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3. Matrix Analysis of Graphs

In Chapter 2 we studied basic concepts of consensus and cooperative control for dynamic agents connected by a communication graph topology. It was seen that the graph properties and the properties of the individual node dynamics interact in intriguing ways that are not at all obvious to a casual inspection. The graph topology can have beneficial effects on individual node and team behaviors, such as inducing consensus under certain conditions, or detrimental effects, such as destabilizing the overall team dynamics though the individual node dynamics are stable.

In Chapter 3 we discussed different types of graphs, including random graphs, small world networks, scale free networks, and distance graphs. We discussed graphs that are balanced, periodic, and that have spanning trees, are strongly connected, and so on. At this point, the key concepts now in increasing our understanding of these ideas are various ideas from the theory of matrices. These ideas also provide the mathematical tools for further analysis of dynamic systems on graphs. The key references in this chapter are [Qu, Wu, Berman, Horn and Johnson].

3.1 Graph Matrices

Certain types of matrices are important in the analysis of graphs and will allow us to better formalize the notions we started to develop in Chapter 1. These include the nonnegative, row stochastic, Metzler, and M matrices. Certain properties of matrices are important for graph analysis. Especially important are matrices that are diagonally dominant, irreducible, primitive, cyclic, and ergodic or stochastically indecomposable and aperiodic (SIA). Certain canonical forms are important in analyzing graphs, including Jordan normal form, Frobenius form, and cyclic form.

Certain matrices are naturally associated with a graph. A graph $G=(V,E)$ with vertices $V=\{v_1,\cdots,v_N\}$ and edges $(v_i,v_j)\in E$ can be represented by an adjacency or connectivity matrix $A=[a_{ij}]\in R^{N\times N}$ with weights $a_{ij}>0$ *if* $(v_j,v_i)\in E$ and $a_{ij}=0$ otherwise. Define the diagonal in-degree matrix $D=diag\{d_i\}$ where the (weighted) in-degree of node v_i is the i-th row sum of A

$$d_i=\sum_{j=1}^{N}a_{ij} \tag{3.1.1}$$

The graph Laplacian matrix is $L=D-A$. Note that L has row sums equal to zero. Many properties of a graph may be studied in terms of its graph Laplacian, as we have seen.

Nonnegative Matrices

In the study of graphs, the algebras of nonnegative matrices and row stochastic matrices are instrumental. We say a matrix E is nonnegative, $E\succeq 0$, if all its elements are nonnegative. Matrix E is positive, $E\succ 0$, if all its elements are positive. The set of all nonnegative $N\times M$ matrices is denoted $R_+^{N\times M}$. The product of two nonnegative matrices is nonnegative. The product of two nonnegative matrices with positive diagonal elements is nonnegative with positive diagonal elements. Given two nonnegative matrices *E*,*F* we say that $E \succeq F$ if $(E-F) \succeq 0$. This is equivalent to saying that each element of *E* is greater than or equal to the corresponding element of *F*.

Let $G_A = (V_A, E_A), G_B = (V_B, E_B)$ be two graphs. We say G_B is a subgraph of G_A, $G_B \subset G_A$, if $V_B \subset V_A$ and E_B is a subset of the edges E_A restricted to V_B. Given any nonnegative matrix $A = [a_{ij}] \succeq 0$, we define the graph G_A corresponding to *A* as the graph having edges $e_{ji}, i \neq j$ when $a_{ij} > 0, i \neq j$, and corresponding edge weights a_{ij}. Note that the diagonal elements of *A* play no role in defining G_A since we assume graphs are simple, that is, have no loops (i.e. no edges (v_i, v_i)) and no multiple edges between the same points. Thus, the adjacency matrix and the Laplacian matrix define the same graph. For convenience we may refer simply to "the graph *A*." Let *A* be the adjacency matrix of a graph G_A and *B* be the adjacency matrix of a graph G_B, with *A, B* indexed by the same vertex set. Then $A \succeq \alpha B$ for some scalar $\alpha > 0$ implies that $G_B \subset G_A$, because the matrix inequality implies that for every nonzero entry b_{ij} of *B* there is a corresponding nonzero entry a_{ij} of *A*. That is, for every edge in G_B there is an edge in G_A.

The set of all nonnegative $N \times M$ matrices is denoted $R_+^{N\times M}$. Note that $|Ex| \leq |E||x|$ for a matrix *E* and vector *x*. Matrix *E* is diagonally positive if $e_{ii} > 0, \forall i$. Matrices *E* and *F* are of the same type if all their negative, zero, and positive elements are in the same respective locations.

3.2 Reachability in Graphs

Denote element (i, j) of a matrix *M* by $[M]_{ij}$. Adjacency matrix *A* shows which nodes can be reached from each other in one step, for $a_{ij} > 0 \ \mathit{if} \ (v_j, v_i) \in E$. Matrix A^2 shows which nodes can be reached from each other in two steps, for $[A^2]_{ij} > 0$ means there are edges $(v_j, v_k) \in E, (v_k, v_i) \in E$ form some node *k*. Generally A^k has element (i, j) nonzero if there is a path of length *k* from node *j* to node *i*.

