

Independent study - Stochastic Differential Equations

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Introduction

This document collects a selection of exercises I worked through from Bernt Øksendal, *Stochastic Differential Equations: An Introduction with Applications*. These solutions were completed as part of my MATH 199 independent study with Professor Ang at UC San Diego. The document serves as a brief record of the topics and problem-solving work I carried out during the course.

Exercise 2.9

To illustrate that the (finite-dimensional) distributions alone do not give all the information regarding the continuity properties of a process, consider the following example:

Let $(\Omega, \mathcal{F}, P) = ([0, \infty), \mathcal{B}, \mu)$ where \mathcal{B} denotes the Borel σ -algebra on $[0, \infty)$ and μ is a probability measure on $[0, \infty)$ with no mass on single points. Define

$$X_t(\omega) = \begin{cases} 1 & \text{if } t = \omega, \\ 0 & \text{otherwise} \end{cases}$$

and

$$Y_t(\omega) = 0 \quad \text{for all } (t, \omega) \in [0, \infty) \times [0, \infty).$$

Prove that $\{X_t\}$ and $\{Y_t\}$ have the same distributions and that X_t is a version of Y_t . And yet we have that $t \mapsto Y_t(\omega)$ is continuous for all ω , while $t \mapsto X_t(\omega)$ is discontinuous for all ω .

Solution. By definition 2.2.2 we have that $\{X_t\}$ and $\{Y_t\}$ have the same finite-dimensional distribution if

$$\mathbb{P}(\{\omega : X_t(\omega) = Y_t(\omega)\}) = 1, \forall t$$

we then can in fact show this is true for our case so that will be how we show that they equal in distribution.

$$\mathbb{P}(\{\omega : X_t(\omega) = Y_t(\omega)\}) = 1 - \mathbb{P}(\{\omega : X_t(\omega) \neq Y_t(\omega)\})$$

Then we use that

$$\mathbb{P}(\{\omega : X_t(\omega) \neq Y_t(\omega)\}) = \mathbb{P}(\{\omega : X_t(\omega) \neq 0\}) = \mathbb{P}(\{\omega : X_t(\omega) = 1\}) = \mathbb{P}(\{t\}) = 0$$

This holds since it had no mass in a single point. This means that for any t we have that $\mathbb{P}(\{\omega : X_t(\omega) \neq Y_t(\omega)\}) = 0$. This gives that

$$\mathbb{P}(\{\omega : X_t(\omega) = Y_t(\omega)\}) = 1 - \mathbb{P}(\{\omega : X_t(\omega) \neq Y_t(\omega)\}) = 1 - 0 = 1, \forall t$$

Specifically we proved something stronger which is that $\{X_t\}$ is a version of $\{Y_t\}$. \square

Exercise 5.1

Verify that the given processes solve the given corresponding stochastic differential equations (B_t denotes 1-dimensional Brownian motion).

(i) $X_t = e^{B_t}$ solves

$$dX_t = \frac{1}{2}X_t dt + X_t dB_t.$$

(ii) $X_t = \frac{B_t}{1+t}$, $B_0 = 0$ solves

$$dX_t = -\frac{1}{1+t}X_t dt + \frac{1}{1+t}dB_t; \quad X_0 = 0.$$

(iii) $X_t = \sin(B_t)$ with $B_0 = a \in (-\frac{\pi}{2}, \frac{\pi}{2})$ solves

$$dX_t = -\frac{1}{2}X_t dt + \sqrt{1-X_t^2} dB_t \quad \text{for } t < \inf\{s > 0 : B_s \notin [-\frac{\pi}{2}, \frac{\pi}{2}]\}.$$

(iv) $(X_1(t), X_2(t)) = (t, e^t B_t)$ solves

$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} 1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} 0 \\ e^{X_1} \end{bmatrix} dB_t.$$

(v) $(X_1(t), X_2(t)) = (\cosh(B_t), \sinh(B_t))$ solves

$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} X_2 \\ X_1 \end{bmatrix} dB_t.$$

