Chapter 2

Reachability and Connectivity

1. Introduction

In the previous chapter it was mentioned that the communication system of an organization can be considered in terms of a graph where people are represented by vertices, and communication channels by arcs. A natural question to ask of such a system, is as to whether information held by a given individual x_i can be communicated to another individual x_j; that is, whether there is a path leading from vertex x_i to vertex x_j. If such a path exists we say that x_j is *reachable* from x_i. If the reachability is restricted to paths of limited cardinality, we would like to know if x_j is still reachable from x_i. The purpose of the present Chapter is to discuss some fundamental concepts relating to the reachability and connectivity properties of graphs and introduce some very basic algorithms.

In terms of the graph representing an organization, the present chapter considers the questions:

(i) What is the least number of people from which every other person in the organization can be reached?

(ii) What is the largest number of people who are mutually reachable?

(iii) How are (i) and (ii) above related?

2. Reachability and Reaching Matrices

The reachability matrix $\mathbf{R} = [r_{ij}]$ is defined as follows:

$$r_{ij} = 1 \text{ if vertex } x_j \text{ is reachable from vertex } x_i.$$
$$= 0 \text{ otherwise.}$$

The set of vertices $R(x_i)$ of the graph G which can be reached from a given vertex x_i consists of those elements x_j for which there is an entry of 1 in cell (x_i, x_j) of the reachability matrix. Obviously all the diagonal elements of $\mathbf{R}$ are 1 since every vertex is reachable from itself, by a path of cardinality 0.

Since $\Gamma(x_i)$ is that set of vertices x_j which are reachable from x_i along a path of cardinality 1 (i.e. that set of vertices for which the arcs (x_i, x_j) exist in the graph) and since $\Gamma(x_j)$ is that set of vertices reachable from x_j along a path of cardinality 1; the set of vertices $\Gamma(\Gamma(x_i)) = \Gamma^2(x_i)$ consists of those vertices reachable from x_i along a path of cardinality 2. Similarly $\Gamma^p(x_i)$ is that set of vertices which are reachable from vertex x_i along a path of cardinality p.

Since any vertex of the graph G which is reachable from x_i must be reachable along a path (or paths) of cardinality 0 or 1 or 2, ... or p for some finite but suitably large value of p, the reachable set of vertices from vertex x_i is:

$$R(x_i) = \{x_i\} \cup \Gamma(x_i) \cup \Gamma^2(x_i) \cup \ldots \cup \Gamma^p(x_i) \tag{2.1}$$

Thus, the reachable set $R(x_i)$ can be obtained from eqn (2.1) by performing the union operations from left to right until such time when the current total set is not increased in size by the next union. When this occurs any subsequent unions will obviously not add any new members to the set and this is then the reachable set $R(x_i)$. The number of unions that may have to be performed depends on the graph, although it is quite clear that p is bounded from above by the number of vertices in the graph minus one.

The reachability matrix can therefore be constructed as follows. Find the reachable sets $R(x_i)$ for all vertices $x_i \in X$ as mentioned above. Set $r_{ij} = 1$ if $x_j \in R(x_i)$, otherwise set $r_{ij} = 0$. The resulting matrix $\mathbf{R}$ is the reachability matrix.

The reaching matrix $\mathbf{Q} = [q_{ij}]$ is defined as follows:

$$\begin{aligned} q_{ij} &= 1 \text{ if vertex } x_j \text{ can reach vertex } x_i \\ &= 0 \text{ otherwise.} \end{aligned}$$

The reaching set $Q(x_i)$ of the graph G is that set of vertices which can reach

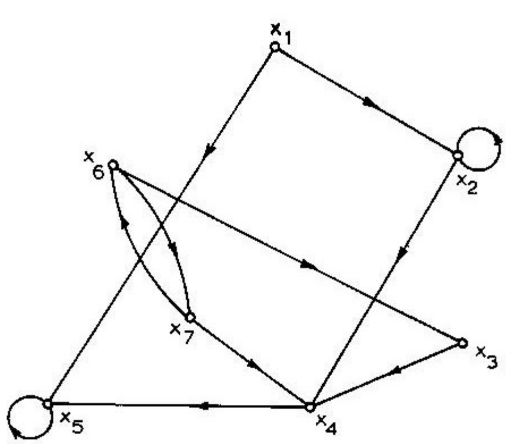

FIG. 2.1.

vertex x_i. In a manner analogous to the calculation of the reachable set $R(x_i)$ from eqn (2.1), the set $Q(x_i)$ can be calculated as:

$$Q(x_i) = \{x_i\} \cup \Gamma^{-1}(x_i) \cup \Gamma^{-2}(x_i) \cup \ldots \cup \Gamma^{-p}(x_i) \tag{2.2}$$

where $\Gamma^{-2}(x_i) = \Gamma^{-1}(\Gamma^{-1}(x_i))$ etc.

