Algorithms
COMP3121/3821/9101/9801
2. DIVIDE-AND-CONQUER

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Another balance puzzle

- **Setting:** We are given 27 coins of the same denomination; we know that one of them is counterfeit and that it is lighter than the others.
- **Task:** Find the counterfeit coin by weighing coins on a pan balance only three times.

- This method is called "divide-and-conquer".
- We have already seen a prototypical "serious" algorithm designed using such a method: the MERGE-SORT.
- We split the array into two, sort the two parts recursively and then merge the two sorted arrays.
- We now look at a closely related but more interesting problem of counting inversions in an array.

Counting the number of inversions

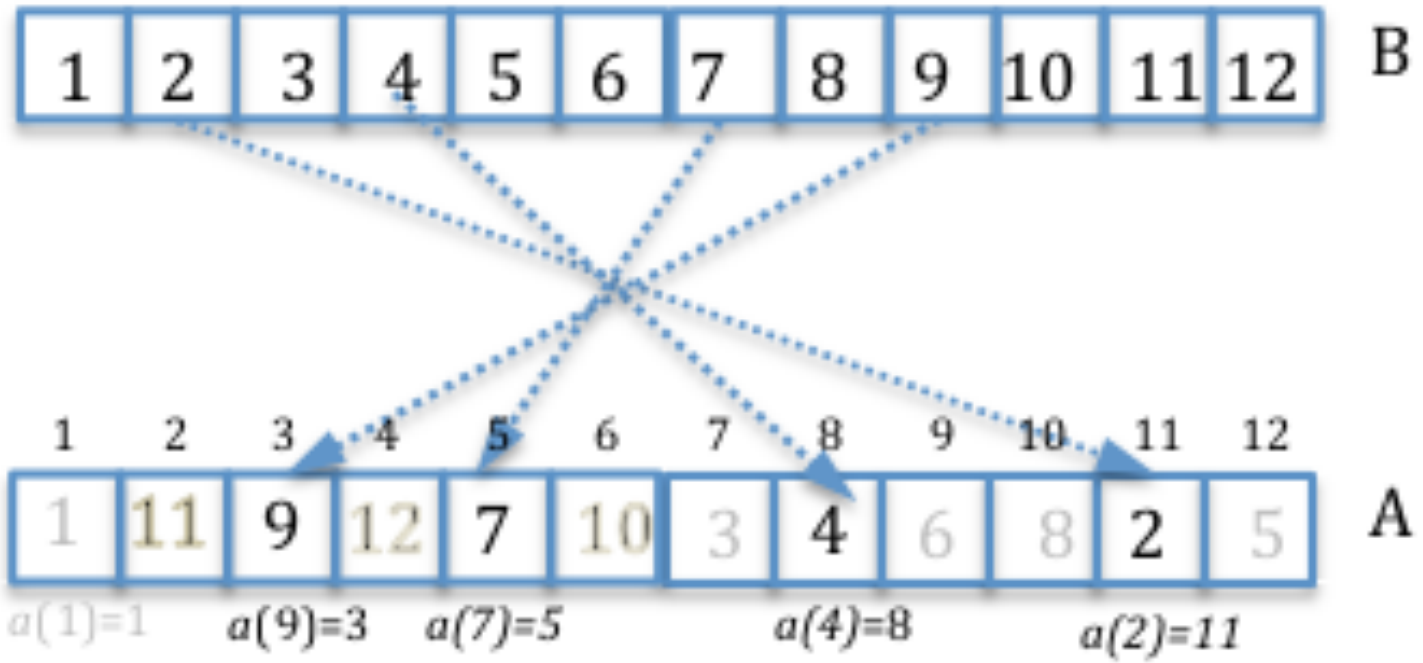
- Assume that you have m users ranking the same set of n movies. You want to determine for any two users A and B how similar their tastes are (*e.g.*, in order to make a recommender system).
- How can we measure the degree of similarity of two users A and B?
- Lets enumerate the movies on the ranking list of user B by assigning to the top choice of user B index 1, assign to his second choice index 2 and so on.

- For the i^{th} movie on $B's$ list we can now look at the position (i.e., index) of that movie on $A's$ list, denoted by $a(i)$.

Counting the number of inversions

- Thus, if the first listed movie on $B's$ list is the second movie on $A's$ list, then $a(1) = 2$; if the second choice of user B is the fifth choice of user A, then $a(2) = 5$, and so on.

