Extended Algorithms Midterm Solutions

1. We implement a version of QuickSort. Uniformly at random, we choose a pair of shoes S. Then, for each kid, we determine if the shoes are too large, too small or if they fit. We separate these kids into three categories based on the results. This takes n shoe trials.

   We know for sure that there is at least one kid K matching these pair of shoes, so for each pair of shoes, we compare them with this kid. Once again, we split the shoes into three categories. This takes n shoe trials.

   We can match each kid fitting shoe S with any shoe matching kid K. Then, we can recursively match the kids with smaller/larger feet with the smaller/larger shoes. Since the splitting behaviour is identical to that of Quicksort, this algorithm also uses $O(n \log n)$ shoe trials, under expectation.

2. (a) Suppose a clause contains both x_i and **not** x_i for some i. Then, this clause is always satisfied, regardless of the assignment. Hence, this clause is satisfied with probability $1 \geq 7/8$.

   If this is not the case, then each l_i in the clause derives from a different variable. The clause is not satisfied if every l_i is not satisfied, and since all variables are assigned independently, this occurs with probability $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$. Hence, the clause is satisfied with probability $1 - 1/8 = 7/8$.

   Thus, every clause is satisfied with probability at least $7/8$.

   (b) Let I_j be the random variable that is 1 if clause j is satisfied, or 0, otherwise. Then

$$\begin{aligned}\mathbb{E}[\text{ number of satisfied clauses }] &= \mathbb{E}\left[\sum_{j=1}^{k} I_j\right] \\ &= \sum_{j=1}^{k} \mathbb{E}[I_j]\end{aligned}$$

   by Linearity of Expectations. Hence,

$$\begin{aligned}\mathbb{E}[\text{ number of satisfied clauses }] &= \sum_{j=1}^{k}(0 \times \Pr[I_j = 0] + 1 \times \Pr[I_j = 1]) \\ &= \sum_{j=1}^{k} \Pr[I_j = 1] \\ &\geq \sum_{j=1}^{k} \frac{7}{8} \\ &= 7k/8.\end{aligned}$$

(c) The expected value of the number of satisfied clauses is $7k/8$. Since the expected value of a random variable is always at most its maximum outcome having positive probability, the maximum number of satisfied clauses among all possible assignments is at least $7k/8$. Hence, there is some assignment satisfying at least $7k/8$ clauses.