The operations are performed from left to right until the next set union operation does not change the set $Q(x_i)$.

It is quite apparent from the definitions that column x_i of the matrix $\mathbf{Q}$ (found by setting $q_{ij} = 1$ if $x_j \in Q(x_i)$, and $q_{ij} = 0$ otherwise), is the same as row x_i of the matrix $\mathbf{R}$; i.e. $\mathbf{Q} = \mathbf{R}^t$, the transpose of the reachability matrix.

2.1 Example

Find the reachability and reaching matrices of the graph G shown in Fig. 2.1. The adjacency matrix of G is:

$$\mathbf{A} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

The reachable sets are calculated from eqn (2.1) as:

$$R(x_1) = \{x_1\} \cup \{x_2, x_5\} \cup \{x_2, x_4, x_5\} \cup \{x_2, x_4, x_5\}$$
$$= \{x_1, x_2, x_4, x_5\}$$

$$R(x_2) = \{x_2\} \cup \{x_2, x_4\} \cup \{x_2, x_4, x_5\} \cup \{x_2, x_4, x_5\}$$
$$= \{x_2, x_4, x_5\}$$

$$R(x_3) = \{x_3\} \cup \{x_4\} \cup \{x_5\} \cup \{x_5\}$$
$$= \{x_3, x_4, x_5\}$$

$$R(x_4) = \{x_4\} \cup \{x_5\} \cup \{x_5\}$$
$$= \{x_4, x_5\}$$

$$R(x_5) = \{x_5\} \cup \{x_5\}$$
$$= \{x_5\}$$
$$R(x_6) = \{x_6\} \cup \{x_3, x_7\} \cup \{x_4, x_6\} \cup \{x_3, x_5, x_7\} \cup \{x_4, x_5, x_6\}$$
$$= \{x_3, x_4, x_5, x_6, x_7\}$$
$$R(x_7) = \{x_7\} \cup \{x_4, x_6\} \cup \{x_3, x_5, x_7\} \cup \{x_4, x_5, x_6\}$$
$$= \{x_3, x_4, x_5, x_6, x_7\}$$

The reachability matrix is therefore given by:

$$\mathbf{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

and the reaching matrix is given by:

$$\mathbf{Q} = \mathbf{R}^t = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

as can easily be checked.

It should be mentioned here that since all entries in the **R** and **Q** matrices are either 0 or 1, each row can be stored in binary form using one (or more) computer words. Thus, finding the **R** and **Q** matrices becomes a computationally very simple task, since the union of sets indicated by eqns (2.1) and (2.2) and the comparison after each union—in order to determine if it is necessary to continue—can then each be done by a single logical operation on a computer.†

Since $R(x_i)$ is the set of vertices which can be reached from x_i and $Q(x_j)$ the set of vertices which can reach x_j, the set $R(x_i) \cap Q(x_j)$ is then the set of vertices which are on at least one path going from x_i to x_j. These vertices are called *essential* with respect to the two end vertices x_i and x_j [7]. All other vertices $x_k \notin R(x_i) \cap Q(x_j)$ are called *inessential* or *redundant* since their removal does not affect any path from x_i to x_j.

The reachability and reaching matrices defined above are absolute, in the sense that the number of steps in the paths reaching from x_i to x_j was not restricted. On the other hand a *limited* reachability and reaching matrices can be defined where the cardinality of the path must not exceed a certain number. These limited matrices can also be calculated from eqns (2.1) and (2.2) in an exactly analogous manner to that described previously where p would now be the upper bound on the allowed path cardinality.

