Hoofdstuk 5

Integratietechnieken

5.1 De onbepaalde integraal

5.1.1 De onbepaalde integraal, fundamentele integralen
5.1.2 Integratie door splitsing

5.2 Integratie door substitutie

5.2.1 Substitutieregel
5.2.2 Enkele typevoorbeelden
5.2.3 Bijkomende fundamentele integralen
5.2.4 Substitutie bij bepaalde integralen

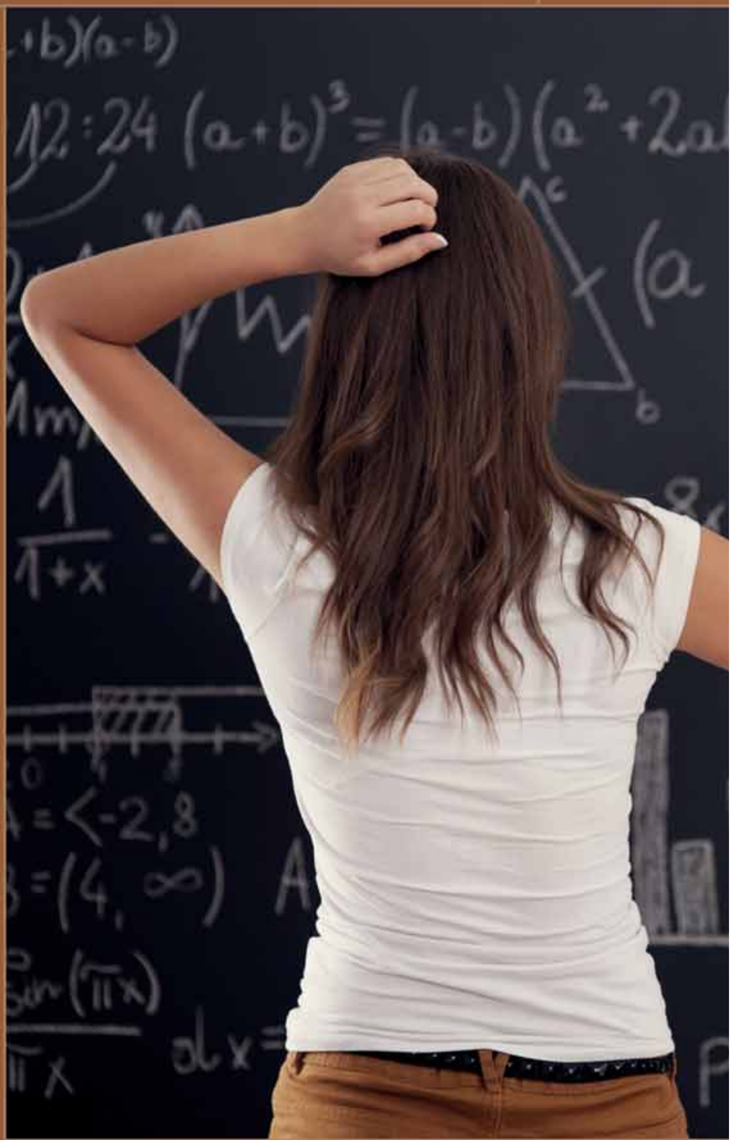
5.3 Partiële integratie

5.3.1 Methode van de partiële integratie
5.3.2 Enkele typevoorbeelden
5.3.3 Partiële integratie bij bepaalde integralen

U **5.4 Integratie van rationale functies**

5.4.1 De noemer is van de eerste graad
5.4.2 De noemer is van de tweede graad en ontbindbaar in factoren van de eerste graad
5.4.3 De noemer is van de tweede graad en niet ontbindbaar in factoren van de eerste graad
5.4.4 De graad van de noemer is groter dan twee

U **5.5 Integratie van irrationale functies**



Opdracht 1 bladzijde 84

Bepaal alle primitieve functies van

1 $f: x \mapsto x^2$

$$F(x) = \frac{x^3}{3} + c \quad \text{want} \quad \frac{d}{dx}\left(\frac{x^3}{3}\right) = \frac{1}{3} \cdot 3x^2 = x^2$$

2 $f: x \mapsto \cos x$

$$F(x) = \sin x + c \quad \text{want} \quad \frac{d}{dx}(\sin x) = \cos x$$

3 $f: x \mapsto e^x + 3^x$

$$F(x) = e^x + \frac{3^x}{\ln 3} + c \quad \text{want} \quad \frac{d}{dx}\left(e^x + \frac{3^x}{\ln 3}\right) = e^x + \frac{1}{\ln 3} \cdot 3^x \cdot \ln 3 = e^x + 3^x$$

Opdracht 2 bladzijde 86

Bereken de onbepaalde integralen.

1 $\int dx = \int 1 \cdot dx = x + c$

2 $\int \frac{1}{x^5}\, dx = \int x^{-5}\, dx = \frac{x^{-4}}{-4} + c = -\frac{1}{4x^4} + c$

3 $\int 2^x\, dx = \frac{2^x}{\ln 2} + c$

4 $\int x^{-1}\, dx = \int \frac{1}{x}\, dx = \ln|x| + c$

5 $\int \sqrt[3]{x}\, dx = \int x^{\frac{1}{3}}\, dx = \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + c = \frac{3}{4}\sqrt[3]{x^4} + c$

6 $\int \frac{x^2}{\sqrt{x}}\, dx = \int x^{2-\frac{1}{2}}\, dx = \int x^{\frac{3}{2}}\, dx = \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + c = \frac{2}{5}\sqrt{x^5} + c$

Opdracht 3 bladzijde 86

Bepaal het voorschrift van f als

1 $f'(x) = x^3$ en $f(1) = 0$

$$f(x) = \frac{x^4}{4} + c$$

$$f(1) = 0 \Leftrightarrow \frac{1}{4} + c = 0 \Leftrightarrow c = -\frac{1}{4}$$

$$\Rightarrow f(x) = \frac{x^4}{4} - \frac{1}{4}$$

2 $f'(x) = \dfrac{1}{\cos^2 x}$ en $f\left(\dfrac{\pi}{4}\right) = 1$

$$f(x) = \int \frac{1}{\cos^2 x}\,dx = \tan x + c$$

$$f\left(\frac{\pi}{4}\right) = 1 \Leftrightarrow \tan\frac{\pi}{4} + c = 1 \Leftrightarrow 1 + c = 1 \Leftrightarrow c = 0$$

$$\Rightarrow f(x) = \tan x$$

Opdracht 4 bladzijde 87

Bereken de onbepaalde integralen.

1 $\displaystyle\int\left(x\sqrt{x} + \frac{1}{x^2}\right)dx$

$$= \int\left(x^{\frac{3}{2}} + x^{-2}\right)dx = \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{x^{-1}}{-1} + c = \frac{2}{5}\sqrt{x^5} - \frac{1}{x} + c$$

2 $\displaystyle\int \frac{1 + \cos^2 x}{\cos^2 x}\,dx$

$$= \int\left(\frac{1}{\cos^2 x} + 1\right)dx = \tan x + x + c$$

3 $\displaystyle\int \cot^2 x\,dx$

$$= \int \frac{\cos^2 x}{\sin^2 x}\,dx = \int \frac{1 - \sin^2 x}{\sin^2 x}\,dx = \int\left(\frac{1}{\sin^2 x} - 1\right)dx = -\cot x - x + c$$

Opdracht 5 bladzijde 89

Bewijs: $\int r \cdot f(x)\,dx = r \cdot \int f(x)\,dx$ met $r \in \mathbb{R}_0$.

Bewijs:

$$\frac{d}{dx}\left[r \cdot \int f(x)\,dx\right]$$

$= r \cdot \dfrac{d}{dx}\left(\int f(x)\,dx\right)$ afgeleide van een veelvoud

$= r \cdot f(x)$ gevolg definitie onbepaalde integraal

Hieruit volgt dat $r \cdot \int f(x)\,dx$ een primitieve functie is van $r \cdot f(x)$.

Volgens de definitie van de onbepaalde integraal geldt dan: $\int r \cdot f(x)\,dx = r \cdot \int f(x)\,dx + c$.

Omdat $r \cdot \int f(x)\,dx$ al een integratieconstante bevat, geldt dus: $\int r \cdot f(x)\,dx = r \cdot \int f(x)\,dx$.

Opdracht 6 bladzijde 89

Bereken

1 $\int \left(2x^7 - \frac{5}{x} - 4\sin x\right) dx = 2 \cdot \frac{x^8}{8} - 5\ln|x| + 4\cos x + c = \frac{x^8}{4} - 5\ln|x| + 4\cos x + c$

2 $\int \left(e^x + x^e\right) dx = e^x + \frac{x^{e+1}}{e+1} + c$

Opdracht 7 bladzijde 89

1 Toon met voorbeelden aan dat

a $\int f(x) \cdot g(x)\, dx \neq \int f(x)\, dx \cdot \int g(x)\, dx$

Voorbeeld

$$\int x(x+1)\,dx = \int (x^2 + x)\,dx = \frac{x^3}{3} + \frac{x^2}{2} + c$$

$$\int x\,dx \cdot \int (x+1)\,dx = \left(\frac{x^2}{2} + c_1\right) \cdot \left(\frac{x^2}{2} + x + c_2\right)$$

$$= \frac{x^4}{4} + \frac{x^3}{2} + c_2\frac{x^2}{2} + c_1\frac{x^2}{2} + c_1x + c_1c_2$$

$$\neq \int x(x+1)\,dx$$

b $\int \frac{f(x)}{g(x)}\, dx \neq \frac{\int f(x)\, dx}{\int g(x)\, dx}$

Voorbeeld

$$\int \frac{x^2}{x}\,dx = \int x\,dx = \frac{x^2}{2} + c$$

$$\frac{\int x^2\,dx}{\int x\,dx} = \frac{\frac{x^3}{3} + c_1}{\frac{x^2}{2} + c_2} \neq \int \frac{x^2}{x}\,dx$$

2 Bereken de onbepaalde integralen.

a $\int (3 - x)\sqrt{x}\, dx = \int \left(3x^{\frac{1}{2}} - x^{\frac{3}{2}}\right) dx = 3\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + c$

$$= 2\sqrt{x^3} - \frac{2}{5}\sqrt{x^5} + c$$

b $\displaystyle\int \frac{x^4 - 3x^3 + 5x - 1}{x^2}\,dx = \int\left(x^2 - 3x + \frac{5}{x} - x^{-2}\right)dx$

$$= \frac{x^3}{3} - \frac{3}{2}x^2 + 5\ln|x| + \frac{1}{x} + c$$

c $\displaystyle\int (x+2)(x^2 - 2x + 4)\,dx = \int (x^3 + 8)\,dx = \frac{x^4}{4} + 8x + c$

Opdracht 8 bladzijde 89

Bepaal het voorschrift van f als

1 $f''(x) = x - 3$, $f'(0) = 1$ en $f(0) = 4$

$$f'(x) = \frac{x^2}{2} - 3x + c$$

$$f'(0) = 1 \iff c = 1$$

$$\Rightarrow f'(x) = \frac{x^2}{2} - 3x + 1$$

$$\Rightarrow f(x) = \frac{x^3}{6} - \frac{3}{2}x^2 + x + c$$

$$f(0) = 4 \iff c = 4$$

$$\Rightarrow f(x) = \frac{x^3}{6} - \frac{3}{2}x^2 + x + 4$$

2 $f''(x) = \sin x$, $f'(\pi) = 1$ en $f\left(\dfrac{3\pi}{2}\right) = -1$

$$f'(x) = -\cos x + c$$

$$f'(\pi) = 1 \iff -(-1) + c = 1 \iff c = 0$$

$$\Rightarrow f'(x) = -\cos x$$

$$\Rightarrow f(x) = -\sin x + c$$

$$f\left(\frac{3\pi}{2}\right) = -1 \iff 1 + c = -1 \iff c = -2$$

$$\Rightarrow f(x) = -\sin x - 2$$

Opdracht 9 bladzijde 90

1 We weten dat $\int x^2\,dx = \dfrac{x^3}{3} + c$.

Toon aan dat $\int (13x+5)^2\,dx \neq \dfrac{(13x+5)^3}{3} + c$.

$$\frac{d}{dx}\left(\frac{(13x+5)^3}{3}\right) = \frac{1}{3}\cdot 3(13x+5)^2\cdot 13 = 13\,(13x+5)^2,$$

$$\text{dus}\quad \int (13x+5)^2\,dx \neq \frac{(13x+5)^3}{3} + c$$

2 Bepaal $f(x)$ als $\int f(x)\,dx = \dfrac{(13x+5)^3}{3} + c$.

$f(x) = 13(13x+5)^2$

want $\dfrac{(13x+5)^3}{3}$ is een primitieve functie van $13(13x+5)^2$,

zie vraag 1

Opdracht 10 bladzijde 90

Bepaal $f(x)$ als $\int f(x)\,dx = \sin(x^2) + c$.

$$\frac{d}{dx}\left[\sin(x^2)\right] = 2x\cos(x^2)$$

dus $f(x) = 2x\cos(x^2)$

Opdracht 11 bladzijde 92

Bereken

1 $\int \cos 4x\,dx$ $\qquad u = 4x \Rightarrow du = 4dx \Rightarrow dx = \frac{1}{4}\,du$

$$= \frac{1}{4}\int \cos u\;du = \frac{1}{4}\sin u + c = \frac{1}{4}\sin 4x + c$$

2 $\int (7x-5)^4\,dx$ $\qquad u = 7x-5 \Rightarrow du = 7dx \Rightarrow dx = \frac{1}{7}\,du$

$$= \frac{1}{7}\int u^4\,du = \frac{1}{7}\cdot\frac{u^5}{5} + c = \frac{1}{35}(7x-5)^5 + c$$

3 $\int \dfrac{dx}{-4x+5}$ $\qquad u = -4x+5 \Rightarrow du = -4\,dx \Rightarrow dx = -\frac{1}{4}\,du$

$$= -\frac{1}{4}\int\frac{du}{u} = -\frac{1}{4}\ln|u| + c = -\frac{1}{4}\ln|-4x+5| + c$$

4 $\displaystyle\int \frac{dx}{1+81x^2} = \int \frac{dx}{1+(9x)^2}$ $\qquad u = 9x \Rightarrow du = 9dx \Rightarrow dx = \frac{1}{9}du$

$$= \frac{1}{9}\int \frac{du}{1+u^2} = \frac{1}{9}\,\text{Bgtan}\,u + c = \frac{1}{9}\,\text{Bgtan}\,(9x) + c$$

5 $\displaystyle\int e^{-5x+3}\,dx$ $\qquad u = -5x+3 \Rightarrow du = -5dx \Rightarrow dx = -\frac{1}{5}du$

$$= -\frac{1}{5}\int e^u\,du = -\frac{1}{5}e^u + c = -\frac{1}{5}e^{-5x+3} + c$$

6 $\displaystyle\int \frac{dx}{\sqrt{1-6x^2}} = \int \frac{dx}{\sqrt{1-\left(\sqrt{6}x\right)^2}}$ $\qquad u = \sqrt{6}\,x \Rightarrow du = \sqrt{6}\,dx \Rightarrow dx = \frac{1}{\sqrt{6}}du$

$$= \frac{1}{\sqrt{6}}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{\sqrt{6}}\,\text{Bgsin}\,u + c = \frac{1}{\sqrt{6}}\,\text{Bgsin}\left(\sqrt{6}x\right) + c$$

Opdracht 12 bladzijde 95

De volgende integralen zijn van het type $\int (f(x))^r \cdot f'(x)\,dx$. Bereken ze met een geschikte substitutie.

1 $\displaystyle\int \frac{dx}{(2x-5)^4} = \int (2x-5)^{-4}\,dx$ $\qquad u = 2x-5 \Rightarrow du = 2dx$

$$= \frac{1}{2}\int u^{-4}\,du = \frac{1}{2}\,\frac{u^{-3}}{-3} + c = -\frac{1}{6u^3} + c = -\frac{1}{6(2x-5)^3} + c$$

2 $\displaystyle\int \frac{x^3}{\sqrt{x^4+2}}\,dx$ $\qquad u = x^4+2 \Rightarrow du = 4x^3dx$

$$= \frac{1}{4}\int \frac{du}{\sqrt{u}} = \frac{1}{4}\int u^{-\frac{1}{2}}\,du = \frac{1}{4}\,\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c = \frac{1}{2}\sqrt{u} + c$$

$$= \frac{1}{2}\sqrt{x^4+2} + c$$

3 $\displaystyle\int \frac{\sin x}{\cos^5 x}\,dx$ $\qquad u = \cos x \Rightarrow du = -\sin x\,dx$

$$= -\int u^{-5}\,du = -\frac{u^{-4}}{-4} + c = \frac{1}{4u^4} + c = \frac{1}{4\cos^4 x} + c$$

4 $\displaystyle\int \frac{\text{Bgtan}\,x}{1+x^2}\,dx$ $\qquad u = \text{Bgtan}\,x \Rightarrow du = \frac{dx}{1+x^2}$

$$= \int u\,du = \frac{u^2}{2} + c = \frac{1}{2}(\text{Bgtan}\,x)^2 + c$$

Opdracht 13 bladzijde 96

De volgende integralen zijn van het type $\int \frac{f'(x)}{f(x)}\,dx$. Bereken ze met een geschikte substitutie.

1 $\int \frac{x^2}{x^3+5}\,dx \qquad u = x^3 + 5 \Rightarrow du = 3x^2 dx$

$$= \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|u| + c = \frac{1}{3}\ln\left|x^3+5\right| + c$$

2 $\int \frac{dx}{x\ln x} \qquad u = \ln x \Rightarrow du = \frac{1}{x}\,dx$

$$= \int \frac{du}{u} = \ln|u| + c = \ln|\ln x| + c$$

3 $\int \frac{dx}{(1+x^2)\,\mathrm{Bgtan}\,x} \qquad u = \mathrm{Bgtan}\,x \Rightarrow du = \frac{dx}{1+x^2}$

$$= \int \frac{du}{u} = \ln|u| + c = \ln|\mathrm{Bgtan}\,x| + c$$

4 $\int \frac{e^x}{e^x+1}\,dx \qquad u = e^x + 1 \Rightarrow du = e^x dx$

$$= \int \frac{du}{u} = \ln|u| + c = \ln\left|e^x+1\right| + c \underset{e^x+1>0}{=} \ln(e^x+1) + c$$

Opdracht 14 bladzijde 96

Bereken met een geschikte substitutie.

1 $\int \frac{\ln^2 x}{x}\,dx \qquad u = \ln x \Rightarrow du = \frac{dx}{x}$

$$= \int u^2\,du = \frac{u^3}{3} + c = \frac{\ln^3 x}{3} + c$$

2 $\int \tan^2 2x\,dx \underset{1+\tan^2 x = \frac{1}{\cos^2 x}}{=} \int\left(\frac{1}{\cos^2 2x} - 1\right)dx = \int \frac{dx}{\cos^2 2x} - x$

$$\underset{\substack{u = 2x \\ du = 2dx}}{=} \frac{1}{2}\int \frac{du}{\cos^2 u} - x$$

$$= \frac{1}{2}\tan u - x + c$$

$$= \frac{1}{2}\tan 2x - x + c$$

3 $\int \sin^2 x\,dx \underset{\cos 2x = 1 - 2\sin^2 x}{=} \int \frac{1-\cos 2x}{2}\,dx = \frac{1}{2}x - \frac{1}{2}\int \cos 2x\,dx$

$$= \frac{1}{2}x - \frac{1}{4}\int \cos 2x\,d(2x) = \frac{1}{2}x - \frac{1}{4}\sin 2x + c$$

4 $\displaystyle\int \frac{dx}{4x^2+12x+9} = \int \frac{dx}{(2x+3)^2} = \frac{1}{2}\int \frac{d(2x+3)}{(2x+3)^2}$

$\displaystyle = -\frac{1}{2}(2x+3)^{-1} + c = -\frac{1}{2\,(2x+3)} + c$

5 $\displaystyle\int 3^{\sin^2 x} \sin 2x\, dx \qquad u = \sin^2 x \Rightarrow du = 2\sin x \cos x\, dx = \sin 2x\, dx$

$\displaystyle = \int 3^u\, du = \frac{3^u}{\ln 3} + c = \frac{3^{\sin^2 x}}{\ln 3} + c$

6 $\displaystyle\int e^x \sqrt{5-3e^x}\, dx \qquad u = 5-3e^x \Rightarrow du = -3e^x dx$

$\displaystyle = -\frac{1}{3}\int u^{\frac{1}{2}}\, du = -\frac{1}{3}\,\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = -\frac{2}{9}\sqrt{(5-3e^x)^3} + c$

Opdracht 15 bladzijde 99

Bereken

1 $\displaystyle\int \frac{dx}{3x^2+2}$

$\displaystyle = \int \frac{dx}{(\sqrt{3}x)^2+2} = \frac{1}{\sqrt{3}}\int \frac{d(\sqrt{3}x)}{(\sqrt{3}x)^2+2} = \frac{1}{\sqrt{2}}\,\frac{1}{\sqrt{3}}\,\text{Bgtan}\left(\frac{\sqrt{3}}{\sqrt{2}}x\right) + c = \frac{1}{\sqrt{6}}\,\text{Bgtan}\left(\frac{\sqrt{3}}{\sqrt{2}}x\right) + c$

2 $\displaystyle\int \frac{dx}{\sqrt{4-5x^2}}$

$\displaystyle = \int \frac{dx}{\sqrt{4-(\sqrt{5}x)^2}} = \frac{1}{\sqrt{5}}\int \frac{d(\sqrt{5}x)}{\sqrt{4-(\sqrt{5}x)^2}} = \frac{1}{\sqrt{5}}\,\text{Bgsin}\left(\frac{\sqrt{5}}{2}x\right) + c$

3 $\displaystyle\int \frac{dx}{\sqrt{2x^2-5}}$

$\displaystyle = \int \frac{dx}{\sqrt{(\sqrt{2}x)^2-5}} = \frac{1}{\sqrt{2}}\int \frac{d(\sqrt{2}x)}{\sqrt{(\sqrt{2}x)^2-5}} = \frac{1}{\sqrt{2}}\ln\left|\sqrt{2}x + \sqrt{2x^2-5}\right| + c$

4 $\displaystyle\int \frac{x}{4x^4+9}\, dx = \int \frac{x\, dx}{(2x^2)^2+9} \qquad u = 2x^2 \Rightarrow du = 4x\, dx$

$\displaystyle = \frac{1}{4}\int \frac{du}{u^2+9} = \frac{1}{3}\cdot\frac{1}{4}\,\text{Bgtan}\,\frac{u}{3} + c$

$\displaystyle = \frac{1}{12}\,\text{Bgtan}\left(\frac{2}{3}x^2\right) + c$

5 $\displaystyle\int \frac{\sin 2x}{\sqrt{9-\cos^2 2x}}\,dx$ $\quad u = \cos 2x \;\Rightarrow\; du = -2\sin 2x\,dx$

$$= -\frac{1}{2}\int \frac{du}{\sqrt{9-u^2}} = -\frac{1}{2}\,\text{Bgsin}\,\frac{u}{3} + c = -\frac{1}{2}\,\text{Bgsin}\left(\frac{\cos 2x}{3}\right) + c$$

6 $\displaystyle\int \frac{e^x}{\sqrt{e^{2x}+1}}\,dx$ $\quad u = e^x \;\Rightarrow\; du = e^x dx$

$$= \int \frac{du}{\sqrt{u^2+1}} = \ln\left|u+\sqrt{u^2+1}\right| + c = \ln\left(e^x+\sqrt{e^{2x}+1}\right) + c$$

Opdracht 16 bladzijde 101

Bereken

1 $\displaystyle\int_2^4 \left(-\frac{1}{2}x+1\right)^3 dx$

$u = -\frac{1}{2}x + 1$

$\Rightarrow\; du = -\frac{1}{2}\,dx$

$x = 2 \;\Rightarrow\; u = 0$

$x = 4 \;\Rightarrow\; u = -1$

$$= -2\int_0^{-1} u^3\,du = -2\left[\frac{u^4}{4}\right]_0^{-1} = -\frac{1}{2}\,(1-0) = -\frac{1}{2}$$

2 $\displaystyle\int_{\frac{1}{3}}^{2} \sqrt{10-3x}\,dx$

$u = 10 - 3x$

$\Rightarrow\; du = -3\,dx$

$x = \frac{1}{3} \;\Rightarrow\; u = 9$

$x = 2 \;\Rightarrow\; u = 4$

$$= -\frac{1}{3}\int_9^4 \sqrt{u}\,du = -\frac{1}{3}\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_9^4 = -\frac{2}{9}\left(4^{\frac{3}{2}} - 9^{\frac{3}{2}}\right)$$

$$= -\frac{2}{9}\,(8-27) = \frac{38}{9}$$

3 $\displaystyle\int_{\frac{1}{2}}^{\frac{1}{\sqrt{3}}} \frac{dx}{\sqrt{1-3x^2}} = \int_{\frac{1}{2}}^{\frac{1}{\sqrt{3}}} \frac{dx}{\sqrt{1-(\sqrt{3}x)^2}} = \frac{1}{\sqrt{3}}\int_{\frac{1}{2}}^{\frac{1}{\sqrt{3}}} \frac{d(\sqrt{3}x)}{\sqrt{1-(\sqrt{3}x)^2}}$

$$= \frac{1}{\sqrt{3}}\left[\text{Bgsin}\,(\sqrt{3}x)\right]_{\frac{1}{2}}^{\frac{1}{\sqrt{3}}} = \frac{1}{\sqrt{3}}\left(\text{Bgsin}\,1 - \text{Bgsin}\,\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{\sqrt{3}}\left(\frac{\pi}{2}-\frac{\pi}{3}\right) = \frac{1}{\sqrt{3}}\cdot\frac{\pi}{6} = \frac{\pi}{6\sqrt{3}}$$

4 $\int_{-1}^{1} x\sqrt{x^2+3}\,dx$

$u = x^2 + 3$

$\Rightarrow\ du = 2x\,dx$

$x = -1 \ \Rightarrow\ u = 4$

$x = 1 \ \Rightarrow\ u = 4$

$= \frac{1}{2}\int_{4}^{4}\sqrt{u}\,du = 0 \qquad \left(\int_{a}^{a} f(x)\,dx = 0\right)$

5 $\int_{0}^{\frac{\pi}{2}} \cos x \sin^3 x\,dx$

$u = \sin x$

$\Rightarrow\ du = \cos x\,dx$

$x = 0 \ \Rightarrow\ u = 0$

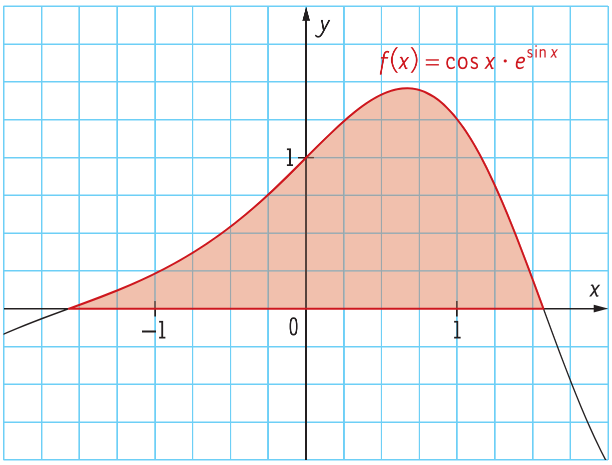
$x = \frac{\pi}{2} \ \Rightarrow\ u = 1$

$$= \int_{0}^{1} u^3\,du = \left[\frac{u^4}{4}\right]_0^1 = \frac{1}{4}$$

Opdracht 17 bladzijde 102

Gegeven is de functie $f\colon x \mapsto \cos x \cdot e^{\sin x}$.

Bereken de oppervlakte van het gekleurde gebied.



- snijpunten met de x-as:

  $\cos x \cdot e^{\sin x} = 0 \ \Leftrightarrow\ \cos x = 0 \ \Leftrightarrow\ x = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$

  $\Rightarrow\ -\frac{\pi}{2}$ en $\frac{\pi}{2}$ zijn de onder- en de bovengrens van de te berekenen integraal

- $A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\, e^{\sin x}\,dx$

  $u = \sin x$

  $\Rightarrow\ du = \cos x\,dx$

  $x = -\frac{\pi}{2} \ \Rightarrow\ u = -1$

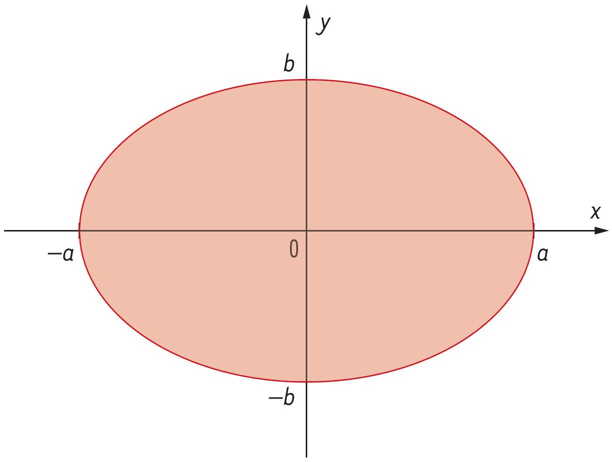
  $x = \frac{\pi}{2} \ \Rightarrow\ u = 1$

$$= \int_{-1}^{1} e^u\,du = \left[e^u\right]_{-1}^{1} = e - e^{-1} = \frac{e^2-1}{e}$$

Opdracht 18 bladzijde 102

De vergelijking $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (met $a>0$ en $b>0$) hoort bij een **ellips** met middelpunt de oorsprong en waarbij de betekenis van a en b kan afgelezen worden op de afbeelding hieronder.

Toon aan dat de oppervlakte van een ellips gelijk is aan πab.