Fact 1. A graph has a path from node *j* to node *i* if and only if its adjacency matrix *A* has a positive element (i, j) of A^k for some integer *k>0*.

A more convenient matrix test for connectedness is given in terms of the reachability matrix. A node *i* is said to be reachable from node *j* if there is a path from node *i* to node *j*. Define the reachability matrix as

$$R_k = (I + A)^k \tag{3.2.1}$$

This matrix has a positive element (i, j) if node *i* can be reached from node *j* in *k* steps or less. Note that

$$R_2 = (I+A)^2 = I + 2A + A^2 \succeq A + A^2 \qquad (3.2.2)$$

So that $(A + A^2)$ is a subgraph of R_2. Therefore, by induction, R_k contains the information about A^k and all lower powers of *A*.

Fact 2. A graph has a path from node *j* to node *i* if and only if $[R_k]_{ij} > 0$ for some integer *k>0*. A graph is strongly connected if and only if $R_k \succ 0$ for some *k*.

Example 3.2-1: Reachability. Consider the graph in Figure *, which has adjacency matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

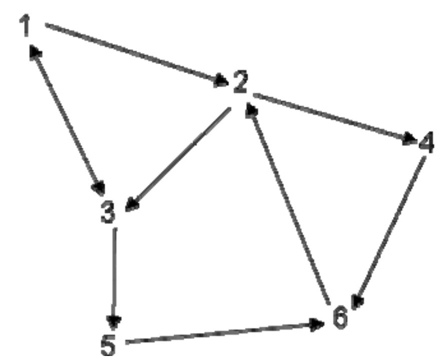


Figure 1. A directed graph.

One has

A2, A3, etc

Reachability matrix

$$R_1 = I + A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

shows which nodes are reachable from each other within 1 step. Note that there are 1's on the diagonal, indicating that every node is reachable from itself within 1 step, in fact, in zero steps. Note that

$$R_4 = R_3 = (I+A)^3 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

is a positive matrix. This shows that the all nodes can be reached from all others within 3 steps. Therefore, the graph is strongly connected and the length of the longest path from one to another, that is the graph diameter, is equal to 3. ■

Note that the definition given for the reachability matrix allows the R_k to have entries larger than 1. In fact, a more appropriate definition for the reachability matrix uses Boolean operations in (3.2.1), where additions are replaced by 'or' operations and multiplications are replaced by 'and' operations. Then, the reachability matrix R_k is a logical matrix having entries of 1 in element (i, j) if node *i* can be reached from *j* in *k* steps or fewer, and zero otherwise.

3.3 Irreducible Matrices and Frobenius Form

In this section we introduce some concepts that are important for analysis of graphs and cooperative control performance using matrix techniques. We cover irreducibility, Frobenius form, hierarchical structure of graphs, diagonal dominance, and spectral properties.

4.4

Irreducibility

An arbitrary square matrix *E* is reducible if it can be brought by a row/column permutation matrix *T* to lower block triangular (LBT) form

$$F = TET^T = \begin{bmatrix} * & 0 \\ * & * \end{bmatrix} \tag{3.3.1}$$

Two matrices that are similar using permutation matrices are said to be cogredient. Note that, if T is a permutation matrix then $T^{-1} = T^T$.

The next result ties irreducible matrices to graph theory.

Thm 1. A graph *G* is strongly connected if and only if its adjacency matrix *A* is irreducible.

Proof?

An irreducible matrix has its elements 'tightly coupled' in some way, as illustrated by the next two examples.

Example 3.3-1: Irreducible Sets of Equations. Consider the linear equations

$$Ex = b \tag{3.3.2}$$

with *E* a matrix, not necessarily square, and *x* and *b* vectors. If *E* is nonsingular, there is a unique solution *x*. Yet, *E* may be singular and the system of equations can still enjoy some important properties. Let *T* be a permutation matrix and write

$$Ex = T^T FTx = b, \quad FTx = Tb, \quad F\bar{x} = \bar{b} \tag{3.3.3}$$

where $\bar{x} = Tx, \bar{b} = Tb$. Suppose now that *E* is reducible and write

$$\begin{bmatrix} F_{11} & 0 \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \end{bmatrix} \tag{3.3.4}$$

or

$$\bar{x}_1 = \bar{b}_1, \quad F_{22}\bar{x}_2 = \bar{b}_2 - F_{21}\bar{x}_1 \tag{3.3.5}$$

This means the equations can be solved by first solving a reduced set of equations for $\bar{x}_1$ and then solving for $\bar{x}_2$. On the other hand, if *E* is irreducible, the equations must be solved simultaneously for all variables. ■

Example 3.3-2: Interlaced Elements of Irreducible Matrices. Consider the graph in Example 3.2-1. It is strongly connected, so its adjacency matrix A is irreducible. To see graphically what this means and get some intuition, start at any node, say node 3. Go down the third column of *A* until reaching an entry of ‘1’, then go left along that row, in this case row 1. This is illustrated by arrows in Figure 3.3-1. Now go down column 1 until encountering an entry of ‘1’. Then go left along that row, in this example row 2. Irreducibility means that by proceeding in this fashion, one will eventually pass through all entries of 1 in the matrix, and go through every row and every column.

