Independent study - Stochastic Differential Equations

Johan Jönsson

January 2026

Introduction

This document collects a selection of exercises I worked through from Bernt Øksendal, *Stochastic Differential Equations: An Introduction with Applications*. These solutions were completed as part of my MATH 199 independent study with Professor Ang at UC San Diego. The document serves as a brief record of the topics and problem-solving work I carried out during the course.

Exercise 2.9

To illustrate that the (finite-dimensional) distributions alone do not give all the information regarding the continuity properties of a process, consider the following example:

Let $(\Omega, \mathcal{F}, P) = ([0,\infty), \mathcal{B}, \mu)$ where $\mathcal{B}$ denotes the Borel σ-algebra on $[0,\infty)$ and μ is a probability measure on $[0,\infty)$ with no mass on single points. Define

$$X_t(\omega) = \begin{cases} 1 & \text{if } t = \omega, \\ 0 & \text{otherwise} \end{cases}$$

and

$$Y_t(\omega) = 0 \qquad \text{for all } (t,\omega) \in [0,\infty) \times [0,\infty).$$

Prove that $\{X_t\}$ and $\{Y_t\}$ have the same distributions and that X_t is a version of Y_t. And yet we have that $t \mapsto Y_t(\omega)$ is continuous for all ω, while $t \mapsto X_t(\omega)$ is discontinuous for all ω.

Solution. By definition 2.2.2 we have that $\{X_t\}$ and $\{Y_t\}$ the same finite-dimensional distribution if

$$\mathbb{P}(\{\omega : X_t(\omega) = Y_t(\omega)\}) = 1, \, \forall t$$

we then can in fact show this is true for our case so that will be how we show that they equal in distribution.

$$\mathbb{P}(\{\omega : X_t(\omega) = Y_t(\omega)\}) = 1 - \mathbb{P}(\{\omega : X_t(\omega) \neq Y_t(\omega)\})$$

Then we use that

$$\mathbb{P}(\{\omega : X_t(\omega) \neq Y_t(\omega)\}) = \mathbb{P}(\{\omega : X_t(\omega) \neq 0\}) = \mathbb{P}(\{\omega : X_t(\omega) = 1\}) = \mathbb{P}(\{t\}) = 0$$

This holds since it had no mass in a single point. This means that for any t we have that $\mathbb{P}(\{\omega : X_t(\omega) \neq Y_t(\omega)\}) = 0$. This gives that

$$\mathbb{P}(\{\omega : X_t(\omega) = Y_t(\omega)\}) = 1 - \mathbb{P}(\{\omega : X_t(\omega) \neq Y_t(\omega)\}) = 1 - 0 = 1, \, \forall t$$

Specifically we proved something stronger which is that $\{X_t\}$ is a version of $\{Y_t\}$. $\square$

Exercise 5.1

Verify that the given processes solve the given corresponding stochastic differential equations (B_t denotes 1-dimensional Brownian motion).

(i) $X_t = e^{B_t}$ solves
$$dX_t = \frac{1}{2} X_t \, dt + X_t \, dB_t.$$

(ii) $X_t = \dfrac{B_t}{1+t}$, $B_0 = 0$ solves
$$dX_t = -\frac{1}{1+t} X_t \, dt + \frac{1}{1+t} \, dB_t; \qquad X_0 = 0.$$

(iii) $X_t = \sin(B_t)$ with $B_0 = a \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ solves
$$dX_t = -\frac{1}{2} X_t \, dt + \sqrt{1 - X_t^2} \, dB_t \qquad \text{for } t < \inf\{s > 0 : B_s \notin \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right]\}.$$

(iv) $(X_1(t), X_2(t)) = (t, e^t B_t)$ solves
$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} 1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} 0 \\ e^{X_1} \end{bmatrix} dB_t.$$

(v) $(X_1(t), X_2(t)) = (\cosh(B_t), \sinh(B_t))$ solves
$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} X_2 \\ X_1 \end{bmatrix} dB_t.$$