- $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \iff y^2=b^2\left(1-\frac{x^2}{a^2}\right)$

  De bovenste helft van de ellips heeft als vergelijking $y=b\sqrt{1-\frac{x^2}{a^2}}$

  $$=\frac{b}{a}\sqrt{a^2-x^2}$$

- $A=4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$

  $$=4\,\frac{b}{a}\int_0^a\sqrt{a^2-x^2}\,dx$$

  $x=a\cos t$ en $t\in I \Rightarrow dx=-a\sin t\,dt$

  $x=0 \Rightarrow \cos t=0 \Rightarrow t=\frac{\pi}{2}$

  $x=a \Rightarrow \cos t=1 \Rightarrow t=0$

  $$=4\,\frac{b}{a}\int_{\frac{\pi}{2}}^{0}\sqrt{a^2\,(1-\cos^2 t)}(-a\sin t)\,dt$$

  $$=4\,\frac{b}{a}\int_{\frac{\pi}{2}}^{0}a\sin t(-a\sin t)\,dt$$

  $$=-4ab\int_{\frac{\pi}{2}}^{0}\sin^2 t\,dt$$

  $$=4ab\int_0^{\frac{\pi}{2}}\frac{1-\cos 2t}{2}\,dt$$

  $$=2ab\left[t-\frac{1}{2}\sin 2t\right]_0^{\frac{\pi}{2}}$$

  $$=2ab\left(\frac{\pi}{2}-\frac{1}{2}\sin\pi-(0-0)\right)$$

  $$=\pi ab$$

Opdracht 19 bladzijde 104

Bereken $\frac{d}{dx}(x \cdot e^x)$ en gebruik dit om $\int x \cdot e^x\, dx$ te berekenen.

$$\frac{d}{dx}(xe^x) = e^x + x\,e^x$$

Hieruit volgt dat

$$\int \frac{d}{dx}(xe^x)dx = \int e^x\,dx + \int xe^x\,dx$$

en dus:

$$\int xe^x\,dx = xe^x - \int e^x\,dx$$

$$= xe^x - e^x + c$$

$$(= e^x(x-1) + c)$$

Opdracht 20 bladzijde 107

Bereken $\int x \cdot 2^x\, dx$.

$$\int x \cdot 2^x\,dx$$

$$u = x \;\Rightarrow\; du = dx$$

$$dv = 2^x dx \;\Rightarrow\; v = \frac{2^x}{\ln 2}$$

$$= x \cdot \frac{2^x}{\ln 2} - \frac{1}{\ln 2}\int 2^x\,dx$$

$$= \frac{1}{\ln 2}\,x \cdot 2^x - \frac{1}{\ln^2 2} \cdot 2^x + c$$

Opdracht 21 bladzijde 108

Bereken

1 $\int x \cos 2x\, dx$

$$u = x \;\Rightarrow\; du = dx$$

$$dv = \cos 2x\,dx \;\Rightarrow\; v = \frac{1}{2}\sin 2x$$

$$= \frac{1}{2}x\sin 2x - \frac{1}{2}\int \sin 2x\,dx$$

$$= \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + c$$

2 $\int x^3 e^x \, dx$

$$u = x^3 \;\Rightarrow\; du = 3x^2 dx$$
$$dv = e^x \, dx \;\Rightarrow\; v = e^x$$

$$= x^3 e^x - 3\int x^2 e^x \, dx$$

$$u = x^2 \;\Rightarrow\; du = 2x dx$$
$$dv = e^x \, dx \;\Rightarrow\; v = e^x$$

$$= x^3 e^x - 3\left(x^2 e^x - 2\int x e^x \, dx\right)$$

$$= x^3 e^x - 3x^2 e^x + 6\int x e^x \, dx$$

$$u = x \;\Rightarrow\; du = dx$$
$$dv = e^x \, dx \;\Rightarrow\; v = e^x$$

$$= x^3 e^x - 3x^2 e^x + 6\left(x e^x - \int e^x \, dx\right)$$

$$= x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + c$$

3 $\int \ln x \, dx$

$$u = \ln x \;\Rightarrow\; du = \frac{1}{x} dx$$
$$dv = dx \;\Rightarrow\; v = x$$

$$= x \ln x - \int x \cdot \frac{1}{x} dx$$

$$= x \ln x - \int dx$$

$$= x \ln x - x + c$$

4 $\int e^{-2x} \sin 2x \, dx$

$$u = e^{-2x} \;\Rightarrow\; du = -2e^{-2x}$$
$$dv = \sin 2x \, dx \;\Rightarrow\; v = -\frac{1}{2} \cos 2x$$

$$= -\frac{1}{2} e^{-2x} \cos 2x - \int e^{-2x} \cos 2x \, dx$$

$$u = e^{-2x} \;\Rightarrow\; du = -2e^{-2x}$$
$$dv = \cos 2x \, dx \;\Rightarrow\; v = \frac{1}{2} \sin 2x$$

$$= -\frac{1}{2} e^{-2x} \cos 2x - \left(\frac{1}{2} e^{-2x} \sin 2x + \int e^{-2x} \sin 2x \, dx\right)$$

$$= -\frac{1}{2} e^{-2x} \cos 2x - \frac{1}{2} e^{-2x} \sin 2x - \int e^{-2x} \sin 2x \, dx$$

$$\Rightarrow\ 2\int e^{-2x}\sin 2x\,dx = -\frac{1}{2}e^{-2x}\cos 2x - \frac{1}{2}e^{-2x}\sin 2x + c$$

$$\Rightarrow\ \int e^{-2x}\sin x\,dx = -\frac{1}{4}e^{-2x}\cos 2x - \frac{1}{4}e^{-2x}\sin 2x + c$$

Opdracht 22 bladzijde 109

Bereken

1 $\int_0^{\pi} x \sin 3x\, dx$

$$u = x \Rightarrow du = dx$$
$$dv = \sin 3x\,dx \Rightarrow v = -\frac{1}{3}\cos 3x$$

$$= \left[-\frac{1}{3}x\cos 3x\right]_0^{\pi} + \frac{1}{3}\int_0^{\pi}\cos 3x\,dx$$

$$= \left[-\frac{1}{3}x\cos 3x\right]_0^{\pi} + \left[\frac{1}{9}\sin 3x\right]_0^{\pi}$$

$$= -\frac{1}{3}\pi\cos 3\pi - 0 + 0 - 0$$

$$= \frac{1}{3}\pi$$

2 $\int_1^{e} x \ln x\, dx$

$$u = \ln x \Rightarrow du = \frac{1}{x}dx$$
$$dv = x\,dx \Rightarrow v = \frac{x^2}{2}$$

$$= \left[\frac{1}{2}x^2\ln x\right]_1^{e} - \frac{1}{2}\int_1^{e} x\,dx$$

$$= \left[\frac{1}{2}x^2\ln x\right]_1^{e} - \left[\frac{1}{4}x^2\right]_1^{e}$$

$$= \frac{1}{2}e^2\ln e - \frac{1}{2}\ln 1 - \left(\frac{1}{4}e^2 - \frac{1}{4}\right)$$

$$= \frac{1}{2}e^2 - \frac{1}{4}e^2 + \frac{1}{4}$$

$$= \frac{1}{4}(e^2 + 1)$$

3 $\int_0^{\pi} e^{-x} \sin 4x\, dx$

We bereken eerst $\int e^{-x} \sin 4x\ dx$.

$\int e^{-x} \sin 4x\ dx$

$$u = e^{-x} \Rightarrow du = -e^{-x}dx$$

$$dv = \sin 4x\, dx \Rightarrow v = -\frac{1}{4} \cos 4x$$

$$= -\frac{1}{4} e^{-x} \cos 4x - \frac{1}{4} \int e^{-x} \cos 4x\, dx$$

$$u = e^{-x} \Rightarrow du = -e^{-x}dx$$

$$dv = \cos 4x\, dx \Rightarrow v = \frac{1}{4} \sin 4x$$

$$= -\frac{1}{4} e^{-x} \cos 4x - \frac{1}{4} \left(\frac{1}{4} e^{-x} \sin 4x + \frac{1}{4} \int e^{-x} \sin 4x\ dx \right)$$

$$= -\frac{1}{4} e^{-x} \cos 4x - \frac{1}{16} e^{-x} \sin 4x - \frac{1}{16} \int e^{-x} \sin 4x\ dx$$

$$\Rightarrow \frac{17}{16} \int e^{-x} \sin 4x\ dx = -\frac{1}{16} e^{-x} (4 \cos 4x + \sin 4x)$$

$$\Rightarrow \int e^{-x} \sin 4x\ dx = -\frac{1}{17} e^{-x}(4 \cos 4x + \sin 4x) + c$$

$$\Rightarrow \int_0^{\pi} e^{-x} \sin 4x\, dx = -\frac{1}{17} e^{-\pi}\ (4 \cos 4\pi + \sin 4\pi) - \left(-\frac{1}{17} (4 \cos 0 + \sin 0) \right)$$

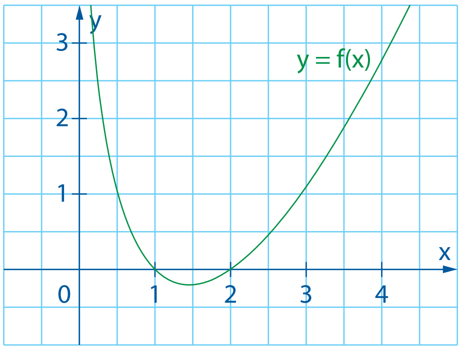
$$= -\frac{4}{17} e^{-\pi} + \frac{4}{17}$$

$$= \frac{4}{17} (1 - e^{-\pi})$$

$$= \frac{4}{17} \cdot \frac{e^{\pi} - 1}{e^{\pi}}$$

Opdracht 23 bladzijde 109

Bereken de oppervlakte van het gebied begrensd door de grafiek van de functie met voorschrift $f(x) = (x - 2)\ \ln x$ en de x-as.



- Nulpunten:

  $(x - 2) \ln x = 0 \iff x = 2 \quad \text{of} \quad x = 1$

- Tussen 1 en 2 ligt de grafiek onder de x-as, zodat

  $A = -\int_1^2 (x-2) \ln x \, dx$

$$u = \ln x \Rightarrow du = \frac{1}{x}dx$$

$$dv = (x-2)dx \Rightarrow v = \frac{x^2}{2} - 2x$$

$$= -\left(\left[\left(\frac{x^2}{2} - 2x\right)\ln x\right]_1^2 - \int_1^2 \left(\frac{x}{2} - 2\right) dx\right)$$

$$= -\left[\left(\frac{x^2}{2} - 2x\right) \ln x\right]_1^2 + \left[\frac{x^2}{4} - 2x\right]_1^2$$

$$= -(-2 \ln 2 - 0) + \left(-3 + \frac{7}{4}\right)$$

$$= 2 \ln 2 - \frac{5}{4}$$

Opdracht 24 bladzijde 109

Bepaal de waarde van $k \neq 0$ zodat $\int_1^k x^2 \ln x \, dx = \frac{1}{9}$.

We berekenen eerst $\int x^2 \ln x \, dx$.

$$u = \ln x \Rightarrow du = \frac{1}{x}dx$$

$$dv = x^2\,dx \Rightarrow v = \frac{x^3}{3}$$

$$= \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2\,dx$$

$$= \frac{x^3}{3} \ln x - \frac{1}{9} x^3 + c$$

$$= \frac{1}{9} x^3 (3 \ln x - 1) + c$$

$$\Rightarrow \int_1^k x^2 \ln x \, dx = \frac{1}{9} \iff \left[\frac{1}{9} x^3 (3 \ln x - 1)\right]_1^k = \frac{1}{9}$$

$$\iff \frac{1}{9} k^3 (3 \ln k - 1) + \frac{1}{9} = \frac{1}{9}$$

$$\Leftrightarrow \; k^3(3\ln k - 1) = 0$$

$$\underset{k \neq 0}{\Leftrightarrow} \; \ln k = \frac{1}{3}$$

$$\Leftrightarrow \; k = e^{\frac{1}{3}}$$

$$\Leftrightarrow \; k = \sqrt[3]{e}$$

Opdracht 25 bladzijde 112

Bereken

1 $\displaystyle\int \frac{x^2 - 4x + 2}{x + 1}\,dx$

$$\begin{array}{rl|l}
 & x^2 - 4x + 2 & x + 1 \\ \cline{3-3}
 & \mp x^2 \mp x & x - 5 \\ \cline{2-2}
 & -5x + 2 & \\
 & \pm 5x \pm 5 & \\ \cline{2-2}
 & 7 &
\end{array}$$

$$= \int \left(x - 5 + \frac{7}{x + 1} \right) dx$$

$$= \frac{x^2}{2} - 5x + 7 \int \frac{d(x + 1)}{x + 1}$$

$$= \frac{x^2}{2} - 5x + 7 \ln |x + 1| + c$$

2 $\displaystyle\int \frac{3x}{1 - \frac{1}{2}x}\,dx$

$$\begin{array}{rl|l}
 & 3x & -\frac{1}{2}x + 1 \\ \cline{3-3}
 & \mp 3x \pm 6 & -6 \\ \cline{2-2}
 & 6 &
\end{array}$$

$$= \int \left(-6 + \frac{6}{1 - \frac{1}{2}x} \right) dx$$

$$= -6x + 6 \cdot (-2) \int \frac{d\left(1 - \frac{1}{2}x\right)}{1 - \frac{1}{2}x}$$

$$= -6x - 12 \ln \left| 1 - \frac{1}{2}x \right| + c$$

Opdracht 26 bladzijde 112

1 Bepaal A en B zodat $\dfrac{3x+1}{x^2-1} = \dfrac{A}{x+1} + \dfrac{B}{x-1}$ voor alle $x \in \mathbb{R} \setminus \{-1, 1\}$.

$$\frac{A}{x+1} + \frac{B}{x-1} = \frac{A(x-1)+B(x+1)}{(x+1)(x-1)}$$
$$= \frac{(A+B)x - A + B}{x^2-1}$$

$$\frac{3x+1}{x^2-1} = \frac{(A+B)x - A + B}{x^2-1}$$

$$\Leftrightarrow \begin{cases} A+B=3 \\ B-A=1 \end{cases} \Leftrightarrow \begin{cases} A=1 \\ B=2 \end{cases}$$

2 Bereken $\displaystyle\int \frac{3x+1}{x^2-1}\,dx$ door gebruik te maken van **1**.

$$\int \frac{3x+1}{x^2-1}\,dx = \int \frac{1}{x+1}\,dx + \int \frac{2}{x-1}\,dx$$
$$= \int \frac{d(x+1)}{x+1} + 2\int \frac{d(x-1)}{x-1}$$
$$= \ln|x+1| + 2\ln|x-1| + c$$

Opdracht 27 bladzijde 115

Bereken

1 $\displaystyle\int \frac{2x+3}{x^2-6x-7}\,dx$

- $x^2 - 6x - 7 = (x-7)(x+1)$
- $\dfrac{2x+3}{x^2-6x-7} = \dfrac{A}{x-7} + \dfrac{B}{x+1} = \dfrac{(A+B)x + A - 7B}{(x-7)(x+1)}$

$$\Rightarrow \begin{cases} A+B=2 \\ A-7B=3 \end{cases} \Leftrightarrow \begin{cases} A = \dfrac{17}{8} \\ B = -\dfrac{1}{8} \end{cases}$$

- $\displaystyle\int \frac{2x+3}{x^2-6x-7}\,dx = \frac{17}{8}\int \frac{d(x-7)}{x-7} - \frac{1}{8}\int \frac{d(x+1)}{x+1}$

$$= \frac{17}{8}\ln|x-7| - \frac{1}{8}\ln|x+1| + c$$

2 $\displaystyle\int \frac{x-5}{x^2+2x+1}\,dx$

- $x^2+2x+1=(x+1)^2$

- $$\frac{x-5}{x^2+2x+1}=\frac{A}{x+1}+\frac{B}{(x+1)^2}$$
  $$=\frac{Ax+A+B}{(x+1)^2}$$

  $$\Rightarrow \begin{cases} A=1 \\ A+B=-5 \end{cases} \Leftrightarrow \begin{cases} A=1 \\ B=-6 \end{cases}$$

- $$\int \frac{x-5}{x^2+2x+1}\,dx=\int \frac{d(x+1)}{x+1}-6\int \frac{d(x+1)}{(x+1)^2}$$
  $$=\ln|x+1|+\frac{6}{x+1}+c$$

3 $\displaystyle\int \frac{x^3+3}{2x^2+5x+3}\,dx$

- euclidische deling

$$\begin{array}{rrrr|l}
x^3 & & & +\ 3 & 2x^2+5x+3 \\ \cline{5-5}
\mp x^3 & \mp \frac{5}{2}x^2 & \mp \frac{3}{2}x & & \frac{1}{2}x-\frac{5}{4} \\ \cline{1-4}
 & -\frac{5}{2}x^2 & -\frac{3}{2}x & +\ 3 & \\
 & \pm\frac{5}{2}x^2 & \pm\frac{25}{4}x & \pm\frac{15}{4} & \\ \cline{2-4}
 & & \frac{19}{4}x & +\frac{27}{4} &
\end{array}$$

- $$\int \frac{x^3+3}{2x^2+5x+3}\,dx=\int\left(\frac{1}{2}x-\frac{5}{4}+\frac{\frac{19}{4}x+\frac{27}{4}}{2x^2+5x+3}\right)dx$$
  $$=\frac{1}{4}x^2-\frac{5}{4}x+\int \frac{\frac{19}{4}x+\frac{27}{4}}{2x^2+5x+3}\,dx$$

  - $$2x^2+5x+3=2(x+1)\left(x+\frac{3}{2}\right)$$
    $$=(x+1)(2x+3)$$

  - $$\frac{\frac{19}{4}x+\frac{27}{4}}{2x^2+5x+3}=\frac{A}{x+1}+\frac{B}{2x+3}$$
    $$=\frac{(2A+B)x+3A+B}{(x+1)\,(2x+3)}$$

$$\Rightarrow \begin{cases} 2A + B = \dfrac{19}{4} \\ 3A + B = \dfrac{27}{4} \end{cases} \Leftrightarrow \begin{cases} A = 2 \\ B = \dfrac{3}{4} \end{cases}$$

- $$\int \frac{19x + \frac{27}{4}}{2x^2 + 5x + 3}dx = 2\int \frac{d(x+1)}{x+1} + \frac{3}{4} \cdot \frac{1}{2} \int \frac{d(2x+3)}{2x+3}$$

$$= 2\ \ln|x+1| + \frac{3}{8}\ln|2x+3| + c$$

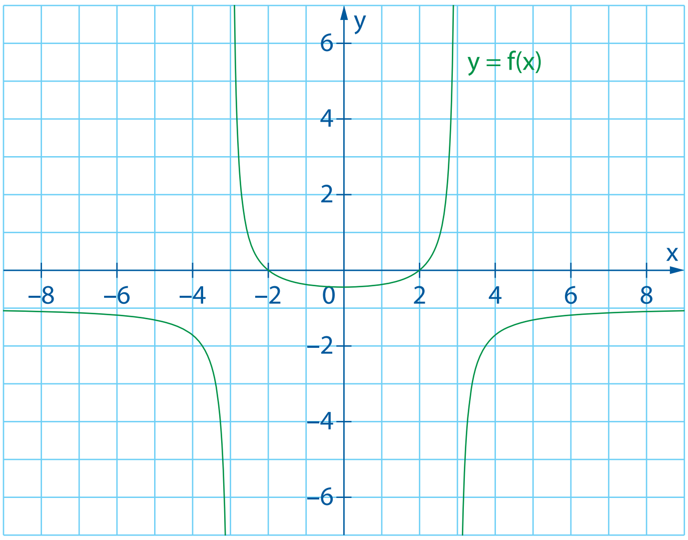
- $$\int \frac{x^3 + 3}{2x^2 + 5x + 3}dx = \frac{1}{4}x^2 - \frac{5}{4}x + 2\ \ln|x+1| + \frac{3}{8}\ \ln|2x+3| + c$$

Opdracht 28 bladzijde 115

Bereken de oppervlakte van het gebied tussen de grafiek van de functie met voorschrift $f(x) = \dfrac{-x^2 + 4}{x^2 - 9}$ en de x-as.

- Tekenverloop van $f(x) = \dfrac{-x^2 + 4}{x^2 - 9}$:

x		–3		–2		2		3	
t(x)	–	–	–	0	+	0	–	–	–
n(x)	+	0	–	–	–	–	–	0	+
f(x)	–	|	+	0	–	0	+	|	–



- $$A = -\int_{-2}^{2} \frac{-x^2 + 4}{x^2 - 9}dx$$

$$= \int_{-2}^{2} \frac{x^2 - 4}{x^2 - 9}dx \underset{\text{symmetrie}}{=} 2\int_{0}^{2} \frac{x^2 - 4}{x^2 - 9}dx$$

- $$\int \frac{x^2 - 4}{x^2 - 9}dx$$

  - euclidische deling:

$$\begin{array}{r|l} x^2 - 4 & x^2 - 9 \\ \hline \mp x^2 \pm 9 & 1 \\ \hline 5 & \end{array}$$

- $$\int \frac{x^2-4}{x^2-9}dx = \int \left(1+\frac{5}{x^2-9}\right)dx$$
$$= x + 5\int \frac{dx}{x^2-9}$$

- $$\frac{1}{x^2-9} = \frac{A}{x-3} + \frac{B}{x+3}$$
$$= \frac{(A+B)x+3A-3B}{(x-3)(x+3)}$$

$$\Rightarrow \begin{cases} A+B=0 \\ 3A-3B=1 \end{cases} \Leftrightarrow \begin{cases} A=\frac{1}{6} \\ B=-\frac{1}{6} \end{cases}$$

- $$\int \frac{dx}{x^2-9} = \frac{1}{6}\int \frac{d(x-3)}{x-3} - \frac{1}{6}\int \frac{d(x+3)}{x+3}$$
$$= \frac{1}{6}\ln|x-3| - \frac{1}{6}\ln|x+3| + c$$
$$= \frac{1}{6}\ln\left|\frac{x-3}{x+3}\right| + c$$

$$\Rightarrow A = 2\left[x+\frac{5}{6}\ln\left|\frac{x-3}{x+3}\right|\right]_0^2$$
$$= 2\left(2+\frac{5}{6}\ln\frac{1}{5}-0\right)$$
$$= 4+\frac{5}{3}\ln\frac{1}{5}$$
$$= 4-\frac{5}{3}\ln 5$$

Opdracht 29 bladzijde 117

Bereken

1 $$\int \frac{dx}{\underbrace{x^2-4x+13}_{D<0}} = \int \frac{dx}{(x-2)^2+9} = \int \frac{d(x-2)}{(x-2)^2+9} = \frac{1}{3}\,\text{Bgtan}\left(\frac{x-2}{3}\right)+c$$

2 $$\int \frac{16x+8}{\underbrace{4x^2+3}_{D<0}}dx = \int \frac{2\cdot 8x+8}{4x^2+3}dx = 2\int \frac{8x}{4x^2+3}dx + 8\int \frac{dx}{(2x)^2+3}$$
$$= 2\ln(4x^2+3) + \frac{8}{2}\int \frac{d(2x)}{(2x)^2+3} = 2\ln(4x^2+3) + \frac{4}{\sqrt{3}}\,\text{Bgtan}\left(\frac{2}{\sqrt{3}}x\right)+c$$

3 $$\int \frac{x}{x^2+6x+10}\,dx = \int \frac{\frac{1}{2}(2x+6)-3}{x^2+6x+10}\,dx = \frac{1}{2}\int \frac{2x+6}{x^2+6x+10}\,dx - 3\int \frac{dx}{x^2+6x+10}$$

$\hookrightarrow D < 0$

$$= \frac{1}{2}\ln(x^2+6x+10) - 3\int \frac{d(x+3)}{(x+3)^2+1}$$

$$= \frac{1}{2}\ln(x^2+6x+10) - 3\,\mathrm{Bgtan}(x+3) + c$$

4 $$\int \frac{2x+3}{4x^2+x+1}\,dx = \int \frac{\frac{1}{4}(8x+1)+\frac{11}{4}}{4x^2+x+1}\,dx$$

$\hookrightarrow D < 0$

$$= \frac{1}{4}\int \frac{8x+1}{4x^2+x+1}\,dx + \frac{11}{4}\cdot\frac{1}{2}\int \frac{d\left(2x+\frac{1}{4}\right)}{\left(2x+\frac{1}{4}\right)^2+\frac{15}{16}}$$

$$= \frac{1}{4}\ln(4x^2+x+1) + \frac{11}{8}\cdot\frac{4}{\sqrt{15}}\,\mathrm{Bgtan}\left(\frac{4}{\sqrt{15}}\left(2x+\frac{1}{4}\right)\right) + c$$

$$= \frac{1}{4}\ln(4x^2+x+1) + \frac{11}{2\sqrt{15}}\,\mathrm{Bgtan}\left(\frac{8x+1}{\sqrt{15}}\right) + c$$

Opdracht 30 bladzijde 120

Bereken

1 $$\int \frac{x+1}{x^3+x^2-6x}\,dx$$

- $x^3+x^2-6x = x(x^2+x-6) = x(x-2)(x+3)$
- $$\frac{x+1}{x^3+x^2-6x} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+3}$$

$$= \frac{A(x^2+x-6) + B(x^2+3x) + C(x^2-2x)}{x(x-2)(x+3)}$$

$$= \frac{(A+B+C)x^2 + (A+3B-2C)x - 6A}{x(x-2)(x+3)}$$

$$\Rightarrow \begin{cases} A+B+C=0 \\ A+3B-2C=1 \\ -6A=1 \end{cases} \Leftrightarrow \begin{cases} A=-\frac{1}{6} \\ B=\frac{3}{10} \\ C=-\frac{2}{15} \end{cases}$$

$$\int \frac{x+1}{x^3+x^2-6x}\,dx = -\frac{1}{6}\int \frac{dx}{x} + \frac{3}{10}\int \frac{d(x-2)}{x-2} - \frac{2}{15}\int \frac{d(x+3)}{x+3}$$

$$= -\frac{1}{6}\ln|x| + \frac{3}{10}\ln|x-2| - \frac{2}{15}\ln|x+3| + c$$

2 $\displaystyle\int \frac{x+2}{x^4-1}\,dx$

- $$\frac{x+2}{x^4-1} = \frac{x+2}{(x-1)(x+1)(x^2+1)}$$
$$= \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$
$$= \frac{A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x^2-1)}{(x-1)(x+1)(x^2+1)}$$
$$= \frac{(A+B+C)x^3 + (A-B+D)x^2 + (A+B-C)x + A-B-D}{x^4-1}$$

$$\Rightarrow \begin{cases} A+B+C=0 \\ A-B+D=0 \\ A+B-C=1 \\ A-B-D=2 \end{cases} \Leftrightarrow \begin{cases} A=\frac{3}{4} \\ B=-\frac{1}{4} \\ C=-\frac{1}{2} \\ D=-1 \end{cases}$$

- $$\int \frac{x+2}{x^4-1}dx = \frac{3}{4}\int \frac{d(x-1)}{x-1} - \frac{1}{4}\int \frac{d(x+1)}{x+1} + \int \frac{-\frac{1}{2}x-1}{x^2+1}dx$$
$$= \frac{3}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| - \frac{1}{2}\cdot\frac{1}{2}\int \frac{d(x^2+1)}{x^2+1} - \operatorname{Bgtan} x$$
$$= \frac{3}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| - \frac{1}{4}\ln(x^2+1) - \operatorname{Bgtan} x + c$$

3 $\displaystyle\int \frac{2x-1}{x^4+x^2}\,dx$

- $$\frac{2x-1}{x^4+x^2} = \frac{2x-1}{x^2(x^2+1)}$$
$$= \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$
$$= \frac{Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2}{x^4+x^2}$$
$$= \frac{(A+C)x^3 + (B+D)x^2 + Ax + B}{x^4+x^2}$$

$$\Rightarrow \begin{cases} A+C=0 \\ B+D=0 \\ A=2 \\ B=-1 \end{cases} \Leftrightarrow \begin{cases} A=2 \\ B=-1 \\ C=-2 \\ D=1 \end{cases}$$

- $$\int \frac{2x-1}{x^4+x^2}dx = 2\int \frac{dx}{x} - \int \frac{dx}{x^2} + \int \frac{-2x+1}{x^2+1}dx$$
$$= 2\ln|x| + \frac{1}{x} - \int \frac{d(x^2+1)}{x^2+1} + \text{Bgtan}\, x$$
$$= 2\ln|x| + \frac{1}{x} - \ln(x^2+1) + \text{Bgtan}\, x + c$$

Opdracht 31 bladzijde 120

Bereken

1 $$\int \frac{dx}{(x+1)\sqrt{x}}$$

$t^2 = x$ met $t > 0 \;\Rightarrow\; 2t\,dt = dx$

$$= \int \frac{2t\,dt}{(t^2+1)t} = 2\int \frac{dt}{t^2+1} = 2\,\text{Bgtan}\, t + c = 2\,\text{Bgtan}\sqrt{x} + c$$

2 $$\int \frac{x}{1+\sqrt[3]{x}}\,dx$$

$t^3 = x \;\Rightarrow\; 3t^2\,dt = dx$

$$= \int \frac{t^3}{1+t} \cdot 3t^2\,dt$$

$$= 3\int \frac{t^5}{t+1}dt$$

$$\begin{array}{r|l}
t^5 & t+1 \\ \hline
\mp t^5 \mp t^4 & t^4 - t^3 + t^2 - t + 1 \\ \hline
-t^4 & \\
\pm t^4 \pm t^3 & \\ \hline
t^3 & \\
\mp t^3 \mp t^2 & \\ \hline
-t^2 & \\
\pm t^2 \pm t & \\ \hline
t & \\
\mp t \mp 1 & \\ \hline
-1 &
\end{array}$$

$$= 3\int \left(t^4 - t^3 + t^2 - t + 1 - \frac{1}{t+1}\right)dt$$

$$= 3\left(\frac{t^5}{5} - \frac{t^4}{4} + \frac{t^3}{3} - \frac{t^2}{2} + t - \ln|t+1|\right) + c$$

$$= \frac{3}{5}\sqrt[3]{x^5} - \frac{3}{4}\sqrt[3]{x^4} + x - \frac{3}{2}\sqrt[3]{x^2} + 3\sqrt[3]{x} - 3\ln\left|\sqrt[3]{x}+1\right| + c$$

