Section V.6: Warshall's Algorithm to find Transitive Closure

Definition V.6.1: Let S be the finite set $\{v_1, ..., v_n\}$, R a relation on S. The **adjacency matrix** A of R is an n x n Boolean (zero-one) matrix defined by

$$A_{i,j} \quad = \quad \begin{cases} 1 \text{ if the digraph } D \text{ has an edge from } v_i \text{ to } v_j \\ 0 \text{ if the digraph } D \text{ has no edge from } v_i \text{ to } v_j \end{cases}$$

(This is a special case of the adjacency matrix M of a directed graph in Epp p. 642. Her definition allows for more than one edge between two vertices. But the digraph of a relation has at most **one** edge between any two vertices).

Warshall's algorithm is an efficient method of finding the adjacency matrix of the transitive closure of relation R on a finite set S from the adjacency matrix of R. It uses properties of the digraph D, in particular, walks of various lengths in D.

The definition of walk, transitive closure, relation, and digraph are all found in Epp.

Definition V.6.2: We let A be the adjacency matrix of R and T be the adjacency matrix of the transitive closure of R. T is called the **reachability matrix** of digraph D due to the property that $T_{i,j} = 1$ if and only if v_j can be reached from v_i in D by a sequence of arcs (edges).

Digraph Implementation

Definition V.6.3: If a, v_1, v_2, ... , v_n, b is a walk in a digraph D, $a \neq v_1$, $b \neq v_n$, $n > 2$, then v_1, v_2, ... and v_n are the **interior vertices** of this walk (path).

In **Warshall's algorithm** we construct a sequence of Boolean matrices $A = W^{[0]}$, $W^{[1]}, W^{[2]}, \ldots, W^{[n]} = T$, where A and T are as above. This can be done from digraph D as follows.

$[W^{[1]}]_{i,j} = 1$ if and only if there is a walk from v_i to v_j with elements of a subset of $\{v_1\}$ as interior vertices.

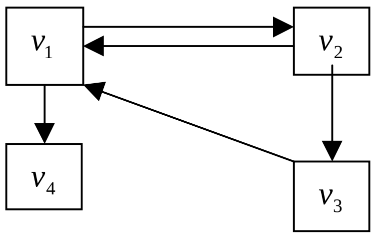
$[W^{[2]}]_{i,j} = 1$ if and only if there is a walk from v_i to v_j with elements of a subset of $\{v_1, v_2\}$ as interior vertices.

Continuing this process, we generalize to

$[W^{[k]}]_{i,j} = 1$ if and only if there is a walk from v_i to v_j with elements of a subset of $\{v_1, v_2, \ldots, v_k\}$ as interior vertices.

Note: In constructing $W^{[k]}$ from $W^{[k-1]}$ we shall either keep zeros or change some zeros to ones. No ones ever get changed to zeros. Example V.6.1 illustrates this process.

Example V.6.1: Get the transitive closure of the relation represented by the digraph below. Use the method described above. Indicate what arcs must be added to this digraph to get the digraph of the transitive closure, and draw the digraph of the transitive closure.



Solution: $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Walks $\{v_2, v_1, v_2\}$, $\{v_2, v_1, v_4\}$ and $\{v_3, v_1, v_2\}$ have elements of $\{v_1\}$as interior vertices. Therefore $[W^{[1]}]_{2,2} = 1$, $[W^{[1]}]_{2,4} = 1$, and $[W^{[1]}]_{3,2} = 1$

$$W^{[1]} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & \underline{1} & 1 & \underline{1} \\ 1 & \underline{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ The new "ones" are underlined.

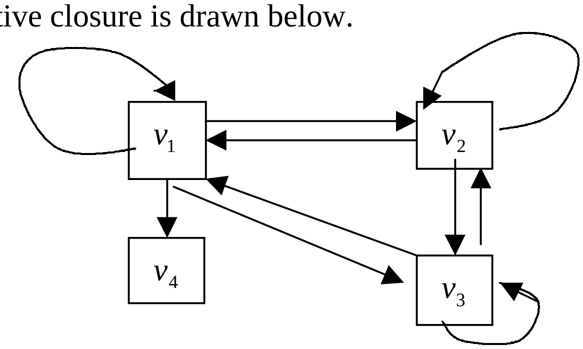
We need only consider walks with v_2 or v_1 and v_2 as interior vertices since walks with v_1 as an interior vertex have already been considered. Walks $\{v_1, v_2, v_3\}$, $\{v_1, v_2, v_1\}$, and $\{v_3, v_1, v_2, v_3\}$ have elements of $\{v_1, v_2\}$ as interior vertices. Therefore $[W^{[2]}]_{1,1} = 1$, $[W^{[2]}]_{1,3}$ =1, and $[W^{[2]}]_{3,3} = 1$. There are other walks with elements of subsets of $\{v_1, v_2\}$as interior vertices, but they do not contribute any new "ones" to $W^{[2]}$.

$$W^{[2]} = \begin{bmatrix} 0 & 1 & \underline{1} & 1 \\ 1 & 1 & 1 & 1 \\ 1 & \underline{1} & \underline{1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ The new “ones” are underlined.

