COMP3121/3821/9101/9801 (10s1)
Midterm Exam – 3 hrs

PLEASE write **CLEARLY**, show **ALL** of your work and **EXPLAIN** everything in reasonable detail

<u>The Master Theorem:</u>

Let a, b be constants such that $a \geq 1$ and $b > 1$, let $f(n)$ be a function and let $T(n)$ be defined by the recurrence

$$T(n) = a\, T(\lfloor n/b \rfloor) + f(n).$$

Then $T(n)$ can be estimated asymptotically as follows :

1) If $f(n) = O(n^{\log_b a - \varepsilon})$ for some $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$

2) If $f(n) = \Theta(n^{\log_b a})$ then $T(n) = \Theta(n^{\log_b a} \log n)$

3) If $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some $\varepsilon > 0$, and if $a\, f(n/b) < c\, f(n)$ for some constant $c < 1$ all $n \geq n_0$ for some sufficiently large n_0, then $T(n) = \Theta(f(n))$.

1. In each case determine the exact asymptotic growth rate of the solution $T(n)$ to the recurrence.
   a. **[5pt]** $T(n) = 16\, T(n/4) + n(2 + \sin n)$
   b. **[5pt]** $T(n) = 2T(n/2) + n + 1/n$
   c. **[5pt]** $T(n) = 8T(n/2) + n^{\log n}$
   d. **[10pt]** $T(n) = 2\, T(n/2) + (1 + (-1)^{[\log n]})\, n^2$ (here $[x]$ denotes the integer part of x.)

2. Assume you are given n sorted arrays of different sizes. You are allowed to merge any two arrays into a single new sorted array and proceed in this manner until only one array is left. Design an algorithm that achieves this task and uses minimal total number of moves of elements of the arrays. Give an **informal** justification why your algorithm is optimal. [25pt]

3. Assume that you got a fabulous job and you wish to repay your student loan as quickly as possible. Unfortunately, the bank "Roadrobbery" which gave you the loan has the condition that you must start by paying off $1 and then each subsequent month you must pay either double the amount you paid the previous month, the same amount as the previous month or a half of the amount you paid the previous month. On top of these conditions, your schedule must be such that the last payment is $1. Design an algorithm which, given the size of your loan, produces a payment schedule which minimizes the number of months it will take you to repay your loan while satisfying all of the bank's requirements.[25pt]

4. Assume that P is a polynomial of degree 16 and Q a polynomial of degree 8. You are given values of P at 32 integers lying in the interval [1, 40] and values of Q at 34 integers from the same interval.
    a. Show that you can find the coefficients of *P* and the coefficients of *Q* without any large number multiplications. [10pt]
    b. Show that you can find the coefficients of the product *PQ* using only 25 large number multiplications. (counting only multiplication of numbers that can both be arbitrarily large)[15p]

5. a. Simplify the expressions $(\omega_{64}^{14}\ \omega_{32}^{17})^{128}$ and $\omega_{64}^{13}\ \omega_{32}^{-15}$ [10pt]
   b. Simplify the polynomial
   $P(x) = (x - \omega_{64}^{0})\,(x - \omega_{64}^{1})\,(x - \omega_{64}^{2})\ldots(x - \omega_{64}^{63})$ [15pt]

EXTRA PROBLEM FOR THE EXTENDED CLASSES 3821 & 9801 ONLY!

6. You are given an unfair coin with unequal probabilities of head and tail. Design an algorithm which uses several tosses of this coin to simulate a fair coin. **[20pt]**