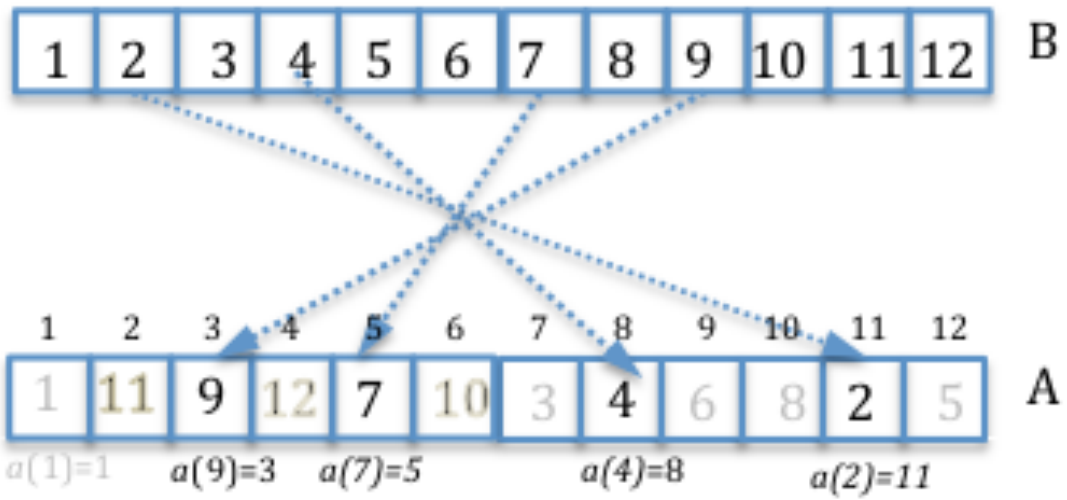
- Note that in this way for all i, $1 \leq i \leq n$, $A[a(i)] = i$.



Counting the number of inversions

- A good measure of how different these two users are, is the total number of *inversions*, i.e., total number of pairs of movies i, j such that movie i precedes movie j on $B's$ list but movie j is higher up on $A's$ list than the movie i.

- In other words, we count the number of pairs of movies i, j such that $i < j$ (movie i precedes movie j on $B's$ list) but $a(i) > a(j)$ (movie i is in the position $a(i)$ on $A's$ list.



- *E.g.*, 1 and 2 do not form an inversion because $a(1) < a(2)$ ($a(1) = 1$ and $a(2) = 11$ because $a(1)$ is on the first and $a(2)$ is on the eleventh place in A);

- However, for example 4 and 7 do form an inversion because $a(7) < a(4)$ ($a(7) = 5$ because $a(7)$ is on the fifth place in A and $a(4) = 8$)

Counting the number of inversions

- An easy way to count the total number of inversions between two lists is by looking at all pairs $i < j$ of movies on one list and determining if they are inverted in the second list, but this would produce a quadratic time algorithm, $T(n) = \Theta(n^2)$.

- We now show that this can be done in a much more efficient way, in time $O(n \log n)$, by applying a DIVIDE-AND-CONQUER strategy.