A graph is said to be *transitive* if the existence of arcs (x_i, x_j) and (x_j, x_k) implies the existence of arc (x_i, x_k). The *transitive closure* of a graph $G = (X, A)$ is the graph $G_{tc} = (X, A \cup A')$ where A' is the minimal number of arcs necessary to make G_{tc} transitive. It is then quite apparent that, since a path from x_i to x_j in G must correspond to an arc (x_i, x_j) in G_{tc}, the reachability matrix $\mathbf{R}$ of G is the same as the adjacency matrix $\mathbf{A}$ of G_{tc} with all its diagonal elements set to 1.

3. Calculation of Strong Components

A strong component (SC) of a graph G has been defined in the previous chapter as being a maximal strongly connected subgraph of G. Since, for a strongly connected graph, vertex x_j is reachable from vertex x_i and *vice versa* for any x_i and x_j, the SC containing a given vertex x_i is unique and x_i will appear in the set of vertices of one and only one SC. This statement is quite obvious since if x_i appears in two or more strong components, then a path from any vertex in one SC to any other vertex in another SC would always exist (via x_i), and hence the union of the strong components would be strongly connected, a fact which is contrary to the definition of the SC.

If vertex x_i is taken to be both the initial and terminal vertex then the set of vertices essential with respect to these two identical ends (i.e. the set of vertices on some circuit containing x_i) is given by $R(x_i) \cap Q(x_i)$. Since all these essential vertices can reach and be reached from x_i, they can also reach and be reached from each other. Moreover, as no other vertex is

† Alternative ways of constructing the sets $R(x_i)$ and $Q(x_i)$ using a vertex labelling procedure starting from x_i, are indicated in Chapter 7 dealing with *trees*.

essential with respect to the ends x_i and x_i, the set $R(x_i) \cap Q(x_i)$—which can be calculated by eqns (2.1) and (2.2)—defines the vertices of the unique SC of G containing vertex x_i.

If these vertices are removed from the graph $G = (X, \Gamma)$, the remaining subgraph $G' = \langle X - R(x_i) \cap Q(x_i) \rangle$ can be treated in the same way by finding a SC containing a vertex $x_j \in X - R(x_i) \cap Q(x_i)$. This process can be repeated until all the vertices of G have been allocated to one SC. When this is done the graph G is said to have been *partitioned* into its strong components [3].

The graph $G^* = (X^*, \Gamma^*)$ defined so that each of its vertices represents the vertex set of a strong component of G, and an arc (x_i^*, x_j^*) exists if and only if there exists an arc (x_i, x_j) in G for some $x_i \in x_i^*$ and some $x_j \in x_j^*$; is called the *condensed* graph of G.

It is quite apparent that the condensed graph G^* contains no circuits, since a circuit would mean that any vertex on that circuit could reach (and be reached from) any other vertex, and hence the union of these vertices of G^* would be a SC of G^* and therefore also a SC of G, a fact which is contrary to the definition that the vertices of G^* correspond to the SC's of G.

3.1 Example

For the graph G given in Fig. 2.2, find the strong components, and the condensed graph G^*.

Let us find the SC of G containing vertex x_1.

From eqns (2.1) and (2.2) we find:

$$R(x_1) = \{x_1, x_2, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$$

and

$$Q(x_1) = \{x_1, x_2, x_3, x_5, x_6\}$$

Therefore the SC containing vertex x_1 is the subgraph

$$\langle R(x_1) \cap Q(x_1) \rangle = \langle \{x_1, x_2, x_5, x_6\} \rangle$$

Similarly, the SC containing vertex x_8, say, is the subgraph $\langle \{x_8, x_{10}\} \rangle$, the SC containing x_7 is $\langle \{x_4, x_7, x_9\} \rangle$, the SC containing vertex x_{11} is $\langle \{x_{11}, x_{12}, x_{13}\} \rangle$ and the SC containing vertex x_3 is $\langle \{x_3\} \rangle$. It should be noted that this last SC contains just a single vertex of G.

The condensed graph G^* is then given by the graph of Fig. 2.3.

