Extended Algorithms Midterm Problems

Two questions, marked out of a total of $10 + 10 =$ **20 marks**.

1. [**10 marks**] Suppose that you are taking care of n kids, who took their shoes off. You have to take the kids out and it is your task to make sure that each kid is wearing a pair of shoes of the right size (not necessarily their own, but one of the same size). All you can do is to try to put a pair of shoes on a kid, and see if they fit, are too large or too small; you are **not** allowed to compare a shoe with another shoe or a foot with another foot.

   Describe an algorithm whose expected number of shoe trials is $O(n \log n)$ which properly fits shoes on every kid.

   Hint: you could try to devise a modification of the QuickSort *algorithm, or, perhaps a "double version" of it, with switching between two kinds of pivots.*

2. Suppose you have n boolean variables $x_1, x_2, \ldots, x_n$. You are given a set of k *clauses*, each of the form (l_1 **or** l_2 **or** l_3) where l_1, l_2 and l_3 are each either x_i or (**not** x_i) for one of the boolean variables x_i. Additionally, l_1, l_2 and l_3 are all different within the same clause.

   For instance, when $n = 4$,

   (x_1 **or** (**not** x_4) **or** (**not** x_3)) as well as (x_3 **or** (**not** x_4) **or** (**not** x_3))

   are both valid clauses, but

   (x_2 **or** (**not** x_4) **or** x_2)

   is not, because x_2 is repeated (note that a valid clause can contain both x_i and (**not** x_i) but it can contain neither x_i twice nor (**not** x_i) twice.

   An *assignment* is simply a mapping of either `true` or `false` to each x_i. We say an assignment *satisfies* a clause if it causes the clause to evaluate to `true`. Note that, if a clause contains both x_i and (**not** x_i) for some i, then such a clause is satisfied by every assignment.

   Suppose we generate an assignment by choosing the value of each x_i **independently** by flipping a fair coin n times (i.e., for each variable x_i we pick either `true` or `false`, each with probability $1/2$).

   (a) [**2 marks**] Show that the probability of satisfying an arbitrary clause is *at least* $7/8$.

   (b) [**4 marks**] Hence, show that the expected number of satisfied clauses is $7k/8$. You may wish to use *linearity of expectations*, which states that $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$ for any random variables X and Y, regardless of whether they are independent or not.

   (c) [**4 marks**] Hence, explain why there always exists some assignment satisfying *at least* $7k/8$ clauses.