Thus, in some sense, the entries of an irreducible matrix are ‘tightly coupled’. In fact, though an irreducible matrix may not be nonsingular, it does not take much to make it so, as will be seen in the discussion below on irreducible diagonal dominance.

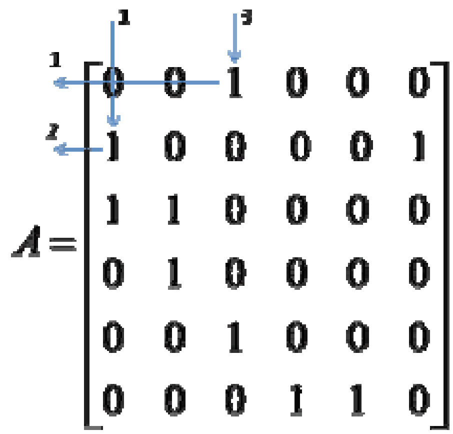


Figure 3.3-1. Pictorial illustration showing that the entries of an irreducible matrix are ‘tightly coupled’.

■

Frobenius Form and Hierarchical Structure of Graphs

A reducible matrix E can be brought by a permutation matrix T to the lower block triangular Frobenius canonical form [Wu 2007, Qu]

$$F = TET^T = \begin{bmatrix} F_{11} & 0 & \cdots & 0 \\ F_{21} & F_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ F_{p1} & F_{p2} & \cdots & F_{pp} \end{bmatrix} \qquad (3.3.6)$$

where F_{ii} is square and irreducible. Note that if F_{ii} is a scalar and irreducible it is equal to 0. The degree of reducibility is defined as $p-1$. The graph has p strongly connected subgraphs corresponding to the graphs of the diagonal blocks.

Given a reducible graph adjacency matrix A, the structure revealed in the Frobenius form is usually not evident. The conversion to LBT form can be done in linear time using search algorithms [Tarjan 1972].

The condensation directed graph of a directed graph G is constructed by assigning a vertex to each strongly connected component of G and an edge between 2 vertices if and only if there exists an edge of the same orientation between corresponding strongly connected components of G. The condensation directed graph H of G contains a spanning directed tree if and only if G contains a spanning directed tree.

Graphs have been used in the solution of large systems of equations, in resource assignment problems and task scheduling in manufacturing, and elsewhere. Steward [1962] and Warfield [1973] defined a graph wherein each node is associated with a task. The edges $(v_j, v_i) \in E$ denote causality relationships in that job j must be performed immediately before job i. In this context the adjacency matrix is called the task sequencing matrix or the design structure matrix.

Example 3.3-3: Hierarchical Structure of Graphs. This example is taken from Warfield [1973]. Consider the graph adjacency matrix shown in Figure 3.3-2a. Any structure in the graph cannot be seen from inspection of this matrix. However, the matrix can be brought by row and column permutations to the Frobenius form in Figure 3.3-2b. This matrix has a block triangular structure, from which the graph shown in Figure 3.3-3 can be immediately drawn. (Note that these arrows are drawn backwards due to Warfield's application in manufacturing.) Now it is seen that the graph has four strongly connected components, with edges between them. This is a hierarchical structure which can be captured by writing the condensation graph adjacency matrix in Figure 3.3-4 and drawing the condensation graph, which exhibits the hierarchical ordering among the components of the original graph.

$$
M_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
$$

(a)

$$
M_2 = \begin{array}{c} \\ 1 \\ 3 \\ 7 \\ 5 \\ 4 \\ 2 \\ 6 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} 1 & 3 & 7 & 5 & 4 & 2 & 6 \end{array} \\
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}
\end{array}.
$$

(b)

Figure 3.3-2. (a) Original graph matrix. (b) Frobenius form obtained by reordering the nodes of the graph.

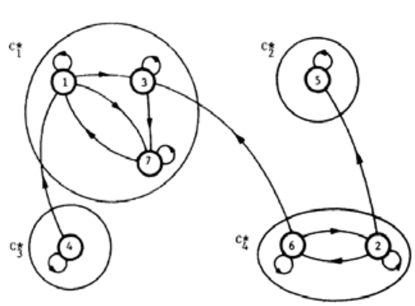
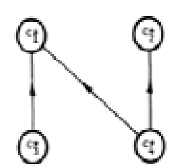


Figure 3.3-3. Hierarchical form of graph drawn from the Frobenius form.

$$
M' = \begin{array}{c} \\ C_1^* \\ C_2^* \\ C_3^* \\ C_4^* \end{array}
\begin{array}{c}
\begin{array}{cccc} C_1^* & C_2^* & C_3^* & C_4^* \end{array} \\
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0
\end{bmatrix}
\end{array}.
$$

(a)



(b)

Figure 3.3-4. Condensation graph. (a) Adjacency matrix. (b) Diagram.