Solution. (i) We begin to observe that

$$X_t = g(t, B_t) = e^{B_t} \implies \frac{\partial g}{\partial t}(t, x) = 0, \frac{\partial g}{\partial x}(t, B_t) = e^{B_t}, \frac{\partial^2 g}{\partial x^2}(t, B_t) = e^{B_t}$$

$$\begin{aligned} dX_t &= \frac{\partial g}{\partial t}(t, B_t) dt + \frac{\partial g}{\partial x}(t, B_t) dB_t + \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(t, B_t) \cdot (dB_t)^2 = \\ &e^{B_t} dB_t + \frac{1}{2} e^{B_t} \cdot (dB_t)^2 = e^{B_t} dB_t + \frac{1}{2} e^{B_t} dt = \frac{1}{2} X_t dt + X_t dB_t \end{aligned}$$

(ii) We begin to observe that

$$\begin{aligned} X_t &= g(t, B_t) = \frac{B_t}{1+t}, B_0 = 0 \implies \frac{\partial g}{\partial t}(t, x) = \frac{-B_t}{(1+t)^2}, \frac{\partial g}{\partial x}(t, B_t) = \frac{1}{1+t}, \frac{\partial^2 g}{\partial x^2}(t, B_t) = 0 \\ dX_t &= \frac{\partial g}{\partial t}(t, B_t) dt + \frac{\partial g}{\partial x}(t, B_t) dB_t + \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(t, B_t) \cdot (dB_t)^2 = \\ &\frac{-B_t}{(1+t)^2} dt + \frac{1}{1+t} \cdot dB_t = -\frac{1}{1+t} X_t dt + \frac{1}{1+t} dB_t \end{aligned}$$

(iii) We begin to observe that

$$\begin{aligned}
X_t &= g(t, B_t) = \sin(B_t), B_0 = a \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \implies \\
\frac{\partial g}{\partial t}(t, x) &= 0, \frac{\partial g}{\partial x}(t, B_t) = \cos(B_t), \frac{\partial^2 g}{\partial x^2}(t, B_t) = -\sin(B_t) \\
dX_t &= \frac{\partial g}{\partial t}(t, B_t) dt + \frac{\partial g}{\partial x}(t, B_t) dB_t + \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(t, B_t) \cdot (dB_t)^2 = \\
\cos(B_t) \cdot dB_t - \frac{1}{2} \sin(B_t) \cdot (dB_t)^2 &= \sqrt{1 - \sin(B_t)^2} \cdot dB_t - \frac{1}{2} \sin(B_t) dt = -\frac{1}{2} X_t dt + \sqrt{1 - X_t^2} dB_t
\end{aligned}$$

Which holds for $t < \inf\{s > 0 : B_s \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\}$.

(iv) We begin to observe that

$$\begin{aligned}
(X_1(t), X_2(t)) &= (g_1(t, B_t), g_2(t, B_t)) = (t, e^t B_t) \implies \\
\frac{\partial g_1}{\partial t}(t, X) &= 1, \frac{\partial g_2}{\partial t}(t, B_t) = e^t B_t, \frac{\partial g_1}{\partial x}(t, B_t) = 0, \frac{\partial g_2}{\partial x}(t, B_t) = e^t, \\
\frac{\partial g_1}{\partial x^2}(t, B_t) &= 0, \frac{\partial g_2}{\partial x^2}(t, B_t) = 0
\end{aligned}$$

Then we get that

$$dX_1 = 1 dt, dX_2 = e^t B_t dt + e^t dB_t = X_2 dt + e^{X_1} dB_t$$

(v) We begin to observe that

$$\begin{aligned}
(X_1(t), X_2(t)) &= (g_1(t, B_t), g_2(t, B_t)) = (\cosh(B_t), \sinh(B_t)) = \left(\frac{e^{B_t} + e^{-B_t}}{2}, \frac{e^{B_t} - e^{-B_t}}{2}\right) \implies \\
\frac{\partial g_1}{\partial t}(t, X) &= 0, \frac{\partial g_2}{\partial t}(t, B_t) = 0, \frac{\partial g_1}{\partial x}(t, B_t) = \sinh(B_t), \frac{\partial g_2}{\partial x}(t, B_t) = \cosh(B_t) \\
\frac{\partial g_1}{\partial x^2}(t, B_t) &= \cosh(B_t), \frac{\partial g_2}{\partial x^2}(t, B_t) = \sinh(B_t)
\end{aligned}$$