Solution. (i) We begin to observe that
$$X_t = g(t, B_t) = e^{B_t} \implies \frac{\partial g}{\partial t}(t,x) = 0, \; \frac{\partial g}{\partial x}(t, B_t) = e^{B_t}, \; \frac{\partial^2 g}{\partial x^2}(t, B_t) = e^{B_t}$$

$$dX_t = \frac{\partial g}{\partial t}(t, B_t) \, dt + \frac{\partial g}{\partial x}(t, B_t) \, dB_t + \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(t, B_t) \cdot (dB_t)^2 =$$
$$e^{B_t} dB_t + \frac{1}{2} e^{B_t} \cdot (dB_t)^2 = e^{B_t} dB_t + \frac{1}{2} e^{B_t} dt = \frac{1}{2} X_t \, dt + X_t \, dB_t$$

(ii) We begin to observe that
$$X_t = g(t, B_t) = \frac{B_t}{1+t}, \; B_0 = 0 \implies \frac{\partial g}{\partial t}(t,x) = \frac{-B_t}{(1+t)^2}, \; \frac{\partial g}{\partial x}(t, B_t) = \frac{1}{1+t}, \; \frac{\partial^2 g}{\partial x^2}(t, B_t) = 0$$

$$dX_t = \frac{\partial g}{\partial t}(t, B_t) \, dt + \frac{\partial g}{\partial x}(t, B_t) \, dB_t + \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(t, B_t) \cdot (dB_t)^2 =$$
$$\frac{-B_t}{(1+t)^2} dt + \frac{1}{1+t} \cdot dB_t = -\frac{1}{1+t} X_t \, dt + \frac{1}{1+t} \, dB_t$$

(iii) We begin to observe that

$$X_t = g(t, B_t) = \sin(B_t),\ B_0 = a \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \implies$$

$$\frac{\partial g}{\partial t}(t,x) = 0,\ \frac{\partial g}{\partial x}(t,B_t) = \cos(B_t),\ \frac{\partial^2 g}{\partial x^2}(t,B_t) = -\sin(B_t)$$

$$dX_t = \frac{\partial g}{\partial t}(t,B_t)\,dt + \frac{\partial g}{\partial x}(t,B_t)\,dB_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,B_t)\cdot(dB_t)^2 =$$

$$\cos(B_t)\cdot dB_t - \frac{1}{2}\sin(B_t)\cdot(dB_t)^2 = \sqrt{1-\sin(B_t)^2}\cdot dB_t - \frac{1}{2}\sin(B_t)dt = -\frac{1}{2}X_t\,dt + \sqrt{1-X_t^2}\,dB_t$$

Which holds for $t < \inf\{s > 0 : B_s \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\}$.

(iv) We begin to observe that

$$(X_1(t), X_2(t)) = (g_1(t,B_t), g_1(t,B_t)) = (t, e^t B_t) \implies$$

$$\frac{\partial g_1}{\partial t}(t,X) = 1,\ \frac{\partial g_2}{\partial t}(t,B_t) = e^t B_t,\ \frac{\partial g_1}{\partial x}(t,B_t) = 0,\ \frac{\partial g_2}{\partial x}(t,B_t) = e^t,$$
$$\frac{\partial g_1}{\partial x^2}(t,B_t) = 0,\ \frac{\partial g_2}{\partial x^2}(t,B_t) = 0$$

Then we get that

$$dX_1 = 1\,dt,\ dX_2 = e^t B_t\,dt + e^t\,dB_t = X_2\,dt + e^{X_1}dB_t$$

(v) We begin to observe that

$$(X_1(t), X_2(t)) = (g_1(t,B_t), g_1(t,B_t)) = (\cosh(B_t), \sinh(B_t)) = \left(\frac{e^{B_t}+e^{-B_t}}{2}, \frac{e^{B_t}-e^{-B_t}}{2}\right) \implies$$

$$\frac{\partial g_1}{\partial t}(t,X) = 0,\ \frac{\partial g_2}{\partial t}(t,B_t) = 0,\ \frac{\partial g_1}{\partial x}(t,B_t) = \sinh(B_t),\ \frac{\partial g_2}{\partial x}(t,B_t) = \cosh(B_t)$$
$$\frac{\partial g_1}{\partial x^2}(t,B_t) = \cosh(B_t),\ \frac{\partial g_2}{\partial x^2}(t,B_t) = \sinh(B_t)$$