3 $\displaystyle\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx$

$$t^6 = x \;\Rightarrow\; 6t^5\,dt = dx$$

$$= \int \frac{t^3}{1+t^2}\cdot 6t^5\,dt$$

$$= 6\int \frac{t^8}{t^2+1}\,dt$$

$$\begin{array}{rrrrrrrr|l}
t^8 & & & & & & & & t^2+1 \\
\mp t^8 & \mp t^6 & & & & & & & t^6 - t^4 + t^2 - 1 \\
\hline
 & -t^6 & & & & & & & \\
 & \pm t^6 & \pm t^4 & & & & & & \\
\hline
 & & t^4 & & & & & & \\
 & & \mp t^4 & \mp t^2 & & & & & \\
\hline
 & & & -t^2 & & & & & \\
 & & & \pm t^2 & \pm 1 & & & & \\
\hline
 & & & & 1 & & & &
\end{array}$$

$$= 6\int\left(t^6 - t^4 + t^2 - 1 + \frac{1}{t^2+1}\right)dt$$

$$= \frac{6}{7}t^7 - \frac{6}{5}t^5 + 2t^3 - 6t + 6\,\mathrm{Bgtan}\,t + c$$

$$= \frac{6}{7}\sqrt[6]{x^7} - \frac{6}{5}\sqrt[6]{x^5} + 2\sqrt{x} - 6\sqrt[6]{x} + 6\,\mathrm{Bgtan}\sqrt[6]{x} + c$$

4 $\displaystyle\int \frac{x^2}{\sqrt{9-4x^2}}\,dx$

$$u = x \;\Rightarrow\; du = dx$$

$$dv = \frac{x}{\sqrt{9-4x^2}}\,dx \;\Rightarrow\; v = -\frac{1}{8}\int \frac{d(9-4x^2)}{\sqrt{9-4x^2}} = -\frac{1}{4}\sqrt{9-4x^2}$$

$$= -\frac{1}{4}x\sqrt{9-4x^2} + \frac{1}{4}\int \sqrt{9-4x^2}\,dx$$

$$= -\frac{1}{4}x\sqrt{9-4x^2} + \frac{1}{4}\int \frac{\sqrt{9-4x^2}\,\sqrt{9-4x^2}}{\sqrt{9-4x^2}}\,dx$$

$$= -\frac{1}{4}x\sqrt{9-4x^2} + \frac{1}{4}\int \frac{9-4x^2}{\sqrt{9-4x^2}}\,dx$$

$$= -\frac{1}{4}x\sqrt{9-4x^2} + \frac{9}{4}\int \frac{dx}{\sqrt{9-4x^2}} - \int \frac{x^2}{\sqrt{9-4x^2}}\,dx$$

$$\Rightarrow\; 2\int \frac{x^2}{\sqrt{9-4x^2}}\,dx = -\frac{1}{4}x\sqrt{9-4x^2} + \frac{9}{4}\cdot\frac{1}{2}\int \frac{d(2x)}{\sqrt{9-(2x)^2}}$$

$$= -\frac{1}{4}x\sqrt{9-4x^2} + \frac{9}{8}\,\mathrm{Bgsin}\left(\frac{2x}{3}\right) + c$$

$$\Rightarrow\; \int \frac{x^2}{\sqrt{9-4x^2}}\,dx = -\frac{1}{8}x\sqrt{9-4x^2} + \frac{9}{16}\,\mathrm{Bgsin}\left(\frac{2x}{3}\right) + c$$

5 $\displaystyle\int \frac{1}{x\sqrt{x^2-25}}\,dx$

$\displaystyle = \int \frac{x}{x^2\sqrt{x^2-25}}\,dx$

$t^2 = x^2 - 25$ met $t > 0 \;\Rightarrow\; 2t\,dt = 2x\,dx$

$\displaystyle = \int \frac{t}{(t^2+25)\,t}\,dt$

$\displaystyle = \int \frac{dt}{t^2+25}$

$\displaystyle = \frac{1}{5}\,\text{Bgtan}\,\frac{t}{5} + c$

$\displaystyle = \frac{1}{5}\,\text{Bgtan}\,\frac{\sqrt{x^2-25}}{5} + c$

Opdracht 32 bladzijde 126

Bereken

1 $\displaystyle\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3}\sqrt{x^3} + c$

2 $\displaystyle\int \frac{dx}{x^3} = \int x^{-3}\,dx = \frac{x^{-2}}{-2} + c = -\frac{1}{2x^2} + c$

3 $\displaystyle\int \frac{x^2\sqrt{x}}{\sqrt[3]{x}}\,dx = \int \frac{x^2 \cdot x^{\frac{1}{2}}}{x^{\frac{1}{3}}}\,dx = \int x^{\frac{13}{6}}\,dx$

$\displaystyle = \frac{x^{\frac{19}{6}}}{\frac{19}{6}} + c = \frac{6}{19}\sqrt[6]{x^{19}} + c$

4 $\displaystyle\int \frac{dx}{4+4x^2} = \frac{1}{4}\int \frac{dx}{1+x^2} = \frac{1}{4}\,\text{Bgtan}\,x + c$

5 $\displaystyle\int \frac{dx}{\cos^2 x - 1} = -\int \frac{dx}{1-\cos^2 x} = -\int \frac{dx}{\sin^2 x} = \cot x + c$

Opdracht 33 bladzijde 126

Bereken

1 $\int (-2x^2+5x-6)\,dx = -\frac{2}{3}x^3+\frac{5}{2}x^2-6x+c$

2 $\int (2x-3)(3x+2)\,dx = \int (6x^2-5x-6)\,dx$

$$= 2x^3-\frac{5}{2}x^2-6x+c$$

3 $\int (x^2-1)(x^2+1)(x^4+1)\,dx = \int (x^4-1)(x^4+1)\,dx = \int (x^8-1)\,dx = \frac{x^9}{9}-x+c$

4 $\int \frac{3x-4}{2x}\,dx = \int\left(\frac{3}{2}-\frac{2}{x}\right)dx = \frac{3}{2}x-2\ln|x|+c$

5 $\int \frac{x^2-4}{\sqrt{3x}}\,dx = \frac{1}{\sqrt{3}}\int\left(\frac{x^2}{\sqrt{x}}-\frac{4}{\sqrt{x}}\right)dx = \frac{1}{\sqrt{3}}\int\left(x^{\frac{3}{2}}-4x^{-\frac{1}{2}}\right)dx$

$$= \frac{1}{\sqrt{3}}\left(\frac{x^{\frac{5}{2}}}{\frac{5}{2}}-4\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right)+c = \frac{1}{\sqrt{3}}\left(\frac{2}{5}\sqrt{x^5}-8\sqrt{x}\right)+c$$

6 $\int \frac{x^2+5}{x^2+1}\,dx = \int \frac{(x^2+1)+4}{x^2+1}\,dx = \int\left(1+\frac{4}{x^2+1}\right)dx$

$$= x+4\,\mathrm{Bgtan}\,x+c$$

Opdracht 34 bladzijde 126

Bepaal het voorschrift van f als

1 $f'(x)=x^2-x$ en $f(3)=4$

- $f(x)=\frac{1}{3}x^3-\frac{1}{2}x^2+c$
- $f(3)=4 \Leftrightarrow 9-\frac{9}{2}+c=4 \Leftrightarrow c=-\frac{1}{2}$

$\Rightarrow f(x)=\frac{1}{3}x^3-\frac{1}{2}x^2-\frac{1}{2}$

2 $f''(x) = x + 1,\ f'(0) = 0$ en $f(0) = 5$

- $f'(x) = \frac{1}{2}x^2 + x + c$
- $f'(0) = 0 \ \Leftrightarrow\ c = 0$

  $\Rightarrow\ f'(x) = \frac{1}{2}x^2 + x$
- $f(x) = \frac{1}{6}x^3 + \frac{1}{2}x^2 + c$
- $f(0) = 5 \ \Leftrightarrow\ c = 5$

  $\Rightarrow\ f(x) = \frac{1}{6}x^3 + \frac{1}{2}x^2 + 5$

3 $f''(x) = \frac{1}{x^3},\ f'(-1) = 1$ en $f(1) = 0$

- $f'(x) = \frac{x^{-2}}{-2} + c = -\frac{1}{2x^2} + c$
- $f'(-1) = 1 \ \Leftrightarrow\ -\frac{1}{2} + c = 1 \ \Leftrightarrow\ c = \frac{3}{2}$

  $\Rightarrow\ f'(x) = -\frac{1}{2x^2} + \frac{3}{2}$
- $f(x) = \frac{x^{-1}}{2} + \frac{3}{2}x + c = \frac{1}{2x} + \frac{3}{2}x + c$
- $f(1) = 0 \ \Leftrightarrow\ \frac{1}{2} + \frac{3}{2} + c = 0 \ \Leftrightarrow\ c = -2$

$\Rightarrow\ f(x) = \frac{1}{2x} + \frac{3}{2}x - 2$

Opdracht 35 bladzijde 126

Bereken

1 $$\int \frac{2^{x+1}}{3^{x+2}}\,dx = \frac{2}{9}\int \frac{2^x}{3^x}dx = \frac{2}{9}\int\left(\frac{2}{3}\right)^x dx$$

$$= \frac{2}{9}\cdot\frac{\left(\frac{2}{3}\right)^x}{\ln\frac{2}{3}} + c = \frac{2}{9\ln\frac{2}{3}}\cdot\left(\frac{2}{3}\right)^x + c$$

2 $$\int \frac{6^{3x}}{4^{3x-5}}\,dx = 4^5\int\left(\frac{6}{4}\right)^{3x} dx = 4^5\int\left(\left(\frac{3}{2}\right)^3\right)^x dx$$

$$= 4^5\int\left(\frac{27}{8}\right)^x dx = \frac{1024}{\ln\frac{27}{8}}\cdot\left(\frac{27}{8}\right)^x + c = \frac{1024}{3\ln\frac{3}{2}}\cdot\left(\frac{3}{2}\right)^{3x} + c$$

3 $\displaystyle\int \frac{1+\cos^2 x}{1+\cos 2x}\,dx = \int \frac{1+\cos^2 x}{2\cos^2 x}\,dx = \frac{1}{2}\int\left(\frac{1}{\cos^2 x}+1\right)dx = \frac{1}{2}(\tan x + x) + c$

$\cos 2x = 2\cos^2 x - 1$

4 $\displaystyle\int \frac{\cos 2x}{\sin^2 x \cdot \cos^2 x}\,dx = \int \frac{\cos^2 x - \sin^2 x}{\sin^2 x \cos^2 x}\,dx = \int\left(\frac{1}{\sin^2 x} - \frac{1}{\cos^2 x}\right)dx = -\cot x - \tan x + c$

Opdracht 36 bladzijde 127

Bereken

1 $\displaystyle\int (-3x+5)^4\,dx$

$u = -3x+5 \;\Rightarrow\; du = -3dx$

$\displaystyle = -\frac{1}{3}\int u^4\,dt = -\frac{1}{3}\cdot\frac{u^5}{5} + c = -\frac{1}{15}(-3x+5)^5 + c$

of $\displaystyle\int (-3x+5)^4\,dt = -\frac{1}{3}\int(-3x+5)^4\,d(-3x+5) = -\frac{1}{3}\frac{(-3x+5)^5}{5} + c$

$\displaystyle = -\frac{1}{15}(-3x+5)^5 + c$

2 $\displaystyle\int \frac{dx}{-2x+9} = -\frac{1}{2}\int\frac{d(-2x+9)}{-2x+9} = -\frac{1}{2}\ln|-2x+9| + c$

3 $\displaystyle\int e^{3x+14}\,dx = \frac{1}{3}\int e^{3x+14}\,d(3x+14) = \frac{1}{3}e^{3x+14} + c$

4 $\displaystyle\int \frac{dx}{\sin^2(2-x)} = -\int\frac{d(2-x)}{\sin^2(2-x)} = \cot(2-x) + c$

5 $\displaystyle\int \cos[\pi(x-1)]\,dx = \frac{1}{\pi}\int\cos[\pi(x-1)]\,d(\pi(x-1)) = \frac{1}{\pi}\sin[\pi(x-1)] + c$

6 $\displaystyle\int 4^{-2x+3}\,dx = -\frac{1}{2}\int 4^{-2x+3}\,d(-2x+3) = -\frac{1}{2}\cdot\frac{4^{-2x+3}}{\ln 4} + c = -\frac{4^{-2x+3}}{4\ln 2} = -\frac{64\cdot 2^{-4x}}{4\ln 2} + c$

$\displaystyle = -\frac{16}{\ln 2}\cdot 2^{-4x} + c$

Opdracht 37 bladzijde 127

Bereken

1 $\displaystyle\int \frac{dx}{\sqrt{1-4x^2}} = \frac{1}{2}\int\frac{d(2x)}{\sqrt{1-(2x)^2}} = \frac{1}{2}\,\text{Bgsin}(2x) + c$

2 $\displaystyle\int \frac{dx}{9x^2+1} = \frac{1}{3}\int\frac{d(3x)}{1+(3x)^2} = \frac{1}{3}\,\text{Bgtan}(3x) + c$

3 $\displaystyle\int \frac{dx}{\sqrt{3x^2+7}} = \frac{1}{\sqrt{3}} \int \frac{d\left(\sqrt{3}x\right)}{\sqrt{\left(\sqrt{3}x\right)^2+7}} = \frac{1}{\sqrt{3}} \ln\left|\sqrt{3}x + \sqrt{3x^2+7}\right| + c$

4 $\displaystyle\int \frac{dx}{\sqrt{25-36x^2}} = \frac{1}{6} \int \frac{d(6x)}{\sqrt{25-(6x)^2}} = \frac{1}{6}\,\text{Bgsin}\left(\frac{6x}{5}\right) + c$

5 $\displaystyle\int \frac{dx}{\sqrt{100x^2-49}} = \frac{1}{10} \int \frac{d(10x)}{\sqrt{(10x)^2-49}} = \frac{1}{10} \ln\left|10x + \sqrt{100x^2-49}\right| + c$

6 $\displaystyle\int \frac{dx}{81x^2+64} = \frac{1}{9} \int \frac{d(9x)}{(9x)^2+64} = \frac{1}{9} \cdot \frac{1}{8}\,\text{Bgtan}\left(\frac{9x}{8}\right) + c$

$$= \frac{1}{72}\,\text{Bgtan}\left(\frac{9x}{8}\right) + c$$

Opdracht 38 bladzijde 127

Bereken

1 $\displaystyle\int (x^3-1)^3 x^2\, dx$

$$u = x^3 - 1 \;\Rightarrow\; du = 3x^2 dx$$

$$= \frac{1}{3}\int u^3\, dt = \frac{1}{3}\,\frac{u^4}{4} + c = \frac{1}{12}(x^3-1)^4 + c$$

2 $\displaystyle\int x^2 e^{2x^3+3}\, dx$

$$u = 2x^3 + 3 \;\Rightarrow\; du = 6x^2 dx$$

$$= \frac{1}{6}\int e^u\, du = \frac{1}{6}e^u + c = \frac{1}{6}e^{2x^3+3} + c$$

3 $\displaystyle\int e^{\cos x} \sin x\, dx$

$$u = \cos x \;\Rightarrow\; du = -\sin x\, dx$$

$$= -\int e^u\, du = -e^u + c = -e^{\cos x} + c$$

4 $\displaystyle\int \frac{dx}{x \ln^2 x}$

$$u = \ln x \;\Rightarrow\; du = \frac{1}{x}\, dx$$

$$= \int \frac{du}{u^2} = \int u^{-2}\, du = \frac{u^{-1}}{-1} + c = -\frac{1}{u} + c = -\frac{1}{\ln x} + c$$

5 $\displaystyle\int \frac{\sqrt{3+\ln x}}{x}\,dx$

$$u = 3 + \ln x \;\Rightarrow\; du = \frac{1}{x}\,dx$$

$$= \int \sqrt{u}\,du = \int u^{\frac{1}{2}}\,du = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3}\sqrt{(3+\ln x)^3} + c$$

6 $\displaystyle\int \frac{3x+1}{(3x^2+2x)^3}\,dx$

$$u = 3x^2 + 2x \;\Rightarrow\; du = (6x+2)\,dx = 2(3x+1)\,dx$$

$$= \frac{1}{2}\int u^{-3}\,du = \frac{1}{2}\cdot\frac{u^{-2}}{-2} + c = -\frac{1}{4(3x^2+2x)^2} + c$$

7 $\displaystyle\int \frac{\cos x - \sin x}{(\cos x + \sin x)^2}\,dx$

$$u = \cos x + \sin x \;\Rightarrow\; du = (-\sin x + \cos x)\,dx$$

$$= \int \frac{du}{u^2} = \frac{u^{-1}}{-1} + c = -\frac{1}{\cos x + \sin x} + c$$

8 $\displaystyle\int \frac{\cos x - x\sin x}{x\cos x}\,dx$

$$u = x\cos x \;\Rightarrow\; du = (-x\sin x + \cos x)\,dx$$

$$= \int \frac{du}{u} = \ln|u| + c = \ln|x\cos x| + c$$

Opdracht 39 bladzijde 127

De onbepaalde integraal $\displaystyle\int x e^{\frac{2}{3}x^2}\,dx$ is gelijk aan

A $\frac{4}{3}e^{\frac{2}{3}x^2} + c$ **B** $\frac{3}{4}e^{\frac{2}{3}x^2} + c$ **C** $\frac{2}{3}e^{\frac{2}{3}x^2} + c$ **D** $\frac{3}{2}e^{\frac{2}{3}x^2} + c$

(Bron © Toelatingsproef arts-tandarts)

$$u = \frac{2}{3}x^2 \Rightarrow du = \frac{4}{3}x\,dx$$

$$= \frac{3}{4}\int e^u\,du = \frac{3}{4}e^u + c = \frac{3}{4}e^{\frac{2}{3}x^2} + c$$

Antwoord B is juist.

Opdracht 40 bladzijde 128

Bereken

1 $\displaystyle\int_4^5 \frac{2}{(x-3)^3}\,dx = 2\int_4^5 (x-3)^{-3}\,d(x-3) = 2\left[\frac{(x-3)^{-2}}{-2}\right]_4^5$

$$= -\left[\frac{1}{(x-3)^2}\right]_4^5 = -\left(\frac{1}{4}-1\right) = \frac{3}{4}$$

2 $\displaystyle\int_{-2}^{2} x\sqrt{4-x^2}\,dx = -\frac{1}{2}\int_0^0 \sqrt{u}\,du = 0$

$u = 4 - x^2 \Rightarrow du = -2x\,dx$

$x = -2 \Rightarrow u = 0$

$x = 2 \Rightarrow u = 0$

3 $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cot x\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cos x}{\sin x}\,dx = \int_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} \frac{du}{u} = \left[\ln|u|\right]_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}}$

$u = \sin x \Rightarrow du = \cos x\,dx$

$x = \dfrac{\pi}{6} \Rightarrow u = \dfrac{1}{2}$

$x = \dfrac{\pi}{4} \Rightarrow u = \dfrac{\sqrt{2}}{2}$

$$= \ln\frac{\sqrt{2}}{2} - \ln\frac{1}{2} = \ln\sqrt{2} - \ln 2 + \ln 2 = \ln\sqrt{2}$$

4 $\displaystyle\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\sin x}{1+\cos x}\,dx = -\int_{\frac{3}{2}}^{\frac{1}{2}} \frac{du}{u} = \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{du}{u} = \left[\ln|u|\right]_{\frac{1}{2}}^{\frac{3}{2}}$

$u = 1 + \cos x \Rightarrow du = -\sin x\,dx$

$x = \dfrac{\pi}{3} \Rightarrow u = \dfrac{3}{2}$

$x = \dfrac{2\pi}{3} \Rightarrow u = \dfrac{1}{2}$

$$= \ln\frac{3}{2} - \ln\frac{1}{2} = \ln 3$$

5 $\int_0^3 (x-2)\,e^{-x^2+4x-4}\,dx$

$u = -x^2 + 4x - 4 \;\Rightarrow\; du = (-2x+4)\,dx = -2(x-2)\,dx$

$x = 0 \;\Rightarrow\; u = -4$

$x = 3 \;\Rightarrow\; u = -1$

$$= -\frac{1}{2}\int_{-4}^{-1} e^u\,du = -\frac{1}{2}\left[e^u\right]_{-4}^{-1} = -\frac{1}{2}\left[e^{-1} - e^{-4}\right]$$

$$= -\frac{1}{2}\left[\frac{1}{e} - \frac{1}{e^4}\right] = \frac{1}{2}\cdot\frac{1-e^3}{e^4} = \frac{1-e^3}{2e^4}$$

6 $\displaystyle\int_{-\frac{\sqrt{2}}{2}}^{0} \frac{x}{\sqrt{1-x^4}}\,dx = \frac{1}{2}\int_{\frac{1}{2}}^{0} \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\left[\text{Bgsin } u\right]_{\frac{1}{2}}^{0}$

$u = x^2 \;\Rightarrow\; du = 2x\,dx$

$x = -\frac{\sqrt{2}}{2} \;\Rightarrow\; u = \frac{1}{2}$

$x = 0 \;\Rightarrow\; u = 0$

$$= \frac{1}{2}\left[\text{Bgsin } 0 - \text{Bgsin } \frac{1}{2}\right] = -\frac{\pi}{12}$$

Opdracht 41 bladzijde 128

$\int_0^{\pi} \sin^2 x\,dx$ is gelijk aan

A $-\pi$ **B** $-\frac{\pi}{2}$ **C** $\frac{\pi}{2}$ **D** π

(Bron © Toelatingsproef arts-tandarts)

$\cos 2x = 1 - 2\sin^2 x$

$$= \int_0^{\pi} \frac{1-\cos 2x}{2}\,dx = \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\pi} = \frac{1}{2}\pi - 0 - (0-0) = \frac{\pi}{2}$$

Antwoord C is juist.

Opdracht 42 bladzijde 128

Bereken de volgende integralen door gebruik te maken van de omgekeerde formules van Simpson.

1 $\int \cos 5x \cos 2x \, dx$

2 $\int \sin 2x \cos 3x \, dx$

3 $\int \sin 3x \sin x \, dx$

4 $\int \sin x \cos x \cos 2x \, dx$

Omgekeerde formules van Simpson:

$$\sin x \cos y = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$$

$$\cos x \cos y = \frac{1}{2}\left[\cos(x+y) + \cos(x-y)\right]$$

$$\sin x \sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$$

1 $\int \cos 5x \cos 2x \, dx = \int \frac{1}{2}(\cos 7x + \cos 3x)\,dx = \frac{1}{14}\sin 7x + \frac{1}{6}\sin 3x + c$

2 $\int \sin 2x \cos 3x \, dx = \int \frac{1}{2}(\sin 5x + \sin(-x))dx = -\frac{1}{10}\cos 5x + \frac{1}{2}\cos x + c$

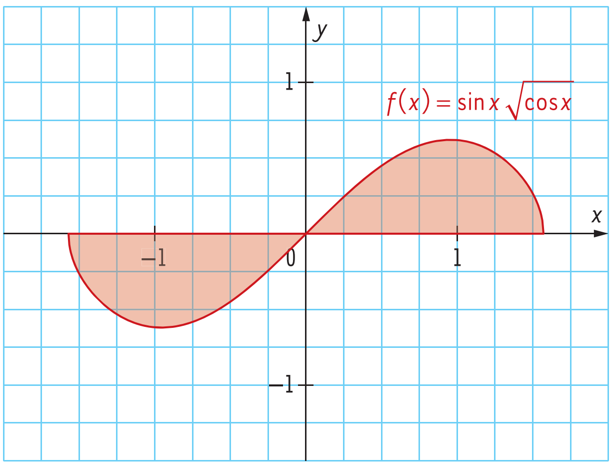
3 $\int \sin 3x \sin x \, dx = \int \frac{1}{2}(\cos 2x - \cos 4x)\,dx = \frac{1}{4}\sin 2x - \frac{1}{8}\sin 4x + c$

4 $\int \sin x \cos x \cos 2x \, dx = \int \frac{1}{2}\sin 2x \cos 2x\,dx = \int \frac{1}{4}\sin 4x\,dx = -\frac{1}{16}\cos 4x + c$

Opdracht 43 bladzijde 128

Gegeven is de grafiek van de functie met voorschrift $f(x) = \sin x\sqrt{\cos x}$ over $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Bereken de oppervlakte van het gekleurde gebied.



Wegens symmetrie geldt:

$$A = 2\int_0^{\frac{\pi}{2}} \sin x\sqrt{\cos x}\,dx$$

$u = \cos x \Rightarrow du = -\sin x\,dx$

$x = 0 \Rightarrow u = 1$

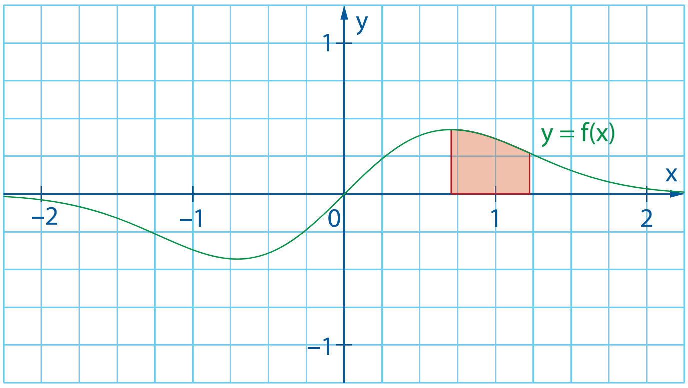
$x = \frac{\pi}{2} \Rightarrow u = 0$

$$= -2\int_1^0 \sqrt{u}\,du = 2\int_0^1 u^{\frac{1}{2}}\,du = 2\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 = \frac{4}{3}$$

Opdracht 44 bladzijde 129

Bereken de oppervlakte van het gebied begrensd door de grafiek van de functie met voorschrift $f(x) = xe^{-x^2}$, de x-as en de verticale rechten door het maximum en het buigpunt met strikt positieve x-waarde.

$f(x) = xe^{-x^2}$



- $f'(x) = e^{-x^2} - 2x^2e^{-x^2} = (1-2x^2)e^{-x^2}$

  nulpunten: $\pm\dfrac{1}{\sqrt{2}}$

- $f''(x) = -4xe^{-x^2} - 2x(1-2x^2)e^{-x^2}$

  $= (4x^3 - 6x)e^{-x^2}$

  nulpunten: $0, -\sqrt{\dfrac{3}{2}}, \sqrt{\dfrac{3}{2}}$

$\Rightarrow$ verticale rechten door $\dfrac{1}{\sqrt{2}}$ en door $\sqrt{\dfrac{3}{2}}$

- f is positief over $\left[\dfrac{1}{\sqrt{2}}, \sqrt{\dfrac{3}{2}}\right]$

$$\Rightarrow A = \int_{\frac{1}{\sqrt{2}}}^{\sqrt{\frac{3}{2}}} x\,e^{-x^2}\,dx$$

$u = -x^2 \Rightarrow du = -2x\,dx$

$x = \dfrac{1}{\sqrt{2}} \Rightarrow u = -\dfrac{1}{2}$

$x = \sqrt{\dfrac{3}{2}} \Rightarrow u = -\dfrac{3}{2}$

$$= -\frac{1}{2}\int_{-\frac{1}{2}}^{-\frac{3}{2}} e^u\,du = \frac{1}{2}\left[e^u\right]_{-\frac{3}{2}}^{-\frac{1}{2}} = \frac{1}{2}\left[e^{-\frac{1}{2}} - e^{-\frac{3}{2}}\right]$$

$$= \frac{1}{2}\left(\frac{1}{\sqrt{e}} - \frac{1}{e\sqrt{e}}\right) = \frac{e-1}{2e\sqrt{e}}$$

Opdracht 45 bladzijde 129

Bereken

1 $\displaystyle\int \frac{1+\tan^2 x}{\cos^2 x}\,dx = \int (1+u^2)\,du = u + \frac{u^3}{3} + c$

$u = \tan x \Rightarrow du = \dfrac{1}{\cos^2 x}\,dx$

$= \tan x + \dfrac{\tan^3 x}{3} + c$

2 $\displaystyle\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx = 2\int \sqrt{u}\;du = 2\cdot\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{4}{3}\sqrt{\left(1+\sqrt{x}\right)^3} + c$

$$u = 1+\sqrt{x} \;\Rightarrow\; du = \frac{1}{2\sqrt{x}}dx$$

3 $\displaystyle\int \frac{e^{2x}}{\sqrt{3-4e^{4x}}}\,dx = \frac{1}{4}\int \frac{du}{\sqrt{3-u^2}} = \frac{1}{4}\,\text{Bgsin}\,\frac{u}{\sqrt{3}} + c$

$$u = 2\,e^{2x} \;\Rightarrow\; du = 4\,e^{2x}\,dx$$

$$= \frac{1}{4}\,\text{Bgsin}\,\frac{2e^{2x}}{\sqrt{3}} + c$$

4 $\displaystyle\int \frac{\cos(\ln x)}{x}\,dx = \int \cos u\,du = \sin u + c = \sin(\ln x) + c$

$$u = \ln x \;\Rightarrow\; du = \frac{1}{x}dx$$

5 $\displaystyle\int \frac{dx}{\cos^2 x\sqrt{1-4\tan^2 x}} = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\,\text{Bgsin}\,u + c$

$$u = 2\tan x \;\Rightarrow\; du = \frac{2}{\cos^2 x}dx$$

$$= \frac{1}{2}\,\text{Bgsin}(2\tan x) + c$$

6 $\displaystyle\int \frac{\cos 3x}{8+\sin^2 3x}\,dx = \frac{1}{3}\int \frac{du}{8+u^2} = \frac{1}{3}\cdot\frac{1}{2\sqrt{2}}\,\text{Bgtan}\,\frac{u}{2\sqrt{2}} + c$

$$u = \sin 3x \;\Rightarrow\; du = 3\cos 3x\,dx$$

$$= \frac{1}{6\sqrt{2}}\,\text{Bgtan}\,\frac{\sin 3x}{2\sqrt{2}} + c$$

7 $\displaystyle\int \frac{\sqrt{\text{Bgsin}\,5x}}{\sqrt{1-25x^2}}\,dx = \frac{1}{5}\int \sqrt{u}\;du = \frac{1}{5}\cdot\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c$

$$u = \text{Bgsin}\,5x \;\Rightarrow\; du = \frac{5}{\sqrt{1-25x^2}}dx$$

$$= \frac{2}{15}\sqrt{(\text{Bgsin}\,5x)^3} + c$$

8 $\displaystyle\int \frac{x^2}{x^6+9}\,dx = \int \frac{x^2}{(x^3)^2+9}dx = \frac{1}{3}\int \frac{du}{u^2+9} = \frac{1}{9}\,\text{Bgtan}\,\frac{x^3}{3} + c$

$$u = x^3 \;\Rightarrow\; du = 3x^2\,dx$$

9 $\displaystyle\int \frac{dx}{x\sqrt{4-25\ln^2 x}}$

$$u = 5\ln x \;\Rightarrow\; du = \frac{5}{x}dx$$

$$= \frac{1}{5}\int \frac{du}{\sqrt{4-u^2}} = \frac{1}{5}\,\text{Bgsin}\,\frac{u}{2} + c = \frac{1}{5}\,\text{Bgsin}\left(\frac{5}{2}\ln x\right) + c$$

10 $\displaystyle\int \frac{\cos x}{2-\cos^2 x}\,dx = \int \frac{\cos x}{1+\sin^2 x}dx = \int \frac{du}{1+u^2} = \text{Bgtan}\,(\sin x) + c$

$$u = \sin x \;\Rightarrow\; du = \cos x\,dx$$

11 $\displaystyle\int \cot x\,\ln(\sin x)\,dx = \int \frac{\cos x}{\sin x}\ln(\sin x)dx = \int u\,du = \frac{u^2}{2} + c$

$$u = \ln(\sin x) \;\Rightarrow\; du = \frac{1}{\sin x}\cdot\cos x\,dx$$

$$= \frac{\ln^2(\sin x)}{2} + c$$

12 $\displaystyle\int x^3(1+x^2)^4\,dx = \int x^2\cdot x\cdot(1+x^2)^4\,dx$

$$u = 1+x^2 \;\Rightarrow\; du = 2x\,dx$$

$$= \frac{1}{2}\int (u-1)\cdot u^4\,du = \frac{1}{2}\int (u^5-u^4)\,du = \frac{u^6}{12} - \frac{u^5}{10} + c$$

$$= \frac{(1+x^2)^6}{12} - \frac{(1+x^2)^5}{10} + c$$

13 $\displaystyle\int \frac{xe^{-2x^2}}{3-e^{-2x^2}}\,dx = \frac{1}{4}\int \frac{du}{u} = \frac{1}{4}\ln|u| + c = \frac{1}{4}\ln\left|3-e^{-2x^2}\right| + c$

$$u = 3-e^{-2x^2} \;\Rightarrow\; du = 4x\,e^{-2x^2}\,dx$$

14 $\displaystyle\int \frac{\cos 3x}{\sqrt{4-\sin^2 3x}}\,dx = \frac{1}{3}\int \frac{du}{\sqrt{4-u^2}} = \frac{1}{3}\,\text{Bgsin}\,\frac{u}{2} + c$

$$u = \sin 3x \;\Rightarrow\; du = 3\cos 3x\,dx$$

$$= \frac{1}{3}\,\text{Bgsin}\,\frac{\sin 3x}{2} + c$$

Opdracht 46 bladzijde 129

De integraal $\int \sin x \cos x \, dx$ kun je op drie manieren berekenen:

$$\int \sin x \cos x \, dx = \int \sin x \, d(\sin x) = \frac{1}{2}\sin^2 x + c$$

$$\int \sin x \cos x \, dx = -\int \cos x \, d(\cos x) = -\frac{1}{2}\cos^2 x + c$$

$$\int \sin x \cos x \, dx = \frac{1}{2}\int \sin 2x \, dx = \frac{1}{2}\cdot\frac{1}{2}\int \sin 2x \, d(2x) = -\frac{1}{4}\cos 2x + c$$

Verklaar dit verschil in resultaten.