The operations described above in order to find the SC's of a graph can also be done most conveniently by the direct use of the **R** and **Q** matrices defined in the previous section. Thus, if we write $\mathbf{R} \otimes \mathbf{Q}$ to mean the element-by-element multiplication of the two matrices, then it is immediately apparent that row x_i of the matrix $\mathbf{R} \otimes \mathbf{Q}$ contains values of 1 in those columns x_j for

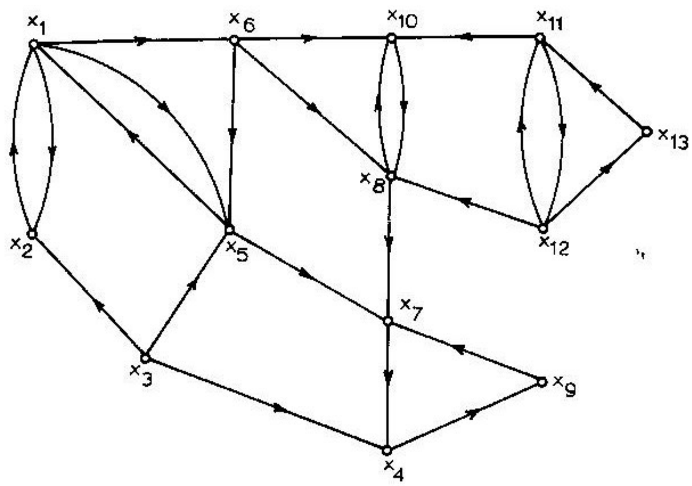

FIG. 2.2. The graph G

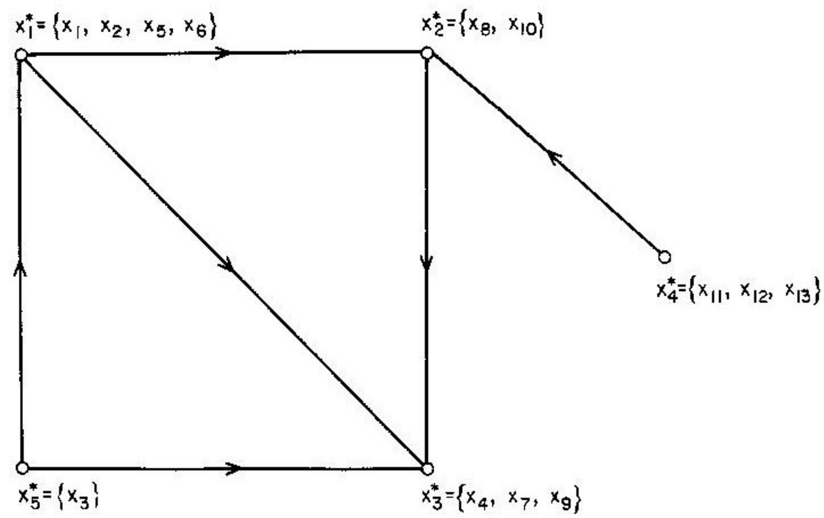

FIG. 2.3. The condensed graph G^*

which x_i and x_j are mutually reachable, and values of 0 in all other places. Thus, two vertices are in the same SC if and only if their corresponding rows (or columns) are identical. The vertices whose corresponding rows contain an entry of 1 under column x_j, then form the vertex set of the SC containing x_j. It is quite apparent that the matrix $\mathbf{R} \otimes \mathbf{Q}$ can be transformed by transposition of rows and columns into block diagonal form; each of the diagonal submatrices corresponding to a SC of G and having entries of 1's, all other entries being 0. For the above example the matrix $\mathbf{R} \otimes \mathbf{Q}$ arranged in this form becomes:

$$
\mathbf{R} \otimes \mathbf{Q} = \begin{array}{c|cccc|cc|ccc|ccc|c|}
 & x_1 & x_2 & x_5 & x_6 & x_8 & x_{10} & x_4 & x_7 & x_9 & x_{11} & x_{12} & x_{13} & x_3 \\ \hline
x_1 & 1 & 1 & 1 & 1 & & & & & & & & & \\
x_2 & 1 & 1 & 1 & 1 & 0 & & & 0 & & & 0 & & 0 \\
x_5 & 1 & 1 & 1 & 1 & & & & & & & & & \\
x_6 & 1 & 1 & 1 & 1 & & & & & & & & & \\ \hline
x_8 & & & & & 1 & 1 & & 0 & & & 0 & & 0 \\
x_{10} & & & & & 1 & 1 & & & & & & & \\ \hline
x_4 & & & & & & & 1 & 1 & 1 & & & & \\
x_7 & & 0 & & & 0 & & 1 & 1 & 1 & & 0 & & 0 \\
x_9 & & & & & & & 1 & 1 & 1 & & & & \\ \hline
x_{11} & & & & & & & & & & 1 & 1 & 1 & \\
x_{12} & & 0 & & & 0 & & & 0 & & 1 & 1 & 1 & 0 \\
x_{13} & & & & & & & & & & 1 & 1 & 1 & \\ \hline
x_3 & & 0 & & & 0 & & & 0 & & & 0 & & 1 \\ \hline
\end{array}
$$