Then we get that

$$dX_1 = \sinh(B_t)\,dB_t + \frac{1}{2}\cosh(B_t)\,dB_t\,dB_t = \sinh(B_t)\,dB_t + \frac{1}{2}\cosh(B_t)\,dt = X_2\,dB_t + \frac{1}{2}X_1\,dt$$

$$dX_1 = \cosh(B_t)\,dB_t + \frac{1}{2}\sinh(B_t)\,dB_t\,dB_t = \cosh(B_t)\,dB_t + \frac{1}{2}\sinh(B_t)\,dt = X_1\,dB_t + \frac{1}{2}X_2\,dt$$

□

Exercise 5.4

Solve the following stochastic differential equations:

(i)

$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} dt + \begin{bmatrix} 1 & 0 \\ 0 & X_1 \end{bmatrix} \begin{bmatrix} dB_1 \\ dB_2 \end{bmatrix}.$$

(ii)

$$dX_t = X_t\,dt + dB_t.$$

(Hint: Multiply both sides with “the integrating factor” e^{-t} and compare with $d(e^{-t}X_t)$)

(iii)

$$dX_t = -X_t\,dt + e^{-t}\,dB_t.$$

Solution. (i) We take the integral form

$$X_1(t) = \int_0^t dX_1 = \int_0^t ds + \int_0^t dB_1 = X_1(0) + t + B_1(t)$$

Which gives us that

$$X_2(t) = \int_0^t dX_2 = \int_0^t X_1(0) + s + B_1(s) dB_2 = X_2(0) + X_1(0)B_2(t) + \int_0^t s dB_2 + \int_0^t B_1(s) dB_2$$

(ii) We take the hint and rewrite the following

$$e^{-t}dX_t = e^{-t}X_t\,dt + e^{-t}dB_t \implies e^{-t}\,dX_t - e^{-t}X_t\,dt = e^{-t}dB_t$$

We recognize that $d(e^{-t}X_t) = e^{-t}\,dX_t - e^{-t}X_t\,dt$ thus we get that

$$\int_0^t d(e^{-s}X_s)ds = \int_0^t e^{-s}dB_s \implies X_t = X_0e^t + \int_0^t e^{t-s}dB_s$$

(iii)

$$dX_t = -X_t\,dt + e^{-t}\,dB_t \implies e^t dX_t + e^t X_t\,dt = 1\,dB_t$$

We observe that $e^t dX_t + e^t X_t\,dt = d(e^t X_t)$

$$\int_0^t d(e^s X_s)\,ds = \int_0^t 1\,dB_s \implies X_t = (B_t + X_0)e^{-t}$$

□

Exercise 5.5

a) Solve the *Ornstein-Uhlenbeck equation* (or *Langevin equation*)

$$dX_t = \mu X_t\,dt + \sigma\,dB_t$$

where μ, σ are real constants, $B_t \in \mathbb{R}$.

The solution is called the *Ornstein-Uhlenbeck process.* (Hint: See Exercise 5.4 (ii).)

b) Find $\mathbb{E}[X_t]$ and $\mathrm{Var}[X_t] := \mathbb{E}\big[(X_t - \mathbb{E}[X_t])^2\big]$.

Solution.

$$e^{\mu t}dX_t = \mu e^{\mu t}X_t\,dt + \sigma e^{\mu t}\,dB_t \implies d(X_t e^{\mu t}) = \sigma e^{\mu t}\,dB_t$$

Thus we integrate again

$$\int_0^t d(X_t e^{-\mu s})\,ds = X_t e^{\mu t} - X_0 = \int_0^t \sigma e^{-\mu s}\,dB_s \implies$$

$$X_t = X_0 e^{\mu t} + \int_0^t \sigma e^{-\mu(s-t)}\,dB_s$$

Find $\mathbb{E}[X_t]$ and $\mathrm{Var}[X_t] := \mathbb{E}\big[(X_t - \mathbb{E}[X_t])^2\big]$.