We steunen op het feit dat een onbepaalde integraal op een constante na bepaald is.

$$I_1 = \int \sin x \cos x \, dx = \int \sin x \, d(\sin x) = \frac{1}{2}\sin^2 x + c_1$$

$$= \frac{1}{2}(1 - \cos^2 x) + c_1 = -\frac{1}{2}\cos^2 x + \frac{1}{2} + c_1$$

$$I_2 = \int \sin x \cos x \, dx = -\int \cos x \, d(\cos x) = -\frac{1}{2}\cos^2 x + c_2$$

$$\Rightarrow I_1 = I_2 \quad \left(c_2 = \frac{1}{2} + c_1\right)$$

$$I_3 = \int \sin x \cos x \, dx = \frac{1}{2}\int \sin 2x \, dx = -\frac{1}{4}\cos 2x + c_3$$

$$= -\frac{1}{4}(2\cos^2 x - 1) + c_3 = -\frac{1}{2}\cos^2 x + \frac{1}{4} + c_3$$

$$\Rightarrow I_3 = I_2 \quad \left(c_2 = \frac{1}{4} + c_3\right)$$

Opdracht 47 bladzijde 130

$\int \frac{\cos 2x}{(\cos x + \sin x)^2} \, dx$ is gelijk aan

A $\frac{1}{2}\ln(\cos x + \sin x) + c$

B $\frac{1}{2}\ln(1 + \sin 2x) + c$

C $\frac{1}{2}\ln(\cos x - \sin x) + c$

D $\frac{1}{2}\ln(1 - \sin 2x) + c$

(Bron © ACTM State Math Contest, 2009)

$$\int \frac{\cos 2x}{(\cos x + \sin x)^2} \, dx = \int \frac{\cos 2x}{\cos^2 x + 2\cos x \sin x + \sin^2 x} \, dx$$

$$= \int \frac{\cos 2x}{1 + \sin 2x} \, dx$$

$$u = 1 + \sin 2x \Rightarrow du = 2\cos 2x \, dx$$

$$= \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln|1 + \sin 2x| + c = \frac{1}{2}\ln(1 + \sin 2x) + c$$ (want sin2x ligt tussen –1 en 1)

Antwoord B is juist.

Opdracht 48 bladzijde 130

De functie $f\colon \mathbb{R} \to \mathbb{R}$ heeft als voorschrift $f(t) = ae^{-\frac{t}{T}}$, met a en T constant.
Verder weten we dat $f(0) = e$ en $f(2) = 1$.

Bereken $\int_0^2 f(t)\,dt$.

A $e - 1$ **B** $2e - 2$ **C** $2e$ **D** $2 - \frac{2}{e}$ **E** 1

(Bron © IJkingstoets burgerlijk ingenieur, 2012)

- $f(t) = a\,e^{-\frac{t}{T}}$

  $f(0) = e \;\Leftrightarrow\; a = e$

  $f(2) = 1 \;\underset{a=e}{\Leftrightarrow}\; e \cdot e^{-\frac{2}{T}} = 1 \;\Leftrightarrow\; e^{1-\frac{2}{T}} = 1 \;\Leftrightarrow\; T = 2$

  $\Rightarrow\; f(t) = e \cdot e^{-\frac{t}{2}} = e^{1-\frac{t}{2}}$

- $$\int_0^2 e^{1-\frac{t}{2}}\,dt = -2\int_0^2 e^{1-\frac{t}{2}}\,d\left(1 - \frac{t}{2}\right)$$

  $$= -2\left[e^{1-\frac{t}{2}}\right]_0^2 = -2(1 - e) = 2e - 2$$

Antwoord B is juist.

Opdracht 49 bladzijde 130

Bereken $\int_0^{\frac{\pi}{2}} \sqrt{1 + \sin x}\,dx$.

(Bron © University of Cincinnati Math Contest, 2007)

$$\int_0^{\frac{\pi}{2}} \sqrt{1 + \sin x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)dx$$

cos x en sin x zijn positief in het eerste kwadrant

$$= \left[2\sin\frac{x}{2} - 2\cos\frac{x}{2}\right]_0^{\frac{\pi}{2}}$$

$$= 2\left(\sin\frac{\pi}{4} - \cos\frac{\pi}{4} - (\sin 0 - \cos 0)\right)$$

$$= 2\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} + 1\right)$$

$$= 2$$

Opdracht 50 bladzijde 130

Integralen van de vorm $\int \sin^n x \cos^m x\, dx$ met m of n oneven

Voorbeeld

Om de integraal $\int \sin^2 x \cos^3 x\, dx$ te berekenen, schrijven we $\cos^3 x$ als $\cos^2 x \cos x = (1 - \sin^2 x)\cos x$. Nu kunnen we de substitutie $u = \sin x$ toepassen.

$$\int \sin^2 x \cos^3 x\, dx = \int \sin^2 x\,(1 - \sin^2 x)\cos x\, dx \qquad u = \sin x \;\Rightarrow\; du = \cos x\, dx$$
$$= \int u^2(1 - u^2)\, du$$
$$= \int (u^2 - u^4)\, du$$
$$= \frac{u^3}{3} - \frac{u^5}{5} + c$$
$$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + c$$

Bereken op analoge manier.

1 $\int \sin^3 2x \cos^2 2x\, dx = \int \sin^2 2x \cdot \cos^2 2x \cdot \sin 2x\, dx = \int (1 - \cos^2 2x) \cos^2 2x \cdot \sin 2x\, dx$

$$u = \cos 2x \;\Rightarrow\; du = -2 \sin 2x\, dx$$
$$= -\frac{1}{2} \int (1 - u^2)\, u^2\, du = -\frac{1}{2} \int (u^2 - u^4)\, du$$
$$= -\frac{1}{2}\left(\frac{u^3}{3} - \frac{u^5}{5}\right) + c$$
$$= -\frac{1}{6} \cos^3 2x + \frac{1}{10} \cos^5 2x + c$$

2 $\int \sin^5 x\, dx = \int \sin^4 x \cdot \sin x\, dx = \int (1 - \cos^2 x)^2 \sin x\, dx$

$$u = \cos x \;\Rightarrow\; du = -\sin x\, dx$$
$$= -\int (1 - u^2)^2\, du = -\int (1 - 2u^2 + u^4)\, du$$
$$= -\frac{u^5}{5} + \frac{2}{3}u^3 - u + c$$
$$= -\frac{1}{5} \cos^5 x + \frac{2}{3} \cos^3 x - \cos x + c$$

Opdracht 51 bladzijde 131

Integralen van de vorm $\int \sin^n x \cos^m x\,dx$ met m en n even

Voorbeeld

Om de integraal $\int \sin^2 x \cos^2 x\,dx$ te berekenen, moeten we beroep doen op goniometrische formules.

$$\int \sin^2 x \cos^2 x\,dx = \int (\sin x \cos x)^2\,dx \qquad \sin 2x = 2 \sin x \cos x$$

$$= \int \left(\frac{1}{2}\sin 2x\right)^2 dx$$

$$= \frac{1}{4}\int \sin^2 2x\,dx \qquad \sin^2 x = \frac{1-\cos 2x}{2}$$

$$= \frac{1}{4}\int \frac{1-\cos 4x}{2}\,dx$$

$$= \frac{1}{8}\int dx - \frac{1}{8}\int \cos 4x\,dx$$

$$= \frac{x}{8} - \frac{\sin 4x}{32} + c$$

Bereken op analoge manier.

1 $\int \cos^4 x\,dx$

$$= \int \left(\frac{1+\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)\,dx$$

$$= \frac{1}{4}x + \frac{1}{2}\cdot\frac{1}{2}\int \cos 2x\,d\,2x + \frac{1}{4}\int \frac{1+\cos 4x}{2}\,dx$$

$$= \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{8}x + \frac{1}{8}\cdot\frac{1}{4}\int \cos 4x\,d(4x)$$

$$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c$$

2 $\int_0^{\frac{\pi}{2}} \sin^4 x\,dx$

- $$\int \sin^4 x\,dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\,dx$$

$$= \frac{1}{4}x - \frac{1}{2}\cdot\frac{1}{2}\int \cos 2x\,d\,2x + \frac{1}{4}\int \frac{1+\cos 4x}{2}\,dx$$

$$= \frac{1}{4}x - \frac{1}{4}\sin 2x + \frac{1}{8}x + \frac{1}{8}\cdot\frac{1}{4}\int \cos 4x\,d(4x)$$

$$= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c$$

- $$\int_0^{\frac{\pi}{2}} \sin^4 x\,dx = \left[\frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{3\pi}{16}$$

Opdracht 52 bladzijde 131

De *t*-formules

Sommige goniometrische integralen kunnen berekend worden met de volgende substitutie:

$$t = \tan\frac{x}{2} \;\Rightarrow\; dt = \frac{1}{\cos^2\frac{x}{2}} \cdot \frac{1}{2}\,dx = \frac{1}{2}\left(1 + \tan^2\frac{x}{2}\right)dx = \frac{1+t^2}{2}\,dx$$

$$\Rightarrow\; dx = \frac{2}{1+t^2}\,dt$$

Dan is:

$$\tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}} = \frac{2t}{1-t^2}$$

$$\cos x = 2\cos^2\frac{x}{2} - 1 = \frac{2}{1+\tan^2\frac{x}{2}} - 1 = \frac{2}{1+t^2} - 1 = \frac{1-t^2}{1+t^2}$$

$$\sin x = \cos x \tan x = \frac{2t}{1+t^2}$$

Voorbeeld

$$\int \frac{dx}{\sin x} = \int \frac{1+t^2}{2t} \cdot \frac{2}{1+t^2}\,dt \qquad t = \tan\frac{x}{2}$$

$$= \int \frac{dt}{t}$$

$$= \ln|t| + c$$

$$= \ln\left|\tan\frac{x}{2}\right| + c$$

Bereken door gebruik te maken van de *t*-formules.

1 $$\int \frac{dx}{1+\sin x} = \int \frac{\frac{2}{1+t^2}}{1+\frac{2t}{1+t^2}}\,dt \qquad t = \tan\frac{x}{2}$$

$$= \int \frac{2}{1+t^2+2t}\,dt = 2\int \frac{d(t+1)}{(t+1)^2}$$

$$= 2 \cdot \frac{(t+1)^{-1}}{-1} + c = -\frac{2}{t+1} + c$$

$$= -\frac{2}{1+\tan\frac{x}{2}} + c$$

2 $\displaystyle\int \frac{dx}{\sin x + \cos x + 1}$ $\qquad t = \tan \frac{x}{2}$

$$= \int \frac{\dfrac{2}{1+t^2}}{\dfrac{2t}{1+t^2} + \dfrac{1-t^2}{1+t^2} + 1}\, dt$$

$$= \int \frac{2}{2t + 1 - t^2 + 1 + t^2}\, dt = \int \frac{1}{t+1}\, d(t+1)$$

$$= \ln|t+1| + c = \ln\left|\tan \frac{x}{2} + 1\right| + c$$

Opdracht 53 bladzijde 132

Bereken de volgende integralen door ze te herleiden tot $\displaystyle\int \frac{du}{\sqrt{k-u^2}}$ of $\displaystyle\int \frac{du}{\sqrt{u^2+k}}$.

1 $\displaystyle\int \frac{dx}{\sqrt{4x^2+4x+5}} = \int \frac{dx}{\sqrt{(2x+1)^2+4}} = \frac{1}{2}\int \frac{d(2x+1)}{\sqrt{(2x+1)^2+4}}$

$$= \frac{1}{2} \ln\left|2x + 1 + \sqrt{4x^2+4x+5}\right| + c$$

2 $\displaystyle\int \frac{dx}{\sqrt{-25x^2-20x}} = \int \frac{dx}{\sqrt{4-(5x+2)^2}} = \frac{1}{5}\int \frac{d(5x+2)}{\sqrt{4-(5x+2)^2}}$

$$= \frac{1}{5} \operatorname{Bgsin}\left(\frac{5}{2}x + 1\right) + c$$

3 $\displaystyle\int \frac{dx}{\sqrt{25x^2+20x}} = \int \frac{dx}{\sqrt{(5x+2)^2-4}} = \frac{1}{5}\int \frac{d(5x+2)}{\sqrt{(5x+2)^2-4}}$

$$= \frac{1}{5} \ln\left|5x + 2 + \sqrt{25x^2+20x}\right| + c$$

4 $\displaystyle\int \frac{dx}{\sqrt{12-12x-9x^2}} = \int \frac{dx}{\sqrt{16-(3x+2)^2}} = \frac{1}{3}\int \frac{d(3x+2)}{\sqrt{16-(3x+2)^2}}$

$$= \frac{1}{3} \operatorname{Bgsin} \frac{3x+2}{4} + c$$

Opdracht 54 bladzijde 132

1 Bereken $\displaystyle\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx$.

$$\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx = -\int_0^{\pi} \frac{d(\cos x)}{1+\cos^2 x} = -\left[\operatorname{Bgtan}(\cos x)\right]_0^{\pi}$$

$$= -(\operatorname{Bgtan}(-1) - \operatorname{Bgtan} 1) = -\left(-\frac{\pi}{4} - \frac{\pi}{4}\right) = \frac{\pi}{2}$$

2 Toon aan dat voor een functie f, die continu is in $[0, a]$, geldt:

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$\int_0^a f(a-x)\,dx$$

$$u = a - x \Rightarrow du = -dx$$
$$x = 0 \Rightarrow u = a$$
$$x = a \Rightarrow u = 0$$

$$= -\int_a^0 f(u)\,du = \int_0^a f(u)\,du = \int_0^a f(x)\,dx$$

3 Maak gebruik van **1** en **2** om $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx$ te berekenen.

$$\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}\,dx$$

$$= \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx - \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$$

$\downarrow$

$\sin(\pi - x) = \sin x$

$\cos(\pi - x) = -\cos x$

$$\Rightarrow 2\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx = \pi \cdot \frac{\pi}{2}$$

$$\Rightarrow \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx = \frac{\pi^2}{4}$$

Opdracht 55 bladzijde 132

Als $I(n) = \int_0^{\pi} \sin(nx)\,dx$, bepaal dan $\sum_{n=0}^{+\infty} I(5^n)$.

(Bron © Rice University Mathematics Tournament, 2007)

- $$I(n) = \int_0^{\pi} \sin(nx)\,dx = \frac{1}{n}\int_0^{\pi} \sin(nx)\,d(nx)$$
$$= -\frac{1}{n}\left[\cos(nx)\right]_0^{\pi} = -\frac{1}{n}(\cos(n\pi) - 1)$$
$$= -\frac{1}{n}(\pm 1 - 1)$$
$$= \begin{cases} 0 & \text{als n even is} \\ \frac{2}{n} & \text{als n oneven is} \end{cases}$$

- $$\sum_{n=0}^{+\infty} I(5^n) = I(5^0) + I(5^1) + I(5^2) + \ldots$$
$$= I(1) + I(5) + I(25) + \ldots$$
$$= 2 + \frac{2}{5} + \frac{2}{25} + \ldots$$
$\Downarrow$ som van een oneindige MR, $u_1 = 2, q = \frac{1}{5}$
$$= 2 \cdot \frac{1}{1 - \frac{1}{5}} = \frac{5}{2}$$

Opdracht 56 bladzijde 133

Bereken

1 $\int x \sin 3x \, dx$

$$u = x \Rightarrow du = dx$$

$$dv = \sin 3x \, dx \Rightarrow v = \frac{1}{3} \int \sin 3x \, d(3x) = -\frac{1}{3} \cos 3x$$

$$= -\frac{1}{3} x \cos 3x + \frac{1}{3} \int \cos 3x \, dx$$

$$= -\frac{1}{3} x \cos 3x + \frac{1}{9} \int \cos 3x \, d(3x)$$

$$= -\frac{1}{3} x \cos 3x + \frac{1}{9} \sin 3x + c$$

2 $\int \ln(4x) \, dx$

$$u = \ln(4x) \Rightarrow du = \frac{1}{4x} \cdot 4 \, dx = \frac{1}{x} dx$$

$$dv = dx \Rightarrow v = x$$

$$= x \ln(4x) - \int x \cdot \frac{1}{x} \, dx$$

$$= x \ln(4x) - x + c$$

3 $\int x^2 e^{-\frac{1}{2}x} \, dx$

$$u = x^2 \Rightarrow du = 2x \, dx$$

$$dv = e^{-\frac{1}{2}x} \Rightarrow v = -2 \int e^{-\frac{1}{2}x} \, d\left(-\frac{1}{2}x\right) = -2e^{-\frac{1}{2}x}$$

$$= -2x^2 \, e^{-\frac{1}{2}x} + 4 \int x \, e^{-\frac{1}{2}x} \, dx$$

$$u = x \Rightarrow du = dx$$

$$dv = e^{-\frac{1}{2}x} \Rightarrow v = -2 \, e^{-\frac{1}{2}x}$$

$$= -2x^2 \, e^{-\frac{1}{2}x} + 4\left(-2x \, e^{-\frac{1}{2}x} + 2 \int e^{-\frac{1}{2}x} \, dx\right)$$

$$= -2x^2 \, e^{-\frac{1}{2}x} + 4\left(-2x \, e^{-\frac{1}{2}x} - 4 \, e^{-\frac{1}{2}x}\right) + c$$

$$= -2x^2 \, e^{-\frac{1}{2}x} - 8x \, e^{-\frac{1}{2}x} - 16 \, e^{-\frac{1}{2}x} + c$$

4 $\int e^{-x} \cos 4x \, dx$

$$u = \cos 4x \;\Rightarrow\; du = -4 \sin 4x \, dx$$
$$dv = e^{-x} dx \;\Rightarrow\; v = \int e^{-x} dx = -\int e^{-x} \, d(-x) = -e^{-x}$$

$$= -e^{-x} \cos 4x - 4 \int e^{-x} \sin 4x \, dx$$

$$u = \sin 4x \;\Rightarrow\; du = 4 \cos 4x \, dx$$
$$dv = e^{-x} dx \Rightarrow\; v = -e^{-x}$$

$$= -e^{-x} \cos 4x - 4\left(-e^{-x} \sin 4x + 4 \int e^{-x} \cos 4x \, dx\right)$$
$$= -e^{-x} \cos 4x + 4e^{-x} \sin 4x - 16 \int e^{-x} \cos 4x \, dx$$

$$\Rightarrow\; 17 \int e^{-x} \cos 4x \, dx = -e^{-x} \cos 4x + 4e^{-x} \sin 4x + c$$

$$\Rightarrow\; \int e^{-x} \cos 4x \, dx = -\frac{1}{17} e^{-x} \cos 4x + \frac{4}{17} e^{-x} \sin 4x + c$$

5 $\int x^4 \ln x \, dx$

$$u = \ln x \;\Rightarrow\; du = \frac{1}{x} dx$$
$$dv = x^4 dx \;\Rightarrow\; v = \int x^4 \, dx = \frac{x^5}{5}$$

$$= \frac{x^5}{5} \ln x - \frac{1}{5} \int \frac{1}{x} \cdot x^5 \, dx$$
$$= \frac{x^5}{5} \ln x - \frac{1}{5} \int x^4 \, dx$$
$$= \frac{x^5}{5} \ln x - \frac{1}{25} x^5 + c$$
$$= \frac{1}{25} x^5 (5 \ln x - 1) + c$$

6 $\int (\ln x)^2 \, dx$

$$u = (\ln x)^2 \;\Rightarrow\; du = 2(\ln x) \cdot \frac{1}{x} dx$$
$$dv = dx \;\Rightarrow\; v = x$$

$$= x(\ln x)^2 - 2 \int \ln x \, dx$$

$$u = \ln x \;\Rightarrow\; du = \frac{1}{x} dx$$
$$dv = dx \;\Rightarrow\; v = x$$

$$= x(\ln x)^2 - 2\left(x \ln x - \int dx\right)$$
$$= x(\ln x)^2 - 2x \ln x + 2x + c$$

7 $\int 2^x \sin x \, dx$

$$u = 2^x \Rightarrow du = 2^x \cdot (\ln 2)dx$$
$$dv = \sin x \, dx \Rightarrow v = -\cos x$$

$$= -2^x \cos x + (\ln 2)\int 2^x \cos x \, dx$$

$$u = 2^x \Rightarrow du = 2^x \cdot (\ln 2)dx$$
$$dv = \cos x \, dx \Rightarrow v = \sin x$$

$$= -2^x \cos x + (\ln 2)\left(2^x \sin x - (\ln 2)\int 2^x \sin x \, dx\right)$$

$$= -2^x \cos x + 2^x \cdot (\ln 2) \cdot \sin x - (\ln^2 2) \cdot \int 2^x \sin x \, dx$$

$$\Rightarrow (1 + \ln^2 2)\int 2^x \sin x \, dx = -2^x \cos x + 2^x (\ln 2)\sin x + c$$

$$\Rightarrow \int 2^x \sin x \, dx$$

$$= \frac{-2^x}{1 + \ln^2 2} \cos x + \frac{2^x}{1 + \ln^2 2} (\ln 2)\sin x + c$$

8 $\int \text{Bgsin}\, x \, dx$

$$u = \text{Bgsin}\, x \Rightarrow du = \frac{1}{\sqrt{1 - x^2}} dx$$

$$dv = dx \Rightarrow v = x$$

$$= x \,\text{Bgsin}\, x - \int \frac{x}{\sqrt{1 - x^2}} dx$$

$$= x \,\text{Bgsin}\, x + \frac{1}{2} \int \frac{d(1 - x^2)}{\sqrt{1 - x^2}}$$

$$= x \,\text{Bgsin}\, x + \sqrt{1 - x^2} + c$$

Opdracht 57 bladzijde 133

Bereken

1 $\int_{-1}^{1} x^2 e^x \, dx$

- $\int x^2 e^x \, dx$

$$u = x^2 \Rightarrow du = 2x \, dx$$
$$dv = e^x dx \Rightarrow v = e^x$$

$$= x^2 e^x - 2\int x e^x \, dx$$

$$u = x \Rightarrow du = dx$$
$$dv = e^x dx \Rightarrow v = e^x$$

$$= x^2 e^x - 2\left(x e^x - \int e^x \, dx\right)$$

$$= x^2 e^x - 2x e^x + 2e^x + c$$

$$= e^x (x^2 - 2x + 2) + c$$

- $\int_{-1}^{1} x^2 e^x \, dx = \left[e^x (x^2 - 2x + 2)\right]_{-1}^{1} = e - \frac{1}{e} \cdot 5 = \frac{e^2 - 5}{e}$

2 $\int_1^e \ln x \, dx$

$$u = \ln x \;\Rightarrow\; du = \frac{1}{x}\, dx$$

$$dv = dx \;\Rightarrow\; v = x$$

$$= \left[x \ln x\right]_1^e - \int_1^e dx$$

$$= \left[x \ln x\right]_1^e - \left[x\right]_1^e$$

$$= e - (e - 1)$$

$$= 1$$

3 $\int_0^\pi x^2 \sin 3x \, dx$

- $\int x^2 \sin 3x \, dx$

$$u = x^2 \;\Rightarrow\; du = 2x\, dx$$

$$dv = \sin 3x\, dx \;\Rightarrow\; v = \frac{1}{3}\int \sin 3x \, d(3x) = -\frac{1}{3}\cos 3x$$

$$= -\frac{1}{3}x^2 \cos 3x + \frac{2}{3}\int x \cos 3x \, dx$$

$$u = x \;\Rightarrow\; du = dx$$

$$dv = \cos 3x\, dx \;\Rightarrow\; v = \frac{1}{3}\int \cos 3x \, d(3x) = \frac{1}{3}\sin 3x$$

$$= -\frac{1}{3}x^2 \cos 3x + \frac{2}{3}\left(\frac{1}{3}x \sin 3x - \frac{1}{9}\int \sin 3x \, d(3x)\right)$$

$$= -\frac{1}{3}x^2 \cos 3x + \frac{2}{9}x \sin 3x + \frac{2}{27}\cos 3x + c$$

- $\int_0^\pi x^2 \sin 3x \, dx = \left[-\frac{1}{3}x^2 \cos 3x + \frac{2}{9}x \sin 3x + \frac{2}{27}\operatorname{cox} 3x\right]_0^\pi$

$$= \frac{1}{3}\pi^2 - \frac{2}{27} - \frac{2}{27}$$

$$= \frac{1}{3}\pi^2 - \frac{4}{27}$$

$$= \frac{9\pi^2 - 4}{27}$$

4 $\displaystyle\int_0^{\frac{\pi}{2}} e^{2x}\sin\frac{1}{2}x\,dx$

- $\displaystyle\int e^{2x}\sin\frac{1}{2}x\,dx$

$$u=\sin\frac{1}{2}x \Rightarrow du=\frac{1}{2}\cos\frac{1}{2}x\,dx$$

$$dv=e^{2x}dx \Rightarrow v=\frac{1}{2}e^{2x}$$

$$=\frac{1}{2}e^{2x}\sin\frac{1}{2}x-\frac{1}{4}\int e^{2x}\cos\frac{1}{2}x\,dx$$

$$u=\cos\frac{1}{2}x \Rightarrow du=-\frac{1}{2}\sin\frac{1}{2}x$$

$$dv=e^{2x}dx \Rightarrow v=\frac{1}{2}e^{2x}$$

$$=\frac{1}{2}e^{2x}\sin\frac{1}{2}x-\frac{1}{4}\left(\frac{1}{2}e^{2x}\cos\frac{1}{2}x+\frac{1}{4}\int e^{2x}\sin\frac{1}{2}x\,dx\right)$$

$$=\frac{1}{2}e^{2x}\sin\frac{1}{2}x-\frac{1}{8}e^{2x}\cos\frac{1}{2}x-\frac{1}{16}\int e^{2x}\sin\frac{1}{2}x\,dx$$

$$\Rightarrow \frac{17}{16}\int e^{2x}\sin\frac{1}{2}x\,dx=\frac{1}{8}e^{2x}\left(4\sin\frac{1}{2}x-\cos\frac{1}{2}x\right)+c$$

$$\Rightarrow \int e^{2x}\sin\frac{1}{2}x\,dx=\frac{2}{17}e^{2x}\left(4\sin\frac{1}{2}x-\cos\frac{1}{2}x\right)+c$$

- $\displaystyle\int_0^{\frac{\pi}{2}} e^{2x}\sin\frac{1}{2}x\,dx=\frac{2}{17}\left[e^{2x}\left(4\sin\frac{1}{2}x-\cos\frac{1}{2}x\right)\right]_0^{\frac{\pi}{2}}$

$$=\frac{2}{17}\left(e^{\pi}\left(4\cdot\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\right)-(-1)\right)$$

$$=\frac{2}{17}\left(\frac{3\sqrt{2}}{2}e^{\pi}+1\right)$$

$$=\frac{3\sqrt{2}e^{\pi}+2}{17}$$

Opdracht 58 bladzijde 133

De bepaalde integraal $\int_0^{\pi} x(\sin x + \cos x)\, dx$ is gelijk aan

A $\frac{\pi}{2}$ **B** $-\frac{\pi}{2}$ **C** $\pi + 2$ **D** $\pi - 2$

(Bron © Toelatingsproef arts-tandarts)

$u = x \Rightarrow du = dx$
$dv = (\sin x + \cos x)\, dx \Rightarrow v = -\cos x + \sin x$

$= \left[x(\sin x - \cos x)\right]_0^{\pi} - \int_0^{\pi} (\sin x - \cos x)\, dx$

$= \left[x(\sin x - \cos x)\right]_0^{\pi} - \left[-\cos x - \sin x\right]_0^{\pi}$

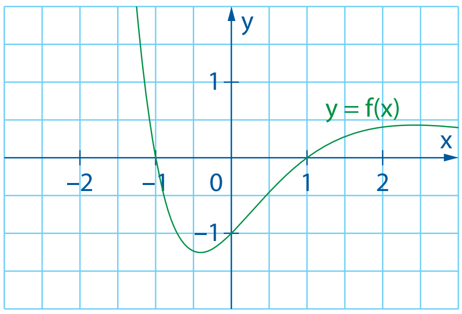
$= \pi(0 + 1) + (-1 - (-1))$

$= \pi - 2$

Antwoord D is juist.