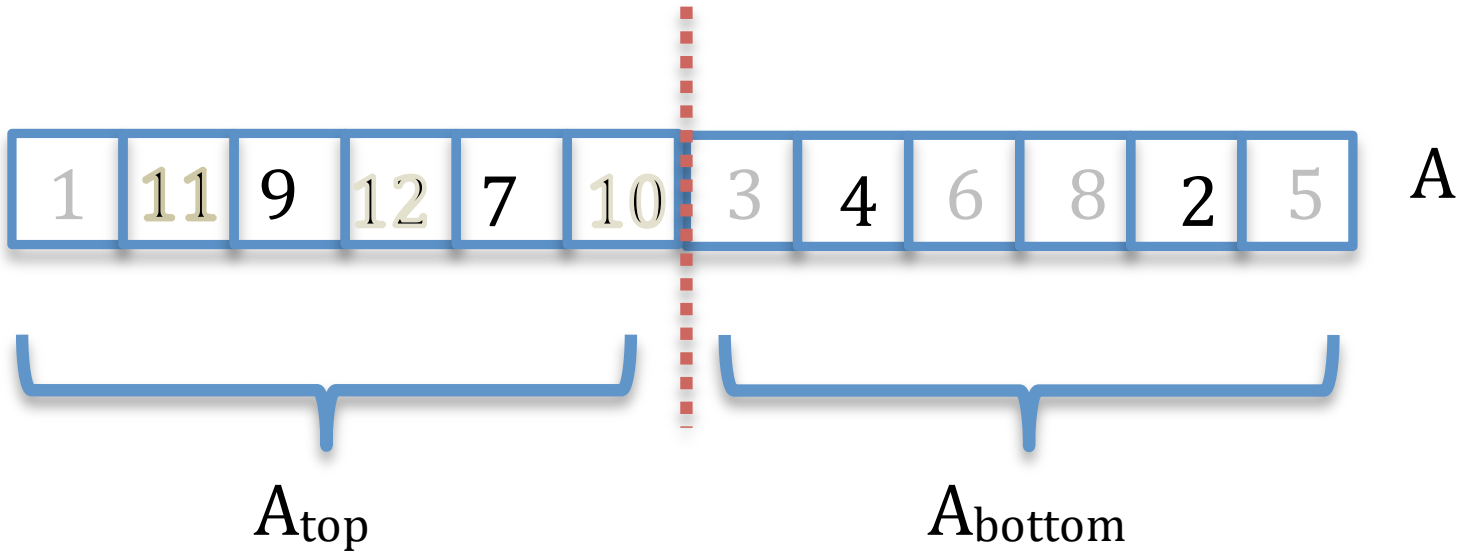
- Clearly, since the total number of pairs is quadratic in n, we cannot afford to inspect all possible pairs.

- The main idea is to tweak the MERGE-SORT algorithm, by extending it to recursively both sort an array A **and** determine the number of inversions in A.

Counting the number of inversions

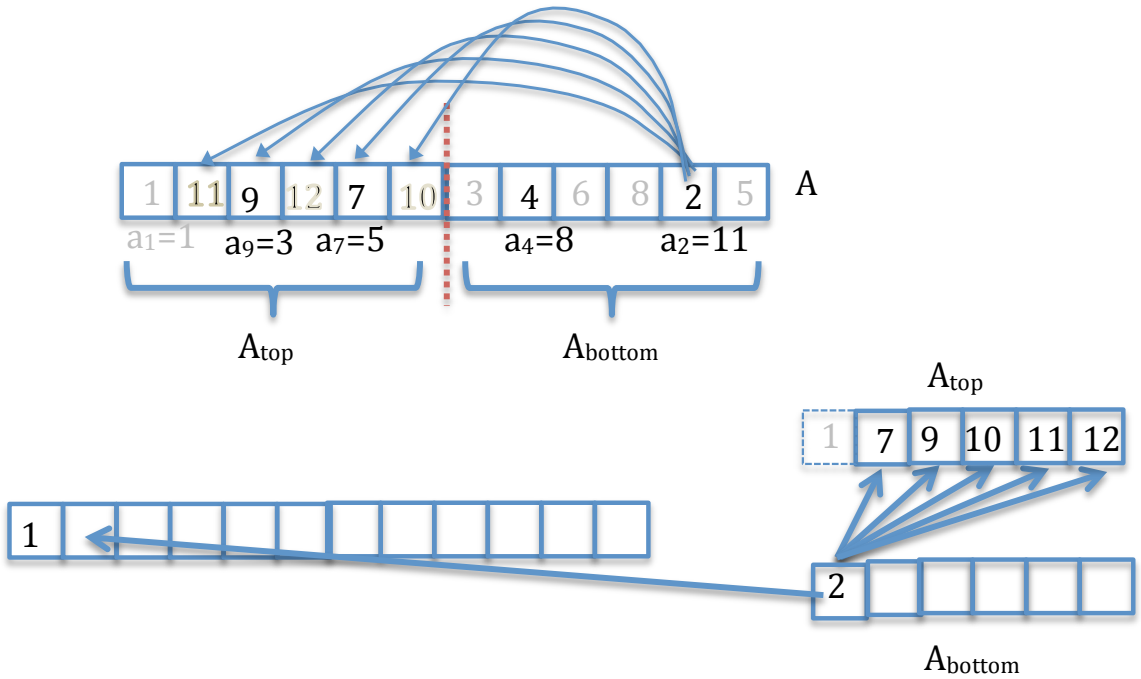
- We split the array A into two (approximately) equal parts $A_{top} = A[1 \ldots \lfloor n/2 \rfloor]$ and $A_{bottom} = A[\lfloor n/2 \rfloor + 1 \ldots n]$.