■

Change arrows in this example.

Change inequality signs. Check.

Properties of Irreducible Matrices E:

1. If E is irreducible. If vector $z \in R_+^n$ has $1 \le k < n$ positive entries, then $(I+E)z$ has more than k positive entries
2. Matrix $E \succeq 0$ is irreducible if and only if $\gamma z \succeq Ez,\ for\ \gamma > 0, z \ge 0$ implies *z=0* or *z>0*
3. Matrix $E \succeq 0$ is irreducible if and only if $e_{ij}^{(k)} > 0$ for every *(i,j)* and some integer *k>0*, where $e_{ij}^{(k)}$ denotes the *(i,j)* element of E^k.
4. Matrix $E \succeq 0$ is irreducible if and only if $(cI+E)^k \succ 0$ for some integer *k* and any scalar *c>0*.

See Berman 79, p. 40ff for more information.

Let $E_k \in R_+^{n\times n}$. If all matrices in the sequence $\{E_k\}$ are irreducible and diagonally positive, then for some $1 \le \eta < r$ and for all k one has $E_{k+\eta} \cdots E_{k+1} E_k \succ 0$ [Wolfowitz].

Lemma. *A* is irreducible if and only if *D-A* is irreducible for any diagonal *D*.

Proof: $TAT^T = \begin{bmatrix} * & 0 \\ * & * \end{bmatrix}$ if and only if $T(D-A)T^T = \begin{bmatrix} * & 0 \\ * & * \end{bmatrix}$ since TDT^T is diagonal for all permutation matrices *T*. ■

Therefore, the adjacency matrix and Laplacian matrix are both irreducible if and only if the graph is strongly connected.

The Frobenius form (3.3.6) is said to be lower triangularly complete if in every block row *i* there exists at least one $j < i$ such that $F_{ij} \ne 0$. That is, every block row has at has least one nonzero entry. It is said to be lower triangularly positive if $F_{ij} \succ 0, \forall j < i$.

Thm. A graph has a spanning tree if and only if the adjacency matrix Frobenius form is lower triangularly complete.

Proof: Zhihua p. 164. ■

If the graph has a spanning tree, then the nodes having a directed path to all other nodes are roots of spanning trees, and members of the leader group. These correspond to the graph of leader block F_{11} in Frobenius form.

[Wu 2007 p. 16] The zero eigenvalue of a reducible Laplacian matrix of a digraph has multiplicity *m* if and only if *m* is the minimum number of directed trees which together span the

the digraph.

The spanning forest of a graph is the set of spanning trees [Wu 2007]

[Wu 2007 p. 16] The zero eigenvalue of the Laplacian matrix of a graph has multiplicity *m* if and only if *m* is the number of connected components in the graph.

A graph contains a spanning tree if and only if adjacency matrix *A* is irreducible or 1-reducible.

[Wu 2007 p. 18] Consider a Laplacian matrix $L \in R^{N\times N}$ whose graph contains a spanning directed tree. Let $w = \begin{bmatrix} p_1 & \cdots & p_N \end{bmatrix}^T$ be a nonnegative vector such that $w^T L = 0$. Then $p_i = 0$ for all vertices *i* that do not have directed paths to all other vertices in the graph and $p_i > 0$ otherwise.

Proof:

Diagonal Dominance

Thm. Gerschgorin Circle Criterion. All eigenvalues of matrix $E = [e_{ij}] \in R^{n\times n}$ are located within the following union of *n* discs

$$\bigcup_{i=1}^{n} \left\{ z \in C : \left| z - e_{ii} \right| \le \sum_{j\ne i} \left| e_{ij} \right| \right\} \tag{3.3.7}$$

Matrix $E = [e_{ij}] \in R^{n\times n}$ is:

1. Diagonally dominant if, for all *i*,

$$e_{ii} \ge \sum_{j\ne i} \left| e_{ij} \right| \tag{3.3.8}$$

2. Strictly diagonally dominant if these inequalities are all strict.
3. Strongly diagonally dominant if at least one of the inequalities is strict [Serre 2000]
4. Irreducibly diagonally dominant if it is irreducible and at least one of the inequalities is strict.

Moreover, let *E* be irreducible with all row sums equal to zero. Then eigenvalue $\lambda = 0$ has multiplicity of 1.

If *E* is diagonally dominant (resp.strictly diagonally dominant) then its eigenvalues have nonnegative (resp. positive) real parts.