4. Bases

A *basis* B of a graph is a set of vertices from which every vertex of the graph can be reached and which is minimal in the sense that no subset of B has this property. Thus, if we write $R(B)$—the reachable set of B—to mean:

$$R(B) = \bigcup_{x_i \in B} R(x_i) \tag{2.3}$$

then B is a basis if and only if:

$$R(B) = X \qquad \text{and} \qquad \forall S \subset B, R(S) \neq X \tag{2.4}$$

The second condition ($R(S) \neq X$, $\forall S \subset B$) of eqn (2.4) is equivalent to the condition $x_j \notin R(x_i)$ for any two distinct x_i, $x_j \in B$, i.e. a vertex in B cannot be reached from another vertex also in B. This can be shown as follows:

Since for any two sets H and $H' \subseteq H$ we have $R(H') \subseteq R(H)$, the condition $R(S) \neq X$, $\forall S \subset B$ is equivalent to $R(B - \{x_j\}) \neq X$ for all $x_j \in B$, in other words $R(x_j) \nsubseteq R(B - \{x_j\})$. This last condition can be satisfied if and only if $x_j \notin R(B - \{x_j\})$, i.e. if and only if $x_j \notin R(x_i)$ for any x_i, $x_j \in B$.

A basis is, therefore, a set B of vertices which satisfies the following two conditions:

(i) All vertices of G are reachable from some vertex in B and

(ii) No vertex in B can reach another vertex also in B.

From the above two conditions we can immediately state that:

(a) No two vertices in B can be in the same SC of the graph G.

and (b) For any graph without circuits there is a unique basis consisting of of the set of points with indegree 0.

The proofs of these statements are very simple and follow directly from the definitions. (See problems P2.3 and P2.4).

Thus, according to (a) and (b) above, since the condensed graph G^* of a graph G has no circuits, its basis B^* say, is the set of vertices of G^* with indegree 0. The bases of G itself can then be found by forming sets containing one vertex from each one of the vertex-sets in B^*, i.e. if $B^* = \{S_1, S_2, \ldots, S_m\}$ —m being the number of vertex-sets S_j in the basis B^* of G^*—then B is any set $\{x_{i_1}, x_{i_2}, \ldots, x_{i_m}\}$ where $x_{i_j} \in S_j$.

4.1 Example

For the graph G shown in Fig. 2.2 the condensed graph G^* is given in Fig. 2.3. The basis of this graph is $\{x_4^*, x_5^*\}$ since x_4^* and x_5^* are the only two vertices of G^* with indegree 0. The bases of G are then given by $\{x_3, x_{11}\}$, $\{x_3, x_{12}\}$ and $\{x_3, x_{13}\}$.

A concept dual to that of the basis can be defined in terms of the reaching sets $Q(x_i)$ as follows:

A *contra-basis* $\bar{B}$ is a set of vertices of the graph $G = (X, \Gamma)$, so that

$$\left.\begin{aligned} Q(\bar{B}) = \bigcup_{x_i \in \bar{B}} Q(x_i) = X \\ \text{and} \qquad \forall S \subseteq \bar{B}, \quad Q(S) \neq X \end{aligned}\right\} \tag{2.5}$$

i.e. $\bar{B}$ is a minimal set of vertices which can be reached from every other vertex. The properties of $\bar{B}$ are analogous to those of B where directed concepts are replaced by the opposite counterparts. For example, the definition of eqn (2.5) is equivalent to two conditions similar to (i) and (ii) above but with B replaced by $\bar{B}$ and the words "are reachable from" replaced by "can reach" and *vice versa*.

Thus, the contra-basis of a condensed graph G^* is the set of vertices of G^* with outdegree 0, and the contra-bases of G itself can then be found by constructing sets taking one vertex from each vertex-set in the contra-basis of G^*, similar to what was done previously for the bases.

