2]{R_2+(-1)R_1} \begin{pmatrix} -1 & -1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$(2\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0} \Rightarrow \boldsymbol{x} = (-t, t, t, t) = t(-1, 1, 1, 1)$.

So a basis for the eigenspace for eigenvalue 2 is $(-1, 1, 1, 1)$.

$$\boldsymbol{I} - \boldsymbol{A} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ -1 & 0 & -2 & 0 \\ -1 & -1 & -1 & 0 \end{pmatrix} \xrightarrow[R_4+(-1)R_1]{\substack{R_2+(-1)R_1 \\ R_3+(-1)R_2}} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & -1 & -1 & 0 \end{pmatrix}.$$

$(\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0} \Rightarrow \boldsymbol{x} = (0, 0, 0, t) = t(0, 0, 0, 1)$.

So a basis for the eigenspace for eigenvalue 1 is $(0, 0, 0, 1)$.

Question 2

Continue your working below.

Since there are four linearly independent eigenvectors for this 4×4 matrix, the matrix $\boldsymbol{A}$ is diagonalizable.

Let $\boldsymbol{P} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ -1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Continue on page 14 and 15 if you need more writing space.

Question 3 [10 marks]

Let $V = \text{span}\{(1,1,0,1),(3,2,1,1),(-1,0,2,-2)\}$.

(i) [3 marks] Using the Gram-Schmidt process, find an orthogonal basis for V.

(ii) [2 marks] Using the result in (i), find the projection of $(2,-2,-2,3)$ onto V.

(iii) [2 marks] Extend your basis in (i) to an orthogonal basis for $\mathbb{R}^4$.

(iv) [3 marks] Find a least squares solution of the system $\boldsymbol{Ax} = \boldsymbol{b}$ where

$$\boldsymbol{A} = \begin{pmatrix} 1 & 3 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -2 \end{pmatrix} \text{ and } \boldsymbol{b} = \begin{pmatrix} 2 \\ -2 \\ -2 \\ 3 \end{pmatrix}.$$

Show your working below.

(i) Let $\boldsymbol{u}_1 = (1,1,0,1)$, $\boldsymbol{u}_2 = (3,2,1,1)$ and $\boldsymbol{u}_3 = (-1,0,2,-2)$.

$$\begin{aligned}
\boldsymbol{v}_1 &= \boldsymbol{u}_1 = (1,1,0,1),\\
\boldsymbol{v}_2 &= \boldsymbol{u}_2 - \frac{\boldsymbol{u}_2\cdot\boldsymbol{v}_1}{\boldsymbol{v}_1\cdot\boldsymbol{v}_1}\boldsymbol{v}_1 = (3,2,1,1) - \frac{6}{3}(1,1,0,1) = (1,0,1,-1),\\
\boldsymbol{v}_3 &= \boldsymbol{u}_3 - \frac{\boldsymbol{u}_3\cdot\boldsymbol{v}_1}{\boldsymbol{v}_1\cdot\boldsymbol{v}_1}\boldsymbol{v}_1 - \frac{\boldsymbol{u}_3\cdot\boldsymbol{v}_2}{\boldsymbol{v}_2\cdot\boldsymbol{v}_2}\boldsymbol{v}_2\\
&= (-1,0,2,-2) - \frac{-3}{3}(1,1,0,1) - \frac{3}{3}(1,0,1,-1) = (-1,1,1,0).
\end{aligned}$$

An orthogonal basis for V is $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.

(ii) The projection of $\boldsymbol{v} = (2,-2,-2,3)$ onto V is

$$\begin{aligned}
\boldsymbol{p} &= \frac{(\boldsymbol{v}\cdot\boldsymbol{v}_1)}{(\boldsymbol{v}_1\cdot\boldsymbol{v}_1)}\boldsymbol{v}_1 + \frac{(\boldsymbol{v}\cdot\boldsymbol{v}_2)}{(\boldsymbol{v}_2\cdot\boldsymbol{v}_2)}\boldsymbol{v}_2 + \frac{(\boldsymbol{v}\cdot\boldsymbol{v}_3)}{(\boldsymbol{v}_3\cdot\boldsymbol{v}_3)}\boldsymbol{v}_3\\
&= \frac{3}{3}(1,1,0,1) + \frac{-3}{3}(1,0,1,-1) + \frac{-6}{3}(-1,1,1,0) = (2,-1,-3,2).
\end{aligned}$$

(iii) Note that $\boldsymbol{v} - \boldsymbol{p} = (0,-1,1,1)$ is orthogonal to each $\boldsymbol{v}_i$. Then

$$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, (0,-1,1,1)\}$$

is an orthogonal basis for $\mathbb{R}^4$.