- Note that the total number of inversions in array A is equal to the sum of the number of inversions $I(A_{top})$ in A_{top} (such as 9 and 7) plus the number of inversions $I(A_{bottom})$ in A_{bottom} (such as 8 and 2) plus the number of inversions $I(A_{top}, A_{bottom})$ across the two halves (such as 7 and 4).



Counting the number of inversions

- We now recursively sort arrays A_{top} and A_{bottom} and obtain the number of inversions $I(A_{top})$ in the sub-array A_{top} and the number of inversions $I(A_{bottom})$ in the sub-array A_{bottom}.
- We now merge the two sorted arrays A_{top} and A_{bottom} while counting the number of inversions $I(A_{top}, A_{bottom})$ which are across the two sub-arrays.
- When the next smallest element among all elements in both arrays is an element in A_{bottom}, such an element clearly is in an inversion with all the remaining elements in A_{top} and we add the total number of elements remaining in A_{top} to the current value of the number of inversions across A_{top} and A_{bottom}.



Counting the number of inversions

- Whenever the next smallest element among all elements in both arrays is an element in A_{top}, such an element clearly is not involved in any inversions across the two arrays (such as 1, for example).

- After the merging operation is completed, we obtain the total number of inversions $I(A_{top}, A_{bottom})$ across A_{top} and A_{bottom}.

- The total number of inversions $I(A)$ in array A is finally obtained as:

$$I(A) = I(A_{top}) + I(A_{bottom}) + I(A_{top}, A_{bottom})$$

- **Next:** we study applications of divide and conquer to arithmetic of very large integers.

Basics revisited: how do we add two numbers?

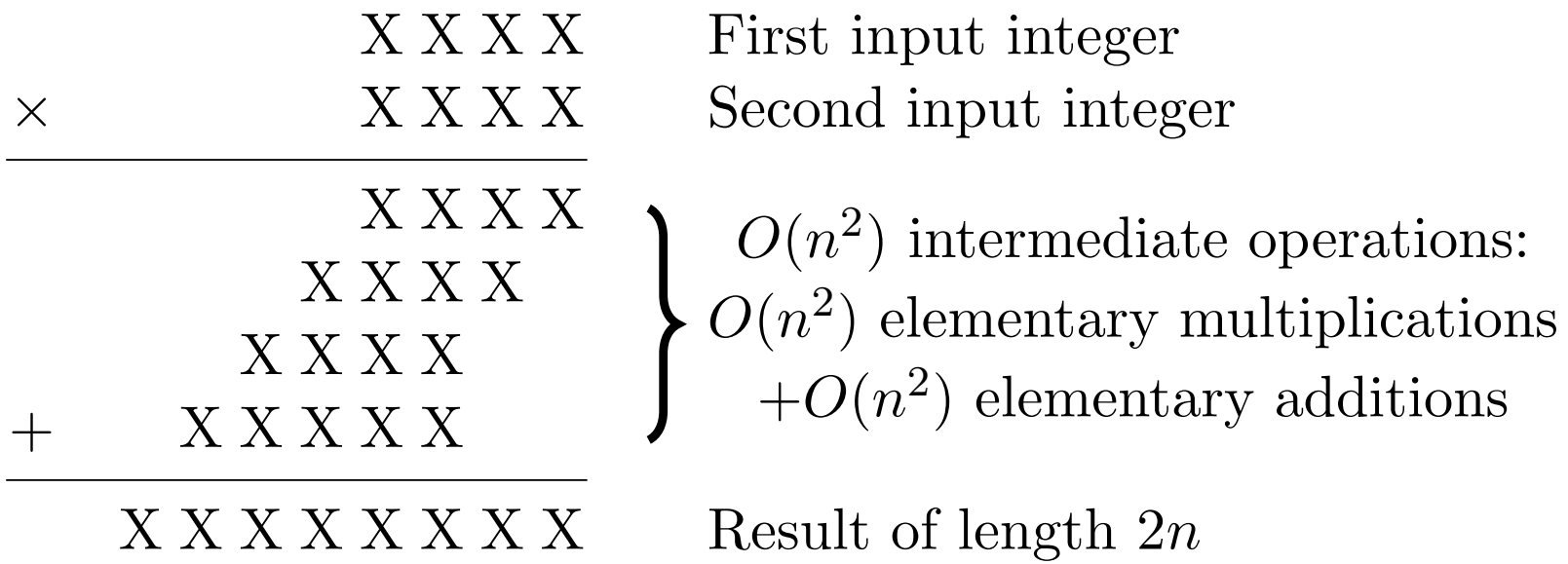
	C C C C C	Carry
	X X X X X	First integer
+	X X X X X	Second integer
	X X X X X X	Result

- adding 3 bits can be done in constant time;
- the whole algorithm runs in linear time i.e., $O(n)$ many steps.

