Independent study - Stochastic Differential Equations

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Introduction

This document collects a selection of exercises I worked through from Bernt Øksendal, *Stochastic Differential Equations: An Introduction with Applications*. These solutions were completed as part of my MATH 199 independent study with Professor Ang at UC San Diego. The document serves as a brief record of the topics and problem-solving work I carried out during the course.

Exercise 2.9

To illustrate that the (finite-dimensional) distributions alone do not give all the information regarding the continuity properties of a process, consider the following example:

Let $(\Omega, \mathcal{F}, P) = ([0,\infty), \mathcal{B}, \mu)$ where $\mathcal{B}$ denotes the Borel σ-algebra on $[0,\infty)$ and μ is a probability measure on $[0,\infty)$ with no mass on single points. Define

$$X_t(\omega) = \begin{cases} 1 & \text{if } t = \omega, \\ 0 & \text{otherwise} \end{cases}$$

and

$$Y_t(\omega) = 0 \qquad \text{for all } (t,\omega) \in [0,\infty) \times [0,\infty).$$

Prove that $\{X_t\}$ and $\{Y_t\}$ have the same distributions and that X_t is a version of Y_t. And yet we have that $t \mapsto Y_t(\omega)$ is continuous for all ω, while $t \mapsto X_t(\omega)$ is discontinuous for all ω.

Solution. By definition 2.2.2 we have that $\{X_t\}$ and $\{Y_t\}$ the same finite-dimensional distribution if

$$\mathbb{P}(\{\omega : X_t(\omega) = Y_t(\omega)\}) = 1, \, \forall t$$

we then can in fact show this is true for our case so that will be how we show that they equal in distribution.

$$\mathbb{P}(\{\omega : X_t(\omega) = Y_t(\omega)\}) = 1 - \mathbb{P}(\{\omega : X_t(\omega) \neq Y_t(\omega)\})$$

Then we use that

$$\mathbb{P}(\{\omega : X_t(\omega) \neq Y_t(\omega)\}) = \mathbb{P}(\{\omega : X_t(\omega) \neq 0\}) = \mathbb{P}(\{\omega : X_t(\omega) = 1\}) = \mathbb{P}(\{t\}) = 0$$

This holds since it had no mass in a single point. This means that for any t we have that $\mathbb{P}(\{\omega : X_t(\omega) \neq Y_t(\omega)\}) = 0$. This gives that

$$\mathbb{P}(\{\omega : X_t(\omega) = Y_t(\omega)\}) = 1 - \mathbb{P}(\{\omega : X_t(\omega) \neq Y_t(\omega)\}) = 1 - 0 = 1, \, \forall t$$

Specifically we proved something stronger which is that $\{X_t\}$ is a version of $\{Y_t\}$. $\square$

Exercise 5.1

Verify that the given processes solve the given corresponding stochastic differential equations (B_t denotes 1-dimensional Brownian motion).

(i) $X_t = e^{B_t}$ solves
$$dX_t = \frac{1}{2}X_t\,dt + X_t\,dB_t.$$

(ii) $X_t = \dfrac{B_t}{1+t}$, $B_0 = 0$ solves
$$dX_t = -\frac{1}{1+t}X_t\,dt + \frac{1}{1+t}\,dB_t; \qquad X_0 = 0.$$

(iii) $X_t = \sin(B_t)$ with $B_0 = a \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ solves
$$dX_t = -\frac{1}{2}X_t\,dt + \sqrt{1-X_t^2}\,dB_t \qquad \text{for } t < \inf\{s>0 : B_s \notin \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right]\}.$$

(iv) $(X_1(t), X_2(t)) = (t, e^t B_t)$ solves
$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} 1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} 0 \\ e^{X_1} \end{bmatrix} dB_t.$$

(v) $(X_1(t), X_2(t)) = (\cosh(B_t), \sinh(B_t))$ solves
$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} X_2 \\ X_1 \end{bmatrix} dB_t.$$

Solution. (i) We begin to observe that
$$X_t = g(t, B_t) = e^{B_t} \implies \frac{\partial g}{\partial t}(t,x) = 0,\ \frac{\partial g}{\partial x}(t,B_t) = e^{B_t},\ \frac{\partial^2 g}{\partial x^2}(t,B_t) = e^{B_t}$$

$$\begin{gathered} dX_t = \frac{\partial g}{\partial t}(t,B_t)\,dt + \frac{\partial g}{\partial x}(t,B_t)\,dB_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,B_t)\cdot(dB_t)^2 = \\ e^{B_t}dB_t + \frac{1}{2}e^{B_t}\cdot(dB_t)^2 = e^{B_t}dB_t + \frac{1}{2}e^{B_t}dt = \frac{1}{2}X_t\,dt + X_t\,dB_t \end{gathered}$$