Then we get that

$$\begin{aligned}
dX_1 &= \sinh(B_t) dB_t + \frac{1}{2} \cosh(B_t) dB_t dB_t = \sinh(B_t) dB_t + \frac{1}{2} \cosh(B_t) dt = X_2 dB_t + \frac{1}{2} X_1 dt \\
dX_2 &= \cosh(B_t) dB_t + \frac{1}{2} \sinh(B_t) dB_t dB_t = \cosh(B_t) dB_t + \frac{1}{2} \sinh(B_t) dt = X_1 dB_t + \frac{1}{2} X_2 dt
\end{aligned}$$

□

Exercise 5.4

Solve the following stochastic differential equations:

(i)

$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} dt + \begin{bmatrix} 1 & 0 \\ 0 & X_1 \end{bmatrix} \begin{bmatrix} dB_1 \\ dB_2 \end{bmatrix}.$$

(ii)

$$dX_t = X_t dt + dB_t.$$

(Hint: Multiply both sides with “the integrating factor” e^{-t} and compare with $d(e^{-t} X_t)$)

(iii)

$$dX_t = -X_t dt + e^{-t} dB_t.$$

Solution. (i) We take the integral form

$$X_1(t) = \int_0^t dX_1 = \int_0^t ds + \int_0^t dB_1 = X_1(0) + t + B_1(t)$$

Which gives us that

$$X_2(t) = \int_0^t dX_2 = \int_0^t X_1(0) + s + B_1(s) dB_2 = X_2(0) + X_1(0)B_2(t) + \int_0^t s dB_2 + \int_0^t B_1(s) dB_2$$

(ii) We take the hint and rewrite the following

$$e^{-t} dX_t = e^{-t} X_t dt + e^{-t} dB_t \implies e^{-t} dX_t - e^{-t} X_t dt = e^{-t} dB_t$$

We recognize that $d(e^{-t} X_t) = e^{-t} dX_t - e^{-t} X_t dt$ thus we get that

$$\int_0^t d(e^{-s} X_s) ds = \int_0^t e^{-s} dB_s \implies X_t = X_0 e^t + \int_0^t e^{t-s} dB_s$$

(iii)

$$dX_t = -X_t dt + e^{-t} dB_t \implies e^t dX_t + e^t X_t dt = 1 dB_t$$

We observe that $e^t dX_t + e^t X_t dt = d(e^t X_t)$

$$\int_0^t d(e^s X_s) ds = \int_0^t 1 dB_s \implies X_t = (B_t + X_0) e^{-t}$$

□

Exercise 5.5

a) Solve the *Ornstein-Uhlenbeck equation* (or *Langevin equation*)

$$dX_t = \mu X_t dt + \sigma dB_t$$

where μ, σ are real constants, $B_t \in \mathbb{R}$.

The solution is called the *Ornstein-Uhlenbeck process*. (Hint: See Exercise 5.4 (ii).)

b) Find $\mathbb{E}[X_t]$ and $\text{Var}[X_t] := \mathbb{E}[(X_t - \mathbb{E}[X_t])^2]$.

Solution.

$$e^{\mu t} dX_t = \mu e^{\mu t} X_t dt + \sigma e^{\mu t} dB_t \implies d(X_t e^{\mu t}) = \sigma e^{\mu t} dB_t$$

Thus we integrate again

$$\begin{aligned} \int_0^t d(X_t e^{-\mu s}) ds &= X_t e^{\mu t} - X_0 = \int_0^t \sigma e^{-\mu s} dB_s \implies \\ X_t &= X_0 e^{\mu t} + \int_0^t \sigma e^{-\mu(s-t)} dB_s \end{aligned}$$

Find $\mathbb{E}[X_t]$ and $\text{Var}[X_t] := \mathbb{E}[(X_t - \mathbb{E}[X_t])^2]$.

$$\mathbb{E}[X_t] = \mathbb{E}[X_0 e^{\mu t} + \int_0^t \sigma e^{-\mu(s-t)} dB_s] = \mathbb{E}[X_0] e^{\mu t} + \sigma \mathbb{E}\left[\int_0^t e^{-\mu(s-t)} dB_s\right] = \mathbb{E}[X_0] e^{\mu t}$$

Then we want to find

$$\text{Var}[X_t] := \mathbb{E}[(X_t - \mathbb{E}[X_t])^2] = \mathbb{E}[X_t^2] - (\mathbb{E}[X_t])^2$$