$$\mathbb{E}[X_t] = \mathbb{E}[X_0 e^{\mu t} + \int_0^t \sigma e^{-\mu(s-t)}\, dB_s] = \mathbb{E}[X_0] e^{\mu t} + \sigma \mathbb{E}[\int_0^t e^{-\mu(s-t)}\, dB_s] = \mathbb{E}[X_0] e^{\mu t}$$

Then we want to find

$$\mathrm{Var}[X_t] := \mathbb{E}\big[(X_t - \mathbb{E}[X_t])^2\big] = \mathbb{E}\big[X_t^2\big] - (\mathbb{E}[X_0] e^{\mu t})^2$$

Thus we solve the

$$\mathbb{E}\big[X_t^2\big] = \mathbb{E}\big[(X_0 e^{\mu t} + \int_0^t \sigma e^{-\mu(s-t)}\, dB_s)^2\big] =$$

$$\mathbb{E}\big[X_0^2\big] e^{2\mu t} + 2\mathbb{E}\big[X_0 e^{\mu t} \int_0^t \sigma e^{-\mu(s-t)}\, dB_s\big] + \mathbb{E}\big[(\int_0^t \sigma e^{-\mu(s-t)}\, dB_s)^2\big] =$$

$$\mathbb{E}\big[X_0^2\big] e^{2\mu t} + 2\mathbb{E}\big[X_0 e^{\mu t} \int_0^t \sigma e^{-\mu(s-t)}\, dB_s\big] + \mathbb{E}\big[\int_0^t (\sigma e^{-\mu(s-t)})^2\, dt\big] =$$

$$\mathbb{E}\big[X_0^2\big] e^{2\mu t} + 2\mathbb{E}\big[X_0 e^{\mu t} \int_0^t \sigma e^{-\mu(s-t)}\, dB_s\big] + \frac{\sigma^2}{2\mu}\big(e^{2\mu t} - 1\big)$$

We know analyze this

$$\mathbb{E}\big[X_0 e^{\mu t} \int_0^t \sigma e^{-\mu(s-t)}\, dB_s\big] = \mathbb{E}\big[\mathbb{E}\big[X_0 e^{\mu t} \int_0^t \sigma e^{-\mu(s-t)}\, dB_s \mid \mathcal{F}_0\big]\big] =$$

$$\mathbb{E}\big[X_0 \mathbb{E}\big[e^{\mu t} \int_0^t \sigma e^{-\mu(s-t)}\, dB_s \mid \mathcal{F}_0\big]\big] = \mathbb{E}\big[X_0 \cdot 0\big] = 0$$

Thus we get that

$$\mathrm{Var}[X_t] = (\mathbb{E}\big[X_0^2\big] - \mathbb{E}[X_0]^2) e^{2\mu t} + \frac{\sigma^2}{2\mu}\big(e^{2\mu t} - 1\big) = \mathrm{Var}\big[X_0 e^{\mu t}\big] + \frac{\sigma^2}{2\mu}\big(e^{2\mu t} - 1\big)$$

□

Exercise 5.6

Solve the stochastic differential equation

$$dY_t = r\, dt + \alpha Y_t\, dB_t$$

where r, α are real constants, $B_t \in \mathbb{R}$. (Hint: Multiply the equation by the 'integrating factor')

$$F_t = \exp\Big(-\alpha B_t + \tfrac{1}{2}\alpha^2 t\Big)$$

Solution.

$$\exp\Big(-\alpha B_t + \tfrac{1}{2}\alpha^2 t\Big) dY_t = \exp\Big(-\alpha B_t + \tfrac{1}{2}\alpha^2 t\Big) r\, dt + \exp\Big(-\alpha B_t + \tfrac{1}{2}\alpha^2 t\Big) \alpha Y_t\, dB_t \implies$$

$$d(\exp\Big(-\alpha B_t + \tfrac{1}{2}\alpha^2 t\Big) Y_t) = \exp\Big(-\alpha B_t + \tfrac{1}{2}\alpha^2 t\Big) r\, dt \implies$$

$$\exp\Big(-\alpha B_t + \tfrac{1}{2}\alpha^2 t\Big) Y_t = exp\Big(\alpha B_t - \tfrac{1}{2}\alpha^2 t\Big)(Y_0 + \int_0^t \exp\Big(-\alpha B_s + \tfrac{1}{2}\alpha^2 s\Big) r\, ds$$