Thm. [Taussky 1949, Wu 2007]. Let *E* be strictly diagonally dominant or irreducibly diagonally dominant. Then *E* is nonsingular. If in addition, the diagonal elements of E are all positive real numbers, then $\operatorname{Re}\lambda_i(E) > 0,\ 1 \le i \le n$.

The usefulness of these constructions is illustrated by the proof of the following result, which uses matrices to prove graph properties.

Thm. [Wu]. Let graph of *A* contain a spanning tree. Then $\lambda_1 = 0$ is a simple eigenvalue of the

Laplacian matrix $L=D-A$ with a nonnegative left eigenvalue $w^T = \begin{bmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_N \end{bmatrix} \ge 0$. Moreover, the entries γ_i are positive for any node v_i which has a directed path to all other nodes, and zero for all other nodes.

Proof: The graph Laplacian $L=D-A$ is cogredient to the Frobenius form

$$F = TLT^T = \begin{bmatrix} F_{11} & 0 & \cdots & 0 \\ F_{21} & F_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ F_{p1} & F_{p2} & \cdots & F_{pp} \end{bmatrix}$$

with all F_{ii} square and irreducible.

There exists a spanning tree iff L has a simple eigenvalue $\lambda_1 = 0$, or equivalently if F is lower triangularly complete. Then, there is only one leader group, which corresponds to F_{11}, which is irreducible with row sum zero. Therefore F_{11} has a simple eigenvalue $\lambda_0 = 0$ with right eigenvector $\underline{1}$ and left eigenvector $w_1 > 0$. PROVE THIS???

Matrix F has all row sums zero, and in every row i there exists at least one $j < i$ such $F_{ij} \ne 0$ (i.e. F_{ij} has least one nonzero entry). Therefore $F_{ii}, i \ge 2$ are diagonally dominant irreducible M matrices with row sums greater than zero in at least one row. Therefore, F_{ii}, $i \ge 2$ are nonsingular.

Now write

$$\begin{bmatrix} w_1^T & w_2^T & \cdots & w_p^T \end{bmatrix} \begin{bmatrix} F_{11} & 0 & \cdots & 0 \\ F_{21} & F_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ F_{p1} & F_{p2} & \cdots & F_{pp} \end{bmatrix} = 0$$

where w_i are vectors of appropriate dimension. Therefore

$$\begin{aligned} & w_p^T F_{pp} = 0 \\ & w_{p-1}^T F_{(p-1)(p-1)} + w_p^T F_{p(p-1)} = 0 \\ & \vdots \\ & w_1^T F_{11} + w_2^T F_{21} + \cdots + w_p^T F_{p1} = 0 \end{aligned}$$

Since $F_{ii}, i \ge 2$ are nonsingular, this implies $w_i = 0, i \ge 2$ and $w_1^T F_{11} = 0$, with $w_1 > 0$.

■

Spectral Properties

The spectral radius of matrix $E \in R^{n \times n}$ is $\rho(E) = \max_i |\lambda_i|$, the maximum eigenvalue magnitude.

In discrete-time (z-plane) analysis, the spectral gap generally refers to the magnitude of the difference between the two largest eigenvalues. In continuous-time (s-plane) analysis, the spectral gap generally refers to $\min_i |\lambda_i|$, the minimum eigenvalue magnitude.

Thm. Perron-Frobenius. Let $E \in R_+^{n \times n}$. Then:

1. Spectral radius $\rho(E) \geq 0$ is an eigenvalue and its eigenvector is nonnegative.
2. If $E \succ 0$, then $\rho(E) > 0$ is a simple eigenvalue of multiplicity 1, and its eigenvector is positive.
3. The graph of E contains a spanning tree if and only if there is a positive vector v such that $Ev = \rho(E)v$, and $\rho(E) > 0$ is a simple eigenvalue
4. If E is irreducible, there is a positive vector v such that $Ev = \rho(E)v$, and $\rho(E) > 0$ is a simple eigenvalue

Note- properties 3 and 4 mean that both the left eigenvector and right eigenvector of E for $\rho(E)$ are nonnegative.

Lemma. Matrix $E \succeq 0$ is irreducible if and only if $\rho(E)$ is a simple eigenvalue and both its right and left eigenvectors are positive [Berman 79. p. 42].

Properties.

1. If $0 \succeq A \preceq B$ then $\rho(A) \leq \rho(B)$ [Berman 79, p. 27]
2. If $0 \succeq A \preceq B$ and $A+B$ is irreducible, then $\rho(A) < \rho(B)$ [Berman 79]
3. If B is a principal submatrix of $A \succeq 0$, then $\rho(B) \leq \rho(A)$ [Berman 79]
4. If $A \succeq 0$ is irreducible and $B \succeq 0$, then $A+B$ is irreducible [Berman 79]

3.4 Stochastic, Primitive, Cyclic, and Ergodic Matrices

Important in the study of graphs are the stochastic, primitive, and ergodic or stochastically indecomposable and aperiodic (SIA) matrices. We have already used SIA matrices in Chapter 2 in the study of time-varying graph topologies.