Opdracht 59 bladzijde 133

Bereken de oppervlakte van het gebied begrensd door de grafiek van de functie met voorschrift $f(x) = (x^2 - 1)e^{-x}$ en de x-as.



- nulpunten van f: –1 en 1
- f is negatief over [–1, 1]
- $A = -\int_{-1}^{1} (x^2 - 1)\, e^{-x}\, dx$
- $\int (x^2 - 1)\, e^{-x}\, dx$

  $u = x^2 - 1 \Rightarrow du = 2x\, dx$
  $dv = e^{-x}\, dx \Rightarrow v = -e^{-x}$

  $= -(x^2 - 1)\, e^{-x} + 2\int x e^{-x}\, dx$

  $u = x \Rightarrow du = dx$
  $dv = e^{-x}\, dx \Rightarrow v = -e^{-x}$

  $= -(x^2 - 1)e^{-x} + 2\left(-xe^{-x} + \int e^{-x}\, dx\right)$

  $= -(x^2 - 1)e^{-x} - 2xe^{-x} - 2e^{-x} + c$

  $= -e^{-x}(x^2 - 1 + 2x + 2) + c$

  $= -e^{-x}(x^2 + 2x + 1) + c$

  $= -e^{-x}(x + 1)^2 + c$

- $A = \left[e^{-x}\,(x + 1)^2\right]_{-1}^{1}$

  $= \frac{4}{e}$

Opdracht 60 bladzijde 133

Bereken

1 $\int \cos(\ln x)\, dx$

$$u = \cos(\ln x) \;\Rightarrow\; du = -\sin(\ln x) \cdot \frac{1}{x}\, dx$$
$$dv = dx \;\Rightarrow\; v = x$$
$$= x\cos(\ln x) + \int \sin(\ln x)\, dx$$
$$u = \sin(\ln x) \;\Rightarrow\; du = \cos(\ln x) \cdot \frac{1}{x}\, dx$$
$$dv = dx \;\Rightarrow\; v = x$$
$$= x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)\, dx$$
$$\Rightarrow\; 2\int \cos(\ln x)\, dx = x\cos(\ln x) + x\sin(\ln x)) + c$$
$$\Rightarrow\; \int \cos(\ln x)\, dx = \frac{1}{2} x\cos(\ln x) + \frac{1}{2} x\sin(\ln x) + c$$

2 $\int \frac{x-1}{x^2} \ln x\, dx$

$$= \int \frac{\ln x}{x}\, dx - \int \frac{1}{x^2} \ln x\, dx$$
$$u = \ln x \;\Rightarrow\; du = \frac{1}{x}\, dx \qquad u = \ln x \;\Rightarrow\; du = \frac{1}{x}\, dx$$
$$dv = \frac{1}{x^2}\, dx \;\Rightarrow\; v = -\frac{1}{x}$$
$$= \int u\, du - \left(-\frac{1}{x} \ln x + \int \frac{1}{x^2}\, dx\right)$$
$$= \frac{u^2}{2} + \frac{1}{x} \ln x + \frac{1}{x} + c$$
$$= \frac{\ln^2 x}{2} + \frac{1}{x} \ln x + \frac{1}{x} + c$$

3 $\int \frac{x}{\cos^2 x}\, dx$

$$u = x \;\Rightarrow\; du = dx$$
$$dv = \frac{1}{\cos^2 x}\, dx \;\Rightarrow\; v = \tan x$$
$$= x\tan x - \int \tan x\, dx$$
$$= x\tan x - \int \frac{\sin x}{\cos x}\, dx$$
$$= x\tan x + \int \frac{d(\cos x)}{\cos x}$$
$$= x\tan x + \ln|\cos x| + c$$

4 $\int (2x + e^x)^2\, dx$

$$= \int (4x^2 + 4xe^x + e^{2x})\,dx$$

$$= \frac{4x^3}{3} + 4\int xe^x\, dx + \frac{1}{2}\int e^{2x}\, d(2x)$$

$$u = x \Rightarrow du = dx$$
$$dv = e^x\, dx \Rightarrow v = e^x$$

$$= \frac{4}{3}x^3 + 4\left(xe^x - \int e^x\, dx\right) + \frac{1}{2}e^{2x}$$

$$= \frac{4}{3}x^3 + 4xe^x - 4e^x + \frac{1}{2}e^{2x} + c$$

5 $\int x^3 e^{x^2}\, dx$

- We voeren eerst de substitutie $t = x^2$ uit:

  $$t = x^2 \Rightarrow dt = 2x\, dx$$

  $$\int x^3 e^{x^2}\, dx = \int x^2 \cdot e^{x^2} \cdot x\, dx$$

  $$= \frac{1}{2}\int t\, e^t\, dt$$

- $\frac{1}{2}\int t\, e^t\, dt$

  $$u = t \Rightarrow du = dt$$
  $$dv = e^t\, dt \Rightarrow v = e^t$$

  $$= \frac{1}{2}\left(te^t - \int e^t\, dt\right)$$

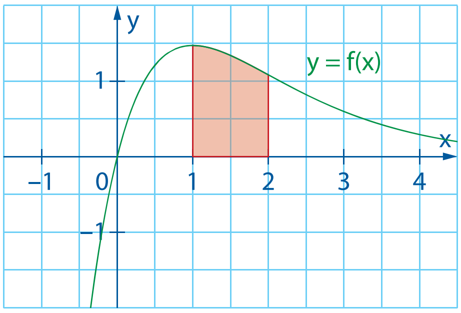
  $$= \frac{1}{2}(te^t - e^t) + c$$

  $$= \frac{1}{2}e^t(t - 1) + c$$

- $\int x^3 e^{x^2}\, dx = \frac{1}{2}e^{x^2}(x^2 - 1) + c$

Opdracht 61 bladzijde 134

Bereken de oppervlakte van het gebied begrensd door de grafiek van de functie met voorschrift $f(x) = 4xe^{-x}$, de x-as en de verticale rechten door het maximum en het buigpunt van de grafiek van f.



- $f(x) = 4xe^{-x}$

  $f'(x) = 4(e^{-x} - xe^{-x}) = 4e^{-x}(1 - x)$

  $\Rightarrow$ nulpunt: 1 (met tekenwissel)

  $f''(x) = -4e^{-x}(1 - x) - 4e^{-x} = -4e^{-x}(2 - x)$

  $\Rightarrow$ nulpunt: 2 (met tekenwissel)

  De verticale rechten hebben als vergelijking $x = 1$ en $x = 2$.

- f is positief over [1, 2]

$$\Rightarrow A = 4\int_1^2 xe^{-x}\,dx$$

$$u = x \Rightarrow du = dx$$
$$dv = e^{-x}\,dx \Rightarrow v = -e^{-x}$$

$$= 4\left(-\left[xe^{-x}\right]_1^2 + \int_1^2 e^{-x}\,dx\right)$$

$$= 4\left(-\left(\frac{2}{e^2} - \frac{1}{e}\right) - \left[e^{-x}\right]_1^2\right)$$

$$= 4\left(-\frac{2}{e^2} + \frac{1}{e} - \frac{1}{e^2} + \frac{1}{e}\right)$$

$$= 4 \cdot \frac{2e - 3}{e^2}$$

Opdracht 62 bladzijde 134

Door partiële integratie wordt

$$\int_a^b \text{Bgsin}\sqrt{x}\,dx = b\,\text{Bgsin}\sqrt{b} - a\,\text{Bgsin}\sqrt{a} - \int_a^b \sqrt{q(x)}\,dx$$

Bepaal de rationale functie $q(x)$.

$$\int \text{Bgsin}\sqrt{x}\,dx$$

$$u = \text{Bgsin}\sqrt{x} \Rightarrow du = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}\,dx$$
$$dv = dx \Rightarrow v = x$$

$$= x\,\text{Bgsin}\sqrt{x} - \int \frac{x}{2\sqrt{1-x} \cdot \sqrt{x}}\,dx$$

$$= x\,\text{Bgsin}\sqrt{x} - \int \sqrt{\frac{x^2}{4x - 4x^2}}\,dx$$

$$\Rightarrow q(x) = \frac{x^2}{4x - 4x^2}$$

Opdracht 63 bladzijde 134

Bereken $\int x \text{ Bgsin } x \, dx$.

$$\int x \text{ Bgsin } x \text{ dx}$$

$$u = \text{Bgsin } x \;\Rightarrow\; du = \frac{1}{\sqrt{1-x^2}} \text{ dx}$$

$$dv = x \text{ dx} \;\Rightarrow\; v = \frac{1}{2} x^2$$

$$= \frac{1}{2} x^2 \text{ Bgsin } x - \frac{1}{2} \int \frac{x^2}{\sqrt{1-x^2}} \text{ dx}$$

$$u = x \;\Rightarrow\; du = dx$$

$$dv = \frac{x}{\sqrt{1-x^2}} \text{ dx} \;\Rightarrow\; v = -\frac{1}{2} \int \frac{d(1-x^2)}{\sqrt{1-x^2}} = -\sqrt{1-x^2}$$

$$= \frac{1}{2} x^2 \text{ Bgsin } x - \frac{1}{2} \left(-x\sqrt{1-x^2} + \int \sqrt{1-x^2} \text{ dx} \right)$$

$$\int \sqrt{1-x^2} \text{ dx} = \int \frac{1-x^2}{\sqrt{1-x^2}} \text{ dx} = \text{Bgsin } x - \int \frac{x^2}{\sqrt{1-x^2}} \text{ dx}$$

$$\Rightarrow\; 2 \int \frac{x^2}{\sqrt{1-x^2}} \text{ dx} = -x\sqrt{1-x^2} + \text{Bgsin } x + c$$

$$\Rightarrow\; \int \frac{x^2}{\sqrt{1-x^2}} \text{ dx} = -\frac{1}{2} x\sqrt{1-x^2} + \frac{1}{2} \text{Bgsin } x + c$$

$$= \frac{1}{2} x^2 \text{ Bgsin } x + \frac{1}{4} x\sqrt{1-x^2} - \frac{1}{4} \text{Bgsin } x + c$$

Opdracht 64 bladzijde 134

Gegeven: $I_n = \int_0^1 x^n \sqrt{1-x}\,dx \qquad (n \in \mathbb{N})$

1 Toon aan dat $I_0 = \frac{2}{3}$.

$$\begin{aligned} I_0 &= \int_0^1 \sqrt{1-x}\,dx \\ &= -\int_0^1 \sqrt{1-x}\,d(1-x) \\ &= -\frac{2}{3}\left[(1-x)^{3/2}\right]_0^1 \\ &= -\frac{2}{3}(0-1) \\ &= \frac{2}{3} \end{aligned}$$

2 Toon met behulp van partiële integratie aan dat $I_n = \frac{2n}{2n+3} \cdot I_{n-1}$.

$$\int x^n \sqrt{1-x}\,dx$$

$$u = x^n \Rightarrow du = nx^{n-1}dx$$

$$dv = \sqrt{1-x}\,dx \Rightarrow v = -\int (1-x)^{\frac{1}{2}}\,d(1-x) = -\frac{2}{3}(1-x)^{\frac{3}{2}}$$

$$= -\frac{2}{3}x^n\sqrt{(1-x)^3} + \frac{2}{3}n\int x^{n-1}(1-x)^{\frac{3}{2}}\,dx$$

$$= -\frac{2}{3}x^n\sqrt{(1-x)^3} + \frac{2}{3}n\int x^{n-1}(1-x)\sqrt{1-x}\,dx$$

$$= -\frac{2}{3}x^n\sqrt{(1-x)^3} + \frac{2}{3}n\int x^{n-1}\sqrt{1-x}\,dx - \frac{2}{3}n\int x^n\sqrt{1-x}\,dx$$

$$\Rightarrow \left(1+\frac{2}{3}n\right)\int x^n\sqrt{1-x}\,dx = -\frac{2}{3}x^n\sqrt{(1-3)^3} + \frac{2}{3}n\int x^{n-1}\sqrt{1-x}\,dx$$

$$\Rightarrow \int x^n\sqrt{1-x}\,dx = \frac{3}{2n+3}\left(-\frac{2}{3}x^n\sqrt{(1-x)^3} + \frac{2}{3}n\int x^{n-1}\sqrt{1-x}\,dx\right)$$

$$\text{Dus: } \int_0^1 x^n\sqrt{1-x}\,dx = \frac{3}{2n+3}\left(0-0+\frac{2}{3}n\int_0^1 x^{n-1}\sqrt{1-x}\,dx\right)$$

$$= \frac{2n}{2n+3}\,I_{n-1}$$

3 Maak gebruik van **1** en **2** om aan te tonen dat $I_2 = \frac{16}{105}$.

$$I_2 = \frac{4}{7}\,I_1 = \frac{4}{7}\cdot\frac{2}{5}\,I_0 = \frac{8}{35}\cdot\frac{2}{3} = \frac{16}{105}$$

Opdracht 65 bladzijde 134

Gegeven: $G(n) = \int_0^{+\infty} x^n e^{-x}\, dx$ met $n \in \mathbb{N}_0$

1 Toon aan dat $G(1) = 1$. Hou rekening met de eigenschap $\lim\limits_{x \to +\infty} x^n e^{-x} = 0$.

$$G(1) = \int_0^{+\infty} x\, e^{-x}\, dx$$

$$= \lim_{a \to +\infty} \int_0^a x\, e^{-x}\, dx$$

$$u = x \Rightarrow du = dx$$
$$dv = e^{-x}\, dx \Rightarrow v = -e^{-x}$$

$$\int_0^a x\, e^{-x}\, dx = -\left[x\, e^{-x}\right]_0^a + \int_0^a e^{-x}\, dx$$

$$= -\left[x\, e^{-x}\right]_0^a - \left[e^{-x}\right]_0^a$$

$$= -ae^{-a} - (e^{-a} - 1)$$

$$= -ae^{-a} - e^{-a} + 1$$

$$G(1) = \lim_{a \to +\infty} (-ae^{-a} - e^{-a} + 1)$$

$$= -\lim_{a \to +\infty} ae^{-a} - \lim_{a \to +\infty} e^{-a} + 1$$

$$= 0 - 0 + 1$$

$$= 1$$

2 Toon met behulp van partiële integratie aan dat $G(n+1) = (n+1)G(n)$.

$$G(n+1) = \int_0^{+\infty} x^{n+1}\, e^{-x}\, dx$$

- $\int x^{n+1}\, e^{-x}\, dx$

$$u = x^{n+1} \Rightarrow du = (n+1)x^n\, dx$$
$$dv = e^{-x}\, dx \Rightarrow v = -e^{-x}$$

$$= -x^{n+1}e^{-x} + (n+1)\int x^n e^{-x}\, dx$$

- $\int_0^{+\infty} x^{n+1}\, e^{-x}\, dx = \lim\limits_{a \to +\infty} (-a^{n+1}e^{-a}) + (n+1)\int_0^{+\infty} x^n e^{-x}\, dx$

$$= 0 + (n+1)\int_0^{+\infty} x^n\, e^{-x}\, dx$$

$\Rightarrow G(n+1) = (n+1)\, G(n)$

3 Maak gebruik van **1** en **2** om aan te tonen dat $\int_0^{+\infty} x^n\, e^{-x}\, dx = n!$.

$G(1) = 1!$	uit **1**
$G(2) = 2 \cdot G(1) = 2 \cdot 1 = 2!$	uit **2**
$G(3) = 3 \cdot G(2) = 3 \cdot 2 \cdot 1 = 3!$	uit **2**

…

$\Rightarrow G(n) = \int_0^{+\infty} x^n\, e^{-x}\, dx = n!$ voor elk natuurlijk getal n

Opmerking:
We hebben hier een bewijs door inductie gebruikt (zie opdracht 42 hoofdstuk 4)

Opdracht 66 bladzijde 135

$\displaystyle\int \frac{x^5}{1+x^2}\,dx$ is gelijk aan

A $\dfrac{x^4}{4} - \dfrac{x^2}{2} + \ln\sqrt{1+x^2} + c$

B $\dfrac{x^4}{4} - \dfrac{3x^2}{2} + \ln\sqrt{1+x^2} + c$

C $\dfrac{x^4}{4} + \dfrac{3x^2}{2} + \ln\sqrt{1+x^2} + c$

D $\dfrac{x^4}{4} + \dfrac{x^2}{2} + \ln\sqrt{1+x^2} + c$

E geen van de voorgaande

(Bron © University of Central Arkansas Regional Math Contest, 2009)

$$\int \frac{x^5}{1+x^2}\,dx$$

$$\begin{array}{rr|l}
x^5 & & x^2+1 \\ \hline
\mp x^5 \mp x^3 & & x^3 - x \\ \hline
-x^3 & & \\
\pm x^3 \pm x & & \\ \hline
x & &
\end{array}$$

$$= \int\left(x^3 - x + \frac{x}{1+x^2}\right)dx$$

$$= \frac{x^4}{4} - \frac{x^2}{2} + \frac{1}{2}\int \frac{d(1+x^2)}{1+x^2}$$

$$= \frac{x^4}{4} - \frac{x^2}{2} + \frac{1}{2}\ln(1+x^2) + c$$

$$= \frac{x^4}{4} - \frac{x^2}{2} + \ln(1+x^2)^{\frac{1}{2}} + c$$

$$= \frac{x^4}{4} - \frac{x^2}{2} + \ln\sqrt{1+x^2} + c$$

Antwoord A is juist.

Opdracht 67 bladzijde 135

Bereken

1 $\displaystyle\int \frac{2x^2+3x}{2x-1}\,dx$

$$\begin{array}{rr|l}
2x^2 + 3x & & 2x-1 \\ \hline
\mp 2x^2 \pm x & & x+2 \\ \hline
4x & & \\
\mp 4x \pm 2 & & \\ \hline
2 & &
\end{array}$$

$$= \int\left(x + 2 + \frac{2}{2x-1}\right)dx$$

$$= \frac{x^2}{2} + 2x + 2\cdot\frac{1}{2}\int\frac{d(2x-1)}{2x-1}$$

$$= \frac{x^2}{2} + 2x + \ln|2x-1| + c$$

2 $\displaystyle\int \frac{x^3}{3x-1}\,dx$

$$\begin{array}{l|l} x^3 & 3x-1 \\ \cline{2-2} \mp x^3 \pm \frac{1}{3}x^2 & \frac{1}{3}x^2+\frac{1}{9}x+\frac{1}{27} \\ \cline{1-1} \frac{1}{3}x^2 & \\ \mp\frac{1}{3}x^2 \pm \frac{1}{9}x & \\ \cline{1-1} \frac{1}{9}x & \\ \mp\frac{1}{9}x \pm \frac{1}{27} & \\ \cline{1-1} \frac{1}{27} & \end{array}$$

$$= \int\left(\frac{1}{3}x^2+\frac{1}{9}x+\frac{1}{27}+\frac{\frac{1}{27}}{3x-1}\right)dx$$

$$= \frac{1}{9}x^3+\frac{1}{18}x^2+\frac{1}{27}x+\frac{1}{27}\cdot\frac{1}{3}\int\frac{d(3x-1)}{3x-1}$$

$$= \frac{1}{9}x^3+\frac{1}{18}x^2+\frac{1}{27}x+\frac{1}{81}\ln|3x-1|+c$$

3 $\displaystyle\int \frac{2x^4}{x^2+1}\,dx$

$$= 2\int\frac{x^4}{x^2+1}dx$$

$$\begin{array}{l|l} x^4 & x^2+1 \\ \cline{2-2} \mp x^4 \mp x^2 & x^2-1 \\ \cline{1-1} -x^2 & \\ \pm x^2 \pm 1 & \\ \cline{1-1} 1 & \end{array}$$

$$= 2\int\left(x^2-1+\frac{1}{x^2+1}\right)dx$$

$$= \frac{2}{3}x^3-2x+2\,\mathrm{Bgtan}\,x+c$$

4 $\displaystyle\int \frac{dx}{x(x+1)}$

$$\frac{1}{x(x+1)}=\frac{A}{x}+\frac{B}{x+1}=\frac{(A+B)x+A}{x(x+1)}$$

$$\Rightarrow \begin{cases} A+B=0 \\ A=1 \end{cases} \Leftrightarrow \begin{cases} A=1 \\ B=-1 \end{cases}$$

$$= \int\frac{dx}{x}-\int\frac{dx}{x+1}$$

$$= \ln|x|-\ln|x+1|+c$$

$$= \ln\left|\frac{x}{x+1}\right|+c$$

5 $\displaystyle\int \frac{dx}{x^2-2x}$

$$\frac{1}{x^2-2x} = \frac{1}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2} = \frac{(A+B)x-2A}{x(x-2)}$$

$$\Rightarrow \begin{cases} A+B=0 \\ -2A=1 \end{cases} \Leftrightarrow \begin{cases} A=-\dfrac{1}{2} \\ B=\dfrac{1}{2} \end{cases}$$

$$= -\frac{1}{2}\int\frac{dx}{x} + \frac{1}{2}\int\frac{dx}{x-2}$$

$$= -\frac{1}{2}\ln|x| + \frac{1}{2}\ln|x-2| + c$$

$$= \frac{1}{2}\ln\left|\frac{x-2}{x}\right| + c$$

6 $\displaystyle\int \frac{2x+1}{x^2-2x+1}\,dx$

$$\frac{2x+1}{x^2-2x+1} = \frac{2x+1}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} = \frac{Ax-A+B}{(x-1)^2}$$

$$\Rightarrow \begin{cases} A=2 \\ -A+B=1 \end{cases} \Leftrightarrow \begin{cases} A=2 \\ B=3 \end{cases}$$

$$= 2\int\frac{dx}{x-1} + 3\int\frac{dx}{(x-1)^2}$$

$$= 2\ln|x-1| - \frac{3}{x-1} + c$$

7 $\displaystyle\int \frac{x^2+2x-3}{x^2+2x+2}\,dx$

$$= \int\frac{x^2+2x+2-5}{x^2+2x+2}dx \qquad \text{(of euclidische deling)}$$

$$= \int dx - 5\int\frac{dx}{\underbrace{x^2+2x+2}_{D<0}}$$

$$= x - 5\int\frac{dx}{(x+1)^2+1}$$

$$= x - 5\,\text{Bgtan}(x+1) + c$$

8 $\displaystyle\int \frac{x+1}{\underbrace{2x^2-9x+4}_{D>0}}\,dx$

$$\frac{x+1}{2x^2-9x+4}=\frac{x+1}{(x-4)(2x-1)}=\frac{A}{x-4}+\frac{B}{2x-1}=\frac{(2A+B)x-A-4B}{(x-4)(2x-1)}$$

$$\Rightarrow \begin{cases}2A+B=1\\ -A-4B=1\end{cases} \Leftrightarrow \begin{cases}A=\dfrac{5}{7}\\ B=-\dfrac{3}{7}\end{cases}$$

$$=\frac{5}{7}\int\frac{dx}{x-4}-\frac{3}{7}\int\frac{dx}{2x-1}$$

$$=\frac{5}{7}\int\frac{d(x-4)}{x-4}-\frac{3}{7}\cdot\frac{1}{2}\int\frac{d(2x-1)}{2x-1}$$

$$=\frac{5}{7}\ln|x-4|-\frac{3}{14}\ln|2x-1|+c$$

9 $\displaystyle\int \frac{x^3-4}{\underbrace{x^2+4x+4}_{D>0}}\,dx$

$$\begin{array}{rrrr|l} x^3 & & & -\ 4 & x^2+4x+4 \\ \cline{5-5} \mp x^3 & \mp 4x^2 & \mp\ 4x & & x-4 \\ \cline{1-4} & -4x^2 & -\ 4x & -\ 4 & \\ & \pm 4x^2 & \pm 16x & \pm 16 & \\ \cline{2-4} & & 12x & +\ 12 & \end{array}$$

$$=\int\left(x-4+\frac{12x+12}{x^2+4x+4}\right)dx$$

$$=\frac{x^2}{2}-4x+\int\frac{12x+12}{x^2+4x+4}dx$$

$$\frac{12x+12}{x^2+4x+4}=\frac{12x+12}{(x+2)^2}=\frac{A}{x+2}+\frac{B}{(x+2)^2}=\frac{Ax+2A+B}{(x+2)^2}$$

$$\Rightarrow \begin{cases}A=12\\ 2A+B=12\end{cases} \Leftrightarrow \begin{cases}A=12\\ B=-12\end{cases}$$

$$=\frac{x^2}{2}-4x+12\int\frac{dx}{x+2}-12\int\frac{dx}{(x+2)^2}$$

$$=\frac{x^2}{2}-4x+12\ln|x+2|+\frac{12}{x+2}+c$$

10 $\displaystyle\int \frac{2x+3}{\underbrace{x^2+2x+4}_{D<0}}\,dx$

$$=\int\frac{(2x+2)+1}{x^2+2x+4}dx$$

$$=\int\frac{2x+2}{x^2+2x+4}dx+\int\frac{1}{x^2+2x+4}dx$$

$$=\ln(x^2+2x+4)+\int\frac{1}{(x+1)^2+3}dx$$

$$=\ln(x^2+2x+4)+\frac{1}{\sqrt{3}}\,\mathrm{Bgtan}\,\frac{x+1}{\sqrt{3}}+c$$

Opdracht 68 bladzijde 135

Bereken

1 $\displaystyle\int \frac{dx}{x^3 - x}$

$$\frac{1}{x^3 - x} = \frac{1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

$$= \frac{(A+B+C)x^2 + (B-C)x - A}{x^3 - x}$$

$$\Rightarrow \begin{cases} A+B+C=0 \\ B-C=0 \\ -A=1 \end{cases} \Leftrightarrow \begin{cases} A=-1 \\ B=\dfrac{1}{2} \\ C=\dfrac{1}{2} \end{cases}$$

$$= -\int \frac{dx}{x} + \frac{1}{2}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{dx}{x+1}$$

$$= -\ln|x| + \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + c$$

2 $\displaystyle\int \frac{x^4 - x^3 - x - 1}{x^3 - x^2}\,dx$

$$\begin{array}{r|l} x^4 - x^3 - x - 1 & x^3 - x^2 \\ \hline \mp x^4 \pm x^3 & x \\ \hline -x-1 & \end{array}$$

$$= \int \left(x - \frac{x+1}{x^3 - x^2} \right) dx$$

$$= \frac{x^2}{2} - \int \frac{x+1}{x^3 - x^2}\,dx$$

$$\frac{x+1}{x^3 - x^2} = \frac{x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} = \frac{Ax(x-1) + B(x-1) + CX^2}{x^3 - x^2}$$

$$= \frac{(A+C)x^2 + (-A+B)x - B}{x^3 - x^2}$$

$$\Rightarrow \begin{cases} A+C=0 \\ -A+B=1 \\ -B=1 \end{cases} \Leftrightarrow \begin{cases} A=-2 \\ B=-1 \\ C=2 \end{cases}$$

$$= \frac{x^2}{2} - \left(-2\int \frac{dx}{x} - \int \frac{dx}{x^2} + 2\int \frac{dx}{x-1} \right)$$

$$= \frac{x^2}{2} + 2\ln|x| - \frac{1}{x} - 2\ln|x-1| + c$$

$$= \frac{x^2}{2} - \frac{1}{x} + 2\ln\left|\frac{x}{x-1}\right| + c$$

3 $\int \frac{4x-3}{x^3-3x^2}\,dx$

$$\frac{4x-3}{x^3-3x^2} = \frac{4x-3}{x^2\,(x-3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-3}$$

$$= \frac{Ax(x-3)+B(x-3)+Cx^2}{x^3-3x^2} = \frac{(A+C)x^2+(-3A+B)x-3B}{x^3-3x^2}$$

$$\Rightarrow \begin{cases} A+C=0 \\ -3A+B=4 \\ -3B=-3 \end{cases} \Leftrightarrow \begin{cases} A=-1 \\ B=1 \\ C=1 \end{cases}$$

$$= -\int \frac{dx}{x} + \int \frac{dx}{x^2} + \int \frac{dx}{x-3}$$

$$= -\ln|x| - \frac{1}{x} + \ln|x-3| + c$$

$$= -\frac{1}{x} + \ln\left|\frac{x-3}{x}\right| + c$$

4 $\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx$

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$$

$$= \frac{A(x-1)(x^2+1)+B(x^2+1)+(Cx+D)(x-1)^2}{(x^2+1)(x-1)^2}$$

$$= = \frac{(A+C)x^3+(-A+B-2C+D)x^2+(A+C-2B)x-A+B+D}{(x^2+1)(x-1)^2}$$

$$\Rightarrow \begin{cases} A+C=0 \\ -A+B-2C+D=0 \\ A+C-2B=-2 \\ -A+B+D=4 \end{cases} \Leftrightarrow \begin{cases} A=-2 \\ B=1 \\ C=2 \\ D=1 \end{cases}$$

$$= -2\int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2} + \int \frac{2x+1}{x^2+1}dx$$

$$= -2\ln|x-1| - \frac{1}{x-1} + \int \frac{d(x^2+1)}{x^2+1} + \int \frac{dx}{x^2+1}$$

$$= -2\ln|x-1| - \frac{1}{x-1} + \ln(x^2+1) + \mathrm{Bgtan}\,x + c$$

5 $\displaystyle\int \frac{dx}{x^4-16}$

$$\frac{1}{x^4-16} = \frac{1}{(x-2)(x+2)(x^2+4)}$$
$$= \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+4}$$
$$= \frac{A(x+2)(x^2+4) + B(x-2)(x^2+4) + (Cx+D)(x^2-4)}{x^4-16}$$
$$= \frac{(A+B+C)x^3 + (2A-2B+D)x^2 + (4A+4B-4C)x + 8A-8B-4D}{x^4-16}$$

$$\Rightarrow \begin{cases} A+B+C=0 \\ 2A-2B+D=0 \\ 4A+4B-4C=0 \\ 8A-8B-4D=1 \end{cases} \Leftrightarrow \begin{cases} A=\frac{1}{32} \\ B=-\frac{1}{32} \\ C=0 \\ D=-\frac{1}{8} \end{cases}$$

$$= \frac{1}{32}\int\frac{dx}{x-2} - \frac{1}{32}\int\frac{dx}{x+2} - \frac{1}{8}\int\frac{dx}{x^2+4}$$
$$= \frac{1}{32}\ln|x-2| - \frac{1}{32}\ln|x+2| - \frac{1}{16}\,\text{Bgtan}\,\frac{x}{2} + c$$
$$= \frac{1}{32}\ln\left|\frac{x-2}{x+2}\right| - \frac{1}{16}\,\text{Bgtan}\,\frac{x}{2} + c$$

6 $\displaystyle\int \frac{x^3+x^2+x+2}{x^4+3x^2+2}\,dx$

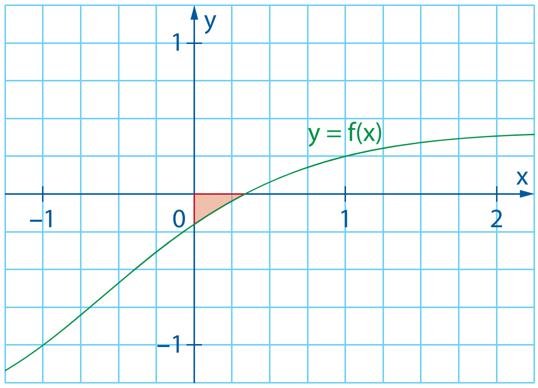
$$\frac{x^3+x^2+x+2}{x^4+3x^2+2} = \frac{x^3+x^2+x+2}{(x^2+1)(x^2+2)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2}$$
$$= \frac{(A+C)x^3 + (B+D)x^2 + (2A+C)x + 2B+D}{x^4+3x^2+2}$$

$$\Rightarrow \begin{cases} A+C=1 \\ B+D=1 \\ 2A+C=1 \\ 2B+D=2 \end{cases} \Leftrightarrow \begin{cases} A=0 \\ B=1 \\ C=1 \\ D=0 \end{cases}$$

$$= \int\frac{dx}{x^2+1} + \int\frac{x}{x^2+2}dx$$
$$= \text{Bgtan}\,x + \frac{1}{2}\ln(x^2+2) + c$$

Opdracht 69 bladzijde 135

Bereken de oppervlakte van het gebied begrensd door de grafiek van de functie met voorschrift $f(x) = \dfrac{3x-1}{x^2+2x+5}$, de x-as en de y-as.