Question 3

Show your working below.

(iv) $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^T\boldsymbol{v}$ is $\begin{pmatrix} 3 & 6 & -3 \\ 6 & 15 & -3 \\ -3 & -3 & 9 \end{pmatrix}\boldsymbol{x} = \begin{pmatrix} 3 \\ 3 \\ -12 \end{pmatrix}$.

$$\left(\begin{array}{ccc|c} 3 & 6 & -3 & 3 \\ 6 & 15 & -3 & 3 \\ -3 & -3 & 9 & -12 \end{array}\right) \xrightarrow[R_3+R_1]{R_2+(-2)R_1} \left(\begin{array}{ccc|c} 3 & 6 & -3 & 3 \\ 0 & 3 & 3 & -3 \\ 0 & 3 & 6 & -9 \end{array}\right) \xrightarrow{R_3+(-1)R_2} \left(\begin{array}{ccc|c} 3 & 6 & -3 & 3 \\ 0 & 3 & 3 & -3 \\ 0 & 0 & 3 & -6 \end{array}\right)$$

$$\xrightarrow[\frac{1}{3}R_3]{\substack{\frac{1}{3}R_1 \\ \frac{1}{3}R_2}} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right) \xrightarrow[R_2+(-1)R_3]{R_1+R_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right) \xrightarrow{R_1+(-2)R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right).$$

So a least squares solution to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{v}$ is $\begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix}$

Alternatively, the answer can be obtained by solving $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{p}$ where $\boldsymbol{p}$ is the projection in (ii) above.

Continue on page 14 and 15 if you need more writing space.

Question 4 [10 marks]

Let $\boldsymbol{A}$ be a 3×2 matrix and $\boldsymbol{B}$ be a 2×3 matrix such that

$$\boldsymbol{AB} = \begin{pmatrix} -2 & -14 & 14 \\ 5 & 15 & -10 \\ 4 & 8 & -3 \end{pmatrix}.$$

(i) [3 marks] Find a basis for the row space of $\boldsymbol{AB}$ and state the rank of $\boldsymbol{AB}$.

(ii) [2 marks] Show that $(\boldsymbol{AB})^2 = 5\boldsymbol{AB}$.

(iii) [2 marks] What is the rank of $\boldsymbol{BA}$? Justify your answer.

(iv) [3 marks] Find $\boldsymbol{BA}$. Show clearly how you derive your answer.

Show your working below.

(i) We check that

$$\begin{pmatrix} -2 & -14 & 14 \\ 5 & 15 & -10 \\ 4 & 8 & -3 \end{pmatrix} \xrightarrow[\frac{1}{5}R_2]{-\frac{1}{2}R_1} \begin{pmatrix} 1 & 7 & -7 \\ 1 & 3 & -2 \\ 4 & 8 & -3 \end{pmatrix} \xrightarrow[R_3 - 4R_1]{R_2 - R_1} \begin{pmatrix} 1 & 7 & -7 \\ 0 & -4 & 5 \\ 0 & -20 & 25 \end{pmatrix} \xrightarrow{R_3 - 5R_2} \begin{pmatrix} 1 & 7 & -7 \\ 0 & -4 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

So a basis for the row space is given by $\{(1, 7, -7), (0, -4, 5)\}$ and $\text{rank}(\boldsymbol{AB}) = 2$.

(ii) One verifies that:

$$(\boldsymbol{AB})^2 = \begin{pmatrix} -10 & -70 & 70 \\ 25 & 75 & -50 \\ 20 & 40 & -15 \end{pmatrix} = 5\boldsymbol{AB}.$$

(iii) $\text{rank}(\boldsymbol{BA}) \geq \text{rank}(\boldsymbol{A}(\boldsymbol{BA})\boldsymbol{B}) = \text{rank}((\boldsymbol{AB})^2) = 2$.

Since $\boldsymbol{BA}$ is 2×2, so $\text{rank}(\boldsymbol{BA}) = 2$.