In the example of the graph G shown in Fig. 2.2, the condensed graph G^*, (shown in Fig. 2.3), contains only one vertex x_3^* with out-degree 0. Thus the contra-basis of G^* is $\{x_3^*\}$ and the contra-bases of G are $\{x_4\}$, $\{x_7\}$ and $\{x_9\}$.

4.2 Application to organizational structure

If the graph G represents the authority or influence structure of an organization, then members of a strong component of G would have equal authority and influence over each other such as could, for example, exist in a committee. A basis of G could then be interpreted as a "coalition" involving the least number of people which would have authority over every person in the organization [2, 3].

On the set of vertices representing people of the same organization, let a new graph G' be constructed to represent channels of communication, so that arc (x_i, x_j) implies that x_i can communicate with x_j. G' will of course be related to G but in which way is not necessarily obvious. The least number of people who know or can obtain all the facts about the organization then form one of the contra-bases of G'. One may then be justified in saying that an effective coalition to control the organization would be the set H of people given by:

$$H = B(G) \cup \bar{B}(G') \tag{2.6}$$

where $B(G)$ and $\bar{B}(G')$ are one of the bases and contra-bases of G and G' respectively chosen so that $|H|$, the number of people in H, is a minimum.

The above graph-theoretic description of an organization is, of course, greatly oversimplified. One of the shortcomings which spring immediately to mind is that it may not be so desirable for a person outside the basis to have authority over a person who is inside.

One can, therefore, define a *power-basis* as the set of vertices $B_p \subseteq X$, so that,

$$R(B_p) = X, \qquad Q(B_p) = B_p \tag{2.7a}$$

and

$$R(S) \neq X \qquad \forall S \subset B_p \tag{2.7b}$$

The second part of condition (2.7a) expresses the fact that only people within B_p can have authority over other people also in B_p, and could be replaced by the equivalent condition $R(X - B_p) \cap B_p = \emptyset$. This condition implies that if a vertex in a SC of G is in B_p, then every other vertex in the same SC must also be in B_p. Since the basis of G^* is the set of those of its vertices with indegree 0 i.e. those not reachable from any other vertex, the power-basis of G is then the union of the vertex-sets in the basis of G^* i.e.

$$B_p = \bigcup_{S_i \in B^*} S_i \tag{2.8}$$

For the graph of example 4.1 (Figs. 2.2 and 2.3), the power-basis of G is $\{x_3, x_{11}, x_{12}, x_{13}\}$. It may be noted that if this graph represents an organization, x_3 may be regarded as its top boss having authority over the sets of people x_1^*, x_2^* and x_3^*, whereas $\{x_{11}, x_{12}, x_{13}\}$ may be regarded as a committee having authority over the two sets of people x_2^* and x_3^*.

5. Problems Associated with Limiting the Reachability

The basis was defined in the last section from the unrestricted reachability sets. If the reachability is restricted to those vertices which are reached along paths of limited cardinality (as mentioned earlier in Section 2), and a restricted basis is defined in terms of these reachabilities, two complications arise.

(a) The concepts of strong components and condensed graphs, which simplified the problem of finding bases in the case of unrestricted reachability, cannot now be used. Extensions of these concepts to the case of restricted reachability do not lead to any significant reduction in problem complexity.

In the case where reachability is limited to single arcs, the restricted bases are called *minimal dominating sets*, and they are considered in greater detail in Chapter 3. In the case where reachability is limited to, say, q arcs, a graph G' may be defined whose vertex set is the same as that of G and where arc (x_i, x_j) exists if and only if there is a path of cardinality less than or equal to q leading from x_i to x_j (see eqn 2.1). The restricted reachability matrix of G will then correspond to the adjacency matrix of G' which is a graph that can be called the restricted transitive closure of G: according to the definition of the transitive closure given in Section 2. The problem of finding the restricted bases of G is then equivalent to the problem of finding the minimal dominating sets of G'.

(b) In the unrestricted case, all the different bases contain the same number of elements; this number being given by the number of vertices of the condensed graph having indegree 0. In the restricted case, however, the restricted bases may contain different numbers of elements and what may now be required is that particular restricted basis with the smallest number of elements. Alternatively, if the restriction on the reachability is not given, one may require that restricted basis which contains exactly p (say) vertices and which can reach all other vertices of G within the smallest possible restricted reachability. These problems are very closely related to the problem of finding a *p-centre* and which is discussed in much greater detail in Chapter 4.