Can we do it faster than in linear time?

- no, because we have to read every bit of the input.
- $\rightarrow$ no asymptotically faster algorithm.

Basics revisited: how do we multiply two numbers?



- We assume that two `X`'s can be multiplied in $O(1)$ time (each `X` could be a bit or a digit in some other base).
- Thus the above procedure runs in time $O(n^2)$.
- Can we do it in **LINEAR** time, like addition?
- **No one knows!**
- "Simple" problems can actually turn out to be difficult!

Can we do multiplication faster than $O(n^2)$?

Let us try a divide-and-conquer algorithm:
take our two input numbers A and B, and split them into two halves:

$$A = A_1 2^{\frac{n}{2}} + A_0 \qquad \underbrace{XX\ldots X}_{\frac{n}{2}}\underbrace{XX\ldots X}_{\frac{n}{2}}$$

$$B = B_1 2^{\frac{n}{2}} + B_0$$

- A_0, B_0 - the least significant bits; A_1, B_1 the most significant bits.
- AB can now be calculated as follows:

$$AB = A_1 B_1 2^n + (A_1 B_0 + B_1 A_0) 2^{\frac{n}{2}} + A_0 B_0 \qquad (1)$$

What we mean is that the product AB can be calculated recursively by the following program:

```
mult(A, B)
    if |A| = |B| = 1 then
        return AB
    else
        A_1 ← MoreSignificantPart(A)
        A_0 ← LessSignificantPart(A)
        B_1 ← MoreSignificantPart(B)
        B_0 ← LessSignificantPart(B)
        X ← mult(A_0, B_0)
        Y ← mult(A_0, B_1)
        Z ← mult(A_1, B_0)
        W ← mult(A_1, B_1)
        return W 2^n + (Y + Z) 2^{n/2} + X
```

How many steps does this algorithm take?

Each multiplication of two n digit numbers is replaced by four multiplications of $n/2$ digit numbers: A_1B_1, A_1B_0, B_1A_0, A_0B_0, plus we have a **linear** overhead to shift and add:

$$T(n) = 4T\left(\frac{n}{2}\right) + c\,n \tag{2}$$

Can we do multiplication faster than $O(n^2)$?

Claim:

If $T(n)$ satisfies $T(n) = 4T\left(\frac{n}{2}\right) + cn$ then $T(n) = (c+1)n^2 - cn$

Proof by induction.

Base step: $n = 1$

$\rightarrow$ Indeed $T(1) = 1 = (c+1) \times 1^2 - c \times 1$

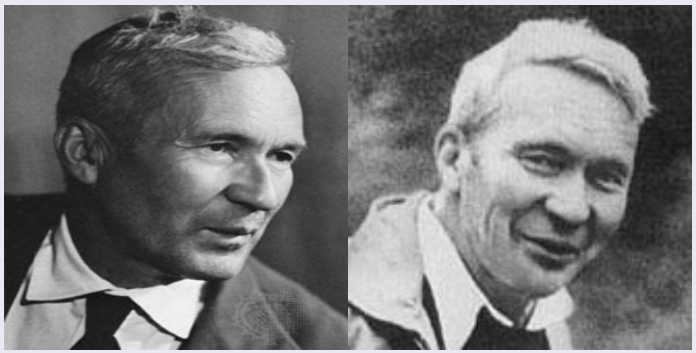
Heredity step: assume it holds for $\lfloor n/2 \rfloor$ and prove it also holds for n:

$$
\begin{array}{lll}
\text{If} & T\left(\frac{n}{2}\right) & = (c+1)\left(\frac{n}{2}\right)^2 - c\frac{n}{2} \\
\text{and} & T(n) & = 4T\left(\frac{n}{2}\right) + cn \\
\text{Then} & T(n) & = 4\left((c+1)\left(\frac{n}{2}\right)^2 - c\frac{n}{2}\right) + cn \\
 & & = (c+1)n^2 - 2cn + cn \\
 & & = (c+1)n^2 - cn
\end{array}
$$