(iv) From (ii),

$(\boldsymbol{BA})^3 = \boldsymbol{BABABA} = \boldsymbol{B}(\boldsymbol{AB})^2\boldsymbol{A} = \boldsymbol{B}(5\boldsymbol{AB})\boldsymbol{A} = 5(\boldsymbol{BA})^2$.

From (iii), we have $\boldsymbol{BA}$ is full rank and invertible.

It follows that $\boldsymbol{BA} = 5\boldsymbol{I} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$.

Question 4

Show your working below.

Continue on page 14 and 15 if you need more writing space.

Question 5 [10 marks]

(a) [4 marks]

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation and

$$\boldsymbol{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \boldsymbol{u}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \boldsymbol{u}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

a basis for $\mathbb{R}^3$.

Suppose $T(\boldsymbol{u}_1) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, T(\boldsymbol{u}_2) = \begin{pmatrix} 0 \\ 6 \end{pmatrix}, T(\boldsymbol{u}_3) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

Write down the standard matrix of T and find $\mathrm{Ker}(T)$ explicitly.

Show your working below.

Observe that:

$\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2}(\boldsymbol{u}_1 + \boldsymbol{u}_2 - \boldsymbol{u}_3)$.

So $T(\boldsymbol{e}_1) = \frac{1}{2}(T(\boldsymbol{u}_1) + T(\boldsymbol{u}_2) - T(\boldsymbol{u}_3)) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

$\boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}(\boldsymbol{u}_2 + \boldsymbol{u}_3 - \boldsymbol{u}_1)$.

So $T(\boldsymbol{e}_2) = \frac{1}{2}(T(\boldsymbol{u}_2) + T(\boldsymbol{u}_3) - T(\boldsymbol{u}_1)) = \begin{pmatrix} 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$.

$\boldsymbol{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{2}(\boldsymbol{u}_1 + \boldsymbol{u}_3 - \boldsymbol{u}_2)$.

So $T(\boldsymbol{e}_3) = \frac{1}{2}(T(\boldsymbol{u}_1) + T(\boldsymbol{u}_3) - T(\boldsymbol{u}_2)) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$.

Hence the standard matrix of T is given by $\begin{pmatrix} 0 & 0 & 2 \\ 2 & 4 & -2 \end{pmatrix}$.

By solving $\begin{pmatrix} 0 & 0 & 2 \\ 2 & 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$, we get the general solution $x = -2t, y = t, z = 0$. So $\mathrm{Ker}(T) = \{(-2t, t, 0) \mid t \in \mathbb{R}\}$.

Question 5

(b) [6 marks]

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator such that its standard matrix is diagonalisable.

Prove that $\mathrm{R}(T) = \mathrm{R}(T \circ T)$ and $\mathrm{Ker}(T) = \mathrm{Ker}(T \circ T)$.

Show your working below.

Let $\boldsymbol{A}$ be the standard matrix for T.

Then $\boldsymbol{A}$ has n linearly independent eigenvectors, say $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$, associated to eigenvalues $\lambda_1, \dots, \lambda_n$, respectively.

Suppose that $\lambda_1 = \cdots = \lambda_k = 0$, and $\lambda_i \neq 0$ if $i > k$.

For each i, $\boldsymbol{A}\boldsymbol{v}_i = \lambda \boldsymbol{v}_i$, and thus $\boldsymbol{A}^2 v_i = \lambda^2 \boldsymbol{v}_i$; so $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ are the eigenvectors of $\boldsymbol{A}^2$ associated to eigenvalues $\lambda_1^2, \dots, \lambda_n^2$.

Note that $\lambda_i^2 = 0$ if $i \leq k$ and $\lambda_i^2 \neq 0$ if $i > k$. Then

$$\begin{aligned}\mathrm{Ker}(T) &= \text{nullspace of } \boldsymbol{A} = \mathrm{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\} \\ &= \text{nullspace of } \boldsymbol{A}^2 = \mathrm{Ker}(T \circ T).\end{aligned}$$

$R(T \circ T) = \text{column space of } \boldsymbol{A}^2 \subseteq \text{column space of } \boldsymbol{A} = R(T)$, and

$$\dim R(T \circ T) = n - \dim \mathrm{Ker}(T) = n - \dim \mathrm{Ker}(T \circ T) = \dim R(T).$$

We conclude that $R(T \circ T) = R(T)$.

Continue on page 14 and 15 if you need more writing space.