6. Problems P2

1. For the graph of Fig. 2.4 find the reachability and reaching matrices.

2. For the graph of Fig. 2.4 calculate the strong components, draw the condensed graph and find all the bases.

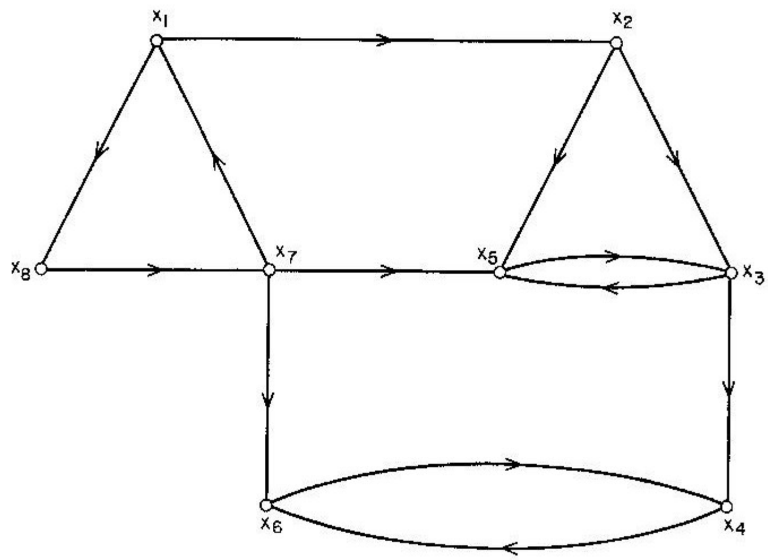

FIG. 2.4

3. Prove that in any graph G all vertices of indegree 0 are in every basis.

4. Prove that in any graph without circuits there is a unique basis consisting of all points of indegree 0.

5. Show that any two bases of a graph G have the same number of vertices.

6. Prove that a vertex x_i is in both the basis B and contrabasis $\bar{B}$ of a graph G if and only if the strong component containing x_i corresponds to an isolated vertex of the condensed graph G^*.

7. Show that if $\hat{x}_i$ is a vertex of the graph (X, A) which produces a maximum to the expression:

$$\max_{x_i \in X} |R(x_i)|,$$

then $\hat{x}_i$ is in a basis.

8. With all arithmetic operations taken to be boolean, (i.e. $0 + 0 = 0$, $0 + 1 = 1 + 1 = 1$, $0 \cdot 0 = 0 \cdot 1 = 0$ and $1 \cdot 1 = 1$), show that—with reachability restricted to paths of q arcs or less—the restricted reachability matrix $\mathbf{R}_q$ is given by:

$$\mathbf{R}_q = \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \ldots + \mathbf{A}^q = (\mathbf{I} + \mathbf{A})^q$$

where $\mathbf{I}$ is the $n \times n$ unity matrix.

9. Show that the entry $r_{ii}^{(2)}$ of the matrix $\mathbf{R}^2$ is equal to the number of vertices in the strong component containing vertex x_i.

7. References

1. Chen, Y. C. and O. Wing (1964), Connectivity of directed graphs, Proc. of Allerton Conference on Circuit and System Theory, Univ. of Illinois.
2. Flament, C. (1963), "Applications of graph theory to group structure", Prentice-Hall, Engelwood Cliffs, New Jersey.

3. Harary, F., Norman, R. Z. and Cartwright, D. (1965), "Structural models: An introduction to the theory of directed graphs", Wiley, New York.
4. Leifman, L. Ya. (1966), An efficient algorithm for partitioning an oriented graph into bicomponents, *Kibernetika*, **2**, p. 19 (in Russian). English translation: Cybernetics, Plenum Press, New York.
5. Marimont, R. B. (1959), A new method of checking the consistency of precedence matrices, *Jl. of ACM*, **6**, p. 164.
6. Moyles, D. M. and Thompson, G. L. (1969), An algorithm for finding a minimum equivalent graph of a digraph. *Jl. of ACM*, **16**, p. 455.
7. Ramamoorthy, C. V. (1966), Analysis of graphs by connectivity considerations, *Jl. of ACM*, **13**, p. 211.