Thus we solve the

$$\begin{aligned} \mathbb{E}[X_t^2] &= \mathbb{E}\left[(X_0 e^{\mu t} + \int_0^t \sigma e^{-\mu(s-t)} dB_s)^2\right] = \\ \mathbb{E}[X_0^2] e^{2\mu t} &+ 2\mathbb{E}\left[X_0 e^{\mu t} \int_0^t \sigma e^{-\mu(s-t)} dB_s\right] + \mathbb{E}\left[\left(\int_0^t \sigma e^{-\mu(s-t)} dB_s\right)^2\right] = \\ \mathbb{E}[X_0^2] e^{2\mu t} &+ 2\mathbb{E}\left[X_0 e^{\mu t} \int_0^t \sigma e^{-\mu(s-t)} dB_s\right] + \mathbb{E}\left[\int_0^t (\sigma e^{-\mu(s-t)})^2 dt\right] = \\ \mathbb{E}[X_0^2] e^{2\mu t} &+ 2\mathbb{E}\left[X_0 e^{\mu t} \int_0^t \sigma e^{-\mu(s-t)} dB_s\right] + \frac{\sigma^2}{2\mu}(e^{2\mu t} - 1) \end{aligned}$$

We know analyze this

$$\begin{aligned} \mathbb{E}\left[X_0 e^{\mu t} \int_0^t \sigma e^{-\mu(s-t)} dB_s\right] &= \mathbb{E}\left[\mathbb{E}\left[X_0 e^{\mu t} \int_0^t \sigma e^{-\mu(s-t)} dB_s \mid \mathcal{F}_0\right]\right] = \\ \mathbb{E}\left[X_0 \mathbb{E}\left[e^{\mu t} \int_0^t \sigma e^{-\mu(s-t)} dB_s \mid \mathcal{F}_0\right]\right] &= \mathbb{E}\left[X_0 \cdot 0\right] = 0 \end{aligned}$$

Thus we get that

$$\text{Var}[X_t] = (\mathbb{E}[X_0^2] - \mathbb{E}[X_0]^2)e^{2\mu t} + \frac{\sigma^2}{2\mu}(e^{2\mu t} - 1) = \text{Var}[X_0 e^{\mu t}] + \frac{\sigma^2}{2\mu}(e^{2\mu t} - 1)$$

□

Exercise 5.6

Solve the stochastic differential equation

$$dY_t = r dt + \alpha Y_t dB_t$$

where r, α are real constants, $B_t \in \mathbb{R}$. (Hint: Multiply the equation by the ‘integrating factor’)

$$F_t = \exp\left(-\alpha B_t + \frac{1}{2}\alpha^2 t\right)$$

Solution.

$$\begin{aligned} \exp\left(-\alpha B_t + \frac{1}{2}\alpha^2 t\right) dY_t &= \exp\left(-\alpha B_t + \frac{1}{2}\alpha^2 t\right) r dt + \exp\left(-\alpha B_t + \frac{1}{2}\alpha^2 t\right) \alpha Y_t dB_t \implies \\ d(\exp\left(-\alpha B_t + \frac{1}{2}\alpha^2 t\right) Y_t) &= \exp\left(-\alpha B_t + \frac{1}{2}\alpha^2 t\right) r dt \implies \\ \exp\left(-\alpha B_t + \frac{1}{2}\alpha^2 t\right) Y_t &= \exp\left(\alpha B_t - \frac{1}{2}\alpha^2 t\right) (Y_0 + \int_0^t \exp\left(-\alpha B_s + \frac{1}{2}\alpha^2 s\right) r ds) \end{aligned}$$

□

Exercise 5.9

Show that there is a unique strong solution X_t of the 1-dimensional stochastic differential equation

$$dX_t = \ln(1 + X_t^2) dt + \mathbb{1}_{\{X_t > 0\}} X_t dB_t, \quad X_0 = a \in \mathbb{R}.$$

Solution. We then have to check that the functions described satisfy Theorem 5.2.1. Where

$$\sigma(t, x) = \begin{cases} x, & x > 0 \\ 0, & x \leq 0 \end{cases} \quad b(t, x) = \ln(1 + x^2)$$