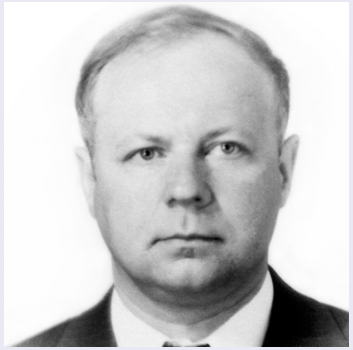
Can we do multiplication faster than $O(n^2)$?

Thus, if $T(n) = 4\,T\left(\frac{n}{2}\right) + c\,n$ then $T(n) = (c+1)n^2 - c\,n = O(n^2)$
i.e., we gained **nothing** with our divide-and-conquer!

Some history



In 1952, one of the most famous



mathematicians of the 20^{th} century, Kolmogorov, conjectured that you cannot multiply in less than $\Omega(n^2)$ elementary operations. In 1960, Karatsuba, then a 23-year-old student, found an algorithm (later it was called "divide and conquer") that multiplies two n-digit numbers in $\Theta(n^{\log_2 3}) \approx \Theta(n^{1.58\ldots})$ elementary steps, thus disproving the conjecture! Kolmogorov was "surprised"! A bit later, Kolmogorov wrote a paper about the result, listing Karatsuba as an author without informing him. In 1962, the paper was published and Karatsuba

The Karatsuba trick

How did Karatsuba do it?

Take again our two input numbers A and B, and split them into halves:

$$A = A_1 2^{\frac{n}{2}} + A_0 \qquad \underbrace{XX\ldots X}_{\frac{n}{2}}\underbrace{XX\ldots X}_{\frac{n}{2}}$$

$$B = B_1 2^{\frac{n}{2}} + B_0$$

The product can now be calculated as follows:

$$\begin{aligned} AB &= A_1B_1 2^n + (A_1B_0 + A_0B_1)2^{\frac{n}{2}} + A_0B_0 \\ &= A_1B_1 2^n + ((A_1 + A_0)(B_1 + B_0) - A_1B_1 - A_0B_0)2^{\frac{n}{2}} + A_0B_0 \\ &= A_1B_1 2^n + (UV - A_1B_1 - A_0B_0)2^{\frac{n}{2}} + A_0B_0 \end{aligned}$$

where $U = A_1 + A_0$ and $V = (B_1 + B_0)$

Thus, the algorithm will look like this:

```
mult(A, B)
    if |A| = |B| = 1 then
        return AB
    else
        A_1 ← MoreSignificantPart(A)
        A_0 ← LessSignificantPart(A)
        B_1 ← MoreSignificantPart(B)
        B_0 ← LessSignificantPart(B)
        U ← A_0 + A_1
        V ← B_0 + B_1
        X ← mult(A_0, B_0)
        W ← mult(A_1, B_1)
        Y ← mult(U, V)
        return W 2^n + (Y − X − W) 2^{n/2} + X
```

How many multiplications does this take? (addition is in linear time!)
We need A_1B_1, A_0B_0 and $(A_1 + A_0)(B_1 + B_0)$; thus

$$T(n) = 3\,T\left(\frac{n}{2}\right) + c\,n$$

The Karatsuba trick

$$\begin{aligned}
\text{Since} \quad & T(n) = 3\,T\left(\tfrac{n}{2}\right) + c\,n \\
\text{and} \quad & T\left(\tfrac{n}{2}\right) = 3\,T\left(\tfrac{n}{2^2}\right) + c\,\tfrac{n}{2} \\
\text{and} \quad & T\left(\tfrac{n}{2^2}\right) = 3\,T\left(\tfrac{n}{2^3}\right) + c\,\tfrac{n}{2^2} \\
\text{and} \quad & \ldots
\end{aligned}$$

