COMP3121/3821/9101/9801 (06s1)

Midterm Exam – 3 hrs for 4 problems

PLEASE write **CLEARLY**, show **ALL** of your work and **EXPLAIN** everything in detail!

The Master Theorem:

Let a, b be constants such that $a \geq 1$ and $b > 1$, let $f(n)$ be a function and let $T(n)$ be defined on natural numbers by the recurrence $T(n) = a\,T(\lfloor n/b \rfloor) + f(n)$. Then $T(n)$ can be bounded asymptotically as follows :

1) If $f(n) = O(n^{\log_b a - \varepsilon})$ for some $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$

2) If $f(n) = \Theta(n^{\log_b a})$ then $T(n) = \Theta(n^{\log_b a} \log n)$

3) If $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some $\varepsilon > 0$, and if $a\,f(n/b) \leq c\,f(n)$ for some constant $c < 1$ all $n \geq n_0$ for some sufficiently large n_0, then $T(n) = \Theta(f(n))$.

1. In each case determine the asymptotic growth rate of the solution $T(n)$ to the recurrence.
   a. **[3pt]** $T(n) = 16\,T(n/4) + n$
   b. **[3pt]** $T(n) = T(3n/8) + n$
   c. **[10pt]** $T(n) = \sqrt{n}\,T(\sqrt{n}) + n \log_2 n$
   d. **[10pt]** $T(n) = T(n-1) + n + \log_2 n + 1$

2. Assume you are given an array A[$1..n$] with n distinct integers.

   a. **[20pt]** Design an algorithm which runs in time $O(n^2 \log n)$ (worst case) which determines if there are numbers x, y and z in the array A such that $x^2 + y^2 = z^2$.

   b. **[30pt]** Design an algorithm which determines if there are four ***distinct*** numbers x, y, z and u in the array such that $x^2 + y^2 = z^2 + u^2$ that also runs in (worst case) time $O(n^2 \log n)$.

3. [**20pt**] Design an algorithm that multiplies two cubic polynomials (polynomials of degree 3, like $2x^3 + 6x^2 - x + 3$) using only seven large number multiplications. Note that a multiplication of a number by a constant can be done in linear time and thus we do not count it; we only count multiplications that involve operands that ***both*** depend on the coefficients of the two polynomials. You do **NOT** have to invert particular matrices, just describe your algorithm in detail.

4. [**20pt**] Consider the polynomial $x^{16} + 8x^8 + 1$. We want to evaluate it at all roots of unity of order 32. How many distinct values do we get? Explain!