(compiled by Nina Narodytska)

Try solving each problem without looking at the hints. After you put a serious effort, if you cannot solve it, try again after reading Hints but not reading More Hints. After reading More Hints you should be able to do it if you try hard.

1. (*Kvant 2007* a Russian math journal for the high school kids)

   Alice and Bob perform a trick for the audience. Alice leaves the room. Someone from the audience writes an arbitrary sequence $[s_1, \ldots, s_{101}]$ of 101 decimal digits on a blackboard. Bob covers two consecutive digits of this sequence so that they are not visible any more. Alice enters the room and guesses the two closed digits by inspecting the remaining sequence.

   (a) Propose a strategy for Alice and Bob so that Alice can always reproduce the hidden digits.

   (b) Prove that no deterministic strategy exists for sequences of length < 101.

   Hint: After Bob covers two consecutive numbers, Alice is left with two pieces of information: the remaining sequence and the position of covered digits. Come up with a strategy for Alice and Bob such that these two pieces of information uniquely encode the starting sequence.

   More Hint: Try splitting the sequence into even and odd index sub sequences, and encode the $\bmod (10)$ values of the sums of the two sub sequences by choosing appropriately which two indices in the arrays you should cover.

   (b) Assume that the length N of the array is less than 101. There are 10^{N-2} sequences of length $N-2$, and for each such sequence there are altogether $N-1$ positions where the covered digits could be (in front, on the back or somewhere in between $N-2$ numbers - all together $N-1$ positions). Thus, there at most $(n-1)10^{N-2}$ partially covered distinct sequences obtained from sequences of length N by covering two consecutive digits. To be able to identify uniquely which of 10^N sequences of length N such partly covered sequence comes from, total number of sequences of length N, i.e., 10^N must be smaller than total number of resulting partly covered sequences. Draw your conclusion from there.

2. (*Kvant, 1971*)

   Alice and Bob play a game. They start with a sequence of consecutive 4097 numbers, $0, \ldots 4096$. At the first step Alice removes any 2048 numbers. At the second step Bob removes any 1024 numbers from the remaining sequence of numbers. At the third step Alice removes 512 numbers. At the fourth step Bob removes 256 numbers and so on. At the 11th step Alice removes one number from the remaining three numbers. Let $a < b$ be the last two numbers. Bob pays the difference of $b - a$ to Alice.

   (a) Propose a strategy for Alice that guarantees the biggest worst-case gain, regardless of Bob's strategy.

   (b) Propose a strategy for Bob that guarantees the lowest worst-case loss, regardless of Alice's strategy.

Hints: Try the greedy strategy both for Alice and for Bob: at each stage Alice tries to at least double the minimal difference between any two remaining numbers. What do you think, what is the easiest way to do it? At each stage Bob tries to minimize the maximal difference between any of the remaining numbers by making sure maximal difference between the remaining numbers will be at least halved. What is the simplest way for Bob to achieve this?

More Hints: What happens with the minimal difference between any two remaining numbers if at each stage Alice crosses out every other number of the remaining sequence? What happens with the maximal difference between any two remaining numbers if at each stage Bob splits the sequence into two, such that every number in the first half is smaller than every number in the second half, and takes the one where the difference between the smallest and the largest number is smaller?

3. Let $M(i,j), \quad i = 1 \ldots n, \quad j = 1 \ldots n$ be a matrix of distinct integers. Each row and each column of the matrix is in an increasing order, so that for each row i:
$M(i,1) < M(i,2) < \ldots < M(i,n),\ i = 1 \ldots n;$
and for each column j:
$M(1,j) < M(2,j) < \ldots < M(n,j),\ j = 1 \ldots n.$

   You need to determine whether M contains an integer X in O(n) time.

   Hint: The number which is the last number in the first row is the largest in the first row and the smallest in the last column.

4. Two children, A and B wish to split n cakes. A always cuts the cakes in two and then B can either choose first or let A chose first. B gets to chose first n-1 times, and must let A chose first once. How should A cut the cakes to ensures he gets as much as possible.

   Hint: Try to solve the problem for n = 2 and n = 3 first, and then generalize.

5. Cut a rectangular cake with icing on the top and on each side into n equal pieces that contain equal amount of cake and equal amount of icing.

6. You are playing cards with a group of n-1 other people and you are dealing cards in the usual circular order. Then the phone rings and you stop dealing to answer it; when you get back no one can remember who got the last card. Can you ensure everyone gets the same number of cards WITHOUT counting any cards? Can you ensure that not only everyone gets the same number of cards but also exactly the SAME cards that he would have gotten if the dealing was not interrupted?

7. Order cards $1, 2, 3, \ldots 10$ in a stack such that if you turn the top card in the stack face up and put it on the table, then put the next card in the stack on the bottom of the stack and keep repeating that procedure, the card on the table appear in the order $1, 2, 3, \ldots 10$

An interesting additional problem about recursion:

(a) Assume that you are in a 100 floor building and that you are given 2 identical balls which you can throw 14 times altogether. Show that you can determine what the highest floor is in this building from which you can throw such balls without breaking them. Show that 13 throws might not be enough.

(b) What is the largest number of floors a building can have and still always be possible to determine what the highest floor is in such a building from which you can throw identical balls without breaking them, if you have k balls and can throw them n times? You are allowed to break all available balls to find such a floor.

Hint: Try to think recursively, rather than trying to see how to do it directly!