Calculus, Part 2: Foundations

Chapter Summaries

Chapter 3: Functions

The chapter begins with a provisional definition of a function as: a rule which assigns, to certain real numbers, some other real number. A function is any rule, not just one that can be expressed by an algebraic formula...nor is it necessarily a rule can be applied in practice."

The set of numbers to which a function applies is the **domain**.

Polynomial functions are introduced, and the degree is defined as the nonzero coefficient of the highest power.

Rational functions are defined as quotients of polynomial functions.

Elementary operations (+, -, *, /) are defined over functions. The domain of a function built by elementary operations from other functions is the intersection of the domains of all the functions. There is a caveat when '/' is used that for example if h = f/g, then g $\neq$ 0.

Associativity, and commutativity over addition and multiplication of functions are shown to be easy to prove.

Function composition is introduced. Note that composition is not generally commutative, although it is associative.

A more precise definition for a function is given: A **function** is a collection of pairs of numbers such that if (a, b) and (a, c) are both in the collection, then b=c.

Similarly a more precise definition is given for domain: The domain of f is the set of all a for which there is some b such that (a, b) is in f.

Chapter 4: Graphs

Spivak notes: "[the] method of 'drawing' numbers is intended solely as a method of picturing certain abstract ideas, and our proofs will never rely on these pictures."

The set $x : a < x < b$ is denoted by (a, b) and called the open interval from a to b. The set $x : a \leq x \leq b$ is denoted by $[a, b]$ and called the closed interval from a to b.

Distance between points (a, b) and (c, d) is defined $\sqrt{(a-c)^2 + (b-d)^2}$.

Functions of the form $f(x) = cx + d$ are called linear functions, and functions of the form $f(x) = x^n$ are called power functions.

It is noted that a polynomial function of degree n will have at most n-1 local minima and maxima, though it may have fewer. Spivak notes that "Although these assertions are easy to make, we will not even contemplate giving proofs until Part 3 (once the powerful methods of part 3 are available)."

Various graphs are shown to demonstrate that some functions are impossible to accurately convey in a drawing.

Some graphs are discussed which are not functions, namely the circle. A circle with center (a, b) and radius $r > 0$ contains all points (x, y) with $\sqrt{(x-a)^2 + (y-b)^2} = r$. Also discussed is the ellipse which is defined as the set of points the sum of whose distances from 2 foci is a constant, and the hyperbola is defined analogously, except that we require the difference of the 2 distances to be a constant. Though these are not functions, they can be reduced to the functions of which they are composed.

Spivak concludes by noting: "A mathematical definition of this concept [the reasonableness of a function] is by no means easy, and a great deal of this book may be viewed as successive attempts to impose more and more conditions that a 'reasonable' function must sastisfy."