- nulpunt teller: $\dfrac{1}{3}$

  nulpunt noemer: /

x		$\frac{1}{3}$	
f(x)	–	0	+

- $A = -\displaystyle\int_0^{\frac{1}{3}} \frac{3x-1}{x^2+2x+5}\,dx$

- $\displaystyle\int \frac{3x-1}{x^2+2x+5}\,dx = \int \frac{\frac{3}{2}(2x+2)-4}{\underbrace{x^2+2x+5}_{D<0}}\,dx$

  $= \dfrac{3}{2}\ln(x^2+2x+5) - 4\displaystyle\int \frac{dx}{(x+1)^2+4}$

  $= \dfrac{3}{2}\ln(x^2+2x+5) - 2\,\text{Bgtan}\,\dfrac{x+1}{2} + c$

- $A = \left[2\,\text{Bgtan}\,\dfrac{x+1}{2} - \dfrac{3}{2}\ln(x^2+2x+5)\right]_0^{\frac{1}{3}}$

  $= 2\,\text{Bgtan}\,\dfrac{2}{3} - \dfrac{3}{2}\ln\dfrac{52}{9} - \left(2\,\text{Bgtan}\,\dfrac{1}{2} - \dfrac{3}{2}\ln 5\right)$

  $= 2\left(\text{Bgtan}\,\dfrac{2}{3} - \text{Bgtan}\,\dfrac{1}{2}\right) + \dfrac{3}{2}\ln\dfrac{45}{52}$

Opdracht 70 bladzijde 136

Bereken

1 $\displaystyle\int \frac{dx}{(x-1)\sqrt{x}}$

$t^2 = x$ met $t > 0 \;\Rightarrow\; 2t\,dt = dx$

$$= \int \frac{2t\,dt}{(t^2-1)t} = 2\int \frac{dt}{t^2-1}$$

$$\frac{1}{t^2-1} = \frac{A}{t-1} + \frac{B}{t+1} = \frac{(A+B)t + A - B}{t^2-1}$$

$$\Rightarrow \begin{cases} A + B = 0 \\ A - B = 1 \end{cases} \Leftrightarrow \begin{cases} A = \dfrac{1}{2} \\ B = -\dfrac{1}{2} \end{cases}$$

$$= \int \frac{dt}{t-1} - \int \frac{dt}{t+1}$$

$$= \ln|t-1| - \ln|t+1| + c = \ln\left|\frac{t-1}{t+1}\right| + c = \ln\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right| + c$$

2 $\displaystyle\int \frac{dx}{(x+2)\sqrt{x+1}}$

$t^2 = x + 1$ met $t > 0 \;\Rightarrow\; 2t\,dt = dx$

$$= \int \frac{2t\,dt}{(t^2+1)t} = 2\int \frac{dt}{t^2+1} = 2\,\text{Bgtan}\,t + c$$

$$= 2\,\text{Bgtan}\,\sqrt{x+1} + c$$

3 $\displaystyle\int \frac{1-\sqrt{3x+2}}{1+\sqrt{3x+2}}\,dx$

$t^2 = 3x + 1$ met $t > 0 \;\Rightarrow\; 2t\,dt = 3\,dx$

$$= \frac{2}{3}\int \frac{1-t}{1+t}\cdot t\,dt = \frac{2}{3}\int \frac{-t^2+t}{1+t}\,dt$$

$$\begin{array}{rr|l}
-t^2 + & t & t+1 \\ \cline{3-3}
\pm t^2 \pm & t & -t+2 \\ \cline{1-2}
 & 2t & \\
 & \mp 2t \mp 2 & \\ \cline{2-2}
 & -2 &
\end{array}$$

$$= \frac{2}{3}\int \left(-t + 2 - \frac{2}{t+1}\right) dt$$

$$= -\frac{t^2}{3} + \frac{4}{3}t - \frac{4}{3}\ln|t+1| + c$$

$$= -\frac{1}{3}(3x+2) + \frac{4}{3}\sqrt{3x+2} - \frac{4}{3}\ln\left|\sqrt{3x+2}+1\right| + c$$

4 $\displaystyle\int \frac{x^3}{\sqrt{x-1}}\,dx$

$t^2 = x - 1$ met $t > 0 \Rightarrow 2t\,dt = dx$

$$= 2\int \frac{(t^2+1)^3}{t}\,t\,dt$$

$$= 2\int (t^6 + 3t^4 + 3t^2 + 1)\,dt$$

$$= \frac{2}{7}t^7 + \frac{6}{5}t^5 + 2t^3 + 2t + c$$

$$= \frac{2}{7}\sqrt{(x-1)^7} + \frac{6}{5}\sqrt{(x-1)^5} + 2\sqrt{(x-1)^3} + 2\sqrt{x-1} + c$$

5 $\displaystyle\int \frac{dx}{x + \sqrt[3]{x}}$

$t^3 = x \Rightarrow 3t^2\,dt = dx$

$$= 3\int \frac{t^2}{t^3+t}\,dt = 3\int \frac{t}{t^2+1}\,dt = \frac{3}{2}\int \frac{d(t^2+1)}{t^2+1}$$

$$= \frac{3}{2}\ln(t^2+1) + c = \frac{3}{2}\ln\left(\sqrt[3]{x^2}+1\right) + c$$

6 $\displaystyle\int \frac{dx}{\sqrt{x} - \sqrt[4]{x}}$

$t^4 = x$ met $t > 0 \Rightarrow 4t^3\,dt = dx$

$$= 4\int \frac{t^3}{t^2-t}\,dt = 4\int \frac{t^2}{t-1}\,dt$$

$$\begin{array}{r|l}
t^2 & t-1 \\ \hline
\mp t^2 \pm t & t+1 \\ \hline
t & \\
\mp t \pm 1 & \\ \hline
1 &
\end{array}$$

$$= 4\int \left(t + 1 + \frac{1}{t-1}\right)dt$$

$$= 2t^2 + 4t + 4\ln|t-1| + c$$

$$= 2\sqrt{x} + 4\sqrt[4]{x} + 4\ln\left|\sqrt[4]{x} - 1\right| + c$$

Opdracht 71 bladzijde 136

$\int x^3\sqrt{1+x^2}\,dx$ is gelijk aan

A $\dfrac{x^4\sqrt{1+x^2}}{4}+c$

B $\dfrac{x^4}{4}\sqrt{1+\dfrac{x^3}{3}}+c$

C $\dfrac{(1+x^2)^{\frac{3}{2}}(3x^2-2)}{15}+c$

D $x^3\sqrt{1+\dfrac{x^3}{3}}+c$

E geen van de voorgaande

(Bron © ACTM State Math Contest, 2008)

$$\int x^3\sqrt{1+x^2}\,dx=\int x^2\sqrt{1+x^2}\cdot x\,dx$$

$$t=1+x^2 \;\Rightarrow\; dt=2x\,dx$$

$$=\frac{1}{2}\int(t-1)\sqrt{t}\,dt=\frac{1}{2}\int\left(t^{\frac{3}{2}}-t^{\frac{1}{2}}\right)dt$$

$$=\frac{1}{2}\left(\frac{t^{\frac{5}{2}}}{\frac{5}{2}}-\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right)+c$$

$$=\frac{\sqrt{t^5}}{5}-\frac{\sqrt{t^3}}{3}+c=\frac{\sqrt{t^3}}{15}(3t-5)+c$$

$$=\frac{(1+x^2)^{\frac{3}{2}}}{15}(3+3x^2-5)+c=\frac{(1+x^2)^{\frac{3}{2}}\cdot(3x^2-2)}{15}+c$$

Antwoord C is juist.

Opdracht 72 bladzijde 136

Bereken

1 $\displaystyle\int\frac{dx}{x\sqrt{1-x}}$

$$t^2=1-x \text{ met } t>0 \;\Rightarrow\; 2t\,dt=-dx$$

$$=-\int\frac{2t\,dt}{(1-t^2)t}=2\int\frac{dt}{t^2-1}$$

$$\frac{1}{t^2-1}=\frac{A}{t-1}+\frac{B}{t+1}=\frac{(A+B)t+A-B}{t^2-1}$$

$$\Rightarrow\begin{cases}A+B=0\\A-B=1\end{cases}\Leftrightarrow\begin{cases}A=\dfrac{1}{2}\\[2mm]B=-\dfrac{1}{2}\end{cases}$$

$$= \int \frac{dt}{t-1} - \int \frac{dt}{t+1}$$

$$= \ln|t-1| - \ln|t+1| + c$$

$$= \ln\left|\frac{t-1}{t+1}\right| + c$$

$$= \ln\left|\frac{\sqrt{1-x}-1}{\sqrt{1-x}+1}\right| + c$$

2 $\displaystyle\int \frac{\sqrt{x}}{\sqrt[4]{x^3}+1}\,dx$

$t^4 = x$ met $t > 0 \;\Rightarrow\; 4t^3\,dt = dx$

$$= 4\int \frac{t^2}{t^3+1}\,t^3\,dt = 4\int \frac{t^5}{t^3+1}dt$$

$$\begin{array}{r|l} t^5 & t^3+1 \\ \hline \mp t^5 \mp t^2 & t^2 \\ \hline -t^2 & \end{array}$$

$$= 4\int \left(t^2 - \frac{t^2}{t^3+1}\right)dt$$

$$= 4\frac{t^3}{3} - \frac{4}{3}\int \frac{d(t^3+1)}{t^3+1}$$

$$= \frac{4}{3}\left(\sqrt[4]{x^3} - \ln\left|\sqrt[4]{x^3}+1\right|\right) + c$$

3 $\displaystyle\int \frac{x^3}{(4-9x^2)^{\frac{3}{2}}}\,dx$

$t^2 = 4 - 9x^2$ met $t > 0 \;\Rightarrow\; 2t\,dt = -18x\,dx$

$\Rightarrow\; t\,dt = -9x\,dx$

$$= \int \frac{\frac{4-t^2}{9}}{t^3}\cdot\left(-\frac{1}{9}t\right)dt$$

$$= \frac{1}{81}\int \frac{t^2-4}{t^2}\,dt$$

$$= \frac{1}{81}\int (1-4t^{-2})\,dt$$

$$= \frac{1}{81}\left(t + \frac{4}{t}\right) + c$$

$$= \frac{1}{81}\,\frac{t^2+4}{t} + c$$

$$= \frac{1}{81}\,\frac{8-9x^2}{\sqrt{4-9x^2}} + c$$

$$= \frac{8-9x^2}{81\sqrt{4-9x^2}} + c$$

4 $\displaystyle\int \frac{\sqrt{2x+1}}{x}\,dx$

$t^2 = 2x + 1$ met $t > 0 \;\Rightarrow\; 2t\,dt = 2\,dx$

$$= \int \frac{t}{\frac{t^2-1}{2}} \cdot t\,dt = 2\int \frac{t^2}{t^2-1}dt = 2\int \frac{t^2-1+1}{t^2-1}dt$$

$$= 2\int dt + 2\int \frac{dt}{t^2-1}$$

$$\frac{1}{t^2-1} = \frac{A}{t-1} + \frac{B}{t+1} = \frac{(A+B)t + A - B}{t^2-1}$$

$$\Rightarrow \begin{cases} A+B=0 \\ A-B=1 \end{cases} \Leftrightarrow \begin{cases} A = \frac{1}{2} \\ B = -\frac{1}{2} \end{cases}$$

$$= 2t + \int \frac{dt}{t-1} - \int \frac{dt}{t+1}$$

$$= 2t + \ln|t-1| - \ln|t+1| + c$$

$$= 2\sqrt{2x+1} + \ln\left|\frac{\sqrt{2x+1}-1}{\sqrt{2x+1}+1}\right| + c$$

5 $\displaystyle\int \frac{dx}{x\sqrt{x^3-1}}$

$t^2 = x^3 - 1$ met $t > 0 \;\Rightarrow\; 2t\,dt = 3x^2\,dx$

$$= \int \frac{x^2\,dx}{x^3\sqrt{x^3-1}} = \frac{2}{3}\int \frac{t}{(t^2+1)t}dt = \frac{2}{3}\int \frac{dt}{t^2+1} = \frac{2}{3}\text{Bgtan}\,t + c$$

$$= \frac{2}{3}\,\text{Bgtan}\,\sqrt{x^3-1} + c$$

6 $\displaystyle\int \frac{dx}{x(x^2-1)\sqrt{x^2-1}}$

$t^2 = x^2 - 1$ met $t > 0 \Rightarrow 2t\,dt = 2x\,dx$

$$= \int \frac{x}{x^2(x^2-1)\sqrt{x^2-1}}\,dx = \int \frac{t}{(1+t^2)\,t^2 \cdot t}\,dt$$

$$= \int \frac{dt}{t^2(1+t^2)}$$

$$\frac{1}{t^2(1+t^2)} = \frac{A}{t} + \frac{B}{t^2} + \frac{Ct+D}{t^2+1}$$

$$= \frac{At(t^2+1)+B(t^2+1)+(Ct+D)t^2}{t^2(1+t^2)}$$

$$= \frac{(A+C)t^3+(B+D)t^2+At+B}{t^2(1+t^2)}$$

$$\Rightarrow \begin{cases} A+C=0 \\ B+D=0 \\ A=0 \\ B=1 \end{cases} \Leftrightarrow \begin{cases} A=0 \\ B=1 \\ C=0 \\ D=-1 \end{cases}$$

$$= \int \frac{dt}{t^2} - \int \frac{dt}{t^2+1}$$

$$= -\frac{1}{t} - \text{Bgtan}\, t + c$$

$$= -\frac{1}{\sqrt{x^2-1}} - \text{Bgtan}\sqrt{x^2-1} + c$$

Opdracht 73 bladzijde 137

Bereken

1 $\displaystyle\int_0^3 \frac{dx}{1+\sqrt{x+1}}$

- $\displaystyle\int \frac{dx}{1+\sqrt{x+1}}$

$t^2 = x + 1$ met $t > 0 \Rightarrow 2t\,dt = dx$

$$= 2\int \frac{t}{1+t}\,dt = 2\int \frac{t+1-1}{t+1}\,dt = 2\left(\int dt - \int \frac{dt}{t+1}\right)$$

$$= 2(t - \ln|t+1|) + c = 2\left(\sqrt{x+1} - \ln\left|\sqrt{x+1}+1\right|\right) + c$$

- $\displaystyle\int_0^3 \frac{dx}{1+\sqrt{x+1}} = 2\left[\sqrt{x+1} - \ln\left|\sqrt{x+1}+1\right|\right]_0^3$

$$= 2(2 - \ln 3 - (1 - \ln 2)) = 2\left(1 + \ln\frac{2}{3}\right)$$

2 $\displaystyle\int_1^3 \frac{dx}{\sqrt{x}\sqrt{4-x}}$

- $\displaystyle\int \frac{dx}{\sqrt{x}\ \sqrt{4-x}}$

$t^2 = 4 - x$ met $t > 0 \;\Rightarrow\; 2t\,dt = -dx$

$$= -2\int \frac{t}{\sqrt{4-t^2}\ t}\,dt = -2\int \frac{dt}{\sqrt{4-t^2}} = -2\ \text{Bgsin}\ \frac{t}{2} + c$$

$$= -2\ \text{Bgsin}\ \frac{\sqrt{4-x}}{2} + c$$

$$\int_1^3 \frac{dx}{\sqrt{x}\ \sqrt{4-x}} = -2\left[\text{Bgsin}\ \frac{\sqrt{4-x}}{2}\right]_1^3$$

$$= -2\left[\text{Bgsin}\ \frac{1}{2} - \text{Bgsin}\ \frac{\sqrt{3}}{2}\right]$$

- $\displaystyle = -2\left(\frac{\pi}{6} - \frac{\pi}{3}\right)$

$$= \frac{1}{3}\pi$$

Opdracht 74 bladzijde 137

Bereken

1 $\displaystyle\int \sqrt{\frac{x+9}{x-9}}\,dx = \int \sqrt{\frac{(x+9)^2}{x^2-81}}\,dx = \int \frac{x+9}{\sqrt{x^2-81}}\,dx$

$$= \int \frac{x}{\sqrt{x^2-81}}\,dx + 9\int \frac{dx}{\sqrt{x^2-81}}$$

$$= \frac{1}{2}\int \frac{d(x^2-81)}{\sqrt{x^2-81}} + 9\int \frac{dx}{\sqrt{x^2-81}}$$

$$= \sqrt{x^2-81} + 9\ln\left|x + \sqrt{x^2-81}\right| + c$$

2 $\displaystyle\int \sqrt{x^2+2x-3}\,dx = \int \frac{x^2+2x-3}{\sqrt{x^2+2x-3}}\,dx$

$$= \int \frac{(x+1)^2-4}{\sqrt{(x+1)^2-4}}\,dx$$

$t = x + 1 \;\Rightarrow\; dt = dx$

$$= \int \frac{t^2-4}{\sqrt{t^2-4}}\,dt = \int \frac{t^2}{\sqrt{t^2-4}}\,dt - 4\ln\left|t + \sqrt{t^2-4}\right|$$

$$\int \frac{t^2}{\sqrt{t^2-4}}\,dt$$

$$u = t \;\Rightarrow\; du = dt$$

$$dv = \frac{t}{\sqrt{t^2-4}}\,dt \;\Rightarrow\; v = \frac{1}{2}\int \frac{d(t^2-4)}{\sqrt{t^2-4}} = \sqrt{t^2-4}$$

$$= t\sqrt{t^2-4} - \int \sqrt{t^2-4}\,dt$$

$$= t\sqrt{t^2-4} - \int \frac{t^2-4}{\sqrt{t^2-4}}\,dt$$

$$= t\sqrt{t^2-4} - \int \frac{t^2}{\sqrt{t^2-4}}\,dt + 4\ln\left|t+\sqrt{t^2-4}\right|$$

$$\Rightarrow\; 2\int \frac{t^2}{\sqrt{t^2-4}}\,dt = t\sqrt{t^2-4} + 4\ln\left|t+\sqrt{t^2-4}\right| + c$$

$$\Rightarrow\; \int \frac{t^2}{\sqrt{t^2-4}}\,dt = \frac{1}{2}\,t\sqrt{t^2-4} + 2\ln\left|t+\sqrt{t^2-4}\right| + c$$

$$= \frac{1}{2}\,t\sqrt{t^2-4} - 2\ln\left|t+\sqrt{t^2-4}\right| + c$$

$$= \frac{1}{2}\,(x+1)\sqrt{x^2+2x-3} - 2\ln\left|x+1+\sqrt{x^2+2x-3}\right| + c$$

Opdracht 75 bladzijde 137

Bereken met de gegeven substitutie.

1 $\displaystyle\int \frac{\sqrt{x+1}}{(x-1)^{\frac{5}{2}}}\,dx \qquad\qquad x-1 = \frac{1}{u} \;\Rightarrow\; dx = -\frac{1}{u^2}du$

$$= -\int \frac{\sqrt{\frac{1}{u}+2}}{\left(\frac{1}{u}\right)^{\frac{5}{2}}}\cdot\frac{1}{u^2}\,du$$

$$= -\int \frac{\sqrt{1+2u}}{\sqrt{u}}\cdot\frac{u^{\frac{5}{2}}}{u^2}\,du$$

$$= -\frac{1}{2}\int \sqrt{1+2u}\;d(1+2u)$$

$$= -\frac{1}{2}\cdot\frac{2}{3}(1+2u)^{\frac{3}{2}} + c$$

$$= -\frac{1}{3}\sqrt{\left(1+\frac{2}{x-1}\right)^3} + c$$

$$= -\frac{1}{3}\sqrt{\left(\frac{x+1}{x-1}\right)^3} + c$$

2 $\displaystyle\int \frac{dx}{x^2\sqrt{49-x^2}}$ $\qquad x = 7\sin t \;$ met $\; t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\;\Rightarrow\; dx = 7\cos t\, dt$

$$= 7\int \frac{\cos t\, dt}{49\sin^2 t \cdot 7\cos t}$$

$$= \frac{1}{49}\int \frac{dt}{\sin^2 t} = -\frac{1}{49}\cot t + c$$

$$= -\frac{1}{49}\,\frac{\cos t}{\sin t} + c = -\frac{1}{49}\,\frac{\sqrt{1-\frac{x^2}{49}}}{\frac{x}{7}} + c$$

$$= -\frac{1}{49}\,\frac{\sqrt{49-x^2}}{x} + c$$

$$= -\frac{\sqrt{49-x^2}}{49x} + c$$

3 $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+4}}$ $\qquad x = 2\tan t \;$ met $\; t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\;\Rightarrow\; dx = \frac{2}{\cos^2 t}\, dt$

$$= 2\int \frac{\frac{dt}{\cos^2 t}}{4\tan^2 t\sqrt{4(1+\tan^2 t)}} = \frac{1}{2}\int \frac{\frac{dt}{\cos^2 t}}{\tan^2 t \cdot \frac{2}{\cos t}}$$

$$= \frac{1}{4}\int \frac{dt}{\cos t \cdot \tan^2 t} = \frac{1}{4}\int \frac{\cos t}{\sin^2 t}\, dt$$

$$= \frac{1}{4}\int \frac{d(\sin t)}{\sin^2 t} = -\frac{1}{4\sin t} + c = -\frac{1}{4\cos t\tan t} + c$$

$$= -\frac{\sqrt{1+\tan^2 t}}{4\tan t} + c = -\frac{\sqrt{1+\frac{x^2}{4}}}{4\cdot\frac{x}{2}} = -\frac{\sqrt{4+x^2}}{4x} + c$$

4 $\displaystyle\int_{2\sqrt{3}}^{6} \frac{dx}{x^2\sqrt{x^2-9}}$ $\qquad x = \frac{3}{\cos t} \;$ met $\; t \in \left]0, \frac{\pi}{2}\right[$

$$\Rightarrow \bullet\; x^2 - 9 = 9\left(\frac{1}{\cos^2 t} - 1\right) = 9\tan t^2$$

$$\bullet\; dx = -3(\cos t)^{-2}(-\sin t)dt = \frac{3\sin t}{\cos^2 t}dt$$

$$\bullet\; x = 2\sqrt{3} \;\Leftrightarrow\; 2\sqrt{3} = \frac{3}{\cos t}$$

$$\Leftrightarrow\; \cos t = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2} \;\Rightarrow\; t = \frac{\pi}{6}$$

$$x = 6 \;\Leftrightarrow\; 6 = \frac{3}{\cos t} \;\Leftrightarrow\; \cos t = \frac{1}{2} \;\Rightarrow\; t = \frac{\pi}{3}$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\frac{3 \sin t}{\cos^2 t}}{\frac{9}{\cos^2 t} \cdot 3 \tan t} dt$$

$$= \frac{1}{9} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos t \, dt$$

$$= \frac{1}{9} \left[\sin t\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{1}{9}\left(\sin \frac{\pi}{3} - \sin \frac{\pi}{6}\right) = \frac{1}{9}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) = \frac{\sqrt{3}-1}{18}$$

5 $\displaystyle\int_{-3}^{-1} \frac{dx}{(x^2+6x+13)^{\frac{3}{2}}}$ $\qquad x = 2\tan t - 3 \;\text{ met }\; t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

$$dx = \frac{2}{\cos^2 t} dt$$

$$x = -3 \;\Rightarrow\; \tan t = 0 \;\Rightarrow\; t = 0$$

$$x = -1 \;\Rightarrow\; \tan t = 1 \;\Rightarrow\; t = \frac{\pi}{4}$$

$$= \int_{-3}^{-1} \frac{dx}{((x+3)^2+4)^{\frac{3}{2}}} = 2 \int_0^{\frac{\pi}{4}} \frac{\frac{1}{\cos^2 t}}{(4\tan^2 t + 4)^{\frac{3}{2}}} dt$$

$$= \frac{2}{8} \int_0^{\frac{\pi}{4}} \frac{dt}{\cos^2 t \cdot \left(\frac{1}{\cos^2 t}\right)^{\frac{3}{2}}} = \frac{1}{4} \int_0^{\frac{\pi}{4}} \cos t \, dt$$

$$= \frac{1}{4}\left[\sin t\right]_0^{\frac{\pi}{4}} = \frac{1}{4}\left(\sin \frac{\pi}{4} - \sin 0\right)$$

$$= \frac{1}{4}\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{8}$$

Opdracht 76 bladzijde 138

Bereken

1 $\displaystyle\int \frac{\sin 2x}{\cos^3 x} dx = \int \frac{2 \sin x \cos x}{\cos^3 x} dx = 2 \int \frac{\sin x}{\cos^2 x} dx$

$$= -2 \int \frac{d(\cos x)}{\cos^2 x} = \frac{2}{\cos x} + c$$

2 $\displaystyle\int \frac{\ln^3 x}{2x} dx = \frac{1}{2} \int \ln^3 x \, d(\ln x) = \frac{\ln^4 x}{8} + c$

3 $\int_0^{\pi}(\sin x+\cos x)^2\,dx=\int_0^{\pi}(\sin^2 x+2\sin x\cos x+\cos^2 x)\,dx$

$$=\int_0^{\pi}(1+\sin 2x)\,dx=\left[x-\frac{1}{2}\cos 2x\right]_0^{\pi}=\pi-\frac{1}{2}-\left(0-\frac{1}{2}\right)=\pi$$

4 $\displaystyle\int\frac{\sin 2x}{3\sin^2 x+5\cos^2 x}\,dx=\int\frac{\sin 2x}{3+2\cos^2 x}\,dx$

$$u=3+2\cos^2 x\ \Rightarrow\ du=4\cos x(-\sin x)dx=-2\sin 2x\,dx$$

$$=-\frac{1}{2}\int\frac{du}{u}=-\frac{1}{2}\ln|u|+c=-\frac{1}{2}\ln|3+2\cos^2 x|+c$$

$$=-\frac{1}{2}\ln(3+2\cos^2 x)+c$$

5 $\displaystyle\int\frac{3x^2+5}{x^2+1}\,dx$

$$\begin{array}{r|l} 3x^2+5 & x^2+1 \\ \hline \mp 3x^2\mp 3 & 3 \\ \hline 2 & \end{array}$$

$$=\int\left(3+\frac{2}{x^2+1}\right)dx=3x+2\,\text{Bgtan}\,x+c$$

6 $\displaystyle\int_1^5\frac{e^{\sqrt{2x-1}}}{\sqrt{2x-1}}\,dx$

$$u=\sqrt{2x-1}\ \Rightarrow\ du=\frac{1}{\sqrt{2x-1}}dx$$

$$x=1\ \Rightarrow\ u=1$$

$$x=5\ \Rightarrow\ u=3$$

$$=\int_1^3 e^u\,du=\left[e^u\right]_1^3=e^3-e=e\,(e^2-1)$$

7 $\displaystyle\int\frac{x}{\sqrt{25x^2+81}}\,dx=\frac{1}{50}\int\frac{d(25x^2+81)}{\sqrt{25x^2+81}}=\frac{1}{25}\sqrt{25x^2+81}+c$

8 $\displaystyle\int\frac{dx}{4x^2+4x+5}\underset{D<0}{=}\int\frac{dx}{(2x+1)^2+4}=\frac{1}{2}\int\frac{d(2x+1)}{(2x+1)^2+4}=\frac{1}{4}\,\text{Bgtan}\,\frac{2x+1}{2}+c$