(ii) We begin to observe that
$$X_t = g(t,B_t) = \frac{B_t}{1+t},\ B_0 = 0 \implies \frac{\partial g}{\partial t}(t,x) = \frac{-B_t}{(1+t)^2},\ \frac{\partial g}{\partial x}(t,B_t) = \frac{1}{1+t},\ \frac{\partial^2 g}{\partial x^2}(t,B_t) = 0$$

$$\begin{gathered} dX_t = \frac{\partial g}{\partial t}(t,B_t)\,dt + \frac{\partial g}{\partial x}(t,B_t)\,dB_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,B_t)\cdot(dB_t)^2 = \\ \frac{-B_t}{(1+t)^2}dt + \frac{1}{1+t}\cdot dB_t = -\frac{1}{1+t}X_t\,dt + \frac{1}{1+t}\,dB_t \end{gathered}$$

(iii) We begin to observe that

$$X_t = g(t, B_t) = \sin(B_t),\ B_0 = a \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \implies$$

$$\frac{\partial g}{\partial t}(t,x) = 0,\ \frac{\partial g}{\partial x}(t,B_t) = \cos(B_t),\ \frac{\partial^2 g}{\partial x^2}(t,B_t) = -\sin(B_t)$$

$$dX_t = \frac{\partial g}{\partial t}(t,B_t)\,dt + \frac{\partial g}{\partial x}(t,B_t)\,dB_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,B_t)\cdot(dB_t)^2 =$$

$$\cos(B_t)\cdot dB_t - \frac{1}{2}\sin(B_t)\cdot(dB_t)^2 = \sqrt{1-\sin(B_t)^2}\cdot dB_t - \frac{1}{2}\sin(B_t)dt = -\frac{1}{2}X_t\,dt + \sqrt{1-X_t^2}\,dB_t$$

Which holds for $t < \inf\{s > 0 : B_s \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\}$.

(iv) We begin to observe that

$$(X_1(t), X_2(t)) = (g_1(t,B_t), g_1(t,B_t)) = (t, e^t B_t) \implies$$

$$\frac{\partial g_1}{\partial t}(t,X) = 1,\ \frac{\partial g_2}{\partial t}(t,B_t) = e^t B_t,\ \frac{\partial g_1}{\partial x}(t,B_t) = 0,\ \frac{\partial g_2}{\partial x}(t,B_t) = e^t,$$
$$\frac{\partial g_1}{\partial x^2}(t,B_t) = 0,\ \frac{\partial g_2}{\partial x^2}(t,B_t) = 0$$

Then we get that

$$dX_1 = 1\,dt,\ dX_2 = e^t B_t\,dt + e^t\,dB_t = X_2\,dt + e^{X_1}dB_t$$

(v) We begin to observe that

$$(X_1(t), X_2(t)) = (g_1(t,B_t), g_1(t,B_t)) = (\cosh(B_t), \sinh(B_t)) = \left(\frac{e^{B_t}+e^{-B_t}}{2}, \frac{e^{B_t}-e^{-B_t}}{2}\right) \implies$$

$$\frac{\partial g_1}{\partial t}(t,X) = 0,\ \frac{\partial g_2}{\partial t}(t,B_t) = 0,\ \frac{\partial g_1}{\partial x}(t,B_t) = \sinh(B_t),\ \frac{\partial g_2}{\partial x}(t,B_t) = \cosh(B_t)$$
$$\frac{\partial g_1}{\partial x^2}(t,B_t) = \cosh(B_t),\ \frac{\partial g_2}{\partial x^2}(t,B_t) = \sinh(B_t)$$

Then we get that

$$dX_1 = \sinh(B_t)\,dB_t + \frac{1}{2}\cosh(B_t)\,dB_t\,dB_t = \sinh(B_t)\,dB_t + \frac{1}{2}\cosh(B_t)\,dt = X_2\,dB_t + \frac{1}{2}X_1\,dt$$

$$dX_1 = \cosh(B_t)\,dB_t + \frac{1}{2}\sinh(B_t)\,dB_t\,dB_t = \cosh(B_t)\,dB_t + \frac{1}{2}\sinh(B_t)\,dt = X_1\,dB_t + \frac{1}{2}X_2\,dt$$

□

Exercise 5.4

Solve the following stochastic differential equations:

(i)

$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} dt + \begin{bmatrix} 1 & 0 \\ 0 & X_1 \end{bmatrix} \begin{bmatrix} dB_1 \\ dB_2 \end{bmatrix}.$$

(ii)

$$dX_t = X_t\,dt + dB_t.$$

(Hint: Multiply both sides with "the integrating factor" e^{-t} and compare with $d(e^{-t}X_t)$)

(iii)

$$dX_t = -X_t\,dt + e^{-t}\,dB_t.$$

Solution. (i) We take the integral form

$$X_1(t) = \int_0^t dX_1 = \int_0^t ds + \int_0^t dB_1 = X_1(0) + t + B_1(t)$$

Which gives us that

$$X_2(t) = \int_0^t dX_2 = \int_0^t X_1(0)+s+B_1(s)dB_2 = X_2(0)+X_1(0)B_2(t)+\int_0^t sdB_2+\int_0^t B_1(s)dB_2$$

(ii) We take the hint and rewrite the following

$$e^{-t}dX_t = e^{-t}X_t\,dt + e^{-t}dB_t \implies e^{-t}\,dX_t - e^{-t}X_t\,dt = e^{-t}dB_t$$

