COMP3121/3821/9101/9801 (07s1)
April 24, 2007

Midterm Exam – 3 hrs

PLEASE write **CLEARLY**, show **ALL** of your work and **EXPLAIN** everything in detail!

<u>The Master Theorem:</u>
Let a, b be constants such that $a \geq 1$ and $b > 1$, let $f(n)$ be a function and let $T(n)$ be defined on natural numbers by the recurrence $T(n) = a\,T(\lfloor n/b \rfloor) + f(n)$. Then $T(n)$ can be bounded asymptotically as follows :

1) If $f(n) = O(n^{\log_b a - \varepsilon})$ for some $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$

2) If $f(n) = \Theta(n^{\log_b a})$ then $T(n) = \Theta(n^{\log_b a} \log n)$

3) If $f(n) = \Omega(n^{\log_b a+\varepsilon})$ for some $\varepsilon > 0$, and if $a\,f(n/b) \leq c\,f(n)$ for some constant $c < 1$ all $n \geq n_0$ for some sufficiently large n_0, then $T(n) = \Theta(f(n))$.

1. In each case determine the asymptotic growth rate of the solution $T(n)$ to the recurrence. (each case 5 points)
    a. $T(n)=4T(n/4) + n \log n^3$
    b. $T(n) = 3T(n/4)+ n$
    c. $T(n)=2T(n/2)+n \log n$
    d. $T(n)=T(3n/8) +6 \sqrt{n}$
    e. $T(n)=T(n-2) + \log_2 \sqrt{n}$

2. There is a line of 111 stalls, some of which need to be covered with boards. You can use up to 11 boards, each of which may cover any number of consecutive stalls. Cover all the necessary stalls, while covering as few total stalls as possible. (30 pts)

3. Given two sequences of letters A and B, find if B is a subsequence of A in the sense that one can delete some letters from A and obtain the sequence B. (20 pts)

4. Square a complex number $a+ib$ using only two multiplications of real numbers. (15 pts)

5. Fibonacci numbers are defined by : $F(0)=0$, $F(1)=1$, $F(n)=F(n-1)+F(n-2)$ for $n \geq 2$. Thus, the sequence of $F(n)$ looks as follows: 0 1 1 2 3 5 8 13 21 …

    a. Show that $\begin{pmatrix} F(n+1) & F(n) \\ F(n) & F(n-1) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n$ (20 pts)

    b. Find $F(n)$ in $\Theta(\log_2 n)$ steps. Note that any addition or multiplication are considered a single step. (20 pts)