9 $\int x^2 \ln 3x\, dx$

$$u = \ln 3x \Rightarrow du = \frac{1}{x}\, dx$$

$$dv = x^2 dx \Rightarrow v = \int x^2\, dx = \frac{x^3}{3}$$

$$= \frac{x^3}{3} \ln 3x - \frac{1}{3} \int x^2\, dx$$

$$= \frac{x^3}{3} \ln 3x - \frac{x^3}{9} + c$$

$$= \frac{x^3}{9} (3 \ln 3x - 1) + c$$

10 $\int \frac{\sin x \cos x}{1 - \cos x}\, dx$

$$u = 1 - \cos x \Rightarrow du = \sin x\, dx$$

$$= \int \frac{1-u}{u}\, du = \int \left(\frac{1}{u} - 1\right) du = \ln|u| - u + c = \ln|1 - \cos x| - 1 + \cos x + c$$

$$= \ln|1 - \cos x| + \cos x + c$$

$$= \ln(1 - \cos x) + \cos x + c$$

11 $\int x^3 \sin 2x\, dx$

$$u = x^3 \Rightarrow du = 3x^2\, dx$$

$$dv = \sin 2x\, dx \Rightarrow v = \frac{1}{2}\int \sin 2x\, d(2x) = -\frac{1}{2}\cos 2x$$

$$= -\frac{1}{2}x^3 \cos 2x + \frac{3}{2} \int x^2 \cos 2x\, dx$$

$$u = x^2 \Rightarrow du = 2x\, dx$$

$$dv = \cos 2x\, dx \Rightarrow v = \frac{1}{2}\int \cos 2x\, d(2x) = \frac{1}{2}\sin 2x$$

$$= -\frac{1}{2}x^3 \cos 2x + \frac{3}{2} \left(\frac{1}{2}x^2 \sin 2x - \int x \sin 2x\, dx\right)$$

$$= -\frac{1}{2}x^3 \cos 2x + \frac{3}{4}x^2 \sin 2x - \frac{3}{2}\int x \sin 2x\, dx$$

$$u = x \Rightarrow du = dx$$

$$dv = \sin 2x\, dx \Rightarrow v = -\frac{1}{2} \cos 2x$$

$$= -\frac{1}{2}x^3 \cos 2x + \frac{3}{4}x^2 \sin 2x - \frac{3}{2} \left(-\frac{1}{2}x \cos 2x + \frac{1}{2} \cdot \frac{1}{2} \int \cos 2x\, d(2x)\right)$$

$$= -\frac{1}{2}x^3 \cos 2x + \frac{3}{4}x^2 \sin 2x + \frac{3}{4}x \cos 2x - \frac{3}{8} \sin 2x + c$$

$$= \frac{1}{4}x\,(3 - 2x^2) \cos 2x + \frac{3}{8}\,(2x^2 - 1) \sin 2x + c$$

12 $\displaystyle\int \frac{\sqrt[3]{x}-\sqrt{x}}{x^2}\,dx = \int\left(x^{-\frac{5}{3}} - x^{-\frac{3}{2}}\right)dx = \frac{x^{-\frac{2}{3}}}{-\frac{2}{3}} - \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + c$

$$= -\frac{3}{2\sqrt[3]{x^2}} + \frac{2}{\sqrt{x}} + c$$

13 $\displaystyle\int 4^{\tan x}(1+\tan^2 x)\,dx = \int \frac{4^{\tan x}}{\cos^2 x}\,dx = \int 4^{\tan x}\,d(\tan x) = \frac{4^{\tan x}}{\ln 4} + c$

14 $\displaystyle\int_{-2}^{2} \ln(x+3)\,dx$

$t = x+3 \Rightarrow dt = dx$
$x = -2 \Rightarrow t = 1$
$x = 2 \Rightarrow t = 5$

$= \displaystyle\int_1^5 \ln t\,dt$

$u = \ln t \Rightarrow du = \dfrac{1}{t}dt$
$dv = dt \Rightarrow v = t$

$= [t\ln t]_1^5 - \displaystyle\int_1^5 dt$

$= 5\ln 5 - 0 - [t]_1^5$

$= 5\ln 5 - 4$

15 $\displaystyle\int \frac{dx}{\sin^2 x\sqrt{9-\cot^2 x}} = -\int \frac{d(\cot x)}{\sqrt{9-\cot^2 x}}$

$$= -\text{Bgsin}\,\frac{\cot x}{3} + c$$

16 $\displaystyle\int \frac{e^{\sin x}\cos x}{\sqrt{4+e^{2\sin x}}}\,dx$

$u = e^{\sin x} \Rightarrow du = e^{\sin x}\cdot\cos x\,dx$

$= \displaystyle\int \frac{du}{\sqrt{4+u^2}} = \ln\left|u+\sqrt{4+u^2}\right| + c$

$= \ln\left|e^{\sin x} + \sqrt{4+e^{2\sin x}}\right| + c$

$= \ln\left(e^{\sin x} + \sqrt{4+e^{2\sin x}}\right) + c$

17 $\displaystyle\int \frac{dx}{\sqrt{1-x^2}\,\text{Bgsin}\,x} = \int \frac{d(\text{Bgsin}\,x)}{\text{Bgsin}\,x} = \ln|\text{Bgsin}\,x| + c$

18 $$\int_2^3 \frac{2}{(7-4x)^3}\,dx = \frac{-2}{4}\int_2^3 (7-4x)^{-3}\,d(7-4x)$$
$$= -\frac{1}{2}\left[\frac{(7-4x)^{-2}}{-2}\right]_2^3 = \frac{1}{4}\left[\frac{1}{(7-4x)^2}\right]_2^3$$
$$= \frac{1}{4}\left(\frac{1}{25}-1\right)$$
$$= -\frac{6}{25}$$

19 $$\int_{\frac{1}{3}}^{\frac{\sqrt{3}}{3}} \frac{\text{Bgtan}\,3x}{9x^2+1}\,dx = \frac{1}{3}\int_{\frac{1}{3}}^{\frac{\sqrt{3}}{3}} \text{Bgtan}\,3x\,d(\text{Bgtan}\,3x) = \frac{1}{3}\left[(\text{Bgtan}\,3x)^2\right]_{\frac{1}{3}}^{\frac{\sqrt{3}}{3}}$$
$$= \frac{1}{6}\left(\left(\frac{\pi}{3}\right)^2 - \left(\frac{\pi}{4}\right)^2\right) = \frac{7\pi^2}{864}$$

20 $$\int \frac{e^x}{e^{2x}+36}\,dx = \int \frac{d(e^x)}{(e^x)^2+36} = \frac{1}{6}\,\text{Bgtan}\,\frac{e^x}{6} + c$$

21 $$\int \frac{x^3}{\sqrt{4-x^8}}\,dx = \int \frac{x^3}{\sqrt{4-(x^4)^2}}\,dx = \frac{1}{4}\int \frac{d(x^4)}{\sqrt{4-(x^4)^2}} = \frac{1}{4}\,\text{Bgsin}\,\frac{x^4}{2} + c$$

22 $$\int x e^{4x^2}\,dx = \frac{1}{8}\int e^{4x^2}\,d(4x^2) = \frac{1}{8}e^{4x^2} + c$$

23 $$\int \frac{3x-\sqrt{\text{Bgsin}\,x}}{\sqrt{1-x^2}}\,dx = 3\int \frac{x}{\sqrt{1-x^2}}\,dx - \int \frac{\sqrt{\text{Bgsin}\,x}}{\sqrt{1-x^2}}\,dx$$
$$= -\frac{3}{2}\int \frac{d(1-x^2)}{\sqrt{1-x^2}} - \int \sqrt{\text{Bgsin}\,x}\;d(\text{Bgsin}\,x)$$
$$= -3\sqrt{1-x^2} - \frac{2}{3}\sqrt{(\text{Bgsin}\,x)^3} + c$$

24 $$\int \frac{1-\cos x}{1+\cos x}\,dx$$

$$\cos 2x = 1 - 2\sin^2 x = 2\cos^2 x - 1$$
$$= \int \frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}\,dx = \int \frac{1-\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}}\,dx = 2\int \frac{d\,\frac{x}{2}}{\cos^2\frac{x}{2}} - \int dx = 2\tan\frac{x}{2} - x + c$$

25 $$\int e^x \tan(1+e^x)\,dx$$

$$u = 1+e^x \Rightarrow du = e^x\,dx$$
$$= \int \tan u\,du = \int \frac{\sin u}{\cos u}\,du = -\int \frac{d(\cos u)}{\cos u} = -\ln|\cos u| + c$$
$$= -\ln|\cos(1+e^x)| + c$$

26 $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\ln(\tan x)}{\cos^2 x}\,dx$

- $\displaystyle\int \frac{\ln(\tan x)}{\cos^2 x}\,dx$

$$t = \tan x \ \Rightarrow\ dt = \frac{dx}{\cos^2 x}$$

$$= \int \ln t\,dt$$

$$u = \ln t \ \Rightarrow\ du = \frac{1}{t}\,dt$$

$$dv = dt \ \Rightarrow\ v = t$$

$$= t\ln t - \int dt = t(\ln t - 1) + c$$

$$= \tan x(\ln \tan x - 1) + c$$

- $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\ln(\tan x)}{\cos^2 x}\,dx = \Big[\tan x\,(\ln \tan x - 1)\Big]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$

$$= \sqrt{3}\,\left(\ln\sqrt{3} - 1\right) + 1$$

27 $\displaystyle\int \frac{\sin 2x}{(2+\sin x)^2}\,dx = \int \frac{2\sin x\cos x}{(2+\sin x)^2}\,dx$

$$u = 2 + \sin x \ \Rightarrow\ du = \cos x\,dx$$

$$= 2\int \frac{u-2}{u^2}\,du = 2\int\left(\frac{1}{u} - 2u^{-2}\right)du$$

$$= 2\left(\ln|u| + \frac{2}{u}\right) + c$$

$$= 2\left(\ln(2+\sin x) + \frac{2}{2+\sin x}\right) + c$$

28 $\displaystyle\int_0^1 \left(x - \frac{\tan x}{\cos x}\right) dx$

$$= \int_0^1 \left(x - \frac{\sin x}{\cos^2 x}\right) dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 + \int_0^1 \frac{d(\cos x)}{\cos^2 x}$$

$$= \frac{1}{2} - \left[\frac{1}{\cos x}\right]_0^1$$

$$= \frac{1}{2} - \left(\frac{1}{\cos 1} - 1\right)$$

$$= \frac{3}{2} - \frac{1}{\cos 1}$$

29 $\displaystyle\int \frac{\ln(\ln x)}{x}\,dx$

$$t = \ln x \;\Rightarrow\; dt = \frac{dx}{x}$$

$$= \int \ln t\,dt$$

$$u = \ln t \;\Rightarrow\; du = \frac{dt}{t}$$

$$dv = dt \;\Rightarrow\; v = t$$

$$= t\ln t - \int dt$$

$$= t\ln t - t + c$$

$$= t(\ln t - 1) + c$$

$$= \ln x(\ln(\ln x) - 1) + c$$

30 $\displaystyle\int x^2\,\mathrm{Bgtan}\,x\,dx$

$$u = \mathrm{Bgtan}\,x \;\Rightarrow\; du = \frac{1}{1+x^2}\,dx$$

$$dv = x^2\,dx \;\Rightarrow\; v = \frac{x^3}{3}$$

$$= \frac{x^3}{3}\,\mathrm{Bgtan}\,x - \frac{1}{3}\int \frac{x^3}{x^2+1}\,dx$$

$$\begin{array}{r|l} x^3 & x^2+1 \\ \cline{2-2} \mp x^3 \mp x & x \\ \cline{1-1} -x & \end{array}$$

$$= \frac{x^3}{3}\mathrm{Bgtan}\,x - \frac{1}{3}\int\left(x - \frac{x}{x^2+1}\right)dx$$

$$= \frac{x^3}{3}\mathrm{Bgtan}\,x - \frac{1}{6}x^2 + \frac{1}{3}\cdot\frac{1}{2}\int\frac{d(x^2+1)}{x^2+1}$$

$$= \frac{x^3}{3}\mathrm{Bgtan}\,x - \frac{1}{6}x^2 + \frac{1}{6}\ln(x^2+1) + c$$

Opdracht 77 bladzijde 139

We beschouwen twee uitdrukkingen:

1 $\int \ln x \, dx = \ln x + x + c$

2 $\int \sin^2(2x) \, dx = \frac{1}{2}x + \frac{1}{8}\cos(4x) + c$

Kies het juiste antwoord.

A Uitdrukkingen 1 en 2 zijn juist.

B Uitdrukkingen 1 en 2 zijn fout.

C Uitdrukking 1 is juist en uitdrukking 2 is fout.

D Uitdrukking 1 is fout en uitdrukking 2 is juist.

(Bron © Toelatingsproef arts-tandarts)

Uitdrukking 1:

$$\int \ln x \, dx \neq \ln x + x + c$$

$$\text{want } \frac{d}{dx}(\ln x + x) = \frac{1}{x} + 1 \neq \ln x$$

Uitdrukking 2:

$$\int \sin^2 2x \, dx \neq \frac{1}{2}x + \frac{1}{8}\cos(4x) + c \quad \text{want}$$

$$\int \sin^2 2x \, dx = \int \frac{1 - \cos 4x}{2} dx = \frac{1}{2}x - \frac{1}{2} \cdot \frac{1}{4} \int \cos 4x \, d(4x)$$

$$= \frac{1}{2}x - \frac{1}{8}\sin 4x + c$$

$\Rightarrow$ Antwoord B is juist.

Opdracht 78 bladzijde 139

$\int_1^4 \frac{\ln x}{x} dx$ is gelijk aan

A $\ln 4$ **B** $0{,}5 \ln 4$ **C** $2 \ln^2 4$ **D** $2 \ln^2 2$

(Bron © Toelatingsproef arts-tandarts)

$$\int_1^4 \frac{\ln x}{x} dx = \int_1^4 \ln x \, d(\ln x) = \left[\frac{\ln^2 x}{2}\right]_1^4$$

$$= \frac{1}{2}(\ln^2 4 - \ln^2 1) = \frac{1}{2}(2 \ln 2)^2 = 2 \ln^2 2$$

Antwoord D is juist.

Opdracht 79 bladzijde 139

$\displaystyle\int_0^{e-1} \frac{x-1}{x+1}\,dx$ is gelijk aan

A $e-5$ **B** $e-3$ **C** 0 **D** $e-1$

(Bron © Toelatingsproef arts-tandarts)

$$\int_0^{e-1} \frac{x-1}{x+1}\,dx$$
$$= \int_0^{e-1} \frac{x+1-2}{x+1}\,dx$$
$$= \int_0^{e-1} \left(1-\frac{2}{x+1}\right)dx$$
$$= [x]_0^{e-1} - 2\int_0^{e-1} \frac{d(x+1)}{x+1}$$
$$= [x]_0^{e-1} - 2\left[\ln|x+1|\right]_0^{e-1}$$
$$= e-1-2(\ln e - \ln 1)$$
$$= e-1-2$$
$$= e-3$$

$\Rightarrow$ Antwoord B is juist.

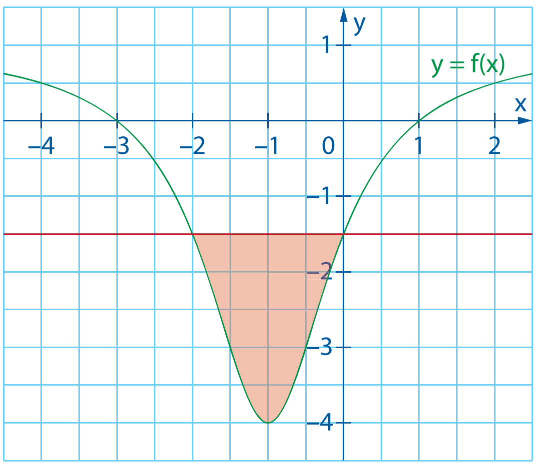
Opdracht 80 bladzijde 140

Bereken de oppervlakte van het gebied begrensd door de grafiek van

$f: x \mapsto \dfrac{x^2+2x-3}{x^2+2x+2}$ en de horizontale rechte door het snijpunt van de grafiek van f en de y-as.

- snijpunt van de grafiek van f en de y-as:

  $\left(0, -\dfrac{3}{2}\right)$



- snijpunten van de grafiek van f en de rechte met vergelijking $y = -\dfrac{3}{2}$:

$$\frac{x^2+2x-3}{x^2+2x+2} = -\frac{3}{2} \Leftrightarrow 2x^2+4x-6 = -3x^2-6x-6$$
$$\Leftrightarrow 5x^2+10x = 0$$
$$\Leftrightarrow 5x(x+2) = 0$$
$$\Leftrightarrow x = 0 \text{ of } x = -2$$

$$A = \int_{-2}^{0} \left(-\frac{3}{2} - \frac{x^2 + 2x - 3}{x^2 + 2x + 2} \right) dx$$

$$= \int_{-2}^{0} \left(-\frac{3}{2} - \frac{x^2 + 2x + 2 - 5}{\underbrace{x^2 + 2x + 2}_{D<0}} \right) dx$$

$$= \int_{-2}^{0} \left(-\frac{3}{2} - 1 + \frac{5}{(x+1)^2 + 1} \right) dx$$

$$= -\frac{5}{2}\left[x\right]_{-2}^{0} + 5 \int_{-2}^{0} \frac{d(x+1)}{(x+1)^2 + 1}$$

$$= -5 + 5\left[\text{Bgtan}\,(x+1)\right]_{-2}^{0}$$

$$= -5 + 5(\text{Bgtan}\,1 - \text{Bgtan}(-1))$$

$$= -5 + 5\left(\frac{\pi}{4} + \frac{\pi}{4}\right)$$

$$= 5\left(\frac{\pi}{2} - 1\right)$$

Opdracht 81 bladzijde 140

Bereken

1 $$\int \frac{1 - \tan^2 2x}{1 + \tan^2 2x}\, dx = \int \frac{\dfrac{\cos^2 2x - \sin^2 2x}{\cos^2 2x}}{\dfrac{1}{\cos^2 2x}}\, dx = \frac{1}{4} \int \cos 4x \, d(4x) = \frac{1}{4} \sin 4x + c$$

2 $$\int \frac{5}{\underbrace{3x^2 - 2x - 1}_{D>0}}\, dx$$

$$\frac{5}{3x^2 - 2x - 1} = \frac{5}{(x-1)(3x+1)} = \frac{A}{x-1} + \frac{B}{3x+1}$$

$$= \frac{(3A+B)x + A - B}{3x^2 - 2x - 1}$$

$$\Rightarrow \begin{cases} 3A + B = 0 \\ A - B = 5 \end{cases} \Leftrightarrow \begin{cases} A = \dfrac{5}{4} \\ B = -\dfrac{15}{4} \end{cases}$$

$$= \frac{5}{4} \int \frac{d(x-1)}{x-1} - \frac{15}{4} \cdot \frac{1}{3} \int \frac{d(3x+1)}{3x+1}$$

$$= \frac{5}{4} \ln|x-1| - \frac{5}{4} \ln|3x+1| + c$$

$$= \frac{5}{4} \ln\left|\frac{x-1}{3x+1}\right| + c$$

3 $\int x^2 \cos^2 x\, dx = \int x^2 \cdot \frac{1+\cos 2x}{2} dx = \frac{1}{2}\int x^2\, dx + \frac{1}{2}\int x^2 \cos 2x\, dx$

$$= \frac{1}{6}x^3 + \frac{1}{2}\int x^2 \cos 2x\, dx$$

$$u = x^2 \Rightarrow du = 2x\, dx$$

$$dv = \cos 2x\, dx \Rightarrow v = \frac{1}{2}\sin 2x$$

$$= \frac{1}{6}x^3 + \frac{1}{2}\left(\frac{1}{2}x^2 \sin 2x - \int x \sin 2x\, dx\right)$$

$$u = x \Rightarrow du = dx$$

$$dv = \sin 2x\, dx \Rightarrow v = -\frac{1}{2}\cos 2x$$

$$= \frac{1}{6}x^3 + \frac{1}{4}x^2 \sin 2x - \frac{1}{2}\left(-\frac{1}{2}x \cos 2x + \frac{1}{2}\int \cos 2x\, dx\right)$$

$$= \frac{1}{6}x^3 + \frac{1}{4}x^2 \sin 2x + \frac{1}{4}x \cos 2x - \frac{1}{4}\cdot\frac{1}{2}\sin 2x + c$$

$$= \frac{1}{6}x^3 + \frac{1}{4}x \cos 2x + \frac{2x^2 - 1}{8}\sin 2x + c$$

4 $\int \frac{x+1}{\underbrace{x^2+x+1}_{D<0}} dx$

$$= \int \frac{\frac{1}{2}(2x+1) + \frac{1}{2}}{x^2+x+1} dx$$

$$= \frac{1}{2}\int \frac{2x+1}{x^2+x+1} dx + + \frac{1}{2}\int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}$$

$$= \frac{1}{2}\ln(x^2+x+1) + \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\,\text{Bgtan}\,\frac{2\left(x+\frac{1}{2}\right)}{\sqrt{3}} + c$$

$$= \frac{1}{2}\ln(x^2+x+1) + \frac{1}{\sqrt{3}}\,\text{Bgtan}\,\frac{2x+1}{\sqrt{3}} + c$$

5 $\displaystyle\int \frac{x^2+3x-4}{x^2-2x-8}\,dx$

$$\begin{array}{r|l} x^2+3x-4 & x^2-2x-8 \\ \hline \mp x^2 \pm 2x \pm 8 & 1 \\ \hline 5x+4 & \end{array}$$

$$= \int\left(1+\frac{5x+4}{x^2-2x-8}\right)dx$$

$$= x+\int\frac{5x+4}{x^2-2x-8}dx$$

$$\frac{5x+4}{x^2-2x-8} = \frac{5x+4}{(x-4)\,(x+2)} = \frac{A}{x-4} + \frac{B}{x+2} = \frac{(A+B)x+2A-4B}{(x-4)\,(x+2)}$$

$$\Rightarrow \begin{cases} A+B=5 \\ 2A-4B=4 \end{cases} \Leftrightarrow \begin{cases} A=4 \\ B=1 \end{cases}$$

$$= x+4\int\frac{d(x-4)}{x-4}+\int\frac{d(x+2)}{x+2}$$

$$= x+4\ln|x-4|+\ln|x+2|+c$$

6 $\displaystyle\int \sin 5x \sin 9x\,dx = \int \sin 9x \sin 5x\;dx$

$$= \frac{1}{2}\int(\cos 4x - \cos 14x)\,dx$$

$$= \frac{1}{8}\sin 4x - \frac{1}{28}\sin 14x + c$$

7 $\displaystyle\int e^{ax}\sin bx\,dx$

$$u = e^{ax} \;\Rightarrow\; du = ae^{ax}\,dx$$

$$dv = \sin bx\,dx \;\Rightarrow\; v = -\frac{1}{b}\cos bx$$

$$= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\;dx$$

$$u = e^{ax} \;\Rightarrow\; du = ae^{ax}\,dx$$

$$dv = \cos bx\,dx \;\Rightarrow\; v = -\frac{1}{b}\sin bx$$

$$= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\left(\frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\int e^{ax}\sin bx\;dx\right)$$

$$= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}\int e^{ax}\sin bx\;dx$$

$$\Rightarrow \frac{a^2+b^2}{b^2}\int e^{ax}\sin bx\,dx = \frac{1}{b^2}e^{ax}(a\sin bx - b\cos bx)+c$$

$$\Rightarrow \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)+c$$

8 $\int \ln(x^2+4)\,dx$

$$u = \ln(x^2+4) \Rightarrow du = \frac{2x}{x^2+4}dx$$
$$dv = dx \Rightarrow v = x$$
$$= x\ln(x^2+4) - 2\int \frac{x^2}{x^2+4}dx$$
$$= x\ln(x^2+4) - 2\int \frac{x^2+4-4}{x^2+4}dx$$
$$= x\ln(x^2+4) - 2x + 8\int \frac{dx}{x^2+4}$$
$$= x\ln(x^2+4) - 2x + 4\,\text{Bgtan}\,\frac{x}{2} + c$$

9 $\int (1-x^3)^{50}x^5\,dx = \int (1-x^3)^{50}x^3 \cdot x^2\,dx$

$$t = 1-x^3 \Rightarrow dt = -3x^2dx$$
$$= -\frac{1}{3}\int t^{50}(1-t)dt = -\frac{1}{3}\int (t^{50}-t^{51})dt$$
$$= -\frac{1}{3}\left(\frac{t^{51}}{51} - \frac{t^{52}}{52}\right) + c$$
$$= -\frac{1}{3}\left(\frac{(1-x^3)^{51}}{51} - \frac{(1-x^3)^{52}}{52}\right) + c$$

10 $\int \frac{e^{\cos x}\sin x}{\sqrt{4e^{2\cos x} - 12e^{\cos x} + 1}}\,dx$

$$u = e^{\cos x} \Rightarrow du = e^{\cos x}(-\sin x)dx$$
$$= -\int \frac{du}{\sqrt{4u^2-12u+1}} = -\int \frac{du}{\sqrt{(2u-3)^2-8}}$$
$$= -\frac{1}{2}\int \frac{d(2u-3)}{\sqrt{(2u-3)^2-8}}$$
$$= -\frac{1}{2}\ln\left|2u-3+\sqrt{4u^2-12u+1}\right| + c$$
$$= -\frac{1}{2}\ln\left|2e^{\cos x} - 3 + \sqrt{4e^{2\cos x} - 12e^{\cos x} + 1}\right| + c$$

11 $\displaystyle\int_{\frac{1}{12}}^{\frac{1}{4}} \frac{2}{\sqrt{x}+4x\sqrt{x}}\,dx$

$t^2 = x$ met $t > 0 \;\Rightarrow\; 2t\,dt = dx$

- $$\int \frac{2}{\sqrt{x}+4x\sqrt{x}}\,dx = 4\int \frac{t}{t+4t^2\cdot t}\,dt$$

$$= 4\int \frac{dt}{1+4t^2} = \frac{4}{2}\int \frac{d(2t)}{1+(2t)^2}$$

$$= 2\,\text{Bgtan}(2t) + c = 2\,\text{Bgtan}\left(2\sqrt{x}\right) + c$$

- $$\int_{\frac{1}{12}}^{\frac{1}{4}} \frac{2}{\sqrt{x}+4x\sqrt{x}}\,dx = 2\left[\text{Bgtan}\,2\sqrt{x}\right]_{\frac{1}{12}}^{\frac{1}{4}}$$

$$= 2\left(\text{Bgtan}\,1 - \text{Bgtan}\,\frac{\sqrt{3}}{3}\right)$$

$$= 2\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

$$= \frac{\pi}{6}$$

12 $\displaystyle\int \ln\left(x+\sqrt{1+x^2}\right)dx$

$$u = \ln\left(x+\sqrt{1+x^2}\right) \;\Rightarrow\; du = \frac{1+\dfrac{x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}}\,dx$$

$$= \frac{\dfrac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}}\,dx = \frac{1}{\sqrt{1+x^2}}\,dx$$

$dv = dx \;\Rightarrow\; v = x$

$$= x\ln\left(x+\sqrt{1+x^2}\right) - \int \frac{x}{\sqrt{1+x^2}}\,dx$$

$$= x\ln\left(x+\sqrt{1+x^2}\right) - \frac{1}{2}\int \frac{d(1+x^2)}{\sqrt{1+x^2}}$$

$$= x\ln\left(x+\sqrt{1+x^2}\right) - \sqrt{1+x^2} + c$$

13 $\displaystyle\int \frac{dx}{e^x+e^{-x}} = \int \frac{e^x dx}{e^{2x}+1} = \int \frac{d(e^x)}{(e^x)^2+1} = \text{Bgtan}(e^x) + c$

14 $\displaystyle\int e^{\sqrt{x}}\,dx = 2\int te^t\,dt$

$t^2 = x$ met $t > 0$ $\qquad$ $u = t \;\Rightarrow\; du = dt$

$\Rightarrow\; 2t\,dt = dx$ $\qquad$ $dv = e^t\,dt \;\Rightarrow\; v = e^t$

$$= 2\left(te^t - \int e^t\,dt\right) = 2e^t(t-1) + c = 2e^{\sqrt{x}}\left(\sqrt{x}-1\right) + c$$

15 $\displaystyle\int \frac{\sqrt{3x-1}}{x+1}\,dx$

$$t^2 = 3x - 1 \Rightarrow 2t\,dt = 3\,dx$$

$$= \frac{2}{3}\int \frac{t^2}{\frac{t^2+1}{3}+1}\,dt = 2\int \frac{t^2+4-4}{t^2+4}\,dt = 2\left(\int dt - 4\int \frac{dt}{t^2+4}\right)$$

$$= 2\left(t - 2\,\mathrm{Bgtan}\frac{t}{2}\right) + c = 2\left(\sqrt{3x-1} - 2\,\mathrm{Bgtan}\frac{\sqrt{3x-1}}{2}\right) + c$$