□

Exercise 5.9

Show that there is a unique strong solution X_t of the 1-dimensional stochastic differential equation

$$dX_t = \ln(1+X_t^2)\,dt + \mathbb{1}_{\{X_t>0\}}X_t\,dB_t, \qquad X_0 = a \in \mathbb{R}.$$

Solution. We then have to check that the functions described satisfy Theorem 5.2.1. Where

$$\sigma(t,x) = \begin{cases} x,\ x>0 \\ 0,\ x\le 0 \end{cases} \qquad b(t,x) = \ln(1+x^2)$$

It then has to satisfy the two stated conditions below

$$\exists C \text{ s.t } |\sigma(t,x)| + |b(t,x)| \le C(1+|x|),\ \forall x\in\mathbb{R},\ t\in[0,T]$$

$$\exists D \text{ s.t } |b(t,x)-b(t,y)| + |\sigma(t,x)-\sigma(t,x)| \le D|x-y|,\ \forall x\in\mathbb{R},\ t\in[0,T]$$

We observe that tat for σ we have that

$$|\sigma(t,x)|\le|x|,\ \sigma(t,x)-\sigma(t,x) = \begin{cases} x-y,\ x>0 \text{ and } y>0 \\ x,\ x>0 \text{ and } y\le 0 \\ -y,\ x\le 0 \text{ and } y>0 \end{cases} \implies |\sigma(t,x)-\sigma(t,x)|\le|x-y|$$

Then for b we have that

$$|b(t,x)|\le|x| \text{ since } \forall t\ge 0,\ f(t) = t-\ln(1+t^2) \implies f'(t) = 1-\frac{2t}{1+t^2} = \frac{(t+1)^2}{1+t^2} \ge 0,\ f(0)=0$$

We will use mean value theorem and observe that $\frac{d}{dx}\ln(1+x^2) = \frac{2x}{1+x^2}$ where $|\frac{2x}{1+x^2}|\le 1 \implies$

$$|\ln(1+x^2)-\ln(1+y^2)| = |f'(c)(x-y)| \le 1|x-y|$$

Which means that we can take $C=D=2$ so we can conclude by the Theorem 5.2.1 that the solution is adapted to $\mathcal{F}_t^Z \implies$ strong solution.