Stochastic Matrices

A matrix $E \succeq 0$ is row stochastic if all its row sums equal to 1, i.e. if $E \in M_1(1)$

A matrix $E \succeq 0$ is doubly stochastic if all its row sums and column sums equal to 1.

The product of two row stochastic matrices *E,F* is matrices is row stochastic because $EF\underline{1} = E\underline{1} = \underline{1}$.

The maximum eigenvalue of a stochastic matrix is 1. A matrix $E \succeq 0$ is row stochastic if and only if $\underline{1}$ is an eigenvector for the eigenvalue 1.

Let $E \in R_+^{n \times n}$ have all row sums equal to a constant $c > 0$. Then [Beard book 2008]:

1. $\rho(E) = c$, and is an eigenvalue of E with eigenvector $\underline{1}$.
2. $\lambda_1 = \rho(E) = c$ is simple if and only if E has a spanning tree. Then $rank(E) = n-1$.
3. If $e_{ii} > 0$ for all i, then $|\lambda| < c$ for all eigenvectors $\lambda \neq c$.
4. If E has a spanning tree and if $e_{ii} > 0$ for all i, then $\lambda_1 = \rho(E) = c$ is the unique eigenvalue of maximum modulus.

Primitive Matrices

A nonnegative matrix $E \in R_+^{n\times n}$ is primitive if $E^k \succ 0$ for some integer $k>0$. The smallest such k is the index of primitivity.

If $A \succeq 0$ is primitive and $B \succeq 0$, then $A+B$ is primitive [Berman 79, p. 28]

Thm. Let E be primitive with a nonnegative eigenvector v for eigenvalue λ. Then $\lambda = \rho(E)$ and v is the unique positive eigenvector.

Thm. $E \in R_+^{n\times n}$ is irreducible if and only if $cI + E$ is primitive for every scalar $c>0$.

Thm. E is irreducible and has only one eigenvalue of maximum modulus if and only if E is primitive.

Thm. If E is irreducible and diagonally positive, it is also primitive.

Thm. If E is irreducible and its trace is positive, it is also primitive [Berman p. 34].

If E is irreducible it may not be that $E^k \succ 0$ for some k.
If E is irreducible and p-cyclic, then E^p is block diagonal.
If E is irreducible and p-cyclic for $p>1$, then E is not primitive.

Thm. Let E be primitive with left eigenvector w and right eigenvector v for $\rho(E)$. Then

$$\lim_{k\to\infty} \frac{E^k}{\rho^k(E)} = \frac{vw^T}{w^T v}$$

and the convergence is exponential.

Cyclic or Periodic Matrices

Thm. Perron-Frobenius. Let $E \in R_+^{n\times n}$. Then if E is irreducible and if $p \geq 1$ eigenvalues of E are of modulus $\rho(E)$, they are distinct roots of polynomial equation $\lambda^p - [\rho(E)]^p = 0$ and there exists a permutation matrix T such that E can be expressed in the cyclic form

$$F = TET^T = \begin{bmatrix} 0 & F_{12} & 0 & \cdots & 0 \\ 0 & 0 & F_{23} & \cdots & 0 \\ 0 & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & F_{(p-1)p} \\ F_{p1} & 0 & 0 & 0 & 0 \end{bmatrix}$$

where the diagonal zero blocks are square. Then, matrix E is called p-cyclic or p-periodic. The index of cyclicity or periodicity is p.

The cyclicity period p of E is the greatest common divisor of the lengths of all the loops in the graph of E. Matrix E is acyclic or aperiodic if the cyclicity period is 1. i.e. if E is 1-cyclic, $p=1$.

If graph A is a cycle of length p, then $A^p = I$ the identity matrix.

Thm. E is irreducible and aperiodic if and only if E is primitive.

Example. periodic graph.

Thm. Given matrix F in this form, matrix F (and hence E) is irreducible if $F_{12}F_{23}\cdots E_{(p-1)p}F_{p1}$ is irreducible.
The nonzero eigenvalues of F are the p-th roots of the eigenvalues of $F_{12}F_{23}\cdots E_{(p-1)p}F_{p1}$ [Serre 2000, p. 86]

Ergodic or Stochastically Indecomposable and Aperiodic (SIA) Matrices

Let matrix $E \in R^{N\times N}$ be row stochastic. Matrix E is said to be stochastically irreducible and aperiodic (SIA), or ergodic, if

$$\lim_{k\to\infty} E^k = \underline{1}w^T \qquad (4.1)$$

for some $w \in R^N$, with $\underline{1} \in R^N$ the vector of 1's.

Let E be a row stochastic matrix. If E has a simple eigenvalue $\lambda_1 = 1$ and all other eigenvalues satisfy $|\lambda| < 1$, then E is SIA.

Let E be irreducible and aperiodic, e.g. primitive. Then E is SIA.