16 $\displaystyle\int \frac{4x^4+2x^3-12x^2+9}{x^3-3x+2}\,dx$

$$\begin{array}{rrrr|l}
4x^4 + 2x^3 & -12x^2 & & +9 & x^3-3x+2 \\ \cline{5-5}
\mp 4x^4 & \pm 12x^2 & \mp 8x & & 4x+2 \\ \cline{1-4}
2x^3 & & -8x & +9 & \\
\mp 2x^3 & & \pm 6x & \mp 4 & \\ \cline{1-4}
& & -2x & +5 &
\end{array}$$

$$= \int\left(4x+2+\frac{-2x+5}{x^3-3x+2}\right)dx$$

$$= 2x^2 + 2x + \int \frac{-2x+5}{x^3-3x+2}\,dx$$

$$\frac{-2x+5}{x^3-3x+2} = \frac{-2x+5}{(x-1)^2\,(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

$$= \frac{A\,(x-1)\,(x+2) + B\,(x+2) + C\,(x-1)^2}{x^3-3x+2}$$

$$= \frac{(A+C)x^2 + (A+B-2C)x - 2A + 2B + C}{x^3-3x+2}$$

$$\Rightarrow \begin{cases} A + C = 0 \\ A + B - 2C = -2 \\ -2A + 2B + C = 5 \end{cases} \Leftrightarrow \begin{cases} A = -1 \\ B = 1 \\ C = 1 \end{cases}$$

$$= 2x^2 + 2x - \int \frac{d(x-1)}{x-1} + \int \frac{d(x-1)}{(x-1)^2} + \int \frac{d(x+2)}{x+2}$$

$$= 2x^2 + 2x - \ln|x-1| - \frac{1}{x-1} + \ln|x+2| + c$$

$$= 2x^2 + 2x - \frac{1}{x-1} + \ln\left|\frac{x+2}{x-1}\right| + c$$

17 $\displaystyle\int \frac{x^4}{x^3+x^2-x-1}\,dx$

$$\begin{array}{r|l} x^4 & x^3+x^2-x-1 \\ \cline{2-2} \mp x^4 \mp x^3 \pm x^2 \pm x & x-1 \\ \cline{1-1} -x^3+x^2+x & \\ \pm x^3 \pm x^2 \mp x \mp 1 & \\ \cline{1-1} 2x^2 \qquad -1 & \end{array}$$

$$= \int\left(x-1+\frac{2x^2-1}{x^3+x^2-x-1}\right)dx$$

$$= \frac{x^2}{2}-x+\int\frac{2x^2-1}{x^3+x^2-x-1}dx$$

$$\frac{2x^2-1}{x^3+x^2-x-1} = \frac{2x^2-1}{(x-1)(x+1)^2} = \frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

$$= \frac{A\,(x+1)^2+B(x-1)(x+1)+C(x-1)}{(x-1)\,(x+1)^2}$$

$$= \frac{(A+B)x^2+(2A+C)x+A-B-C}{(x-1)\,(x+1)^2}$$

$$\Rightarrow \begin{cases} A+B=2 \\ 2A+C=0 \\ A-B-C=-1 \end{cases} \Leftrightarrow \begin{cases} A=\frac{1}{4} \\ B=\frac{7}{4} \\ C=-\frac{1}{2} \end{cases}$$

$$= \frac{x^2}{2}-x+\frac{1}{4}\int\frac{d(x-1)}{x-1}+\frac{7}{4}\int\frac{d(x+1)}{x+1}-\frac{1}{2}\int\frac{d(x+1)}{(x+1)^2}$$

$$= \frac{x^2}{2}-x+\frac{1}{4}\ln|x-1|+\frac{7}{4}\ln|x+1|+\frac{1}{2(x+1)}+c$$

18 $\displaystyle\int \frac{(1+\sin x)\cos x}{2\sin^2 x + 3\sin x + 5}\,dx$

$t = \sin x \;\Rightarrow\; dt = \cos x\, dx$

$$= \int \frac{1+t}{\underbrace{2t^2+3t+5}_{D<0}}dt$$

$$= \int \frac{\frac{1}{4}(4t+3)+\frac{1}{4}}{2t^2+3t+5}dt$$

$$= \frac{1}{4}\ln(2t^2+3t+5) + \frac{1}{8}\int \frac{d\left(t+\frac{3}{4}\right)}{\underbrace{t^2+\frac{3}{2}t+\frac{5}{2}}_{\left(t+\frac{3}{4}\right)^2+\frac{31}{16}}}$$

$$= \frac{1}{4}\ln(2t^2+3t+5) + \frac{1}{8}\cdot\frac{4}{\sqrt{31}}\,\text{Bgtan}\,\frac{4}{\sqrt{31}}\left(t+\frac{3}{4}\right)+c$$

$$= \frac{1}{4}\ln(2\sin^2 x+3\sin x+5) + \frac{1}{2\sqrt{31}}\text{Bgtan}\frac{4\sin x+3}{\sqrt{31}}+c$$

19 $\displaystyle\int x^3\sqrt{9-4x^2}\,dx$

$$= \int x^2\sqrt{9-4x^2}\,x\,dx$$

$t^2 = 9-4x^2$ met $t>0 \;\Rightarrow\; 2t\,dt = -8x\,dx$

$$= -\frac{1}{4}\int\frac{9-t^2}{4}\cdot t\cdot t\,dt$$

$$= \frac{1}{16}\int(t^4-9t^2)\,dt = = \frac{1}{16}\left(\frac{t^5}{5}-3t^3\right)+c$$

$$= \frac{1}{80}t^3\,(t^2-15)+c$$

$$= \frac{1}{80}\sqrt{(9-4x^2)^3}\,(9-4x^2-15)+c$$

$$= -\frac{1}{40}\sqrt{(9-4x^2)^3}\,(3+2x^2)+c$$

20 $\displaystyle\int \frac{dx}{\cos^2 x\,(\tan^3 x + \tan^2 x)}$

$$t = \tan x \;\Rightarrow\; dt = \frac{1}{\cos^2 x}\,dx$$

$$= \int \frac{dt}{t^3 + t^2} = \int \frac{dt}{t^2(t+1)}$$

$$\frac{1}{t^2(t+1)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t+1} = \frac{At\,(t+1) + B\,(t+1) + Ct^2}{t^2(t+1)}$$

$$= \frac{(A+C)t^2 + (A+B)t + B}{t^3 + t^2}$$

$$\Rightarrow \begin{cases} A + C = 0 \\ A + B = 0 \\ B = 1 \end{cases} \Leftrightarrow \begin{cases} A = -1 \\ B = 1 \\ C = 1 \end{cases}$$

$$= -\int \frac{dt}{t} + \int \frac{dt}{t^2} + \int \frac{d(t+1)}{t+1}$$

$$= -\ln|t| - \frac{1}{t} + \ln|1+t| + c$$

$$= \ln\left|\frac{1+\tan x}{\tan x}\right| - \cot x + c$$

21 $\displaystyle\int \frac{\text{Bgsin}\sqrt{x}}{\sqrt{1-x}}\,dx$

$$t = \text{Bgsin}\sqrt{x} \;\Rightarrow\; dt = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}\,dx \;\Rightarrow\; 2\sin t\,dt = \frac{dx}{\sqrt{1-x}}$$

$$\Downarrow$$

$$\sin t = \sqrt{x} \qquad\qquad \sqrt{x} = \sin t$$

$$= 2\int t \sin t\,dt$$

$$u = t \;\Rightarrow\; du = dt$$
$$dv = \sin t\,dt \;\Rightarrow\; v = -\cos t$$

$$= 2\left(-t\cos t + \int \cos t\,dt\right)$$

$$= 2(-t\cos t + \sin t) + c$$

$$= 2\left(-\left(\text{Bgsin}\sqrt{x}\right) \cdot \sqrt{1-x} + \sqrt{x}\right) + c$$

$$= 2\left(\sqrt{x} - \sqrt{1-x}\;\text{Bgsin}\sqrt{x}\right) + c$$

22 $\displaystyle\int_1^{\sqrt{3}} \frac{3x^2 + x + 4}{x^3 + x}\,dx$

- $\displaystyle\frac{3x^2 + x + 4}{x^3 + x} = \frac{3x^2 + x + 4}{x(x^2+1)} = \frac{A}{x} + \frac{Bx + C}{x^2+1} = \frac{A(x^2+1) + (Bx+C)x}{x(x^2+1)}$

$$= \frac{(A+B)x^2 + Cx + A}{x^3 + x}$$

$$\Rightarrow \begin{cases} A + B = 3 \\ C = 1 \\ A = 4 \end{cases} \Leftrightarrow \begin{cases} A = 4 \\ B = -1 \\ C = 1 \end{cases}$$

$$\Rightarrow \int \frac{3x^2 + x + 4}{x^3 + x}dx = 4\int \frac{dx}{x} + \int \frac{-x+1}{x^2+1}dx$$

$$= 4\ln|x| - \frac{1}{2}\int \frac{d(x^2+1)}{x^2+1} + \text{Bgtan}\,x$$

$$= 4\ln|x| - \frac{1}{2}\ln(x^2+1) + \text{Bgtan}\,x + c$$

- $\displaystyle\int_1^{\sqrt{3}} \frac{3x^2 + x + 4}{x^3 + x}\,dx = 4\ln\sqrt{3} - \frac{1}{2}\ln 4 + \frac{\pi}{3} - \left(-\frac{1}{2}\ln 2 + \frac{\pi}{4}\right)$

$$= 4\ln\sqrt{3} - \frac{1}{2}\ln 2 + \frac{1}{12}\pi$$

$$= 2\ln 3 - \frac{1}{2}\ln 2 + \frac{\pi}{12}$$

23 $\displaystyle\int \frac{dx}{x\sqrt{9 + 4x^2}} = \int \frac{x\,dx}{x^2\sqrt{9+4x^2}} = \frac{1}{4}\int \frac{t}{\frac{t^2-9}{4}\cdot t}dt = \int \frac{dt}{t^2-9}$

$t^2 = 9 + 4x^2$ met $t > 0 \;\Rightarrow\; 2t\,dt = 8x\,dx$

$$\frac{1}{t^2-9} = \frac{A}{t-3} + \frac{B}{t+3} = \frac{(A+B)t + 3A - 3B}{(t-3)(t+3)}$$

$$\Rightarrow \begin{cases} A + B = 0 \\ 3A - 3B = 1 \end{cases} \Leftrightarrow \begin{cases} A = \frac{1}{6} \\ B = -\frac{1}{6} \end{cases}$$

$$= \frac{1}{6}\int \frac{d(t-3)}{t-3} - \frac{1}{6}\int \frac{d(t+3)}{t+3}$$

$$= \frac{1}{6}\ln|t-3| - \frac{1}{6}\ln|t+3|$$

$$= \frac{1}{6}\ln\left|\frac{\sqrt{9+4x^2} - 3}{\sqrt{9+4x^2} + 3}\right| + c$$

24 $\displaystyle\int \frac{\tan x}{\sqrt{2\cos x - 1}}\,dx = \int \frac{\sin x}{\cos x\,\sqrt{2\cos x - 1}}\,dx$

$t^2 = 2\cos x - 1$ met $t > 0 \;\Rightarrow\; 2t\,dt = -2\sin x\,dx$

$$= -\int \frac{t}{\dfrac{t^2+1}{2}\cdot t}\,dt = -2\int \frac{dt}{t^2+1} = -2\,\text{Bgtan}\,t + c$$

$$= -2\,\text{Bgtan}\left(\sqrt{2\cos x - 1}\right) + c$$

25 $\displaystyle\int \frac{\sin x}{4+\cos^2 x}\,dx$

$u = \cos x \;\Rightarrow\; du = -\sin x\,dx$

$$= -\int \frac{du}{4+u^2}$$

$$= -\frac{1}{2}\,\text{Bgtan}\frac{u}{2} + c$$

$$= -\frac{1}{2}\,\text{Bgtan}\left(\frac{\cos x}{2}\right) + c$$

26 $\displaystyle\int e^{\text{Bgsin}\,x}\,dx$

$u = e^{\text{Bgsin}\,x} \;\Rightarrow\; du = \dfrac{e^{\text{Bgsin}\,x}}{\sqrt{1-x^2}}\,dx$

$dv = dx \;\Rightarrow\; v = x$

$$= xe^{\text{Bgsin}\,x} - \int \frac{e^{\text{Bgsin}\,x}\cdot x}{\sqrt{1-x^2}}\,dx$$

$u = e^{\text{Bgsin}\,x} \;\Rightarrow\; du = \dfrac{e^{\text{Bgsin}\,x}}{\sqrt{1-x^2}}\,dx$

$dv = \dfrac{x}{\sqrt{1-x^2}}\,dx \;\Rightarrow\; v = -\dfrac{1}{2}\displaystyle\int \frac{d(1-x^2)}{\sqrt{1-x^2}} = -\sqrt{1-x^2}$

$$= xe^{\text{Bgsin}\,x} + e^{\text{Bgsin}\,x}\sqrt{1-x^2} - \int e^{\text{Bgsin}\,x}\,dx$$

$$\Rightarrow\; 2\int e^{\text{Bgsin}\,x}\,dx = \left(x+\sqrt{1-x^2}\right)e^{\text{Bgsin}\,x} + c$$

$$\Rightarrow\; \int e^{\text{Bgsin}\,x}\,dx = \frac{1}{2}e^{\text{Bgsin}\,x}\left(x+\sqrt{1-x^2}\right) + c$$

27 $\displaystyle\int \frac{\ln x}{x\sqrt{1-4\ln x-\ln^2 x}}\,dx$

$$t=\ln x \;\Rightarrow\; dt=\frac{dx}{x}$$

$$=\int\frac{t}{\sqrt{1-4t-t^2}}\,dt=\int\frac{t\,dt}{\sqrt{1-(t^2+4t)}}=\int\frac{t\,dt}{\sqrt{5-(t+2)^2}}$$

$$=\int\frac{t+2-2}{\sqrt{5-(t+2)^2}}\,dt=\int\frac{t+2}{\sqrt{5-(t+2)^2}}\,dt-2\int\frac{dt}{\sqrt{5-(t+2)^2}}$$

$$\rightarrow u=5-(t+2)^2$$
$$\Rightarrow du=-2(t+2)\,dt$$

$$=-\frac{1}{2}\int\frac{du}{\sqrt{u}}-2\int\frac{d(t+2)}{\sqrt{5-(t+2)^2}}$$

$$=-\sqrt{u}-2\,\text{Bgsin}\,\frac{t+2}{\sqrt{5}}+c$$

$$=-\sqrt{5-(\ln x+2)^2}-2\,\text{Bgsin}\frac{\ln x+2}{\sqrt{5}}+c$$

$$=-\sqrt{1-4\ln x-\ln^2 x}-2\,\text{Bgsin}\frac{\ln x+2}{\sqrt{5}}+c$$

28 $\displaystyle\int \frac{x^5+x^4-8}{x^3-4x}\,dx$

$$\begin{array}{rrrr|l}
x^5+x^4 & & & -8 & x^3-4x \\ \cline{5-5}
\mp x^5 & \pm 4x^3 & & & x^2+x+4 \\ \cline{1-4}
x^4 & +4x^3 & & -8 & \\
\mp x^4 & & \pm 4x^2 & & \\ \cline{1-4}
 & 4x^3 & +4x^2 & -8 & \\
 & \mp 4x^3 & & \pm 16x & \\ \cline{2-4}
 & & 4x^2 & +16x-8 &
\end{array}$$

$$=\int\left(x^2+x+4+\frac{4x^2+16x-8}{x^3-4x}\right)dx$$

$$=\frac{x^3}{3}+\frac{x^2}{2}+4x+4\int\frac{x^2+4x-2}{x^3-4x}\,dx$$

$$\frac{x^2+4x-2}{x^3-4x}=\frac{x^2+4x-2}{x\,(x-2)\,(x+2)}=\frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+2}$$

$$=\frac{A\,(x^2-4)+Bx\,(x+2)+Cx\,(x-2)}{x^3-4x}$$

$$=\frac{(A+B+C)x^2+(2B-2C)x-4A}{x^3-4x}$$

$$\Rightarrow \begin{cases} A+B+C=1 \\ 2B-2C=4 \\ -4A=-2 \end{cases} \Leftrightarrow \begin{cases} A=\dfrac{1}{2} \\ B=\dfrac{5}{4} \\ C=-\dfrac{3}{4} \end{cases}$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + 4x + 4\left(\frac{1}{2}\int\frac{dx}{x} + \frac{5}{4}\int\frac{d(x-2)}{x-2} - \frac{3}{4}\int\frac{d(x+2)}{x+2}\right)$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + 4x + 2\ln|x| + 5\ln|x-2| - 3\ln|x+2| + c$$

29 $$\int\frac{dx}{(x-2)\sqrt{x+2}} = 2\int\frac{t\,dt}{(t^2-4)\,t} = 2\int\frac{dt}{t^2-4}$$

$$t^2 = x+2 \text{ met } t>0 \Rightarrow 2t\,dt = dx$$

$$\frac{1}{t^2-4} = \frac{A}{t-2} + \frac{B}{t+2} = \frac{(A+B)t+2A-2B}{t^2-4}$$

$$\Leftrightarrow \begin{cases} A+B=0 \\ 2A-2B=1 \end{cases} \Leftrightarrow \begin{cases} A=\dfrac{1}{4} \\ B=-\dfrac{1}{4} \end{cases}$$

$$= \frac{1}{2}\int\frac{d(t-2)}{t-2} - \frac{1}{2}\int\frac{d(t+2)}{t+2}$$

$$= \frac{1}{2}\ln|t-2| - \frac{1}{2}\ln|t+2| + c$$

$$= \frac{1}{2}\ln\left|\frac{\sqrt{x+2}-2}{\sqrt{x+2}+2}\right| + c$$

30 $$\int\frac{\sin 2x}{\sqrt{-\cos^4 x + 4\cos^2 x + 1}}\,dx$$

$$t = \cos^2 x \Rightarrow dt = 2\cos x(-\sin x)dx = -\sin 2x\,dx$$

$$= -\int\frac{dt}{\sqrt{-t^2+4t+1}} = -\int\frac{dt}{\sqrt{5-(t-2)^2}}$$

$$= -\int\frac{d(t-2)}{\sqrt{5-(t-2)^2}} = -\text{Bgsin}\frac{t-2}{\sqrt{5}} + c$$

$$= -\text{Bgsin}\frac{\cos^2 x - 2}{\sqrt{5}} + c$$

Opdracht 82 bladzijde 140

Bereken de oppervlakte begrensd door de parabool met vergelijking $y = \dfrac{x^2}{2p}$ en de kromme met vergelijking $y = \dfrac{p^3}{x^2 + p^2}$ met $p > 0$.

- snijpunten:

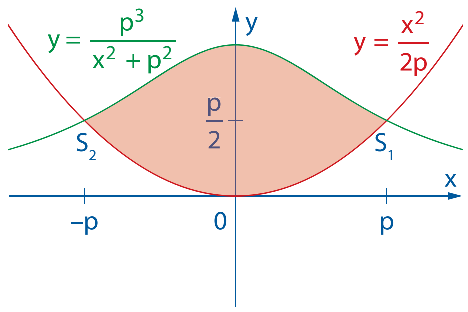
$$\frac{x^2}{2p} = \frac{p^3}{x^2 + p^2} \Leftrightarrow x^4 + p^2x^2 = 2p^4 \Leftrightarrow x^4 + p^2x^2 - 2p^4 = 0$$

$$D = 9p^4$$

$$x^2 = \frac{-p^2 \pm 3p^2}{2} \begin{cases} \nearrow -2p^2 \\ \searrow p^2 \end{cases}$$

$$\Leftrightarrow x = \pm p$$

$$\Rightarrow S_1\left(p, \frac{p}{2}\right), S_2\left(-p, \frac{p}{2}\right)$$



- Wegens symmetrie:

$$A = 2\int_0^p \left(\frac{p^3}{x^2 + p^2} - \frac{x^2}{2p}\right)dx$$

$$= 2\int_0^p \frac{2p^4 - x^4 - p^2x^2}{2p\,(x^2 + p^2)}\,dx$$

$$= \frac{-1}{p}\int_0^p \frac{x^4 + p^2x^2 - 2p^4}{x^2 + p^2}\,dx$$

$$\begin{array}{r|l} x^4 + p^2x^2 - 2p^4 & x^2 + p^2 \\ \cline{2-2} \mp x^4 \mp p^2x^2 & x^2 \\ \hline -2p^4 & \end{array}$$

$$= -\frac{1}{p}\int_0^p \left(x^2 - \frac{2p^4}{x^2 + p^2}\right)dx$$

$$= -\frac{1}{3p}\left[x^3\right]_0^p + \frac{2p^3}{p}\left[\text{Bgtan}\,\frac{x}{p}\right]_0^p$$

$$= -\frac{p^2}{3} + 2p^2(\text{Bgtan}\,1 - \text{Bgtan}\,0)$$

$$= -\frac{p^2}{3} + 2p^2 \cdot \frac{\pi}{4}$$

$$= p^2\left(\frac{\pi}{2} - \frac{1}{3}\right)$$

Opdracht 83 bladzijde 140

Bereken

1 $\displaystyle\int \frac{xe^x}{(1+x)^2}\,dx$

$$u = xe^x \Rightarrow du = (e^x + xe^x)dx = e^x(x+1)dx$$

$$dv = \frac{dx}{(1+x)^2} \Rightarrow v = -\frac{1}{x+1}$$

$$= -\frac{xe^x}{x+1} + \int e^x\,dx$$

$$= -\frac{xe^x}{x+1} + e^x + c$$

$$= e^x\left(\frac{-x+x+1}{x+1}\right) + c$$

$$= \frac{e^x}{x+1} + c$$

2 $\displaystyle\int \sqrt{x^2 + 2x + 10}\,dx = \int \sqrt{(x+1)^2 + 9}\,dx$

$$t = x + 1 \Rightarrow dt = dx$$

$$= \int \sqrt{t^2 + 9}\,dt$$

$$u = \sqrt{t^2 + 9} \Rightarrow du = \frac{t}{\sqrt{t^2+9}}\,dt$$

$$dv = dt \Rightarrow v = t$$

$$= t\sqrt{t^2+9} - \int \frac{t^2}{\sqrt{t^2+9}}\,dt$$

$$= t\sqrt{t^2+9} - \int \frac{t^2+9-9}{\sqrt{t^2+9}}\,dt$$

$$= t\sqrt{t^2+9} - \int \sqrt{t^2+9}\,dt + 9\int \frac{dt}{\sqrt{t^2+9}}$$

$$\Rightarrow 2\int \sqrt{t^2+9}\,dt = t\sqrt{t^2+9} + 9\ln\left|t + \sqrt{t^2+9}\right| + c$$

$$\Rightarrow \int \sqrt{t^2+9}\,dt = \frac{t}{2}\sqrt{t^2+9} + \frac{9}{2}\ln\left|t + \sqrt{t^2+9}\right| + c$$

$$\Rightarrow \int \sqrt{x^2+2x+10}\,dx = \frac{x+1}{2}\sqrt{x^2+2x+10} + \frac{9}{2}\ln\left|x + 1 + \sqrt{x^2+2x+10}\right| + c$$

3 $\int (\text{Bgcos } x)^2 \, dx$

$$u = (\text{Bgcos x})^2 \;\Rightarrow\; du = 2\,\text{Bgcos x}\left(-\frac{1}{\sqrt{1-x^2}}\right)dx$$

$$dv = dx \;\Rightarrow\; v = x$$

$$= x\,(\text{Bgcos x})^2 + 2\int \text{Bgcos x} \cdot \frac{x}{\sqrt{1-x^2}}\,dx$$

$$u = \text{Bgcos x} \;\Rightarrow\; du = -\frac{1}{\sqrt{1-x^2}}\,dx$$

$$dv = \frac{x}{\sqrt{1-x^2}}\,dx \;\Rightarrow\; v = -\frac{1}{2}\int \frac{d(1-x^2)}{\sqrt{1-x^2}} = -\sqrt{1-x^2}$$

$$= x(\text{Bgcos x})^2 - 2\sqrt{1-x^2}\,\text{Bgcos x} - 2\int dx$$

$$= x(\text{Bgcos x})^2 - 2\sqrt{1-x^2}\,\text{Bgcos x} - 2x + c$$

4 $\int \frac{x \,\text{Bgtan } 2x}{\sqrt{4x^2+1}}\, dx$

$$u = \text{Bgtan } 2x \;\Rightarrow\; du = \frac{2}{1+4x^2}dx$$

$$dv = \frac{x}{\sqrt{4x^2+1}}dx \;\Rightarrow\; v = \frac{1}{8}\int \frac{d(4x^2+1)}{\sqrt{4x^2+1}} = \frac{1}{4}\sqrt{4x^2+1}$$

$$= \frac{1}{4}\sqrt{4x^2+1}\,\text{Bgtan } 2x - \frac{1}{2}\int \frac{\sqrt{4x^2+1}}{4x^2+1}dx$$

$$= \frac{1}{4}\sqrt{4x^2+1}\,\text{Bgtan } 2x - \frac{1}{2}\int \frac{dx}{\sqrt{4x^2+1}}$$

$$= \frac{1}{4}\sqrt{4x^2+1}\,\text{Bgtan } 2x - \frac{1}{4}\int \frac{d(2x)}{\sqrt{(2x)^2+1}}$$

$$= \frac{1}{4}\sqrt{4x^2+1}\,\text{Bgtan } 2x - \frac{1}{4}\ln\left|2x + \sqrt{4x^2+1}\right| + c$$

5 $\int \cos^2(\ln x)\, dx$

$$u = \cos^2(\ln x) \;\Rightarrow\; du = 2\cos(\ln x)(-\sin(\ln x)) \cdot \frac{1}{x}\,dx$$
$$dv = dx \;\Rightarrow\; v = x$$
$$= x\cos^2(\ln x) + 2\int \cos(\ln x)\sin(\ln x)\,dx$$
$$= x\cos^2(\ln x) + \int \sin(2\ln x)\,dx$$

$$u = \sin(2\ln x) \;\Rightarrow\; du = \cos(2\ln x) \cdot \frac{2}{x}\,dx$$
$$dv = dx \;\Rightarrow\; v = x$$
$$\Rightarrow \int \sin(2\ln x)\,dx = x\sin(2\ln x) - 2\int \cos(2\ln x)\,dx$$
$$u = \cos(2\ln x) \;\Rightarrow\; du = -\sin(2\ln x) \cdot \frac{2}{x}\,dx$$
$$dv = dx \;\Rightarrow\; v = x$$
$$= \sin(2\ln x) - 2x\cos(2\ln x) - 4\int \sin(2\ln x)\,dx$$
$$\Rightarrow 5\int \sin(2\ln x)\,dx = x\sin(2\ln x) - 2x\cos(2\ln x) + c$$
$$\Rightarrow \int \sin(2\ln x)\,dx = \frac{1}{5}x\sin(2\ln x) - \frac{2}{5}x\cos(2\ln x) + c$$
$$= x\cos^2(\ln x) + \frac{1}{5}x(\sin(2\ln x) - 2\cos(2\ln x)) + c$$

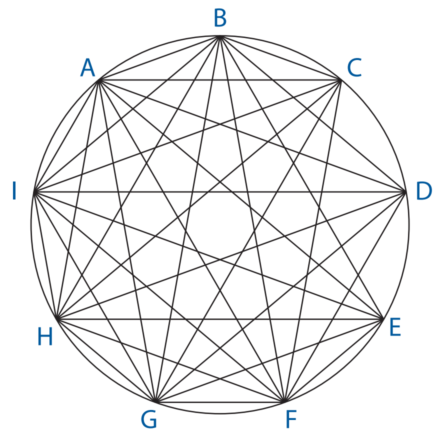
6 $\int \frac{\ln 2x}{x\ln 4x}\, dx = \int \frac{\ln 2 + \ln x}{x(\ln 4 + \ln x)}\,dx$

$$t = \ln 4 + \ln x \;\Rightarrow\; dt = \frac{1}{x}\,dx$$
$$= \int \frac{\ln 2 + t - \ln 4}{t}\,dt$$
$$= \int \frac{\ln\frac{1}{2} + t}{t}\,dt$$
$$= \ln\frac{1}{2} \cdot \ln|t| + t + c$$
$$= \ln\frac{1}{2} \cdot \ln|\ln 4 + \ln x| + \ln 4 + \ln x + c$$
$$= \ln 4x - (\ln 2) \cdot \ln|\ln 4x| + c$$

Hersenbreker 1 bladzijde 142

In de figuur zie je een regelmatige negenhoek met al zijn diagonalen. Hoeveel gelijkbenige driehoeken zijn er waarvan de drie verschillende hoekpunten ook hoekpunten van de negenhoek zijn?

(Een driehoek is gelijkbenig als twee of drie zijden dezelfde lengte hebben.)



A 27 **B** 30 **C** 33 **D** 36 **E** 39

(Bron © NWO eerste ronde, 2012)

Noem de hoekpunten A, B, ...

Elk hoekpunt vormt met het dichtste hoekpunt links en rechts een gelijkbenige driehoek. Zo zijn er 9 gelijkbenige driehoeken: ΔABI, ΔBCA ...

Er zijn ook 9 gelijkbenige driehoeken van de vorm ΔACH (steeds 2 hoekpunten verder).

Van de gelijkbenige driehoeken van de vorm ΔADG (steeds 3 hoekpunten verder) zijn er maar 3: ΔADG, ΔBEH en ΔCFI.

Van de gelijkbenige driehoeken van de vorm ΔAEF (4 hoekpunten verder) zijn er dan weer 9.

In totaal zijn er $9 + 9 + 3 + 9 = 30$ gelijkbenige driehoeken.

Antwoord B is juist.

Hersenbreker 2 bladzijde 142

Een getal bestaat uit drie verschillende cijfers. De som van de vijf andere getallen met drie verschillende cijfers die men met deze cijfers kan vormen, is 2003.

Bepaal dat getal.

(Bron © VWO finale, 2003)

Stel het gevraagde getal gelijk aan $A = 100x + 10y + z$.

Er geldt dan dat
$$\begin{aligned} A + 2003 &= (100x + 10y + z) + (100x + 10z + y) + (100y + 10x + z) \\ &\quad + (100y + 10z + x) + (100z + 10x + y) + (100z + 10y + x) \\ &= 222(x + y + z). \end{aligned}$$

Aangezien A ligt tussen 102 en 987, zijn de mogelijkheden voor $x + y + z$ enkel 10, 11, 12 en 13. (Immers: $9 \times 222 < 2003$, $10 \times 222 = 2220$, ..., $14 \times 222 = 3108$ en $3108 - 2003 = 1105$ is een getal van 4 cijfers.)

Verschillende mogelijkheden:

$x + y + z$	$222(x + y + z)$	$A = 222(x + y + z) - 2003$	
10	2220	217	
11	2442	439	
12	2664	~~661~~	(gelijke cijfers)
13	2886	~~883~~	(gelijke cijfers)

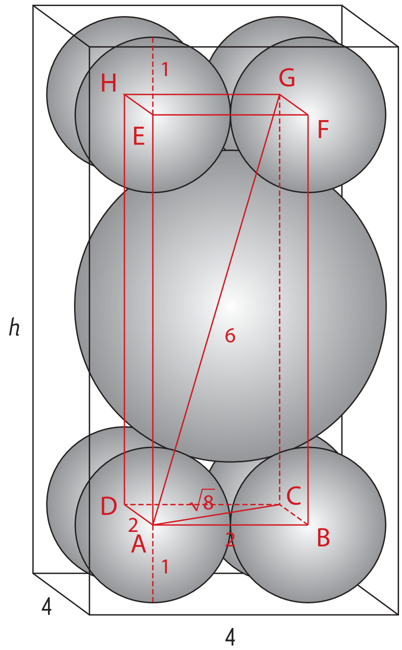
- $217 + 271 + 127 + 172 + 712 + 721 = 2220 = 2003 + 217$
- $439 + 493 + 349 + 394 + 934 + 943 = 3552 = 2003 + 1549$

$\Rightarrow$ Het gevraagde getal is 217.

Hersenbreker 3 bladzijde 142

Een balk met afmetingen $4 \times 4 \times h$ bevat een bol met straal 2 en acht kleinere bollen met straal 1. De kleine bollen raken telkens drie zijden van de balk en de grote bol raakt de kleine bollen.

Bereken h.



Verbind je de middelpunten van de klein bollen, dan vind je een balk met als grondvlak een vierkant met zijde 2.

In de rechthoekige driehoek ACG is

$$|CG| = \sqrt{6^2 - 8} = \sqrt{28}$$
$$= 2\sqrt{7}$$

De hoogte is dan gelijk aan $2 + 2\sqrt{7}$.