We recognize that $d(e^{-t}X_t) = e^{-t}\,dX_t - e^{-t}X_t\,dt$ thus we get that

$$\int_0^t d(e^{-s}X_s)ds = \int_0^t e^{-s}dB_s \implies X_t = X_0e^t + \int_0^t e^{t-s}dB_s$$

(iii)

$$dX_t = -X_t\,dt + e^{-t}\,dB_t \implies e^t dX_t + e^t X_t\,dt = 1\,dB_t$$

We observe that $e^t dX_t + e^t X_t\,dt = d(e^t X_t)$

$$\int_0^t d(e^s X_s)\,ds = \int_0^t 1\,dB_s \implies X_t = (B_t + X_0)e^{-t}$$

□

Exercise 5.5

a) Solve the *Ornstein-Uhlenbeck equation* (or *Langevin equation*)

$$dX_t = \mu X_t\,dt + \sigma\,dB_t$$

where μ, σ are real constants, $B_t \in \mathbb{R}$.

The solution is called the *Ornstein-Uhlenbeck process.* (Hint: See Exercise 5.4 (ii).)

b) Find $\mathbb{E}[X_t]$ and $\mathrm{Var}[X_t] := \mathbb{E}\big[(X_t - \mathbb{E}[X_t])^2\big]$.

Solution.

$$e^{\mu t}dX_t = \mu e^{\mu t}X_t\,dt + \sigma e^{\mu t}\,dB_t \implies d(X_t e^{\mu t}) = \sigma e^{\mu t}\,dB_t$$

Thus we integrate again

$$\int_0^t d(X_t e^{-\mu s})\,ds = X_t e^{\mu t} - X_0 = \int_0^t \sigma e^{-\mu s}\,dB_s \implies$$

$$X_t = X_0 e^{\mu t} + \int_0^t \sigma e^{-\mu(s-t)}\,dB_s$$

Find $\mathbb{E}[X_t]$ and $\mathrm{Var}[X_t] := \mathbb{E}\big[(X_t - \mathbb{E}[X_t])^2\big]$.

$$\mathbb{E}[X_t] = \mathbb{E}[X_0 e^{\mu t} + \int_0^t \sigma e^{-\mu(s-t)}\, dB_s] = \mathbb{E}[X_0]e^{\mu t} + \sigma\mathbb{E}[\int_0^t e^{-\mu(s-t)}\, dB_s] = \mathbb{E}[X_0]e^{\mu t}$$

Then we want to find

$$\mathrm{Var}[X_t] := \mathbb{E}\big[(X_t - \mathbb{E}[X_t])^2\big] = \mathbb{E}\big[X_t^2\big] - (\mathbb{E}[X_0]e^{\mu t})^2$$

Thus we solve the

$$\mathbb{E}\big[X_t^2\big] = \mathbb{E}\big[(X_0 e^{\mu t} + \int_0^t \sigma e^{-\mu(s-t)}\, dB_s)^2\big] =$$

$$\mathbb{E}\big[X_0^2\big]e^{2\mu t} + 2\mathbb{E}\big[X_0 e^{\mu t}\int_0^t \sigma e^{-\mu(s-t)}\, dB_s\big] + \mathbb{E}\big[(\int_0^t \sigma e^{-\mu(s-t)}\, dB_s)^2\big] =$$

$$\mathbb{E}\big[X_0^2\big]e^{2\mu t} + 2\mathbb{E}\big[X_0 e^{\mu t}\int_0^t \sigma e^{-\mu(s-t)}\, dB_s\big] + \mathbb{E}\big[\int_0^t (\sigma e^{-\mu(s-t)})^2\, dt\big] =$$

$$\mathbb{E}\big[X_0^2\big]e^{2\mu t} + 2\mathbb{E}\big[X_0 e^{\mu t}\int_0^t \sigma e^{-\mu(s-t)}\, dB_s\big] + \frac{\sigma^2}{2\mu}\big(e^{2\mu t} - 1\big)$$

We know analyze this

$$\mathbb{E}\big[X_0 e^{\mu t}\int_0^t \sigma e^{-\mu(s-t)}\, dB_s\big] = \mathbb{E}\big[\mathbb{E}\big[X_0 e^{\mu t}\int_0^t \sigma e^{-\mu(s-t)}\, dB_s \mid \mathcal{F}_0\big]\big] =$$

$$\mathbb{E}\big[X_0\mathbb{E}\big[e^{\mu t}\int_0^t \sigma e^{-\mu(s-t)}\, dB_s \mid \mathcal{F}_0\big]\big] = \mathbb{E}\big[X_0 \cdot 0\big] = 0$$

Thus we get that

$$\mathrm{Var}[X_t] = (\mathbb{E}\big[X_0^2\big] - \mathbb{E}[X_0]^2)e^{2\mu t} + \frac{\sigma^2}{2\mu}\big(e^{2\mu t} - 1\big) = \mathrm{Var}\big[X_0 e^{\mu t}\big] + \frac{\sigma^2}{2\mu}\big(e^{2\mu t} - 1\big)$$

$\square$