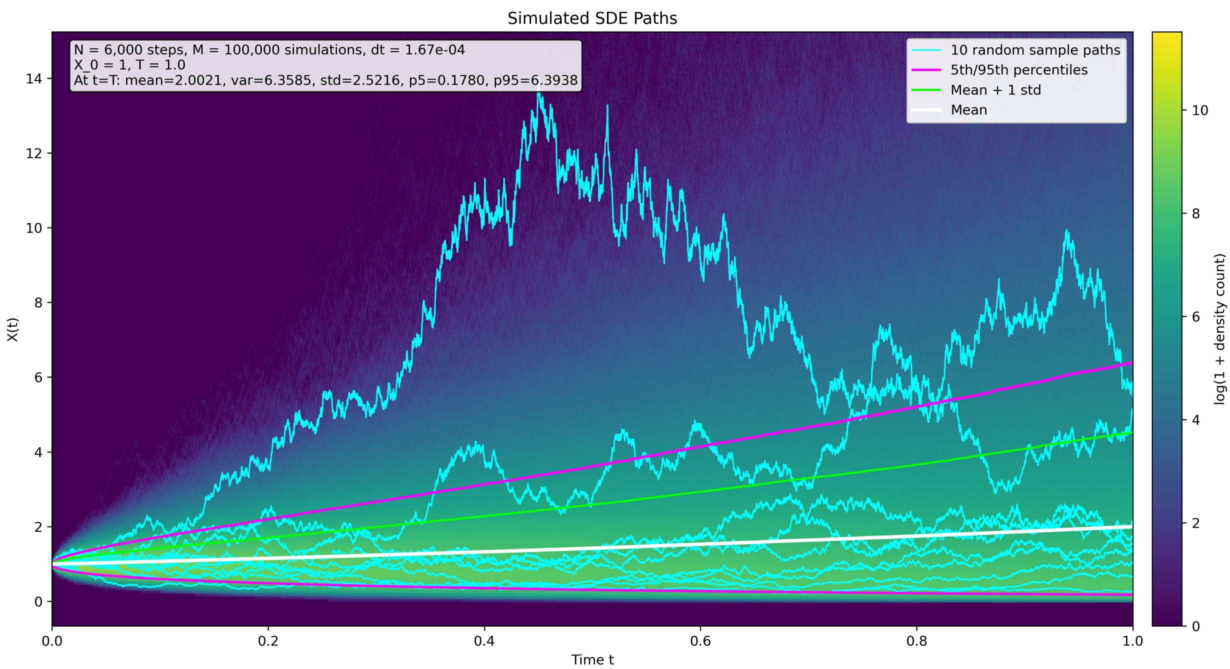

Figure 1: Simulated SDE paths with density, percentiles, and +1 standard deviation band.

□

Exercise 7.1

Find the generator of the following Itô diffusions:

a) $dX_t = \mu X_t\,dt + \sigma\,dB_t$ (The Ornstein-Uhlenbeck process) ($B_t \in \mathbb{R}$; μ, σ constants).

b) $dX_t = rX_t\,dt + \alpha X_t\,dB_t$ (The geometric Brownian motion) ($B_t \in \mathbb{R}$; r, α constants).

c) $dY_t = r\,dt + \alpha Y_t\,dB_t$ ($B_t \in \mathbb{R}$; r, α constants).

d)
$$dY_t = \begin{bmatrix} dt \\ dX_t \end{bmatrix}, \qquad \text{where } X_t \text{ is as in a).}$$

e)
$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} 1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} 0 \\ e^{X_1} \end{bmatrix} dB_t, \qquad (B_t \in \mathbb{R}).$$

f)
$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} dt + \begin{bmatrix} 1 & 0 \\ 0 & X_1 \end{bmatrix} \begin{bmatrix} dB_1 \\ dB_2 \end{bmatrix}.$$

g) $X(t) = (X_1, \ldots, X_n)$, where
$$dX_k(t) = r_k X_k\,dt + X_k \cdot \sum_{j=1}^{n} \alpha_{kj}\,dB_j, \qquad 1 \le k \le n,$$
and $(B_1, \ldots, B_n)$ is Brownian motion in $\mathbb{R}^n$, and r_k, α_{kj} are constants.

Solution. We will use Theorem 7.3.3. Let X_t be the Itô diffusion
$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t.$$
If $f \in C_0^2(\mathbb{R}^n)$, then $f \in D_A$ and the generator A is given by
$$Af(x) = \sum_i b_i(x) \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j} (\sigma\sigma^T)_{i,j}(x) \frac{\partial^2 f}{\partial x_i\,\partial x_j}.$$

a) We apply the formula from above
$$Af(x) = \mu x \frac{df}{dx} + \frac{1}{2}\sigma^2 \frac{d^2 f}{dx^2}.$$

b) We apply the formula from above
$$Af(x) = rx \frac{df}{dx} + \frac{1}{2}\alpha^2 x^2 \frac{d^2 f}{dx^2}.$$

c) We apply the formula from above
$$Af(x) = rx \frac{df}{dy} + \frac{1}{2}\alpha^2 y^2 \frac{d^2 f}{dy^2}.$$

d)
$$Af(x) = \frac{\partial f}{\partial t} + (\mu x)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2}.$$

Why is is $\frac{\partial f}{\partial t}$ not $\frac{\partial f}{\partial x}$?

e)

$$Af(x) = \frac{\partial f}{\partial x_1} + x\frac{\partial f}{\partial x_2} + \frac{1}{2}e^{2x}\frac{\partial^2 f}{\partial^2 x_2}.$$

f)

$$Af(x) = \frac{\partial f}{\partial x_1} + \frac{1}{2}\frac{\partial^2 f}{\partial^2 x_1} + \frac{1}{2}x^2\frac{\partial^2 f}{\partial^2 x_2}$$

g)

$\square$