It then has to satisfy the two stated conditions below

$$\exists C \text{ s.t } |\sigma(t, x)| + |b(t, x)| \leq C(1 + |x|), \forall x \in \mathbb{R}, t \in [0, T]$$

$$\exists D \text{ s.t } |b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq D|x - y|, \forall x, y \in \mathbb{R}, t \in [0, T]$$

We observe that for σ we have that

$$|\sigma(t, x)| \leq |x|, \sigma(t, x) - \sigma(t, y) = \begin{cases} x - y, & x > 0 \text{ and } y > 0 \\ x, & x > 0 \text{ and } y \leq 0 \\ -y, & x \leq 0 \text{ and } y > 0 \end{cases} \implies |\sigma(t, x) - \sigma(t, y)| \leq |x - y|$$

Then for b we have that

$$|b(t, x)| \leq |x| \text{ since } \forall t \geq 0, f(t) = t - \ln(1 + t^2) \implies f'(t) = 1 - \frac{2t}{1 + t^2} = \frac{(t+1)^2}{1+t^2} \geq 0, f(0) = 0$$

We will use mean value theorem and observe that $\frac{d}{dx} \ln(1 + x^2) = \frac{2x}{1+x^2}$ where $|\frac{2x}{1+x^2}| \leq 1 \implies$

$$|\ln(1 + x^2) - \ln(1 + y^2)| = |f'(c)(x - y)| \leq 1|x - y|$$

Which means that we can take $C = D = 2$ so we can conclude by the Theorem 5.2.1 that the solution is adapted to $\mathcal{F}_t^Z \implies$ strong solution.

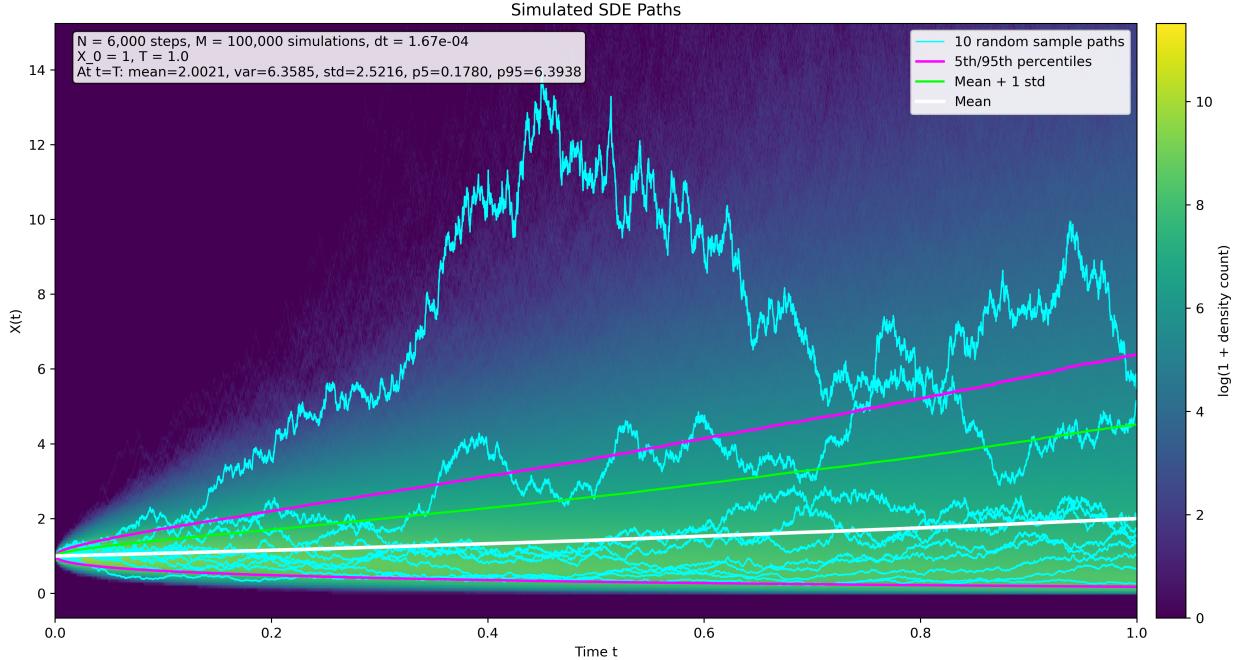


Figure 1: Simulated SDE paths with density, percentiles, and $+1$ standard deviation band.

□