$$\begin{aligned}
\text{we get} \quad T(n) &= 3\,\underbrace{T\left(\frac{n}{2}\right)}+c\,n \\
&= 3\left(3T\left(\tfrac{n}{2^2}\right) + c\tfrac{n}{2}\right) + c\,n = 3^2 T\left(\tfrac{n}{2^2}\right) + c\tfrac{3n}{2} + c\,n = 3^2\,\underbrace{T\left(\frac{n}{2^2}\right)}+cn\left(\tfrac{3}{2}+1\right) \\
&= 3^2\left(3T\left(\tfrac{n}{2^3}\right) + c\tfrac{n}{2^2}\right) + cn\left(\tfrac{3}{2}+1\right) = 3^3\,\underbrace{T\left(\frac{n}{2^3}\right)}+cn\left(\tfrac{3^2}{2^2}+\tfrac{3}{2}+1\right) \\
&= 3^3\left(3T\left(\tfrac{n}{2^4}\right) + c\tfrac{n}{2^3}\right) + cn\left(\tfrac{3^2}{2^2}+\tfrac{3}{2}+1\right) = 3^4 T\left(\tfrac{n}{2^4}\right) + cn\left(\tfrac{3^3}{2^3}+\tfrac{3^2}{2^2}+\tfrac{3}{2}+1\right) \\
&= \ldots \\
&= 3^{\lfloor \log_2 n \rfloor} T\left(\frac{n}{\lfloor 2^{\log_2 n} \rfloor}\right) + cn \sum_{k=0}^{\lfloor \log_2 n \rfloor - 1} \left(\frac{3}{2}\right)^k
\end{aligned}$$

Reminder: geometric series

Claim:

$$\text{If } r \neq 1 \text{ then} \qquad \sum_{k=0}^{m} r^k = \frac{r^{m+1} - 1}{r - 1}$$

Proof.

$$\begin{aligned}(r-1)\sum_{k=0}^{m} r^k &= (r-1)(r^m + r^{m-1} + \cdots + r + 1) \\ &= (r^{m+1} + r^m + \cdots + r^2 + r) - (r^m + r^{m-1} + \cdots + r + 1) \\ &= r^{m+1} - 1\end{aligned}$$

□

In our case, $r = \frac{3}{2}$ and $m = \log_2 n - 1$.

Reminder: basic logarithm identity

Claim:

$$\text{If } a, b, c > 0 \text{ then} \qquad a^{\log_b c} = c^{\log_b a}$$

Proof.

$$\log_b c \cdot \log_b a = \log_b a \cdot \log_b c \qquad \text{(because } \times \text{ is commutative)}$$
$$\log_b(a^{\log_b c}) = \log_b(c^{\log_b a}) \qquad \text{(because } y \log_b x = \log_b x^y\text{)}$$
$$a^{\log_b c} = c^{\log_b a} \qquad \text{($\log_b$ is injective: } \log_b y = \log_b x \Rightarrow y = x\text{)}$$

□

In our case, we will use $a = 3$, $b = 2$ and $c = n$, so $3^{\log_2 n} = n^{\log_2 3}$
and $a = \frac{3}{2}$, $b = 2$ and $c = n$, so $\left(\frac{3}{2}\right)^{\log_2 n} = n^{\log_2 \frac{3}{2}} = n^{-1+\log_2 3}$

The Karatsuba trick

$$
\begin{aligned}
T(n) &= 3^{\lfloor \log_2 n \rfloor} T\left(\frac{n}{\lfloor 2^{\log_2 n} \rfloor}\right) + cn \sum_{k=0}^{\lfloor \log_2 n \rfloor - 1} \left(\frac{3}{2}\right)^k \\
&\approx 3^{\log_2 n} T(1) + c\, n \frac{\left(\frac{3}{2}\right)^{\log_2 n} - 1}{\frac{3}{2} - 1} \\
&\approx 3^{\log_2 n} T(1) + 2c\, n \left(\frac{3}{2}\right)^{\log_2 n} - 2cn \\
&\approx n^{\log_2 3} T(1) + 2c\, n^{\log_2 3} - 2cn \\
&= O(n^{\log_2 3}) = O(n^{1.58\ldots}) \ll n^2
\end{aligned}
$$

Please review the basic properties of logarithms and the asymptotic notation from the review material (the first item at the class webpage under "class resources".)

A Karatsuba style trick also works for matrices: Strassen's algorithm for faster matrix multiplication

- If we want to multiply two $n \times n$ matrices A and B, the product will be a matrix C also of size $n \times n$. To obtain each of n^2 entries in C we do n multiplications, so matrix product by brute force is $\Theta(n^3)$.