E is SIA if and only if the Frobenius form (3.3.6) is lower triangularly complete and F_{11} is aperiodic. That is, if and only if the graph contains a spanning tree and the leader subgraph is aperiodic. Matrices E that are lower triangularly complete are said to be stochastically indecomposable [Qu].

3.5 M Matrices

Metzler, Z, and M Matrices

A matrix with off-diagonal elements nonnegative is called a Metzler matrix. Pictorially,

$$Metzler = \begin{bmatrix} * & \geq 0 \\ \geq 0 & * \end{bmatrix}$$

$M_1(c)$ denotes the set of real matrices having all row sums equal to c. Let $M_2(c)$ be the set of Metzler matrices in $M_1(c)$.

D is row stochastic if $D \succeq 0$ and $D \in M_1(1)$

Any Metzler matrix can be expressed as

$$E = -sI + A, \quad s \geq 0, A \succeq 0$$

Thm. Perron-Frobenius. Consider two matrices $E_1 \in M_1(c), \quad E_2 \in M_2(c)$. Then

1. c is an eigenvalue of E_1 with right eigenvector $\underline{1}$.
2. The real parts of all eigenvalues of E_2 are less than or equal to c, and those eigenvalues with real part equal to c are real.
3. E_2 has a spanning tree if and only if c is a simple eigenvalue [Beard].
4. If E_2 is irreducible, then c is a simple eigenvalue.

Thm. [Berman 79, p. 37]. Let $E \in R_+^{n \times n}$ be irreducible. Let S be the maximum row sum and s be the minimum row sum. Let v be a positive eigenvector and let $\gamma = \max_{i,j}(x_i / x_j)$. Then

$$s \leq \rho(E) \leq S$$

$$\sqrt{S/s} \leq \gamma$$

and equality holds if and only if S=s.

Let Z denote the set of matrices whose off-diagonal elements are nonpositive. Pictorially,

$$Z = \begin{bmatrix} * & \leq 0 \\ \leq 0 & * \end{bmatrix}$$

Note that the negative of a Metzler matrix is a Z matrix.

A Z matrix is called an (singular) M matrix if all its principal minors are nonnegative. That is, a matrix is an M matrix if all off diagonal elements are nonpositive and all leading minors are nonnegative. It is a nonsingular M matrix if all the principal minors are positive. Pictorially,

$$M = \begin{bmatrix} + & \le 0 \\ \le 0 & + \end{bmatrix}$$

An M matrix can be written as $E = sI - A, \quad s > 0, A \succeq 0$.

If E is a singular M matrix, then $s \ge \rho(A)$ and $-E$ is a Metzler matrix.

If E is a nonsingular M matrix, then $s > \rho(A)$, and $-E$ is both Metzler and Hurwitz.

A symmetric singular (nonsingular) M matrix is positive semidefinite (definite).

Let E be a diagonally dominant M matrix (i.e. nonpositive elements off the diagonal, nonnegative elements on the diagonal). Then $\lambda = 0$ is an eigenvalue of E if and only if all row sums are equal to 0.

If E is diagonally dominant (resp. strictly diagonally dominant) then its eigenvalues have nonnegative (resp. positive) real parts. Additionally, if E is a Z matrix, then E is a singular (resp. nonsingular) M matrix.

Properties of M Matrices

Thm. Properties of nonsingular M matrices [Qu]. Let $E \in Z$. Then the following conditions are equivalent:

1. E is a nonsingular M matrix
2. the leading principal minors of E are all positive.
3. The eigenvalues of E have positive real parts.
4. E^{-1} exists and is nonnegative.
5. There exist vectors $w, v > 0$ such that $Ev, E^T w$ are both positive.
6. There exists a positive diagonal matrix S such that $ES + SE^T$ is strictly diagonally dominant and hence also positive definite.

Thm. Properties of singular M matrices [Qu]. Let $E \in Z$. Then the following conditions are equivalent:

1. E is a singular M matrix
2. the leading principal minors of E are all nonnegative.
3. the eigenvalues of E have nonnegative real parts.
4. $(D+E)^{-1}$ exists and is nonnegative for all positive diagonal matrices $D \succ 0$.
5. There exist vectors $w, v > 0$ such that $Ev, E^T w$ are both nonnegative.
6. There exists a nonnegative diagonal matrix S such that $ES + SE^T$ is diagonally dominant and hence also positive semidefinite.

Thm [Qu]. Property c. Let $E \in Z$. Then the following are equivalent:

1. E is a singular M matrix with 'Property c'.
2. $rank(E) = rank(E^2)$

3. There exists a symmetric positive definite matrix P such that $PE + E^T P$ is positive semidefinite, that is, matrix $-E$ is Lyapunov stable.

Moreover, these properties are guaranteed if $E \in Z$ and there exists a vector *v>0* such that $Ex \geq 0$.