Question 6 [10 marks]

(a) [4 marks]

The nullspace of a 3×4 matrix $\boldsymbol{A}$ is given by span $\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$.

Determine whether each of the following is true or false:

(i) The first two columns of $\boldsymbol{A}$ are linearly independent.

(ii) The second and fourth columns of $\boldsymbol{A}$ are identical.

Show your working below.

$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ being in the nullspace of $\boldsymbol{A}$

means the homogeneous system $\boldsymbol{Ax} = \boldsymbol{0}$ has a general solutions $\left\{ \begin{array}{rcl} x & = & 0 \\ y & = & t \\ z & = & s+t \\ w & = & s \end{array} \right.$

which gives two equations $x = 0$ and $y - z + w = 0$.

Since the nullity of $\boldsymbol{A}$ is 2, by dimension theorem, the rank of $\boldsymbol{A}$ is also 2.

Hence the rref $\boldsymbol{R}$ of $\boldsymbol{A}$ is given by

$$\boldsymbol{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We see that the first two columns of $\boldsymbol{R}$ are linearly independent. So the corresponding columns in $\boldsymbol{A}$ must also be linearly independent.

Hence (i) is true.

We also observe that the second and fourth columns of $\boldsymbol{R}$ are identical. So the corresponding columns in $\boldsymbol{A}$ must also be identical.

Hence (ii) is true.

Question 6

(b) [6 marks]

Let $V = \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ be a vector space such that $\boldsymbol{v}_i$ are unit vectors for all i and $\boldsymbol{v}_i \cdot \boldsymbol{v}_j < 0$ if $i \neq j$.

(i) Show that no two vectors among $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ are linearly dependent.

(ii) Prove that $\dim V \geq 3$.

Show your working below.

Clearly $\boldsymbol{v}_i \neq 0$.

(i) Assume that $\boldsymbol{v}_1, \boldsymbol{v}_2$ are linearly dependent.

Let $\boldsymbol{v}_1 = c\boldsymbol{v}_2$. Then $0 > \boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = c(\boldsymbol{v}_1 \cdot \boldsymbol{v}_1) = c$. We would have

$$0 > \boldsymbol{v}_1 \cdot \boldsymbol{v}_3 = c(\boldsymbol{v}_2 \cdot \boldsymbol{v}_3) > 0,$$

which is a contradiction.

Hence, $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly independent. Similarly for any other two vectors.

(ii) Assume that $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are linearly dependent. Then any vector in $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ is a linear combination of the other two. Write $\boldsymbol{v}_3 = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2$.

$$0 > \boldsymbol{v}_1 \cdot \boldsymbol{v}_3 = c_1 + c_2(\boldsymbol{v}_1 \cdot \boldsymbol{v}_2) \quad \text{and} \quad 0 > \boldsymbol{v}_2 \cdot \boldsymbol{v}_3 = c_1(\boldsymbol{v}_1 \cdot \boldsymbol{v}_2) + c_2.$$

If $c_1 > 0$, then $c_2(\boldsymbol{v}_1 \cdot \boldsymbol{v}_2) < -c_1 < 0$, and thus $c_2 > 0$.

By Cauhy-Schwarz inequality, $0 < -\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 \leq \|\boldsymbol{v}_1\| \, \|\boldsymbol{v}_2\| = 1$. We would have

$$c_1 < c_2(-\boldsymbol{v}_1 \cdot \boldsymbol{v}_2) \leq c_2 \quad \text{and} \quad c_2 < c_1(-\boldsymbol{v}_1 \cdot \boldsymbol{v}_2) \leq c_1$$

which is a contradiction (based on our assumption that $c_1 > 0$).

Therefore, $c_1 < 0$ and hence $c_2 < 0$.

However, this would imply that $\boldsymbol{v}_3 \cdot \boldsymbol{v}_4 = c_1(\boldsymbol{v}_1 \cdot \boldsymbol{v}_4) + c_2(\boldsymbol{v}_2 \cdot \boldsymbol{v}_4) > 0$, a contradiction.

Hence, $\boldsymbol{v}_1, \boldsymbol{v}_2$ and $\boldsymbol{v}_3$ are linearly independent. So $\dim V \geq 3$.

(Additional working spaces for ALL questions - indicate your question numbers clearly.)

(Additional working spaces for ALL questions - indicate your question numbers clearly.)

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