There are walks with elements of subsets of $\{v_1, v_2, v_3\}$ as interior vertices. We need only consider walks with v_3 (possibly along with v_1 or v_2 or both) as interior vertices since walks with v_1, v_2 (but not v_3) as interior vertices have already been considered. However, none of these walks create any new “ones” in $W^{[3]}$. We continue this process to obtain $W^{[4]}$. However, any walks we construct with v_4 as an interior vertex contributes no new “ones”. Therefore, $T = W^{[4]} = W^{[3]} = W^{[2]}$. Therefore,

$$T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

One must add arcs from v_1 to v_1, v_1 to v_3, v_2 to v_2, v_3 to v_2, and v_3 to v_3. The graph of the transitive closure is drawn below.



PseudoCode Implementation

No algorithm is practical unless it can be implemented for a large data set. The following version of Warshall’s algorithm is found in Bogart’s text (pp. 470-471). The algorithm immediately follows from definition V.6.4.

Definition V.6.4: If *A* is an m x n matrix, then the **Boolean OR operation** of row *i* and row *j* is defined as the *n*-tuple $x = (x_1, x_2, \cdots, x_n)$ where each $x_k = a_{ik} \vee a_{jk}$. We do componentwise OR on row *i* and row *j*.

Notation: Let a_i and a_j denote the *i*-th and *j*-th rows of *A*, respectively**.** Then we say $x = a_i \vee a_j$**.** (Italic *x* represents an *n*-tuple**)**

Algorithm Warshall
Input: Adjacency matrix A of relation R on a set of n elements
Output: Adjacency matrix T of the transitive closure of R.
Algorithm Body:

```
T := A [initialize T to A]
for j := 1 to n
    for i := 1 to n
        if T_{i,j} = 1 then
            a_i := a_i ∨ a_j [form the Boolean OR of row i and row j, store it in a_i]
    next i
next j
```

end Algorithm Warshall

Note: The matrix T at the end of each iteration of j is the same as $W^{[j]}$ in the digraph implementation of Warshall's algorithm.

Example V.6.2: Let $A = T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Trace the pseudocode implementation of Warshall's algorithm on A, showing the details of each Boolean OR between rows

Solution:

$j = 1$ $\quad i = 1 \quad T_{i,j} = 0$ no action
$\quad i = 2 \quad T_{i,j} = 0$ no action
$\quad i = 3 \quad T_{i,j} = 0$ no action Therefore $W^{[1]} = T = A$

$j = 2$ $\quad i = 1 \quad T_{i,j} = 1$ (0 1 0) OR (0 1 1) = $(0 \vee 0, 1 \vee 1, 0 \vee 1)$ = (0 1 1)
row 1 of T becomes (0 1 1)
$\quad i = 2 \quad T_{i,j} = 1$ row 2 OR row 2 is computed and put into row 2
however, row 2 OR row 2 = row 2
$\quad i = 3 \quad T_{i,j} = 0$ no action

At this stage we have $T = W^{[2]} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$j = 3$ $i = 1$ $T_{i,j} = 0$ no action

$i = 2$ $T_{i,j} = 1$ (0 1 1) OR (0 0 0) = $(0 \vee 0, 1 \vee 0, 1 \vee 0)$ = (0 1 1)

result is put into row 2, which is unchanged

$i = 3$ $T_{i,j} = 0$ no action

At this stage $T = W^{[3]} = W^{[2]}$ above. We now have the transitive closure.

Exercises:

(1) For each of the adjacency matrices A given below, **(a)** draw the corresponding digraph and **(b)** find the matrix T of the transitive closure using the digraph implementation of Warshall's algorithm. Show all work (see example V.6.1). **(c)** Indicate what arcs must be added to the digraph for A to get the digraph of the transitive closure, and draw the digraph of the transitive closure.

(i) $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

(2) For the matrix A in example V.6.1, compute all the Boolean OR operations that occur in the pseudocode version of Warshall's algorithm. Write separately the matrices that result from a OR operation in the inner loop. Also convince yourself that the matrix T at the end of each iteration of j is the same as $W^{[j]}$ in the digraph implementation of Warshall's algorithm.