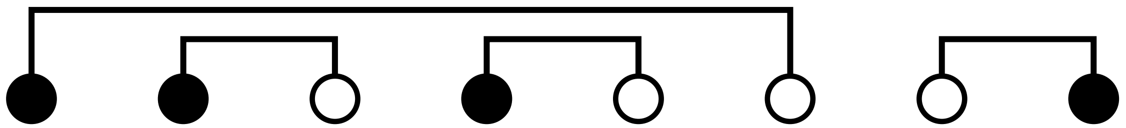
1. (a) Assume that you are given $n$ white and $n$ black dots, lying on a line, equally spaced. The dots appear in any order of black and white, see the picture below. Design a greedy algorithm which connects each black dot with a (different) white dot, so that the total length of wires used to form such connected pairs is minimal. The length of wire used to connect two dots is equal to their distance along the line. (20 pt)



(b) Is your greedy solution optimal? (10 pts)

2. (a) Assume that you are given a collection of sets $X_1, X_2, \ldots, X_n$ of natural numbers from 1 to M. Every natural number from 1 to M is contained in at least one of these sets $X_i$, but some numbers might be contained in several such sets. Design a greedy algorithm which attempts to select minimal number of sets $X_i$ such that their union still contains all numbers from 1 to M (i.e., every number from 1 to M is contained in at least one of these selected sets. (20 pts)

(b) Give an example of a collection of sets for which the greedy algorithm does not produce optimal solution, i.e., when it is possible to select fewer sets than what greedy algorithm selects. (10 pts)

3. Multiply the following pairs of polynomials using as few multiplications of large numbers as possible (large numbers are those which depend on the coefficients $a_i$ Your end result should be in the standard coefficient form like $P$ and $Q$ in case (a).

(a) $P(x) = a_0 + a_2x^2 + a_4x^4 + a_6x^6;$
$Q(x) = b_0 + b_4x^4 + b_6x^6 + b_8x^8 \quad$ (10pts);

(b) $P(x) = a_0 + a_{17}x^{17} + a_{19}x^{19} + a_{21}x^{21};$
$Q(x) = b_0 + b_{17}x^{17} + b_{19}x^{19} + b_{21}x^{21}; \quad$ (10pts)

(c) $P(x) = a_0 + a_{100}x^{100}; \quad Q(x) = a_1x + a_{99}x^{99} \quad$ (10pts).

4. Apply the Master Theorem to determine the asymptotic growth rate of the solutions to the following recurrences. (turn other page)

**The Master Theorem:**

Let $a$, $b$ be constants such that $a \geq 1$ and $b > 1$, let $f(n)$ be a function and let $T(n)$ be defined on natural numbers by the recurrence $T(n) = a\,T(\lfloor n/b \rfloor) + f(n)$. Then $T(n)$ can be bounded asymptotically as follows :

1) If $f(n) = O(n^{\log_b a - \epsilon})$ for some $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$

2) If $f(n) = \Theta(n^{\log_b a})$ then $T(n) = \Theta(n^{\log_b a} \log n)$

3) If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some $\epsilon > 0$ and if there exist a constant $c < 1$ and a sufficiently large $n_0$ such that $a\,f(n/b) \leq c\,f(n)$ for all $n \geq n_0$, then $T(n) = \Theta(f(n))$.

(a) $T(n) = 2T(n/2) + n + \sin \pi n$; (10 pts)
(b) $T(n) = T(1/8n) + 8$; (10 pts)
(c) $T(n) = 64T(n/4) + n^{\log n}$; (10 pts)

5. Every square of an $8 \times 8$ chess board contains an integer. You start from the top left corner and on each move you can go either one square down or one square to the right (if possible); you stop when you reach any square on the bottom row. You get as much money as the sum of all numbers in the squares which you have visited. Design an algorithm which, given such a table, determines a sequence of moves which maximizes your pay off. (30 pts)

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6. Show that in the Master Theorem condition that $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some $\epsilon > 0$ is redundant because it follows from the condition that there exist a constant $c < 1$ and a sufficiently large $n_0$ such that $a\,f(n/b) \leq c\,f(n)$ for all $n \geq n_0$.