- However, we can do it faster using Divide-And-Conquer;

- We split each matrix into four blocks of (approximate) size $n/2 \times n/2$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \tag{3}$$

- We obtain: $a\,e + b\,g = r$ $\qquad$ $a\,f + b\,h = s$
  $c\,e + d\,g = t$ $\qquad$ $c\,f + d\,h = u$

- Prima facie, there are 8 matrix multiplications, each running in time $T\left(\frac{n}{2}\right)$ and 4 matrix additions, each running in time $O(n^2)$, so such a direct calculation would result in time complexity governed by the recurrence

$$T(n) = 8T\left(\frac{n}{2}\right) + c\,n^2$$

- The first case of the Master Theorem gives $T(n) = \Theta(n^3)$, so nothing gained.

Strassen's algorithm for faster matrix multiplication

- However, we can instead evaluate:

$$A = a\,(f - h); \qquad B = (a + b)\,h; \qquad C = (c + d)\,e \qquad D = d\,(g - e);$$
$$E = (a + d)(e + h); \quad F = (b - d)(g + h); \quad H = (a - c)(e + f).$$

- We now obtain

$$\begin{aligned} E + D - B + F &= (ae + de + ah + dh) + (dg - de) - (ah + bh) + (bg - dg + bh - dh) \\ &= a\,e + b\,g = r; \\ A + B &= (a\,f - a\,h) + (a\,h + b\,h) = a\,f + b\,h = s; \\ C + D &= (c\,e + d\,e) + (d\,g - d\,e) = c\,e + d\,g = t; \\ E + A - C - H &= (ae + de + ah + dh) + (af - ah) - (ce + de) - (ae - ce + af - cf) \\ &= c\,f + d\,h = u. \end{aligned}$$

- We have obtained all 4 components of C using only 7 matrix multiplications and 18 matrix additions/subtractions.
- Thus, the run time of such recursive algorithm satisfies $T(n) = 7T(n/2) + O(n^2)$ and the Master Theorem yields $T(n) = \Theta(n^{\log_2 7}) = O(n^{2.808})$.
- In practice, this algorithm beats the ordinary matrix multiplication for $n > 32$.

Next time:

1. Can we multiply large integers faster than $O\left(n^{\log_2 3}\right)$??
2. Can we avoid messy computations like:

$$
\begin{aligned}
T(n) = \quad & 3T\left(\frac{n}{2}\right) + c\,n \\
= \quad & 3\left(3T\left(\frac{n}{2^2}\right) + c\frac{n}{2}\right) + c\,n = 3^2 T\left(\frac{n}{2^2}\right) + c\frac{3n}{2} + c\,n = 3^2 T\left(\frac{n}{2^2}\right) + cn\left(\frac{3}{2}+1\right) \\
= \quad & 3^2\left(3T\left(\frac{n}{2^3}\right) + c\frac{n}{2^2}\right) + cn\left(\frac{3}{2}+1\right) = 3^3 T\left(\frac{n}{2^3}\right) + cn\left(\frac{3^2}{2^2}+\frac{3}{2}+1\right) \\
= \quad & 3^3\left(3T\left(\frac{n}{2^4}\right) + c\frac{n}{2^3}\right) + cn\left(\frac{3^2}{2^2}+\frac{3}{2}+1\right) = 3^4 T\left(\frac{n}{2^4}\right) + cn\left(\frac{3^3}{2^3}+\frac{3^2}{2^2}+\frac{3}{2}+1\right) \\
= \quad & \ldots \\
= \quad & 3^{\lfloor \log_2 n \rfloor} T\left(\frac{n}{\lfloor 2^{\log_2 n} \rfloor}\right) + cn \sum_{k=0}^{\lfloor \log_2 n \rfloor - 1} \left(\frac{3}{2}\right)^k \\
\approx \quad & 3^{\log_2 n} T(1) + c\,n\,\frac{\left(\frac{3}{2}\right)^{\log_2 n} - 1}{\frac{3}{2} - 1} \\
\approx \quad & 3^{\log_2 n} T(1) + 2c\,n\left(\frac{3}{2}\right)^{\log_2 n} - 2cn \\
\approx \quad & n^{\log_2 3} T(1) + 2c\,n^{\log_2 3} - 2cn \\
= \quad & O(n^{\log_2 3}) = O(n^{1.58\ldots}) \ll n^2
\end{aligned}
$$

That's All, Folks!!