Thm. Properties of irreducible M matrices [Qu]. Let $E = sI - A$ be an irreducible M matrix, that is, $A \succeq 0$ and is irreducible. Then:

1. *E* has rank $n-1$
2. there exists a vector *v>0* such that *Ev=0*.
3. $Ex \geq 0$ implies *Ex=0*.
4. Each principal submatrix of order less that *n* is a nonsingular M matrix
5. $(D+E)^{-1}$ exists and is nonnegative for all nonnegative diagonal matrices *D* with at least one positive element.
6. Matrix E has Property c.
7. There exists a positive diagonal matrix *P* such that $PE + E^T P$ is positive semidefinite, that is, matrix *E* is pseudo-diagonally dominant, that is, matrix -*E* is Lyapunov stable.

Thm. Let $E \in Z$ be a singular but irreducible M matrix. Then $\bar{E} = E + diag\{0, 0, \cdots, \varepsilon\}$ is a nonsingular M matrix for any $\varepsilon > 0$. Moreover, $-\bar{E}$ is stable.

Thm. Let $E \in Z$ be a singular but irreducible M matrix. Let $\varepsilon_i \geq 0$ and at least one of them be positive. Then $\bar{E} = E + diag\{\varepsilon_i\}$ is a nonsingular M matrix. Moreover, $-\bar{E}$ is stable.

Thm. Discrete-time. Let *F* be a nonnegative and irreducible matrix with $\rho(F) < 1$ ($\rho(F) = 1$). Then there exists a positive diagonal matrix *P* such that $P - A^T PA$ is positive definite (positive semidefinite).

Mapping M Matrices to Row Stochastic Matrices

(Laplacian be L=D-A is an M matrix with row sums equal to zero.)

Let *L* be an M matrix with row sums equal to zero. Then [Beard book 2008]:

L has an eigenvalue at zero with eigenvector $\underline{1}$, and all nonzero eigenvalues are in the open-right half plane.

$\lambda_1 = 0$ is simple if and only if L contains a spanning tree.

L has rank *n-1*.

Let *L* be an M matrix with row sums equal to zero. Then [Beard book 2008]:

$E = e^{-Lt}$ is row stochastic with positive diagonal elements.

Let L have a simple eigenvalue $\lambda_1 = 0$. Then

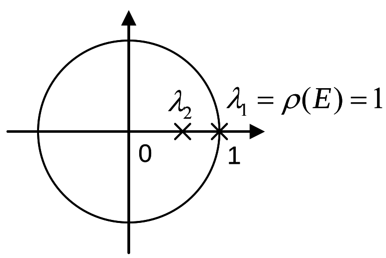
$$\lim_{t\to\infty} e^{-Lt} = \underline{1}w^T \text{, where } w > 0 \text{ is the left eigenvalue of } L.$$

Let $E \geq 0$ be row stochastic. Then $A=I-E$ is an M matrix with row sums zero.

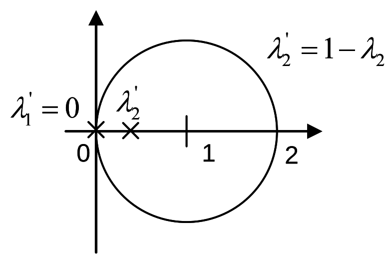
E is irreducible if and only if $I-E$ is irreducible.

Let E have eigenvalues at λ_i with eigenvectors v_i. Then:

1. $E - \mu I$ has eigenvalues at $\lambda_i - \mu$ with eigenvectors v_i.
2. $\mu I - E$ has eigenvalues at $\mu - \lambda_i$ with eigenvectors v_i.



Eigenvalue region of stochastic matrix.
Example: $E = D^{-1}A$



Eigenvalue region of normalized M matrix w/ row sum= 0 and diag. entries =1. Example: $L = I - D^{-1}A$

Let L be an M matrix with row sum zero. Then:
$G = I - L$
is a Metzler matrix with row sum of one. $-G$ is an M matrix in $M_2(1)$.

Laplacian $L=D-A$ is an M matrix with row sum zero. Then:

1. $G = I - L = I - D + A$
is a Metzler matrix with row sum of one. $-G$ is an M matrix in $M_2(1)$.
2. $G = I - D^{-1}L = D^{-1}A \geq 0$
is a row stochastic matrix with diagonal elements equal to 0.
3. $F = I - (I+D)^{-1}L = (I+D)^{-1}(I+D-L) = (I+D)^{-1}(I+A)$
is a row stochastic matrix with diagonal elements equal to 1.

References

Horn and Johnson
Berman
Tausky
Zhihua Qu

Wu 2007

D.V. Steward, “On an approach to techniques for the analysis of the structure of large systems of equations,” SIAM Review, vol. 4, pp. 321-342, Oct. 1962.

J.W. Warfield, “Binary matrices in system modeling,” IEEE Trans. Systems, Man, Cybernetics, vol. SMC-3, no. 5, pp. 441-449, Sept. 1973.